Acta Mathematicae Applicatae Sinica, English Series
Vol. 26, No. 3 (2010) 387–394
DOI: 10.1007/s10255-010-0005-2
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematica Applicatae Sinica, English Series

Multiplicity of Solutions for a Class of Kirchhoff Type Problems

Xiao-ming He[1], **Wen-ming Zou**[2]

[1]School of Science, Minzu University of China, Beijing 100081, China (E-mail: xmhe923@126.com)
[2]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
(E-mail:wzou@math.tsinghua.edu.cn)

Abstract In this paper we apply the (variant) fountain theorems to study the symmetric nonlinear Kirchhoff nonlocal problems. Under the Ambrosetti-Rabinowitz's 4-superlinearity condition, or no Ambrosetti-Rabinowitz's 4-superlinearity condition, we present two results of existence of infinitely many large energy solutions, respectively.

Keywords Kirchhoff nonlocal problems, Multiple solutions, Fountain theorems
2000 MR Subject Classification 35J60; 35J25

1 Introduction

In this paper we consider the problem

$$\begin{cases} -\Big(a+b\displaystyle\int_\Omega |\nabla u|^2\Big)\Delta u = f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N=1,2$ or 3), $a,b>0$ and $f:\ \Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function, with $F(x,u)=\int_0^u f(x,s)ds$, $x\in\Omega$.

Problem (1) is related to the stationary analogue of the equations

$$u_{tt}-\Big(a+b\int_\Omega |\nabla u|^2\Big)\Delta u = g(x,u), \tag{2}$$

proposed by Kirchhoff[11] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. We refer to [6, 16] for early classical studies. Equation (2) received much attention only after Lions[12] proposed an abstract framework to the problem. Some important and interesting results can be found in [3, 5, 7], for example. More recently, Alves et al.[1] and Ma and Rivera[13] obtained positive solutions of such problems by variational methods. Similar nonlocal problems also model several physical and biological systems where u describes a process which depends on the average of itself, for example the population density, see [4, 8, 9].

Recently, problems like type (1) have been investigated by several authors. Perrero and Zhang[15] obtained a nontrivial solution of (1) by using the Yang index and critical group. In [18] they revisited (1) via invariant sets of descent flow and obtained the existence of a positive

Manuscript received July 22, 2008. Revised November 6, 2008.

solution, a negative solution, and a sign changing solutions. In [14] Mao and Zhang obtained a similar result as [18].

When $b = 0$, the existence of infinitely many solutions of (1) has been studied extensively. Most results are obtained by applying some variant of the classical Lusternik-Schnirelmann theory, the symmetric mountain pass theorem, or the fountain theorem, see [2, 17], for example. The purpose of this paper is to prove the existence of infinitely many weak solutions for (1). Our approach is based on the classical fountain theorem of [17] and a recent variant fountain theorem due to [19]. To our best knowledge, these methods have not been used to search infinitely many solutions of (1) before.

Before stating our main results, we first introduce some preliminary nations. Let E be a Banach space with the norm $||\cdot||$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=0}^{k} X_j$; $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$ and

$$B_k = \{u \in Y_k : \ ||u|| \le \rho_k\}, \quad N_k = \{u \in Z_k : \ ||u|| = \gamma_k\} \ \text{ for } \rho_k > \gamma_k > 0.$$

Theorem 1 ([17], Fountain theorem). *Let $\varphi \in C^1(E,\mathbb{R})$ be a even functional. If, for every $k \in \mathbb{N}$, there exist $\rho_k > \gamma_k > 0$ such that*

(A_1) $a_k = \max_{u\in Y_k, ||u||=\rho_k} \varphi(u) \le 0,$

(A_2) $b_k = \inf_{u\in Z_k, ||u||=\gamma_k} \varphi(u) \to \infty, \ k \to \infty,$

(A_3) φ satisfies the $(PS)_c$ condition for every $c > 0$,

then φ has an unbounded sequence of critical values.

Now we consider the C^1-functional $\Phi_\lambda : \ E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \qquad \lambda \in [1, 2].$$

We assume that

(B_1) Φ_λ maps bounded sets into bounded sets uniformly for $\lambda \in [1, 2]$. Furthermore, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$.

(B_2) $B(u) \ge 0$ for all $u \in E$; $A(u) \to \infty$ or $B(u) \to \infty$ as $||u|| \to \infty$; or

(B_3) $B(u) \le 0$ for all $u \in E$; $B(u) \to -\infty$ as $||u|| \to \infty$.

