

THE IMAGE OF COLMEZ'S MONTREAL FUNCTOR

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ABSTRACT

We prove a conjecture of Colmez concerning the reduction modulo p of invariant lattices in irreducible admissible unitary p -adic Banach space representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ with $p \geq 5$. This enables us to restate nicely the p -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ and deduce a conjecture of Breuil on irreducible admissible unitary completions of locally algebraic representations.

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1. Introduction

In this paper we study p -adic and mod- p representation theory of $G := \mathrm{GL}_2(\mathbf{Q}_p)$. Our results complement the work of Berger, Breuil and Colmez on the p -adic and mod- p Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, see [5] for an overview. Let L be a finite extension of \mathbf{Q}_p with a ring of integers \mathcal{O} , uniformizer ϖ and residue field k .

Theorem 1.1. — *Assume $p \geq 5$ and let Π be a unitary admissible absolutely irreducible L -Banach space representation of G with a central character and let Θ be an open bounded G -invariant lattice in Π . Then $\Theta \otimes_{\mathcal{O}} k$ is of finite length. Moreover, one of the following holds:*

- (i) $\Theta \otimes_{\mathcal{O}} k$ is absolutely irreducible supersingular;
- (ii) $(\Theta \otimes_{\mathcal{O}} k)^{ss} \subseteq (\mathrm{Ind}_{\mathbf{p}}^G \chi_1 \otimes \chi_2 \omega^{-1})^{ss} \oplus (\mathrm{Ind}_{\mathbf{p}}^G \chi_2 \otimes \chi_1 \omega^{-1})^{ss}$ for some smooth characters $\chi_1, \chi_2 : \mathbf{Q}_p^\times \rightarrow k^\times$, where $\omega(x) = x|x| \pmod{p}$.

Further, the inclusion in (ii) is not an isomorphism if and only if Π is ordinary.

We say that Π is *ordinary* if it is a subquotient of a parabolic induction of a unitary character, so that Π is either a unitary character $\Pi \cong \eta \circ \det$, a twist of the universal unitary completion of the smooth Steinberg representation by a unitary character $\Pi \cong \widehat{\mathrm{Sp}} \otimes \eta \circ \det$ or Π is a unitary parabolic induction of a unitary character. An irreducible smooth k -representation is *supersingular* if it is not a subquotient of any principal series representation. The theorem answers affirmatively for $p \geq 5$ a question of Colmez denoted (Q3) in [23].

Let Z be the centre of G , we fix a continuous character $\zeta : Z \rightarrow \mathcal{O}^\times$. Let $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ be the category of \mathcal{O} -torsion modules with a continuous G -action for the discrete topology on the module, let $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$, consisting of representations on which the centre Z acts by (the image of) ζ , and let $\mathrm{Mod}_{G,\zeta}^{\mathrm{fin}}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(\mathcal{O})$, consisting of representations, which are of finite length as $\mathcal{O}[G]$ -modules. In his Montreal lecture Colmez has defined an exact covariant \mathcal{O} -linear functor $\mathbf{V} : \mathrm{Mod}_{G,\zeta}^{\mathrm{fin}}(\mathcal{O}) \rightarrow \mathrm{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ to the category of \mathcal{O} -modules with a continuous action of $\mathcal{G}_{\mathbf{Q}_p}$, the absolute Galois group of \mathbf{Q}_p . Given Π as in Theorem 1.1 one may choose an open bounded G -invariant lattice Θ in Π and define $\mathbf{V}(\Pi) := L \otimes \varprojlim \mathbf{V}(\Theta / \varpi^n \Theta)$. Since all open bounded lattices in Π are commensurable the definition does not depend on the choice of Θ .

Corollary 1.2. — *Let Π be a unitary admissible absolutely irreducible L -Banach space representation of G with a central character then $\dim_L \mathbf{V}(\Pi) \leq 2$. Moreover, $\dim_L \mathbf{V}(\Pi) < 2$ if and only if Π is ordinary.*

Once one has this, the results of Berger-Breuil [7], Colmez [23] and Kisin [41] imply:

Theorem 1.3. — Assume $p \geq 5$, the functor \mathbf{V} induces a bijection between isomorphism classes of

- (i) absolutely irreducible admissible unitary non-ordinary L -Banach space representations of G with the central character¹ ζ , and
- (ii) absolutely irreducible 2-dimensional continuous L -representations of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant equal to $\zeta \varepsilon$,

where ε is the cyclotomic character, and we view ζ as a character of $\mathcal{G}_{\mathbf{Q}_p}$ via class field theory.²

In [23] Colmez has also defined a characteristic 0 construction, which to every 2-dimensional continuous L -representation of $\mathcal{G}_{\mathbf{Q}_p}$ associates an admissible unitary L -Banach space representation $\Pi(V)$ of G , such that $\mathbf{V}(\Pi(V)) \cong V$. Colmez has calculated³ locally algebraic vectors in $\Pi(V)$ in terms of p -adic Hodge theoretic data attached to V . As a consequence of Theorem 1.3 we know that for every Π in (i) there exists a unique V such that $\Pi \cong \Pi(V)$. Using this in Section 12 we determine admissible absolutely irreducible completions of absolutely irreducible locally algebraic representations. In particular, we show that $\mathrm{Sp} \otimes |\det|^{k/2} \otimes \mathrm{Sym}^k L^2$ admits precisely $\mathbf{P}^1(L)$ non-isomorphic absolutely irreducible admissible unitary completions, where k is a positive integer and Sp is the smooth Steinberg representation of G over L . This confirms a conjecture of Breuil. However, our main result can be summed up as:

1.1. *The correspondence is an equivalence of categories.* — Let $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)$ be the category of unitary admissible L -Banach space representations of G with central character ζ and let $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm},\mathrm{fl}}(L)$ be the full subcategory consisting of objects of finite length. Let $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(\mathcal{O})$ consisting of those objects which are locally of finite length, that is (τ, V) is an object of $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})$ if and only if for every $v \in V$ the $\mathcal{O}[G]$ -module $\mathcal{O}[G]v$ is of finite length. We obtain Bernstein-centre-like⁴ results for the categories $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})$ and $\mathrm{Ban}_{G,\zeta}^{\mathrm{adm},\mathrm{fl}}(L)$. That is we decompose them into a direct product of subcategories and show that each subcategory is naturally equivalent to the category of modules over the rings related to deformation theory of 2-dimensional $\mathcal{G}_{\mathbf{Q}_p}$ -representations.

¹ Dospinescu and Schraen have shown recently in [27] that every absolutely irreducible unitary admissible L -Banach representation of a p -adic Lie group admits a central character.

² We normalize it the same way as Colmez in [23], see Section 5, so that the uniformizers correspond to geometric Frobenii.

³ In [23] some cases are conditional on the results of Emerton, which have now appeared in [32, §7.4].

⁴ Since we work in the category of locally finite representations, our rings are analogous to the completions of the rings in Bernstein's theory [8] at maximal ideals.

To fix ideas let Π be as in Theorem 1.3(i) so that $V := \mathbf{V}(\Pi)$ is an absolutely irreducible 2-dimensional continuous L -representation of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant $\zeta\varepsilon$. Let $R_V^{\zeta\varepsilon}$ be the deformation ring representing the deformation problem of V with determinant $\zeta\varepsilon$ to local artinian L -algebras, and let V^u be the universal deformation of V with the determinant $\zeta\varepsilon$. Let $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(L)_\Pi$ be the full subcategory of $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(L)$ consisting of the representations with all irreducible subquotients isomorphic to Π .

Theorem 1.4. — *The category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(L)_\Pi$ is a direct summand of the category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(L)$ and it is naturally equivalent to the category of $R_V^{\zeta\varepsilon}$ -modules of finite length.*

The first assertion in Theorem 1.4 means that any finite length admissible unitary L -Banach space representation Π_1 of G with a central character ζ can be canonically decomposed $\Pi_1 \cong \Pi_2 \oplus \Pi_3$, such that all the irreducible subquotients of Π_2 are isomorphic to Π and none of the irreducible subquotients of Π_3 are isomorphic to Π . The equivalence of categories in Theorem 1.4 is realized as follows: to each B in $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(L)_\Pi$ we let $m(B) = \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(V^u, \check{\mathbf{V}}(B))$, where $\check{\mathbf{V}}(B) = \mathbf{V}(B)^*(\varepsilon\zeta)$, and then show in Theorem 11.7 that $\check{\mathbf{V}}(B) \cong m(B) \otimes_{R_V^{\zeta\varepsilon}} V^u$. So at least in some sense we may describe what kind of representations of $\mathcal{G}_{\mathbf{Q}_p}$ lie in the image of \mathbf{V} , which explains the title of our paper.

We will discuss now what happens with $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$ and recall that we assume $p \geq 5$. We may define an equivalence relation on the set of (isomorphism classes of) irreducible objects of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$, where $\tau \sim \pi$ if and only if there exists a sequence of irreducible representations $\tau = \tau_0, \tau_1, \dots, \tau_n = \pi$ such that $\tau_i = \tau_{i+1}$, $\text{Ext}_G^1(\tau_i, \tau_{i+1}) \neq 0$ or $\text{Ext}_G^1(\tau_{i+1}, \tau_i) \neq 0$ for each i . An equivalence class is called a block. To a block \mathfrak{B} we associate $\pi_{\mathfrak{B}} := \bigoplus_{\pi \in \mathfrak{B}} \pi$, $\pi_{\mathfrak{B}} \hookrightarrow J_{\mathfrak{B}}$ an injective envelope of $\pi_{\mathfrak{B}}$ in $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$ and $\tilde{E}_{\mathfrak{B}} := \text{End}_G(J_{\mathfrak{B}})$. One may show that $\tilde{E}_{\mathfrak{B}}$ is naturally a pseudo-compact ring, see Section 2. By a general result of Gabriel on locally finite categories [35, §IV] we have a decomposition of categories:

$$\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}},$$

where the product is taken over all the blocks \mathfrak{B} and $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ denotes a full subcategory of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$ consisting of those representations, such that all the irreducible subquotients lie in \mathfrak{B} . Moreover, the functor $\tau \mapsto \text{Hom}_G(\tau, J_{\mathfrak{B}})$ induces an anti-equivalence of categories between $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ and the category of compact $\tilde{E}_{\mathfrak{B}}$ -modules. In this paper we explicitly work out the rings $\tilde{E}_{\mathfrak{B}}$.

We are going to describe the blocks. Since we are working over a coefficient field which is not algebraically closed, not every irreducible k -representation τ of G is absolutely irreducible. However, we show that given a block \mathfrak{B} there exists a finite extension l of k such that for all $\tau \in \mathfrak{B}$, $\tau \otimes_k l$ is isomorphic to a finite direct sum of absolutely irre-

ducible representations. The blocks containing an absolutely irreducible representation are given by:

- (i) $\mathfrak{B} = \{\pi\}$, supersingular;
- (ii) $\mathfrak{B} = \{\mathrm{Ind}_p^G \chi_1 \otimes \chi_2 \omega^{-1}, \mathrm{Ind}_p^G \chi_2 \otimes \chi_1 \omega^{-1}\}$, $\chi_1 \chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$;
- (iii) $\mathfrak{B} = \{\mathrm{Ind}_p^G \chi \otimes \chi \omega^{-1}\}$;
- (iv) $\mathfrak{B} = \{\eta \circ \det, \mathrm{Sp} \otimes \eta \circ \det, (\mathrm{Ind}_p^G \omega \otimes \omega^{-1}) \otimes \eta \circ \det\}$.

To prove this, one has to compute all the Ext^1 groups between irreducible representations of G , which have been classified by Barthel-Livne [1] and Breuil [16]. In many cases these computations have been dealt with by Breuil and the author [20], Colmez [23] and Emerton [31] by different methods, and the computation was completed in [56]. To each \mathfrak{B} we may associate a 2-dimensional semi-simple k -representation ρ of $\mathcal{G}_{\mathbf{Q}_p}$ using the semi-simple mod- p correspondence of Breuil, [16], [17]: (i) $\rho := \mathbf{V}(\pi)$ is absolutely irreducible; (ii) $\rho = \chi_1 \oplus \chi_2$; (iii) $\rho = \chi \oplus \chi$; (iv) $\rho := \eta \oplus \eta\omega$. Let $\mathbf{R}_{\mathrm{tr}\rho}^{\mathrm{ps}, \zeta\varepsilon}$ be the universal deformation ring representing the deformation problem of 2-dimensional pseudocharacters with determinant $\zeta\varepsilon$ lifting the trace of ρ , see Section A for a definition.

Theorem 1.5. — *Let \mathfrak{B} be as above then the centre of $\tilde{\mathbf{E}}_{\mathfrak{B}}$ and hence the centre of the category $\mathrm{Mod}_{G, \zeta}^{\mathrm{lin}}(\mathcal{O})^{\mathfrak{B}}$ is naturally isomorphic to $\mathbf{R}_{\mathrm{tr}\rho}^{\mathrm{ps}, \zeta\varepsilon}$.*

Recall that the centre of an abelian category is the ring of endomorphisms of the identity functor. In particular, it acts naturally on every object in the category. We also show that $\tilde{\mathbf{E}}_{\mathfrak{B}}$ is finitely generated as a module over its centre and after localizing away from the reducible locus it is isomorphic to a matrix algebra. In cases (i), (ii) we have a nice characterization of $\tilde{\mathbf{E}}_{\mathfrak{B}}$ in terms of the Galois side. We may extend \mathbf{V} to the category $\mathrm{Mod}_{G, \zeta}^{\mathrm{lin}}(\mathcal{O})$ since every object is a union of subobjects of finite length. If $\mathfrak{B} = \{\pi\}$ with π supersingular then $\mathbf{J}_{\mathfrak{B}}$ is simply an injective envelope of π . Let $\rho = \mathbf{V}(\pi)$, $\mathbf{R}_{\rho}^{\zeta\varepsilon}$ be the deformation ring representing the deformation problem of ρ with determinant equal to $\zeta\varepsilon$ and let ρ^{un} be the universal deformation with determinant $\zeta\varepsilon$.

Theorem 1.6. — *If $\mathfrak{B} = \{\pi\}$ with π supersingular then $\mathbf{V}(\mathbf{J}_{\mathfrak{B}})^{\vee}(\zeta\varepsilon) \cong \rho^{\mathrm{un}}$ and $\tilde{\mathbf{E}}_{\mathfrak{B}} \cong \mathbf{R}_{\rho}^{\zeta\varepsilon}$, where \vee denotes the Pontryagin dual.*

Thus to every τ in $\mathrm{Mod}_{G, \zeta}^{\mathrm{lin}}(\mathcal{O})^{\mathfrak{B}}$ we may associate a compact $\tilde{\mathbf{E}}_{\mathfrak{B}}$ -module $\mathfrak{m}(\tau) := \mathrm{Hom}_G(\tau, \mathbf{J}_{\mathfrak{B}})$ and then $\mathbf{V}(\tau)^{\vee}(\zeta\varepsilon) \cong \mathfrak{m}(\tau) \widehat{\otimes}_{\tilde{\mathbf{E}}_{\mathfrak{B}}} \rho^{\mathrm{un}}$.

In the generic reducible case (ii), $\mathrm{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\chi_1, \chi_2)$ and $\mathrm{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\chi_2, \chi_1)$ are both 1-dimensional. Thus there exists unique up to isomorphism non-split extension ρ_1 of χ_1 by χ_2 and ρ_2 of χ_2 by χ_1 . Since $\chi_1 \neq \chi_2$ the deformation problems of ρ_1 and ρ_2 with determinant equal to $\zeta\varepsilon$ are represented by $\mathbf{R}_{\rho_1}^{\zeta\varepsilon}$ and $\mathbf{R}_{\rho_2}^{\zeta\varepsilon}$ respectively. Let ρ_1^{un} and ρ_2^{un} be the universal deformation of ρ_1 and ρ_2 with determinant $\zeta\varepsilon$, respectively.

Theorem 1.7. — If \mathfrak{B} is as in (ii) then $\mathbf{V}(\mathbf{J}_{\mathfrak{B}})^{\vee}(\zeta\varepsilon) \cong \rho_1^{\text{un}} \oplus \rho_2^{\text{un}}$ and $\widetilde{\mathbf{E}}_{\mathfrak{B}} \cong \text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{\text{un}} \oplus \rho_2^{\text{un}})$, where \vee denotes the Pontryagin dual.

Again one may describe the image of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ under \mathbf{V} as follows: to every τ in $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ we may associate a compact $\widetilde{\mathbf{E}}_{\mathfrak{B}}$ -module $\mathfrak{m}(\tau) := \text{Hom}_G(\tau, \mathbf{J}_{\mathfrak{B}})$ and then $\mathbf{V}(\tau)^{\vee}(\zeta\varepsilon) \cong \mathfrak{m}(\tau) \widehat{\otimes}_{\widetilde{\mathbf{E}}_{\mathfrak{B}}}(\rho_1^{\text{un}} \oplus \rho_2^{\text{un}})$. For non-generic cases, (iii) and (iv) see the introductions to Section 9 and Section 10.

Proposition 1.8. — The category $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})$ decomposes into a direct sum of categories:

$$\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L}) \cong \bigoplus_{\mathfrak{B}} \text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})^{\mathfrak{B}},$$

where objects of $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})^{\mathfrak{B}}$ are those Π in $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})$ such that for every open bounded G -invariant lattice Θ in Π the irreducible subquotients of $\Theta \otimes_{\mathcal{O}} k$ lie in \mathfrak{B} .

Let $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}}$ be the full subcategory of $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})^{\mathfrak{B}}$ consisting of objects of finite length.

Corollary 1.9. — Suppose that \mathfrak{B} contains an absolutely irreducible representation. We have a natural equivalence of categories

$$\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{n} \in \text{MaxSpec } \mathbf{R}_{\text{tr } \rho}^{\text{ps}, \zeta \varepsilon}[1/p]} \text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}.$$

The category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length over the \mathfrak{n} -adic completion of $\widetilde{\mathbf{E}}_{\mathfrak{B}}[1/p]$.

To explain the last equivalence let Π be an object of $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}}$ and choose an open bounded lattice Θ in Π . For each $n \geq 1$, $\Theta/\varpi^n\Theta$ is an object of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$. Since $\mathbf{R}_{\text{tr } \rho}^{\text{ps}, \zeta \varepsilon}$ is naturally isomorphic to the centre of the category $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$, it acts on $\Theta/\varpi^n\Theta$. By passing to the limit and inverting p we obtain an action of $\mathbf{R}_{\text{tr } \rho}^{\text{ps}, \zeta \varepsilon}[1/p]$ on Π . By definition Π is an object of $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ if and only if it is killed by a power of the maximal ideal \mathfrak{n} . The corollary is essentially an application of the Chinese remainder theorem. If \mathfrak{n} corresponds to the trace of an absolutely irreducible representation, defined over the residue field of \mathfrak{n} , then one may show that the \mathfrak{n} -adic completion of $\widetilde{\mathbf{E}}_{\mathfrak{B}}[1/p]$ is isomorphic to a matrix algebra over the \mathfrak{n} -adic completion of $\mathbf{R}_{\text{tr } \rho}^{\text{ps}, \zeta \varepsilon}[1/p]$. We obtain:

Theorem 1.10. — Let \mathfrak{n} be a maximal ideal of $\mathbf{R}_{\text{tr } \rho}^{\text{ps}, \zeta \varepsilon}[1/p]$ with residue field \mathbf{L} and suppose that the corresponding pseudocharacter $\mathbf{T}_{\mathfrak{n}}$ is the trace of an absolutely irreducible representation \mathbf{V} , defined over \mathbf{L} . Then the category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is naturally equivalent to the category of modules of

finite length over the \mathfrak{n} -adic completion of $\mathbf{R}_{\mathrm{tr},\rho}^{\mathrm{ps},\zeta\varepsilon}[1/p]$. In particular it contains only one irreducible object $\Pi_{\mathfrak{n}}$. The Banach space representation $\Pi_{\mathfrak{n}}$ is non-ordinary, and is the unique irreducible admissible unitary \mathbf{L} -Banach space representation of \mathbf{G} with a central character satisfying $\mathbf{V}(\Pi_{\mathfrak{n}}) \cong \mathbf{V}$.

