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Existence of at Least Two Periodic Solutions of a Ratio-dependent Predator-prey Model with Exploited Term

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Abstract In this paper, we study a non-autonomous ratio-dependent predator-prey model with exploited term. By means of the coincidence degree theory, we establish a sufficient condition for the existence of at least two positive periodic solutions of this model.

Keywords Predator-prey model, ratio-dependent, exploited term, periodic solution, coincidence degree
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1 Introduction

In recent years, the existence of periodic solutions of the predator-prey model is widely studied. Models with exploited terms are often considered. Generally, the model with exploited terms is described as follows:

$$\dot{x} = xf(x,y) - h, \qquad \dot{y} = yg(x,y) - k,$$

where x and y are functions of time representing densities of prey and predator, respectively; h and k are exploited terms standing for the harvests (see [5]). Particularly, a non-autonomous ratio-dependent predator-prey model with an exploited term is described by the following system of ordinary differential equations

$$\dot{x} = x\Big(a - bx - \frac{cy}{my+x}\Big) - h, \qquad \dot{y} = y\Big(-d + \frac{fx}{my+x}\Big), \tag{1.1}$$

where a, c, d, f, m are the prey intrinsic growth rate, capture rate, death rate of predator, conversion rate, half saturation-parameter, respectively. Moreover, on account of biological background of Model (1.1), we always assume that all of the parameters are positive constants. For the detailed biological meanings, we refer to [1–3] and references cited therein.

Since realistic models require the inclusion of the effect of changing environment, it motivates us to consider the following model:

$$\begin{cases} x'(t) = x(t)\Big(a(t) - b(t)x(t) - \dfrac{c(t)y(t)}{m(t)y(t)+x(t)}\Big) - h(t), \\ y'(t) = y(t)\Big(-d(t) + \dfrac{f(t)x(t)}{m(t)y(t)+x(t)}\Big). \end{cases} \tag{1.2}$$

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Correspondingly, we assume the parameters in (1.2) are positive $\omega-$periodic functions. The assumption of periodicity of the parameters is a way of incorporating periodicity of the environment (e.g. seasonal effects of weather, food supplies, mating habits etc).

Our aim of this paper is to establish a sufficient condition for the existence of at least two positive $\omega-$periodic solutions of System (1.2) by applying the continuation theorem of coincidence degree theory, which was proven in [4] by Gaines and Mawhin. Now we briefly introduce the coincidence degree theorem.

Let X, Z be real Banach spaces, $L : \mathrm{dom}L \subset X \to Z$ a Fredholm oprator of index zero and $P : X \to X, Q : Z \to Z$ continuous projects such that $\mathrm{Im}P = \mathrm{Ker}L, \mathrm{Ker}Q = \mathrm{Im}L, X = \mathrm{Ker}L \bigoplus \mathrm{Ker}P$ and $Z = \mathrm{Im}L \bigoplus \mathrm{Im}Q$. Denote the generalized inverse (of L) by $K_P : \mathrm{Im}L \to \mathrm{Ker}P \cap \mathrm{dom}L$ and an isomorphism of $\mathrm{Im}Q$ onto $\mathrm{Ker}L$ by $J : \mathrm{Im}Q \to \mathrm{Ker}L$. The continuation theorem is as follows.

Theorem A. *Let L, P, Q and K_P be defined as above, and let $\Omega \subset X$ be an open bounded set and $N : X \to Z$ be a continuous mapping, which is $L-$compact on $\overline{\Omega}$ (i.e., $QN : \overline{\Omega} \to Z$ and $K_p(i-Q)N : \Omega \to X$ are compact). Assume*
(i) for each $\lambda \in (0,1), x \in \partial\Omega \cap \mathrm{dom}L, Lx \neq \lambda Nx$;
(ii) for each $x \in \partial\Omega \cap \mathrm{Ker}L, QNx \neq 0$;
(iii) $deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$.
Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

2 Main Result

In the following discussions, we shall use the notations

$$\overline{g} = \frac{1}{\omega}\int_0^\omega g(t)dt, \qquad g^L = \min_{t\in[0,\omega]} g(t), \qquad g^M = \max_{t\in[0,\omega]} g(t),$$

where g is a continuous $\omega-$periodic function.

We now state and prove our main result.

Theorem 2.1. *Assume that the following*

$$\overline{f} > \overline{d} \quad and \quad \left(a - \frac{c}{m}\right)^L > 2\sqrt{b^M h^M}.$$

Then system (1.2) has at least two positive $\omega-$periodic solutions.

