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Solitary Wave Solutions and Kink Wave Solutions for a Generalized PC Equation

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Abstract We present in this paper a generalised PC (GPC) equation which includes several known models. The corresponding traveling wave system is derived and we show that the homoclinic orbits of the traveling wave system correspond to the solitary waves of GPC equation, and the heteroclnic orbits correspond to the kink waves. Under some parameter conditions, the existence of above two types of orbits is demonstrated and the explicit expressions of the two solutions are worked out.

Keywords GPC equation, solitary waves, kink waves
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1 Introduction

In this paper we consider the following generalized PC equation.

$$u_{tt} - u_{ttxx} - (a_1 u + a_2 u^{p+1} + a_3 u^{2p+1})_{xx} = 0. \tag{1.1}$$

When $a_1 = 1,\ a_2 = \frac{1}{p+1}$ and $a_3 = 0$, (1.1) becomes the equation

$$u_{tt} - u_{ttxx} - \Big(u + \frac{1}{p+1} u^{p+1}\Big)_{xx} = 0, \tag{1.2}$$

which was studied in [1,3]. Bogolubsky[1] and Clarkson et al[3] gave some solitary-wave solutions of (1.2) with $p = 1, 2, 4$ and studied the interaction of two solitary waves numerically. When $a_1 = 0, a_2 = -\frac{1}{2}, a_3 = 0$, and $p = 1$, (1.1) reduces to the equation

$$u_{tt} - u_{ttxx} + \frac{1}{2}(u^2)_{xx} = 0. \tag{1.3}$$

Parker[6] also studied the solitary wave and exact solutions of (1.3). If $p = 1$ or $p = 2$, then (1.1) becomes the equation

$$u_{tt} - u_{ttxx} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0, \tag{1.4}$$

or

$$u_{tt} - u_{ttxx} - (a_1 u + a_2 u^3 + a_3 u^5)_{xx} = 0, \tag{1.5}$$

which were studied in [5,7]. Zhang and Ma[7] gave some explicit solitary wave solutions of (1.4) and (1.5). Li and Zhang[5] obtained more explicit formulae of solitary wave solutions and kink wave solutions.

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In this paper, we study solitary wave solutions and kink wave solutions of Eq.(1.1). To this end, substituting $u=\varphi(\xi)$ with $\xi=x-ct$ into (1.1), we have

$$c^2\varphi''-c^2\varphi''''-(a_1\varphi+a_2\varphi^{p+1}+a_3\varphi^{2p+1})''=0. \tag{1.6}$$

Note that the solitary wave solutions and kink wave solutions tend to some constants when $|\xi|$ tends to ∞. This implies that $\varphi'(\xi),\varphi''(\xi)$ and $\varphi'''(\xi)$ tend to zero when $|\xi|$ tends to ∞. Therefore integrating (1.6), once we get

$$c^2\varphi'-c^2\varphi'''-\left(a_1\varphi+a_2\varphi^{p+1}+a_3\varphi^{2p+1}\right)'=0. \tag{1.7}$$

Further integrating (1.7) once we have

$$\varphi''=\frac{1}{c^2}\left[(c^2-a_1)\varphi-a_2\varphi^{p+1}-a_3\varphi^{2p+1}+g\right]. \tag{1.8}$$

Letting $y=\varphi'$, we obtain a planar system

$$\frac{d\varphi}{d\xi}=y,\qquad \frac{dy}{d\xi}=\frac{[(c^2-a_1)\varphi-a_2\varphi^{p+1}-a_3\varphi^{2p+1}+g]}{c^2}. \tag{1.9}$$

It is well known that if $\varphi(\xi)$ has a extreme value and $\lim_{\xi\to-\infty}\varphi(\xi)=\lim_{\xi\to\infty}\varphi(\xi)=A$ which is a constant, then $u=\varphi(x-ct)$ is called a solitary wave solution; if $\varphi(\xi)$ is monotonic and $\lim_{\xi\to-\infty}\varphi(\xi)=B\neq\lim_{\xi\to\infty}\varphi(\xi)=D$, where B and D are two constants, then $u=\varphi(x-ct)$ is called a kink wave solution. These two types of traveling wave solutions will be called long-wave solutions.