Theorem 1.11. — *Let \mathfrak{n} be a maximal ideal of $\mathbf{R}_{\mathrm{tr},\rho}^{\mathrm{ps},\zeta\varepsilon}[1/p]$ with residue field \mathbf{L} and suppose that the corresponding pseudocharacter is equal to $\psi_1 + \psi_2$, where $\psi_1, \psi_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{L}^\times$ are continuous group homomorphisms. Then the irreducible objects of $\mathrm{Ban}_{\mathbf{G},\zeta}^{\mathrm{adm},\mathrm{fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ are subquotients of $(\mathrm{Ind}_{\mathbf{p}}^{\mathbf{G}} \psi_1 \otimes \psi_2 \varepsilon^{-1})_{\mathrm{cont}}$ and $(\mathrm{Ind}_{\mathbf{p}}^{\mathbf{G}} \psi_2 \otimes \psi_1 \varepsilon^{-1})_{\mathrm{cont}}$, where we consider ψ_1, ψ_2 as characters of \mathbf{Q}_p^\times via the class field theory and $\varepsilon(x) := x|x|$, for all $x \in \mathbf{Q}_p^\times$.*

1.2. A sketch of proof. — Let \mathbf{G} be any p -adic analytic group. Let $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{sm}}(\mathcal{O})$ be the category of smooth representations of \mathbf{G} on \mathcal{O} -torsion modules and let $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{lfm}}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{sm}}(\mathcal{O})$ consisting of locally finite representations. Let $\mathrm{Mod}_{\mathbf{G}}^2(\mathcal{O})$ be a full subcategory of $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{lfm}}(\mathcal{O})$ closed under arbitrary direct sums and subquotients in $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{lfm}}(\mathcal{O})$. An example of such category $\mathrm{Mod}_{\mathbf{G}}^2(\mathcal{O})$ can be $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{lfm}}(\mathcal{O})$ itself, or $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{lfm}}(k)$ the full subcategory consisting of objects killed by ϖ , or $\mathrm{Mod}_{\mathbf{G},\zeta}^{\mathrm{lfm}}(\mathcal{O})$ the full subcategory consisting of objects on which \mathbf{Z} , the centre of \mathbf{G} , acts by a fixed character ζ , but there are lots of such categories. It follows from [35, §II] that every object in $\mathrm{Mod}_{\mathbf{G}}^2(\mathcal{O})$ has an injective envelope.

Instead of working with torsion modules we prefer to work dually with compact modules. Let \mathbf{H} be a compact open subgroup of \mathbf{G} and let $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{pro\,aug}}(\mathcal{O})$ be the category of profinite $\mathcal{O}[[\mathbf{H}]]$ -modules with an action of $\mathcal{O}[\mathbf{G}]$ such that the two actions are the same when restricted to $\mathcal{O}[\mathbf{H}]$. This category has been introduced by Emerton in [30]. Sending π to its Pontryagin dual π^\vee , see Section 2, induces an anti-equivalence of categories between $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{sm}}(\mathcal{O})$ and $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{pro\,aug}}(\mathcal{O})$. Let $\mathfrak{C}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_{\mathbf{G}}^{\mathrm{pro\,aug}}(\mathcal{O})$ anti-equivalent to $\mathrm{Mod}_{\mathbf{G}}^2(\mathcal{O})$ via Pontryagin duality and let $\mathfrak{C}(k)$ be the full subcategory consisting of objects killed by ϖ . Since an anti-equivalence reverses the arrows every object in $\mathfrak{C}(\mathcal{O})$ has a projective envelope.

Let π be an irreducible object of $\mathrm{Mod}_{\mathbf{G}}^2(\mathcal{O})$ such that $\mathrm{End}_{\mathbf{G}}(\pi) = k$. Let $\mathbf{S} := \pi^\vee$ and $\tilde{\mathbf{P}} \rightarrow \mathbf{S}$ a projective envelope of \mathbf{S} in $\mathfrak{C}(\mathcal{O})$. We assume the existence of an object \mathbf{Q} in $\mathfrak{C}(k)$ of finite length, satisfying the following hypotheses:

- (H1) $\mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{Q}, \mathbf{S}') = 0, \forall \mathbf{S}' \in \mathrm{Irr}(\mathfrak{C}(k)), \mathbf{S} \not\cong \mathbf{S}'$;
- (H2) \mathbf{S} occurs as a subquotient in \mathbf{Q} with multiplicity 1;
- (H3) $\mathrm{Ext}_{\mathfrak{C}(k)}^1(\mathbf{Q}, \mathbf{S}') = 0, \forall \mathbf{S}' \in \mathrm{Irr}(\mathfrak{C}(k)), \mathbf{S} \not\cong \mathbf{S}'$;
- (H4) $\mathrm{Ext}_{\mathfrak{C}(k)}^1(\mathbf{Q}, \mathbf{S})$ is finite dimensional;
- (H5) $\mathrm{Ext}_{\mathfrak{C}(k)}^2(\mathbf{Q}, \mathbf{R}) = 0$, where $\mathbf{R} = \mathrm{rad} \mathbf{Q}$ is the maximal proper subobject of \mathbf{Q} ;
- (H0) $\mathrm{Hom}_{\mathfrak{C}(k)}(\tilde{\mathbf{P}}[\varpi], \mathbf{R}) = 0$,

where $\mathrm{Irr}(\mathfrak{C}(k))$ denotes the set of irreducible objects in $\mathfrak{C}(k)$ (equivalently in $\mathfrak{C}(\mathcal{O})$), and $\tilde{\mathbf{P}}[\varpi]$ denotes the kernel of multiplication by ϖ . We encourage the reader for the sake of this introduction to assume that $\mathbf{Q} = \mathbf{S}$ then the only real hypotheses are (H3)

and (H4). As an example one could take G a pro- p group and π the trivial representation, or $G = \mathbf{Q}_p^\times$ and π a continuous character from G to k^\times and $\text{Mod}_G^2(\mathcal{O}) = \text{Mod}_G^{\text{fin}}(\mathcal{O})$.

The ring $\tilde{\mathbb{E}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}})$ can be naturally equipped with a topology with respect to which it is a pseudo-compact ring. It can be shown, see Section 2, that $\tilde{\mathbb{E}}$ is a local (possibly non-commutative) ring with residue field $\text{End}_{\mathfrak{C}(k)}(\mathbb{S}) = k$. Since k is assumed to be finite $\tilde{\mathbb{E}}$ is in fact compact. In Proposition 3.8 and its Corollaries we show:

Proposition 1.12. — *If the hypotheses are satisfied then the natural topology on $\tilde{\mathbb{E}}$ coincides with the topology defined by the maximal ideal; $\tilde{\mathbb{P}}$ is a flat $\tilde{\mathbb{E}}$ -module and $k \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} \cong \mathbb{Q}$.*

Remark 1.13. — Let us comment on the rigidity of the setup. There always exists an object of $\mathfrak{C}(\mathcal{O})$ satisfying (H1), (H2) and (H3). Moreover, it is uniquely determined up to isomorphism and is isomorphic to $k \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}$, which is the maximal quotient of $\tilde{\mathbb{P}}$ containing \mathbb{S} with multiplicity one. So once we impose (H1), (H2) and (H3) we have no flexibility about (H4) and (H5), moreover $k \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}$ need not be of finite length in general. If either (H4) or (H5) is not satisfied, one might try and replace $\mathfrak{C}(\mathcal{O})$ by a different category, for example a full subcategory or, as we do in Section 10, by a quotient category and hope that the hypotheses hold there.

Using Proposition 1.12 one can do deformation theory with non-commutative coefficients. Let \mathfrak{A} be the category of finite local (possibly non-commutative) artinian augmented \mathcal{O} -algebras with residue field k . The ring $\tilde{\mathbb{E}}$ is a pro-object in this category. A deformation of \mathbb{Q} to A is a pair (M, α) , where M is an object of $\mathfrak{C}(\mathcal{O})$ together with the map of \mathcal{O} -algebras $A \rightarrow \text{End}_{\mathfrak{C}(\mathcal{O})}(M)$, which makes M into a flat A -module and $\alpha : k \widehat{\otimes}_A M \cong \mathbb{Q}$ is an isomorphism in $\mathfrak{C}(k)$. Let $\text{Def}_{\mathbb{Q}} : \mathfrak{A} \rightarrow \text{Sets}$ be the functor associating to A the set of isomorphism classes of deformations of \mathbb{Q} to A . We show in Theorem 3.26 that:

Theorem 1.14. — *If the hypotheses are satisfied then the map which sends $\varphi : \tilde{\mathbb{E}} \rightarrow A$ to $A \widehat{\otimes}_{\tilde{\mathbb{E}}, \varphi} \tilde{\mathbb{P}}$ induces a bijection between A^\times -conjugacy classes of $\text{Hom}_{\widehat{\mathfrak{A}}}(\tilde{\mathbb{E}}, A)$ and $\text{Def}_{\mathbb{Q}}(A)$.*

If we restrict the functor $\text{Def}_{\mathbb{Q}}$ to \mathfrak{A}^{ab} , the full subcategory of \mathfrak{A} consisting of commutative algebras, then we recover the usual deformation theory with commutative coefficients.

Corollary 1.15. — $\text{Def}_{\mathbb{Q}}^{ab}(A) = \text{Hom}_{\widehat{\mathfrak{A}}}(\tilde{\mathbb{E}}^{ab}, A)$, where $\tilde{\mathbb{E}}^{ab}$ is the maximal commutative quotient of $\tilde{\mathbb{E}}$.

Let Π be an admissible unitary Banach space representation of G in the sense of Schneider-Teitelbaum [61] and let Θ be an open bounded G -invariant lattice in Π . We denote by Θ^d its Schikhof dual, $\Theta^d := \text{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})$ equipped with the topology of pointwise convergence. We have shown in [57] that there exists a natural topological

isomorphism $\Theta^d \cong \varprojlim (\Theta / \varpi^n \Theta)^\vee$. Thus Θ^d is an object of $\text{Mod}_G^{\text{proaug}}(\mathcal{O})$. Let $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ denote the full subcategory of the category of admissible Banach space representations of G , such that for some (equivalently every) open bounded G -invariant lattice Θ , Θ^d is an object of $\mathfrak{C}(\mathcal{O})$. One may show that, since $\mathfrak{C}(\mathcal{O})$ is assumed to be closed under subquotients in $\text{Mod}_G^{\text{proaug}}(\mathcal{O})$, the category $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ is abelian. The idea is instead of studying Banach space representations study $\tilde{\mathbb{E}}$ -modules $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d)$ and $\mathfrak{m}(\Pi) := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d) \otimes_{\mathcal{O}} L$.

Lemma 1.16. — *The (possibly infinite) dimension of $\mathfrak{m}(\Pi)$ is equal to the multiplicity with which π occurs in $\Theta \otimes_{\mathcal{O}} k$.*

This is the *cde*-triangle of Serre, see §15 of [63].

Proposition 1.17. — *Suppose that the centre \mathcal{Z} of $\tilde{\mathbb{E}}$ is noetherian and $\tilde{\mathbb{E}}$ is a finitely generated \mathcal{Z} -module. If Π in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ is irreducible then $\mathfrak{m}(\Pi)$ is finite dimensional.*

Let $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ denote the full subcategory of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ consisting of objects of finite length. Let $\text{Ker } \mathfrak{m}$ be the full subcategory of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ consisting of those Π such that $\mathfrak{m}(\Pi) = 0$. It follows from Lemma 1.16 that Π is an object of $\text{Ker } \mathfrak{m}$ if and only if π does not appear as a subquotient of the reduction of Θ modulo ϖ . Since $\tilde{\mathbb{P}}$ is projective one may show that the functor \mathfrak{m} is exact and so $\text{Ker } \mathfrak{m}$ is a thick subcategory. We denote the quotient category by $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m}$.

Theorem 1.18. — *Suppose that the hypotheses (H0)–(H5) hold and \mathcal{Q} is a finitely generated $\mathcal{O}[[\mathbb{H}]]$ -module for an open compact subgroup \mathbb{H} of G . Assume further that the centre of $\tilde{\mathbb{E}}$ is noetherian and $\tilde{\mathbb{E}}$ is a finitely generated module over its centre. Then the functor \mathfrak{m} induces an anti-equivalence of categories between $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m}$ and the category of finite dimensional L -vector spaces with a right $\tilde{\mathbb{E}}[1/\rho]$ -action.*

Corollary 1.19. — *Under the assumptions of Theorem 1.18 the functor \mathfrak{m} induces a bijection between isomorphism classes of:*

- (i) *irreducible right $\tilde{\mathbb{E}}[1/\rho]$ -modules, finite dimensional over L ;*
- (ii) *irreducible Π in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ such that π occurs as a subquotient of $\Theta / \varpi \Theta$ for some open bounded G -invariant lattice Θ in Π .*

Moreover, Π is absolutely irreducible if and only if $\mathfrak{m}(\Pi)$ is absolutely irreducible as $\tilde{\mathbb{E}}[1/\rho]$ -module.

The inverse functor to \mathfrak{m} in Theorem 1.18 is constructed as follows. Let \mathfrak{m} be a finite dimensional $\tilde{\mathbb{E}}[1/\rho]$ -module. Let \mathfrak{m}^0 be any finitely generated $\tilde{\mathbb{E}}$ -submodule of \mathfrak{m} , which contains an L -basis of \mathfrak{m} . Our assumptions imply that $\tilde{\mathbb{E}}$ is compact and noetherian, thus \mathfrak{m}^0 is an open bounded \mathcal{O} -lattice in \mathfrak{m} . Since $\tilde{\mathbb{P}}$ is a flat $\tilde{\mathbb{E}}$ -module by Proposition 1.12 we deduce that $\mathfrak{m}^0 \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}$ is \mathcal{O} -torsion free. Let $\Pi(\mathfrak{m}) := \text{Hom}_{\mathcal{O}}^{\text{cont}}(\mathfrak{m}^0 \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}, L)$

with the topology induced by the supremum norm. One may show that the natural map $\Pi \rightarrow \Pi(\mathfrak{m}(\Pi))$ in $\text{Ban}_{\mathfrak{e}(\mathcal{O})}^{\text{adm,fl}}$ is an isomorphism in the quotient category. If Π is irreducible and $\mathfrak{m}(\Pi) \neq 0$ we deduce that the natural map $\Pi \rightarrow \Pi(\mathfrak{m}(\Pi))$ is an injection. Let m be the multiplicity with which π occurs as a subquotient of $\Theta/\varpi\Theta$. Lemma 1.16 says that $\dim_{\mathbb{L}} \mathfrak{m}(\Pi) = m$ and thus \mathfrak{m}^0 is a free \mathcal{O} -module of rank m and so $\mathfrak{m}^0 \otimes_{\mathcal{O}} k$ is an m -dimensional k -vector space. It follows from the Proposition 1.12 that the semisimplification of $(\mathfrak{m}^0 \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{P}}) \otimes_{\mathcal{O}} k \cong (\mathfrak{m}^0 \otimes_{\mathcal{O}} k) \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{P}}$ is isomorphic to the semisimplification of $\mathbb{Q}^{\oplus m}$. Using this we obtain:

Corollary 1.20. — *Suppose the assumptions of Theorem 1.18 are satisfied. Let Π in $\text{Ban}_{\mathfrak{e}(\mathcal{O})}^{\text{adm}}$ be irreducible and suppose that π occurs as a subquotient of $\Theta/\varpi\Theta$ then*

$$\overline{\Pi} \subseteq ((\mathbb{Q}^{\oplus m})^{\vee})^{ss},$$

where $\overline{\Pi}$ denotes the semi-simplification of $\Theta/\varpi\Theta$ and m the multiplicity with which π occurs in $\overline{\Pi}$.

From the hypotheses one may deduce that $\text{Ext}_{\mathfrak{e}(k)}^1(\mathbb{Q}, \mathbb{Q})$ is finite dimensional and so Corollary 1.15 implies that the tangent space of $\widetilde{\mathbb{E}}$ is finite dimensional. Thus if $\widetilde{\mathbb{E}}$ is commutative then it is noetherian. The irreducible modules of $\widetilde{\mathbb{E}}[1/p]$ correspond to the maximal ideals and the absolutely irreducible modules correspond to the maximal ideals of $\widetilde{\mathbb{E}}[1/p]$ with residue field \mathbb{L} . In particular, the absolutely irreducible modules are 1-dimensional. Hence, we obtain:

Corollary 1.21. — *Suppose that the hypotheses (H0)–(H5) hold and \mathbb{Q} is a finitely generated $\mathcal{O}[[\mathbb{H}]]$ -module for an open compact subgroup \mathbb{H} of \mathbb{G} and $\widetilde{\mathbb{E}}$ is commutative. Then for every absolutely irreducible Π in $\text{Ban}_{\mathfrak{e}(\mathcal{O})}^{\text{adm}}$ such that π is a subquotient in $\Theta/\varpi\Theta$ we have $\overline{\Pi} \subseteq (\mathbb{Q}^{\vee})^{ss}$.*

In Theorem 3.39 we devise a criterion for commutativity of $\widetilde{\mathbb{E}}$.

Theorem 1.22. — *Let $d := \dim \text{Ext}_{\mathfrak{e}(k)}^1(\mathbb{Q}, \mathbb{Q})$ and $r = \lfloor \frac{d}{2} \rfloor$. Suppose that the hypotheses (H0)–(H5) are satisfied and there exists a surjection $\widetilde{\mathbb{E}} \twoheadrightarrow \mathcal{O}[[x_1, \dots, x_d]]$. Further, suppose that for every exact sequence*

$$(1) \quad 0 \rightarrow \mathbb{Q}^{\oplus r} \rightarrow \mathbb{T} \rightarrow \mathbb{Q} \rightarrow 0$$

with $\dim \text{Hom}_{\mathfrak{e}(k)}(\mathbb{T}, \mathbb{S}) = 1$ we have $\dim \text{Ext}_{\mathfrak{e}(k)}^1(\mathbb{T}, \mathbb{S}) \leq \frac{r(r-1)}{2} + d$ then $\widetilde{\mathbb{E}} \cong \mathcal{O}[[x_1, \dots, x_d]]$.

Recall that up to now \mathbb{G} was an arbitrary p -adic analytic group and the category $\text{Mod}_{\mathbb{G}}^2(\mathcal{O})$ was any full subcategory of $\text{Mod}_{\mathbb{G}}^{\text{lfm}}(\mathcal{O})$ closed under arbitrary direct sums and subquotients in $\text{Mod}_{\mathbb{G}}^{\text{lfm}}(\mathcal{O})$. Now we apply the formalism to $\mathbb{G} = \text{GL}_2(\mathbb{Q}_p)$ and $\text{Mod}_{\mathbb{G}}^2(\mathcal{O}) = \text{Mod}_{\mathbb{G}, \zeta}^{\text{lfm}}(\mathcal{O})$, where ζ is a fixed central character. We show in Proposition 5.16 that injective objects in $\text{Mod}_{\mathbb{G}, \zeta}^{\text{lfm}}(\mathcal{O})$ are also injective in $\text{Mod}_{\mathbb{G}, \zeta}^{\text{sm}}(\mathcal{O})$ and this

implies that they are p -divisible and hence projective objects in $\mathfrak{C}(\mathcal{O})$ are \mathcal{O} -torsion free. Thus $\tilde{\mathbb{P}}[\varpi] = 0$ and so the hypothesis (H0) is satisfied. Results of Breuil [16] and Barthel-Livne [1] imply that any object of finite length in $\text{Mod}_{G,\zeta}^{\text{sm}}(\mathcal{O})$ is admissible, dually this means that every object of finite length in $\mathfrak{C}(\mathcal{O})$ is a finitely generated $\mathcal{O}[[H]]$ -module, where H is any open compact subgroup of G . Thus to make the formalism work we only need to find Q and be able to compute Ext-groups.