For $k \ge 2$, let

$$\begin{aligned}
\Gamma_k &:= \{\gamma \in C(B_k, E) : \ \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\},\\
c_k(\lambda) &:= \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda(\gamma(u)),\\
b_k(\lambda) &:= \inf_{u\in Z_k, ||u||=\gamma_k} \Phi_\lambda(u),\\
a_k(\lambda) &:= \max_{u\in Y_k, ||u||=\rho_k} \Phi_\lambda(u).
\end{aligned}$$

Now we recall the following variant fountain theorem (see [19, 20], Theorem 2.1).

Theorem 2 ([19], Variant fountain theorem). *Assume (B_1) and (B_2) (or (B_3)). If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1, 2]$, then $c_k(\lambda) \ge b_k(\lambda)$ for all $\lambda \in [1, 2]$. Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that*

$$\sup_n ||u_n^k(\lambda)|| < \infty, \ \Phi'_\lambda(u_n^k(\lambda)) \to 0 \quad \textit{and} \quad \Phi_\lambda(u_n^k(\lambda)) \ \to c_k(\lambda), \qquad \textit{as } n \to \infty.$$

In this paper we consider $E = H_0^1(\Omega)$ endowed with the norm $||u|| := \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}$. In the sequel by $|\cdot|_q$ we denote the usual L^q-norm. Since Ω is a bounded domain, it is well known that $E \hookrightarrow L^q(\Omega)$ continuously for $q \in [1, 2^*]$, and compactly for $q \in [1, 2^*)$. Moreover, there exists $c_q > 0$ such that

$$|u|_q \leq c_q ||u||, \qquad \forall u \in E.$$

Denote by $0 < \mu_1 < \mu_2 < \cdots$ the distinct eigenvalues of $-\Delta$ in $L^2(\Omega)$ with zero Dirichlet boundary conditions, by $e_1, e_2, e_3, \cdots$ the eigenfunctions corresponding to eigenvalues, respectively.

We recall that a weak solution of (1) is any $u \in E$ such that

$$(a + b||u||^2) \int_\Omega \nabla u \nabla v = \int_\Omega f(x, u)v, \qquad \forall v \in E,$$

and the weak solutions of (1) are precisely the critical points of the functional

$$\Phi(u) = \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - \int_\Omega F(x, u), \qquad \in E.$$

From now on, the letter c will be repeatedly used to denote various positive constants whose exact value is irrelevant. The main results of this paper are the following:

Theorem 3. *Assume that*

(F_1) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ *and for some* $4 < p < 2^* = \begin{cases} \frac{2n}{n-2}, & n \geq 3, \\ \infty, & n = 1, 2, \end{cases}$

$$|f(x, u)| \leq c(1 + |u|^{p-1}),$$

(F_2) there exists $\mu > 4$ *and* $R > 0$ *such that*

$$|u| \geq R \Rightarrow 0 < \mu F(x, u) \leq u f(x, u),$$

(F_3) $f(x, -u) = -f(x, u), \ \forall x \in \Omega, \ \forall u \in \mathbb{R}$.

Then problem (1) has a sequence of solutions $\{u_k\}$ *such that* $\Phi(u_k) \to \infty$ *as* $k \to \infty$.

Theorem 4. *In addition to conditions (F_1), (F_3), suppose that the following conditions are satisfied:*

(G_1) $f(x, u)u \geq 0$ *for* $u > 0$*; and* $\lim \inf\limits_{|u| \to \infty} \frac{F(x,u)}{|u|^4} \to \infty$ *uniformly for* $x \in \Omega$,

(G_2) $f(x, u) = o(|u|)$ *as* $|u| \to \infty$ *uniformly in* $x \in \Omega$; $\frac{f(x,u)}{u^3}$ *is an increasing function of* $u \geq 0$ *for every* $x \in \Omega$.

(G_3) $\widetilde{F}(x, u) := \frac{1}{4} f(x, u)u - F(x, u) \to \infty$ *as* $|u| \to \infty$ *uniformly in* $x \in \Omega$.

Then problem (1) has infinitely many solutions $\{u_n\}$ *satisfying* $\Phi_1(u_n) \to \infty$.