We will show the relationship between long-wave solutions and some orbits of System (1.9) in Section 2. In Section 3 we will give some bifurcation phase portraits. In Section 4 we will derive some explicit expressions of long-wave solutions. Moreover, We also give some conditions under which Eq.(1.1) has no long-wave solution. A short conclusion will be given in Section 5.

2 The Relationship of Long-wave and Some Orbits

In this section, we show the relationship of long-wave of Eq.(1.1) and some orbits of System (1.9).

Proposition 1. *Consider Eq.(1.1) and System (1.9).*

(i) If $\varphi=\varphi(\xi)$ and $y=y(\xi)$ is the parameter expression of a homoclinic orbit of System (1.9), then $u=\varphi(x-ct)$ is a solitary wave solution of Eq.(1.1).

(ii) If $\varphi=\varphi(\xi)$ and $y=y(\xi)$ is the parameter expression of a heteroclinic orbit of system (1.9), then $u=\varphi(x-ct)$ is a kink wave solution of Eq.(1.1).

Proof. It is easily seen that all singular points of System (1.9) are on the φ axis. From the qualitative theory of dynamical systems we know that if $\varphi(\xi)$ and $y(\xi)$ denote a homoclinic orbit of System (1.9), then certainly this homoclinic orbit starts and returns to one of the singular points of System (1.9). Thus $\lim_{\xi\to-\infty}\varphi(\xi)=\lim_{\xi\to\infty}\varphi(\xi)=\alpha$ which is a constant and $\lim_{|\xi|\to\infty}y(\xi)=\lim_{|\xi|\to\infty}\varphi'(\xi)=0$. On the other hand, since $\varphi(\xi)$ and $y(\xi)$ satisfy System (1.9), $\varphi(\xi)$ satisfies Eq.(1.8). Further, $u=\varphi(x-ct)$ satisfies Eq.(1.1). This implies that $u=\varphi(x-ct)$ is a solitary wave solution of Eq.(1.1).

Similarly, if $\varphi(\xi)$ and $y(\xi)$ denote a heteroclinic orbit of System (1.9), then certainly this heteroclinic orbit starts from one of singular points of System (1.9) and goes into some one of others. This implies that $\lim_{\xi\to-\infty}\varphi(\xi)=\beta$ and $\lim_{\xi\to\infty}\varphi(\xi)=\gamma$, where $\beta\neq\gamma$ are constants. Meanwhile we have $\lim_{|\xi|\to\infty}y(\xi)=\lim_{|\xi|\to\infty}\varphi'(\xi)=0$. On the other hand, since $\varphi(\xi)$ and $y(\xi)$ satisfy System (1.9), $u=\varphi(\xi)$ satisfies Eq.(1.8). Further, $u=\varphi(x-ct)$ satisfies Eq.(1.1). This implies that $u=\varphi(x-ct)$ is a kink wave solution of Eq.(1.1). This completes the proof.

From Proposition 1 we see that the homoclinic orbits and heteroclinic orbits can be used to seek the long-wave solutions of Eq.(1.1). In other words, the problem is converted into seeking expressions of homoclinic orbits and heteroclinic orbits. In next section we will discuss some bifurcation phase portraits.

3 Some Bifurcation Phase Portraits

In this section, we discuss some bifurcation phase portraits of System (1.9). Suppose

$$f(\varphi)=(c^2-a_1)\varphi-a_2\varphi^{p+1}-a_3\varphi^{2p+1}, \tag{3.1}$$

(1.9) is reduced to

$$\frac{d\varphi}{d\xi}=y, \qquad \frac{dy}{d\xi}=\frac{f(\varphi)+g}{c^2}. \tag{3.2}$$

Thus, to study the distribution of singular points of (1.9), we need to investigate the zero points of $f(\varphi)+g$. As mentioned in Section 2, the case $a_3=0$ is discussed in [1,2]. Now suppose $a_3\neq 0$. From $f(\varphi)=0$ we get

$$\varphi_0=0, \qquad \varphi_\pm=\pm\Big(\frac{-a_2\pm\sqrt{a_2^2-4a_3(a_1-c^2)}}{2a_3}\Big)^{1/p}. \tag{3.3}$$

