FOLIATIONS WITH POSITIVE SLOPES AND BIRATIONAL STABILITY OF ORBIFOLD COTANGENT BUNDLES

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ABSTRACT

Let X be a smooth connected projective manifold, together with an snc orbifold divisor Δ, such that the pair (X, Δ) is log-canonical. If $K_X + \Delta$ is pseudo-effective, we show, among other things, that any quotient of its orbifold cotangent bundle has a pseudo-effective determinant. This improves considerably our previous result (Campana and Păun in Ann. Inst. Fourier. 65:835, 2015), where generic positivity instead of pseudo-effectivity was obtained. One of the new ingredients in the proof is a version of the Bogomolov-McQuillan algebraicity criterion for holomorphic foliations whose minimal slope with respect to a movable class (instead of an ample complete intersection class) is positive.

CONTENTS

1. Introduction

In the present text we evaluate the *positive directions* of the tangent bundle of a projective manifold by means of the slope of its subsheaves with respect to classes of movable

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curves. The crucial property is the birational nature of this notion. We show that the positive directions of the tangent bundle are given by fibering-type contractions birationally preserved. They are the same ones as those appearing in the Log-minimal model program, which rest on much more delicate notions and arguments, in some sense dual to those presented here.

Our basic tool and starting point is Theorem 1.1 below. This result is also valid in the orbifold context, considerably extending its range of applicability. The arguments we use lead to positivity/negativity properties of tensor powers of orbifold cotangent bundles. In this article, we work in characteristic zero exclusively. Classical results on rational curves are known, and thus quoted here, only when the orbifold divisor is zero.

Let X be a projective manifold, and let $\mathcal{F} \subset \mathrm{TX}$ be a holomorphic foliation. Given a movable class $\alpha \in \mathrm{H}^{n-1,n-1}(\mathrm{X}, \mathbf{R})$ on X, the condition: $\mu_{\alpha,\min}(\mathcal{F}) > 0$ means that the inequality of intersection numbers:

$$c_1(\mathcal{Q}).\alpha > 0, \tag{1}$$

holds for any non-zero quotient $\mathcal{F} \to \mathcal{Q} \to 0$.

Our first main result is:

Theorem **1.1**. — *Let* X *be projective smooth, and let* $\mathcal{F} \subset \mathrm{TX}$ *be a foliation such that* $\mu_{\alpha,\min}(\mathcal{F}) > 0$ *for some movable class* α. *Then* $\mathcal{F}$ *is an algebraic foliation and the closure of its leaves are rationally connected.*

The algebraicity statement is the *movable version* of the Bogomolov-McQuillan algebraicity criterion [7], where the class α is a complete intersection class $[\mathrm{C}] = [\mathrm{H}]^{n-1}$ of (very) ample hypersurfaces. The condition (1) in this case means that the restriction $\mathcal{F}|_{\mathrm{C}}$ is ample. The proof given here follows the ideas from [7], strengthened by the theory of semi-stability with respect to movable classes introduced and developed in [18]. A main difference with [7] is that we do not restrict to movable curves of the given class. The failure of Mehta-Ramanathan in this context would anyhow prevent from doing this.

The rational connectedness statement is obtained by a simple and direct combination of several results: the existence of a 'relative rational quotient' for any fibration, the pseudo-effectivity of the canonical bundle of its base by [25], and Theorem 3.4 below, asserting (in a more refined version) the pseudo-effectivity of the relative canonical bundle of a fibration having generic fibres with pseudo-effective canonical bundle. The slope considerations are central in this proof, as well as their birational preservation in the case of movable classes.

This proof radically differs from the previous ones given in [7] and [35] in the special case of α a complete intersection class.

Theorem 1.1 plays a crucial role in the proof of the next statement (we refer to Theorem 4.9 in Section 4 for a more complete statement).

Theorem **1.2**. — *Let* X *be a non-singular projective manifold, and let* $\mathcal{F}$ *be a foliation on* X *having canonical singularities and such that* $K_{\mathcal{F}}$ *pseudo-effective. For any* $m > 0$, *the determinant bundle of any coherent, torsion-free quotient of* $\otimes^m \Omega^1_{\mathcal{F}}$ *is pseudo-effective.*

When K_X is pseudo-effective, Theorem 1.2 follows directly from Theorem 1.3 below, since $\Omega^1_{\mathcal{F}}$ is a quotient of Ω^1_X. Remark however that it is interesting in its own right, since foliations $\mathcal{F}$ with $K_{\mathcal{F}}$ pseudo-effective on some uniruled X′*s* do exist (cf. [19]). In fact, Theorem 1.2 does not seem provable with the methods of [19], even using also [5].

Our original motivation was to establish the birational stability of the cotangent bundle $\Omega^1(X, \Delta)$ of smooth log-canonical orbifold pairs (X, Δ) for which $K_X + \Delta$ is pseudo-effective (this last condition being essentially necessary).

Recall (cf. [19]) that the orbifold cotangent bundle is defined by lifting logarithmic differentials with denominators of fractionary exponents to suitable ramified covers $\pi : X_\Delta \to X$ adapted to the pair (X, Δ). The ramified cover π is Galois, and we denote by G the corresponding group.

Theorem **1.3**. — *Let* (X, Δ) *be a smooth projective log-canonical pair, with pseudo-effective canonical bundle* $K_X + \Delta$. *Let* $\mathcal{Q}$ *be any quotient of the tensor power* $\otimes^m \pi^\star \Omega^1(X, \Delta)$, $m \geq 1$ *being any integer.*

For any movable class α *on* X, *we then have, on* X_Δ*:*

$$c_1(\mathcal{Q}).\pi^\star\alpha \geq 0 \tag{2}$$

If $\Delta = 0$, and if K_X is pseudo-effective, this says that the determinant of any quotient of $\otimes^m \Omega^1_X$ is pseudo-effective, strengthening a fundamental result of Y. Miyaoka [41] stating that $\Omega^1_X|_C$ is *nef* for any sufficiently generic complete intersection curve $C \subset X$. In [18], Theorem 1.3 is stated[1] as Theorem 1.4 when $\Delta = 0$. In [19] we obtained the analog of Miyaoka's theorem for log-canonical orbifolds.

In order to illustrate the main ideas, we now sketch the proof of Theorem 1.3 in case $\Delta = 0$.

Assume by contradiction the existence of a sheaf $\mathcal{Q}$ and a movable class α as above, such that the inequality (2) is not satisfied. By dualising, this means that the maximal destabilising subsheaf $\mathcal{F}$ of T_X has a positive α-slope. The algebraicity criterion of Theorem 1.1 shows that $\mathcal{F}$ defines a rational map $p : X \dashrightarrow Z$, the generic fibre of which is the Zariski-closure of a leaf of $\mathcal{F}$, and rationally connected. This contradicts the pseudo-effectivity of K_X.

If $\Delta \neq 0$, then the proof of Theorem 1.3 requires several constructions of foundational nature. They are related to the notion of holomorphic orbifold tensors, which

[1] As pointed out by A. Langer, there is a gap in the proof of Theorem 1.4 of [18]. On page 49, the reference to Y. Miyaoka's paper in Comment. Math. Univ. St Paul. 42 (1993), pp. 1–7 is indeed used in a context which is not covered by this text. This does not affect the results of Sections 3 and 5 of [18]. In fact, all statements of [18] are true, as special cases of the ones in the present text.

is exposed in detail in Section 5. As we have already mentioned, the holomorphic tensors corresponding to (X, Δ) are defined on suitable covers $\pi : X_\Delta \to X$. An equally important technical tool is the orbifold version of Lie bracket. We show that the orbifold tangent bundle is closed with respect to this operation, and we derive an orbifold version of Frobenius integrability criteria. The inverse image $\pi^\star T_X$ is not a sub-sheaf of T_{X_Δ}, and additional arguments are needed in this broader context.

The difference in the conclusions of Theorem 1.4 and Theorem 1.1 is due to the absence of a theory of rationally connected objects in the category of orbifold pairs.

A consequence of the results we develop in orbifold setting is the following version of Theorem 1.1, as follows.

Theorem **1.4**. — *Let* (X, Δ) *be a smooth projective log-canonical pair. Let* $\mathcal{F} \subset \pi^* T(X, \Delta)$ *be a saturated* G*-invariant subsheaf such that:*

(1) $\mu_{\alpha,\min}(\mathcal{F}) > 0$,
(2) $\mu_{\alpha,\min}(\mathcal{F}) > \frac{1}{2}\mu_{\alpha,\max}(\pi^* T(X, \Delta)/\mathcal{F})$.

The saturation of $\mathcal{F}$ *in* $\pi^*(TX)$ *then is equal to the* π*-inverse image of a coherent sheaf* $\mathcal{F}' \subset T_X$*. Moreover,* $\mathcal{F}'$ *defines an algebraic foliation on* X *such that the restriction of* $K_X + \Delta$ *to the closure* F′ *of the generic leaf of* $\mathcal{F}'$ *is not pseudo-effective.*

Structure of the text. — Section 2 recalls the notions and results needed here about the *stability with respect to a movable class,* introduced in [18].

Section 3 studies the positivity properties of the *relative canonical bundle* of a rational map. In particular, its degree on lifts of movable classes is preserved under modifications.This permits a reduction to 'neat' models of arbitrary rational fibrations.

Section 4 establishes Theorems 1.1 and 1.2.

In the next two sections we treat the orbifold version of these results.

Section 5 reviews the definition of the orbifold (co)tangent bundles. For the smooth log-canonical pairs (X, Δ) considered here, these objects admit an explicit simple description on suitable ramified covers introduced by Y. Kawamata.

The notion of *Lie derivative* in orbifold setting is introduced here. This operator is deduced from the lift of the Lie derivative of T_X. We establish a version of the classical Frobenius integrability criteria, in the following sense. If $\mathcal{F}_\Delta \subset \pi^\star T(X, \Delta)$ is a saturated and G-invariant subsheaf for which the orbifold Lie bracket vanishes, then $\mathcal{F}_\Delta$ is the π-inverse image of a holomorphic foliation $\mathcal{F}$ on X.

Section 6 gives the proofs of Theorems 1.4 and 1.3, by combining the previous preparatory results.

Section 7 deals with the *birational stability* of the orbifold cotangent bundle of (X, Δ) if $K_X + \Delta$ is pseudo-effective. This means that the numerical dimension of any sub-line bundle L of $\otimes^m \pi^* \Omega^1(X, \Delta)$ is bounded by the numerical dimension of $K_X + \Delta$.

Combined in Section 8 with the work of Viehweg-Zuo [50], these results permit to compare the variation of families of projective manifolds with ample canonical bundles to the canonical bundle on the base of the family. Related results by B. Taji and Popa-Schnell are mentioned ([49] and [47]).

2. Slope and semi-stability with respect to movable classes

We will collect in this section a few results concerning the notion of slope stability of a sheaf with respect to a movable class. They were introduced in [18]. These results play a crucial role in the proof of our algebraicity criteria, cf. section four. See [26] for a detailed and extended presentation.

2.1. *The movable cone.* — To start with, let $N_1(X)_{\mathbf{R}}$ be the space of numerical curves classes on X. We recall the following notion.

Definition **2.1**. — *A class* $\alpha \in N_1(X)_{\mathbf{R}}$ *is called movable if we have* $\alpha \cdot D \geq 0$ *for any effective divisor* D. *The set of such classes form a closed convex cone denoted by* Mov(X) *and called the movable cone.*

A movable class is said to be rational if it belongs to $N_1(X)_{\mathbf{Q}}$.

By the main result in [6], the set Mov(X) is the closed convex cone in $N_1(X)_{\mathbf{R}}$ generated by the classes [C] of 'movable curves', where an irreducible curve is said to be 'movable' if it is a member of a covering algebraic family of curves on X parametrised by an irreducible projective variety. This is also the closed cone generated by the classes of curves of the form $\pi_\star(H_1 \cap \dots \cap H_{n-1})$, where $\pi : \widehat{X} \to X$ is a modification of the manifold X, and the H_j's are hyperplane sections of $\widehat{X}$.

Let $p : Y \to X$ is be a generically finite map between two smooth projective manifolds. If $\beta \in N_1(Y)_{\mathbf{R}}$ is a movable class on Y, then the *p-inverse image of* β is denoted by $p^\star(\beta) \in N_1(X)_{\mathbf{R}}$ and it is defined in such a way that the projection formula holds true. Alternatively, β can be represented by a closed real form of bi-degree $(n-1, n-1)$ and then $p^\star(\beta)$ is the class defined by the pull-back of the said form. It is clear that $p^\star(\beta) \in \mathrm{Mov}(X)$ as soon as we have $\beta \in \mathrm{Mov}(Y)$.

2.2. *Slopes associated to a movable class.* — Let $\mathcal{E} \neq 0$ be a coherent, torsion-free sheaf on X; let $\det \mathcal{E}$ its determinant, that is, the bi-dual of its top power. This is a line bundle on X with first Chern class $c_1(\mathcal{E})$. If $\alpha \in \mathrm{Mov}(X)$ is a movable class the α-slope $\mu_\alpha(\mathcal{E})$ of $\mathcal{E}$ is:

$$\mu_\alpha(\mathcal{E}) := \frac{c_1(\mathcal{E}).\alpha}{\mathrm{rk}(\mathcal{E})} \tag{3}$$

The α-semi-stability is defined as usual.

Definition **2.2**. — *The torsion-free coherent sheaf* $\mathcal{E}$ *is* α*-semistable if*

$$\mu_\alpha(\mathcal{G}) \leq \mu_\alpha(\mathcal{E}) \tag{4}$$

for any non-trivial coherent subsheaf $\mathcal{G} \subset \mathcal{E}$.

The α-stability (not used here) is defined in a similar manner, the inequality (4) being strict if the rank of $\mathcal{G}$ is strictly smaller than the rank of $\mathcal{E}$.

As showed in [18], essentially all of the properties of the classical slope-stability theory still hold in this extended setting. A crucial exception is the Mehta-Ramanathan theorem (see Example 4.3 below).

The construction of Harder-Narasimhan filtrations with respect to movable classes also holds, with the same properties. We only state the results for smooth projective manifolds since this is the only case needed here. As observed in [26], the theory adapts immediately when the variety X is $\mathbf{Q}$-factorial.

Definition **2.3**. — *Let* X *be a non-singular manifold, and let* $\mathcal{E}$ *be a coherent, torsion-free sheaf of positive rank on* X. *We define:*

$$\mu_{\alpha,\max}(\mathcal{E}) := \sup\left\{\mu_\alpha(\mathcal{F}) : \mathcal{F} \subset \mathcal{E},\ \textit{any nonzero coherent subsheaf}\right\} \tag{5}$$

as well as its dual version:

$$\mu_{\alpha,\min}(\mathcal{E}) := \inf\left\{\mu_\alpha(\mathcal{Q}) : \mathcal{E} \to \mathcal{Q} \to 0\right\} \tag{6}$$

where the quotient sheaf $\mathcal{Q}$ *in* (6) *is coherent, non-zero and torsion-free.*

We quote next the following result.

Proposition **2.4**. — *[18] There exists a non-zero, coherent sheaf, unique and maximal for the inclusion* $\mathcal{F} \subset \mathcal{E}$ *such that we have*

$$\mu_\alpha(\mathcal{F}) = \mu_{\alpha,\max}(\mathcal{E}). \tag{7}$$

The supremum in (5) *is thus a maximum. Moreover, the sheaf* $\mathcal{F}$ *is saturated in* $\mathcal{E}$.

The sheaf $\mathcal{F}$ in Proposition 2.4 is obviously α-semistable; it is called the maximal destabilising subsheaf.

The following simple vanishing criterion for sections of coherent sheaves in terms of the slope function will be used here.

Lemma **2.5** *([18]).* — *Let* $\mathcal{E}$ *be a coherent, torsion-free sheaf.*
If $\mu_{\alpha,\max}(\mathcal{E}) < 0$ *for some movable class* α, *then* $H^0(X, \mathcal{E}) = 0$.
More generally: $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') = 0$, *if* $\mu_{\alpha,\min}(\mathcal{E}) > \mu_{\alpha,\max}(\mathcal{E}')$.

For example, the first claim of this lemma applies if $\mathcal{E}$ is α-semistable and of negative slope.

We will use the following in the proof of Theorem 2.10:

Proposition **2.6**. — *Let* $\pi : X' \to X$ *a finite Galois ramified cover of group* G *and degree* d *between complex and connected projective manifolds. Let* α *be a movable class on* X, *and let* $\mathcal{E}$ *be a torsion* *free coherent sheaf on* X. *We define* $\mathcal{E}' := \pi^*(\mathcal{E})$ *and* $\alpha' := \pi^*(\alpha)$ *the respective inverse images. Then we have*

$$\mu_{\alpha,\max}(\mathcal{E}) = \frac{1}{d}\mu_{\alpha',\max}(\mathcal{E}'),$$

and $\mathcal{F}' := \pi^*(\mathcal{F})$ *is the maximal destabilising subsheaf of* $\mathcal{E}'$, *if* $\mathcal{F}$ *is the maximal destabilising subsheaf of* $\mathcal{E}$.

Proof. — Let $\mathcal{F}'$ be the maximal destabilising subsheaf of $\mathcal{E}'$ (with respect to α'). It is G-invariant, since so is $\mathcal{E}'$. Moreover, $\mathcal{F}'$ is saturated in $\mathcal{E}'$ by Proposition 2.4. Then we have $\mathcal{F}' = \pi^*(\mathcal{F})$ for some torsion free coherent sheaf $\mathcal{F}$ on X, cf. [26] and thus we infer the inequality $d \cdot \mu_{\alpha,\max}(\mathcal{E}) \geq \mu_{\alpha',\max}(\mathcal{E}')$.

Since we clearly have $d \cdot \mu_{\alpha,\max}(\mathcal{E}) \leq \mu_{\alpha',\max}(\mathcal{E}')$, the claim follows. □

2.3. *Tensor products.* — If $\mathcal{E}_1, \mathcal{E}_2$ be two coherent, torsion-free sheaves on X. We denote by $\mathcal{E}_1 \widehat{\otimes} \mathcal{E}_2$ the reflexive hull $(\mathcal{E}_1 \otimes \mathcal{E}_2)^{\star\star}$.