Remark 1. In theorem 3 we adopt the well known Ambrosetti-Rabinowitz type 4-superlinear condition (F_2) to guarantee the boundedness of all (PS) sequences of the corresponding functional Φ. However, there are many functions which are 4-superlinear but it is not possible to satisfy (F_2) for any $\mu > 4$, so, without (F_2), it becomes more complicated. In Theorem 4 we use Theorem 2 without (PS)-type assumption, to obtain infinitely many solutions of (1) under no Ambrosetti-Rabinowitz type 4-superlinear condition (F_2). An example is given at the end of the proofs.

2 Proofs of the Main Results

Proof of Theorem 3. It is evident that $\Phi(u) \in C^1(X, \mathbb{R})$. After integrating, we obtain from (F_2) that

$$c(|u|^\mu - 1) \le F(x,u). \tag{3}$$

Let $\beta \in (\mu^{-1}, 4^{-1})$ and $\{u_n\} \subset E$ be a $(PS)_c$-sequence of $\Phi(u)$. Then for n large enough, we have,

$$\begin{aligned}
c + 1 + ||u_n|| \ge & \Phi(u_n) - \beta\Phi'(u_n)u_n \\
= & a\Big(\frac{1}{2} - \beta\Big)||u_n||^2 + b\Big(\frac{1}{4} - \beta\Big)||u_n||^4 + \int_\Omega (\beta f(x,u_n)u_n - F(x,u_n)) \\
\ge & a\Big(\frac{1}{2} - \beta\Big)||u_n||^2 + b\Big(\frac{1}{4} - \beta\Big)||u_n||^4 + (\beta\mu - 1)\int_\Omega F(x,u_n) - c \\
\ge & a\Big(\frac{1}{2} - \beta\Big)||u_n||^2 + b\Big(\frac{1}{4} - \beta\Big)||u_n||^4 + c(\beta\mu - 1)|u_n|_\mu^\mu - c \\
\ge & a\Big(\frac{1}{2} - \beta\Big)(||z_n||^2 + \lambda_1|y_n|_2^2) + b\Big(\frac{1}{4} - \beta\Big)(||z_n||^2 + \lambda_1|y_n|_2^2)^2 \\
& + c(\beta\mu - 1)|u_n|_\mu^\mu - c,
\end{aligned}$$

where $u_n = y_n + z_n, y_n \in Y_n, z_n \in Z_n$. It is easy to verify that $\{u_n\}$ is bounded in E using the fact that $\dim Y_n$ is finite.

Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. By the Rellich theorem, $u_n \to u$ in $L^p(\Omega)$. Then Theorem A.2[17] implies that $f(x,u_n) \to f(x,u)$ in $L^q(\Omega)$ with $q = p/(p-1)$. We next prove that $\{u_n\}$ has a convergent subsequence. Observe that

$$\begin{aligned}
& \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \\
= & a\int_\Omega |\nabla(u_n - u)|^2 + b||u_n||^2 \int_\Omega \nabla u_n \nabla(u_n - u) \\
& + b||u||^2 \int_\Omega \nabla u \nabla(u - u_n) - \int_\Omega (f(x,u_n) - f(x,u))(u_n - u) \\
= & a||u_n - u||^2 - \int_\Omega (f(x,u_n) - f(x,u))(u_n - u) + b||u_n||^2 \int_\Omega \nabla u_n \nabla(u_n - u) \\
& + b||u||^2 \int_\Omega \nabla u \nabla(u - u_n) + b||u_n||^2 \int_\Omega \nabla u \nabla(u - u_n) - b||u_n||^2 \int_\Omega \nabla u \nabla(u - u_n) \\
= & a||u_n - u||^2 - \int_\Omega (f(x,u_n) - f(x,u))(u_n - u) \\
& + b||u_n||^2 \cdot ||u_n - u||^2 + b(||u_n||^2 - ||u||^2)\int_\Omega \nabla u \cdot \nabla(u_n - u) \\
= & (a + b||u_n||^2)||u_n - u||^2 + b(||u_n||^2 - ||u||^2)\int_\Omega \nabla u \cdot \nabla(u_n - u) \\
& - \int_\Omega (f(x,u_n) - f(x,u))(u_n - u).
\end{aligned} \tag{4}$$

It is clear that the left hand side of (4) and the middle term of the right hand side of (4) tend to zero as $n \to \infty$. It follows from the Höld inequality that

$$\int_\Omega |f(x,u_n) - f(x,u))(u_n - u)| \leq |f(x,u_n - f(x,u)|_q |u_n - u|_p \to 0,$$

as $n \to \infty$. Thus we deduce that $||u_n - u|| \to 0$, as $n \to \infty$. Hence condition (A_3) is satisfied.