(3.3) may be used to determine the number of zero points of $f(\varphi)$. For example, when p is even and

$$a_3<0, \qquad a_2>0, \qquad \frac{a_2^2}{4a_3}<a_1-c^2<0, \tag{3.4}$$

$f(\varphi)$ has five zero points $\varphi_0, \pm\varphi_1, \pm\varphi_2$ and their expressions are given as follows

$$\varphi_0=0, \qquad \varphi_1=\Big(\frac{-a_2-\sqrt{a_2^2-4a_3(a_1-c^2)}}{2a_3}\Big)^{1/p}, \tag{3.5}$$

$$\varphi_2=\Big(\frac{-a_2+\sqrt{a_2^2-4a_3(a_1-c^2)}}{2a_3}\Big)^{1/p}. \tag{3.6}$$

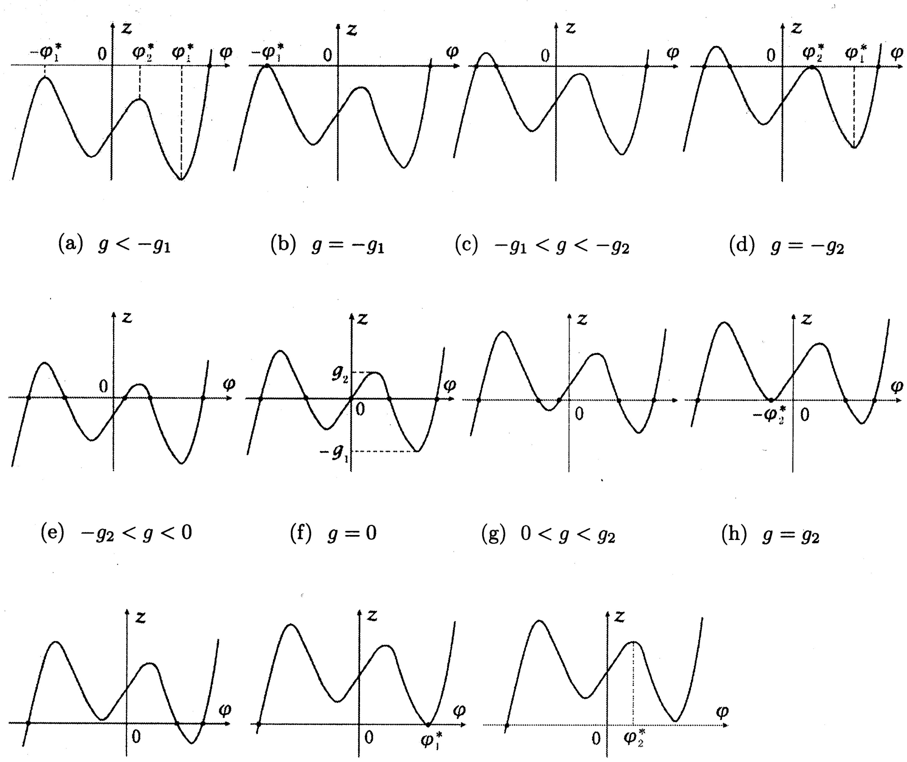
Suppose

$$\varphi_1^*=\Big(\frac{-(p+1)a_2-\sqrt{(p+1)^2a_2^2-4(2p+1)a_3(a_1-c^2)}}{2(2p+1)a_3}\Big)^{1/p}, \tag{3.7}$$

$$\varphi_2^*=\Big(\frac{-(p+1)a_2+\sqrt{(p+1)^2a_2^2-4(2p+1)a_3(a_1-c^2)}}{2(2p+1)a_3}\Big)^{1/p}, \tag{3.8}$$

$$g_i=|f(\varphi_i^*)|, \qquad i=1,2, \tag{3.9}$$

then it is easily seen that g_1 and g_2 are the extreme values of $f(\varphi)$. Assume that $g_1>g_2$ (for the case $g_1\leq g_2$ is similar). Using the above information the graph of $F(\varphi)=f(\varphi)+g$ can be drawn easily. For example, subject to (3.4) and p is even, the graph of $z=F(\varphi)$ is as that in Fig.1.