The following fundamental result was established in [18] if the class α is either rational or in the interior of Mov(X). When $\mathcal{E}_j$ are vector bundles, the proof given by Matei Toma relies on the deep analytic Kobayashi-Hitchin correspondance for Gauduchon metrics established by Li-Yau [39]. For α arbitrary, the proof given in [26], is treated by reduction to this basic case.

Theorem **2.7** *([18, 26]).* — *Let* α *be a movable class on* X; *if* $\mathcal{E}_j$ *as above are both* α*-semistable, then so is* $\mathcal{E}_1 \widehat{\otimes} \mathcal{E}_2$.

The slope behaves well under tensor operations. We only mention next the few properties used here, and we refer to the articles quoted above for a complete proof.

Proposition **2.8**. — *Let* $\mathcal{E}, \mathcal{G}$ *be two torsion-free coherent sheaves, and let* α *be a movable class. Then we have the following properties:*

(1) *The slope of the tensor product equals*

$$\mu_\alpha(\mathcal{E} \widehat{\otimes} \mathcal{G}) = \mu_\alpha(\mathcal{E}) + \mu_\alpha(\mathcal{G}).$$

(2) *For each* $m \geq 1$ *we have*

$$\mu_\alpha\big(\mathrm{Sym}^m(\mathcal{E})\big)^{\star\star} = m\mu_\alpha(\mathcal{E}).$$

(3) *The slope of the exterior product equals*

$$\mu_\alpha\big(\wedge^2(\mathcal{E})\big)^{\star\star} = 2\mu_\alpha(\mathcal{E}).$$

(4) *Moreover, if* $\mathcal{E}$ *and* $\mathcal{G}$ *are semistable with respect to* α, *then the sheaves* $\mathcal{E} \widehat{\otimes} \mathcal{G}$, $(\mathrm{Sym}^m(\mathcal{E}))^{\star\star}$ *and* $(\wedge^2(\mathcal{E}))^{\star\star}$ *are equally* α*-semistable.*

The following statement is established in [26] as consequence of the existence of the Harder-Narasimhan filtration with respect to a mobile class.

Theorem **2.9** *([18, 26]). — Let $\mathcal{E}, \mathcal{G}$ be two torsion-free coherent sheaves, and let α be a movable class. Then we have*

$$\mu_{\alpha,\max}(\mathcal{E}\widehat{\otimes}\mathcal{G}) = \mu_{\alpha,\max}(\mathcal{E}) + \mu_{\alpha,\max}(\mathcal{G}), \tag{8}$$

together with the corresponding relations for $\mu_{\alpha,\min}$. Similar identities hold true if we replace the reflexive tensor product in (8) with $(\mathrm{Sym}^m\,\mathcal{E})^{\star\star}$ or with $(\wedge^2\mathcal{E})^{\star\star}$.

The results collected here are relevant in the context of the vanishing criterion we discuss next.

2.4. *A vanishing criterion: from exterior to tensor powers.* — We consider the following situation: $\pi : X' \to X$ is a finite ramified Galois cover of group G between two connected complex projective manifolds. Let E′ be a G-invariant holomorphic vector bundle on X′, and L′ be any numerically trivial line bundle on X′. We finally consider also ample movable classes $\alpha = H^{n-1}$, for H varying in an non-empty open subset of the polarisation classes on X, with $\alpha' := \pi^*(\alpha)$ their inverse images on X′. By [27], Proposition 6.5, these $\alpha' s$ thus cover a nonempty open subset U in the cone of movable classes on X. The proof of [27] consists in differentiating the map $p : H \to H^{n-1}$, the Hard Lefschetz theorem implying that it is submersive at any point of H (moreover, the authors also show the injectivity of p by a ingenious use of Khovanskii-Teissier inequalities).

Theorem **2.10**. — *We assume that the following holds*

(1) $\mu_{\pi^*(\gamma),\max}(E') \leq 0$, *for any* $\gamma \in U$.
(2) $H^0(X', \wedge^q E' \otimes L') = 0$, *for any* $q > 0$, *and* $L' \equiv 0$ *on* X′.

Then we have

$$H^0\big(X', \otimes^m E' \otimes L'\big) = 0,$$

for any $m > 0$ *and* $L' \equiv 0$ *on* X′.

Before proceeding to the proof, we remark that if $E' = \pi^*(E)$ for some vector bundle E on X, and if $H_1(X, \mathbf{Z}) = 0$,[2] then the hypothesis (2) above can be replaced by the weaker hypothesis:

(2′) $H^0(X, \wedge^q E) = 0$, for any $q > 0$.

and obtain the same conclusion.

[2] In particular, if the algebraic fundamental group $\widehat{\pi}_1(X)$ of X is trivial.

Proof of Theorem 2.10. — Assume by contradiction that we have a non-zero section of $\otimes^m E' \otimes L'$ for some $L' \equiv 0$. Then $\mu_{\alpha',\max}(\otimes^m E' \otimes L') = m.\mu_{\alpha',\max}(E') = 0$, for every $\alpha \in U$. There thus exists an α for which the maximal α'-destabilising subsheaf $\mathcal{F}' \subset E'$ has maximum rank $q > 0$. Because $\mu_{\beta',\max}(E') = 0$ for $\beta' = \pi^*(\beta)$ with β close to α, we see[3] that $\mathcal{F}'$ is also the β'-maximal destabilising subsheaf of E' (just write $\beta = \alpha + t.\gamma$, $t > 0$, and replace t by $-t$, just as in the proof of the vanishing of a derivative in a local maximum). Since $\mathcal{F}'$ is G-invariant, so is $\det(\mathcal{F}')$, and so $N.\det(\mathcal{F}') = \pi^*(L)$, for some $L \in \mathrm{Pic}(X)$, if $N := \mathrm{Card}(G) = \deg(\pi)$. Thus $N.\det(\mathcal{F}').\pi^*(\alpha) = \pi^*(L).\pi^*(\alpha) = N.L.\alpha = 0, \forall \alpha \in U$, and: $L \equiv 0$, so that, also: $\det(\mathcal{F}') \equiv 0$. Since $\det(\mathcal{F}') \subset \wedge^q E'$, we get that $H^0(E', \wedge^q E' \otimes L') \neq 0$, if $L' = -\det(\mathcal{F}')$. This contradicts the hypothesis 2, and proves the theorem.

Let us show how to modify the proof in order to get the conclusion from the hypothesis $2'$ if $E' = \pi^*(E)$ and if $H_1(X, \mathbf{Z}) = 0$: in this case indeed, by Proposition 2.6, $\mu_{\pi^*(\alpha),\max}(\pi^*(E)) = N.\mu_{\alpha,\max}(E)$ and the maximal $\pi^*(\alpha)$-destabilising subsheaf $\mathcal{F}'$ of $\pi^*(E)$ is the inverse image by π of the maximal destabilising subsheaf $\mathcal{F}$ of E. We thus obtain $\mathcal{F} \subset E$ such that $\det(\mathcal{F}) \equiv 0$, so that $\mathcal{O}_X \cong \det(\mathcal{F})$, because $H_1(X, \mathbf{Z}) = 0$, and so: $\mathcal{O}_X \cong \det(\mathcal{F}) \subset \wedge^q(E)$, contradicting $2'$. □

An illustration of the applications of this result is the following statement.

Corollary **2.11** *([21]). — Let* (X, D) *be a smooth orbifold pair with* X *projective smooth and* D *a reduced divisor on* X *with simple normal crossings. Assume that:*

(1) (X, D) *is Fano (i.e.:* $-(K_X + D)$ *is ample);*
(2) $H^0(X, \Omega^q_X(\mathrm{Log}(D)) \otimes L) = 0$, *for* $q > 0$ *and any* $L \equiv 0$ *in* $\mathrm{Pic}(X)$.

Then we have $H^0(X, \otimes^m \Omega^1_X(\mathrm{Log}(D)) \otimes L) = 0$ *for any* $m > 0$ *and* $L \equiv 0$ *on* X.

For the proof we refer to [21].

Remark that, in general, $H^0(X, \Omega^q_X(\mathrm{Log}(D))) \neq 0$ for Fano pairs (X, D) as above (as shown by $(\mathbf{P}^n, D)$, if D is a union of $k \leq n$ hyperplanes in general position, for which $h^0(X, \Omega^1_X(\mathrm{Log}(D))) = k$.)

2.5. *Birational invariance of slope-positive foliations.* — We consider a saturated distribution $\mathcal{F} \subset TX$, and a birational morphism $\pi : \widehat{X} \to X$, where $\widehat{X}$ is also non-singular. Then we get an induced distribution on $\widehat{\mathcal{F}} \subset T\widehat{X}$, as follows. The tangent bundle of $\widehat{X}$ can be seen as subsheaf of the π-inverse image $\pi^\star(T_X)$ of the tangent bundle of X, and we define $\widehat{\mathcal{F}} := \pi^\star(\mathcal{F}) \cap T_{\widehat{X}}$.

We establish next the preservation of the slopes under birational modifications. Although very simple, this observation is fundamental. It is also noticed in the very recent article [23]. It was already stated in [18], Section 5, but not used in the context of foliations.

[3] This clever observation was communicated to us by Matei Toma.

Lemma **2.12**. — *Let* $\pi : \widehat{X} \to X$ *be a birational morphism between two smooth and connected complex projective manifolds.*

Let $\mathcal{F} \subset TX$ *be a saturated distribution, and* $\widehat{\mathcal{F}} := \pi^*(\mathcal{F}) \cap T\widehat{X}$ *be its inverse image in* $T\widehat{X}$. *Let* α *be a movable class on* X, *and* $\widehat{\alpha} := \pi^\star \alpha$ *be its inverse image on* $\widehat{X}$. *Then* $\widehat{\alpha} \in \mathrm{Mov}(\widehat{X})$ *is a movable class on* $\widehat{X}$. *Moreover, we have*

$$\mu_\alpha(\mathcal{F}) = \mu_{\widehat{\alpha}}(\widehat{\mathcal{F}}) \tag{9}$$

as well as $\mu_{\alpha,\min}(\mathcal{F}) = \mu_{\widehat{\alpha},\min}(\widehat{\mathcal{F}})$.

Proof. — The fact that $\widehat{\alpha}$ is a movable class on $\widehat{X}$ is a direct consequence of the definition. We have $\mu_\alpha(\mathcal{F}) = \mu_{\widehat{\alpha}}(\pi^\star \mathcal{F})$; on the other hand, $\det(\widehat{\mathcal{F}})$ and $\det(\pi^\star \mathcal{F})$ differ by an (effective) π-exceptional divisor E on $\widehat{X}$. Since $\widehat{\alpha} \cdot E = 0$ for any such divisor, the statement is proved. □

Remark **2.13**. — In particular, both slopes in (9) are simultaneously positive, negative, or zero *provided that the class* $\widehat{\alpha}$ *is the inverse image of the movable class* α *on* X. However, in the other-contracting-direction, this type of preservation of slope-positivity with respect to movable classes β and $\pi_\star \beta$ on $\widehat{X}$ and X respectively may fail, as illustrated by the following example. Let $\mathcal{F}$ be the foliation given by a generic pencil of conics on $\mathbf{P}^2$. The slope of $\mathcal{F}$ is negative, but becomes positive on the blow-up $\widehat{\mathbf{P}}^2$ of the four base-points, if one chooses for β on $\widehat{\mathbf{P}}^2$, for example the class of the strict transform of a generic member of the given pencil of conic, or a suitable ample class. We thank J. Pereira for this observation and for this example. This example will reappear below in the context of Theorem 4.9.

Remark **2.14**. — Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two torsion-free coherent sheaves having the same rank (we recall that in this context, the sheaves $\mathcal{F}_i$ are locally free outside a subset of co-dimension at least two, and their respective rank is defined via the associated vector bundles). If $\mu_{\alpha,\min}(\mathcal{F}_1) > 0$, then we equally have $\mu_{\alpha,\min}(\mathcal{F}_2) > 0$. This is a consequence of the fact that $\det(\mathcal{F}_2) = \det(\mathcal{F}_1) \otimes \mathcal{O}(D)$ for some *effective* divisor D. Notice, by contrast, that if $\mathcal{F}_1$ is semi-stable, $\mathcal{F}_2$ need not be semi-stable (see Remark 4.18).

3. Pseudoeffectivity of relative canonical bundles

Let $p : X \dashrightarrow Z$ be a dominant and connected rational map, where X and Z are non-singular projective manifolds. In this section we define the 'saturated' relative canonical bundle of p, and establish some of its birational positivity properties with respect to a movable class. This relies in particular on the preceding observation that changing both X and the birational model Z, and lifting α, the slopes of the corresponding sheaves are preserved.

Let $X_0 \subset X$ be the largest Zariski open set such that the restriction

$$(\mathbf{10}) \qquad p|_{X_0} : X_0 \to Z$$

of our given rational map p is holomorphic; in particular we have $\mathrm{codim}_X(X \setminus X_0) \geq 2$. The map (10) above will be denoted by p_0 in the sequel.

Definition **3.1**. — *Let* K_Z *be any divisor on* Z *in the canonical class. Let* $p^\star K_Z$ *be the closure* $\overline{p_0^\star K_Z}$ *of the analytic cycle* $p_0^\star K_Z$ *of* X_0. *The relative canonical bundle of* p *is*

$$(\mathbf{11}) \qquad K_{X/Z} := K_X - p^\star K_Z.$$

We introduce next the divisor $D(p)$ on X by:

$$(\mathbf{12}) \qquad D(p) := \sum_k (t_k - 1) F_k,$$

where the hypersurfaces F_k in (12) are all the irreducible divisors of X which, restricted to X_0, are mapped by p_0 to divisors $G_k \subset Z$, such that $p_0^\star G_k$ vanishes to the order $t_k \geq 2$ along F_k. It is a key object in the study of *holomorphic foliations*, whose definition will be recalled next.

Definition **3.2**. — *A foliation on a manifold* X *is a coherent subsheaf* $\mathcal{F} \subset T_X$ *enjoying the following properties:*

(i) $\mathcal{F}$ *is closed under the Lie bracket, and*
(ii) *The quotient* $T_X/\mathcal{F}$ *is torsion-free, i.e.* $\mathcal{F}$ *is saturated in* T_X.

Let $X_0 \subset X$ be the maximal open subset of X such that the restriction $\mathcal{F}|_{X_0}$ is a sub-bundle. We note that the codimension of the complement $X \setminus X_0$ in X is at least two, given that $\mathcal{F}$ is torsion-free. A *leaf* of $\mathcal{F}$ is a connected, locally closed holomorphic sub-manifold $L \subset X_0$ whose tangent bundle coincides with $\mathcal{F}$, i.e. $T_L = \mathcal{F}|_L$. A leaf L is called algebraic if it is open in its Zariski closure.

For example, the kernel of the differential of a rational map $p : X \dashrightarrow Z$ defines a foliation on X, whose leaves are algebraic. Even if $\Delta = 0$, the relevance of the divisor $D(p)$ to the study of foliations is explained by the following remark 3.3. This is certainly well-known to experts. We will not give the proof of this statement here, because the more general orbifold version will be established in Lemma 5.12.

Remark **3.3**. — Let $p : X \dashrightarrow Z$ be a dominant rational fibration, and let $\mathcal{F}$ be a foliation on X such that $\mathcal{F} = \mathrm{Ker}(dp)$. Let $\pi_X : \widehat{X} \to X$ and $\pi_Z : \widehat{Z} \to Z$ be modifications of X and Z, respectively, such that the following properties are satisfied:

(i) The induced map $\widehat{p} : \widehat{X} \to \widehat{Z}$ is regular, its discriminant locus E is a snc divisor and so it is the inverse image $\widehat{p}^{-1}(E)$.

(ii) If a component W of $\widehat{p}^{-1}(\mathrm{E})$ is $\widehat{p}$-exceptional, then it is also π_X-exceptional.

Let $\widehat{\mathcal{F}}$ be the foliation induced by $\mathcal{F}$ on $\widehat{X}$; then we have the equality

$$K_{\widehat{\mathcal{F}}} = K_{\widehat{X}/\widehat{Z}} - D(\widehat{p}). \tag{13}$$

modulo a divisor which is π_X-exceptional, cf. Lemma 5.12 and [23]. In particular, if K_X is pseudo-effective, then so it is $K_{\mathcal{F}}$, by the crucial Theorem 3.4 below.

In the previous remark, we denote by

$$K_{\mathcal{F}} := \det\big(\mathcal{F}^\star\big) \tag{14}$$

the *canonical bundle of a foliation* $\mathcal{F}$.

The main result of this section is the following one. A similar observation is made in [23], Proposition 4.3 and the references there.

Theorem **3.4**. — *Let* (X, Δ) *be a lc pair, such that* X *is smooth and such that* Δ *is snc. Assume that* $K_X + \Delta$ *is pseudo-effective. Then for any rational map* p *as in Remark 3.3, the divisor*

$$K_{X/Z} + \Delta^{\mathrm{hor}} - D(p)$$

is pseudo-effective.

In the statement Theorem 3.4 above we denote by Δ^{hor} the divisor having the same multiplicities as Δ on the irreducible hypersurfaces of X which project onto Y via the map p, and zero for any other hypersurfaces.

Proof. — We shall deduce this statement from Theorem 2.11 in [19].

Consider a holomorphic birational model of p. There exists a modifications $\pi_X : \widehat{X} \to X$ of X for which the next properties hold:

(1) The induced map $\widehat{p} : \widehat{X} \to Z$ is holomorphic.
(2) The π_X inverse image of Δ is snc.

Define the divisor $\widehat{\Delta}$ by the usual formula:

$$E_1 + \pi_X^\star(K_X + \Delta) = K_{\widehat{X}} + \widehat{\Delta} \tag{15}$$

where E_1 is effective and π_X-exceptional and $(\widehat{X}, \widehat{\Delta})$ is lc.

By Definition 3.1 we deduce that we have the equality

$$\widehat{p}^\star K_Z = \pi_X^\star\big(p^\star K_Z\big) + E_2 \tag{16}$$

where E_2 is a π_X-exceptional divisor.