By (3), we have

$$\Phi(u) \leq \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - c|u|_\mu^\mu + c.$$

Since on the finite-dimensional space Y_k all norms are equivalent, therefore, (A_1) is satisfied for every $\rho_k > 0$ large enough.

We next verify condition (A_2). To this end, from (F_1), we have

$$|F(x,u)| \leq c(1 + |u|^p).$$

Define

$$\beta_k := \sup_{u \in Z_k, ||u||=1} |u|_p, \tag{5}$$

so that on Z_k, we have

$$\Phi(u) \geq \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - c|u|_p^p - c \geq \sqrt{\frac{ab}{2}}||u||^3 - c\beta_k^p||u||^p - c.$$

Take $\gamma_k := \Big(\frac{cp\beta_k^p}{\sqrt{(ab)/2}}\Big)^{\frac{1}{3-p}}$. Since $\beta_k \to 0, k \to \infty$ ([17]), we obtain, for $u \in Z_k$ with $||u|| = \gamma_k$,

$$\Phi(u) \geq \sqrt{\frac{ab}{2}}\Big(1 - \frac{1}{p}\Big)\Big(\frac{cp\beta_k^p}{\sqrt{(ab)/2}}\Big)^{\frac{3}{3-p}} - c \to \infty$$

as $k \to \infty$. So, condition (A_2) is proved. Now all conditions of Theorem 1 hold, therefore, problem (1) has a sequence of solutions $\{u_k\}$ such that $\Phi(u_k) \to \infty,\ k \to \infty$. □

Proof of Theorem 4. We consider the functional

$$\Phi_\lambda(u) = \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - \lambda \int_\Omega F(x,u) := A(u) - \lambda B(u),$$

for $\lambda \in [1,2]$. Then $B(u) \geq 0, A(u) \to \infty$ as $||u|| \to \infty$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $\lambda \in [1,2], u \in E$.

Claim 1. There exists $\lambda_n \to 1$ as $n \to \infty$, $c_k^* > b_k^* > 0$ and $\{z_n\}_{n=1}^\infty \subset E$ such that

$$\Phi'_{\lambda_n}(z_n) = 0, \qquad \Phi_{\lambda_n}(z_n) \in [b_k^*, c_k^*].$$

By conditions (G_1), for any $L > 0$, there exists a constant C_L such that $F(x,u) \geq L|u|^4 - C_L$ for all $u \in \mathbb{R}$.

Let $c_{k,r}$ be such that $|u|_r \geq c_{k,r}||u||, \ \ \forall u \in Y_k$. Therefore, for $u \in Y_k$,

$$\begin{aligned}\Phi_\lambda(u) \leq& \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - \lambda L \int_\Omega |u|^4 + \lambda \int_\Omega C_L \\ \leq& \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - L(c_{k,4})^4||u||^4 + c \to -\infty\end{aligned}$$

for $\rho_k > 0$ large enough, provided $L(c_{k,4})^4 > \frac{b}{4}$. Hence we have

$$a_k(\lambda) = \max_{u \in Y_k, ||u||=\rho_k} \Phi_\lambda(u) \leq 0$$

uniformly for $\lambda \in [1,2]$ if $\rho_k > 0$ large enough.

On the other hand, by (F_1), (G_2), we have, for any $\varepsilon > 0$, there exists $D_\varepsilon > 0$, such that

$$|f(x,u)| \leq D_\varepsilon |u|^{p-1} + \varepsilon |u|, \qquad \forall x \in \Omega, \quad \forall u \in \mathbb{R}.$$

Let β_k be defined as (5). Hence, for each $u \in Z_k$ and $\varepsilon > 0$ small enough, one has

$$\begin{aligned}\Phi_\lambda(u) =& \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - \lambda \int_\Omega F(x,u) \\ \geq& \frac{a}{2}||u||^2 + \frac{b}{4}||u||^4 - \frac{\lambda\varepsilon}{2}|u|_2^2 - \frac{\lambda D_\varepsilon}{p}|u|_p^p \\ \geq& \frac{a}{4}||u||^2 + \frac{b}{4}||u||^4 - c|u|_p^p \\ \geq& \frac{a}{4}||u||^2 - c\beta_k^p ||u||^p.\end{aligned}$$