One has to consider four separate cases corresponding to the shape of the block \mathfrak{B} described in Section 1.1. In the generic cases (i) and (ii), Q is the Pontryagin dual of what Colmez calls *atome automorphe*, that is in case (i) $Q = S = \pi^\vee$, in case (ii) $Q = \kappa^\vee$ where κ is the unique non-split extension between the two distinct principal series representations which lie in the block \mathfrak{B} and S is the cosocle of Q . In Section 6 and Section 8 we verify that the hypotheses (H1)–(H5) are satisfied. Thus by Theorem 1.14 the endomorphism ring \tilde{E} of a projective envelope $\tilde{\mathbb{P}}$ of S in $\mathfrak{C}(\mathcal{O})$ represents a deformation problem of Q with non-commutative coefficients. Using the results of Kisin [41] we show that the functor $\check{\mathbf{V}} : \mathfrak{C}(\mathcal{O}) \rightarrow \text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$, $M \mapsto \mathbf{V}(M^\vee)^\vee(\zeta\varepsilon)$ induces a morphism of deformation functors of Q and $\check{\mathbf{V}}(Q)$ and a surjection $\tilde{E}^{ab} \twoheadrightarrow R_{\check{\mathbf{V}}(Q)}^{\varepsilon\zeta}$, where $R_{\check{\mathbf{V}}(Q)}^{\varepsilon\zeta}$ is the ring representing a deformation problem of $\check{\mathbf{V}}(Q)$ with commutative coefficients and determinant equal to $\varepsilon\zeta$. This argument uses the density of crystalline points in the deformation space and essentially is the same as in [41], except that Kisin deforms objects in $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$ and we deform objects in the dual category $\mathfrak{C}(\mathcal{O})$. In the generic cases the ring $R_{\check{\mathbf{V}}(Q)}^{\varepsilon\zeta}$ is formally smooth and thus a further Ext computation enables us to deduce from Theorem 1.22 that $\check{\mathbf{V}}$ induces an isomorphism $\tilde{E} \cong R_{\check{\mathbf{V}}(Q)}^{\varepsilon\zeta}$. In particular, \tilde{E} is commutative and Corollary 1.21 applies.

The non-generic cases are much more involved. Let $\pi = \text{Ind}_{\mathbb{B}}^G \chi \otimes \chi\omega^{-1}$. In case (iii), $\mathfrak{B} = \{\pi\}$ and we show in Section 9 that the hypotheses (H1)–(H5) are satisfied with $Q = S = \pi^\vee$. One may further show that the dimension of $\text{Ext}_{\mathfrak{C}(b)}^1(Q, Q)$ is 2 and there exists a surjection $\tilde{E} \twoheadrightarrow \mathcal{O}[[x, y]]$, but the last condition in Theorem 1.22 fails. However, we can still compute \tilde{E} using the fact that $\check{\mathbf{V}}$ induces a morphism of deformation functors. Let $R_{2\chi}^{\text{ps},\zeta\varepsilon}$ be the universal deformation ring parameterising 2-dimensional pseudocharacters of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant $\zeta\varepsilon$ lifting $\chi + \chi$ and let $T : \mathcal{G}_{\mathbf{Q}_p} \rightarrow R_{2\chi}^{\text{ps},\zeta\varepsilon}$ be the universal pseudocharacter. We show that \tilde{E} is naturally isomorphic to $R_{2\chi}^{\text{ps},\zeta\varepsilon}[[\mathcal{G}_{\mathbf{Q}_p}]]/J$, where J is a closed two-sided ideal generated by $g^2 - T(g)g + \zeta\varepsilon(g)$ for all $g \in \mathcal{G}_{\mathbf{Q}_p}$. This time we use in an essential way that we allow the coefficients in our deformation theory to be non-commutative. We then show that the absolutely irreducible modules of $\tilde{E}[1/p]$ are at most 2-dimensional, thus using Lemma 1.16 and Corollaries 1.19, 1.20 we obtain that if Π is absolutely irreducible and $\overline{\Pi}$ contains π then $\overline{\Pi} \subseteq \pi^{\oplus 2}$. The idea to look for \tilde{E} of this shape was inspired by [10].

The last case when the block contains 3 distinct irreducible representations is the hardest one. The new feature here is that we need to pass to a certain quotient category

for the formalism to work. This reflects that the deformation ring on the Galois side is not formally smooth. We invite the reader to look at the introduction to Section 10 for more details.

If Θ is an open bounded G -invariant lattice in an admissible unitary L -Banach space representation Π of G with a central character ζ then $\Theta/\varpi\Theta$ is an admissible k -representation of G and thus contains an irreducible subquotient. After replacing L with a finite extension we may assume that the subquotient is absolutely irreducible and thus lies in one of the blocks considered above.

A large part of this paper is devoted to calculations of Ext groups between smooth k -representations of $\mathrm{GL}_2(\mathbf{Q}_p)$. These calculations enable us to apply a general formalism developed in Section 3 and Section 4. This is the technical heart of the paper and where the restrictions on the residual characteristic appear. We also use in an essential way that the group is $\mathrm{GL}_2(\mathbf{Q}_p)$. There are two E_2 -spectral sequences at our disposal. One is obtained from the work of Ollivier [51] and Vignéras [67] on the functor of invariants of the pro- p Iwahori subgroup of G , see Section 5.4. The other is due to Emerton [31] and is induced by his functor of ordinary parts, see Section 7.1.

1.3. Organization. — The paper essentially consists of two parts: in Sections 2, 3 and 4 we develop a theory which works for any p -adic analytic group G provided certain conditions are satisfied; in the rest of the paper we show that these conditions are satisfied when $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and $p \geq 5$. The appendix contains some results on deformation theory of 2-dimensional $\mathcal{G}_{\mathbf{Q}_p}$ -representations.

We will now review the sections in more detail. In Section 2 we introduce and recall some facts about locally finite categories. In Section 3 we set up a formalism with which we do deformation theory with non-commutative coefficients in Section 3.1. In Section 3.3 we devise a criterion with the help of which one may show that the deformation rings we obtain in Section 3.1 are in fact commutative. This criterion will be applied in the generic cases when $G = \mathrm{GL}_2(\mathbf{Q}_p)$, that is when the deformation ring on the Galois side is formally smooth. In Section 4 we work out a theory of blocks for admissible unitary Banach space representations of a p -adic analytic group G . Using the work of Schneider-Teitelbaum [61] (and Lazard [45]) one can forget all the functional analytic problems and the theory works essentially the same way as if G was a finite group. This section up to Section 4.1 is independent of Section 3 and the results are somewhat more general than outlined in Section 1.2. In Section 4.1 we establish a relationship between Banach space representations and the generic fibre of a (possibly non-commutative) ring \tilde{E} representing a deformation problem of Section 3.1. In the applications the ring \tilde{E} turns out to be a finitely generated module over its centre and the centre is a noetherian ring. We show in Section 4 that when these conditions are satisfied we obtain nice finiteness conditions on Banach space representations. Starting from Section 5, $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and $p \geq 5$. The Sections 6, 7, 9, 10 correspond to \mathfrak{B} being as in the cases (i), (ii), (iii) and (iv) of Section 1.1. The argument in the generic cases is outlined in Section 5.8.

1.4. A speculation. — It is known, see for example [20], [37], [57], that if $G \neq \mathrm{GL}_2(\mathbf{Q}_p)$ then there are too many representations of G to have a correspondence with Galois representations. One possible purely speculative scenario to remedy this, would be that a global setting, for example a Shimura curve, cuts out a full subcategory $\mathrm{Mod}_G^2(\mathcal{O})$ of $\mathrm{Mod}_G^{\mathrm{lfm}}(\mathcal{O})$, closed under direct sums and subquotients and for this subcategory results similar to those described in Section 1.1 hold. Moreover, different global settings with the same group G at p would give rise to different subcategories $\mathrm{Mod}_G^2(\mathcal{O})$. For this reason we have taken great care in Section 3 and Section 4 to work with an arbitrary p -adic analytic group G and arbitrary full subcategory $\mathrm{Mod}_G^2(\mathcal{O})$ of $\mathrm{Mod}_G^{\mathrm{lfm}}(\mathcal{O})$, closed under direct sums and subquotients.

2. Notation and preliminaries

Let L be a finite extension of \mathbf{Q}_p , with the ring of integers \mathcal{O} , uniformizer ϖ , and $k = \mathcal{O}/\varpi\mathcal{O}$. Let G be a topological group which is locally pro- p . Later on we will assume that G is p -adic analytic and the main application will be to $G = \mathrm{GL}_2(\mathbf{Q}_p)$ with $p \geq 5$.

Let (A, \mathfrak{m}) be a complete local noetherian \mathcal{O} -algebra with residue field k . We denote by $\mathrm{Mod}_G(A)$ the category of $A[G]$ -modules, $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ the full subcategory with objects V such that

$$V = \bigcup_{H, n} V^H[\mathfrak{m}^n],$$

where the union is taken over all open subgroups of G and integers $n \geq 1$ and $V[\mathfrak{m}^n]$ denotes elements of V killed by all elements of \mathfrak{m}^n . We will call such representations *smooth*. Let $\mathrm{Mod}_G^{\mathrm{lfm}}(A)$ be a full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ with objects smooth G -representation which are *locally of finite length*, this means for every $v \in \pi$ the smallest $A[G]$ -submodule of π containing v is of finite length. These categories are abelian and are closed under direct sums, direct limits and subquotients in $\mathrm{Mod}_G(A)$, that is if we have an exact sequence $0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \pi_3 \rightarrow 0$ in $\mathrm{Mod}_G(A)$ with π_2 an object of $\mathrm{Mod}_G^{\mathrm{lfm}}(A)$ then π_1 and π_3 are objects of $\mathrm{Mod}_G^{\mathrm{lfm}}(A)$. It is useful to observe:

Lemma 2.1. — *Let τ be an object of $\mathrm{Mod}_G^{\mathrm{lfm}}(A)$ and $\mathrm{Hom}_{A[G]}(\pi, \tau) = 0$ for all irreducible π in $\mathrm{Mod}_G^{\mathrm{lfm}}(A)$ then τ is zero.*

We note that the lemma fails in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$, for example $\mathrm{c}\text{-Ind}_{\mathrm{GL}_2(\mathbf{Z}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathbf{1}$ does not contain any irreducible subrepresentations. In practice, we will work with a variant of the above categories by fixing a central character. Let Z be the centre of G and $\zeta : Z \rightarrow A^\times$ a continuous character. We will denote by $\mathrm{Mod}_{G, \zeta}^2(A)$ the full subcategory of $\mathrm{Mod}_G^2(A)$ consisting of those objects on which Z acts by a character ζ . If we have a subgroup H of G then the subscript ζ in $\mathrm{Mod}_{H, \zeta}^2(A)$ will indicate that $\mathrm{Mod}_{H, \zeta}^2(A)$ is a full subcategory

of $\text{Mod}_{\mathbf{H}}^2(A)$ with objects precisely those π such that $zv = \zeta(z)v$ for all $z \in Z \cap \mathbf{H}$ and all $v \in \pi$.

We recall some standard facts about injective and projective envelopes, see [35, §II.5]. Let \mathcal{A} be an abelian category. A monomorphism $\iota : N \hookrightarrow M$ is essential if for every non-zero subobject M' of M we have $\iota(N) \cap M'$ is non-zero. An injective envelope of an object M in \mathcal{A} is an essential monomorphism $\iota : M \hookrightarrow I$ with I an injective object of \mathcal{A} . An epimorphism $q : M \twoheadrightarrow N$ in \mathcal{A} is essential if for every morphism $s : P \rightarrow M$ in \mathcal{A} the assertion “ qs is an epimorphism” implies that s is an epimorphism. A projective envelope of an object N of \mathcal{A} is an essential epimorphism $q : P \twoheadrightarrow N$ with P a projective object in \mathcal{A} . If an injective or projective envelope exists then it is unique up to (non-unique) isomorphism. So by abuse of language we will forget the morphism and just say I is an injective envelope of M or P is a projective envelope of M .

Lemma 2.2. — *The categories $\text{Mod}_G^{\text{sm}}(A)$, $\text{Mod}_{G,\zeta}^{\text{sm}}(A)$, $\text{Mod}_G^{\text{lfin}}(A)$, $\text{Mod}_{G,\zeta}^{\text{lfin}}(A)$ have generators and exact inductive limits.*

Proof. — Let $\mathbf{X} := \bigoplus_{\mathcal{P},n} \text{c-Ind}_{\mathcal{P}}^G A/\mathfrak{m}^n$, where the sum is taken over all open pro- p groups of G and positive integers n then for V in $\text{Mod}_G^{\text{sm}}(A)$ we have

$$\text{Hom}_{\text{AI}[G]}(\text{c-Ind}_{\mathcal{P}}^G A/\mathfrak{m}^n, V) \cong V^{\mathcal{P}}[\mathfrak{m}^n].$$

Hence, $\text{Hom}_{\text{AI}[G]}(\mathbf{X}, V) \cong \prod_{\mathcal{P},n} V^{\mathcal{P}}[\mathfrak{m}^n]$. Since V is a smooth representation the above isomorphism implies that \mathbf{X} is a generator for $\text{Mod}_G^{\text{sm}}(A)$.

Let $\zeta : Z \rightarrow A^\times$ be a continuous character and let $\zeta_n : Z \rightarrow (A/\varpi^n A)^\times$ be the reduction of ζ modulo \mathfrak{m}^n . Since ζ is continuous given an open pro- p group \mathcal{P} of G we may find an open subgroup \mathcal{P}' of \mathcal{P} such that ζ_n is trivial on $\mathcal{P}' \cap Z$. In this case it makes sense to consider ζ_n as a character of $Z\mathcal{P}'$. Let $\mathbf{X}_\zeta := \bigoplus_{\mathcal{P},n} \text{c-Ind}_{Z\mathcal{P}'}^G \zeta_n$ where the sum is taken over all $n \geq 1$ and all open pro- p groups \mathcal{P} of G such that ζ_n is trivial on $\mathcal{P} \cap Z$. Then the same argument as above gives that \mathbf{X}_ζ is a generator in $\text{Mod}_{G,\zeta}^{\text{sm}}(A)$.

Let \mathcal{F} (resp. \mathcal{F}_ζ) be the set of quotients of \mathbf{X} (resp. \mathbf{X}_ζ) of finite length. Then \mathcal{F} (resp. \mathcal{F}_ζ) is a set of generators in $\text{Mod}_G^{\text{lfin}}(A)$, (resp. $\text{Mod}_{G,\zeta}^{\text{lfin}}(A)$).

It is clear that all the categories have inductive limits. The exactness of inductive limits follows from [35] Proposition I.6(b). \square

Corollary 2.3. — *The categories in Lemma 2.2 have injective envelopes.*

Proof. — Every object in a category with generators and exact inductive limits has an injective envelope, see Theorem 2 in [35, §II.6]. \square

Lemma 2.4. — *The categories $\text{Mod}_G^{\text{lfin}}(A)$ and $\text{Mod}_{G,\zeta}^{\text{lfin}}(A)$ are locally finite.*

Proof. — Both categories have a set of generators which are of finite length, namely \mathcal{F} and \mathcal{F}_ζ constructed in the proof of Lemma 2.2. Hence they are locally finite, see Section II.4 in [35] for details. \square

An object V of $\text{Mod}_G^{\text{sm}}(A)$ is called *admissible* if $V^H[\mathfrak{m}^i]$ is a finitely generated A -module for every open subgroup H of G and every $i \geq 1$; V is called *locally admissible* if for every $v \in V$ the smallest $A[G]$ -submodule of V containing v is admissible. Let $\text{Mod}_G^{\text{ladm}}(A)$ be a full subcategory of $\text{Mod}_G^{\text{sm}}(A)$ consisting of locally admissible representations. Emerton in [30] shows that if G is p -adic analytic then $\text{Mod}_G^{\text{ladm}}(A)$ is abelian. Moreover, it follows from [30, Thm. 2.3.8] that if $G = \text{GL}_2(\mathbf{Q}_p)$ or G is a torus then $\text{Mod}_{G,\zeta}^{\text{lfin}}(A) = \text{Mod}_G^{\text{ladm}}(A)$. If the conjecture [30, 2.3.7] holds then we would obtain this result in general.

Let H be a compact open subgroup of G and $A[[H]]$ the completed group algebra of H . Let $\text{Mod}_G^{\text{proaug}}(A)$ be the category of profinite linearly topological $A[[H]]$ -modules with an action of $A[G]$ such that the two actions are the same when restricted to $A[H]$ with morphisms G -equivariant continuous homomorphisms of topological $A[[H]]$ -modules. Since any two compact open subgroups of G are commensurable the definition does not depend on the choice of H . Taking Pontryagin duals induces an anti-equivalence of categories between $\text{Mod}_G^{\text{sm}}(A)$ and $\text{Mod}_G^{\text{proaug}}(A)$, see Lemma 2.2.7 in [30]. By Pontryagin dual we mean

$$M^\vee := \text{Hom}_{\mathcal{O}}^{\text{cont}}(M, L/\mathcal{O}),$$

where L/\mathcal{O} carries discrete topology and M^\vee is equipped with compact open topology. We have a canonical isomorphism $M^{\vee\vee} \cong M$.

We note that the duality reverses the arrows, and so if $\text{Mod}_G^{\text{sm}}(A)$ is a full abelian subcategory of $\text{Mod}_G^{\text{sm}}(A)$ then we may define a full subcategory $\mathfrak{C}(A)$ of $\text{Mod}_G^{\text{proaug}}(A)$ by taking the objects to be all M isomorphic to π^\vee for some object π of $\text{Mod}_G^{\text{sm}}(A)$. The category $\mathfrak{C}(A)$ is abelian and if $\text{Mod}_G^{\text{sm}}(A)$ has exact inductive limits and injective envelopes then $\mathfrak{C}(A)$ has exact projective limits and projective envelopes.

Let $\text{Mod}_G^{\text{sm}}(A)$ be a full subcategory of $\text{Mod}_G^{\text{lfin}}(A)$ closed under arbitrary direct sums and subquotients in $\text{Mod}_G^{\text{lfin}}(A)$. Since $\text{Mod}_G^{\text{lfin}}(A)$ has exact inductive limits so does $\text{Mod}_G^{\text{sm}}(A)$. Moreover, $\text{Mod}_G^{\text{sm}}(A)$ has a set of generators of finite length, one may just take a subset of \mathcal{F} constructed in the proof of Lemma 2.2 consisting of objects that lie in $\text{Mod}_G^{\text{sm}}(A)$. Hence, $\text{Mod}_G^{\text{sm}}(A)$ is locally finite and has injective envelopes. We may define a full subcategory $\mathfrak{C}(A)$ of $\text{Mod}_G^{\text{proaug}}(A)$ by taking the objects to be all M isomorphic to π^\vee for some object π of $\text{Mod}_G^{\text{sm}}(A)$. The category $\mathfrak{C}(A)$ is anti-equivalent to $\text{Mod}_G^{\text{sm}}(A)$. In particular, it is abelian, has exact projective limits and projective envelopes.

Let π_1, \dots, π_n be irreducible, pairwise non-isomorphic objects in $\text{Mod}_G^{\text{sm}}(A)$ and let $\iota : \pi_i \hookrightarrow J_i$ be an injective envelope of π_i in $\text{Mod}_G^{\text{sm}}(A)$. Let $S_i := \pi_i^\vee$, $P_i := J_i^\vee$ and $\kappa := \iota^\vee$ then $\kappa : P_i \twoheadrightarrow S_i$ is a projective envelope of S_i in $\mathfrak{C}(A)$. We put $\pi := \bigoplus_{i=1}^n \pi_i$ then $J := \bigoplus_{i=1}^n J_i$ is an injective envelope of π and $P := J^\vee$ is a projective envelope of $S := \pi^\vee \cong \bigoplus_{i=1}^n S_i$ in $\mathfrak{C}(A)$. Let

$$E := \text{End}_{\mathfrak{C}(A)}(P).$$

Each quotient $q : P \rightarrow M$ defines a right ideal of E :

$$(2) \quad \mathfrak{r}(M) := \{\phi \in E : q \circ \phi = 0\}.$$

We define the natural topology on E by taking $\mathfrak{r}(M)$ with M of finite length to be a basis of open neighbourhoods of 0 in E . With respect to the natural topology E is a pseudo-compact ring, see Proposition 13 in [35, §IV.4]. Moreover, $\mathfrak{m} := \mathfrak{r}(S)$ is the Jacobson radical of E and

$$(3) \quad E/\mathfrak{m} \cong \text{End}_{\mathcal{C}(A)}(S) \cong \prod_{i=1}^n \text{End}_{\mathcal{C}(A)}(S_i),$$

see Proposition 12 in [35, §IV.4] for the first isomorphism; the second holds since π_i are irreducible and distinct. Since π_i is irreducible $\text{End}_{\mathcal{C}(A)}(S_i)$ is a skew field over k . We assume for simplicity that $\text{End}_{\mathcal{C}}(\pi_i)$ is finite dimensional for $1 \leq i \leq n$. This holds if π_i are admissible. Since k is a finite field, $k_i := \text{End}_{\mathcal{C}(A)}(S_i)$ is a finite field extension of k . Hence, E/\mathfrak{m} is a finite dimensional k -vector space and, since k is assumed to be finite, E is a compact ring. Thus all the pseudo-compact modules of E will be in fact compact.

