Acta Mathematicae Applicatae Sinica, English Series
Vol. 25, No. 2 (2009) 335–344
DOI: 10.1007/s10255-007-7065-y
www.Applmath.com.cn

Nonconforming H^1-Galerkin Mixed FEM for Sobolev Equations on Anisotropic Meshes

Dong-yang Shi[1], Hai-hong Wang[2]

Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, China
(E-mail: [1]shi_dy@zzu.edu.cn, [2]whh777@sina.com.cn)

Abstract A nonconforming H^1-Galerkin mixed finite element method is analyzed for Sobolev equations on anisotropic meshes. The error estimates are obtained without using Ritz-Volterra projection.

Keywords Nonconforming H^1-Galerkin mixed finite element method, Sobolev equations, anisotropic meshes, error estimates

2000 MR Subject Classification 65N30, 35Q10

1 Introduction

It is well known that Sobolev equations have been found applications in many physical problems, such as the porous theories concerned with percolation into rocks with cracks[4], the transport problems of humidity in soil[16], the heat-conduction problems[19] in different mediums and so on.

However, all of the above studies using finite element methods rely on the regularity assumption or the quasi-uniform assumption[4]. i.e., $\frac{h_K}{\rho_K} \leq C_1$, $\frac{h}{h_K} \leq C_2$, where K is an element, h_K, ρ_K denote the diameter of K and the biggest circle contained in K, respectively. C_1 and C_2 are two constants independent of $h = \max\limits_K\{h_K\}$ and the function considered. But when the domain concerned is very narrow, if we employ the regular partition, the computing cost will be very high. The obvious idea to overcome this difficulty is to use the anisotropic meshes with fewer degrees of freedom[1−3,6,7,17,18].

H^1-Galerkin mixed finite element method was proposed by Pani[13] to solve parabolic partial differential equations. This method allows the approximation spaces to be polynomial spaces with different orders and there is no need to satisfy the LBB condition[8] (see [5,9,10,14]). But so far only the conforming elements with regular meshes are considered and Ritz-Volterra projection is an indispensable tool.

In this paper, we will focus on nonconforming finite element approximation to the Sobolev equations without requiring the above regularity assumption. In next section, we check the anisotropic interpolation properties. In Section 3, based on the good properties of the elements, we derive the error estimates for semidiscrete scheme by using interpolate operator directly, that is, getting rid of the Ritz-Volterra projection. At last, a backward Euler discrete scheme is considered and the same error estimates as those in previous articles are obtained.

Manuscript received March 9, 2007. Revised January 11, 2008.
Supported by the National Natural Science Foundation of China (No. 10671184).

2 Construction of the Element

For the sake of convenience, let $\Omega \subset R^2$ be a convex polygon domain composed by a family of rectangular meshes $\mathcal{J}_h$, which does not need to satisfy the regular conditions. For all $K \in \mathcal{J}_h$, denoted the barycenter of element K by (x_K, y_K), and the length of edges parallel to x-axis and y-axis by $2h_x$ and $2h_y$, respectively. $a_1(x_K - h_x, y_K - h_y)$, $a_2(x_K + h_x, y_K - h_y)$, $a_3(x_K + h_x, y_K + h_y)$ and $a_4(x_K - h_x, y_K + h_y)$ are the four vertices, and $l_1 = \overline{a_1a_2}$, $l_2 = \overline{a_2a_3}$, $l_3 = \overline{a_3a_4}$ and $l_4 = \overline{a_4a_1}$ are the four edges. Let $\widehat{K} = [-1,1] \times [-1,1]$ be the reference element, the four vertices be $\widehat{a}_1 = (-1,-1)$, $\widehat{a}_2 = (1,-1)$, $\widehat{a}_3 = (1,1)$ and $\widehat{a_4} = (-1,1)$. Let $\widehat{l}_1 = \overline{\widehat{a}_1\widehat{a}_2}$, $\widehat{l}_2 = \overline{\widehat{a}_2\widehat{a}_3}$, $\widehat{l}_3 = \overline{\widehat{a}_3\widehat{a}_4}$ and $\widehat{l}_4 = \overline{\widehat{a}_4\widehat{a}_1}$. Then there exists an reversible mapping $F_K : \widehat{K} \to K$

$$x = x_K + h_x\widehat{x}, \qquad y = y_K + h_y\widehat{y}. \tag{2.1}$$

The shape function spaces and interpolation operators of the finite elements on $\widehat{K}$ are defined by (see [12,15,17])

$$\widehat{P}^1 = \mathrm{span}\{1, \xi, \widehat{y}, \varphi(\widehat{x}), \varphi(\widehat{y})\}, \quad \frac{1}{|\widehat{K}|}\int_{\widehat{K}} (\widehat{v} - \widehat{I}^1\widehat{v}) d\widehat{x}d\widehat{y} = 0, \quad \frac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i} (\widehat{v} - \widehat{I}^1\widehat{v}) d\widehat{s} = 0, \tag{2.2}$$

$$\widehat{P}^2 = \mathrm{span}\{1, \widehat{x}, \widehat{y}\} \times \mathrm{span}\{1, \widehat{x}, \widehat{y}\}, \quad \frac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i} \widehat{I}^2\widehat{v} d\widehat{s} = \frac{1}{2}\big(\widehat{v}(\widehat{a}_i) + \widehat{v}(\widehat{a}_{i+1})\big), \tag{2.3}$$