We choose $\gamma_k = \left(\frac{4cp\beta_k^p}{a}\right)^{\frac{1}{2-p}}$, then for $u \in Z_k$ with $||u|| = \gamma_k$,

$$\Phi_\lambda(u) \geq \Big(\frac{4cp\beta_k^p}{a}\Big)^{\frac{2}{2-p}} \Big(\frac{a}{4} - \frac{a}{4p}\Big) := b_k^*,$$

which means that $b_k(\lambda) = \inf_{u \in Z_k, ||u|| = \gamma_k} \Phi_\lambda(u) \geq b_k^* \to \infty$ as $k \to \infty$. So, by Theorem 2, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_k^n(\lambda)\}_{n=1}^\infty$ such that

$$\sup_n ||u_k^n(\lambda)|| < \infty, \qquad \Phi'_\lambda(u_k^n(\lambda)) \to 0$$

and

$$\Phi_\lambda(u_k^n(\lambda)) \to c_k(\lambda) \geq b_k(\lambda) \geq b_k^*$$

as $n \to \infty$. Moreover, since $c_k(\lambda) \leq \sup_{u \in B_k} \Phi_\lambda(u) := c_k^*$, and $H_0^1(\Omega)$ is imbedded compactly into $L^r(\Omega)$ for $2 \leq r < 2^*$. By standard argument, $\{u_k^n(\lambda)\}_{n=1}^\infty$ has a convergent subsequence. Hence, there exists $z^k(\lambda)$ such that $\Phi'_\lambda(z^k(\lambda)) = 0$ and $\Phi_\lambda(z^k(\lambda)) \in [b_k^*, c_k^*]$. Evidently, we can find $\lambda_n \to 1$ and $\{z_n\}$ desired as the claim.

Claim 2. $\{z_n\}_{n=1}^\infty$ is bounded in E.

Assume by contradiction that $||z_n|| \to \infty$ as $n \to \infty$. Put $w_n = \frac{z_n}{||z_n||}$. Then up to a subsequence, we get that

$$\begin{aligned}&w_n \rightharpoonup w \text{ in } H_0^1(\Omega),\\ &w_n \to w \text{ in } L^t(\Omega) \text{ for } 2 \leq t < 2^*,\\ &w_n \to w \text{ a.e. } x \in \Omega.\end{aligned}$$

There are two possible cases: (i) $w \neq 0$ in E; (ii) $w \equiv 0$ in E.

In Case (i), it follows from $\Phi'_{\lambda_n}(z_n) = 0$ that

$$\int_\Omega \frac{f(x,z_n)z_n}{||z_n||^4} \leq c.$$

On the other hand, by Fatou's lemma and conditions (G_1), (G_3), we obtain

$$\int_\Omega \frac{f(x,z_n)z_n}{||z_n||^4} = \int_{\{w_n(x) \neq 0\}} |w_n|^4 \frac{f(x,z_n)z_n}{|z_n|^4} \to \infty,$$

which is a contradiction.

In Case (ii), we can define as in [10] that

$$\Phi_{\lambda_n}(t_n z_n) := \max_{t\in[0,1]} \Phi_{\lambda_n}(tz_n).$$

Let $w_n^* := (4c)^{\frac{1}{2}} w_n$ with $c > 0$, we have for n large enough, that

$$\Phi_{\lambda_n}(t_n z_n) \geq \Phi_{\lambda_n}(w_n^*) = 2ac + 4bc^2 - \lambda_n \int_\Omega F(x, w_n^*) \geq 2ac,$$

which implies that $\lim_{n\to\infty} \Phi_{\lambda_n}(t_n z_n) = \infty$, since $c > 0$ can be large arbitrarily. Obviously, $t_n \in (0,1)$, hence $\langle \Phi'_{\lambda_n}(t_n z_n), t_n z_n\rangle = 0$. It follows from

$$\begin{aligned}\Phi_{\lambda_n}(t_n z_n) =& \Phi_{\lambda_n}(t_n z_n) - \frac{1}{4}\langle \Phi'_{\lambda_n}(t_n z_n), t_n z_n\rangle \\ =& \frac{a}{4}||t_n z_n||^2 + \lambda_n \int_\Omega \Big(\frac{1}{4} f(x, t_n z_n) t_n z_n - F(x, t_n z_n)\Big)\end{aligned}$$

and $\Phi_{\lambda_n}(0) = 0$, that $|t_n z_n|$ must tend to ∞ as $n \to \infty$. Therefore, by (G_3), we have

$$\lambda_n \int_\Omega \Big(\frac{1}{4} f(x, t_n z_n) t_n z_n - F(x, t_n z_n)\Big) \to \infty.$$