Corollary 2.5. — *If S is irreducible then every $\alpha \in E$, $\alpha \notin \mathfrak{m}$ is a unit in E .*

Proof. — Since S is irreducible, it follows from (3) that \mathfrak{m} is maximal. On the other hand, \mathfrak{m} is also the Jacobson radical of E by Proposition 12 in [35, §IV.4]. Hence, E is a local ring. \square

Corollary 2.6. — *If S is irreducible then the centre \mathcal{Z} of E is a local ring with residue field a finite extension of k .*

Proof. — Let \mathfrak{m} be the maximal ideal of E . Let $a \in \mathcal{Z}$ such that $a \notin \mathfrak{m}$. It follows from Lemma 2.5 that a is a unit in E . However, this implies that a is a unit in \mathcal{Z} as for any $c \in E$ we have

$$a^{-1}c - ca^{-1} = (a^{-1}c - ca^{-1})aa^{-1} = (c - c)a^{-1} = 0,$$

as a lies in the centre of E . Hence $(\mathcal{Z}, \mathcal{Z} \cap \mathfrak{m})$ is a local ring. The last assertion follows since $\text{End}_{\mathcal{C}(A)}(S)$ is a finite extension of k and we have injections $\mathcal{O}/\varpi\mathcal{O} \hookrightarrow \mathcal{Z}/(\mathcal{Z} \cap \mathfrak{m}) \hookrightarrow E/\mathfrak{m}$. \square

Lemma 2.7. — *P is a left pseudo-compact E -module.*

Proof. — We will show that there exists a basis of open neighbourhoods of 0 in P consisting of left E -submodules, such that the quotient is an E -module of finite length. Now P is a pseudo-compact A -module, since it is a Pontryagin dual of a discrete A -module, thus it is enough to show that every open A -submodule M of P contains an open

left E -submodule. Since M is open, the quotient P/M is an A -module of finite length, and hence

$$\tau := A[G] \cdot (P/M)^\vee \subset P^\vee$$

is a smooth representation of G of finite length. Dualizing back, we obtain a factorisation $P \twoheadrightarrow \tau^\vee \twoheadrightarrow P/M$. Then $\tau(\tau^\vee)$ is an open right ideal in E for the natural topology.

Since E with the natural topology is a pseudo-compact ring, $E/\tau(\tau^\vee)$ is a right E -module of finite length. Since E modulo its Jacobson radical is a finite dimensional k -vector space by assumption, we may choose $\phi_1, \dots, \phi_m \in E$ such that $\phi_1 + \tau(\tau^\vee), \dots, \phi_m + \tau(\tau^\vee)$ generate $E/\tau(\tau^\vee)$ as an A -module. We may assume that ϕ_1 is the identity map. We claim that

$$(4) \quad \{v \in P : \phi(v) \in M, \forall \phi \in E\} = \bigcap_{i=1}^m \phi_i^{-1}(M).$$

The left hand side of (4) is equal to $\bigcap_{\phi \in E} \phi^{-1}(M)$ and so is contained in the right hand side. Since $P \rightarrow P/M$ factors through $q : P \rightarrow \tau^\vee$ the kernel of q is contained in M . Hence for all $\psi \in \tau(\tau^\vee)$ and all $v \in P$ we have $\psi(v) \in M$. Since M is an A -module and every $\phi \in E$ may be written as $\phi = \sum_{i=1}^m \lambda_i \phi_i + \psi$, where $\lambda_i \in A$ and $\psi \in \tau(\tau^\vee)$, we get the opposite inclusion.

The right hand side of (4) is an open A -submodule of M and the left hand side is a left E -module. Hence, P is a pseudo-compact E -module. \square

Let m be a right pseudo-compact E -module, for definition and properties see §IV.3 of [35]. Let $\{m_i\}_{i \in I}$ be a basis of open neighbourhoods of 0 in m consisting of right E -modules and let $\{P_j\}_{j \in J}$ be a basis of open neighbourhoods of 0 in P consisting of left E -modules. We define the completed tensor product

$$(5) \quad m \widehat{\otimes}_E P := \varprojlim (m/m_i) \otimes_E (P/P_j),$$

where the limit is taken over $I \times J$. Since m/m_i and P/P_j are E -modules of finite length and E modulo its Jacobson radical is a finite dimensional k -vector space, m/m_i and P/P_j are A -modules of finite length and hence the limit exists in the category of pseudo-compact A -modules. By the universality of tensor product we have a natural map $m \otimes_E P \rightarrow m \widehat{\otimes}_E P$, and we denote the image of $m \otimes v$ by $m \widehat{\otimes} v$.

Lemma 2.8. — $m \widehat{\otimes}_E P$ is an object of $\mathfrak{C}(A)$.

Proof. — It follows directly from (5) that $m \widehat{\otimes}_E P$ is a pseudo-compact $A[[H]]$ -module. Since G acts on P by continuous E -linear homomorphisms, it follows from the universal property of the completed tensor product, see [21, §2], that for each pseudo-compact right E -module m we obtain an action of G on $m \widehat{\otimes}_E P$ by continuous, A -linear

homomorphisms. Moreover, the action of $A[G]$ and $A[[H]]$ induce the same action of $A[H]$. Hence, $m \widehat{\otimes}_E P$ is an object of $\text{Mod}_G^{\text{proaug}}(A)$.

If $m = E$ then $E \widehat{\otimes}_E P \cong P$ and so $m \widehat{\otimes}_E P$ is an object of $\mathfrak{C}(A)$. The functor $\widehat{\otimes}_E P$ is right exact and commutes with direct products, see [26, Exp. VII_B], [21, Lem. A.4]. Hence, if $m \cong \prod_{i \in I} E$ for some set I then $m \widehat{\otimes}_E P \cong \prod_{i \in I} P$. Since direct products exist in $\mathfrak{C}(A)$ we deduce that $m \widehat{\otimes}_E P$ is an object of $\mathfrak{C}(A)$. In general, we have an exact sequence of E -modules $\prod_{i \in I} E \rightarrow \prod_{j \in J} E \rightarrow m \rightarrow 0$ for some sets I and J . Since $\widehat{\otimes}_E P$ is right exact we deduce that $m \widehat{\otimes}_E P$ is the cokernel of $\prod_{i \in I} P \rightarrow \prod_{j \in J} P$ and hence is an object of $\mathfrak{C}(A)$. \square

Since $P \rightarrow S$ is essential we have $\text{Hom}_{\mathfrak{C}(A)}(P, S') = 0$ for all irreducible objects of $\mathfrak{C}(A)$ not isomorphic to S_i for $1 \leq i \leq n$, and

$$\text{Hom}_{\mathfrak{C}(A)}(P, S) \cong \text{End}_{\mathfrak{C}(A)}(S) \cong \prod_{i=1}^n k_i.$$

Thus if M is an object of $\mathfrak{C}(A)$ of finite length then $\text{Hom}_{\mathfrak{C}(A)}(P, M)$ is a right E -module of finite length. If M is any object of $\mathfrak{C}(A)$ then we may write $M = \varprojlim M_i$ with M_i of finite length, and $\text{Hom}_{\mathfrak{C}(A)}(P, M) \cong \varprojlim \text{Hom}_{\mathfrak{C}(A)}(P, M_i)$ is a pseudo-compact E -module. Let us also note that, since $E \otimes_E P \cong P \cong E \widehat{\otimes}_E P$, for any finitely presented right pseudo-compact E -module m we have an isomorphism $m \otimes_E P \cong m \widehat{\otimes}_E P$.

Lemma 2.9. — *If m is a pseudo-compact right E -module then*

$$\text{Hom}_{\mathfrak{C}(A)}(P, m \widehat{\otimes}_E P) \cong m.$$

Proof. — If $m \cong \prod_{i \in I} E$ for some set I then $m \widehat{\otimes}_E P \cong \prod_{i \in I} P$ and hence

$$\text{Hom}_{\mathfrak{C}(A)}(P, m \widehat{\otimes}_E P) \cong \text{Hom}_{\mathfrak{C}(A)}\left(P, \prod_{i \in I} P\right) \cong \prod_{i \in I} E \cong m.$$

In general, we have an exact sequence of E -modules $\prod_{i \in I} E \rightarrow \prod_{j \in J} E \rightarrow m \rightarrow 0$ for some sets I and J . Applying $\widehat{\otimes}_E P$ and then $\text{Hom}_{\mathfrak{C}(A)}(P, *)$ to it we get the assertion. \square

Lemma 2.10. — *If M is in $\mathfrak{C}(A)$ such that $\text{Hom}_{\mathfrak{C}(A)}(M, S') = 0$ for all irreducible S' not isomorphic to S_i for $1 \leq i \leq n$, then the natural map*

$$(6) \quad \text{Hom}_{\mathfrak{C}(A)}(P, M) \widehat{\otimes}_E P \rightarrow M$$

is surjective.

Proof. — Let C be the cokernel. Lemma 2.9 and the projectivity of P implies that $\mathrm{Hom}_{\mathfrak{C}(A)}(P, C) = 0$. The exactness of $\mathrm{Hom}_{\mathfrak{C}(A)}(P, *)$ implies that $\mathrm{Hom}_{\mathfrak{C}(A)}(C, S) = 0$. Since C is a quotient of M this implies that $\mathrm{Hom}_{\mathfrak{C}(A)}(C, S') = 0$ for all irreducible objects of $\mathfrak{C}(A)$. This implies $C = 0$ by Lemma 2.1. \square

Lemma 2.11. — *Let $\mathfrak{C}(\mathcal{O}/\varpi^n\mathcal{O})$ be the full subcategory of $\mathfrak{C}(\mathcal{O})$ consisting of objects killed by ϖ^n , let M be an object of $\mathfrak{C}(\mathcal{O}/\varpi^n\mathcal{O})$ and let $q: \tilde{P} \twoheadrightarrow M$ be a projective envelope of M in $\mathfrak{C}(\mathcal{O})$, then $\tilde{P}/\varpi^n\tilde{P} \twoheadrightarrow M$ is a projective envelope of M in $\mathfrak{C}(\mathcal{O}/\varpi^n\mathcal{O})$.*

Remark 2.12. — We note that Pontryagin duality identifies $\mathfrak{C}(\mathcal{O}/\varpi^n\mathcal{O})$ with the full subcategory of $\mathrm{Mod}_G^2(\mathcal{O})$ consisting of objects killed by ϖ^n .

Proof. — Let $q_n: P \twoheadrightarrow M$ be a projective envelope of M in $\mathfrak{C}(\mathcal{O}/\varpi^n\mathcal{O})$. Since \tilde{P} is projective and q_n is essential there exists $\phi: \tilde{P} \rightarrow P$ such that $q = q_n \circ \phi$. Since ϖ^n kills P , ϕ factors through $\tilde{P}/\varpi^n\tilde{P}$, which lies in $\mathfrak{C}(\mathcal{O}/\varpi^n\mathcal{O})$. Since P is projective in this category, the surjection splits and we have $\tilde{P}/\varpi^n\tilde{P} \cong P \oplus N$. Let $\psi: \tilde{P} \rightarrow N$ be the natural map, then the composition $\mathrm{Ker} \psi \hookrightarrow \tilde{P} \twoheadrightarrow M$ is surjective. As q is essential we get $\mathrm{Ker} \psi = \tilde{P}$ and hence $N = 0$, which gives the claim. \square

3. The formalism

Let \mathfrak{C} be a full abelian subcategory of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$ closed under direct products and subquotients in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$. Note that this implies that \mathfrak{C} is closed under projective limits. We further assume that every object M of \mathfrak{C} can be written as $M \cong \varprojlim M_i$, where the limit is taken over all the quotients of finite length. In the sequel \mathfrak{C} will be either the category $\mathfrak{C}(\mathcal{O})$ or its full subcategory $\mathfrak{C}(k)$, introduced in Section 2.

Dually this means that M^\vee is an object of $\mathrm{Mod}_G^{\mathrm{fin}}(\mathcal{O})$. We denote by $\mathrm{Irr}(\mathfrak{C})$ the set of equivalence classes of irreducible objects in \mathfrak{C} and note that the last assumption implies that if M is an object of \mathfrak{C} and $\mathrm{Hom}_{\mathfrak{C}}(M, S') = 0$ for all $S' \in \mathrm{Irr}(\mathfrak{C})$ then M is zero. We denote by $\mathrm{rad} M$ the intersection of all maximal proper subobjects of M .

Let S be an irreducible object of \mathfrak{C} with $\mathrm{End}_{\mathfrak{C}}(S) = k$. We assume the existence of an object Q in \mathfrak{C} of finite length, satisfying the following hypotheses:

- (H1) $\mathrm{Hom}_{\mathfrak{C}}(Q, S') = 0, \forall S' \in \mathrm{Irr}(\mathfrak{C}), S \not\cong S'$;
- (H2) S occurs as a subquotient in Q with multiplicity 1;
- (H3) $\mathrm{Ext}_{\mathfrak{C}}^1(Q, S') = 0, \forall S' \in \mathrm{Irr}(\mathfrak{C}), S \not\cong S'$;
- (H4) $\mathrm{Ext}_{\mathfrak{C}}^1(Q, S)$ is finite dimensional;
- (H5) $\mathrm{Ext}_{\mathfrak{C}}^2(Q, R) = 0$, where $R = \mathrm{rad} Q$ is the maximal proper subobject of Q .

We note that we will introduce an additional hypothesis (H0) in Proposition 3.17 below. Hypotheses (H1) and (H2) imply that $\mathrm{Hom}_{\mathfrak{C}}(Q, S)$ is one dimensional and that Q

has a unique maximal subobject and if we choose a non-zero $\varphi : Q \rightarrow S$ then we obtain an exact sequence:

$$(7) \quad 0 \rightarrow R \rightarrow Q \xrightarrow{\varphi} S \rightarrow 0.$$

We note that φ is essential. Since if we have $\psi : A \rightarrow Q$ such that $\varphi \circ \psi$ is surjective, then $\text{Hom}_{\mathfrak{C}}(\text{Coker } \psi, S') = 0$ for all $S' \in \text{Irr}(\mathfrak{C})$ and so $\text{Coker } \psi = 0$.

Lemma 3.1. — *Equation (7) induces an isomorphism*

$$(8) \quad \text{Ext}_{\mathfrak{C}}^1(Q, Q) \cong \text{Ext}_{\mathfrak{C}}^1(Q, S)$$

and an injection

$$(9) \quad \text{Ext}_{\mathfrak{C}}^2(Q, Q) \hookrightarrow \text{Ext}_{\mathfrak{C}}^2(Q, S).$$

Proof. — We apply $\text{Hom}_{\mathfrak{C}}(Q, *)$ to (7). The injectivity of (9) and surjectivity of (8) follows from (H5). To show the injectivity of (8) it is enough to show that $\text{Ext}_{\mathfrak{C}}^1(Q, R) = 0$. However, a more general statement follows from (H3). Namely, if R' is of finite length and S is not a subquotient of R' then $\text{Ext}_{\mathfrak{C}}^1(Q, R') = 0$. One argues by induction on the number of irreducible subquotients of R' . \square

Lemma 3.2. — *Let $T \in \mathfrak{C}$ be of finite length and suppose that T has a filtration by subobjects T^i , such that $T^0 = T$ and $T^i/T^{i+1} \cong Q^{\oplus n_i}$, for $i \geq 0$. Then (7) induces an isomorphism:*

$$(10) \quad \text{Ext}_{\mathfrak{C}}^1(T, Q) \cong \text{Ext}_{\mathfrak{C}}^1(T, S).$$

Moreover, $\text{Ext}_{\mathfrak{C}}^1(T, S') = 0$ for all $S' \in \text{Irr}(\mathfrak{C})$, $S' \not\cong S$.

Proof. — By devissage and (H3) we have $\text{Ext}_{\mathfrak{C}}^1(T, S') = 0$ for all $S' \in \text{Irr}(\mathfrak{C})$, $S' \not\cong S$. Since (H2) implies that S is not a subquotient of R , we deduce by devissage that $\text{Ext}_{\mathfrak{C}}^1(T, R) = 0$. Further, devissage and (H5) imply that $\text{Ext}_{\mathfrak{C}}^2(T, R) = 0$. Thus applying $\text{Hom}_{\mathfrak{C}}(T, *)$ to (7) we obtain the isomorphism (10). \square

Let $P \xrightarrow{\kappa} S$ be a projective envelope of S in \mathfrak{C} . Note that since κ is essential we have $\text{Hom}_{\mathfrak{C}}(P, S') = 0$ for all $S' \in \text{Irr}(\mathfrak{C})$, $S' \not\cong S$, and $\dim \text{Hom}_{\mathfrak{C}}(P, S) = 1$. Since P is projective the functor $\text{Hom}_{\mathfrak{C}}(P, *)$ is exact, and thus we get:

Lemma 3.3. — *Let $T \in \mathfrak{C}$ be of finite length. Then the length of $\text{Hom}_{\mathfrak{C}}(P, T)$ as an \mathcal{O} -module is equal to the multiplicity, with which S occurs as a subquotient of T .*

We note that since $Q/\text{rad } Q \cong S$ is irreducible and S occurs in Q with multiplicity 1, every non-zero $\phi \in \text{Hom}_{\mathfrak{C}}(Q, Q)$ is an isomorphism. This implies that Q is killed by ϖ . It follows from Lemma 3.3 that (H2) could be reformulated as

$$(H2') \quad \dim \operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}, \mathbf{Q}) = 1.$$

Since $\varphi : \mathbf{Q} \rightarrow \mathbf{S}$ is surjective and \mathbf{P} is projective, there exists $\theta : \mathbf{P} \rightarrow \mathbf{Q}$ such that $\varphi \circ \theta = \kappa$. Moreover, since φ is essential, θ is surjective.