Combining (15) and (16) we get:

$$E_1 + \pi_X^\star(K_{X/Z} + \Delta) = K_{\widehat{X}/Z} + \widehat{\Delta} + E_2, \tag{17}$$

which is preserved when taking into account the multiplicity divisors of the maps p and $\widehat{p}$:

$$E_1 + \pi_X^\star\big(K_{X/Z} + \Delta - D(p)\big) = K_{\widehat{X}/Z} + \widehat{\Delta} - D(\widehat{p}) + E_2. \tag{18}$$

Notice however that the divisors (E_j) in (17) and (18) may be different, but for the notation simplicity we keep the same symbols. The point is that both of them are π_X-exceptional, and can be assumed to be effective.

Next we use the pseudo-effectivity theorem in [19], which implies that the **Q**-line bundle

$$K_{\widehat{X}/Z} + \widehat{\Delta}^{\mathrm{hor}} - D(\widehat{p}) \tag{19}$$

is pseudo-effective on $\widehat{X}$ (we remark that at this point the hypothesis (X, Δ) *is log-canonical* is used in an essential manner). The hypothesis in the statement [19] are indeed satisfied, since for any $z \in Z$ generic the restriction $K_{\widehat{X}_z} + \widehat{\Delta}|_{X_z}$ is pseudo-effective, since so is $K_X + \Delta$, and thus also $K_{\widehat{X}} + \widehat{\Delta}$.

The conclusion follows from the following simple statement.

Lemma **3.5**. — *Let* $\pi : \widehat{X} \to X$ *be a modification between projective manifolds. Let* $\widehat{L}, L$ *be line bundles on* $\widehat{X}$ *and* X *respectively. Assume that:*

$$\widehat{L} = \pi^* L + E_1 \tag{20}$$

for some π*-exceptional divisor* E_1 *on* $\widehat{X}$. *If* $\widehat{L}$ *is pseudo-effective, then so is* L.

Proof. — Let γ be a movable class on X. Then $\pi^\star\gamma$ is a movable class on $\widehat{X}$, and then we have

$$c_1(\widehat{L}).\pi^\star\gamma \geq 0.$$

By relation (20), we deduce that

$$c_1(L).\gamma \geq 0$$

since $E_1 \cdot \pi^\star\gamma = 0$, E_1 being exceptional.

Thus L is pseudo-effective, by [6]. □

The following alternative arguments for Lemma 3.5 were kindly pointed out to us by the referees. We reproduce them here (in arbitrary order), for the benefit of the readers.

• We have $(\pi_\star(\widehat{L}))^{\star\star} = L$, as it follows immediately from the assumptions of 3.5. Since the push-forward of a pseudo-effective class is pseudo-effective, we are done. This has the advantage of avoiding the use of [6].

• We consider an ample line bundle $\widehat{A} := \pi^\star(A) - E$ on $\widehat{X}$, where E is effective and π-exceptional, and A is ample on X. Then for any couple of positive integers the $k \gg m$ the bundle $k\widehat{L} + m\widehat{A}$ has non-identically zero sections. They are induced by the sections of $kL + mA$ (since E_1 is exceptional), and the proof is finished.

The proof of Theorem 3.4 is therefore finished, by (18) combined with Lemma 3.5. □

Remark **3.6**. — In the statement Theorem 3.4 above, if the dominant rational map $p : X \dashrightarrow Z$ is given, then the pseudo-effectivity of the bundle $K_{X/Z} + \Delta^{\rm hor} - D(p)$ is in fact equivalent to the pseudo-effectivity of $K_{\widehat{X}_z} + \Delta|_{\widehat{X}_z}$ for all points z in the complement of a Zariski closed subset of Z, cf. [19] (here we use the notations in the proof of Theorem 3.4). From this perspective, the hypothesis "$K_X + \Delta$ pseudo-effective" of Theorem 3.4 may look abusive. However the point is that this hypothesis insures the pseudo-effectivity of $K_{X/Z} + \Delta^{\rm hor} - D(p)$ even if the rational map p is not given a-priori (and it will be the case in what follows).

4. Algebraicity criteria for foliations

We begin this section by introducing the following notion—which maybe not standard, but it is very convenient for us.

Definition **4.1**. — *Let $\mathcal{F} \subset TX$ be a holomorphic foliation of rank r. We say that the foliation $\mathcal{F}$ is* algebraic *if it is induced by a rational map i.e. $\mathcal{F} = \ker(dp)$ generically on X, for some dominant rational map $p : X \dashrightarrow Z$.*

If this is the case, we see that all the leaves of $\mathcal{F}$ are algebraic subsets of X.

The main result of this section is the following statement.

Theorem **4.2**. — *Let X be a smooth projective manifold, and let $\mathcal{F}$ be a foliation on X such that there exists a movable class α for which we have*

$$\mu_{\alpha,\min}(\mathcal{F}) > 0. \tag{21}$$

The following assertions hold true.

(1) *The foliation $\mathcal{F}$ is algebraic.*
(2) *The closure of every leaf of $\mathcal{F}$ is rationally connected.*

If the class α is a complete intersection of ample hypersurfaces on X, and if $\mathcal{F}$ is α-semistable, Theorem 4.2 is due to Bogomolov-McQuillan cf. [7], as well as Kebekus, Sola-Conde and Toma in [35]. We equally refer to the article by J.-B. Bost, [8], who proves a different, but related, result in an arithmetic context. These results originate in [28], and [7] is motivated by [41]. As already mentioned, the approach of the proof below for claim (1) is the same as in [7]. The main difference is that we work directly on X and not by restricting $\mathcal{F}$ to complete intersection curves. In this way, the Mehta-Ramanathan theorem is not needed, and we avoid the inextricable difficulties generated by both the singularities of $\mathcal{F}$, and the singularities of covering families of movable curves at their base loci. Notice further that the Mehta-Ramanathan theorem fails for movable curves, cf. Example 4.3 below, already mentioned in [18]. The extension from "generic complete intersection class" to "movable class" enlarges considerably the potential applicability.

Example **4.3**. — The Mehta-Ramanathan restriction theorem is open in this movable context, but might fail to hold quite drastically even for 'strongly' covering families of curves on surfaces. It is indeed shown in [6], Section 7, that if S is a smooth K3-surface, then $\mathcal{O}_{\mathrm{P}}(1)$ is not pseudoeffective on $\mathrm{P} := \mathbf{P}(\Omega^1_{\mathrm{S}})$. This means that there exists on S an algebraic family of irreducible curves C_t on S effectively parametrised by a quasi-projective irreducible surface T such that, for each such curve C_t the saturation in TS of the tangent sheaf to C_t has positive degree on C_t. Moreover, for $x \in \mathrm{S}$ generic, all but a finite number of tangent directions of TS at x are realised by the tangent directions to the $\mathrm{C}_t's$ going through x. The proof given in [6] is quite indirect. It were interesting to have concrete realisations of such families C_t even on special $\mathrm{K}3's$.

We shall next prove claim (1); the claim (2) will be established in Section 4.2.

4.1. *Algebraicity.*

Proof. — Let $\mathrm{E} \subset \mathrm{X}$ be the singular set of the foliation $\mathcal{F}$. By definition, it consists of points where $\mathcal{F}$ is not a subbundle ot TX, and in particular:

$$\operatorname{codim}_{\mathrm{X}}(\mathrm{E}) \geq 2. \tag{22}$$

Let $x \in \mathrm{X} \setminus \mathrm{E}$. Since $\mathcal{F}$ is not singular at x, there exists an open set $\Omega_x \subset \mathrm{X} \setminus \mathrm{E}$ together with a submersion $\pi_x : \Omega_x \to \mathbf{C}^{n-r}$ with connected fibres such that for each $y \in \Omega_x$ the intersection $\mathrm{L}_y \cap \Omega_x$ of the leaf of $\mathcal{F}$ passing through y with Ω_x is given by the fibre of π_x containing y. We recall that $n = \dim(\mathrm{X})$ and r is the rank of the foliation $\mathcal{F}$.

Thus we have a cover of the open set $\mathrm{X} \setminus \mathrm{E}$ with open sets Ω_x as above. Let $(\Omega_i)_{i\in \mathrm{I}}$ be a countable, locally finite cover extracted from $(\Omega_x)_{x\in \mathrm{X}\setminus\mathrm{E}}$. We define

$$\widetilde{\Omega} := \cup_{i\in\mathrm{I}} \Omega_i \times \Omega_i \subset (\mathrm{X} \setminus \mathrm{E}) \times (\mathrm{X} \setminus \mathrm{E});$$

it is an open subset.

We define the following $(n+r)$-dimensional locally closed analytic subset $\Lambda \subset \widetilde{\Omega}$ as follows

$$\Lambda := \big\{(z, w) \in X \times X : z \in \Omega_i \text{ and } w \in L_z \cap \Omega_i \text{ for some } i \in I\big\}. \tag{23}$$

We note that the (local) analyticity of Λ is a direct consequence of the fact that $\mathcal{F}$ is a holomorphic foliation.

The set Λ contains the open subset of the diagonal defined by:

$$X_0 := \big\{(z, z) \in X \times X : z \in X \setminus E\big\} \tag{24}$$

and we consider

$$V := \overline{\Lambda}^{\mathrm{Zar}} \tag{25}$$

the Zariski closure of Λ in $X \times X$.

We have $\dim(V) \geq \dim(\Lambda) = n+r$, and we show next that the algebraicity of $\mathcal{F}$ is equivalent to the equality $\dim(V) = n+r$. Indeed, if this holds true, then Λ is open in its Zariski-closure V in $X \times X$. We consider the map $\pi_V : V \to X$ given by the restriction to V of the projection on the first factor $X \times X \to X$. Note that the generic fibres of π_V are irreducible, of dimension equal to r (they correspond to the Zariski closure of the leaves of $\mathcal{F}$).

Let $\tau : \widehat{V} \to V$ be a desingularisation of V, and let $W \subset \widehat{V}$ be the component of $\widehat{V}$ which contains the inverse image of the generic fibres of π_V. We denote by $f : W \to X$ the composed map $\pi_V \circ \tau|_W$; it is surjective and by general results, there exists a constant $d > 0$ such that the degree of each fibre of f is smaller than d.

We consider the Chow scheme $\mathrm{Chow}(W) = \cup_{\delta>0} \mathrm{Chow}_{r,\delta}(W)$ corresponding to r-dimensional cycles of W (where the index δ above stands for the degree of the cycle). The rational map

$$p : X \dashrightarrow \mathrm{Chow}(W) \quad x \to f^{-1}(x) \tag{26}$$

induces the foliation $\mathcal{F}$ generically, and we are done by the compactness of the components of the Barlet-Chow scheme of X.

The algebraicity of $\mathcal{F}$ will then follow from the next standard Riemann-Roch bound on sections:

Lemma **4.4**. — *Let* $X_0 \subset \Lambda \subset V \subset X \times X$ *be defined as above. If, for some ample line bundle* L *on* $X \times X$, *there exists a constant* $C > 0$ *such that* $h^0(V, kL|_V) \leq Ck^{n+r}$ *as* $k \to \infty$, *then the dimension of* V *is equal to* $n+r$.

We show now the existence of such a constant $C_L = C > 0$ for any ample L.

Proposition **4.5**. — *Let* L *be an ample line bundle on* $X \times X$. *There exists a constant* $C > 0$ *such that:* $h^0(V, kL|_V) \leq Ck^{n+r}$, *for any* $k \geq 0$. *As a consequence, the dimension of the algebraic set* V *is equal to* $n+r$.

Proof. — The main ideas in the proof of Proposition 4.5 are the same as in [7]: for any $k \geq 0$, the sections of L^k on V restrict injectively to Λ, since V is its Zariski closure. Next, one considers the restriction of these sections to the formal neighbourhood of X_0 in Λ. In other words, we study the Taylor expansion of sections of $L^k|_\Lambda$ at the points of the diagonal X_0 in the normal directions in Λ.

For any $m > 0$, let X_m be the mth infinitesimal neighbourhood of X_0 in Λ, defined by the structure sheaf: $\mathcal{O}_{X_m} := \mathcal{O}_\Lambda / I_0^{m+1}$, where I_0 is the sheaf of ideals of the diagonal $X_0 \subset \Lambda$. It is enough to produce a bound $C > 0$ independent of m, k such that

$$h^0\big(X_m, L^{\otimes k} \otimes \mathcal{O}_{X_m}\big) \leq Ck^{n+r} \tag{27}$$

for any k, m. Indeed, the space $H^0(X_m, L^{\otimes k} \otimes \mathcal{O}_{X_m})$ is nothing, but the space of all possible Taylor expansions at order m of sections of L^k along X_0 in the directions of $\mathcal{F}$.

For this, remark that over $(X \setminus E)$, we have a natural isomorphism $\mathcal{F} \cong N_{X_0/\Lambda}$, since the normal bundle of X_0 in Λ is naturally isomorphic to the vector bundle corresponding to $\mathcal{F}|_{X\setminus E}$.

The following exact sequence holds over X_0, $\mathcal{F}^*$ being the dual of $\mathcal{F}$:

$$0 \to \mathrm{Sym}^m\big(\mathcal{F}^*\big) \to \mathcal{O}_{X_{m+1}} \to \mathcal{O}_{X_m} \to 0. \tag{28}$$

It shows that it is sufficient to establish that there exists a constant $C > 0$ such that, for any $k \geq 0$:

$$\sum_{m\geq 0} h^0\big(X_0, L^k \otimes \mathrm{Sym}^m\big(\mathcal{F}^*\big)\big) \leq C.k^{n+r}. \tag{29}$$

The estimate (29) will be a consequence of following statement.

Lemma **4.6**. — *Let $\mathcal{F} \subset T_X$ be a coherent sheaf, which is locally free when restricted to the open set $X_0 \subset X$ such that $\mathrm{codim}_X(X \setminus X_0) \geq 2$. Let $\delta_0 > \frac{L.\alpha}{\mu_{\alpha,\min}(\mathcal{F})}$ be any positive integer. The following assertions are true.*

(a) *We have $H^0(X_0, L^k \otimes \mathrm{Sym}^m(\mathcal{F}^*)) = 0$ if $m \geq \delta_0 k$.*

(b) *There exists a non-singular projective manifold Y of dimension $\dim(Y) = \dim(X) + \mathrm{rk}(\mathcal{F}) - 1$ together with a map $p : Y \to X$ and a line bundle $B \to Y$ such that we have*

$$p_\star\big(B^m\big) = \widehat{S^m}\big(\mathcal{F}^\star\big) \tag{30}$$

for any $m \geq 1$. In (30) we denote by $\widehat{S^m}(\mathcal{F}^\star)$ the double dual of the symmetric power $\mathrm{Sym}^m(\mathcal{F}^\star)$.

(c) *For any pair of positive integers k, m we have the equality*

$$h^0\big(X_0, L^k \otimes \mathrm{Sym}^m\big(\mathcal{F}^\star\big)\big) = h^0\big(Y, p^\star\big(L^k\big) \otimes B^m\big). \tag{31}$$

Before proving Lemma 4.6, we notice that it implies almost immediately the inequality (29). Indeed, we have

$$\sum_{m\geq 0} h^0\big(X_0, L^k\otimes \mathrm{Sym}^m(\mathcal{F}^\star)\big) = \sum_{m\leq \delta_0 k} h^0\big(X_0, L^k\otimes \mathrm{Sym}^m(\mathcal{F}^\star)\big) \tag{32}$$

by the point (a) of Lemma 4.6. Next, the point (c), together with the fact that the dimension of Y is equal to $n+r-1$ shows that the right hand side of (32) is $\mathcal{O}(k^{n+r})$. This can be seen as follows: the dimension of the space of global sections of the bundle $p^\star(L^k)\otimes B^m$ is smaller that $h^0(Y, p^\star(L^k)\otimes H^m)$, where H is a very ample bundle on Y such that $H\otimes B^{-1}$ is effective. We therefore have to evaluate the quantity

$$\sum_{m\leq \delta_0 k} h^0\big(Y, p^\star(L^k)\otimes H^m\big) \tag{33}$$

which is smaller than $\delta_0 k\cdot h^0(Y, p^\star(L^k)\otimes H^{\delta_0 k})$ where we recall that δ_0 is a positive integer. By Riemann-Roch theorem, we have the estimate $h^0(Y, p^\star(L^k)\otimes H^{Ck}) = \mathcal{O}(k^{n+r-1})$ as $k\to\infty$, so all in all we have established (29).

In what follows we will identify X with the diagonal of $X\times X$, and X_0 with $X\setminus E$.

Proof. — The point (a) follows from Lemma 2.5 and the slope inequality, if $m\geq k.B$:

$$\mu_{\alpha,\max}\big(L^k\otimes \widehat{S^m}(\mathcal{F}^\star)\big) = k.L.\alpha - m.\mu_{\alpha,\min}(\mathcal{F}) < 0. \tag{34}$$

We remark that here we have used Theorem 2.9.

The point (b) is completely proved in the book by N. Nakayama [45] (cf. Chapter V, Section 3.23), so we will simply recall the construction of (Y, B) for the convenience of the reader.

Let $\pi : \mathbf{P}(\mathcal{F}^\star)\to X$ be the scheme over X associated to the torsion free coherent sheaf $\mathcal{F}^\star$, and let $\mathcal{O}_{\mathcal{F}^\star}(1)$ be the tautological line bundle on $\mathbf{P}(\mathcal{F}^\star)$. Let $\mathbf{P}'(\mathcal{F}^\star)$ be the normalisation of the component of $\mathbf{P}(\mathcal{F}^\star)$ which contains the Zariski open subset $\pi^{-1}(X_0)$ (we recall the crucial fact that the codimension of $X\setminus X_0$ in X is greater than two). Finally, let Y be a smooth projective variety such that there exists a birational morphism $Y\to \mathbf{P}'(\mathcal{F}^\star)$ which is biholomorphic over $\pi^{-1}(X_0)$. We denote by $\mu : Y\to \mathbf{P}(\mathcal{F}^\star)$ the resulting map, and let

$$p : Y\to X \tag{35}$$

be the composition $\pi\circ\mu$. Nakayama shows that we can take

$$B := \mu^\star\big(\mathcal{O}_{\mathcal{F}^\star}(1)\big) + \Lambda, \tag{36}$$

where Λ is an effective p-exceptional divisor. The important fact here (cf. [45]) is that B can be chosen so that (30) holds *for any m*.