On the other hand, we note that $g(t) = \frac{1}{4} t^4 f(x,s)s - F(x,ts)$ is increasing in $t \in (0,1)$, hence, $\frac{1}{4} f(x,s)s - F(x,s)$ is increasing in $s > 0$ since

$$\frac{d}{ds}\Big(\frac{1}{4} f(x,s)s - F(x,s)\Big) = \frac{s^4}{4} \times \frac{d}{ds}\Big(\frac{f(x,s)}{s^3}\Big)$$

and $\frac{f(x,s)}{s^3}$ is increasing for $s > 0$. Now in view of the oddness of f, we get

$$\lambda_n \int_\Omega \Big(\frac{1}{4} f(x, z_n) z_n - F(x, z_n)\Big) \geq \lambda_n \int_\Omega \Big(\frac{1}{4} f(x, t_n z_n) t_n z_n - F(x, t_n z_n)\Big) \to \infty.$$

This is a contradiction since

$$\lambda_n \int_\Omega \widetilde{F}(x, z_n) = \Phi_{\lambda_n}(z_n) - \frac{1}{4}\langle \Phi'_{\lambda_n}(z_n), z_n\rangle - \frac{a}{4}||z_n||^2 \leq \Phi_{\lambda_n}(z_n) \in [b_k^*, c_k^*].$$

At this point, we have obtained a solution z^k such that $\Phi'(z^k) = 0$ and $\Phi(z^k) \in [b_k^*, c_k^*]$. Since $b_k^* \to \infty$ as $k \to \infty$, we may obtain a sequence of solutions $\{z^k\}_{k=1}^\infty$ of problem (1) with $\Phi_1(z^k) \to \infty,\ k \to \infty$. □

Remark 2. Comparing Theorem 4 to Theorem 3.1[19], it is clear that condition (G_1) is weaker than condition (S_2) of Theorem 3.1[19].

Finally, we present an example to illustrate that there is a nonlinear f which satisfies all the conditions of Theorem 4, but does not satisfy the conditions of Theorem 3, especially condition (F_2).

Example 1. Let $f(x,u) = u^3(4 + \ln(1+|u|))$. Simple computation yields that

$$F(x,-u) = F(x,u) = u^4 + \frac{1}{4}u^4 \ln(1+|u|) - \frac{1}{4}\Big(\frac{1}{4}u^4 - \frac{1}{3}|u|^3 + \frac{1}{2}u^2 - |u| + \ln(1+|u|)\Big)$$

and

$$\widetilde{F}(x,u) = \frac{1}{4}\Big(\frac{1}{4}u^4 - \frac{1}{3}|u|^3 + \frac{1}{2}u^2 - |u| + \ln(1+|u|)\Big) \to \infty$$

as $|u| \to \infty$. Therefore, it is easy to see that f satisfies conditions (G_1)–(G_3).

But f does not satisfy (F_2). In fact, assume that there is some $\mu > 4$ such that $\mu F(x,u) \le f(x,u)u$ for $|u|$ large. Then

$$\begin{aligned}\mu F(x,u) =&\mu u^4 + \frac{\mu}{4}u^4\ln(1+|u|) - \frac{\mu}{4}\Big(\frac{1}{4}u^4 - \frac{1}{3}|u|^3 + \frac{1}{2}u^2 - |u| + \ln(1+|u|)\Big)\\ =&\mu u^4 + u^4\ln(1+u|)\Big(\frac{\mu}{4} - \frac{\mu}{16\ln(1+|u|)}\Big) + \frac{\mu}{4}\Big(\frac{1}{3}|u|^3 - \frac{1}{2}u^2 + |u| - \ln(1+|u|)\Big)\\ \le& 4u^4 + u^4\ln(1+|u|)\\ =&uf(x,u),\end{aligned}$$

for $|u|$ large, which is impossible in view of $\mu > 4$.

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