(a) $g < -g_1$ (b) $g = -g_1$ (c) $-g_1 < g < -g_2$ (d) $g = -g_2$

(e) $-g_2 < g < 0$ (f) $g = 0$ (g) $0 < g < g_2$ (h) $g = g_2$

(i) $g_2 < g < g_1$ (j) $g = g_1$ (k) $g > g_1$

Fig.1 The graph $z = f(\varphi) + g$, when p is even and the parameters satisfy (3.4)

On the other hand, when $g = 0$, (1.9) has the following first integral

$$H(\varphi, y) = c^2y^2 + \varphi^2\Big(\frac{a_3}{p+1}\varphi^{2p} + \frac{2a_2}{p+2}\varphi^p + a_1 - c^2\Big) = h. \tag{3.10}$$

From $H(\varphi_2, 0) = 0$, we get

$$a_3^0 = \frac{(p+1)a_2^2}{(p+2)^2(a_1 - c^2)}. \tag{3.11}$$

Suppose $(\bar{\varphi}, 0)$ is a singular point of (3.2), then at $(\bar{\varphi}, 0)$ the eigenvalue of the linearized system of (3.2) is

$$\lambda^2(\bar{\varphi}, 0) = \frac{f'(\bar{\varphi})}{c^2}. \tag{3.12}$$

According to the theory of dynamical systems (e.g. [2,4]), we obtain the following conclusions.

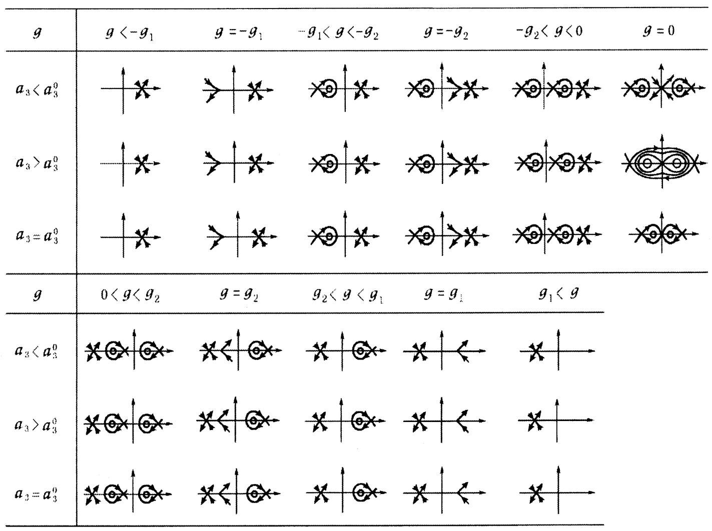
(i) When $f'(\bar{\varphi}) < 0$, $(\bar{\varphi}, 0)$ is a center point.

(ii) When $f'(\bar{\varphi}) > 0$, $(\bar{\varphi}, 0)$ is a saddle point.

(iii) When $f'(\bar{\varphi}) = 0$, $(\bar{\varphi}, 0)$ is a degenerate saddle point.

From the above analysis, subject to (3.4) and p is even, we obtain the bifurcation phase portraits as presented in Table 1.

Table 1. The bifurcation phase portraits of System (1.9), when p is even, $g_1 > g_2$ and the parameters satisfy (3.4)



Using the bifurcation phase portraits and Proposition 1 we can determine the existence of solitary waves and kink waves. Under certain conditions, we can give their explicit representations.

4 Some Explicit Expressions of Long-waves

In this section, we look for some explicit expressions of solitary waves and kink waves. We will always assume $g = 0$ in what follows.

4.1 Solitary Wave Solutions

When $a_3 > a_3^0$, p is even and (3.4) holds, from Table 1 we see that (1.9) has two homoclinic orbits Γ_1 and Γ_2 which connect with singular point $(0, 0)$. To facilitate further discussions, we redraw the $\Gamma_i (i = 1 - 2)$ as Fig.2.

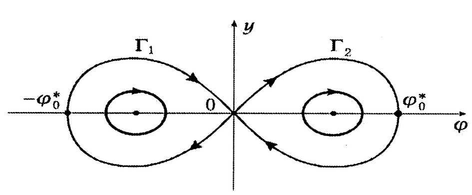


Figure 2. Homoclinic orbits, when $g = 0$, $a_3 > a_3^0$ and condition (3.4) holds

In $\varphi - y$ plane, Γ_1 and Γ_2 are described as

$$y^2 = \frac{\varphi^2(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p - \frac{a_3}{p+1}\varphi^{2p})}{c^2}, \qquad \varphi \in [-\varphi_0^*, \varphi_0^*], \tag{4.1}$$

where

$$\varphi_0^* = \Big(\frac{-(p+1)a_2}{(p+2)a_3} + \frac{p+1}{a_3}\sqrt{\frac{a_2^2}{(p+2)^2} + \frac{a_3(c^2-a_1)}{p+1}}\Big)^{1/p}. \tag{4.2}$$