Proof. Let $x(t) = \exp\{u(t)\}, y(t) = \exp\{v(t)\}$. Then System (1.2) becomes

$$\begin{cases} u'(t) = a(t) - b(t)e^{u(t)} - \dfrac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}} - h(t)e^{-u(t)}, \\ v'(t) = -d(t) + \dfrac{f(t)e^{u(t)}}{m(t)e^{v(t)} + e^{u(t)}}. \end{cases} \tag{2.1}$$

It is easy to see that if System (2.1) has a ω-periodic solution $\big(u^*(t), v^*(t)\big)^T$, then $\big(x^*(t), y^*(t)\big)^T = (\exp\{u^*(t)\},\ \exp\{v^*(t)\})^T$ is a positive ω-periodic solution of System (1.2). So, to prove Theorem 2.1, it suffices to show that System (2.1) has at least two ω-periodic solutions.

For $\lambda \in (0,1)$, we consider the following system

$$\begin{cases} u'(t) = \lambda(a(t) - b(t)e^{u(t)} - \dfrac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}} - h(t)e^{-u(t)}), \\ v'(t) = \lambda\Big(-d(t) + \dfrac{f(t)e^{u(t)}}{m(t)e^{v(t)} + e^{u(t)}}\Big). \end{cases} \tag{2.2}$$

Suppose that $(u(t), v(t))^T$ is a $\omega-$periodic solution of System (2.2) for some $\lambda \in (0, 1)$. Integrating (2.2) over $[0, \omega]$, we obtain

$$\omega\overline{a} = \int_0^\omega \Big(b(t)e^{u(t)} + \frac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}} + h(t)e^{-u(t)}\Big)dt, \tag{2.3}$$

$$\omega\overline{d} = \int_0^\omega \frac{f(t)e^{u(t)}}{m(t)e^{v(t)} + e^{u(t)}}dt. \tag{2.4}$$

From (2.2) – (2.4), it follows that

$$\begin{aligned}\int_0^\omega |u'(t)|dt \leq &\lambda \int_0^\omega \Big(a(t) + b(t)e^{u(t)} + \frac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}} + h(t)e^{-u(t)}\Big)dt \\ <&\omega\overline{a} + \int_0^\omega \Big(b(t)e^{u(t)} + \frac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}} + h(t)e^{-u(t)}\Big)dt = 2\omega\overline{a},\end{aligned} \tag{2.5}$$

and

$$\int_0^\omega |v'(t)|dt \leq \lambda \int_0^\omega \Big(d(t) + \frac{f(t)e^{u(t)}}{m(t)e^{v(t)} + e^{u(t)}}\Big)dt < 2\omega\overline{d}. \tag{2.6}$$

Choose $t_i, \tau_i \in [0, \omega], i = 1, 2$, such that

$$u(t_1) = \min_{t\in[0,\omega]} u(t), \qquad u(\tau_1) = \max_{t\in[0,\omega]} u(t), \tag{2.7}$$

$$v(t_2) = \min_{t\in[0,\omega]} v(t), \qquad v(\tau_2) = \max_{t\in[0,\omega]} v(t). \tag{2.8}$$

From (2.3) and (2.7), we obtain

$$\omega\overline{a} = \int_0^\omega \Big(b(t)e^{u(t)} + \frac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}} + h(t)e^{-u(t)}\Big)dt > \int_0^\omega b(t)e^{u(t)} \geq \omega\overline{b}e^{u(t_1)},$$

which reduces to $u(t_1) < \ln\Big(\frac{\overline{a}}{\overline{b}}\Big)$. This, together with (2.5), gives

$$u(t) \leq u(t_1) + \int_0^\omega |u'(t)|dt < \ln\Big(\frac{\overline{a}}{\overline{b}}\Big) + 2\omega\overline{a} := \delta_1 \tag{2.9}$$

Multiplying the first equality of (2.2) by $e^{u(t)}$, and integrating over $[0, \omega]$, we obtain

$$\int_0^\omega a(t)e^{u(t)}dt = \int_0^\omega b(t)e^{2u(t)}dt + \int_0^\omega \frac{c(t)e^{u(t)+v(t)}dt}{m(t)e^{v(t)} + e^{u(t)}} + \int_0^\omega h(t)dt.$$