where $\varphi(t) = \frac{1}{2}(3t^2 - 1),\ i = 1, 2, 3, 4,\ j = 1, 2.$

It can be easily checked that the interpolations defined above are well-posed. If we denote $\widehat{v}_i = \frac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i} \widehat{v} d\widehat{s}$, $\widehat{v}_5 = \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{v} d\widehat{x}d\widehat{y}, \widehat{v}_{i+5} = \widehat{v}(\widehat{a}_i), i = 1, 2, 3, 4$, the interpolation functions can be expressed as follows

$$\widehat{I}^1\widehat{v} = \widehat{v}_5 + \frac{1}{2}(\widehat{v}_2 - \widehat{v}_4)\widehat{x} + \frac{1}{2}(\widehat{v}_3 - \widehat{v}_1)\widehat{y} + \frac{1}{2}(\widehat{v}_2 + \widehat{v}_4 - 2\widehat{v}_5)\varphi(\widehat{x}) + \frac{1}{2}(\widehat{v}_3 + \widehat{v}_1 - 2\widehat{v}_5)\varphi(\widehat{y}) \tag{2.4}$$

and

$$\widehat{I}^2\widehat{v} = \frac{1}{4}(\widehat{v}_6 + \widehat{v}_7 + \widehat{v}_8 + \widehat{v}_9) + \frac{1}{4}(\widehat{v}_7 + \widehat{v}_8 - \widehat{v}_6 - \widehat{v}_9)\widehat{x} + \frac{1}{4}(\widehat{v}_8 + \widehat{v}_9 - \widehat{v}_6 - \widehat{v}_7)\widehat{y}. \tag{2.5}$$

Then we define the interpolation operators on the general element K as

$$\begin{aligned}
&\Pi_h^1 : H^2(\Omega) \to V_h,\ \Pi_h^1|_K = \Pi_K^1, \Pi_K^1 v = (\widehat{I}^1\widehat{v}) \circ F_K^{-1}, \qquad \forall v \in H^2(K),\\
&\Pi_h^2 : (H^2(\Omega))^2 \to W_h,\\
&\Pi_h^2|_K = \Pi_K^2, \Pi_K^2 q = ((\widehat{I}^2\widehat{q}_1) \circ F_K^{-1}, (\widehat{I}^2\widehat{q}_2) \circ F_K^{-1}), \qquad \forall q = (q_1, q_2) \in (H^2(K))^2,
\end{aligned}$$

and the associated finite element spaces V_h and W_h as

$$V_h = \Big\{v, v|_K = \widehat{v} \circ F_K^{-1}, \widehat{v} \in \widehat{P}^1,\ \int_F [v] ds = 0,\ F \subset \partial K\Big\},$$

$$W_h = \{q = (q_1, q_2),\ q|_K = (\widehat{q}_1 \circ F_K^{-1}, \widehat{q}_2 \circ F_K^{-1}),\ \widehat{q} \in \widehat{P}^2 \times \widehat{P}^2, q(a) = 0,\ \text{for any node } a \in \partial\Omega\},$$

where $[v]$ denote the jump value of v across the boundary F and $[v] = v$ when $F \subset \partial\Omega$.

The following lemma will play an essential role in our forthcoming analysis on anisotropic meshes.

Lemma 2.1. *Operator $\widehat{I}^i (i = 1, 2)$ defined above has the following anisotropic interpolation property, i.e., for multi-index $\alpha = (\alpha_1, \alpha_2)$, when $|\alpha| = 1$, there hold*

$$\|\widehat{D}^\alpha(\widehat{v} - \widehat{I}^1\widehat{v})\|_{0,\widehat{K}} \le C|\widehat{D}^\alpha\widehat{v}|_{1,\widehat{K}}, \qquad \forall \widehat{v} \in H^2(\widehat{K}), \tag{2.6}$$

$$\|\widehat{D}^\alpha(\widehat{q} - \widehat{I}^2\widehat{q})\|_{0,\widehat{K}} \le C|\widehat{D}^\alpha\widehat{q}|_{1,\widehat{K}}, \qquad \forall \widehat{q} \in \big(H^2(\widehat{K})\big)^2. \tag{2.7}$$

Proof. We can obtain (2.6) by [17]. When $\alpha = (1,0)$, for all $\widehat{q} = (\widehat{q}^{(1)}, \widehat{q}^{(2)}) \in (H^2(\widehat{K}))^2$,

$$\begin{aligned}\widehat{D}^\alpha \widehat{I}^2 \widehat{q} =& \Big(\frac{1}{2}(\widehat{q}_2^{(1)} - \widehat{q}_4^{(1)}), \frac{1}{2}(\widehat{q}_2^{(2)} - \widehat{q}_4^{(2)})\Big) = \frac{1}{4}\Big(\int_{\widehat{l}_2} \widehat{q}(1,\widehat{y})d\widehat{y} - \int_{\widehat{l}_4} \widehat{q}(-1,\widehat{y})d\widehat{y}\Big)\\ =& \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \frac{\partial \widehat{q}}{\partial \widehat{x}} d\widehat{x}d\widehat{y}.\end{aligned}$$

For all $\widehat{w} \in (H^1(\widehat{K}))^2$, let

$$F(\widehat{w}) = \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{w} d\widehat{x}d\widehat{y}.$$

Apparently $F \in ((H^1(\widehat{K}))^2)'$, employing the basic anisotropic interpolation theorem [6] yields

$$\|\widehat{D}^\alpha(\widehat{q} - \widehat{I}^2\widehat{q})\|_{0,\widehat{K}} \le C|\widehat{D}^\alpha \widehat{q}|_{1,\widehat{K}}, \qquad \forall \widehat{q} \in (H^2(\widehat{K}))^2.$$

Similarly, we can prove that (2.7) is valid for $\alpha = (0,1)$. The proof is completed.