The equality (31) is a direct consequence of (b), together with the definition of the set X_0, so we do not provide any further explanations. □

Thus, Proposition 4.5 is proved as well. □

Hence, the algebraically criterion is established. □

4.2. *Rational connectedness.* — The following result was proved (by very different arguments) in the case of ample classes in [7] and [35]. Our proof here is using two main techniques: the existence of the relative rational quotient of a map p and the fact that the projective manifolds whose canonical class is not pseudo-effective are uniruled (actually, this is the unique argument in positive characteristic we need in this paper).

Theorem **4.7**. — *Let* X *be projective smooth manifold, and let* $\mathcal{F} \subset TX$ *be a foliation. Assume that there exists a movable class* α *for which* $\mu_{\alpha,\min}(\mathcal{F}) > 0$. *Then* $\mathcal{F}$ *is an algebraic foliation and its leaves are rationally connected.*

Proof. — The fact that $\mathcal{F}$ is an algebraic foliation has been proved. We now treat the last claim of Theorem 4.7 using the relative rational quotient.

Let $p : X \dashrightarrow Z$ be the rational map (26) induced by the application $p_0 : X \dashrightarrow \mathrm{Chow}(W)$, with Z a desingularisation of the image of p_0. We also consider the *relative rational quotient* of p:

$$r : X \dashrightarrow Y \tag{37}$$

This map is constructed in [10] or [36] for the absolute version. The existence of the relative version follows from [12], Appendix. We also have a map $s : Y \dashrightarrow Z$, such that $s \circ r = p$.

Assume by contradiction that the fibres of p are not rationally connected, then:

(a) $\dim Y > \dim Z$.
(b) The canonical bundle of the desingularisation of any generic fibre of s is pseudo-effective by [25].
(c) The generic fibres of r are rationally connected.

We will consider now regular models of the maps defined above: let $\pi_X : \widehat{X} \to X$ and $\pi_Y : \widehat{Y} \to Y$ be smooth modifications of X and Y respectively, such that the applications

$$\widehat{p} := p \circ \pi_X, \quad \widehat{s} := s \circ \pi_Y \tag{38}$$

are regular. We can also assume that there exists a map $\widehat{r} : \widehat{X} \to \widehat{Y}$ such that the equality $\widehat{s} \circ \widehat{r} = \widehat{p}$ is preserved.

Let $\widehat{\mathcal{H}} := \ker(d\widehat{s})$ be the foliation induced by the kernel of the differential of $\widehat{s}$. By formula (13) combined with Remark 3.6 and the property (b) above, we see that

$$\det(\widehat{\mathcal{H}}^\star) \tag{39}$$

is pseudo-effective on $\widehat{Y}$, modulo a divisor which is π_Y-exceptional (we can assume that the transversality conditions (i) and (ii) in Remark 3.3 hold true, modulo changing Z with a birational model—this operation is harmless in our present context). Let $\mathcal{H}$ be the foliation induced by $\widehat{\mathcal{H}}$ on Y; we deduce that $\det(\mathcal{H}^\star)$ is pseudo-effective, by Lemma 3.5.

Let $\widehat{\mathcal{F}}$ be the foliation induced by $\mathcal{F}$ on $\widehat{X}$. Then we have a morphism:

$$\widehat{\mathcal{F}} \to (\pi_Y \circ \widehat{r})^\star \mathcal{H} \tag{40}$$

and we claim that it is generically surjective. The first observation is that the map $\widehat{\mathcal{F}} \to \widehat{r}^\star \widehat{\mathcal{H}}$ is well-defined and generically surjective. This is the case because $\widehat{X}$ and $\widehat{Y}$ are smooth, and for any general enough $z \in Z$ the map in question is induced by the differential of the map $\widehat{X}_z \to \widehat{Y}_z$. The map $\widehat{r}^\star \widehat{\mathcal{H}} \to (\pi_Y \circ \widehat{r})^\star \mathcal{H}$ is an isomorphism at the generic point of $\widehat{Y}$.

We have $\mu_{\pi_X^\star \alpha, \min}(\widehat{\mathcal{F}}) > 0$, since $\mu_{\alpha,\min}(\mathcal{F}) > 0$, cf. Lemma 2.12 and its proof. Hence we infer that

$$\mu_{\pi_X^\star \alpha}\big(\widehat{r}^\star \mathcal{H}\big) > 0, \tag{41}$$

contradicting the pseudo-effectivity of $\det \mathcal{H}^\star$. □

Remark **4.8**. — The discrepancies $K_{\widehat{X}} - \pi^*(K_X)|_F$ of the generic fibre F of $\widehat{p}$ of a 'neat model' of the rational fibration p defined by $\mathcal{F}$ above are of great geometric interest also.

4.3. *Pseudo-effectivity of cotangent sheaves of foliations.* — We establish here a foliated version of Theorem 1.3 when $\Delta = 0$. One of the motivations for this statement is the existence (cf. [19]) of foliations with $K_{\mathcal{F}}$ pseudo-effective on some projective uniruled manifolds. Due to the non-preservation of the pseudo-effectivity of $K_{\mathcal{F}}$ under blow-ups when $\mathcal{F} \neq TX$, new phenomena appear.

Theorem **4.9**. — *Let* X *be a non-singular projective manifold, and let* $\mathcal{F} \subset \mathcal{O}(T_X)$ *be a foliation on* X*, with* $K_{\mathcal{F}}$ *is pseudo-effective. Then one of the following occurs.*

(1) *For any positive integer* $m \geq 1$*, and any coherent, torsion-free sheaf* $\mathcal{Q}$ *such that there exists a generically surjective map*

$$\otimes^m \mathcal{F}^\star \to \mathcal{Q}, \tag{42}$$

$\det \mathcal{Q}$ *is a pseudo-effective line bundle on* X*, or:*

(2) *There exists a proper birational morphism* $\pi : \widehat{X} \to X$ *and a morphism* $p : \widehat{X} \to Z$ *with rationally connected fibres tangent to* $\widehat{\mathcal{F}}$*, the inverse image of* $\mathcal{F}$ *in* $\widehat{X}$*. Moreover,* $\widehat{\mathcal{F}}$ *is the inverse image by* p *of a foliation* $\mathcal{H}$ *on* Z*. In particular,* $K_{\widehat{\mathcal{F}}}$ *is not pseudo-effective in this second case.*

Proof. — Let $\alpha \in \mathrm{Mov}(X)$ be a movable class; we have to prove that, either:

$$c_1(\mathcal{Q}).\alpha \geq 0, \tag{43}$$

or that case (2) occurs. Assume thus that the relation (43) does not hold. Thus $\mu_{\alpha,\min}(\widehat{\otimes}^m \mathcal{F}^\star) < 0$ and by Theorem 2.9 this implies that we have $\mu_{\alpha,\min}(\mathcal{F}^\star) < 0$, which in turns shows the inequality $\mu_{\alpha,\max}(\mathcal{F}) > 0$.

Let $\mathcal{G} \subset \mathcal{F}$ the α-maximal destabilising sheaf of $\mathcal{F}$; then $\mathcal{G}$ is α-semi-stable, and:

$$\mu_\alpha(\mathcal{G}) > 0. \tag{44}$$

It is a simple matter to check that the slope inequalities in Lemma 4.10 below are satisfied, that is to say:

$$2\mu_{\alpha,\min}(\mathcal{G}) > \mu_{\alpha,\max}(\mathcal{F}/\mathcal{G}). \tag{45}$$

This is a well-known consequence of the maximality of $\mathcal{G}$, so we only sketch the argument as follows. We consider a sub-sheaf $\overline{\mathcal{H}} \subset \mathcal{F}/\mathcal{G}$. Then there exists a sub-sheaf $\mathcal{H} \subset \mathcal{F}$, containing $\mathcal{G}$ and inducing $\overline{\mathcal{H}}$. Hence we have $\mu_\alpha(\mathcal{H}) \leq \mu_\alpha(\mathcal{G})$, from which we deduce (after a few standard computations which we skip) that $\mu_{\alpha,\max}(\mathcal{F}/\mathcal{G}) \leq \mu_\alpha(\mathcal{G})$. Now the semi-stability of $\mathcal{G}$ with respect to α implies the inequality (45).

Lemma **4.10**. — *Let* $\mathcal{G} \subset \mathcal{F} \subset \mathrm{TX}$ *be holomorphic (possibly singular) distributions on* X *smooth projective connected. Assume that* $\mathcal{F}$ *is a foliation, and that for some movable class* α *we have:* $\mu_{\alpha,\min}(\mathcal{G}) > 0$ *and also:*

$$2.\mu_{\alpha,\min}(\mathcal{G}) > \mu_{\alpha,\max}(\mathcal{F}/\mathcal{G})). \tag{46}$$

Then $\mathcal{G}$ *is a foliation.*

Proof. — The natural composed map $\wedge^2\mathcal{G} \to \mathrm{TX}/\mathcal{G} \to \mathrm{TX}/\mathcal{F}$ derived from the Lie bracket on X vanishes, since $\mathcal{F}$ is a foliation, and thus defines a section of $\mathrm{H}om(\wedge^2(\mathcal{G}) \to (\mathcal{F}/\mathcal{G}))$ over X. But this vector space vanishes because of the slope conditions. This forces the Lie bracket $\wedge^2\mathcal{G} \to \mathrm{TX}/\mathcal{G}$ to vanish, as claimed. □

In conclusion, $\mathcal{G}$ is integrable. Moreover, by Theorem 4.2 the foliation $\mathcal{G}$ is algebraic, and has rationally connected closure of its leaves. Let $X_0 \subset X$ be the maximal Zariski open set such that $\mathcal{G}|_{X_0}$ is a vector bundle, and such that the singularities of $\mathcal{G}$ are contained in the complement $X \setminus X_0$ (which has co-dimension greater than two).

Thus there exists a rational map

$$p : X \dashrightarrow Z$$

such that $\mathcal{G} = \ker(dp)$ generically on X—thus $\mathcal{G}$ is"algebraic", according to Definition 4.1, beginning of Section 4.

We consider a modification $\pi_X : \widehat{X} \to X$ such that the composed map $\widehat{p} := p \circ \pi_X$ is holomorphic. Since the foliation $\mathcal{G}$ is the kernel of the differential of the map p we infer that *the canonical bundle of the fibres of* $\widehat{p}$ *is not pseudo-effective*, by (44).

Let $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{F}}$ be the foliations induced by $\mathcal{G}$ and $\mathcal{F}$ on $\widehat{X}$, respectively. Then we still have $\widehat{\mathcal{G}} = \ker(d\widehat{p})$ generically on $\widehat{X}$ and $\widehat{\mathcal{G}} \subset \widehat{\mathcal{F}}$.

The following "rigidity lemma" (cf. [2], Lemma 6.7 for similar ideas) will then show that we are in case (2) and this will finish the proof, once we prove that $K_{\widehat{\mathcal{F}}}$ is not pseudo-effective, which is shown below, after the proof of Lemma 4.11.

Lemma **4.11**. — *Let* $\widehat{\mathcal{G}} \subset \widehat{\mathcal{F}}$ *be two foliations on* $\widehat{X}$ *Assume that* $\widehat{\mathcal{G}}$ *is algebraic, defined generically as* $\mathcal{G} = \ker(d\widehat{p})$ *for a dominant map* $\widehat{p} : \widehat{X} \to Z$. *There then exists a foliation* $\mathcal{H}$ *on* Z *such that* $\widehat{\mathcal{F}} = \widehat{p}^{\star}\mathcal{H}$, *generically on* $\widehat{X}$.

Proof. — Let $x_0 \in \widehat{X}$ be such that $\widehat{\mathcal{F}}$ is non-singular at x_0, and such that $y_0 := \widehat{p}(x_0)$ is a regular value of $\widehat{p}$.

Let $\Lambda_0 \subset \widehat{X}$ be a germ a submanifold contained in the leaf L_{x_0} of $\mathcal{F}$ at x_0, transverse to $G_0 := \widehat{p}^{-1}(y_0)$, and such that

$$\mathcal{F}_{x_0} = \mathcal{G}_{x_0} + T_{\Lambda_0, x_0}$$

is a direct sum decomposition. Next $\widehat{p}(\Lambda_0)$ is a germ of a submanifold V_0 of Z at y_0, and $W_0 := \widehat{p}^{-1}(V_0)$ is contained, and hence equal to the germ of the leaf $\mathcal{F}_{x_0}$. Indeed: for each $x \in \Lambda_0$, $\widehat{p}^{-1}(\widehat{p}(x)) = \mathcal{G}(x) \subset \mathcal{F}(x)$, and $\mathcal{F}(x)$ thus contains both $\widehat{p}^{-1}(\widehat{p}(x))$, and Λ_0.

Since this holds for every x_0 having the properties specified above, the lemma follows by analytic continuation. □

We now check that $K_{\widehat{\mathcal{F}}}$ is not pseudo-effective: as a consequence of Lemma 4.11, we have the exact sequence

$$0 \to \widehat{\mathcal{G}}|_U \to \widehat{\mathcal{F}}|_U \to \mathcal{O}_U^{\oplus r} \to 0 \tag{47}$$

where $U := \widehat{p}^{-1}(V)$ and V is a small topological coordinate set centered at a regular value of $\widehat{p}$.

Therefore we have

$$K_F = K_{\widehat{\mathcal{G}}}|_F = K_{\widehat{\mathcal{F}}}|_F \tag{48}$$

where F is a generic fibre of $\widehat{p}$. This shows that $K_{\widehat{\mathcal{F}}}$ is not pseudo-effective. Therefore Theorem 4.9 is proved. □

Remark **4.12**. — The case (2) in Theorem 4.9 actually occurs (see Example 4.15 below). This is due to the fact that, contrarily to the case when $\mathcal{F} = TX$, the pseudo-effectivity of $K_{\mathcal{F}}$ is, in general, not birationally invariant. We thank S. Druel, who called our attention on the fact that $K_{\widehat{\mathcal{F}}}$ in the above proof could, a priori, be non pseudo-effective.

The following corollary is a consequence of the previous rational connectedness statement of Theorem 4.2. Claim (1) generalises in the projective case Brunella's Theorem [9] (which asserts that this result holds in the compact Kähler case as well). This corollary gives an optimal geometric obstruction to the pseudo-effectivity of the canonical bundle of foliations on projective manifolds.

Corollary **4.13**. — *Let* X *be a projective manifold, and let* $\mathcal{F}$ *be a foliation of rank* $0 < r < n := \dim(X)$. *Assume that* $\mathcal{F}$ *is not algebraic. Then:*

(i) *If* $r = 1$, *the bundle* $K_{\mathcal{F}}$ *is pseudo-effective.*
(ii) *For an arbitrary rank* r, *if the bundle* $K_{\mathcal{F}}$ *is not pseudo-effective, there exists a non-trivial algebraic foliation* $\mathcal{G} \subset \mathcal{F}$ *such that* $\mu_{\alpha,\min}(\mathcal{G}) > 0$ *for some movable class* α.

We remark that the converse of the point (ii) does not hold, by Example 4.15 below.

Proof. — Claim (i). Since by assumption $\mathcal{F}$ is not algebraic, Theorem 4.2 implies, that for each movable class α, we have:

$$\mu_{\alpha,\min}(\mathcal{F}) = \mu_{\alpha}(\mathcal{F}) \leq 0. \tag{49}$$

If the rank r of $\mathcal{F}$ is equal to one, then this implies (cf. [6]) that $K_{\mathcal{F}}$ is pseudo-effective, and the point (i) is proved.

The second point has been established during the proof of Theorem 4.9. □

The following statement gives a criteria for the conclusion (1) of Theorem 4.9 to hold true.

Corollary **4.14**. — *In the set-up of Theorem 4.9 the statement (1) holds true provided that one of the following conditions is satisfied.*

1. The foliation $\mathcal{F}$ *is "totally transcendental" i.e. its general leaf does not contains any positive dimensional algebraic submanifold of* X.

2. The singularities of $\mathcal{F}$ *are canonical (this implies that* $K_{\widehat{\mathcal{F}}}$ *is pseudo-effective for any modification* $\widehat{X} \to X$*).*

Example **4.15**. — Let $\mathcal{F}_0$ be the foliation defined on $\mathbf{P}^2$ by the pencil of conics generated by two smooth conics meeting in 4 distinct points. Now take $X := \mathbf{P}^2 \times Y$, where Y is any uniruled projective manifold. Take for $\mathcal{F}$ the inverse image of $\mathcal{F}_0$ on X by the first projection (on $\mathbf{P}^2$). Then $K_{\mathcal{F}}$ is pseudo-effective, since $K_{\mathcal{F}_0}$ is. On the other hand, $\mathcal{F}$ contains $q^*(TY)$, where q is the second projection. Dualising, we get a projection from $\mathcal{F}^*$ onto TY^*, which has a non pseudo-effective determinant. We are then in Case 2 of Theorem 4.9. A modification $\pi : \widehat{X} \to X$ as in the proof is obviously obtained by lifting $\mathcal{F}_0$ to the blow-up of $\mathbf{P}^2$ in the 4 base points of the pencil, and taking the product with Y.

Remark **4.16**. — One can easily see, using Chow cycle spaces and the two preceding lemmas, that any foliation $\mathcal{F}$ on X projective smooth contains a largest algebraic foliation $\mathcal{G}$ given as K*er*(*df*) for a fibration $f : X \dashrightarrow Z$, such that $\mathcal{F} = f^*(\mathcal{H})$ for some 'totally transcendental' foliation $\mathcal{H}$ on Z. And then, $K_{\mathcal{F}}$ is pseudo-effective if and only if $K_{\mathcal{G}}$ is, since $K_{\mathcal{F}} = K_{\mathcal{G}} + f^*(K_{\mathcal{H}})$, and $K_{\mathcal{H}}$ is pseudo-effective. Indeed: if $K_{\mathcal{G}}$ is not pseudo-effective, we have a rationally connected fibration tangent to $\mathcal{G}$, contradicting the pseudo-effectivity of $\mathcal{F}$; the other direction is obvious. We refer to [40], Lemma 2.4 for a more detailed discussion regarding this topic. Moreover, Case 2 of Theorem 4.9 occurs for $\mathcal{F}$ if and only if it occurs for $\mathcal{G}$. In other words: Case 2 of Theorem 4.9 occurs essentially only for algebraic foliations.