Again from (2.7), it follows that $e^{u(\tau_1)}\omega\overline{a} \geq \int_0^\omega a(t)e^{u(t)}dt > \int_0^\omega h(t)dt = \omega\overline{h}$, which reduces to $u(\tau_1) > \ln\Big(\frac{\overline{h}}{\overline{a}}\Big)$. This, together with (2.5), gives

$$u(t) \geq u(\tau_1) - \int_0^\omega |u'(t)|dt > \ln\Big(\frac{\overline{h}}{\overline{a}}\Big) - 2\omega\overline{a} := \delta_2. \tag{2.10}$$

From (2.7) and the first equality of (2.2), we also have

$$a(\tau_1) - b(\tau_1)e^{u(\tau_1)} - \frac{c(\tau_1)e^{v(\tau_1)}}{m(\tau_1)e^{v(\tau_1)} + e^{u(\tau_1)}} - h(\tau_1)e^{-u(\tau_1)} = 0.$$

This implies that

$$b(\tau_1)e^{2u(\tau_1)} - \Big(a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}\Big)e^{u(\tau_1)} + h(\tau_1) > 0.$$

Solving this inequality, we obtain

$$e^{u(\tau_1)} < \frac{\big(a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}\big) - \sqrt{(a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)})^2 - 4b(\tau_1)h(\tau_1)}}{2b(\tau_1)} \le l_-,$$

or

$$e^{u(\tau_1)} > \frac{\big(a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}\big) + \sqrt{(a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)})^2 - 4b(\tau_1)h(\tau_1)}}{2b(\tau_1)} \ge l_+,$$

where

$$l_\pm = \frac{(a - \frac{c}{m})^L \pm \sqrt{((a - \frac{c}{m})^L)^2 - 4b^M h^M}}{2b^M}.$$

Namely,

$$e^{u(\tau_1)} < l_-, \quad \text{or} \quad e^{u(\tau_1)} > l_+.$$

Similarly,

$$e^{u(t_1)} < l_-, \quad \text{or} \quad e^{u(t_1)} > l_+.$$

These, together with (2.9) and (2.10), give

$$\delta_2 < u(t) < \ln l_-, \quad \text{or} \quad \ln l_+ < u(t) < \delta_1. \tag{2.11}$$

From (2.4), (2.8) and (2.9), we have

$$\omega\overline{d} < \int_0^\omega \frac{f(t)e^{u(t)}dt}{m(t)e^{v(t)}} \le \int_0^\omega \frac{f(t)e^{u(t)}dt}{m(t)e^{v(t_2)}} < \int_0^\omega \frac{f(t)e^{\delta_1}dt}{m(t)e^{v(t_2)}} = \frac{1}{e^{v(t_2)}}\frac{\overline{a}}{\overline{b}}\Big(\overline{\frac{f}{m}}\Big)\exp(2\omega\overline{a})\omega,$$

which reduces to $v(t_2) < \ln\Big\{\frac{\overline{a}}{\overline{b}\cdot\overline{d}}\cdot\Big(\overline{\frac{f}{m}}\Big)\Big\} + 2\omega\overline{a}$. This, together with (2.6), therefore gives

$$v(t) \le v(t_2) + \int_0^\omega |v'(t)|dt < \ln\Big\{\frac{\overline{a}}{\overline{b}\cdot\overline{d}}\cdot\Big(\overline{\frac{f}{m}}\Big)\Big\} + 2\omega(\overline{a} + \overline{d}) := \delta_3. \tag{2.12}$$

From (2.4) and (2.10), noticing that $\int_0^\omega \frac{f(t)e^{u(t)}dt}{m(t)e^{v(t)}+e^{u(t)}}$ is increasing with $u(t)$, we have

$$\omega\overline{d} \ge \int_0^\omega \frac{f(t)e^{u(t)}dt}{m^M e^{v(\tau_2)} + e^{u(t)}} > \int_0^\omega \frac{f(t)(\overline{h}/\overline{a})\exp(-2\omega\overline{a})dt}{m^M e^{v(\tau_2)} + (\overline{h}/\overline{a})\exp(-2\omega\overline{a})},$$

which reduces to, $v(\tau_2) > \ln\Big\{\frac{(\overline{f}-\overline{d})\overline{h}}{m^M\overline{a}\overline{d}}\Big\} - 2\omega\overline{a}$.