3 Anisotropic Error Estimates for Sobolev Problem

Now, let us consider the following problem

$$\begin{cases} u_t - \nabla\cdot(a\nabla u_t + b\nabla u) = f(X,t), & X \in \Omega, t \in (0,T],\\ u(X,t) = 0, & X \in \partial\Omega, t \in [0,T],\\ u(X,0) = u_0(X), & X \in \Omega,\end{cases} \tag{3.1}$$

where $X = (x,y)$, $a = a(X,t)$ and $b = b(X,t)$ are continuous functions with bounded derivatives and subject to

$$|b(X,t)| \le a_1, \qquad 0 < a_0 \le |a(X,t)| \le a_1, \qquad X \in \Omega, \tag{3.2}$$

for some positive constants a_0 and a_1.

Introducing the auxiliary variable $q = a\nabla u$, and rewriting (3.1) as:

$$\begin{cases} \nabla u = \alpha q, & X \in \Omega, t \in (0,T],\\ u_t - \nabla\cdot(q_t + \beta q) = f, & X \in \Omega, t \in (0,T],\\ u(X,t) = 0, & X \in \partial\Omega, t \in [0,T],\\ u(X,0) = u_0(X), & X \in \Omega,\end{cases} \tag{3.3}$$

where $\alpha = 1/a, \beta = (b - a_t)/a$.

We denote the natural inner production in $L^2(\Omega)$ by $(\cdot,\cdot)$ and the norm by $||\cdot||_0$, and let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, $H(div;\Omega) = \{q \in (L^2(\Omega))^2, \nabla\cdot q \in L^2(\Omega)\}$. Further, we use the classical Sobolev spaces $W^{m,p}(\Omega)$, $1 \le p \le \infty$, denoted by $W^{m,p}$, with norm $||\cdot||_{m,p}$. When $p = 2$, we simply write $||\cdot||_{m,p}$ as $||\cdot||_m$.

To derive the H^1-Galerkin mixed finite element method, we consider the following weak formulation of (3.3).

Find $\{u,q\} : [0,T] \longrightarrow H_0^1(\Omega) \times H(\mathrm{div};\Omega)$ such that

$$\begin{cases} (\nabla u, \nabla v) = (\alpha q, \nabla v), & \forall v \in H_0^1(\Omega),\\ (\alpha_t q, w) + (\alpha q_t, w) + (\nabla\cdot(q_t + \beta q), \nabla\cdot w) = -(f, \nabla\cdot w), & \forall w \in H(\mathrm{div};\Omega),\\ u(X,0) = u_0(X), & X \in \Omega.\end{cases} \tag{3.4}$$

The discrete problem of (3.4) reads as: $\{u_h, q_h\} : [0,T] \longrightarrow V_h \times W_h$, such that

$$\begin{cases} (\nabla u_h, \nabla v_h) = (\alpha q_h, \nabla v_h), & \forall v_h \in V_h, \\ (\alpha_t q_h, w_h) + (\alpha q_{ht}, w_h) + (\nabla \cdot (q_{ht} + \beta q_h), \nabla \cdot w_h) = -(f, \nabla \cdot w_h), & \forall w_h \in W_h, \\ u_h(0) = \Pi_h^1 u_0(X), & X \in \Omega. \end{cases} \tag{3.5}$$

We define

$$\|q_h\|_{\widetilde{H}(div;\Omega)} = \Big(\sum_{K \in \mathcal{J}_h} \|q_h\|_{0,K}^2 + \|\nabla \cdot q_h\|_{0,K}^2 \Big)^{\frac{1}{2}}, \qquad \forall q_h \in W_h,$$

$$|v_h|_h = \Big(\sum_{K \in \mathcal{J}_h} |v_h|_{1.K}^2 \Big)^{\frac{1}{2}}, \qquad \forall v_h \in V_h,$$

then it is easy to see that $|| \cdot ||_{\widetilde{H}(\mathrm{div};\Omega)}$ and $| \cdot |_h$ are the norms over W_h and V_h, respectively.

Theorem 3.1. *Problem (3.5) has a unique solution.*

Proof. Let $\{\phi_j\}_{j=1}^{r_1} \subset V_h$ and $\{\psi_i\}_{i=1}^{r_2} \subset W_h$ be bases of V_h and W_h, respectively. Then u_h and q_h may be expressed as

$$u_h = \sum_{j=1}^{r_1} h_j(t)\phi_j(x), \qquad q_h = \sum_{i=1}^{r_2} g_i(t)\psi_i(x).$$

Substituting these expressions into (3.5) and choosing $v_h = \phi_i(x)$, $w_h = \psi_j(x)$.
With

$$\begin{array}{lll} H(t) = (h_1(t), \cdots, h_{r_1}(t))', & G(t) = (g_1(t), \cdots, g_{r_2}(t))', & A = ((\nabla\phi_i, \nabla\phi_j))_{r_1 \times r_1}, \\ B = ((\alpha\psi_i, \nabla\phi_j))_{r_1 \times r_2}, & D = ((\alpha_t\psi_i, \psi_j))_{r_2 \times r_2}, & E = ((\alpha\psi_i, \psi_j))_{r_2 \times r_2}, \\ F = ((\nabla \cdot \psi_i, \nabla \cdot \psi_j))_{r_2 \times r_2}, & J = ((\nabla \cdot \beta\psi_i, \nabla \cdot \psi_j))_{r_2 \times r_2}, & M = -(f, \nabla \cdot \psi_j)_{r_2 \times 1}, \end{array}$$

then (3.5) can be stated as follows: Find $\{H(t), G(t)\}$ such that, for all $t \in (0,T)$,

$$\begin{cases} (a) \quad AH(t) = BG(t), \\ (b) \quad (E+M)\dfrac{dG(t)}{dt} + (D+J)G(t) = M. \end{cases} \tag{3.6}$$

Since $(E+F)$, $(D+J)$ and M are Lipschitz continuous, by the theory of differential equations, (3.6) has a unique solution[11], and equivalently (3.5) has a unique solution.