Substituting (4.1) into $\frac{d\varphi}{d\xi} = y$, we have

$$\pm\sqrt{\frac{c^2}{\varphi^2\Big(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p - \frac{a_3}{p+1}\varphi^{2p}\Big)}}\, d\varphi = d\xi. \tag{4.3}$$

Assuming $\varphi(0) = \varphi_0^*$ and integrating (4.3) along homoclinic orbit Γ_2, we also got

$$\int_{\varphi}^{\varphi_0^*} \frac{|c|}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_3}{p+1}s^{2p}}} ds = \int_{\xi}^{0} ds \qquad \text{for} \quad \xi < 0, \tag{4.4}$$

and

$$-\int_{\varphi_0^*}^{\varphi} \frac{|c|}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p - \frac{a_3}{p+1}s^{2p}}} ds = \int_{0}^{\xi} ds \qquad \text{for} \quad \xi \geq 0. \tag{4.5}$$

Completing above two integrals we obtain

$$u_1(x,t) = \varphi(x - ct) = \Big(\frac{(p+2)(c^2-a_1)}{\delta \mathrm{ch}\beta(x-ct) + a_2}\Big)^{1/p}, \tag{4.6}$$

where

$$\delta = \sqrt{\frac{(p+1)a_2^2 + (p+2)^2(c^2-a_1)a_3}{p+1}}, \tag{4.7}$$

$$\beta = \frac{p\sqrt{c^2-a_1}}{|c|}. \tag{4.8}$$

From the above derivations we have the following theorem.

Theorem 1. *Eq.(1.1) has solitary wave solutions $\pm u_1(x,t)$ when p is even, $c^2 > a_1$ and one of the following conditions is satisfied*
(1°) $a_2 > 0$ *and* $a_3 > a_3^0$,
(2°) $a_2 < 0$ *and* $a_3 > 0$.

Proof. For Condition (1°), we have already obtained Equations (4.4) and (4.5). For Condition (2°) and $g = 0$, using (3.1) and (3.6), we know that System (1.9) has three singular points, namely, $(0,0)$ and $(\pm\varphi_2, 0)$. It follows from that from (3.12), $(0,0)$ is saddle point, and $(\pm\varphi_2, 0)$ are center points. Therefore System (1.9) has two homoclinic orbits which are similar to that depicted in Fig.2. Now we study the integrals in (4.4) and (4.5). Let

$$A = c^2 - a_1, \qquad B = -\frac{2a_2}{p+2}, \qquad C_0 = -\frac{a_3}{p+1}, \tag{4.9}$$

and

$$\tau = s^p, \quad \tau_0 = (\varphi_0^*)^p, \quad \tau_1 = \varphi^p, \quad X(\tau) = A + B\tau + C_0\tau^2. \tag{4.10}$$

Thus (4.4) and (4.5) merge into

$$\int_{\tau_1}^{\tau_0} = \frac{d\tau}{\tau\sqrt{X(\tau)}} = \frac{p\ |\xi|}{|c|}. \tag{4.11}$$

By completing the integral in (4.11), we get

$$\frac{\sqrt{X(\tau_1)}+\sqrt{A}}{\tau_1} = \Big(\frac{\sqrt{A}}{\tau_0}+\frac{B}{2\sqrt{A}}\Big)e^{\sqrt{A}\ p|\xi|/|c|} - \frac{B}{2\sqrt{A}}. \tag{4.12}$$

Letting $\eta = \frac{\sqrt{A}\ p|\xi|}{|c|}$, we have

$$\sqrt{A+B\varphi^p+C_0\varphi^{2p}} = \varphi^p\Big[\Big(\frac{\sqrt{A}}{\tau_0}+\frac{B}{2\sqrt{A}}\Big)e^{\eta} - \frac{B}{2\sqrt{A}}\Big] - \sqrt{A}. \tag{4.13}$$