Hence, this together with (2.6), gives

$$v(t) \ge v(\tau_2) + \int_0^\omega |v'(t)|dt > \ln\Big\{\frac{(\overline{f} - \overline{d})\overline{h}}{m^M\overline{a}\overline{d}}\Big\} - 2\omega(\overline{a} + \overline{d}) := \delta_4. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$|v(t)| < |\delta_3| + |\delta_4| + 1 := R_1. \tag{2.14}$$

Clearly, $l_\pm, \delta_1, \delta_2,\ R_1$ are independent of λ.

Now we consider the set of two equations:

$$\begin{cases} \overline{a} - \overline{b}e^u - \frac{1}{\omega}\int_0^\omega \dfrac{c(t)e^v}{m(t)e^v + e^u}dt - \overline{h}e^{-u} = 0, \\ -\overline{d} + \frac{1}{\omega}\int_0^\omega \dfrac{f(t)e^u}{m(t)e^v + e^u}dt = 0, \end{cases} \tag{2.15}$$

where $(u, v)^T$ is a constant vector, and satisfies (2.15).

We point out that (2.15) has two solutions. We introduce the function

$$\varphi(z) = -\overline{d} + \frac{1}{\omega}\int_0^\omega \frac{f(t)}{m(t)z + 1}dt, \qquad z \in [0, +\infty).$$

Clearly, $\varphi(z)$ is decreasing with z and,

$$\varphi(0) = \overline{f} - \overline{d} > 0, \qquad \lim_{z\to+\infty} \varphi(z) = -\overline{d} < 0.$$

Therefore, there exists a unique $z^* > 0$ such that $\varphi(z^*) = 0$.

Substituting $z^* = e^v/e^u$ into the first equation in (2.15), we obtain

$$\overline{a} - \overline{b}e^u - \frac{1}{\omega}\int_0^\omega \frac{c(t)z^*}{m(t)z^* + 1}dt - \overline{h}e^{-u} = 0, \tag{2.16}$$

which obviously has two solutions, denoted by u_1 and u_2 $(u_1 < u_2)$.

From (2.16), we have $\overline{a} - \overline{b}e^u - \overline{\left(\frac{c}{m}\right)} - \overline{h}e^{-u} < 0$. Solving this inequality, we have

$$e^{u_1} < \frac{(\overline{a} - \overline{(\frac{c}{m})}) - \sqrt{(\overline{a} - \overline{(\frac{c}{m})})^2 - 4\overline{b}\cdot\overline{h}}}{2\overline{b}} (\le l_-),$$
$$e^{u_2} > \frac{(\overline{a} - \overline{(\frac{c}{m})}) + \sqrt{(\overline{a} - \overline{(\frac{c}{m})})^2 - 4\overline{b}\cdot\overline{h}}}{2\overline{b}} (\ge l_+).$$

This implies that (2.15) has two solutions, denoted by $(u_1, v_1)^T, (u_2, v_2)^T (v_1 < v_2)$. Further, it follows from (2.16) that

$$\delta_2 < u_1 < \ln l_- \quad \text{and} \quad \ln l_+ < u_2 < \delta_1. \tag{2.17}$$

Take $X = Z = \{(u(t), v(t))^T \in C(\mathbb{R}, \mathbb{R}^2) | u(t+\omega) = u(t), v(t+\omega) = v(t)\}$ and $\|(u(t), v(t))^T\| = \max_{t\in[0,\omega]} |u(t)| + \max_{t\in[0,\omega]} |v(t)|$. Equipped with the norm, X is a Banach space. Let $L : \text{dom}L \subset X \to X$, $L(u(t), v(t))^T = (u'(t), v'(t))^T$, where $\text{dom}L = \{(u(t), v(t))^T \in X : (u(t), v(t))^T \in C^1(\mathbb{R}, \mathbb{R}^2)\}$. Again let $N : X \to X$,

$$N\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} a(t) - b(t)e^{u(t)} - \dfrac{c(t)e^{v(t)}}{m(t)e^{v(t)} + e^{u(t)}} - h(t)e^{-u(t)} \\ -d(t) + \dfrac{f(t)e^{u(t)}}{m(t)e^{v(t)} + e^{u(t)}} \end{pmatrix}.$$

Define projectors P and Q by

$$P\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = Q\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega u(t)dt \\ \frac{1}{\omega}\int_0^\omega v(t)dt \end{pmatrix}, \qquad \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in X.$$