Lemma 3.2. *Suppose that $u_t \in H^2(\Omega)$, then on anisotropic meshes, for all $\psi \in W_h$ we have*

$$\Big| \sum_K \int_{\partial K} u_t \psi \cdot n ds \Big| \leq Ch|u_t|_2 \|\psi\|_0. \tag{3.7}$$

Proof. For all $\psi = (\psi_1, \psi_2) \in W_h$, we have

$$\sum_K \int_{\partial K} u_t(\psi \cdot n) ds = \sum_K \int_{\partial K} (u_t\psi_1 n_1 + u_t\psi_2 n_2) ds = \sum_K (A_1 + A_2 + A_3 + A_4),$$

where

$$
\begin{aligned}
A_1 &= \int_{l_1} -\Big(u_t - \frac{1}{|l_1|}\int_{l_1} u_t dx\Big)\Big(\psi_2 - \frac{1}{|l_1|}\int_{l_1}\psi_2 dx\Big)dx,\\
A_2 &= \int_{l_2} \Big(u_t - \frac{1}{|l_2|}\int_{l_2} u_t dy\Big)\Big(\psi_1 - \frac{1}{|l_2|}\int_{l_2}\psi_1 dy\Big)dy,\\
A_3 &= \int_{l_3} \Big(u_t - \frac{1}{|l_3|}\int_{l_3} u_t dx\Big)\Big(\psi_2 - \frac{1}{|l_3|}\int_{l_3}\psi_2 dx\Big)dx,\\
A_4 &= \int_{l_4} -\Big(u_t - \frac{1}{|l_4|}\int_{l_4} u_t dy\Big)\Big(\psi_1 - \frac{1}{|l_4|}\int_{l_4}\psi_1 dy\Big)dy.
\end{aligned}
$$

On the one hand,

$$
\begin{aligned}
A_1 + A_3 = &-\int_{x_K-h_x}^{x_K+h_x}\Big[u_t(x, y_K - h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} u_t(x, y_K - h_y)\Big]\\
&\cdot\Big[\psi_2(x, y_K - h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\psi_2(x, y_K - h_y)\Big]dx\\
&+\int_{x_K-h_x}^{x_K+h_x}\Big[u_t(x, y_K + h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x} u_t(x, y_K + h_y)\Big]\\
&\cdot\Big[\psi_2(x, y_K + h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\psi_2(x, y_K + h_y)]dx.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\psi_2(x, y_K - h_y) - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\psi_2(x, y_K - h_y)dx\\
=&\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\psi_2(x, y_K - h_y)dt - \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\psi_2(t, y_K - h_y)dt\\
=&\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}[\psi_2(x, y_K - h_y) - \psi_2(t, y_K - h_y)]dt\\
=&\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\Big(\int_t^x \frac{\partial\psi_2}{\partial z}(z, y_K - h_y)dz\Big)dt.
\end{aligned}
$$

Note that $\frac{\partial\psi_2}{\partial x}$ is a constant, by the technique of [17] we have

$$
|A_1 + A_3| \le \frac{4h_x^2}{3}\Big\|\frac{\partial^2 u_t}{\partial x\partial y}\Big\|_{0,K}\Big\|\frac{\partial\psi_2}{\partial x}\Big\|_{0,K}.
$$

Similarly, we get

$$
|A_2 + A_4| \le \frac{4h_y^2}{3}\Big\|\frac{\partial^2 u_t}{\partial x\partial y}\Big\|_{0,K}\Big\|\frac{\partial\psi_1}{\partial y}\Big\|_{0,K}.
$$

Then the desired result (3.7) comes from the following inequalities

$$
\Big\|\frac{\partial\psi_2}{\partial x}\Big\|_{0,K} \le Ch_x^{-1}||\psi_2||_{0,K}, \qquad \Big\|\frac{\partial\psi_1}{\partial y}\Big\|_{0,K} \le Ch_y^{-1}||\psi_1||_{0,K},
$$

which completes the proof.