Lemma 3.4. — *There exists a unique decreasing filtration of \mathbf{P} by subobjects \mathbf{P}^i such that $\mathbf{P} = \mathbf{P}^0$, $\mathbf{P}^i/\mathbf{P}^{i+1} \cong \mathbf{Q}^{\oplus n_i}$, where $n_i \geq 1$, for all $i \geq 0$, and every $\phi : \mathbf{P}^i \rightarrow \mathbf{Q}$ factors through $\mathbf{P}^i/\mathbf{P}^{i+1}$.*

Proof. — If such filtration exists then it is unique as $\mathbf{P}^0 = \mathbf{P}$ and $\mathbf{P}^{i+1} = \bigcap_{\phi} \operatorname{Ker} \phi$, where the intersection is taken over all $\phi \in \operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^i, \mathbf{Q})$. Since $\operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}, \mathbf{Q})$ is 1-dimensional we get that $\mathbf{P}^1 := \operatorname{Ker} \theta$ satisfies the conditions. Suppose that we have defined \mathbf{P}^i , for $0 \leq i \leq n$. Consider the exact sequence:

$$(11) \quad 0 \rightarrow \mathbf{P}^n \rightarrow \mathbf{P} \rightarrow \mathbf{P}/\mathbf{P}^n \rightarrow 0.$$

Let $S' \in \operatorname{Irr}(\mathfrak{C})$, we apply $\operatorname{Hom}_{\mathfrak{C}}(*, S')$ to (11) to get an isomorphism $\operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, S') \cong \operatorname{Ext}_{\mathfrak{C}}^1(\mathbf{P}/\mathbf{P}^n, S')$. Since \mathbf{P} is projective $\operatorname{Ext}_{\mathfrak{C}}^1(\mathbf{P}, *) = 0$. We may apply Lemma 3.2 to $\mathbf{T} = \mathbf{P}/\mathbf{P}^n$ and $\mathbf{T}^i = \mathbf{P}^i/\mathbf{P}^n$. We get

$$(12) \quad \operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, S') = 0, \quad \forall S' \in \operatorname{Irr}(\mathfrak{C}), \quad S' \not\cong \mathbf{S}.$$

Moreover, (H4) implies that $d := \dim \operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, \mathbf{S}) = \dim \operatorname{Ext}_{\mathfrak{C}}^1(\mathbf{P}/\mathbf{P}^n, \mathbf{S})$ is finite. Hence,

$$(13) \quad \mathbf{P}^n / \operatorname{rad} \mathbf{P}^n \cong \mathbf{S}^{\oplus d}.$$

We define $\phi_i : \mathbf{P}^n \rightarrow \mathbf{P}^n / \operatorname{rad} \mathbf{P}^n \rightarrow \mathbf{S}$, where the last map is projection to the i -th component. So ϕ_i form a basis of $\operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, \mathbf{S})$. We apply $\operatorname{Hom}_{\mathfrak{C}}(*, \mathbf{Q})$ and $\operatorname{Hom}_{\mathfrak{C}}(*, \mathbf{S})$ to (11) to get a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, \mathbf{Q}) & \xrightarrow{\cong} & \operatorname{Ext}_{\mathfrak{C}}^1(\mathbf{P}/\mathbf{P}^n, \mathbf{Q}) \\ \downarrow & & \downarrow \cong \\ \operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, \mathbf{S}) & \xrightarrow{\cong} & \operatorname{Ext}_{\mathfrak{C}}^1(\mathbf{P}/\mathbf{P}^n, \mathbf{S}). \end{array} \quad (10)$$

The second vertical arrow is an isomorphism by Lemma 3.2. Hence the first vertical arrow is an isomorphism. Hence there exists $\psi_i \in \operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, \mathbf{Q})$, such that $\varphi \circ \psi_i = \phi_i$. Then ψ_i form a basis of $\operatorname{Hom}_{\mathfrak{C}}(\mathbf{P}^n, \mathbf{Q})$. Let $\theta_n : \mathbf{P}^n \rightarrow \mathbf{Q}^{\oplus d}$ be the map $v \mapsto (\psi_1(v), \dots, \psi_d(v))$. We have a commutative diagram:

$$\begin{array}{ccc} & & \mathbf{Q}^{\oplus d} \\ & \nearrow \theta_n & \downarrow \varphi^{\oplus d} \\ \mathbf{P}^n & \twoheadrightarrow & \mathbf{P}^n / \operatorname{rad} \mathbf{P}^n \end{array}$$

Since the vertical arrow is essential, we get that θ_n is surjective, and define $P^{n+1} := \text{Ker } \theta_n$. Then

$$(14) \quad P^n/P^{n+1} \cong Q^{\oplus d}$$

where $d = \dim \text{Ext}_{\mathcal{C}}^1(P/P^n, S)$. Moreover, we have

$$P^{n+1} = \bigcap_{i=1}^d \text{Ker } \psi_i = \bigcap_{\psi \in \text{Hom}_{\mathcal{C}}(P^n, Q)} \text{Ker } \psi,$$

since ψ_i form a basis of $\text{Hom}_{\mathcal{C}}(P^n, Q)$. \square

Lemma 3.5. — *The natural map $P \rightarrow \varprojlim P/P^n$ is an isomorphism.*

Proof. — Let \mathcal{F} be a set of quotients of P in \mathcal{C} of finite length. Since P is an object of \mathcal{C} , we have $P \cong \varprojlim N$, where the limit is taken over all $N \in \mathcal{F}$. Since P/P^n are of finite length, it is enough to show that for every quotient $q : P \twoheadrightarrow N$ of finite length there exists n such that P^n is contained in the kernel of q . Let N be a counterexample of minimal length m . If N is irreducible then, as $\kappa : P \twoheadrightarrow S$ is essential, we get that $N \cong S$ and $q = \lambda\kappa$ for some $\lambda \in k$. But then P^1 is contained in the kernel of q . Hence, $m > 1$ and we may consider an exact sequence $0 \rightarrow S' \rightarrow N \rightarrow N' \rightarrow 0$, with S' irreducible and N' non-zero. The minimality of m implies that there exists n such that P^n is contained in the kernel of $q' : P \rightarrow N \rightarrow N'$. Since by assumption P^n is not contained in the kernel of q , we obtain a non-zero map $q : P^n \rightarrow S'$. Since S' is irreducible, $\text{rad } P^n$ is contained in the kernel. As by construction P^{n+1} is contained in $\text{rad } P^n$ we obtain a contradiction. \square

Definition 3.6. — *Let $P \xrightarrow{\kappa} S$ be a projective envelope of S in \mathcal{C} . We let*

$$E := \text{End}_{\mathcal{C}}(P), \quad \mathfrak{m} := \{\phi \in E : \kappa \circ \phi = 0\}.$$

A priori \mathfrak{m} is only a right ideal of E . Since P is projective we get a surjection $\text{Hom}_{\mathcal{C}}(P, P) \twoheadrightarrow \text{Hom}_{\mathcal{C}}(P, S)$. Now $\dim \text{Hom}_{\mathcal{C}}(P, S) = 1$, and hence any $\phi \in E$ may be written as $\phi = \lambda + \psi$, where $\lambda \in \mathcal{O}$ and $\psi \in \mathfrak{m}$. Since the image of \mathcal{O} lies in the centre of E , this implies that \mathfrak{m} is a two-sided ideal and $E/\mathfrak{m} \cong \text{End}_{\mathcal{C}}(S) \cong k$.

Lemma 3.7. — *We have $\mathfrak{m}^n P \subseteq P^n$ for $n \geq 0$ and $\mathfrak{m}P = P^1$, so that $P/\mathfrak{m}P \cong Q$.*

Proof. — Recall that $\kappa : P \twoheadrightarrow S$ factors through $\theta : P \twoheadrightarrow Q$. If $\phi \in \mathfrak{m}$ then $\theta \circ \phi$ maps P to $R = \text{rad } Q$. Since $\text{Hom}_{\mathcal{C}}(P, R) = 0$, we obtain $\theta \circ \phi = 0$. Thus

$$(15) \quad \mathfrak{m} = \{\phi \in E : \theta \circ \phi = 0\}.$$

Hence, $\mathfrak{m}P \subseteq P^1 = \ker \theta$. We fix $n \geq 1$ and we claim that if $i \leq n$ then for all $\psi \in \text{Hom}_{\mathcal{C}}(P, P/P^i)$ and $\phi \in \mathfrak{m}^n$ we have $\psi \circ \phi = 0$. The claim applied to the natural map

$P \rightarrow P/P^n$ implies $\mathfrak{m}^n P \subseteq P^n$. We argue by induction on i . If $i = 1$ then $P/P^1 \cong Q$, $\mathrm{Hom}_{\mathcal{C}}(P, Q)$ is 1-dimensional, spanned by θ , and so the claim follows from (15). In general we have an exact sequence:

$$(16) \quad 0 \rightarrow \mathrm{Hom}_{\mathcal{C}}(P, P^{i-1}/P^i) \rightarrow \mathrm{Hom}_{\mathcal{C}}(P, P/P^i) \rightarrow \mathrm{Hom}_{\mathcal{C}}(P, P/P^{i-1}) \rightarrow 0.$$

Let $\psi \in \mathrm{Hom}_{\mathcal{C}}(P, P/P^i)$, $\phi_1 \in \mathfrak{m}^{n-1}$ and $\phi_2 \in \mathfrak{m}$. The image of $\psi \circ \phi_1$ in $\mathrm{Hom}_{\mathcal{C}}(P, P/P^{i-1})$ is zero by the induction hypothesis. Hence, $\psi \circ \phi_1$ induces a map from P to $P^{i-1}/P^i \cong Q^{\oplus d}$. Hence, $\psi \circ \phi_1 \circ \phi_2 = 0$, as $\phi_2 \in \mathfrak{m}$. Now any $\phi \in \mathfrak{m}^n$ can be written as a linear combination of $\phi_1 \circ \phi_2$ with $\phi_1 \in \mathfrak{m}^{n-1}$ and $\phi_2 \in \mathfrak{m}$. Hence, $\psi \circ \phi = 0$.

We know, see (13), that $P^1/\mathrm{rad} P^1 \cong S^{\oplus d}$. Hence, there exists a surjection $P^{\oplus d} \rightarrow P^1$. For $1 \leq i \leq d$ let $X_i : P \rightarrow P^{\oplus d} \rightarrow P^1 \hookrightarrow P$ denote the composition, where the first map is inclusion to the i -th component. Then $X_i \in E$ and $\kappa \circ X_i = 0$. So $X_i \in \mathfrak{m}$ and $P^1 = \sum_{i=1}^d X_i P \subseteq \mathfrak{m}P$. \square

Proposition 3.8. — *For $n \geq 0$ we have:*

- (i) $\mathfrak{m}^n P = P^n$;
- (ii) *the natural map $\mathfrak{m}^n \rightarrow \mathrm{Hom}_{\mathcal{C}}(P, \mathfrak{m}^n P)$ is an isomorphism;*
- (iii) $\dim \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} = \dim \mathrm{Hom}_{\mathcal{C}}(P^{n+1}, S) = \dim \mathrm{Ext}_{\mathcal{C}}^1(P/P^{n+1}, S)$;
- (iv) *the natural map $\mathfrak{m}^n/\mathfrak{m}^{n+1} \widehat{\otimes}_E P \rightarrow \mathfrak{m}^n P/\mathfrak{m}^{n+1} P$ is an isomorphism.*

Proof. — We prove (i) and (ii) by induction on n , and obtain (iii) and (iv) as by-products of the proof. We note (i) and (ii) hold trivially for $n = 0$. Suppose that (i) and (ii) hold for n . Let $d := \dim \mathrm{Hom}_{\mathcal{C}}(P^n, S)$ then $P^n/\mathrm{rad} P^n \cong S^{\oplus d}$, see (13). Since $\mathfrak{m}^n P = P^n$ we get a surjection $P^{\oplus d} \twoheadrightarrow \mathfrak{m}^n P$. For $1 \leq i \leq d$ let

$$X_i : P \rightarrow P^{\oplus d} \twoheadrightarrow \mathfrak{m}^n P \hookrightarrow P$$

denote the composition, where the first map is the inclusion to the i -th component. Then $X_i \in E$ and (ii) implies that $X_i \in \mathfrak{m}^n$. Suppose that $\phi \in \mathfrak{m}$ then $X_i \circ \phi \in \mathfrak{m}^{n+1}$ so the surjection $P^{\oplus d} \twoheadrightarrow \mathfrak{m}^n P \twoheadrightarrow \mathfrak{m}^n P/\mathfrak{m}^{n+1} P$ factors through

$$(17) \quad Q^{\oplus d} \cong (P/\mathfrak{m}P)^{\oplus d} \twoheadrightarrow \mathfrak{m}^n P/\mathfrak{m}^{n+1} P$$

where the first isomorphism follows from Lemma 3.7. On the other hand Lemma 3.7 gives $\mathfrak{m}^{n+1} P \subseteq P^{n+1}$ and since $\mathfrak{m}^n P = P^n$ we have a surjection

$$(18) \quad \mathfrak{m}^n P/\mathfrak{m}^{n+1} P \twoheadrightarrow P^n/P^{n+1} \cong Q^{\oplus d}$$

where the last isomorphism is (14). Since Q is of finite length the composition of (17) and (18) is an isomorphism. Thus $\mathfrak{m}^n P/\mathfrak{m}^{n+1} P \cong P^n/P^{n+1}$ and since $\mathfrak{m}^n P = P^n$ we get $\mathfrak{m}^{n+1} P = P^{n+1}$.

It remains to show that the map $\mathfrak{m}^{n+1} \rightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^{n+1}\mathbb{P})$ is an isomorphism. We apply $\text{Hom}_{\mathcal{E}}(\mathbb{P}, *)$ to the surjection $\mathbb{P}^{\oplus d} \twoheadrightarrow \mathfrak{m}^n\mathbb{P}$ to obtain a surjection:

$$(19) \quad \mathbb{E}^{\oplus d} \twoheadrightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}) \cong \mathfrak{m}^n,$$

where $(\phi_1, \dots, \phi_d) \mapsto \sum_{i=1}^d \mathbf{X}_i \circ \phi_i$. So every $\psi \in \mathfrak{m}^n$ may be written as $\psi = \sum_{i=1}^d \mathbf{X}_i \circ \phi_i$, with $\phi_i \in \mathbb{E}$. Let λ_i be the image of ϕ_i in $\mathbb{E}/\mathfrak{m} \cong k$, then $\phi_i - \lambda_i \in \mathfrak{m}$ and so $\psi \in \sum_{i=1}^d \lambda_i \mathbf{X}_i + \mathfrak{m}^{n+1}$. Hence, $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1} \leq d$. We apply $\text{Hom}_{\mathcal{E}}(\mathbb{P}, *)$ to the surjection $\mathfrak{m}^n\mathbb{P} \twoheadrightarrow \mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P}$ to obtain a surjection:

$$(20) \quad \mathfrak{m}^n \twoheadrightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}) \twoheadrightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P}).$$

Now $\mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P} \cong \mathbb{Q}^{\oplus d}$ and so $\dim \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P}) = d$. The composition in (20) factors through $\mathfrak{m}^n/\mathfrak{m}^{n+1} \twoheadrightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P})$. So $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1} \geq d$. Hence, $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1} = d$ and the surjection is an isomorphism. The commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{m}^{n+1} & \longrightarrow & \mathfrak{m}^n & \longrightarrow & \mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \text{Hom}(\mathbb{P}, \mathfrak{m}^{n+1}\mathbb{P}) & \longrightarrow & \text{Hom}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}) & \longrightarrow & \text{Hom}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P}) \longrightarrow 0 \end{array}$$

implies that $\mathfrak{m}^{n+1} \rightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^{n+1}\mathbb{P})$ is an isomorphism. We have shown that the image of $\{\mathbf{X}_1, \dots, \mathbf{X}_d\}$ in $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a basis of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ as a k -vector space. Thus (17) may be interpreted as an isomorphism $\mathfrak{m}^n/\mathfrak{m}^{n+1} \widehat{\otimes}_{\mathbb{E}} \mathbb{P} \xrightarrow{\cong} \mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P}$, which proves (iv). \square

Corollary 3.9. — *The ideals \mathfrak{m}^n are finitely generated right \mathbb{E} -modules.*

Proof. — This follows from (19). \square

Corollary 3.10. — *We have an isomorphism of \mathcal{O} -modules:*

$$(21) \quad \text{Hom}_{\mathcal{E}}(\mathbb{P}/\mathfrak{m}^n\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P}) \cong \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P})$$

and an isomorphism of rings:

$$(22) \quad \mathbb{E}/\mathfrak{m}^n \cong \text{End}_{\mathcal{E}}(\mathbb{P}/\mathfrak{m}^n\mathbb{P}).$$

Proof. — Application of $\text{Hom}_{\mathcal{E}}(\mathbb{P}, *)$ and $\text{Hom}_{\mathcal{E}}(*, \mathbb{P}/\mathfrak{m}^n\mathbb{P})$ to $0 \rightarrow \mathfrak{m}^n\mathbb{P} \rightarrow \mathbb{P} \rightarrow \mathbb{P}/\mathfrak{m}^n\mathbb{P} \rightarrow 0$ gives exact sequences

$$(23) \quad 0 \rightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathfrak{m}^n\mathbb{P}) \rightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathbb{P}) \rightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P}) \rightarrow 0$$

$$(24) \quad 0 \rightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}/\mathfrak{m}^n\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P}) \rightarrow \text{Hom}_{\mathcal{E}}(\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P}) \rightarrow \text{Hom}_{\mathcal{E}}(\mathfrak{m}^n\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P}).$$

Let $\phi \in \text{Hom}_{\mathfrak{C}}(\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P})$. We may lift it to $\tilde{\phi} \in E$ using (23). Since \mathfrak{m}^n is a two-sided ideal of E , we have $\tilde{\phi}(\mathfrak{m}^n\mathbb{P}) \subseteq \mathfrak{m}^n\mathbb{P}$. Hence, ϕ maps to zero in (24), which implies (21). The last assertion follows from Proposition 3.8(ii). \square

Corollary 3.11. — *We have $E \cong \varprojlim E/\mathfrak{m}^n$. The \mathfrak{m} -adic topology on E coincides with the natural one, defined in Section 2.*

Proof. — Lemma 3.5 and Proposition 3.8(i) imply that $\mathbb{P} \cong \varprojlim \mathbb{P}/\mathfrak{m}^n\mathbb{P}$. Thus

$$(25) \quad E \cong \text{Hom}_{\mathfrak{C}}(\mathbb{P}, \varprojlim \mathbb{P}/\mathfrak{m}^n\mathbb{P}) \cong \varprojlim \text{Hom}_{\mathfrak{C}}(\mathbb{P}, \mathbb{P}/\mathfrak{m}^n\mathbb{P}) \cong \varprojlim E/\mathfrak{m}^n$$

where the last isomorphism follows from Corollary 3.10. It follows from Proposition 3.8(i) and Lemma 3.4 that $\mathbb{P}/\mathfrak{m}^n\mathbb{P}$ is of finite length, hence the ideal $\mathfrak{r}(\mathbb{P}/\mathfrak{m}^n\mathbb{P})$, defined in (2), is an open ideal of E . It follows from Proposition 3.8(iii), that $\mathfrak{r}(\mathbb{P}/\mathfrak{m}^n\mathbb{P}) = \mathfrak{m}^n$. Conversely, if \mathfrak{r} is an open ideal of E then, E/\mathfrak{r} is an E -module of finite length, and so will be annihilated by some power of \mathfrak{m} , which implies that \mathfrak{r} is open in the \mathfrak{m} -adic topology. Hence the two topologies coincide. \square

Corollary 3.12. — *The functor $\widehat{\otimes}_E \mathbb{P}$ is exact.*

Proof. — We will show that if $0 \rightarrow m_1 \rightarrow m_2 \rightarrow m_3 \rightarrow 0$ is an exact sequence of right pseudo-compact E -modules then $0 \rightarrow m_1 \widehat{\otimes}_E \mathbb{P} \rightarrow m_2 \widehat{\otimes}_E \mathbb{P} \rightarrow m_3 \widehat{\otimes}_E \mathbb{P} \rightarrow 0$ is an exact sequence in \mathfrak{C} . Since projective limits commute with the completed tensor product and are exact in \mathfrak{C} , we may assume that m_1, m_2 and m_3 are of finite length. The functor $\widehat{\otimes}_E \mathbb{P}$ is right exact, let $\widehat{\text{Tor}}_E^i(*, \mathbb{P})$ be the i -th left derived functor of $\widehat{\otimes}_E \mathbb{P}$. It is enough to show that $\widehat{\text{Tor}}_E^1(k, \mathbb{P}) = 0$, since by devissage this implies that $\widehat{\text{Tor}}_E^1(m, \mathbb{P}) = 0$ for all pseudo-compact E -modules m , which are of finite length. We apply $\widehat{\otimes}_E \mathbb{P}$ to the exact sequence $0 \rightarrow \mathfrak{m} \rightarrow E \rightarrow k \rightarrow 0$ to obtain an exact sequence:

$$(26) \quad 0 \rightarrow \widehat{\text{Tor}}_E^1(k, \mathbb{P}) \rightarrow \mathfrak{m} \widehat{\otimes}_E \mathbb{P} \rightarrow \mathbb{P} \rightarrow \mathbb{P}/\mathfrak{m}\mathbb{P} \rightarrow 0.$$

It is enough to show that the natural map $\mathfrak{m} \widehat{\otimes}_E \mathbb{P} \rightarrow \mathbb{P}$ is injective. Proposition 3.8(iv) says that the natural map $\mathfrak{m}^n/\mathfrak{m}^{n+1} \widehat{\otimes}_E \mathbb{P} \rightarrow \mathfrak{m}^n\mathbb{P}/\mathfrak{m}^{n+1}\mathbb{P}$ is an isomorphism for all $n \geq 0$. By devissage, we obtain that the natural map $\mathfrak{m}/\mathfrak{m}^n \widehat{\otimes}_E \mathbb{P} \rightarrow \mathfrak{m}\mathbb{P}/\mathfrak{m}^n\mathbb{P}$ is an isomorphism. Passing to the limit we obtain that the natural map $\mathfrak{m} \widehat{\otimes}_E \mathbb{P} \rightarrow \mathfrak{m}\mathbb{P}$ is an isomorphism. \square

We will refer to the result of the Corollary 3.12 as “ \mathbb{P} is E -flat”.