Obviously, $\mathrm{Ker}L = \mathrm{Im}P = \mathbb{R}^2$, $\mathrm{Im}L = \mathrm{Ker}Q = \{(u(t), v(t))^T \in X : \overline{u} = \overline{v} = 0\}$ is closed in X, and dimKer$L$ =dim$(Z/\mathrm{Im}L) = 2$. Thus, L is a Fredholm operator of index zero. Moreover, as usual the inverse K_P of L is as follows:

$$K_P : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P, \qquad K_P \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \int_0^t u(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u(s)dsdt \\ \int_0^t v(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t v(s)dsdt \end{pmatrix}.$$

Again take $R_2 = |v_1| + |v_2|$, and define

$$\Omega_1 = \Big\{(u(t), v(t))^T \in X : \delta_2 < u(t) < \ln l_-, \max_{t\in[0,\omega]} |v(t)| < R_1 + R_2\Big\},$$

$$\Omega_2 = \Big\{(u(t), v(t))^T \in X : \ln l_+ < u(t) < \delta_1, \max_{t\in[0,\omega]} |v(t)| < R_1 + R_2\Big\}.$$

Both Ω_1 and Ω_2 are open bounded subsets of X. Since $l_- < l_+$, we have $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$. From (2.17), we see that $(u_1, v_1)^T \in \Omega_1$ and $(u_2, v_2)^T \in \Omega_2$.

It is easy to show that QN and $K_P(I - Q)N$ are continuous by the Lebesgue convergence theorem, and by Arzela-Ascoli theorem, $QN(\overline{\Omega}_i)$ and $K_P(I - Q)N(\overline{\Omega}_i)$ $(i = 1, 2)$ are compact. Therefore, N is L-compact on each $\overline{\Omega}_i$ for $i = 1, 2$.

Since we are concerned with the periodic solutions, $(u(t), v(t))^T$ confined in dom$L$, System (2.2) can be regarded as the following operator equation $L(u(t), v(t))^T = \lambda N\big(u(t), v(t)\big)^T$, which is System (2.1) when $\lambda = 1$. According to the above estimation of periodic solutions of (2.2), we have proven that

(i) for each $\lambda \in (0, 1)$, $(u(t), v(t))^T \in \partial\Omega_i \cap \mathrm{dom}L$, $L(u(t), v(t))^T \neq \lambda N(u(t), v(t))^T$ for $i = 1, 2$. Namely, Condition (i) in Theorem A is satisfied.

According to Theorem A, next we have to prove the following:

(ii) for any $(u, v)^T \in \partial\Omega_i \cap \mathrm{Ker}L(i = 1, 2)$, $QN(u, v)^T \neq 0$;

(iii) $\deg\{JQN, \Omega_i \cap \mathrm{Ker}L, 0\} \neq 0$.

When $(u, v)^T \in \partial\Omega_i \cap \mathrm{Ker}L = \partial\Omega_i \cap \mathbb{R}^2$ for $i = 1, 2$ i.e., $(u, v)^T$ is a constant vector in $\mathbb{R}^2$, from (2.14) and (2.17) and the fact that (2.15) has two solutions, it follows that

$$QN\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \overline{a} - \overline{b}e^u - \dfrac{1}{\omega}\int_0^\omega \dfrac{c(t)e^v}{m(t)e^v + e^u}dt - \overline{h}e^{-u} \\ -\overline{d} + \dfrac{1}{\omega}\int_0^\omega \dfrac{f(t)e^u}{m(t)e^v + e^u}dt \end{pmatrix} \neq 0.$$

This proves that Condition (ii) in Theorem A is satisfied.

Finally, we will prove that Condition (iii) in Theorem A is satisfied. From (2.16), we see that $e^{u_1} \cdot e^{u_2} = \frac{\overline{h}}{\overline{b}}$. Since $u_1 < u_2$, the above expression implies that $\overline{b}e^{u_i} - \frac{\overline{h}}{e^{u_i}} \neq 0, \quad i = 1, 2.$ Some straightforward calculations further give

$$\deg\{JQN, \Omega_i \cap \mathrm{Ker}L, 0\} = \mathrm{sgn}\Big\{\frac{1}{\omega}\Big(\overline{b}e^{u_i} - \frac{\overline{h}}{e^{u_i}}\Big)\int_0^\omega \frac{f(t)m(t)e^{v_i}dt}{(m(t)e^{v_i} + e^{u_i})^2}\Big\} \neq 0, \qquad i = 1, 2.$$

Summarizing the above discussion, we have proved that each $\Omega_i(i = 1, 2)$ satisfies all the requirements of Theorem A. Hence, System (2.1) has at least one ω-periodic solution in each of Ω_1 and Ω_2. Thus, the proof of Theorem 2.1 is completed.

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