Lemma 3.3. *For all $u \in H^2(\Omega), q, q_t \in (H^2(\Omega))^2$, there hold*

$$|u-\Pi_h^1 u|_h \le Ch|u|_2, \qquad \|q-\Pi_h^2 q\|_0 \le Ch|q|_1, \qquad \|q_t-\Pi_h^2 q_t\|_0 \le Ch|q_t|_1, \tag{3.8}$$

$$\|\nabla\cdot(q-\Pi_h^2 q)\|_0 \le Ch|q|_2, \qquad \|\nabla\cdot(q_t-\Pi_h^2 q_t)\|_0 \le Ch|q_t|_2. \tag{3.9}$$

Proof. On the one hand, by using the Bramble-Hilbert Lemma, we can get

$$||q-\Pi_h^2 q||_0 \le Ch|q|_1, \qquad ||q_t-\Pi_h^2 q_t||_0 \le Ch|q_t|_1.$$

On the other hand, using the property of the affine mapping F_K and the anisotropic interpolate property of operator $\widehat{I}^1$ in Lemma 2.1, we have

$$\begin{aligned}|u-\Pi_h^1 u|_h^2 =& \sum_{K\in J_h}\sum_{|\alpha|=1} h_K^{-2\alpha}(h_x h_y)\|\widehat{D}^\alpha(\widehat{u}-\widehat{I}^1\widehat{u})\|_{0,\widehat{K}}^2\\ \le& C\sum_{K\in J_h}\sum_{|\alpha|=1} h_K^{-2\alpha}(h_x h_y)|\widehat{D}^\alpha\widehat{u}|_{1,\widehat{K}}^2\\ \le& C\sum_{K\in J_h}\sum_{|\alpha|=1}\sum_{|\beta|=1} h_K^{2\beta}\|D^{\alpha+\beta}u\|_{0,K}^2 \le Ch^2|u|_2^2.\end{aligned}$$

Similarly, we obtain

$$|q-\Pi_h^2 q|_h \le Ch|q|_2, \quad |q_t-\Pi_h^2 q_t|_h \le Ch|q_t|_2.$$

Therefore, we have

$$\begin{aligned}&\|\nabla\cdot(q-\Pi_h^2 q)\|_0 \le 2|q-\Pi_h^2 q|_h \le Ch|q|_2,\\ &\|\nabla\cdot(q_t-\Pi_h^2 q_t)\|_0 \le 2|q_t-\Pi_h^2 q_t|_h \le Ch|q_t|_2.\end{aligned}$$

Theorem 3.4. *Assume that $u, u_t, \in H^2(\Omega)$, $q, q_t \in (H^2(\Omega))^2$. Then on anisotropic meshes, there hold*

$$\begin{aligned}&|u-u_h|_h\\ \le& Ch\Big[|u|_2+|q|_1+\Big(\int_0^t(|q(\tau)|_1^2+|q_t(\tau)|_1^2+|q(\tau)|_2^2+|q_t(\tau)|_2^2+|u_t(\tau)|_2^2)d\tau\Big)^{\frac{1}{2}}\Big],\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\|q-q_h\|_{\widetilde{H}(div;\Omega)}\\ \le& Ch\Big[|q|_1+|q|_2+\Big(\int_0^t(|q(\tau)|_1^2+|q_t(\tau)|_1^2+|q(\tau)|_2^2+|q_t(\tau)|_2^2+|u_t(\tau)|_2^2)d\tau\Big)^{\frac{1}{2}}\Big].\end{aligned} \tag{3.11}$$

Proof. Let $u-u_h=(u-\Pi_h^1 u)+(\Pi_h^1 u-u_h)=\eta+\xi, q-q_h=(q-\Pi_h^2 q)+(\Pi_h^2 q-q_h)=\rho+\theta$. From (3.1) to (3.5), we obtain the following error equations

$$(\nabla\xi,\nabla v_h)=(\alpha(\rho+\theta),\nabla v_h)-(\nabla\eta,\nabla v_h), \qquad \forall v_h\in V_h \tag{3.12}$$

and

$$\begin{aligned}&\big((\alpha\theta)_t,w_h\big)+(\nabla\cdot(\theta_t+\beta\theta),\nabla\cdot w_h)\\ =&-(\nabla\cdot(\rho_t+\beta\rho),\nabla\cdot w_h)+\sum_K\int_{\partial K}u_t w_h\cdot n ds-(\alpha_t\rho,w_h)-(\alpha\rho_t,w_h), \qquad \forall w_h\in W_h.\end{aligned} \tag{3.13}$$

Choose $v_h = \xi$ in (3.12), and use the Young's inequality and (3.2) to obtain

$$||\nabla \xi||_0^2 \leq C(||\rho||_0^2 + ||\theta||_0^2 + ||\nabla \eta||_0^2). \tag{3.14}$$

Taking $w_h = \theta$ in (3.13) and noticing that $((\alpha\theta)_t, \theta) = \frac{1}{2}(\alpha_t \theta, \theta) + \frac{1}{2}\frac{d}{dt}(\alpha\theta, \theta)$, we have

$$\begin{aligned} &\frac{1}{2}||\alpha_t^{\frac{1}{2}}\theta||_0^2 + \frac{1}{2}\frac{d}{dt}||\alpha^{\frac{1}{2}}\theta||_0^2 + \frac{d}{dt}||\nabla \cdot \theta||_0^2 \\ =& -(\nabla \cdot \beta\theta, \nabla \cdot \theta) - (\beta \nabla \cdot \theta, \nabla \cdot \theta) - (\alpha_t \rho, \theta) - (\alpha \rho_t, \theta) - (\nabla \cdot \rho_t, \nabla \cdot \theta) \\ &- (\nabla \cdot \beta\rho, \nabla \cdot \theta) - (\beta \nabla \cdot \rho, \nabla \cdot \theta) + \sum_K \int_{\partial K} u_t \theta \cdot n ds. \end{aligned} \tag{3.15}$$