It follows from solving (4.13) that

$$\varphi^p = \frac{2\alpha_0\sqrt{A}\ e^{\eta}}{\alpha_0^2\ e^{2\eta} - 2\alpha_0\delta_0 e^{\eta} + \delta_0^2 - C_0}, \tag{4.14}$$

where

$$\alpha_0 = \sqrt{\frac{(p+1)a_2^2+(p+2)^2a_3A}{(p+1)}}, \tag{4.15}$$

and

$$\delta_0 = -\frac{a_2}{(p+2)\sqrt{A}}. \tag{4.16}$$

From (4.9), (4.15) and (4.16) we see that

$$\delta_0^2 - C_0 = \alpha_0^2. \tag{4.17}$$

Applying (4.17) to (4.14), we find

$$\varphi^p = \frac{\sqrt{A}}{\alpha_0\Big(\frac{e^{\eta}+e^{-\eta}}{2}\Big) - \delta_0}. \tag{4.18}$$

From (4.15), (4.16) and (4.18) we obtain $\varphi(x-ct)$ as that in (4.6). According to above analysis and Proposition 1 we confirm that $\pm u_1(x,t)$ are solitary wave solution of Eq.(1.1). This completes the proof.

4.2 Kink Wave Solutions

Similar to the above approach we can prove the following theorem.

Theorem 2. *If p is even, $a_2 > 0$, $c^2 > a_1$, $a_3 = a_3^0$, then Eq.(1.1) has four kink wave solutions*

$$\bar{u}_{\pm}(x,t) = \pm\Big(\frac{(p+2)(c^2-a_1)}{re^{-\beta(x-ct)}+a_2}\Big)^{1/p}, \tag{4.19}$$

and

$$\hat{u}_{\pm}(x,t) = \pm\Big(\frac{(p+2)(c^2-a_1)}{re^{\beta(x-ct)}+a_2}\Big)^{1/p}, \tag{4.20}$$

where β is as that in (4.8) and $r \neq -a_2$ is an arbitrary constant.

Proof. From Table 1 we see that when $g = 0$, $a_2 > 0, c^2 > a_1$ and $a_3 = a_3^0$, (1.9) has four heteroclinic orbits Γ_i $(i = 3, 4, 5, 6)$ connecting with singular point $(0, 0)$. We redraw them as illus-treated in Fig.3.

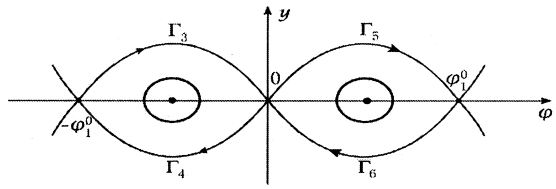


Figure 3. When $g = 0$, $a_2 > 0$, $c^2 > a_1$ and $a_3 = a_3^o$, the four heteroclinic orbits of system (1.9)

On $\varphi - y$ plane, Γ_3 and Γ_4 are given by

$$y^2 = \frac{\varphi^2}{c^2}\Big(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p + \frac{a_2^2}{(p+2)^2(c^2-a_1)}\varphi^{2p}\Big), \quad \text{for} \quad \varphi \in (-\varphi_1^0, 0), \tag{4.21}$$

and Γ_5 and Γ_6 are given by

$$y^2 = \frac{\varphi^2}{c^2}\Big(c^2 - a_1 - \frac{2a_2}{p+2}\varphi^p + \frac{a_2^2}{(p+2)^2(c^2-a_1)}\varphi^{2p}\Big), \quad \text{for} \quad \varphi \in (0, \varphi_1^0). \tag{4.22}$$

Substituting (4.22) into $\frac{d\varphi}{d\xi} = y$ and integrating it along Γ_5 and Γ_6 respectively, we have

$$\int_{\varphi_0}^{\varphi} \frac{ds}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p + \frac{a_2^2}{(p+2)^2(c^2-a_1)}s^{2p}}} = \int_0^{\xi} \frac{ds}{|c|} \quad (\text{along } \Gamma_5), \tag{4.23}$$

$$-\int_{\varphi_0}^{\varphi} \frac{ds}{s\sqrt{c^2 - a_1 - \frac{2a_2}{p+2}s^p + \frac{a_2^2}{(p+2)^2(c^2-a_1)}s^{2p}}} = \int_0^{\xi} \frac{ds}{|c|} \quad (\text{along } \Gamma_6), \tag{4.24}$$

where $\varphi(0) = \varphi_0 \in (0, \varphi_1^0)$ is a constant. Similarly, let

$$A = c^2 - a_1, \qquad B = -\frac{2a_2}{p+2}, \qquad C_1 = \frac{a_2^2}{(p+2)^2(c^2-a_1)}, \tag{4.25}$$

and

$$v = s^p, \qquad v_0 = \varphi_0^p, \qquad v_1 = \varphi^p, \qquad Z(v) = A + Bv + C_1 v^2. \tag{4.26}$$