4.4. *Descent of foliations.* — The following consequence of the preceding theorem will mainly be needed in the proof of Theorem 1.3. But it may have some interest by itself.

Corollary **4.17**. — *Let $\pi : X' \to X$ be a finite surjective holomorphic map of degree $d > 1$ between complex projective manifolds. Let α be a movable class on* X, *and let $\alpha' := \pi^*(\alpha)$ be its pull-back to X'. Let $\mathcal{F}' \subset \pi^*(TX)$ be a torsion-free subsheaf, with $\mathcal{F}^{sat}$ its saturation in $\pi^*(TX)$. Assume moreover that:*

(a) $\mu_{\alpha',\min}(\mathcal{F}') > 0$;
(b) $\mathcal{F}^{sat} = \pi^*(\mathcal{F})$ *for some subsheaf* $\mathcal{F} \subset TX$.

Then $\mu_{\alpha,\min}(\mathcal{F}) > 0$; in particular, if $\mathcal{F}$ is integrable, then the corresponding foliation is algebraic.

Proof. — Since $\mu_{\alpha',\min}(\mathcal{F}') > 0$ we deduce that we have

$$\mu_{\alpha',\min}\big(\mathcal{F}^{sat}\big) > 0 \tag{50}$$

by Remark 2.14. Let $\mathcal{Q}$ be a quotient of $\mathcal{F}$; then $\pi^\star\mathcal{Q}$ is a quotient of $\pi^\star\mathcal{F} = \mathcal{F}^{sat}$. By (50) above, we deduce:

$$\mu_{\alpha'}\big(\pi^\star\mathcal{Q}\big) > 0 \tag{51}$$

and this is equivalent with $\mu_\alpha(\mathcal{Q}) > 0$, which is the claim. The last part of Corollary 4.17 follows from the algebraically criteria. □

Remark **4.18**. — In general, in the situation of the preceding corollary, if $\mathcal{F}'$ is α-semi-stable, $\mathcal{F}^s$ does not need to be α-semi-stable, as shown by the natural injection of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ in $\mathcal{O}(1) \oplus \mathcal{O}(2)$ over $\mathbf{P}^1$.

Remark **4.19**. — These results on foliations immediately extend to logarithmic foliations. We show this in the next section, which will also serve as a simplified model for the case of arbitrary smooth 'orbifold pairs', treated below, and for which additional constructions and definitions are required. We added this short section in order to make the application (through Corollary 8.7) to families of canonically polarised manifolds in Section 8.1 below independent from the general 'orbifold version'. The proof given here of Corollary 8.7 is quite different and shorter from the one given in [48], which showed that the general orbifold pairs could be avoided. Notice however that, once the foundational material are laid, the continuity method used in Theorem 7.11 gives a much more direct alternative proof of Corollary 8.7.

5. Orbifold tensor bundles on Kawamata covers

Let (X, Δ) be a smooth log canonical pair, written as:

$$\Delta = \sum_{j \in J} c_j \Delta_j = \sum_{j \in J} \left(1 - \frac{b_j}{a_j}\right) \Delta_j \tag{52}$$

where J is a finite set, and for each $j \in J$ we have $0 \leq b_j < a_j$ are coprime integers, and the hypersurfaces (Δ_j) are snc. If the coefficient b_j is equal to zero, then we agree that the corresponding denominator a_j is equal to 1.

These orbifold pairs (X, Δ) interpolate between the *compact, or projective case* (i.e. when either $J = \emptyset$) and the *logarithmic, or quasi-projective case*, when $b_j = 0$ for all $j \in J$, respectively. In both cases, the notions of tangent bundle, cotangent bundles and more generally, of holomorphic tensors are classically defined. They play a fundamental role in the study of the geometry of (quasi-)projective manifolds.

We shall introduce the analogous notions corresponding to an arbitrary orbifold pair (X, Δ). Unfortunately they can only be defined on a suitable ramified cover of X adapted to (X, Δ). However, we shall see that they enjoy properties similar to those of the usual ones in the two standard cases (compact, and logarithmic) mentioned above. These properties will turn out to be independent on the cover used to define them.

The underlying idea for the definition is that the local generators as an $\mathcal{O}_X$-module of the orbifold cotangent bundle should "look like":

$$\frac{dz_1}{z_1^{1-b_1/a_1}}, \dots, \frac{dz_r}{z_r^{1-b_r/a_r}}, dz_{n_1+1}, \dots, dz_n, \tag{53}$$

on some coordinate open set $U \subset X$ where the divisor $\lceil \Delta \rceil$ is given by $z_1 \dots z_r = 0$. Unlike in the cases mentioned above, these symbols involve multi-valued functions. Nevertheless, we have the identity

$$\pi^* \left(\frac{dz}{z^{1-b/a}} \right) = N w^{Nb/a} \frac{dw}{w}, \tag{54}$$

where $z = w^{N}$, and we see that the right-hand side is an usual logarithmic differential provided that N/a is an integer.

This suggests that in order to construct the tensor bundle corresponding to the pair (X, Δ), one needs an auxiliary object, namely a map which ramifies along D with divisible enough order. The formal definition will be given in what follows.

5.1. *Ramified coverings.* — We recall in this sub-section a few basic facts concerning global ramified covers associated to an orbifold pair (X, Δ), for which a polarisation is fixed. Our main references are [38], [32] [24], [31]).

Definition **5.1**. — *Let* (X, Δ) *be an orbifold pair as in* (52)*. A ramified cover adapted to* (X, Δ) *is by definition a Galois covering* $\pi : X_\Delta \to X$ *satisfying the following requirements.*

(i) *The variety* X_Δ *is non-singular, and the ramification order of* π *along each component* Δ_i *is equal to* a_i*, i.e.* $\pi^\star(\Delta_i) = a_i \sum_j D_{ji}$.

(ii) *The support of the divisor* $\pi^\star(\Delta) + \mathrm{Ram}(\pi)$ *as well as the branching loci* $\sum H_j$ *of* π *have simple normal crossings.*

Such a map π will be referred to as "Kawamata cover" in what follows, cf. [36], Theorem 1.1.1. The properties which will be relevant for us are stated in the following lemma.

Lemma **5.2**. — *Let* (X, Δ) *be an orbifold pair; then the following assertions are true.*

(a) *The pair* (X, Δ) *admits a Kawamata cover.*

(b) *Let* $\pi : X_\Delta \to X$ *be any Kawamata cover corresponding to* (X, Δ)*, and let* G *be the associated Galois group. For any point* $y \in X_\Delta$ *there exists an open coordinate set* $y \in U$ *which is* G_y*–invariant, and such that the restriction* $\pi|_U$ *has the following shape*

$$\pi(w_1, \dots, w_n) = \left(w_1^{a_1}, \dots, w_k^{a_k}, w_{k+1}, \dots, w_p, w_{p+1}^{m_1}, \dots w_n^{m_n}\right) \tag{55}$$

with respect to co-ordinates (w_i) *and* (z_j) *on* U *and its image, respectively.*

In the definition above we denote by G_y the isotropy group of y. We note that in (55) we assume that the divisor $\lceil \Delta \rceil$ is locally given by the equation $z_1 \dots z_k = 0$. Also, the local hypersurfaces $z_{p+1} = 0, \dots, z_n = 0$ correspond to the extra-ramification of π— which is in general unavoidable, but which will not affect us in any way.

As we see from Lemma 5.2, the map π can be seen as the global version of the standard application $w \to z = w^a$ and we will use it in order to define the orbifold co-tangent bundle and its associated tensor powers.

5.2. *Orbifold tensor bundles.* — Let (X, Δ) be an orbifold pair, and let $\pi : X_\Delta \to X$ be a Kawamata cover. We first introduce here the notion of co-tangent bundle associated to (X, Δ) by following the elegant approach by Y. Miyaoka in [42].

We denote by $\Omega^1_X\langle\lceil\Delta\rceil\rangle$ the logarithmic tangent bundle associated to $(X, \lceil\Delta\rceil)$. Then we have a well-defined residue map

$$\Omega^1_X\big\langle\lceil\Delta\rceil\big\rangle \to \bigoplus_i \mathcal{O}_{\Delta_i} \to 0, \tag{56}$$

which induces a map between the π–inverse images of the sheaves above

$$\pi^\star\Omega^1_X\big\langle\lceil\Delta\rceil\big\rangle \to \bigoplus_i \mathcal{O}_{\pi^\star\Delta_i} \to 0. \tag{57}$$

In (57) we have used the flatness of π in order to identify $\pi^\star\mathcal{O}_\Delta$ with $\mathcal{O}_{\pi^\star\Delta}$. By the properties of the map π, we can write $\pi^\star\Delta_i = a_i D_i$ for some Cartier divisor D_i on X_Δ. Therefore, we have a quotient map of sheaves

$$\mathcal{O}_{\pi^\star\Delta_i} \to \mathcal{O}_{b_i D_i} \to 0 \tag{58}$$

for every i in our set of indices.

All in all, we have a surjective map

$$\pi^\star\Omega^1_X\big\langle\lceil\Delta\rceil\big\rangle \to \bigoplus_i \mathcal{O}_{b_i D_i} \to 0 \tag{59}$$

and we introduce the following notion.

Definition **5.3**. — *The orbifold co-tangent bundle associated to* (X, Δ) *is the kernel of the map* (59). *It is a vector bundle of rank* $n = \dim(X)$, *and it will be denoted in what follows by* $\pi^\star\Omega^1(X, \Delta)$.

Thus, we have the exact sequence

$$0 \to \pi^\star\Omega^1(X, \Delta) \to \pi^\star\Omega^1_X\big\langle\lceil\Delta\rceil\big\rangle \to \bigoplus_i \mathcal{O}_{b_i D_i} \to 0. \tag{60}$$

At this point, a few remarks are in order.

- The bundle $\pi^\star\Omega^1(X, \Delta)$ is G-invariant: this is a direct consequence of the definition.
- With respect to the coordinate system in Lemma 5.2(2), the local frame of this bundle is expressed as

$$w_1^{b_1-1}dw_1, \dots, w_k^{b_k-1}dw_k, dw_{k+1}, \dots, dw_p, w_{p+1}^{m_{p+1}-1}dw_{p+1}, \dots, w_n^{m_n-1}dw_n.$$

- The determinant of the bundle $\pi^\star\Omega^1(X, \Delta)$ is quickly computed from the sequence (60),

$$\det\big(\pi^\star\Omega^1(X, \Delta)\big) = \pi^\star(K_X + \Delta). \tag{61}$$

5.3. *The tangent bundle and the Lie bracket on orbifolds.* — The following definition is natural.

Definition **5.4**. — *The orbifold tangent bundle associated to* (X, Δ) *the dual of* $\pi^\star\Omega^1(X, \Delta)$. *It is a* G*-invariant vector bundle, and it will be denoted in the sequel by* $\pi^\star T(X, \Delta)$.

With respect to the coordinates in Lemma 5.2, the local generators of $\pi^\star T(X, \Delta)$ can be written as follows

$$w_1^{a_1-b_1}e_1, \dots, w_k^{a_k-b_k}e_k, e_{k+1}, \dots, e_n \tag{62}$$

where $e_j := \pi^\star \frac{\partial}{\partial z_j}$ is the local frame of the inverse image $\pi^\star T_X$.

Remark that the local generators of the orbifold tangent bundle can also be written as follows

$$w_j^{1-b_j}\frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_i}, w_l^{1-m_l}\frac{\partial}{\partial w_l} \tag{63}$$

where $j = 1, \dots, k$ as well as $i = k+1, \dots, p$ and $l = p+1, \dots, n$. In this way, the tangent bundle $\pi^\star T(X, \Delta)$ looks more like that dual of $\pi^\star\Omega^1(X, \Delta)$.

Motivation. — Let $\mathcal{F} \subset T_X$ be a coherent subsheaf. The corresponding Lie bracket

$$\Lambda^2\mathcal{F} \to T_X/\mathcal{F} \tag{64}$$

is $\mathcal{O}_X$-linear, and if this map vanishes identically, then $\mathcal{F}$ defines a holomorphic foliation. In the remaining part of this sub-section we will consider the orbifold analogue of these results.

More precisely, let $\mathcal{F}_\Delta \subset \pi^\star T(X, \Delta)$ be a coherent subsheaf of the orbifold tangent bundle. Our objective in what follows is twofold: first we show that under some reasonable hypothesis, we can construct an $\mathcal{O}_{X_\Delta}$-linear map

$$\Lambda^2\mathcal{F}_\Delta \to \pi^\star T(X, \Delta)/\mathcal{F}_\Delta. \tag{65}$$

Then we will show here that if the map (65) vanishes identically, then $\mathcal{F}_\Delta$ is induced by a holomorphic foliation on X by a very explicit procedure.

The first step in this direction is the following statement which permits to recognize the subsheaves of $\pi^\star TX$ which are inverse images of a sheaf on X.

Lemma **5.5**. — *[26], [22] Let* $\mathcal{F} \subset \pi^\star TX$ *be a coherent* $\mathcal{O}_{X_\Delta}$*-module, which is saturated in the inverse image of the tangent sheaf* T_X. *If moreover* $\mathcal{F}$ *is* G*-invariant, then there exists a sheaf* $\mathcal{F}_X$ *of* $\mathcal{O}_X$*-modules on* X *such that*

$$\mathcal{F} = \pi^\star(\mathcal{F}_X). \tag{66}$$

This result Lemma 5.5 is completely proved in the references indicated above, so we will not discuss it here at all.

Let $\mathcal{F}_\Delta \subset \pi^\star \mathrm{T}(\mathrm{X}, \Delta)$ be a coherent G-invariant and saturated subsheaf of the orbifold tangent bundle. We denote by $\mathcal{F}^s$ the saturation of $\mathcal{F}_\Delta$ in $\pi^\star \mathrm{T}_\mathrm{X}$. Then $\mathcal{F}^s$ is equally G-invariant, so by Lemma 5.5 there exists a subsheaf $\mathcal{F}_\mathrm{X} \subset \mathrm{T}_\mathrm{X}$ such that

$$\mathcal{F}^s = \pi^\star(\mathcal{F}_\mathrm{X}). \tag{67}$$

Let

$$\Lambda^2 \mathcal{F}_\mathrm{X} \to \mathrm{T}_\mathrm{X}/\mathcal{F}_\mathrm{X} \tag{68}$$

be the $\mathcal{O}_\mathrm{X}$-linear map induced by the Lie bracket on X. Its π-inverse image composed with the natural map $\Lambda^2 \mathcal{F}_\Delta \to \Lambda^2 \mathcal{F}^2$ gives the $\mathcal{O}_{\mathrm{X}_\Delta}$-linear map

$$\Lambda^2 \mathcal{F}_\Delta \to \pi^\star \mathrm{T}_\mathrm{X}/\mathcal{F}^s. \tag{69}$$

On the other hand, given that $\mathcal{F}_\Delta$ is saturated inside the orbifold tangent bundle, we have the equality $\mathcal{F}_\Delta = \mathcal{F}^s \cap \pi^\star \mathrm{T}(\mathrm{X}, \Delta)$. Thus, we infer that the natural map

$$\pi^\star \mathrm{T}(\mathrm{X}, \Delta)/\mathcal{F}_\Delta \to \pi^\star \mathrm{T}_\mathrm{X}/\mathcal{F}^s \tag{70}$$

is injective.

In this setting, we have the following statement, establishing the existence of the Lie bracket for orbifolds (X, Δ).

Proposition **5.6**. — *Let $\mathcal{F}_\mathrm{X} \subset \pi^\star \mathrm{T}(\mathrm{X}, \Delta)$ be a coherent* G*-invariant and saturated subsheaf of the orbifold tangent bundle. Then the map* (69) *factors through* (70), *i.e. we have an $\mathcal{O}_{\mathrm{X}_\Delta}$-linear map*

$$\Lambda^2 \mathcal{F}_\Delta \to \pi^\star \mathrm{T}(\mathrm{X}, \Delta)/\mathcal{F}_\Delta \tag{71}$$

Our proof will unfold as follows. Let $\mathrm{U} \subset \mathrm{X}_\Delta$ be one of the coordinate subsets provided by Lemma 5.2. We first construct lifting of the usual Lie bracket on X

$$[\cdot, \cdot]_\mathrm{U} : \Lambda^2 \pi^\star \mathrm{TX}|_\mathrm{U} \to \pi^\star \mathrm{TX}|_\mathrm{U}$$

which is only locally defined. Then we show that the orbifold tangent bundle $\pi^\star \mathrm{T}(\mathrm{X}, \Delta)$ is closed under this map.

On the other hand, given any subsheaf $\mathcal{G} \subset \pi^\star \mathrm{TX}$ we show that the map $\Lambda^2 \mathcal{G} \to \pi^\star \mathrm{TX}/\mathcal{G}$ induced by the π-lifting of the usual Lie bracket on X coincides with the one given by $[\cdot, \cdot]_\mathrm{U}$. The former is globally defined and $\mathcal{O}_{\mathrm{X}_\Delta}$-linear. The proposition follows by a linear combination of these facts.

Proof. — Let $\mathcal{L}_X$ be the Lie bracket defined on vector fields on X:

$$\mathcal{L}_X : \Lambda^2 TX \to TX. \tag{72}$$

Let v be a local section of the bundle $\pi^\star TX$. We chose local coordinates $w = (w_1, \dots, w_n)$ and $z = (z_1, \dots, z_n)$ near $y_0 := \pi(x_0)$ given by Lemma 5.2 (we remark that the finite group G is not used in the following definition). Then the map $\pi : X_\Delta \to X$ is locally written as follows

$$\pi(w) = \left(w_1^{a_1}, \dots, w_k^{a_k}, w_{k+1}, \dots, w_p, w_{p+1}^{m_{p+1}}, \dots, w_n^{m_n}\right). \tag{73}$$

In order to simplify the notations, let $c_j := 1 - \frac{b_j}{a_j}$ be the coefficient of D_j in Δ; if the index "j" corresponds to one of the hypersurfaces H_j, then we set $c_j := 0$.

We can write v in a unique manner

$$v = \sum_{I \in \mathcal{E}_{r,a}} w^I \pi^\star v_I \tag{74}$$

where $\mathcal{E}_{r,a}$ is the set of indices $I = (i_1, \dots, i_k, i_{p+1}, \dots i_n)$ such that $0 \leq i_j \leq a_j - 1$ for $j = 1, \dots, k$ and $0 \leq i_\alpha \leq m_\alpha - 1$ for $i \geq p+1$. and we use the multi-index notation $w^I := \prod_j w_1^{i_j}$. The (v_I) above are local vector fields on X.