By the Young's inequality, the positive definiteness of β and Lemma 3.2, we get

$$\begin{aligned} \frac{d}{dt}||\alpha^{\frac{1}{2}}\theta||_0^2 + \frac{d}{dt}||\nabla \cdot \theta||_0^2 \leq& C(||\rho||_0^2 + ||\rho_t||_0^2 + ||\nabla \cdot \rho||_0^2 + ||\nabla \cdot \rho_t||_0^2 \\ &+ ||\theta||_0^2 + ||\nabla \cdot \theta||_0^2 + h^2|u_t|_2^2). \end{aligned} \tag{3.16}$$

Further, integrating (3.16) with respect to time from 0 to t, and noticing $\theta(0) = \nabla \cdot \theta(0) = 0$, we obtain

$$\begin{aligned} ||\theta||_0^2 + ||\nabla \cdot \theta||_0^2 \leq& C\int_0^t (||\rho||_0^2 + ||\rho_t||_0^2 + ||\nabla \cdot \rho||_0^2 + ||\nabla \cdot \rho_t||_0^2 + ||\theta||_0^2 \\ &+ ||\nabla \cdot \theta||_0^2 + h^2|u_t|_2^2) d\tau. \end{aligned} \tag{3.17}$$

Using Gronwall's inequality and Lemma 3.3, we finally get

$$||\theta||_0^2 + ||\nabla \cdot \theta||_0^2 \leq Ch^2 \int_0^t (|q(\tau)|_1^2 + |q_t(\tau)|_1^2 + |q(\tau)|_2^2 + |q_t(\tau)|_2^2 + |u_t(\tau)|_2^2) d\tau. \tag{3.18}$$

The results (3.10) and (3.11) follow from the triangle inequality and Lemma 3.3.

4 The Backward Euler Method for Sobolev Problem

In this section, we will describe the backward Euler method and discuss related error estimates. Let $\triangle t = T/N$ for some integer N, and $t_n = n\triangle t, n = 0, \cdots, N$. For a smooth function ϕ on $[0, T]$, U^n and Q^n are the approximations to $u(t)$ and $q(t)$ at $t = t_n$, respectively. Define $\phi^n = \phi(t_n)$, $\overline{\partial}_t \phi^n = (\phi^n - \phi^{n-1})/\triangle t$.

Given $\{U^{n-1}, Q^{n-1}\} \in V_h \times W_h$, we now determine a pair $\{U^n, Q^n\}$ in $V_h \times W_h$ satisfying

$$\begin{cases} (\nabla U^n, \nabla v_h) = (\alpha^n Q^n, \nabla v_h), & \forall v_h \in V_h, \\ (\overline{\partial}_t \alpha^n Q^n, w_h) + (\alpha^n \overline{\partial}_t Q^n, w_h) + (\nabla \cdot (\overline{\partial}_t Q^n + \beta^n Q^n)), \nabla \cdot w_h) = -(f^n, \nabla \cdot w_h), & \forall w_h \in W_h, \\ U^0 = \Pi_h^1 u_0(x), \end{cases} \tag{4.1}$$

where $\alpha^n = \alpha(X, t^n)$, $\beta^n = \beta(X, t^n)$, $\overline{\partial}_t \alpha^n = (\alpha^n - \alpha^{n-1})/\triangle t$.

In order to get the desired error estimates, we write $U^n - u(t_n) = (U^n - \Pi_h^1 u(t_n)) + (\Pi_h^1 u(t_n) - u(t_n)) = \xi^n + \eta^n$ and $Q^n - q(t_n) = (Q^n - \Pi_h^2 q(t_n)) + (\Pi_h^2 q(t_n) - q(t_n)) = \theta^n + \rho^n$. Since estimates of η^n and ρ^n are given by Lemma 3.3 at $t = t_n$, it is sufficient to estimate ξ^n and

θ^n. Note that the equations for ξ^n and θ^n may be written as

$$(\nabla\xi^n, \nabla v_h) = (\alpha^n(\rho^n + \theta^n), \nabla v_h) - (\nabla\eta^n, \nabla v_h), \qquad \forall v_h \in V_h, \tag{4.2}$$

$$\begin{aligned}&(\overline{\partial}_t\alpha^n(\rho^n + \theta^n), \omega_h) + (\alpha^n(\overline{\partial}_t Q^n - q_t(t_n)), \omega_h) + (\nabla\cdot(\overline{\partial}_t Q^n - q_t(t_n)), \nabla w_h)\\ &+ (\nabla\cdot(\beta^n\rho^n + \beta^n\theta^n), \nabla w_h)\\ =&\sum_K \int_{\partial K} u_t(t_n)\omega_h \cdot n ds, \qquad \forall w_h \in W_h.\end{aligned} \tag{4.3}$$

Theorem 4.1. *Under assumption of the Theorem 3.4, for any notural number* $0 \le n \le N$ *we have*

$$\begin{aligned}|U^n - u(t_n)|_h \le& C(\triangle t)\Big(\int_0^{t_n} |q_{tt}|_0^2 d\tau\Big)^{\frac{1}{2}}\\ &+ Ch\Big[(\triangle t)^{\frac{1}{2}}\Big(\int_0^{t_n}(|u_{tt}(\tau)|_2^2 + |q_t(\tau)|_1^2 + |q_t(\tau)|_2^2)d\tau\Big)^{\frac{1}{2}}\\ &+ |u_0|_2 + |q|_1 + \int_0^{t_n}|u_t(\tau)|_2 d\tau\\ &+ \Big(\int_0^{t_n}(|q(\tau)|_1^2 + |q(\tau)|_2^2 + |u_t(\tau)|_2^2)d\tau\Big)^{\frac{1}{2}}\Big],\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}|Q^n - q(t_n)|_{\widetilde{H}(\mathrm{div};\Omega)} \le& C(\triangle t)\Big(\int_0^{t_n} |q_{tt}|_0^2 d\tau\Big)^{\frac{1}{2}}\\ &+ Ch\Big[(\triangle t)^{\frac{1}{2}}\Big(\int_0^{t_n}(|u_{tt}(\tau)|_2^2 + |q_t(\tau)|_1^2 + |q_t(\tau)|_2^2)d\tau\Big)^{\frac{1}{2}}\\ &+ |q|_1 + |q|_2 + \Big(\int_0^{t_n}(|q(\tau)|_1^2 + |q(\tau)|_2^2 + |u_t(\tau)|_2^2)d\tau\Big)^{\frac{1}{2}}\Big].\end{aligned} \tag{4.5}$$