Corollary 3.13. — *Let $\varphi : E \rightarrow A$ be a map of pseudo-compact rings, which makes A into a pseudo-compact E -module then $A \widehat{\otimes}_{E, \varphi} \mathbb{P}$ is A -flat.*

Proof. — Since A is a pseudo-compact E -module, every pseudo-compact A -module m is also a pseudo-compact E -module via φ . The assertion follows from the isomorphism $m \widehat{\otimes}_A (A \widehat{\otimes}_E P) \cong m \widehat{\otimes}_E P$ and Corollary 3.12. \square

Corollary 3.14. — *Let m be an \mathcal{O} -torsion free, pseudo-compact E -module. Then $m \widehat{\otimes}_E P$ is \mathcal{O} -torsion free.*

Proof. — Since m is \mathcal{O} -torsion free multiplication by ϖ is injective. Since $\widehat{\otimes}_E P$ is exact it remains injective. \square

Remark 3.15. — Let us point out a special case, where our results are particularly easy to prove, and which was the motivation for the formalism. If $\text{Ext}_{\mathcal{C}}^1(S, S') = 0$ for all irreducible S' non-isomorphic to S , and $\text{Ext}_{\mathcal{C}}^1(S, S)$ is finite dimensional, then the hypotheses (H1)–(H5) are satisfied with $Q = S$. The filtration in Lemma 3.4 is simply the radical filtration, which is exhaustive, as by assumption P can be written as projective limit taken over all the quotients of P of finite length. Hence, all the irreducible subquotients of P are isomorphic to S . If m is a pseudo-compact E -module, then $m \widehat{\otimes}_E P$ is a quotient of $\prod_I P$ for some set I , thus all the irreducible subquotients of $m \widehat{\otimes}_E P$ are isomorphic to S . Let $m_1 \hookrightarrow m_2$ be an injection of pseudo-compact E -modules, and let K be the kernel of the induced map $m_1 \widehat{\otimes}_E P \rightarrow m_2 \widehat{\otimes}_E P$. All the irreducible subquotients of K are isomorphic to S , but Lemma 2.9 implies that $\text{Hom}_{\mathcal{C}}(P, K) = 0$. Hence, K is zero and P is E -flat.

3.1. Deformations. — Let $\mathcal{C}(\mathcal{O})$ be a full abelian subcategory of $\text{Mod}_G^{\text{proaug}}(\mathcal{O})$ closed under direct products and subquotients in $\text{Mod}_G^{\text{proaug}}(\mathcal{O})$. We further assume that every object M of $\mathcal{C}(\mathcal{O})$ can be written as $M \cong \varprojlim M_i$, where the limit is taken over all the quotients of finite length. Let $\mathcal{C}(k)$ be a full subcategory of $\mathcal{C}(\mathcal{O})$ consisting of the objects which are killed by ϖ .

Let S and Q be as in the previous section with $\mathcal{C} = \mathcal{C}(k)$. We assume that hypotheses (H1)–(H5) are satisfied in $\mathcal{C} = \mathcal{C}(k)$. Let $P \twoheadrightarrow S$ be a projective envelope of S in $\mathcal{C}(k)$, $E = \text{End}_{\mathcal{C}(k)}(P)$ and \mathfrak{m} the maximal ideal of E defined by 3.6.

Let $\tilde{P} \twoheadrightarrow S$ be a projective envelope of S in $\mathcal{C}(\mathcal{O})$, $\tilde{E} := \text{End}_{\mathcal{C}(\mathcal{O})}(\tilde{P})$ and $\tilde{\mathfrak{m}}$ two sided ideal of \tilde{E} defined by 3.6. For every M in $\mathcal{C}(k)$ we have $\text{Hom}_{\mathcal{C}(\mathcal{O})}(\tilde{P}, M) \cong \text{Hom}_{\mathcal{C}(k)}(\tilde{P}/\varpi\tilde{P}, M)$ thus $\tilde{P}/\varpi\tilde{P}$ is projective in $\mathcal{C}(k)$, and the map $\tilde{P}/\varpi\tilde{P} \rightarrow S$ is essential. Since projective envelopes are unique up to isomorphism, we obtain $P \cong \tilde{P}/\varpi\tilde{P}$. Thus we have an exact sequence:

$$(27) \quad 0 \rightarrow \tilde{P}[\varpi] \rightarrow \tilde{P} \xrightarrow{\varpi} \tilde{P} \rightarrow P \rightarrow 0.$$

Since \tilde{P} is projective applying $\text{Hom}_{\mathcal{C}(\mathcal{O})}(\tilde{P}, *)$ to (27) we obtain an exact sequence

$$(28) \quad 0 \rightarrow \text{Hom}_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \tilde{P}[\varpi]) \rightarrow \tilde{E} \xrightarrow{\varpi} \tilde{E} \rightarrow E \rightarrow 0.$$

Lemma 3.16. — *Let A and B be objects of $\mathfrak{C}(k)$ then there exists an exact sequence*

$$(29) \quad 0 \rightarrow \mathrm{Ext}_{\mathfrak{C}(k)}^1(A, B) \rightarrow \mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^1(A, B) \rightarrow \mathrm{Hom}_{\mathfrak{C}(k)}(A, B).$$

Proof. — Let $0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$ be an extension in $\mathfrak{C}(\mathcal{O})$. Multiplication by ϖ induces an exact sequence

$$(30) \quad 0 \rightarrow B[\varpi] \rightarrow C[\varpi] \rightarrow A[\varpi] \xrightarrow{\partial} B/\varpi B.$$

Since A and B are in $\mathfrak{C}(k)$ we have $B = B[\varpi] = B/\varpi B$ and $A = A[\varpi]$ so ∂ defines an element of $\mathrm{Hom}_{\mathfrak{C}(k)}(A, B)$, which depends only on the class of the extension in $\mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^1(A, B)$. Now $\partial = 0$ if and only if $C = C[\varpi]$, which means if and only if the extension lies in $\mathfrak{C}(k)$. \square

We note that since $\mathfrak{C}(k)$ is a full subcategory of $\mathfrak{C}(\mathcal{O})$, (H1) and (H2) for $\mathfrak{C}(k)$ trivially imply (H1) and (H2) for $\mathfrak{C}(\mathcal{O})$. It follows from the Nakayama's lemma that $\mathfrak{C}(k)$ and $\mathfrak{C}(\mathcal{O})$ have the same irreducible objects. Further, it follows from Lemma 3.16 and (H1) for $\mathfrak{C}(k)$ that (H3) and (H4) for $\mathfrak{C}(k)$ imply (H3) and (H4) for $\mathfrak{C}(\mathcal{O})$.

Proposition 3.17. — *Suppose that the following hypothesis holds:*

$$(H0) \quad \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}[\varpi], \mathrm{rad} Q) = 0,$$

then (H5) for $\mathfrak{C}(k)$ implies (H5) for $\mathfrak{C}(\mathcal{O})$.

Proof. — It follows from (H1) and (H2) that S is not a subquotient of $R = \mathrm{rad} Q$. Thus $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, R) = \mathrm{Hom}_{\mathfrak{C}(k)}(P, R) = 0$, by Lemma 3.3. Since \tilde{P} is projective using (27) we get $\mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^1(P, R) = 0$ and

$$(31) \quad \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}[\varpi], R) \cong \mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^1(\varpi \tilde{P}, R) \cong \mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^2(P, R).$$

Thus (H0) is equivalent to $\mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^2(P, R) = 0$. We apply $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(*, R)$ to $0 \rightarrow \mathfrak{m}P \rightarrow P \rightarrow Q \rightarrow 0$ to get an isomorphism

$$(32) \quad \mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^1(\mathfrak{m}P, R) \cong \mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^2(Q, R).$$

Since P is projective in $\mathfrak{C}(k)$ we also have $\mathrm{Ext}_{\mathfrak{C}(k)}^1(\mathfrak{m}P, R) \cong \mathrm{Ext}_{\mathfrak{C}(k)}^2(Q, R) = 0$ by (H5) for $\mathfrak{C}(k)$. Since $\mathfrak{m}P$ is a quotient of $P^{\oplus d}$ and $\mathrm{Hom}_{\mathfrak{C}(k)}(P, R) = 0$, we get $\mathrm{Hom}_{\mathfrak{C}(k)}(\mathfrak{m}P, R) = 0$. Lemma 3.16 implies $\mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^1(\mathfrak{m}P, R) = 0$ and the assertion follows from (32). \square

For the rest of the section we assume (H1)–(H5) for $\mathfrak{C}(k)$ and (H0). It follows from the Proposition that (H1)–(H5) also hold for $\mathfrak{C}(\mathcal{O})$. Hence, the results of Section 3.1 apply to \tilde{P} , \tilde{E} and $\tilde{\mathfrak{m}}$.

Remark 3.18. — In the application to $G = \mathrm{GL}_2(\mathbf{Q}_p)$ we will show that $\tilde{\mathbb{P}}$ is in fact \mathcal{O} -torsion free, so (H0) will be satisfied.

Definition 3.19. — Let \mathfrak{A} be the category of finite local (possibly non-commutative) artinian \mathcal{O} -algebras (A, \mathfrak{m}_A) such that the image of \mathcal{O} under the structure morphism $\sigma : \mathcal{O} \rightarrow A$ lies in the centre of A , and σ induces an isomorphism $\mathcal{O}/\varpi \mathcal{O} \cong A/\mathfrak{m}_A$. We denote by \mathfrak{A}^{ab} the full subcategory of \mathfrak{A} consisting of commutative algebras.

Remark 3.20. — The category \mathfrak{A} contains genuinely non-commutative rings: for example, every group algebra of a finite p -group over $\mathcal{O}/(\varpi^n)$ is in \mathfrak{A} .

We refer the reader to [43, §19] for basic facts about non-commutative local rings. Let $\widehat{\mathfrak{A}}$ denote the category of local \mathcal{O} -algebras (R, \mathfrak{m}_R) such that for every $n \geq 1$, R/\mathfrak{m}_R^n is an object of \mathfrak{A} and $R \cong \varprojlim R/\mathfrak{m}_R^n$ and morphisms are given by $\mathrm{Hom}_{\widehat{\mathfrak{A}}}(R, S) = \varprojlim \mathrm{Hom}_{\widehat{\mathfrak{A}}}(R, S/\mathfrak{m}_S^m) = \varprojlim \mathrm{Hom}_{\mathfrak{A}}(R/\mathfrak{m}_R^m, S/\mathfrak{m}_S^m)$, where the limit is taken over all $m \geq 1$.

Definition 3.21. — Let (A, \mathfrak{m}_A) be an object of \mathfrak{A} . A deformation of \mathbf{Q} to A is a pair (M, α) , where M is an object of $\mathfrak{C}(\mathcal{O})$ together with the map of \mathcal{O} -algebras $A \rightarrow \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M)$, which makes M into a flat A -module and $\alpha : k \widehat{\otimes}_A M \cong \mathbf{Q}$ is an isomorphism in $\mathfrak{C}(k)$.

Let $(A, \mathfrak{m}_A) \in \mathfrak{A}$, let n be the largest integer such that $\mathfrak{m}_A^n \neq 0$ and (M, α) a deformation of \mathbf{Q} to A . We note that A is finite (as a set). In particular, every finitely generated A -module N is also finitely presented, and for such N we have

$$(33) \quad N \widehat{\otimes}_A M \cong N \otimes_A M.$$

Lemma 3.22. — For $0 \leq i \leq n$ we have

$$(34) \quad \mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \widehat{\otimes}_A M \cong \mathfrak{m}_A^i M / \mathfrak{m}_A^{i+1} M \cong \mathbf{Q}^{\oplus d_i},$$

where $d_i = \dim \mathfrak{m}_A^i / \mathfrak{m}_A^{i+1}$.

Proof. — We argue by induction on i . The statement is true if $i = 0$. In general, by applying $\widehat{\otimes}_A M$ to $0 \rightarrow \mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \rightarrow A / \mathfrak{m}_A^{i+1} \rightarrow A / \mathfrak{m}_A^i \rightarrow 0$, and using flatness of M and (33) we get an isomorphism $\mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \widehat{\otimes}_A M \cong \mathfrak{m}_A^i M / \mathfrak{m}_A^{i+1} M$. Now, $\mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \cong k^{\oplus d_i}$ as an A -module, since $k \widehat{\otimes}_A M \cong \mathbf{Q}$ we obtain the last assertion. \square

Given an \mathcal{O} -module of finite length, we denote by $\ell_{\mathcal{O}}$ its length.

Lemma 3.23. — We have $\ell_{\mathcal{O}}(\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, M)) = \ell_{\mathcal{O}}(A)$.

Proof. — Since $\tilde{\mathbb{P}}$ is projective, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, *)$ is exact and $\dim \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \mathbf{Q}) = \dim \mathrm{Hom}_{\mathfrak{C}(k)}(\tilde{\mathbb{P}}, \mathbf{Q}) = 1$. Hence, $\ell_{\mathcal{O}}(\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, M)) = \sum_{i=0}^n d_i = \ell_{\mathcal{O}}(A)$. \square

Lemma 3.24. — *The natural map*

$$(35) \quad \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \mathbb{M}) \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} \rightarrow \mathbb{M}$$

is an isomorphism of (left) A -modules.

Proof. — Since $\tilde{\mathbb{P}}$ is projective and $\tilde{\mathbb{E}}$ -flat by Corollary 3.12, the functor $F : \mathfrak{C}(\mathcal{O}) \rightarrow \mathfrak{C}(\mathcal{O})$, $F(N) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, N) \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}$ is exact. Moreover, if N is of finite length in $\mathfrak{C}(\mathcal{O})$ then $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, N)$ is an \mathcal{O} -module of finite length, and so the completed and the usual tensor products coincide. Further, we have $F(Q) \cong k \otimes_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} \cong Q$ and (34) gives $F(\mathfrak{m}^i M / \mathfrak{m}^{i+1} M) \cong \mathfrak{m}^i M / \mathfrak{m}^{i+1} M$, $0 \leq i \leq n$. The exactness of F implies $F(M) \cong M$. Since the map $F(N) \rightarrow N$ is functorial, we obtain that the isomorphism in $\mathfrak{C}(\mathcal{O})$ is also an isomorphism of A -modules. \square

Recall that the map $\tilde{\mathbb{P}} \rightarrow S$ factors through $\theta : \tilde{\mathbb{P}} \rightarrow \tilde{Q}$, which induces an isomorphism $\alpha^{univ} : k \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} \cong \tilde{Q}$, $\lambda \widehat{\otimes} v \mapsto \lambda \theta(v)$. We will think of $(\tilde{\mathbb{P}}, \alpha^{univ})$ as the universal deformation of \tilde{Q} .

Lemma 3.25. — *Let (M, α) be a deformation of \tilde{Q} to A . There exists $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\tilde{\mathbb{E}}, A)$ and $\iota : M \cong A \widehat{\otimes}_{\varphi, \tilde{\mathbb{E}}} \tilde{\mathbb{P}}$ such that $\alpha = \alpha^{univ} \circ (k \widehat{\otimes}_A \iota)$.*

Proof. — Since $\tilde{\mathbb{P}}$ is projective, there exists $\psi : \tilde{\mathbb{P}} \rightarrow M$ making the diagram:

$$\begin{array}{ccc} M & \longrightarrow & k \widehat{\otimes}_A M \\ \psi \uparrow & & \alpha \downarrow \cong \\ \tilde{\mathbb{P}} & \longrightarrow & \tilde{Q} \end{array}$$

commute. We claim that the map $A \rightarrow \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, M)$, $a \mapsto a \circ \psi$ induces an isomorphism of A -modules. Lemma 3.23 says that it is enough to prove that the map is injective. Choose $v \in \tilde{\mathbb{P}}$, such that the image of v in \tilde{Q} is non-zero. Suppose $a \in \mathfrak{m}_A^i$ and $a \notin \mathfrak{m}_A^{i+1}$ then (34) gives an isomorphism:

$$\mathfrak{m}_A^i / \mathfrak{m}_A^{i+1} \widehat{\otimes}_k M / \mathfrak{m}_A M \cong \mathfrak{m}_A^i M / \mathfrak{m}_A^{i+1} M.$$

Since $(a + \mathfrak{m}_A^{i+1}) \widehat{\otimes} (\psi(v) + \mathfrak{m}_A M)$ is non-zero, we also obtain $a(\psi(v))$ is non-zero. Hence $a \circ \psi = 0$ if and only if $a = 0$ and so the map is injective. This means that for every $b \in \tilde{\mathbb{E}}$ there exists a unique $\varphi(b) \in A$ such that $\varphi(b) \circ \psi = \psi \circ b$. Uniqueness implies that φ is a homomorphism of algebras. The assertion follows from Lemma 3.24. \square

Let $\mathrm{Def}_Q : \mathfrak{A} \rightarrow \mathrm{Sets}$ be the functor associating to A the set of isomorphism classes of deformations of \tilde{Q} to A . We denote by Def_Q^{ab} the restriction of Def_Q to \mathfrak{A}^{ab} . Let (A, \mathfrak{m}_A) be in \mathfrak{A} , then to $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(\tilde{\mathbb{E}}, A)$ we may associate an isomorphism class of $(A \widehat{\otimes}_{\tilde{\mathbb{E}}, \varphi} \tilde{\mathbb{P}}, \alpha_\varphi)$, where α_φ is the composition of $A \widehat{\otimes}_{\tilde{\mathbb{E}}, \varphi} \tilde{\mathbb{P}} \rightarrow k \widehat{\otimes}_{\tilde{\mathbb{E}}, \varphi} \tilde{\mathbb{P}}$ with α^{univ} . By Corollary 3.13 this gives us a point in $\mathrm{Def}_Q(A)$.

Theorem 3.26. — *The above map induces a bijection between $\text{Def}_Q(A)$ and A^\times -conjugacy classes of $\text{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, A)$.*

Proof. — Lemma 3.25 says that the map $\text{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, A) \rightarrow \text{Def}_Q(A)$ is surjective. Suppose we have $\varphi_1, \varphi_2 \in \text{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, A)$ and an isomorphism $\beta : A \widehat{\otimes}_{\widetilde{E}, \varphi_1} \widetilde{P} \cong A \widehat{\otimes}_{\widetilde{E}, \varphi_2} \widetilde{P}$ in $\mathfrak{C}(A)$ such that the diagram

$$\begin{array}{ccc} A \widehat{\otimes}_{\widetilde{E}, \varphi_1} \widetilde{P} & \longrightarrow & Q \\ \beta \uparrow \cong & \nearrow & \\ A \widehat{\otimes}_{\widetilde{E}, \varphi_2} \widetilde{P} & & \end{array}$$

commutes. For $i \in \{1, 2\}$ define $\psi_i : \widetilde{P} \rightarrow A \widehat{\otimes}_{\widetilde{E}, \varphi_i} \widetilde{P}$ by $\psi_i(v) := 1 \widehat{\otimes} v$. It follows from the proof of Lemma 3.25 that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, A \widehat{\otimes}_{\widetilde{E}, \varphi_i} \widetilde{P})$ is a free A -module of rank 1, and ψ_i is a generator. Since β is an isomorphism, $\beta_* := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\widetilde{P}, \beta)$ is also an isomorphism. Hence there exists $u \in A^\times$ such that $u\psi_1 = \beta_*(\psi_2)$. Since β_* is A -linear, we obtain

$$(36) \quad \beta(a \widehat{\otimes} v) = \beta(a(1 \widehat{\otimes} v)) = [a\beta_*(\psi_2)](v) = au\psi_1(v) = au \widehat{\otimes} v.$$

So for all $b \in \widetilde{E}$, (36) gives

$$(37) \quad \beta(1 \widehat{\otimes} bv) = \beta(\varphi_2(b) \widehat{\otimes} v) = \varphi_2(b)u \widehat{\otimes} v,$$

$$(38) \quad \beta(1 \widehat{\otimes} bv) = u \widehat{\otimes} bv = u\varphi_1(b) \widehat{\otimes} v.$$

It follows from (37) and (38) that $(u\varphi_1(b) - \varphi_2(b)u)\psi_1 = 0$. Hence, $\varphi_2(b) = u\varphi_1(b)u^{-1}$ for all $b \in \widetilde{E}$.

Conversely, suppose that φ_1 and φ_2 lie in the same A^\times -conjugacy class. Since $\mathcal{O} \rightarrow A/\mathfrak{m}_A$ is surjective and the image of \mathcal{O} in A is contained in the centre, there exists $u \in 1 + \mathfrak{m}_A$ such that $\varphi_2 = u\varphi_1u^{-1}$. An easy check shows that $\beta : A \widehat{\otimes}_{\widetilde{E}, \varphi_2} \widetilde{P} \rightarrow A \widehat{\otimes}_{\widetilde{E}, \varphi_1} \widetilde{P}$, $a \widehat{\otimes} v \mapsto au \widehat{\otimes} v$ is the required isomorphism of deformations. \square

Corollary 3.27. — $\text{Def}_Q^{ab}(A) = \text{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}^{ab}, A)$, where \widetilde{E}^{ab} is the maximal commutative quotient of \widetilde{E} .

Proof. — Since A is commutative, every A^\times -conjugacy class consists of one element. Thus $\text{Def}_Q^{ab}(A) = \text{Def}_Q(A) = \text{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}, A) = \text{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{E}^{ab}, A)$. The last equality follows from the universal property of \widetilde{E}^{ab} . \square

Remark 3.28. — If R is an arbitrary non-commutative topological ring then R^{ab} might be the zero ring. This is not the case here, since $\widetilde{E}/\widetilde{\mathfrak{m}} \cong k$ is commutative.