Then we define

$$[v_1, v_2]_U := \sum_{I,J} w^{I+J} \pi^\star\left(\mathcal{L}_X(v_{1I} \wedge v_{2J})\right). \tag{75}$$

We have the following statement, showing that the orbifold tangent bundle is preserved by the local map (75) (we thank B. Claudon for pointing out a slight inaccuracy in the previous version of it).

Proposition **5.7**. — *The orbifold tangent space* $\pi^* T(X, \Delta)$ *is closed under the local bracket* $[\cdot, \cdot]_U$.

Proof. — We consider the restriction of $[\cdot, \cdot]_U$ to the exterior power of the orbifold tangent bundle, composed with the natural projection map

$$[\cdot, \cdot]_{\Delta, U} : \Lambda^2 \pi^* T(X, \Delta) \to \pi^* TX / \pi^* T(X, \Delta); \tag{76}$$

the claim is that this map is identically zero.

By definition, the local generators as $\mathcal{O}_Y$-modules of $\pi^*(TX)$ are

$$\partial_k := \pi^* \frac{\partial}{\partial z_k}, \qquad k = 1, \dots n. \tag{77}$$

As already mentioned, the local generators of $\pi^\star T(X, \Delta)$ can be written explicitly as follows

$$(\mathbf{78}) \qquad w_1^{a_1 c_1} \partial_1, \dots, w_k^{a_k c_k} \partial_k, \partial_{k+1}, \dots \partial_n.$$

Any local function $\varphi \in \mathcal{O}_{X_\Delta}$ can be written in an unique manner $\varphi(w) = \sum_{I \in \mathcal{E}_{r,a}} w^I \psi_I(z)$, for some holomorphic functions (ψ_I) defined locally on X; in this expression we are using the same conventions as in (74).

Let $v = \sum_{j=0}^n \varphi_j(w) \partial_j$ be a local section of $\pi^* T(X, \Delta)$; in particular it can be expressed as follows

$$(\mathbf{79}) \qquad v = \sum_{I \in \mathcal{E}_{r,a}} w^I \pi^\star \rho_I$$

where $\rho_I := \sum_{j=1}^n \psi_{Ij} \frac{\partial}{\partial z_j}$, for each multi-index I.

The main observation now is that *we can assume that the function ψ_{Ij} divisible by z_j provided that the* jth *index of* I *satisfies the inequality* $0 \leq i_j \leq a_j c_j - 1$. This is an immediate consequence of the definitions, and we detail the argument next.

By (78) there exists a family of functions $(\mu_j)_{j=1,\dots,n}$ such that we have

$$(\mathbf{80}) \qquad v = \sum_{j=1}^k \mu_j w^{a_j c_j} \partial_j + \sum_{j=k+1}^n \mu_j \partial_j;$$

we by identifying the coefficients in (80)–(79), we obtain

$$(\mathbf{81}) \qquad \sum_I w^I \psi_{Ij}(z) = \mu_j(w) w_j^{a_j c_j}$$

for each $j = 1, \dots, k$. This clearly proves our assertion, since we have $c_j \leq 1$.

Let $\pi^* \mathcal{V} := \pi^*(T_X \langle \lceil \Delta \rceil \rangle)$ be the inverse image of the logarithmic tangent bundle of corresponding to the pair $(X, \lceil \Delta \rceil)$.

By relation (79) together with the observation above we obtain the decomposition

$$(\mathbf{82}) \qquad v = \sum_{j=1}^n \sum_{i_j=0}^{a_j c_j - 1} w^I \pi^\star V_{Ij} + \sum_{j=1}^n \sum_{i_j = a_j c_j}^{a_j - 1} w^I \pi^\star W_{Ij}$$

where V_{Ij} above are local sections of $\mathcal{V}$, and where W_{Ij} are local holomorphic vector fields on X, multiple of $\frac{\partial}{\partial z_j}$. In (82) we dropped the indexes $i_{p+1}, \dots, i_n$ since they are playing no role.

The proof ends by a case by case analysis.

(a) We have

$$[w^{\mathrm{I}}\pi^{\star}\mathrm{V}_{\mathrm{I}}, w^{\mathrm{J}}\pi^{\star}\mathrm{V}_{\mathrm{J}}]_{\mathrm{U}} = w^{\mathrm{I+J}}\pi^{\star}(\mathcal{L}_{\mathrm{X}}(\mathrm{V}_{\mathrm{I}}, \mathrm{V}_{\mathrm{J}})) \tag{83}$$

so it belongs to $\pi^{\star}\mathrm{T}(\mathrm{X}, \Delta)$, given the fact that the logarithmic tangent bundle is stable by the Lie bracket $\mathcal{L}_{\mathrm{X}}$.

(b) If $j, r \leq k$ then we have

$$[w_j^{a_j c_j} f\partial_j, w_r^{a_r c_r} g\partial_r]_{\mathrm{U}} = w_j^{a_j c_j} w_r^{a_r c_r}\pi^{\star}\left(\mathcal{L}_{\mathrm{X}}\left(f\frac{\partial}{\partial z_j}, g\frac{\partial}{\partial z_r}\right)\right) \tag{84}$$

which clearly belongs to $\pi^{\star}\mathrm{T}(\mathrm{X}, \Delta)$.

(c) If V_{I} is a local section of $\mathcal{V}$ and if $r \leq k$ then we have

$$[w^{\mathrm{I}}\pi^{\star}\mathrm{V}_{\mathrm{I}}, w_r^{a_r c_r} g\partial_r]_{\mathrm{U}} = w^{\mathrm{I}} w_r^{a_r c_r}\pi^{\star}\left(\mathcal{L}_{\mathrm{X}}\left(\mathrm{V}_{\mathrm{I}}, g\frac{\partial}{\partial z_r}\right)\right) \tag{85}$$

and, say, if $r \neq 1$ we have $\mathcal{L}_{\mathrm{X}}(fz_1\frac{\partial}{\partial z_1}, g\frac{\partial}{\partial z_r}) = az_1\frac{\partial}{\partial z_1} + b\frac{\partial}{\partial z_r}$ for some functions a and b whose expressions do not matter: the point is that the π-inverse image of this vector belongs to $\pi^{\star}\mathrm{T}(\mathrm{X}, \Delta)$ when multiplied with $w_r^{a_r c_r}$. If $r = 1$, then no additional explanations are required, because of the factor $w_r^{a_r c_r}$. Also, if $l \geq k+1$ we have $\mathcal{L}_{\mathrm{X}}(f\frac{\partial}{\partial z_l}, g\frac{\partial}{\partial z_r}) = a\frac{\partial}{\partial z_l} + b\frac{\partial}{\partial z_r}$ for some (other) functions a and b, but the result is the same: the π-inverse image of this vector belongs to $\pi^{\star}\mathrm{T}(\mathrm{X}, \Delta)$ when multiplied with $w_r^{a_r c_r}$–remark that there is no vanishing condition imposed for the coefficients $\geq k+1$ in (78).

The Proposition 5.7 is proved. □

Proposition **5.8**. — *Let $\mathcal{G} \subset \pi^{\star}\mathrm{TX}$ be a coherent subsheaf, such that there exists $\mathcal{G}_{\mathrm{X}} \subset \mathrm{TX}$ with the property that $\mathcal{G} = \pi^{\star}\mathcal{G}_{\mathrm{X}}$. Then the following map induced by $[\cdot, \cdot]_{\mathrm{U}}$*

$$\Lambda^2\mathcal{G} \to \pi^{\star}\mathrm{TX}/\mathcal{G} \tag{86}$$

coincides with the π-inverse image of the Lie bracket $\Lambda^2\mathcal{G}_{\mathrm{X}} \to \mathrm{TX}/\mathcal{G}_{\mathrm{X}}$. It is therefore $\mathcal{O}_{\mathrm{X}_{\Delta}}$-linear and globally defined.

Proof. — Let q_j, ρ_j be positive integers, such that $\rho_j \leq a_j - 1$. We denote by $w^{qa+\rho} := \prod w_j^{q_j a_j + \rho_j}$. The calculation required by the Lemma 5.8 is very simple, based on identities of the following type

$$\begin{aligned} w^{qa+\rho}\pi^{\star}(\mathcal{L}_{\mathrm{X}}(v_1, v_2)) &= w^{\rho}\pi^{\star}(z^q\mathcal{L}_{\mathrm{X}}(v_1, v_2)) \\ &= w^{\rho}\pi^{\star}(\mathcal{L}_{\mathrm{X}}(z^q v_1, v_2)) + \psi\pi^{\star}v_1 \end{aligned} \tag{87}$$

where v_j are local sections of $\mathcal{G}_X$, and ψ is a local function on X_Δ. This implies that if V_1, V_2 are local sections of $\pi^\star\mathcal{G}_X$, then we have

$$\pi^\star\mathcal{L}_X\big(\varphi(w)V_1 \wedge V_2\big) \equiv \varphi(w)\pi^\star\mathcal{L}_X(V_1 \wedge V_2) \tag{88}$$

modulo a vector in $\pi^\star\mathcal{G}_X = \mathcal{G}$. This is precisely what we need to prove, given the Definition (75). □

We are now ready for the proof of Proposition 5.6, as follows. By Proposition 5.8, the composed map:

$$\Lambda^2\mathcal{F}_\Delta \to \Lambda^2\mathcal{F}^s \to \pi^\star TX/\mathcal{F}^s \tag{89}$$

is $\mathcal{O}_{X_\Delta}$-linear (recall that $\mathcal{F}^s$ is the saturation of $\mathcal{F}_\Delta$ in $\pi^\star T_X$). This induces a unique factorisation (cf. Proposition 5.7) through:

$$\Lambda^2\mathcal{F}_\Delta \to \pi^\star T(X,\Delta)/\mathcal{F}_\Delta, \tag{90}$$

since the maps $\Lambda^2\mathcal{F}_\Delta \to \Lambda^2\mathcal{F}^s$ and $\pi^\star T(X,\Delta)/\mathcal{F}_\Delta \to \pi^\star TX/\mathcal{F}^s$ are both *injective*. We are using the fact that $\mathcal{F}_\Delta = \mathcal{F}^s \cap \pi^\star T(X,\Delta)$, hence Proposition 5.6 follows from the $\mathcal{O}_{X_\Delta}$-linearity of (89). □

We have the following consequence of these considerations.

Corollary **5.9**. — *Let $\mathcal{F}_\Delta \subset \pi^\star T(X,\Delta)$ be a coherent subsheaf. Assume that $\mathcal{F}_\Delta$ is saturated and* G*-invariant. Let $\mathcal{F}^s$ be the saturation of $\mathcal{F}_\Delta$ in $\pi^\star TX$; by Lemma 5.5 we have $\mathcal{F}^s = \pi^\star\mathcal{F}$. We assume moreover that the orbifold Lie bracket* (90) *vanishes identically. Then the sheaf $\mathcal{F}$ defines a holomorphic foliation on* X.

Proof. — By hypothesis, the linear map (90) is identically zero, so we obtain the following partial conclusion: *let v_1 and v_2 be two local sections of $\mathcal{F}_\Delta$; then* $[v_1, v_2]_U \in \mathcal{F}_\Delta$.

Let V_1, V_2 be two local sections of $\mathcal{F}$: there exists two local holomorphic functions φ_1 and φ_2 on X_Δ such that

$$v_j := \varphi_j\pi^\star V_j \in \mathcal{F}_\Delta \tag{91}$$

for each $j = 1, 2$. Hence we have $\varphi_1\varphi_2\pi^\star\mathcal{L}_X(V_1, V_2) \in \mathcal{F}_\Delta$ and thus

$$\pi^\star\mathcal{L}_X(V_1, V_2) \in \mathcal{F}^s. \tag{92}$$

This implies that we have

$$\mathcal{L}_X(V_1, V_2) \in \mathcal{F} \tag{93}$$

and thus $\mathcal{F}$ defines a foliation on X. □

5.4. *The relative canonical bundle of an orbifold fibration.* — The following results (Theorem 5.10 and Lemma 5.12) have been shown in [19], Proposition 1.9 and Theorem 2.11, by computing the degree of both sides on complete intersection curves of very ample classes. We present here a different proof.

Theorem **5.10**. — *Let* $\mathcal{F}_\Delta \subset \pi^*\mathrm{T}(\mathrm{X}, \Delta)$ *be such that its saturation* $\mathcal{F}_\Delta^{sat}$ *in* $\pi^\star \mathrm{T}_\mathrm{X}$ *is equal to* $\pi^\star(\mathcal{F})$*, where* $\mathcal{F} = \mathrm{Ker}(dp) \subset \mathrm{TX}$ *is an algebraic foliation induced by the rational fibration* $p : \mathrm{X} \dashrightarrow \mathrm{Z}$*. If* $\mathrm{K}_\mathrm{X} + \Delta$ *is pseudo-effective, then:*

$$\mu_{\pi^\star\beta}(\mathcal{F}_\Delta) \leq 0, \tag{94}$$

for any movable class β *on* X.

Proof. — It is based on the following two statements of possibly independent interest.

Lemma **5.11**. — *Let* $\mathcal{F}_\Delta \subset \pi^*\mathrm{T}(\mathrm{X}, \Delta)$ *be such that* $\mathcal{F}_\Delta^{sat} = \pi^*(\mathcal{F})$*, where* $\mathcal{F} = \mathrm{Ker}(dp) \subset \mathrm{TX}$ *is an algebraic foliation induced by the rational fibration* $p : \mathrm{X} \dashrightarrow \mathrm{Z}$*. Then we have*

$$\det \mathcal{F}_\Delta^\star = \pi^\star\big(\mathrm{K}_\mathcal{F} + \Delta^{\mathrm{hor}}\big). \tag{95}$$

As in Theorem 5.10, let $p : \mathrm{X} \dashrightarrow \mathrm{Z}$ be a rational map, and let $\mathcal{F} := \mathrm{Ker}(dp)$ be the foliation induced by the kernel of its differential. The following statement holds true; it appears in [23] in a slightly different form.

Lemma **5.12**. — *Let* $\pi_\mathrm{X} : \widehat{\mathrm{X}} \to \mathrm{X}$ *and let* $\pi_\mathrm{Z} : \widehat{\mathrm{Z}} \to \mathrm{Z}$ *be a modification of* X *and* Z*, respectively, such that the following properties are satisfied.*

(i) *The induced map* $\widehat{p} : \widehat{\mathrm{X}} \to \widehat{\mathrm{Z}}$ *is regular, and let* E *be its discriminant divisor.*
(ii) *The inverse image*

$$\widehat{\mathrm{E}} := \widehat{p}^{-1}\mathrm{E} \tag{96}$$

has normal crossings.
(iii) *Every component of* $\widehat{\mathrm{E}}$ *which is contracted by* $\widehat{p}$ *is equally contracted by* π_X.

Let $\widehat{\mathcal{F}}$ *be the foliation induced by* $\mathcal{F}$ *on* $\widehat{\mathrm{X}}$*; then we claim that the following equality holds true*

$$\mathrm{K}_{\widehat{\mathcal{F}}} = \mathrm{K}_{\widehat{\mathrm{X}}/\widehat{\mathrm{Z}}} - \mathrm{D}(\widehat{p}) \tag{97}$$

modulo a divisor which is π_X*-exceptional.*

Before proving these statements, we show that they imply Theorem 5.10. Let β be a movable class on X; by (97) we have the equality

$$c_1\big(\mathrm{K}_{\widehat{\mathcal{F}}} + \widehat{\Delta}^{\mathrm{hor}}\big).\pi_\mathrm{X}^\star\beta = c_1\big(\mathrm{K}_{\widehat{\mathrm{X}}/\widehat{\mathrm{Z}}} + \widehat{\Delta}^{\mathrm{hor}} - \mathrm{D}(\widehat{p})\big).\pi_\mathrm{X}^\star\beta \tag{98}$$

because $W \cdot \pi_X^\star \beta = 0$ for any π_X–exceptional divisor W. Thus we obtain:

$$c_1\big(K_{\widehat{\mathcal{F}}} + \widehat{\Delta}^{\rm hor}\big).\pi_X^\star \beta \geq 0 \tag{99}$$

by Theorem 3.4. By (iii) of Lemma 5.12 combined with the equality (95) we have

$$c_1\big(K_{\widehat{\mathcal{F}}} + \widehat{\Delta}^{\rm hor}\big).\pi_X^\star \beta = c_1\big(\mathcal{F}_\Delta^\star\big).\pi^\star \beta, \tag{100}$$

proving Theorem 5.10.

• *Proof of* Lemma 5.11

The equality (95) will be shown next to hold by a direct computation in local coordinates. Let $X_1 \subset X$ be a Zariski open set such that the restriction $\mathcal{F}|_{X_1}$ is a non-singular foliation, and such that we have $X_1 \cap D_j \cap D_k = \emptyset$ for each pair of indexes $j \neq k$, cf. (52). We equally assume that X_1 does not contain any of the tangency points of $\mathcal{F}$ with the support of Δ^t, i.e. the set of points $z \in \cup D_j$ such that the tangent space of the divisor at z contains $\mathcal{F}_z$. Here we denote by Δ^t the set of components of Δ which are not invariant by $\mathcal{F}$.

We have

$$\operatorname{codim}_X(X \setminus X_1) \geq 2, \tag{101}$$

hence it would be enough to show that (95) holds true when restricted to $\pi^{-1}(X_1)$, given that the map π is finite.

Let $x_0 \in \pi^{-1}(D_1 \cap X_1)$ be a point; we have to distinguish between two cases.

If D_1 *is not invariant by* $\mathcal{F}$, then in particular D_1 is horizontal with respect to the map p, and moreover we can choose the local coordinates $(z_1, \dots, z_n)$ on an open set U containing the point $\pi(x_0)$ such that

$$D_1 \cap U = (z_1 = 0), \tag{102}$$

and such that $\mathcal{F}|_U$ is generated by

$$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_q}. \tag{103}$$

Near x_0 the map π is given by $(w_1, \dots, w_n) \to (w_1^{a_1}, w_2, \dots, w_n)$ and the intersection $\pi^\star \mathcal{F} \cap \pi^\star T(X, \Delta)$ is generated by

$$w_1^{a_1 - b_1} \pi^\star \frac{\partial}{\partial z_1}, \pi^\star \frac{\partial}{\partial z_2}, \dots, \pi^\star \frac{\partial}{\partial z_q}, \tag{104}$$

(notations as in Section 5.4) and the formula (95) follows.