Thus (4.23) and (4.24) respectively become

$$\int_{v_0}^{v_1} = \frac{dv}{v\sqrt{Z(v)}} = \frac{p\,\xi}{|c|}, \tag{4.27}$$

$$\int_{v_0}^{v_1} = \frac{dv}{v\sqrt{Z(v)}} = -\frac{p\,\xi}{|c|}. \tag{4.28}$$

Completing the integral in (4.27) and (4.28), we have

$$\frac{\sqrt{Z(v_1)} + \sqrt{A}}{v_1} = \alpha_1 e^{-\eta} - \frac{B}{2\sqrt{A}}, \tag{4.29}$$

and

$$\frac{\sqrt{Z(v_1)}+\sqrt{A}}{v_1}=\alpha_1 e^{\eta}-\frac{B}{2\sqrt{A}}, \tag{4.30}$$

where

$$\alpha_1=\frac{\sqrt{Z(v_0)}+\sqrt{A}}{v_0}+\frac{B}{2\sqrt{A}} \quad \text{and} \quad \eta=\frac{p\sqrt{A}\,\xi}{|c|}. \tag{4.31}$$

Solving equations (4.29) and (4.30), we have

$$\varphi^p=\frac{2\sqrt{A}}{\alpha_1 e^{-\eta}-2\delta_0}, \tag{4.32}$$

and

$$\varphi^p=\frac{2\sqrt{A}}{\alpha_1 e^{\eta}-2\delta_0}. \tag{4.33}$$

where δ_0 is in (4.16). From (4.32), (4.33) and Proposition 1 we confirm that $\bar{u}_{\pm}(x,t)$ and $\hat{u}_{\pm}(x,t)$ are kink wave solutions. This completes the proof.

When $p=2, a_2=4\sqrt{\frac{(a_1-c^2)a_3}{3}}$ and $r=a_2$, the four kink wave solutions $\bar{u}_{\pm}(x,t)$ and $\hat{u}_{\pm}(x,t)$ become the solutions (5.9) and (5.10) in [5].

4.3 Non-existence of Long-wave Solution

Similarly we can prove the non-existence of long-wave solution of Eq.(1.1).

Theorem 3. *If p is even, $c^2-a_1>0$ and a_2, a_3 satisfy one of the following conditions,*

(1°) $a_2>0,\ a_3\le \frac{a_2^2}{4(a_1-c^2)}$,

(2°) $a_2<0,\ a_3\le 0$,

then Eq.(1.1) has no long wave solutions.

Proof. Suppose that $u=\varphi(\xi)$ is a long-wave solution, that is, $u=\varphi(\xi)$ is a solitary wave solution or kink wave solution. From the limiting propertis of long wave solution, we konw that $\varphi'(\xi)$, $\varphi''(\xi)$ and $\varphi'''(\xi)$ tend to zero when $|\xi|$ tends to infinity. Thus $\varphi(\xi)$ and $y=\varphi'(\xi)$ satisfy System (1.9). This implies that System (1.9) has a homoclinic orbit which surrounds at least a center point and connects a saddle point, or a heteroclinic orbit which connects two saddle points.

On the other hand, under one of the above conditions, the function

$$f(\varphi)=(c^2-a_1)\varphi-a_2\varphi^{p+1}-a_3\varphi^{2p+1}, \tag{4.34}$$

has only a single zero point. Further, the function $f(\varphi)+g$ has also only a zero point too. This implies that our initial hypothesis is not sound. Thus the proof is completed.

5 Conclusion

In this paper, we have presented a generalized PC equation which includes several known models as special cases. We have also employed bifurcation method in dynamical systems to obtain some explicit expressions of solitary wave and kink wave with some previous results becoming our special cases.

For example, when $p=2, \pm u_1(x,t)$ become the solitary wave solutions which are as that in (5.7) in [5]. The four kink wave solutions $\bar{u}_{\pm}(x,t)$ and $\hat{u}_{\pm}(x,t)$ become the solutions (5.9) and (5.10) in [5] when $p=2, a_2=4\sqrt{\frac{(a_1-c^2)a_3}{3}}$ and $r=a_2$.

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