Lemma 3.29. — *Let $k[\varepsilon]$ be the ring of dual numbers so that $\varepsilon^2 = 0$. Then we have natural isomorphisms*

$$(39) \quad \mathrm{Ext}_{\mathfrak{C}(k)}^1(\mathbf{Q}, \mathbf{Q}) \cong \mathrm{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{\mathbf{E}}^{ab}, k[\varepsilon]) \cong \mathrm{Hom}_{\widehat{\mathfrak{A}}}(\widetilde{\mathbf{E}}, k[\varepsilon]) \cong (\mathfrak{m}/\mathfrak{m}^2)^*,$$

where $*$ denotes k -linear dual.

Proof. — The first isomorphism is classical. The second follows from the fact that $k[\varepsilon]$ is commutative. The third is again classical. \square

Let (A, \mathfrak{m}_A) be in \mathfrak{A} and let $F : \mathfrak{A} \rightarrow \mathrm{Sets}$ be a covariant functor. For each $u \in A^\times$, $\mathrm{ad}(u) : A \rightarrow A$, $a \mapsto uau^{-1}$ is a morphism in \mathfrak{A} , and hence induces a morphism of sets $F(\mathrm{ad}(u)) : F(A) \rightarrow F(A)$. We say that the functor F is *stable under conjugation* if $F(\mathrm{ad}(u)) = \mathrm{id}_{F(A)}$ for all objects A of \mathfrak{A} and all $u \in A^\times$. For \mathbf{R} in $\widehat{\mathfrak{A}}$ we denote $h_{\mathbf{R}} : \mathfrak{A} \rightarrow \mathrm{Sets}$ and $F_{\mathbf{R}} : \mathfrak{A} \rightarrow \mathrm{Sets}$ the functors $h_{\mathbf{R}}(A) := \mathrm{Hom}_{\widehat{\mathfrak{A}}}(\mathbf{R}, A)$ and $F_{\mathbf{R}}(A)$ the set of A^\times -conjugacy classes in $h_{\mathbf{R}}(A)$. We have a variant of Yoneda's lemma.

Lemma 3.30. — *Let $F : \mathfrak{A} \rightarrow \mathrm{Sets}$ be a covariant functor stable under conjugation then the map $\eta \mapsto \eta_{\mathbf{R}}(\{\mathrm{id}_{\mathbf{R}}\})$ induces a bijection between the set of natural transformations $\mathrm{Mor}(F_{\mathbf{R}}, F)$ and $F(\mathbf{R}) := \varprojlim F(\mathbf{R}/\mathfrak{m}_{\mathbf{R}}^n)$.*

Proof. — Mapping a homomorphism to its conjugacy class gives rise to a natural transformation of functors $\kappa : h_{\mathbf{R}} \rightarrow F_{\mathbf{R}}$ and hence a map $\mathrm{Mor}(F_{\mathbf{R}}, F) \rightarrow \mathrm{Mor}(h_{\mathbf{R}}, F)$, $\eta \mapsto \eta \circ \kappa$, which is clearly injective. We claim that it is also surjective. Let $\xi : h_{\mathbf{R}} \rightarrow F$ be a natural transformation, A an object of \mathfrak{A} and $u \in A^\times$. Then we have

$$\xi_A \circ h_{\mathbf{R}}(\mathrm{ad}(u)) = F(\mathrm{ad}(u)) \circ \xi_A = \mathrm{id}_{F(A)} \circ \xi_A = \xi_A.$$

Thus ξ factors through κ and hence the map is surjective. The assertion follows from the usual Yoneda's lemma. \square

Lemma 3.31. — *Let \mathbf{R} and \mathbf{S} be in $\widehat{\mathfrak{A}}$ and suppose that $\eta : F_{\mathbf{R}} \rightarrow F_{\mathbf{S}}$ is an isomorphism of functors then the rings \mathbf{R} and \mathbf{S} are isomorphic. Moreover, η determines the isomorphism up to conjugation.*

Proof. — It follows from Lemma 3.30 that $\mathrm{Mor}(F_{\mathbf{R}}, F_{\mathbf{S}}) \cong \mathrm{Hom}_{\widehat{\mathfrak{A}}}(\mathbf{S}, \mathbf{R})/\mathbf{R}^\times$. Thus we may find $\varphi : \mathbf{S} \rightarrow \mathbf{R}$ such that for each A in \mathfrak{A} , $\eta_A : F_{\mathbf{R}}(A) \rightarrow F_{\mathbf{S}}(A)$ sends the conjugacy class of ψ to the conjugacy class of $\psi \circ \varphi$. Since, η is a bijection for all A , we may find $c \in F_{\mathbf{R}}(\mathbf{S})$ such that $\eta_{\mathbf{R}}(c) = \{\mathrm{id}_{\mathbf{S}}\}$. Choose any $\psi \in c$ then the last equality reads $\psi \circ \varphi = \mathrm{id}_{\mathbf{S}}$, which implies that φ is an isomorphism. The last assertion follows from Lemma 3.30. \square

3.2. Examples. — We give some examples of deformations with possibly non-commutative coefficients. Our coefficients are objects of the category \mathfrak{A} defined in 3.19.

Lemma 3.32. — *Let \mathcal{G} be a finitely generated pro-finite group and $\mathbf{Q} = \mathbf{1}$ the trivial representation. Then $\text{Def}_{\mathbf{1}}(\mathbf{A}) = \text{Hom}_{\widehat{\mathfrak{A}}}(\mathcal{O}[[\mathcal{G}(p)]]^{\text{op}}, \mathbf{A}) / \sim$, where $\mathcal{G}(p)$ is the maximal pro- p quotient of \mathcal{G} , and \sim denotes the equivalence under conjugation by \mathbf{A}^{\times} . Moreover, $\text{Def}_{\mathbf{1}}^{\text{ab}}(\mathbf{A}) = \text{Hom}_{\widehat{\mathfrak{A}}}(\mathcal{O}[[\mathcal{G}(p)^{\text{ab}}]], \mathbf{A})$.*

Proof. — Let (\mathbf{M}, α) be an \mathbf{A} -deformation. Since \mathbf{M} is \mathbf{A} -flat and $k \widehat{\otimes}_{\mathbf{A}} \mathbf{M} \cong k$ we get that \mathbf{M} is a free \mathbf{A} -module of rank 1. Choose $v \in \mathbf{M}$, such that $\alpha(1 \otimes v) = 1$. Then v is a basis vector of \mathbf{M} and for every $g \in \mathcal{G}$ we obtain a unique $a_g \in \mathbf{A}$ such that $gv = a_g v$. Now

$$a_{gh}v = (gh)v = g(hv) = ga_hv = a_hgv = a_h a_g v.$$

Hence, we get a group homomorphism $\mathcal{G}^{\text{op}} \rightarrow 1 + \mathfrak{m}_{\mathbf{A}}$, $g \mapsto a_g$. Since $1 + \mathfrak{m}_{\mathbf{A}}$ is a finite group of p -power order, the map factors through $\mathcal{G}(p)^{\text{op}}$. By extending \mathcal{O} -linearly we obtain a homomorphism $\mathcal{O}[[\mathcal{G}(p)]]^{\text{op}} \rightarrow \mathbf{A}$. A different choice of v would conjugate the homomorphism by $u \in 1 + \mathfrak{m}_{\mathbf{A}}$.

Conversely, $\mathcal{O}[[\mathcal{G}(p)]]$ is a free right $\mathcal{O}[[\mathcal{G}(p)]]^{\text{op}} = \text{End}_{\mathcal{O}(\mathcal{O})}(\mathcal{O}[[\mathcal{G}(p)]])$ module, with the action $b \cdot a := ab$. Thus every $\varphi \in \text{Hom}_{\widehat{\mathfrak{A}}}(\mathcal{O}[[\mathcal{G}(p)]]^{\text{op}}, \mathbf{A})$ defines a deformation $\mathbf{A} \widehat{\otimes}_{\mathcal{O}[[\mathcal{G}(p)]]^{\text{op}}, \varphi} \mathcal{O}[[\mathcal{G}(p)]]$.

If \mathbf{A} is commutative then the map $\mathcal{G} \rightarrow \mathcal{G}(p) \rightarrow 1 + \mathfrak{m}_{\mathbf{A}}$ must further factor through $\mathcal{G}(p)^{\text{ab}}$, and the same argument gives the claim. \square

Lemma 3.33. — *Let $G = \mathbf{Q}_p^{\times}$ and $\chi : \mathbf{Q}_p^{\times} \rightarrow k^{\times}$ a continuous character. If $p \neq 2$ then $\text{Def}_{\chi}(\mathbf{A}) = \text{Hom}_{\widehat{\mathfrak{A}}}(\mathcal{O}[[x, y]], \mathbf{A}) / \mathbf{A}^{\times}$, where $\mathcal{O}[[x, y]]$ denotes the ring of formal (commutative) power series.*

Proof. — We may choose a character $\tilde{\chi} : \mathbf{Q}_p^{\times} \rightarrow \mathcal{O}^{\times}$ lifting χ . After twisting with $\tilde{\chi}$ we may assume that χ is the trivial character. It follows from the proof of Lemma 3.32 that the deformation problem does not change if we replace G with its pro- p completion \widehat{G} . Since $p \neq 2$ we have $G \cong \mathbf{Z} \oplus \mathbf{Z}/(p-1) \oplus \mathbf{Z}_p$ and hence $\widehat{G} \cong \mathbf{Z}_p^2$. Thus $\mathcal{O}[[\widehat{G}]] \cong \mathcal{O}[[x, y]]$ and the assertion follows from the Lemma 3.32. \square

Proposition 3.34. — *Let $G = \mathbf{Q}_p^{\times}$ and $\chi : \mathbf{Q}_p^{\times} \rightarrow k^{\times}$ a continuous character and let $\mathbf{S} := \chi^{\vee}$, then if $p \neq 2$ then $\widetilde{\mathbf{E}} \cong \mathcal{O}[[x, y]]$ and $\mathbf{E} \cong k[[x, y]]$. Moreover, $\widetilde{\mathbf{P}}$ is a free $\widetilde{\mathbf{E}}$ -module of rank 1 and in particular it is \mathcal{O} -torsion free.*

Proof. — We claim that the hypotheses (H1)–(H5) are satisfied for $\mathbf{Q} = \mathbf{S} = \chi^{\vee}$ and note that since in this case $\mathbf{R} = 0$ the hypothesis (H0) is satisfied. Since $\text{Ext}_{\mathbf{G}}^1(\chi, \chi) \cong \text{Hom}^{\text{cont}}(\mathbf{G}, k)$ is 2-dimensional, (H4) holds. Consider a non-split extension $0 \rightarrow \chi \rightarrow$

$\epsilon \rightarrow \tau \rightarrow 0$ in $\text{Mod}_G^{\text{sm}}(k)$ with τ irreducible. Since G is commutative for each $g \in G$ the map $\phi_g : \epsilon \rightarrow \epsilon$, $v \mapsto gv - \chi(g)v$ is G -equivariant. If ϕ_g is non-zero for some g then it induces an isomorphism between τ and χ , if ϕ_g is zero for all g then any k -vector space splitting of the sequence is G -equivariant. Hence, (H3) is satisfied and all the other hypotheses hold trivially, since $\mathbf{R} = 0$. It follows from Lemma 3.33 and Lemma 3.31 that $\tilde{E} \cong \mathcal{O}[[x, y]]$ and hence $E \cong \tilde{E} \otimes_{\mathcal{O}} k \cong k[[x, y]]$. Since \tilde{P} is flat over $\tilde{E} \cong \mathcal{O}[[x, y]]$ by Corollary 3.12 and $k \widehat{\otimes}_{\tilde{E}} \tilde{P} \cong \chi^\vee$ is one dimensional, \tilde{P} is a free \tilde{E} -module of rank 1 and in particular it is also \mathcal{O} -torsion free. \square

Corollary 3.35. — $\dim \text{Ext}_{\mathfrak{C}(k)}^1(\chi^\vee, \chi^\vee) = 2$, $\dim \text{Ext}_{\mathfrak{C}(k)}^2(\chi^\vee, \chi^\vee) = 1$. Moreover, $\text{Ext}_{\mathfrak{C}(k)}^i(\chi^\vee, \chi^\vee) = 0$ for $i \geq 3$ and $\text{Ext}_{\mathfrak{C}(k)}^i(\chi^\vee, S') = 0$ for all $i \geq 0$ and all irreducible $S' \in \mathfrak{C}(k)$ not isomorphic to χ^\vee .

Proof. — Since $E \cong k[[x, y]]$ we apply $\widehat{\otimes}_E P$ to the exact sequence

$$0 \rightarrow k[[x, y]] \rightarrow k[[x, y]] \oplus k[[x, y]] \rightarrow k[[x, y]] \rightarrow k \rightarrow 0$$

where the first arrow is $f \mapsto (xf, yf)$, the second is $(f, g) \mapsto yf - xg$ to get a projective resolution of $\chi^\vee \cong P/\mathfrak{m}P$:

$$0 \rightarrow P \rightarrow P^{\oplus 2} \rightarrow P \rightarrow \chi^\vee \rightarrow 0.$$

The assertions follow from a calculation with this projective resolution. \square

3.3. Criterion for commutativity. — In this section we devise a criterion, see Theorem 3.39, for the ring \tilde{E} to be commutative. When $G = \text{GL}_2(\mathbf{Q}_p)$ we will show that this criterion is satisfied in the generic cases, see Section 5.8, and it will enable us to apply Corollary 4.44. We use the notation of Section 3.1, we assume the hypotheses (H1)–(H5) for $\mathfrak{C}(\mathcal{O})$ or equivalently (H0) and (H1)–(H5) for $\mathfrak{C}(k)$.

Lemma 3.36. — *If there exists a surjection $\tilde{E} \twoheadrightarrow \mathcal{O}[[x_1, \dots, x_d]]$, with $d = \dim \mathfrak{m}/\mathfrak{m}^2$, and the graded ring $\text{gr}_{\mathfrak{m}}^\bullet(E)$ is commutative then $\tilde{E} \cong \mathcal{O}[[x_1, \dots, x_d]]$.*

Proof. — Let $R := k[[x_1, \dots, x_d]]$ and $\mathfrak{m}_1 = (x_1, \dots, x_d)$ be the maximal ideal of R . Applying $\otimes_{\mathcal{O}} k$ we obtain a surjection $E \twoheadrightarrow R$, thus a surjection of graded rings

$$(40) \quad \text{gr}_{\mathfrak{m}}^\bullet(E) \twoheadrightarrow \text{gr}_{\mathfrak{m}_1}^\bullet(R) \cong k[x_1, \dots, x_d].$$

Since $\text{gr}_{\mathfrak{m}}^\bullet(E)$ is commutative and $\dim \mathfrak{m}/\mathfrak{m}^2 = d$, there exists a surjection

$$(41) \quad k[x_1, \dots, x_d] \twoheadrightarrow \text{gr}_{\mathfrak{m}}^\bullet(E).$$

It follows from (40) and (41) that $\text{gr}_{\mathfrak{m}}^\bullet(E) \cong \text{gr}_{\mathfrak{m}_1}^\bullet(R)$. Hence $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}_1^n/\mathfrak{m}_1^{n+1}$ for all $n \geq 1$. By induction we get that $E/\mathfrak{m}^n \cong R/\mathfrak{m}_1^n$ for all $n \geq 1$. Since both rings are complete

we get $E \cong R$. Let K be the kernel of $\tilde{E} \rightarrow \mathcal{O}[[x_1, \dots, x_d]]$. Since $\mathcal{O}[[x_1, \dots, x_d]]$ is \mathcal{O} -flat, we have $K \otimes_{\mathcal{O}} k = 0$ and hence $K = 0$, by Nakayama's lemma for compact \mathcal{O} -modules, [26] Exp. VII_B (0.3.3). \square

Let d be the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a k -vector space and let W be a $(d-r)$ -dimensional k -subspace of $\mathfrak{m}/\mathfrak{m}^2$ then $W + \mathfrak{m}^2$ is a 2-sided ideal of E and the exact sequence of E -modules $0 \rightarrow \mathfrak{m}/(W + \mathfrak{m}^2) \rightarrow E/(W + \mathfrak{m}^2) \rightarrow k \rightarrow 0$ leads by tensoring with P to an exact sequence of G -representations

$$(42) \quad 0 \rightarrow Q^{\oplus r} \rightarrow T \rightarrow Q \rightarrow 0$$

with $T \cong P/(W + \mathfrak{m}^2)P$. Conversely, any T in (42), such that $\text{Hom}_{\mathcal{E}(k)}(T, S)$ is one dimensional, is a quotient $\psi : P \twoheadrightarrow T$, as the cosocle of T is isomorphic to S , and defines a $(d-r)$ -dimensional subspace

$$(43) \quad W := \{a \in \mathfrak{m} : \psi \circ a = 0\}/\mathfrak{m}^2 \subseteq \mathfrak{m}/\mathfrak{m}^2.$$

Lemma 3.37. — *Let T and W be as above then*

$$(44) \quad \dim \text{Ext}_{\mathcal{E}(k)}^1(T, S) = \dim \frac{W + \mathfrak{m}^2}{W\mathfrak{m} + \mathfrak{m}^3} = \dim W + \dim \frac{\mathfrak{m}^2}{W\mathfrak{m} + \mathfrak{m}^3}.$$

Proof. — We have an exact sequence:

$$(45) \quad 0 \rightarrow (W + \mathfrak{m}^2) \widehat{\otimes}_E P \rightarrow P \rightarrow T \rightarrow 0.$$

Since $\dim \text{Hom}_{\mathcal{E}(k)}(T, S) = \dim \text{Hom}_{\mathcal{E}(k)}(P, S) = 1$ and P is projective, by applying $\text{Hom}_{\mathcal{E}(k)}(*, S)$ to (45) we obtain an isomorphism

$$(46) \quad \text{Hom}_{\mathcal{E}(k)}((W + \mathfrak{m}^2) \widehat{\otimes}_E P, S) \cong \text{Ext}_{\mathcal{E}(k)}^1(T, S).$$

Let n be the dimension of $(W + \mathfrak{m}^2)/(W\mathfrak{m} + \mathfrak{m}^3)$ then the exact sequence of right E -modules $0 \rightarrow W\mathfrak{m} + \mathfrak{m}^3 \rightarrow W + \mathfrak{m}^2 \rightarrow k^{\oplus n} \rightarrow 0$ leads to an exact sequence of G -representations:

$$(47) \quad 0 \rightarrow (W\mathfrak{m} + \mathfrak{m}^3) \widehat{\otimes}_E P \rightarrow (W + \mathfrak{m}^2) \widehat{\otimes}_E P \rightarrow Q^{\oplus n} \rightarrow 0.$$

So for the first equality it is enough to show that any $\psi : (W + \mathfrak{m}^2) \widehat{\otimes}_E P \rightarrow Q$ is zero on $(W\mathfrak{m} + \mathfrak{m}^3) \widehat{\otimes}_E P$. Suppose that $\psi(a \widehat{\otimes} v) \neq 0$ for some $a \in W + \mathfrak{m}^2$ and $v \in P$ then the composition $\varphi : P \rightarrow (W + \mathfrak{m}^2) \widehat{\otimes}_E P \rightarrow Q$, $v \mapsto \psi(a \widehat{\otimes} v)$ is non-zero. Since $\text{Hom}_{\mathcal{E}(k)}(P, Q)$ is one dimensional, φ is trivial on $\mathfrak{m}P$ and so for all $b \in \mathfrak{m}$ we have

$$0 = \psi(a \widehat{\otimes} bv) = \psi(ab \widehat{\otimes} v).$$

Hence, ψ is trivial on $(W + \mathfrak{m}^2)\mathfrak{m} \widehat{\otimes}_E P = (W\mathfrak{m} + \mathfrak{m}^3) \widehat{\otimes}_E P$. The last equality follows from the exact sequence $0 \rightarrow \frac{\mathfrak{m}^2}{W\mathfrak{m} + \mathfrak{m}^3} \rightarrow \frac{W + \mathfrak{m}^2}{W\mathfrak{m} + \mathfrak{m}^3} \rightarrow W \rightarrow 0$. \square

Lemma 3.38. — *Let $(\mathbf{R}, \mathfrak{m})$ be a local k -algebra with $\mathbf{R}/\mathfrak{m} \cong k$ and $\mathfrak{m}^3 = 0$. Suppose there exists a surjection*

$$(48) \quad \varphi : \mathbf{R} \twoheadrightarrow k[[x_1, \dots, x_d]]/(x_1, \dots, x_d)^3,$$

where $d = \dim \mathfrak{m}/\mathfrak{m}^2$. Let $r = \lfloor \frac{d}{2} \rfloor$ and further suppose that for every $d - r$ dimensional k -subspace W of $\mathfrak{m}/\mathfrak{m}^2$ we have

$$(49) \quad \dim \frac{\mathfrak{m}^2}{W\mathfrak{m}} \leq \frac{r(r+1)}{2}$$

then (48) is an isomorphism. In particular, \mathbf{R} is commutative.