If D_1 *is invariant by* $\mathcal{F}$, then we first remark that D_1 cannot be horizontal with respect to the map p (given that $\mathcal{F}$ is equal to the kernel of this map generically). An appropriate choice of coordinates will give

$$D_1 \cap U = (z_{q+1} = 0), \tag{105}$$

and such that $\mathcal{F}|_{\mathrm{U}}$ is generated by

$$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_q} \tag{106}$$

and the map π is $(w_1, \dots, w_q, w_{q+1}, \dots, w_n) \to (z_1, \dots z_q, z_{q+1}^{a_{q+1}}, \dots, z_n)$. The intersection $\pi^\star\mathcal{F} \cap \pi^\star\mathrm{T}(\mathrm{X}, \Delta)$ is generated by

$$\pi^\star\frac{\partial}{\partial z_1}, \pi^\star\frac{\partial}{\partial z_2}, \dots, \pi^\star\frac{\partial}{\partial z_q}, \tag{107}$$

which settles Lemma 5.11 in this second case.

If the point x_0 does not belong to the support of $\pi^{-1}(\Delta)$, then the verification of (95) is simpler. Indeed, near such point the orbifold tangent space coincides with the inverse image of the tangent bundle of X, thus we have $\mathcal{F}^{sat}_{\Delta, x_0} = \pi^\star(\mathcal{F})_{x_0}$. The formula (95) follows—we remark that in this case it makes no difference if π is ramified at x_0 or not.

All in all, the lemma is proved. □

• *Proof of* Lemma 5.12

Let $\mathcal{J} \subset \widehat{p}^\star\mathrm{T}_{\widehat{\mathrm{Z}}}$ be the image of the differential of $\widehat{p}$, so that we have

$$0 \to \widehat{\mathcal{F}} \to \mathrm{T}\widehat{\mathrm{X}} \to \mathcal{J} \to 0 \tag{108}$$

outside a set of codimension at least two.

Let x_0 be a generic point of a component W of $\widehat{\mathrm{E}}$ *which is not* $\widehat{p}$*-exceptional*. Then we have a coordinate system centered at x_0, say $(z_1, \dots, z_n)$ with respect to which the map $\widehat{p}$ can be written as follows

$$(z_1, \dots, z_n) \to \left(z_{q+1}, \dots, z_{n-1}, z_n^{k_n}\right) \tag{109}$$

where $\mathrm{W} = (z_n = 0)$ near x_0.

By a direct computation of the differential, we deduce that $\mathcal{J}$ is generated by the vector fields

$$\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots, z_n^{k_n-1}\frac{\partial}{\partial t_{n-q}} \tag{110}$$

near y_0.

Hence the determinant of $\mathcal{J}$ is equal to

$$\det \mathcal{J} = -p^\star\mathrm{K}_{\widehat{\mathrm{Z}}} - \sum_i (k_i - 1)\mathrm{Y}_i \tag{111}$$

where the hypersurfaces appearing in the first sum in (111) correspond to the components of the pre-image of $\mathrm{E} \subset \widehat{\mathrm{Z}}$ which are not exceptional with respect to $\widehat{p}$.

Thus, by the sequence (108) we obtain

$$\det \mathcal{F} - p^\star \mathrm{K}_{\widehat{\mathrm{Z}}} - \sum_i (k_i - 1)\mu^\star \mathrm{Y}_i = -\mathrm{K}_{\widehat{\mathrm{X}}} \tag{112}$$

and after rearranging the terms, this can be reformulated as follows

$$\mathrm{K}_{\widehat{\mathcal{F}}} = \mathrm{K}_{\widehat{\mathrm{X}}/\widehat{\mathrm{Z}}} - \mathrm{D}(\widehat{p}) \tag{113}$$

modulo a divisor which is π_{X}-exceptional. This is what we wanted to prove. □

6. Proof of Theorems 1.4 and 1.3

6.1. *Proof of Theorem 1.4.* — We recall here some notions. Let (X, Δ) be a smooth projective log-canonical orbifold pair. Let $\pi : \mathrm{X}_\Delta \to \mathrm{X}$ be a Kawamata cover adapted to (X, Δ), and let $\pi^*\Omega^1(\mathrm{X}, \Delta)$ and $\pi^*\mathrm{T}(\mathrm{X}, \Delta)$ be its cotangent and tangent bundles, respectively. Let further $\mathcal{F}_\Delta \subset \pi^*\mathrm{T}(\mathrm{X}, \Delta)$ be a coherent, saturated subsheaf. We denote by $\mathcal{F}_\Delta^{sat}$ the saturation of $\mathcal{F}_\Delta$ in $\pi^\star \mathrm{T}_\mathrm{X}$. We assume that we have $\mathcal{F}_\Delta^{sat} = \pi^*(\mathcal{F}_\mathrm{X})$ for some uniquely determined distribution $\mathcal{F}_\mathrm{X} \subset \mathrm{TX}$ on X.

Denote by $\Psi : \wedge^2 \mathcal{F}_\Delta \to \pi^*\mathrm{T}(\mathrm{X}, \Delta)/\mathcal{F}_\Delta$ the associated orbifold Lie bracket (cf. Corollary 5.6).

Definition **6.1**. — *We say that $\mathcal{F}_\Delta$ is a foliation on (X, Δ) if the above map Ψ vanishes identically.*

Remark **6.2**. — We assume that

$$2.\mu_{\pi^*\alpha,\min}(\mathcal{F}_\Delta) > \mu_{\pi^*\alpha,\max}\big(\pi^*\mathrm{T}(\mathrm{X}, \Delta)\big)/\mathcal{F}_\Delta$$

for some movable class α on X. Then $\pi^*\mathcal{L}_\mathrm{X}$ vanishes identically, by the usual slope considerations.

Remark **6.3**. — If $\mu_{\pi^*\alpha,\max}(\pi^*\mathrm{T}(\mathrm{X}, \Delta)) > 0$ then we obtain a foliation $\mathcal{F}_\Delta$ by choosing appropriate pieces of the Harder-Narasimhan filtration of $\pi^*\mathrm{T}(\mathrm{X}, \Delta)$.

Recall the statement of Theorem 1.4:

Theorem **6.4**. — *Let $\mathcal{F}_\Delta \subset \pi^*(\mathrm{T}(\mathrm{X}, \Delta))$ be such that: $\mu_{\pi^\star\alpha,\min}(\mathcal{F}_\Delta) > 0$, and such that $2.\mu_{\alpha,\min}(\mathcal{F}_\Delta) > \mu_{\alpha,\max}(\pi^*(\mathrm{T}(\mathrm{X}, \Delta)/\mathcal{F}_\Delta)$. The saturation of $\mathcal{F}_\Delta$ in $\pi^*(\mathrm{T}_\mathrm{X})$ then defines an algebraic foliation $\mathcal{F}_\mathrm{X}$ on X such that the restriction of $\mathrm{K}_\mathrm{X} + \Delta$ to the closure F of the generic leaf of $\mathcal{F}$ is not pseudo-effective.*

Proof. — By Lemma 5.5 there exists a sheaf $\mathcal{F}_X$ on X such that

$$\pi^\star(\mathcal{F}_X) = \mathcal{F}_\Delta^{\mathrm{sat}}.$$

By Remark 6.2 and Corollary 5.9 we infer that $\mathcal{F}_X$ is a foliation on X. Moreover we have

$$\mu_{\alpha,\min}(\mathcal{F}_X) \geq \mu_{\pi^*\alpha,\min}(\mathcal{F}_\Delta) > 0.$$

Next, by Lemma 5.11 and 5.12 we see that $K_{X/Z} + \Delta$ is not pseudo-effective (we use the same notations as in these statements). We obtain Theorem 6.4 as a consequence of Remark 3.6 (and the references therein). □

6.2. *Proof of Theorem 1.3.*

Proof. — Let (X, Δ) be a smooth lc pair as in Theorem 1.3, and let $\pi : X_\Delta \to X$ be a ramified cover associated to this pair, given by Lemma 5.2. Consider any quotient $\otimes^m \pi^\star \Omega^1(X, \Delta) \to \mathcal{Q} \to 0$.

By contradiction, assume that $c_1(\mathcal{Q}).\pi^\star\alpha < 0$ for some movable class α on X. Then we have $\mu_{\pi^*\alpha,\min}(\otimes^m \pi^\star\Omega^1(X, \Delta)) < 0$. By Theorem 2.9 in Section 2 the inequality $\mu_{\pi^*\alpha,\max}(\pi^\star T(X, \Delta)) > 0$ is equally satisfied.

Let $0 \to \mathcal{F}_\Delta \to \pi^\star T(X, \Delta)$ be the maximal $\pi^\star\alpha$ destabilising subsheaf of the orbifold tangent bundle. The maximality of $\mathcal{F}_\Delta$ induces a few important properties: it is $\pi^\star\alpha$-semistable, G-invariant and saturated in $\pi^\star T(X, \Delta)$.

Moreover, we have: $\mu_{\pi^\star\alpha}(\mathcal{F}_\Delta) > 0$, and

$$2\mu_{\pi^\star\alpha}(\mathcal{F}_\Delta) > \mu_{\pi^\star\alpha,\max}\big(\pi^* T(X, \Delta)/\mathcal{F}_\Delta\big).$$

The conclusion then follows by combining Remark 6.2, Corollary 5.9 and Theorem 5.10. Indeed, the saturation of $\mathcal{F}_\Delta$ in the π-inverse image of T_X is the inverse image of a foliation $\mathcal{F}_X$ on X. It turns out that $\mathcal{F}_X$ is algebraic, and the contradiction follows by applying Theorem 5.10. □

7. Birational stability of the orbifold cotangent bundle

In this section we show the *birational stability* of $\Omega^1(X, \Delta)$ if $K_X + \Delta$ is pseudo-effective, in the sense that, the numerical dimension of any sub-line bundle $\mathcal{L}$ of $\otimes^m(\Omega^1(X, \Delta))$ is bounded from above by the numerical dimension of $K_X + \Delta$, this for any $m \geq 0$. This term was introduced in [13] to express the fact that the positivity of the subsheaves of the cotangent bundles (measured in terms of sections rather than slopes) is at most the same as for the cotangent bundle itself.

If the bundle $\mathcal{L}$ is big, then it turns out that the assumption $K_X + \Delta$ *pseudo-effective* can be dropped. This result was obtained in [19] by delicate arguments using crucially [5]. The strengthening from 'generic semi-positive' to 'pseudo-effective' obtained here permits to give below a short and obvious argument, without using [5].

7.1. *Numerical dimension.* — Let X be smooth and projective, and let $\mathcal{L}$ be a line-bundle (or $\mathbf{Q}$-line-bundle) on X. Let A be any sufficiently ample line bundle on X. Recall from [45]:

Definition **7.1**. — *The numerical dimension* $\nu(X, \mathcal{L})$ *of* $\mathcal{L}$ *is defined by:*

$$\nu(X, \mathcal{L}) := \max\left\{k \in \mathbf{Z} \mid \limsup_{p\to\infty} \frac{h^0(X, p.\mathcal{L} + A)}{p^k} < +\infty\right\}.$$

We have the following properties:

1.1. $\nu(X, \mathcal{L}) \in \{-\infty, 0, 1, \dots, n\}$, and $\nu(X, \mathcal{L}) \geq \kappa(X, \mathcal{L})$.

1.2. $\nu(X, \mathcal{L}) \geq 0$ if and only if $\mathcal{L}$ is pseudo-effective. See [45], Section 5, Lemma 1.4.

1.3. $\nu(X, k.\mathcal{L}) = \nu(X, \mathcal{L})$, for any $k > 0$.

1.4. $\nu(X, k.\mathcal{L} + P) \geq \nu(X, \mathcal{L})$ if P is a pseudo-effective $\mathbf{Q}$-line bundle.

1.5. If $\nu(X, \mathcal{L}) = n$, then $\kappa(X, \mathcal{L}) = n$ (i.e.: $\mathcal{L}$ is 'big').

This variant of the Iitaka-Moishezon dimension of $\mathbf{Q}$-line bundles permits, when applied to adjoint line-bundles, to turn conjectures of Abundance-type into theorems. We define now:

Definition **7.2**. — *Let* (X, Δ) *be a projective log-canonical pair with* X *smooth and* Δ *supported on an snc divisor. Let* $\pi : X_\Delta \to X$ *be an adapted cover. Let*

$$\nu^+(X, \Delta) := \max\left\{\nu(X, \mathcal{L}) \mid \exists\, m \text{ such that } \pi^*(\mathcal{L}) \subset \otimes^m \pi^\star\Omega^1(X, \Delta)\right\}.$$

We define, as usual: $\nu(X, \Delta) := \nu(X, K_X + \Delta)$.

We obviously have the following properties:

P.1. $\nu^+(X, \Delta) \geq \nu(X, \Delta) \geq \kappa(X, \Delta)$. We shall show below the equality: $\nu^+(X, \Delta) = \nu(X, \Delta)$, when $K_X + \Delta$ is pseudo-effective.

P.2. When $\Delta = 0$, we thus have, if K_X is pseudo-efective:

$$\kappa(X) \leq \kappa^+(X) \leq \nu^+(X) = \nu(X),$$

where $\kappa^+(X)$ was defined in [11] as:

$$\kappa^+(X) = \max\left\{\kappa\big(X, \det(F)\big) \mid F \subset \Omega^p(X), p > 0\right\}.$$

P.3. Let $X = \mathbf{P}^d \times Y$, with $\dim(Y) = n - d < n$, and K_Y pseudo-effective. Then $\nu(X) = -\infty$, while $\nu^+(X) = \nu(Y) \geq 0$. These examples show that the restriction K_X pseudo-effective is needed, and explain why the this condition can be dropped when $\nu^+(X) = n$.

P.4. Let $r_X : X \to R_X$ be the 'rational quotient' of X (called also its 'MRC-fibration'). It has rationally connected fibres and non-uniruled base (by [25]). One can easily show that $\nu^+(X) = \nu(R_X)$.

7.2. *Birational stability of orbifold cotangent bundles.*

Theorem **7.3**. — *Let* (X, Δ) *to be smooth projective orbifold pair, such that* $K_X + \Delta$ *is pseudo-effective. Then* $\nu^+(X, \Delta) = \nu(X, \Delta)$. *In other words:*

Let $\mathcal{L}$ *be a line bundle on* X*, together with a non-trivial morphism* $\pi^\star\mathcal{L} \to \otimes^m \pi^\star\Omega^1(X, \Delta)$. *Then:* $\nu(X, \mathcal{L}) \leq \nu(X, K_X + \Delta)$.

Proof. — Let $Q := \otimes^m \pi^\star\Omega^1(X, \Delta)/\pi^\star\mathcal{L}$ the quotient sheaf. Since

$$\det(Q) = n^{m-1}\pi^\star(K_X + \Delta) - \pi^\star\mathcal{L},$$

we have:

$$n^{m-1}(K_X + \Delta) = \mathcal{L} + P.$$

Here P is a **Q**-line bundle on X, whose π-inverse image is equal to $\det(Q)$.

The bundle $\det(Q)$ is non-negative when evaluated on any inverse image of any movable class on X, by Theorem 1.3. Thus P is pseudo-effective, and therefore $\nu(K_X + \Delta) \geq \nu(\mathcal{L})$ by property P.4 above. □

Remark **7.4**. — Assume that the line bundle $\mathcal{L}$ in Theorem 7.3 is big. Then $K_X + \Delta$ is big if pseudo-effective. Indeed we deduce that $\nu(X, K_X + \Delta) = n$, and then the conclusion follows from the property P.5 in Section 7.1 (or, without it, from the fact that the sum of a pseudo-effective and of a big **Q**-line bundle is big). We shall remove the hypothesis "$K_X + \Delta$ pseudo-effective" in the next subsection.

Remark **7.5**. — The following observation has been communicated by Behrouz Taji: if X is smooth projective, n-dimensional, and if $\nu(X) = 0$, with $\chi(X, \mathcal{O}_X) \neq 0$, then $\pi_1(X)$ is finite of cardinality at most 2^{n-1}.

To see this, just apply [11], Theorem 4.1, which says that X has a finite fundamental group of cardinality at most 2^{n-1} if $\kappa^+(X) \leq 0$. But now, observe that: $\kappa^+(X) \leq \nu^+(X) = \nu(X) = 0$. The last equality holds by Theorem 7.3, since K_X is pseudo-effective if $\nu(X) = 0$. Notice that $\nu(X) = 0$ implies $\kappa(X) = 0$, by [33], and that the converse is conjecturally true by Abundance. It was conjectured in [11] that $\kappa^+(X) = \kappa(X)$ if $\kappa(X) \geq 0$.

7.3. *Criteria for pseudoeffectivity and log-general type.*

Theorem **7.6**. — *Let* (X, Δ) *be a smooth orbifold log-canonical pair, and let* L *be a pseudo-effective line bundle on* X*. We assume that there exists a non-zero map*

$$\pi^\star L \to \pi^\star\Omega^{\otimes m}(X, \Delta) \otimes \pi^\star(K_X + \Delta)^p \tag{114}$$

for integers $m \geq 0$ *and* $p > 0$. *Then* $K_X + \Delta$ *is pseudo-effective. (The converse is obvious, taking* $m = 0, p = 1$.*)*

Proof. — Let H be an ample line bundle on X. Let $t_{\min}$ be the minimum of the positive real numbers t such that

$$K_X + \Delta + tH \tag{115}$$

is pseudo-effective. The existence of $t_{\min}$ is guaranteed by the fact that the pseudo-effective cone is closed.