Proof. By Lemma 3.3 at $t = t_n$, we get

$$|\eta^n|_h = |\Pi_h^1 u(t_n) - u(t_n)|_h \le Ch|u(t_n)|_2 \le Ch\Big(|u_0|_2 + \int_0^{t_n}|u_t(\tau)|_2 d\tau\Big). \tag{4.6}$$

Choose $v_h = \xi^n$ in (4.2) to obtain, for $n = 0, \cdots, N$,

$$(\nabla\xi^n, \nabla\xi^n) = (\alpha^n(\rho^n + \theta^n), \nabla\xi^n) - (\nabla\eta^n, \nabla\xi^n).$$

Use the Cauchy-Schwartz's inequality and the Young's inequality to obtain

$$||\nabla\xi^n||_0^2 \le C(||\rho^n||_0^2 + ||\theta^n||_0^2 + ||\nabla\eta^n||_0^2). \tag{4.7}$$

Setting $w_h = \theta^n$ in (4.3), we have

$$\begin{aligned}&(\overline{\partial}_t\alpha^n(\rho^n + \theta^n), \theta^n) + (\alpha^n(\overline{\partial}_t Q^n - q_t(t_n)), \theta^n) + (\nabla\cdot(\overline{\partial}_t Q^n - q_t(t_n)), \nabla\theta^n)\\ &+ (\nabla\cdot(\beta^n\rho^n + \beta^n\theta^n), \nabla\theta^n) = \sum_K\int_{\partial K} u_t(t_n)\theta^n \cdot n ds.\end{aligned} \tag{4.8}$$

Note that

$$\overline{\partial}_t Q^n - q_t(t_n) = \frac{1}{\triangle t}\Big[\rho^n + \theta^n - \rho^{n-1} - \theta^{n-1} + \int_{t_n}^{t_{n-1}} q_{tt}(t_{n-1} - \tau)d\tau\Big]$$

and

$$\Big\|\int_{t_{n-1}}^{t_n} q_{tt}(t_{n-1}-\tau)d\tau\Big\|_0^2 \leq (\triangle t)^2\int_{t_{n-1}}^{t_n}||q_{tt}(\tau)||_0^2 d\tau.$$

By the Young's inequality and the boundaries of $\overline{\partial}_t\alpha^n$ and α^n, we have

$$\begin{aligned}&||\theta^n||_0^2-||\theta^{n-1}||_0^2+||\nabla\cdot\theta^n||_0^2-||\nabla\cdot\theta^{n-1}||_0^2\\ \leq& C(\triangle t)(||\rho^n||_0^2+||\theta^n||_0^2+h^2|u_t(t_n)|_2^2+||\nabla\cdot\theta^n||_0^2+||\nabla\cdot\rho^n||_0^2)\\ &+C(\triangle t)\int_{t_{n-1}}^{t_n}(||\rho_t(\tau)||_0^2+||\nabla\cdot\rho_t(\tau)||_0^2)d\tau+C(\triangle t)^2\int_{t_{n-1}}^{t_n}||q_{tt}(\tau)||_0^2 d\tau. \end{aligned} \tag{4.9}$$

Summing (4.9) in time from 1 to n and using discrete Gronwall's inequality, together with $\theta(0)=0$, we obtain

$$\begin{aligned}&\|\theta^n\|_0^2+\|\nabla\cdot\theta^n\|_0^2\\ \leq& C(\triangle t)\Big[\sum_{i=1}^n(\|\rho^i\|_0^2+h^2|u_t(t_i)|_2^2+\|\nabla\cdot\rho^i\|_0^2)+\int_0^{t_n}(\|\rho_t(\tau)\|_0^2+\|\nabla\cdot\rho_t(\tau)\|_0^2)d\tau\Big]\\ &+C(\triangle t)^2\int_0^{t_n}\|q_{tt}(\tau)\|_0^2 d\tau. \end{aligned} \tag{4.10}$$