We claim that $t_{\min} = 0$. If not, let $(t_k) \subset \mathbf{Q}_+$ be a decreasing sequence of (positive) rational numbers converging to $t_{\min}$. Since H is ample, there exists a smooth $\mathbf{Q}$-divisor still denoted by H in the linear system $|H|$ such that the orbifold

$$(X, \Delta + t_k H) \tag{116}$$

is log-smooth and log-canonical for each $k \geq 1$. If we denote by π_k the corresponding ramified cover, then the map (114) induces an injective morphism of sheaves

$$\pi_k^\star L \otimes \pi_k^\star (K_X + \Delta)^{-p} \to \pi_k^\star \Omega^m(X, \Delta + t_k H) \tag{117}$$

and let Q_k be the co-kernel of (117). As in the proof of 7.11 we infer that we have

$$n^{m-1}(K_X + \Delta + t_k H) = L - p(K_X + \Delta) + P_k \tag{118}$$

where P_k is pseudo-effective. But this implies that

$$K_X + \Delta + t_k \frac{n^{m-1}}{p + n^{m-1}} H \tag{119}$$

is pseudo-effective, for each value of the parameter k.

On the other hand, there exists $k_0 \gg 0$ such that

$$t_{k_0} \frac{c(m,n)}{p + c(m,n)} < t_{\min}$$

since we have assumed that $t_{\min} > 0$ is a strictly positive number. Combined with the fact that the $\mathbf{Q}$-bundle in (119) is pseudo-effective for $k := k_0$, this is in contradiction with the choice of $t_{\min}$. □

Remark **7.7**. — When $m > 0, p = 0$, the above situation occurs with $K_{(X,\Delta)}$ either pseudo-effective, or not pseudo-effective, as one sees by considering $X = \mathbf{P}^k \times Z_{n-k}$, for $0 \leq k < n$, if $\Delta = 0$, K_Z pseudo-effective.

When $p < 0$ instead, we get a lower bound for the existence of L, by the same method.

Theorem **7.8**. — *Let* (X, Δ) *be a smooth orbifold log-canonical pair such that* $K_X + \Delta$ *is pseudo-effective, but not numerically trivial.*

If $p > n^{m-1}$ *is an integer, every map*

$$\pi^\star\big(L \cdot (K_X + \Delta)^p\big) \to \pi^\star\Omega^{\otimes m}(X, \Delta)$$

vanishes, for any pseudo-effective line bundle L *on* X*. In particular we have* $H^0(X_\Delta, \pi^\star\Omega^{\otimes m}(X, \Delta) \otimes \pi^\star(K_X + \Delta)^{-p}) = 0$*, if* $p > n^{m-1}$.

Proof. — Let a non-zero map $\pi^*(L) \to \pi^\star\Omega^{\otimes m}(X, \Delta) \otimes \pi^\star(K_X + \Delta)^{-\otimes p}$ be given. The same arguments as above show that we have

$$L + P + p.(K_X + \Delta) = n^{m-1}.(K_X + \Delta)$$

for some pseudo-effective P. Let now $\alpha \in \mathrm{Mov}(X)$ be such that $(K_X + \Delta).\alpha > 0$ (this is here that the numerical non-triviality of $(K_X + \Delta)$ is used). We get: $(n^{m-1} - p).((K_X + \Delta).\alpha) \geq 0$, and the conclusion by dividing by $(K_X + \Delta).\alpha$. □

Remark **7.9**. — The trivial example of an Abelian variety X together with $\Delta = 0$ shows that for any $(m, p) \in \mathbf{Z}^{\oplus 2}$ the conclusion may fail when $K_{(X,\Delta)}$ is trivial. It however holds for any blow-up of these $X's$. This example illustrates again the fact that our results are stable by blow-ups, but not necessarily by contractions.

Remark **7.10**. — Also, we note that this statement is considerably weaker than the version obtained in [20], where the same conclusion is obtained under the assumption that $p > m$. However, the technical tools needed in [20] for the proof of this sharper result are much more involved than the present arguments.

Theorem **7.11** *([19]).* — *Let* (X, Δ) *be a smooth log-canonical pair, together with a big line bundle* $\mathcal{L} \to X$ *which admits a non-trivial morphism*

$$\pi^\star\mathcal{L} \to \otimes^m \pi^\star\Omega^1(X, \Delta). \tag{120}$$

Then $K_X + \Delta$ *is big.*

Proof. — One proof is essentially the same as the one used for Theorem 7.6 above, and also as the one used in [19], and of Theorem 2.3 in [18], which deals with the case $\Delta = 0$. The statement can also however be directly deduced from the preceding Theorem 7.6 by exactly the same extremely short argument used to deduce Corollary 8.7 from Theorem 8.6, and to which we refer. □

7.4. *Cases* $-(K_X+\Delta)$ *either ample, or numerically trivial.* — We give here a strengthened form of a result in [19]. The proof is exactly the same as the one of Theorem 7.3 above, so we just state the result.

Theorem **7.12**. — *Let* (X, Δ) *be smooth, projective and log-canonical. Assume that* $K_X + \Delta \equiv 0$.

Then $\mu_{\pi^*(\alpha),\max}(\pi^*(\Omega^1(X,\Delta)) \leq 0$.

Let $\pi^*L \to \otimes^m(\pi^*\Omega^1(X,\Delta))$, $m>0$ *be a non-zero sheaf morphism, for some line bundle* L *on* X.

Then: $-L$ *is pseudo-effective. In particular:* $\kappa(X, L) \leq 0$.

Remark **7.13**. — It is proved in [15], Theorem 4.6, that if $c_1(K_X+\Delta)=0$ (resp. if $-(K_X+\Delta)$ is ample), and if the coefficients of Δ are 'standard' (i.e.: of the form $c_j=(1-\frac{1}{m_j})$, with $m_j>0$ integer), the orbifold fundamental group $\pi_1(X,\Delta)$ is almost abelian (resp. finite).

By combining these ideas with adjacent techniques, the following vanishing result is established in [21], using algebro-geometric arguments in characteristic 0 only.

Theorem **7.14**. — *[21] Let* (X, Δ) *be a smooth projective klt orbifold pair. Assume that* $-(K_X+\Delta)$ *is ample. Let* $\pi : X_\Delta \to X$ *be a Kawamata cover adapted to* Δ. *Then, for any* $m>0$ *and any line bundle* $L' \equiv 0$ *on* X_Δ, *we have:* $H^0(X_\Delta, \otimes^m(\pi^*\Omega^1(X,\Delta)) \otimes L')=0$. *Moreover* $\pi_1(X)=\{1\}$.

8. Variation and positivity for quasi-projective families

We mention here an application of Theorem 7.11 in the theory of moduli. An extremely simplified proof of Theorem 7.11 is presented in the next section.

Let $f: V \to B$ a projective submersion with connected fibres between two quasi-projective connected manifolds V, B. The 'variation' $\mathrm{Var}(f) \in \{0,\dots,d:=\dim(B)\}$ of f is the rank of the Kodaira-Spencer map $\kappa\sigma : TB \to R^1f_*(TV/B)$ at the generic point of B. Thus $\mathrm{Var}(f)=0$ if and only if f is isotrivial.

Let $\bar{B}$ be any 'good' smooth projective compatification of B, such that $D=\bar{B}-B$ is an snc divisor.

The following result was conjectured by E. Viehweg, generalising a former hyperbolicity conjecture of I. R. Shafarevich. Special cases where obtained previously by [34], [29], [46].

Theorem **8.1**. — *Let* $f: V \to B$ *be as above. Assume that the fibres of* f *all have an ample canonical bundle and that* $\mathrm{Var}(f)=\dim(B)$. *Then the base* B *is of log-general type, i.e.*

$$\kappa(\bar{B}, K_{\bar{B}}+D)=\dim(B).$$

Proof. — In [50], Viehweg-Zuo have shown that, in this situation, for some $m > 0$, there exists a big sub-line bundle $\mathcal{L}$ of $\mathrm{Sym}^m(\Omega^1_X(\mathrm{Log}(D))$. From Theorem 7.11 we deduce that $K_{\bar{B}} + D$ is big. □

Partial generalisations have been obtained in [47] and [49], also using Theorem 7.11 and variants of the Viehweg-Zuo sheaf.

In [47], it is shown that, for $f : V \to B$ as above, $\bar{B}$ is of Log-general type if the fibres of f are of general type, and if the (birational) variation is maximal (equal to d).

In [49], the 'isotriviality conjecture' formulated in [13] is solved. This conjecture says that if B is 'special', and if the fibres of f are canonically polarised, then f is isotrivial.

Recall that B being 'special' means that $\kappa(\bar{B}, \mathcal{L}) < p$, for any $p > 0$ and any $\mathcal{L} \subset \Omega^p_{\bar{B}}(\mathrm{Log}(D))$. (Very) particular cases of 'special' quasi-projective manifolds are the ones such that $\kappa(\bar{B}, K_{\bar{B}} + D) = 0$ for some (or any) good projective compactification $\bar{B}$ of B. We refer to [13] for more details on 'specialness' and structure results.

When $d = 1$, the only 'special' quasi-projective curves are: $\mathbf{P}^1$, $\mathbf{C}$, $\mathbf{C}^*$, and E, any elliptic curve. I. R. Shafarevich originally formulated his 'hyperbolicity conjecture' as the isotriviality of smooth families of curves of genus at least 2 parametrised by a 'special' quasi-projective curve. Remark that quasi-projective curves are 'special' if and only if non-hyperbolic. In higher dimensions, there are (lots of) 'special' quasi-projective manifolds $\bar{B}$ of all possible log-Kodaira dimensions less than $d := \dim(B)$.

We mention here yet another result, which can be seen as a version of Viehweg conjecture for Calabi-Yau families.

Theorem **8.2**. — *Let $f : V \to B$ be as above. Assume that the first Chern class of the fibres of f is trivial, and that* $\mathrm{Var}(f) = \dim(B)$. *Then the base* B *is of log-general type, i.e.*

$$\kappa(\bar{B}, K_{\bar{B}} + D) = \dim(B).$$

Proof. — It is completely similar to the argument we have provided for Theorem 8.1, except that the existence of the Viehweg-Zuo sheaf is established in [4], Theorem 9.9. □

The preceding results suggest the more general isotriviality question:

Question. — Let $f : V \to B$ be as above[4] Assume that the fibres of f have a pseudo-effective canonical bundle. If B is 'special', is then f is birationally isotrivial? If the birational variation of f is maximal, is then B is of log-general type?

The question is also interesting when f has Fano fibres. A. Kuznetsov in [37], mentions that 'Gushel-Mukai' manifolds (complete intersections in $Gr(2, 5)$ of Plücker

[4] One may even assume only that f be 'quasi-submersive', meaning that the reduction of each of its fibres is smooth. The conclusion should then hold by replacing B with the 'orbifold base' of f, in the sense of [12]. This conjecture was formulated in this form in [1] when the reduced fibres of f have a semi-ample canonical bundle.

hyperplanes and one hyperquadric) provide non-isotrivial families of Fano threefolds with Picard number 1 parametrised by a smooth projective surface. These families are, however, birationally isotrivial.

8.1. *Criteria for pseudoeffectivity and bigness of 'purely' logarithmic cotangent bundles.* — This final subsection is inspired by a very recent and elegant article of C. Schnell cf. [48]. The point in [48] is that Theorem 7.11 can be obtained by combining some of the main results established in the previous sections with induction on the dimension on X. In this way one can avoid using the full force of the results we have in the general orbifold context, provided that all the coefficients of the divisor Δ are equal to one. Nevertheless, the algebraicity criteria (Theorem 1.1) seems indispensable.

Notice however that, even for moduli problems, the treatment of multiple fibres requires the orbifold context.

We change slightly the notations: the orbifold divisor Δ will be denoted here by $D = \sum D_i$, so as to indicate that the pair (X, D) is purely logarithmic. As before, X is non-singular and D is a reduced divisor with simple normal crossings on X. In what follows we will only be concerned with orbifold pairs (X, D) of this type.

In this case the orbifold tangent bundle is the usual logarithmic tangent bundle. This is a vector bundle on X, sometimes denoted by $T_X(-\mathrm{Log}(D))$, but for the consistency's sake, we will conserve the notation $T(X, D)$ here.

The logarithmic tangent bundle $T(X, D)$ is closed under the Lie bracket induced from T_X., i.e. we have

$$\mathcal{L}_D : \Lambda^2 T(X, D) \to T(X, D) \tag{121}$$

given by the restriction of the Lie bracket of X to the subsheaf $T(X, D)$.

Let $\mathcal{F} \subset T(X, D)$ be a coherent subsheaf. We have a $\mathcal{O}_X$-linear map

$$\mathcal{L}_D^{\mathcal{F}} : \Lambda^2 \mathcal{F} \to T(X, D)/\mathcal{F} \tag{122}$$

induced by $\mathcal{L}_D$.

The following statements are particular cases of Corollary 5.9 and of Theorem 6.4, respectively. In the purely logarithmic case, their proofs simplifies considerably, due to the fact that no adapted cover is needed.

Lemma **8.3**. — *Let* $\mathcal{F} \subset T(X, D)$ *be a coherent saturated subsheaf, such that the corresponding Lie bracket* $\mathcal{L}_D^{\mathcal{F}}$ *vanishes identically. We denote by* $\mathcal{F}^s \subset T_X$ *the saturation of* $\mathcal{F}$ *in the tangent bundle of* X. *Then* $\mathcal{F}^s$ *defines a holomorphic foliation.*

Theorem **8.4**. — *Let* $\mathcal{F} \subset T(X, D)$ *be a coherent subsheaf such that the corresponding Lie bracket* $\mathcal{L}_D^{\mathcal{F}}$ *is identically zero. We assume moreover that* $\mu_{\alpha,\min}(\mathcal{F}) > 0$*, for some* $\alpha \in \mathrm{Mov}(X)$. *Then the following are true.*

(1) *The saturation* $\mathcal{F}^s$ *of* $\mathcal{F}$ *in* T_X *defines an algebraic foliation.*
(2) *The restriction of* $K_X + D$ *to the closure of the generic leaf of the algebraic foliation* $\mathcal{F}^s \subset T_X$ *is not pseudo-effective.*

As in the general case of an arbitrary orbifold divisor, the conclusion of the point (2) of Theorem 8.4 means the following. There exists a birational map $p : X' \to X$ such that the support D' of $p^{-1}(D)$ has simple normal crossings, together with a surjective map $f : X' \to Z$ where Z is a non-singular algebraic manifold, such that we have.

• The foliation induced by $\mathcal{F}^s$ on X' coincides generically with $\mathrm{Ker}(f)$,

• The restriction $K_{X'} + D'|_{X'_z}$ is not pseudo-effective, where X'_z is the fibre of f at a generic point $z \in Z$.

Remark **8.5**. — The conclusion here is considerably weaker than in the case where $D = 0$ (rational connectedness being replaced by uniruledness). The analogous result in this generalised situation is established in [21], after suitable equivalent definitions of rational connectedness in this context are given (based on negativity of the orbifold cotangent bundles, but without reference to 'orbifold rational curves').

The next result is the particular case of Theorem 7.6, where Δ is reduced, which permits to give an extremely simple proof (inspired by, and simplifying [48], who observed that one can argue directly in the logarithmic setting).

Theorem **8.6**. — *Let* (X, D) *be a smooth projective connected purely logarithmic orbifold pair. Let* L *be a pseudo-effective line bundle on* X *such that there exists a sheaf embedding*

$$L \to \otimes^m \big(\Omega^1(X, D)\big) \otimes \big(K_X \otimes \mathcal{O}_X(D)\big)^{\otimes p}$$

for some $m \geq 0, p > 0$. *Then* $K_X + D$ *is pseudo-effective.*

Proof. — Assume by contradiction that $K_X + D$ is not pseudo effective. Let $\alpha \in \mathrm{Mov}(X)$ be such that $(K_X + D).\alpha < 0$. Let $\mathcal{F} \subset T(X, D)$ be a maximal destabilising subsheaf, and $f : X \to Z$ be a fibration such that $\mathcal{F} = \mathrm{K}er(df)$ generically (here we replace (X, D) by a suitable birational smooth model in order to make f regular, as explained in the bullets above). The generic orbifold fibre (X_z, D_z) is thus smooth and $K_{X_z} + D_z$ not pseudo-effective.

We proceed now by induction on $\dim(X)$.

If $\dim(Z) = 0$, then $\mathcal{F} = T(X, D)$, and

$$0 < \mu_{\alpha,\min}\big(T(X, D)\big) = -\mu_{\alpha,\max}\big(\Omega(X, D)\big).$$

In the following (in)equalities, we denote $\Omega(X, D) := \Omega$, T is its dual, and $K := K_X + D$. We have

$$0 \leq L.\alpha \leq \mu_{\alpha,\max}\big((\otimes^m \Omega \otimes K^p)\big) = -m.\mu_{\alpha,\min}(T) + p.K.\alpha \leq p.K.\alpha < 0,$$

and thus we obtain a contradiction.

If $\dim(Z) > 0$, we apply the preceding argument to X_z. We have a non-zero morphism

$$L|_{X_z} \to \otimes^k\left(\Omega^1(X_z, D_z)\right) \otimes \left(K_{X_z} \otimes \mathcal{O}_{X_z}(D_z)\right)^{\otimes p}$$

for some $0 \leq k \leq m$, and the important thing is that $p \geq 1$, so we can use induction to conclude that $K_{X_z} + D_z$ is pseudo-effective. This is a contradiction. □

Corollary **8.7**. — *Let* (X, D) *be a smooth projective connected purely logarithmic orbifold pair. Let* L *be a line bundle on* X*, which admits an embedding* $L \subset \otimes^m \Omega^1(X, D)$ *for some* $m > 0$*, and such that the* $\mathbf{Q}$*-bundle* $\varepsilon(K_X + D) + L$ *is big for some rational number* $\varepsilon \geq 0$*. Then* $K_X + D$ *is big.*

Proof. — Since $\varepsilon(K_X + D) + L$ is big, there exists an integer $q > 0$ such that the bundle $L_1 := L^q \otimes K_X \otimes \mathcal{O}_X(D)$ is effective (the number q depends on (X, D), L and ε).

The hypothesis of Corollary 8.7 shows that L_1 admits an injection into $\otimes^{mq}\Omega^1(X, D) \otimes K_X \otimes \mathcal{O}_X(D)$. If so, Theorem 8.6 implies that $K_X + D$ is pseudo-effective. The corollary then follows, since any quotient of $\otimes^m \Omega^1(X, D)$ has pseudo-effective determinant. □

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