By Lemma 3.3, we get

$$\begin{aligned}||\theta^n||_0^2+\|\nabla\cdot\theta^n\|_0^2\leq& Ch^2(\triangle t)\Big[\sum_{i=1}^n(|q(t_i)|_1^2+|q(t_i)|_2^2+|u_t(t_i)|_2^2)+\int_0^{t_n}(|q_t(\tau)|_1^2+|q_t(\tau)|_2^2)d\tau\Big]\\ &+C(\triangle t)^2\int_0^{t_n}||q_{tt}(\tau)||_0^2 d\tau. \end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned}\sum_{i=1}^n|u_t(t_i)|_2^2=&\sum_{i=1}^n\Big|\frac{1}{\triangle t}\Big[\int_{t_{i-1}}^{t_i}(\tau-t_{i-1})u_{tt}(\tau)d\tau+\int_{t_{i-1}}^{t_i}u_t(\tau)d\tau\Big]\Big|_2^2\\ \leq&\int_0^{t_n}\Big(|u_{tt}(\tau)|_2^2+\frac{1}{\triangle t}|u_t(\tau)|_2^2\Big)d\tau, \end{aligned} \tag{4.12}$$

and

$$\sum_{i=1}^n|q(t_i)|_1^2\leq\int_0^{t_n}\Big(|q_t(\tau)|_1^2+\frac{1}{\triangle t}|q(\tau)|_1^2\Big)d\tau,\ \sum_{i=1}^n|q(t_i)|_2^2\leq\int_0^{t_n}\Big(|q_t(\tau)|_2^2+\frac{1}{\triangle t}|q(\tau)|_2^2\Big)d\tau,$$

$$\sum_{i=1}^n|q_t(t_i)|_1^2\leq\int_0^{t_n}\Big(|q_{tt}(\tau)|_1^2+\frac{1}{\triangle t}|q_t(\tau)|_1^2\Big)d\tau,\ \sum_{i=1}^n|q_t(t_i)|_2^2\leq\int_0^{t_n}\Big(|q_{tt}(\tau)|_2^2+\frac{1}{\triangle t}|q_t(\tau)|_2^2\Big)d\tau.$$

Therefore, we have

$$\begin{aligned}||\theta^n||_0^2+\|\nabla\cdot\theta^n\|_0^2\leq& Ch^2(\triangle t)\int_0^{t_n}(|u_{tt}(\tau)|_2^2+|q_t(\tau)|_1^2+|q_t(\tau)|_2^2)d\tau+C(\triangle t)^2\int_0^{t_n}|q_{tt}(\tau)|_0^2 d\tau\\ &+Ch^2\int_0^{t_n}(|q(\tau)|_1^2+|q(\tau)|_2^2+|u_t(\tau)|_2^2)d\tau]. \end{aligned} \tag{4.13}$$

Finally, by the use of the triangle inequality, (4.6), (4.7) and (4.13), we complete the proof.

References

[1] Apel, T. Anisotropic finite elements: Local estimates and Applications. Leipzig: B. G.Teubner Stuttgart, 1999

[2] Apel, T. Dobrowlski, M. Anisotropic interpolation with application to the finite element method. *Computing,* 47(3): 277–293 (1992)

[3] Apel, T., Serge, N., Joachim, S. Crouzeix-Raviart type finite elements on anisotropic meshes. *Numer. Math.,* 89: 193–223 (2001)

[4] Barenblatt, G.I., Zheltov, I.P., Kochian I.V. Basic concepts in the theory of homogeneous liquids in fissured rocks. *J. Appl. Math. Mech.,* 24: 1286–1303 (1960)

[5] Cannon, J., Lin, Y. Non-classical H^1 projection and Galerkin methods for nonlinear parabolic integro-differential equations. Calcolo, 25: 187–201 (1998)

[6] Chen, S.C., Shi, D.Y., Zhao, Y.C. Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes. *IMA J. Numer. Anal.,* 24: 77–95 (2004)

[7] Chen, S.C., Zhao, Y. C., Shi, D.Y. Anisotropic interpolations with application to nonconforming elements. *Appl. Numer. Math.*, 49: 135–152 (2004)

[8] Ciarlet, P.G. The finite element method for elliptic problem. Amsterdam, North-Holland, 1978

[9] Douglas, J.J., Dupont, T. F., Wheeler, M. F. H^1-Galerkin methods for the Laplace and heat equations. Mathematical Aspect of Finite Elements in Partial Differential Equations. *C. de Boor, ed., Academic Press,* New York, 383–415 (1975)

[10] Guo, L., Chen, H.Z. H^1-Galerkin mixed finite element method for the Sobolev equation. *J. Sys. Sci. and Math. Scis.*, 26(3): 301–314 (2006) (in Chinese)

[11] Hale, J. K. Ordinary differential equations. New York, Willey-Interscience, 1969

[12] Lin, Y., Lutz Tobiska, Zhou A.H. Superconvergence and extrapolation of non-conforming low order finite elements applied to the Poisson equation. *IMA J. Numer. Anal.*, 25: 160–181 (2005)

[13] Pani, A.K. An H^1-Galerkin mixed finite element methods for parabolic partial differential equations. *SIAM J. Numer. Anal.,* 35: 712–727 (1998)

[14] Pani, A.K. An H^1-Galerkin mixed finite element methods for parabolic partial integro-differential equations. *SIAM J. Numer. Anal.,* 22: 231–252 (2002)

[15] Park, C.J., Sheen, D.W. P_1-nonconforming quadrilateral finite element methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 41: 624–640 (2003)

[16] Shi, D.M. On the initial boundary value problem of nonlinear the migration of the moisture in soil. *Acta Math. Appl.,* 13: 31–38 (1990) (in Chinese)

[17] Shi, D.Y., Mao, S.P., Chen, S.C. An anisotropic nonconforming finite element with some superconvergence results. *J. Comput. Math.*, 23(3): 261–274 (2005)

[18] Shi, D.Y., Zhu, H.Q. The superconvergence analysis of an anisotropic element. *J. Syst. Scie. & Compl.*, 18(4): 478–487 (2005)

[19] Ying, T.W. A cooling process according to two-temperature theory of heat conduction. *J. Math. Anal. Appl.*, 45: 23–31 (1974)