Proof. — Any commutative local k -algebra (A, \mathfrak{m}_A) with $A/\mathfrak{m}_A = k$, $\mathfrak{m}_A^3 = 0$ and $\dim \mathfrak{m}_A/\mathfrak{m}_A^2 \leq d$ is a quotient of $k[[x_1, \dots, x_d]]/(x_1, \dots, x_d)^3$. Hence,

$$\mathbf{R}^{ab} \cong k[[x_1, \dots, x_d]]/(x_1, \dots, x_d)^3,$$

where \mathbf{R}^{ab} is the maximal commutative quotient of \mathbf{R} . Let \mathfrak{a} be the kernel of φ . Since $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \varphi(\mathfrak{m})/\varphi(\mathfrak{m})^2 = d$, we get that \mathfrak{a} is contained in \mathfrak{m}^2 . Since $\mathfrak{m}^3 = 0$, any k -subspace V of \mathfrak{a} is also a two-sided ideal of \mathbf{R} . Suppose that $\mathfrak{a} \neq 0$ and let $V \subset \mathfrak{a}$ be any k -subspace such that the quotient \mathfrak{a}/V is one dimensional. The surjection $\mathfrak{m}_R^2 \twoheadrightarrow \mathfrak{m}_{R/V}^2$ induces a surjection $\mathfrak{m}_R^2/W\mathfrak{m}_R \twoheadrightarrow \mathfrak{m}_{R/V}^2/W\mathfrak{m}_{R/V}$. Hence, by replacing \mathbf{R} with \mathbf{R}/V we may assume that \mathfrak{a} is a one dimensional k -vector space. We let t be a basis vector of \mathfrak{a} .

If $a, b \in \mathfrak{m}$ then the image of $ab - ba$ in \mathbf{R}^{ab} is zero. Thus there exists $\kappa(a, b) \in k$ such that $ab - ba = \kappa(a, b)t$. If $a \in \mathfrak{m}^2$ or $b \in \mathfrak{m}^2$ then $\kappa(a, b) = 0$, as $\mathfrak{m}^3 = 0$. Hence, κ defines an alternating bilinear form on $\mathfrak{m}/\mathfrak{m}^2$.

We may choose a basis $\mathcal{B} = \{x_1, \dots, x_d\}$ of $\mathfrak{m}/\mathfrak{m}^2$ such that for any two $a, b \in \mathcal{B}$ we have $\kappa(a, b) = 0$, except $\kappa(x_i, x_{d-i+1}) = -\kappa(x_{d-i+1}, x_i) = 1$, $1 \leq i \leq s$, where $d - 2s$ is the dimension of $\{a \in \mathfrak{m}/\mathfrak{m}^2 : \kappa(a, b) = 0, \forall b \in \mathfrak{m}/\mathfrak{m}^2\}$. Let W be the linear span of $\mathcal{S} = \{x_1, \dots, x_{d-r}\}$. The k -subspace of \mathfrak{m}^2 spanned by the set $\mathcal{S} \cdot \mathcal{B} := \{ab : a \in \mathcal{S}, b \in \mathcal{B}\}$ is equal to $W\mathfrak{m}$. The set $\mathcal{S} \cdot \mathcal{B}$ consists of monomials x_i^2 , $1 \leq i \leq d - r$ and $x_i x_j$ with $1 \leq i \leq d - r$, $1 \leq j \leq d$ and $i < j$, since by construction $ab = ba$ for all $a, b \in \mathcal{S}$. We note that $d - r \leq s$, as $d - 2s \geq 0$. It follows from (48) that φ induces a bijection between the sets $\mathcal{S} \cdot \mathcal{B}$ and $\varphi(\mathcal{S}) \cdot \varphi(\mathcal{B})$. Since distinct monomials are linearly independent in \mathbf{R}^{ab} the set $\varphi(\mathcal{S}) \cdot \varphi(\mathcal{B})$ is a basis of $\varphi(W\mathfrak{m}) = \varphi(W)\varphi(\mathfrak{m})$. Hence, the dimension of $W\mathfrak{m}$ is equal to the dimension of $\varphi(W\mathfrak{m})$, which is equal to the cardinality of the set $\varphi(\mathcal{S}) \cdot \varphi(\mathcal{B})$. The latter can be calculated as $|\mathcal{B}| + (|\mathcal{B}| - 1) + \dots + (|\mathcal{B}| - |\mathcal{S}| + 1)$. Since the dimension of $\varphi(\mathfrak{m})^2$ is equal to $|\mathcal{B}| + (|\mathcal{B}| - 1) + \dots + 1$, we deduce that the dimension of $\varphi(\mathfrak{m})^2/\varphi(W\mathfrak{m})$ is equal to $1 + 2 + \dots + (|\mathcal{B}| - |\mathcal{S}|) = \frac{r(r+1)}{2}$. Since we have assumed $\mathfrak{a} \neq 0$ we have $\dim \mathfrak{m}^2 > \dim \varphi(\mathfrak{m})^2$ and hence $\dim \mathfrak{m}^2/W\mathfrak{m} > \dim \varphi(\mathfrak{m})^2/\varphi(W\mathfrak{m})$. This contradicts (49). \square

Theorem 3.39. — *Let $d := \dim \mathfrak{m}/\mathfrak{m}^2$ and $r = \lfloor \frac{d}{2} \rfloor$ and suppose that there exists a surjection $\tilde{E} \twoheadrightarrow \mathcal{O}[[x_1, \dots, x_d]]$. Further, suppose that for every exact sequence*

$$(50) \quad 0 \rightarrow \mathbf{Q}^{\oplus r} \rightarrow \mathbf{T} \rightarrow \mathbf{Q} \rightarrow 0$$

with $\dim \mathrm{Hom}_{\mathfrak{e}(k)}(\mathbf{T}, \mathbf{S}) = 1$ we have $\dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{T}, \mathbf{S}) \leq \frac{r(r-1)}{2} + d$ then $\tilde{E} \cong \mathcal{O}[[x_1, \dots, x_d]]$.

Proof. — The bound on $\dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{T}, \mathbf{S})$ and Lemmas 3.37, 3.38 imply that E/\mathfrak{m}^3 is commutative. Hence, the commutator of any two elements in $\mathrm{gr}_{\mathfrak{m}}^1 E$ is zero in $\mathrm{gr}_{\mathfrak{m}}^{\bullet} E$. Thus the graded ring $\mathrm{gr}_{\mathfrak{m}}^{\bullet}(E)$ is commutative, as it is generated as a ring by $\mathrm{gr}_{\mathfrak{m}}^1 E$ over $\mathrm{gr}_{\mathfrak{m}}^0 E \cong k$, and the result follows from Lemma 3.36. \square

In the applications to $G = \mathrm{GL}_2(\mathbf{Q}_p)$, r will turn out to be equal to 1. We finish the section with lemmas of technical nature tailored for this situation.

Let $a, b \in \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{Q}, \mathbf{Q})$ be equivalence classes of extensions of $0 \rightarrow \mathbf{Q} \rightarrow \mathbf{A} \xrightarrow{\alpha} \mathbf{Q} \rightarrow 0$ and $0 \rightarrow \mathbf{Q} \xrightarrow{\beta} \mathbf{B} \rightarrow \mathbf{Q} \rightarrow 0$, respectively. We denote by $a \circ b \in \mathrm{Ext}_{\mathfrak{e}(k)}^2(\mathbf{Q}, \mathbf{Q})$ the equivalence class of $0 \rightarrow \mathbf{Q} \rightarrow \mathbf{A} \xrightarrow{\beta \circ \alpha} \mathbf{B} \rightarrow \mathbf{Q} \rightarrow 0$. Applying $\mathrm{Hom}_{\mathfrak{e}(k)}(\mathbf{Q}, *)$ we get an exact sequence:

$$(51) \quad \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{Q}, \mathbf{A}) \rightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{Q}, \mathbf{Q}) \xrightarrow{\partial_1} \mathrm{Ext}_{\mathfrak{e}(k)}^2(\mathbf{Q}, \mathbf{Q}).$$

Applying $\mathrm{Hom}_{\mathfrak{e}(k)}(*, \mathbf{Q})$ we get an exact sequence

$$(52) \quad \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{A}, \mathbf{Q}) \rightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{Q}, \mathbf{Q}) \xrightarrow{\partial_2} \mathrm{Ext}_{\mathfrak{e}(k)}^2(\mathbf{Q}, \mathbf{Q}).$$

Then $\partial_1(b) = a \circ b$ and $\partial_2(b) = b \circ a$, [13, §7.6 Prop. 5].

Lemma 3.40. — *The following are equivalent:*

- (i) $\mathrm{Hom}(E, k[x]/(x^3)) \rightarrow \mathrm{Hom}(E, k[x]/(x^2))$ is surjective;
- (ii) $a \circ a = 0$ for all $a \in \mathrm{Ext}_{\mathfrak{e}(k)}^1(\mathbf{Q}, \mathbf{Q})$.

Proof. — By Hom in (i) we mean homomorphisms of local k -algebras. We will show that (i) implies (ii). An extension a may be considered as a deformation of \mathbf{Q} to $k[x]/(x^2)$ and hence as $\varphi \in \mathrm{Hom}(E, k[x]/(x^2))$ by Theorem 3.26. More precisely, a is the equivalence class of

$$(53) \quad 0 \rightarrow k \widehat{\otimes}_E \mathbf{P} \rightarrow k[x]/(x^2) \widehat{\otimes}_{E, \varphi} \mathbf{P} \rightarrow k \widehat{\otimes}_E \mathbf{P} \rightarrow 0.$$

By assumption there exists $\psi \in \mathrm{Hom}(E, k[x]/(x^3))$ lifting φ . This gives an extension

$$(54) \quad 0 \rightarrow k \widehat{\otimes}_E \mathbf{P} \rightarrow k[x]/(x^3) \widehat{\otimes}_{E, \psi} \mathbf{P} \rightarrow k[x]/(x^2) \widehat{\otimes}_{E, \varphi} \mathbf{P} \rightarrow 0.$$

The image of (54) in $\text{Ext}_{\mathcal{E}(k)}^1(\mathbf{Q}, \mathbf{Q})$ via (52) is the extension class of

$$(55) \quad 0 \rightarrow k \widehat{\otimes}_E \mathbf{P} \rightarrow (x)/(x^3) \widehat{\otimes}_{E, \psi} \mathbf{P} \rightarrow k \widehat{\otimes}_E \mathbf{P} \rightarrow 0$$

and is equal to a . Hence, a lies in the kernel of ∂_2 and so $a \circ a = 0$.

Conversely suppose that $a \circ a = 0$ then since $a \circ a = \partial_2(a)$ there exists a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Q} & \longrightarrow & \mathbf{A} & \longrightarrow & \mathbf{Q} & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Q} & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{A} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbf{Q} & \xrightarrow{=} & \mathbf{Q} & & \end{array}$$

Since \mathbf{P} is projective and a is non-split there exists a surjection $\psi : \mathbf{P} \twoheadrightarrow \mathbf{B}$ lifting $\varphi : \mathbf{P} \twoheadrightarrow \mathbf{A}$. It is enough to show that $\mathfrak{a} := \{b \in \mathfrak{m} : \psi \circ b = 0\}$ is a two-sided ideal of E . Since the composition $\mathbf{P} \xrightarrow{\psi} \mathbf{B} \rightarrow \mathbf{Q}$ is trivial on $\mathfrak{m}\mathbf{P}$, the image of $\psi \circ b : \mathbf{P} \rightarrow \mathbf{B}$ is contained in $\mathbf{A} \cong \text{Ker}(\mathbf{B} \twoheadrightarrow \mathbf{Q})$ for all $b \in \mathfrak{m}$. Now $\text{Hom}_{\mathcal{E}(k)}(\mathbf{P}, \mathbf{A})$ is 2-dimensional with basis $\varphi, \varphi_1 : \mathbf{P} \rightarrow \mathbf{Q} \hookrightarrow \mathbf{A}$. For a fixed $b \in \mathfrak{m}$ we may write $\psi \circ b = \lambda\varphi + \mu\varphi_1$. For all $c \in E$ we have $\varphi \circ c \equiv \psi \circ c \pmod{\mathbf{Q}}$ and hence $\varphi \circ c = 0$ if $c \in \mathfrak{a}$. Thus we obtain $\psi \circ b \circ c = \lambda\varphi \circ c + \mu\varphi_1 \circ c = 0$ for all $c \in \mathfrak{a}$. Hence, \mathfrak{a} is a two sided ideal. \square

Lemma 3.41. — *Let $(\mathbf{R}, \mathfrak{m})$ be a commutative local artinian k -algebra with $\mathfrak{m}^3 = 0$ and $\mathbf{R}/\mathfrak{m} = k$. Let d be the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a k -vector space. Then the following are equivalent:*

- (i) $\text{Hom}(\mathbf{R}, k[x]/(x^3)) \rightarrow \text{Hom}(\mathbf{R}, k[x]/(x^2))$ is surjective;
- (ii) $\mathbf{R} \cong k[[x_1, \dots, x_d]]/(x_1, \dots, x_d)^3$.

Proof. — Let $\mathbf{S} := k[[x_1, \dots, x_d]]/(x_1, \dots, x_d)^3$ and $\mathfrak{m}_{\mathbf{S}}$ be the maximal ideal of \mathbf{S} . Since \mathbf{R} is commutative and $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim \mathfrak{m}_{\mathbf{S}}/\mathfrak{m}_{\mathbf{S}}^2$ there exists a surjection $\varphi : \mathbf{S} \twoheadrightarrow \mathbf{R}$, inducing an isomorphism $\mathbf{S}/\mathfrak{m}_{\mathbf{S}}^2 \cong \mathbf{R}/\mathfrak{m}^2$. For $1 \leq i \leq j \leq d$ define $\varphi_{ij} : \mathbf{R} \rightarrow \mathbf{R}/\mathfrak{m}^2 \cong \mathbf{S}/\mathfrak{m}_{\mathbf{S}}^2 \rightarrow k[x]/(x^2)$, where the last arrow is given by sending $x_i \mapsto x, x_j \mapsto x$ and $x_k \mapsto 0$, if $k \neq i$ and $k \neq j$. By assumption there exists $\psi_{ij} : \mathbf{R} \rightarrow k[x]/(x^3)$ lifting φ_{ij} . Let κ be the composition

$$\mathbf{S} \xrightarrow{\varphi} \mathbf{R} \xrightarrow{\prod \psi_{ij}} \prod_{1 \leq i \leq j \leq d} k[x]/(x^3).$$

The kernel of κ is contained in $\mathfrak{m}_{\mathbf{S}}^2$. Any element $y \in \mathfrak{m}_{\mathbf{S}}^2$ maybe written as $y = \sum_{1 \leq i \leq j \leq d} a_{ij} x_i x_j$, and $\psi_{ii}(\varphi(y)) = a_{ii} x^2$ and $\psi_{ij}(\varphi(y)) = (a_{ii} + a_{ij} + a_{jj}) x^2$, if $i < j$. Hence, κ is injective and so φ is injective. The other implication is trivial. \square

Lemma 3.42. — *Assume that $\mathrm{Hom}(E, k[x]/(x^3)) \rightarrow \mathrm{Hom}(E, k[x]/(x^2))$ is surjective and let a be a non-zero extension class of $0 \rightarrow Q \rightarrow T \xrightarrow{\alpha} Q \rightarrow 0$. Then the following are equivalent:*

- (i) *the kernel of $\mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q) \rightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^2(Q, Q)$, $b \mapsto b \circ a$ is at most 1-dimensional;*
- (ii) $\dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(T, Q) = \dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(T, S) \leq \dim \mathfrak{m}/\mathfrak{m}^2$;
- (iii) *the kernel of $\mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q) \rightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^2(Q, Q)$, $b \mapsto a \circ b$ is at most 1-dimensional;*
- (iv) $\dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, T) \leq \dim \mathfrak{m}/\mathfrak{m}^2$.

If the conditions hold then all the inequalities above are in fact equalities.

Proof. — Since \circ is bilinear, Lemma 3.40 gives $a \circ b = -b \circ a$. Thus (i) is equivalent to (iii). We show the equivalence of (i) and (ii). Let Υ be the kernel of $\mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q) \rightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^2(Q, Q)$. Since $Q/\mathrm{rad} Q \cong S$ is irreducible and occurs with multiplicity 1 we have $\dim \mathrm{Hom}_{\mathfrak{e}(k)}(Q, Q) = 1$. Since a is non-split, we also have $\dim \mathrm{Hom}_{\mathfrak{e}(k)}(Q, T) = \dim \mathrm{Hom}_{\mathfrak{e}(k)}(T, Q) = 1$. Since $\dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q) = \dim \mathfrak{m}/\mathfrak{m}^2$ the exact sequence

$$\mathrm{Hom}_{\mathfrak{e}(k)}(Q, Q) \hookrightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q) \rightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^1(T, Q) \twoheadrightarrow \Upsilon$$

gives $\dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(T, Q) = \dim \mathfrak{m}/\mathfrak{m}^2 + \dim \Upsilon - 1$. Lemma 3.2 implies $\mathrm{Ext}_{\mathfrak{e}(k)}^1(T, Q)$ and $\mathrm{Ext}_{\mathfrak{e}(k)}^1(T, S)$ have the same dimension, so (i) is equivalent to (ii). It follows from Lemma 3.40 that $a \circ a = 0$ and so $a \in \Upsilon$, which implies that $\dim \Upsilon \geq 1$ and so $\dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(T, Q) = \dim \mathrm{Ext}_{\mathfrak{e}(k)}^1(T, S) \geq \dim \mathfrak{m}/\mathfrak{m}^2$. This implies that if (i) or (ii) hold then the inequalities are in fact equalities. The same proof shows that (iii) is equivalent to (iv). \square

Lemma 3.43. — *Assume that $\mathrm{Hom}(E, k[x]/(x^3)) \rightarrow \mathrm{Hom}(E, k[x]/(x^2))$ is surjective and that there exists a $(d-1)$ -dimensional subspace V of $\mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q)$ such that the equivalent conditions of Lemma 3.42 hold for every non-zero $a \in V$. Then they hold for every non-zero $a \in \mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q)$.*

Proof. — Let $\varphi_a : \mathrm{Ext}_{\mathfrak{e}(k)}^1(Q, Q) \rightarrow \mathrm{Ext}_{\mathfrak{e}(k)}^2(Q, Q)$ be the map $b \mapsto b \circ a$. Lemma 3.40 implies that a lies in $\mathrm{Ker} \varphi_a$. Thus (i) in Lemma 3.42 holds if and only if a spans $\mathrm{Ker} \varphi_a$. If $a \in V$ then the conditions hold by assumption and so $\mathrm{Ker} \varphi_a = \langle a \rangle$. If $a \notin V$ then using $\varphi_a(b) = -\varphi_b(a)$ we deduce that the restriction of φ_a to V is injective. Thus the image of φ_a is at least $d-1$ dimensional, and so the kernel is at most 1-dimensional. Hence, the conditions of Lemma 3.42 hold for a . \square

4. Banach space representations

From now on we assume that G is a p -adic analytic group. The following fact is essential: for every compact open subgroup H of G the completed group ring $\mathcal{O}[[H]]$ is noetherian. An L -Banach space representation Π of G is an L -Banach space Π together with a

G -action by continuous linear automorphisms such that the map $G \times \Pi \rightarrow \Pi$ describing the action is continuous. A Banach space representation Π is called *unitary*, if there exists a G -invariant norm defining the topology on Π . The existence of such norm is equivalent to the existence of an open bounded G -invariant \mathcal{O} -lattice Θ in Π . A unitary L -Banach space representation is *admissible* if $\Theta \otimes_{\mathcal{O}} k$ is an admissible (smooth) representation of G , this means that the space of invariants $(\Theta \otimes_{\mathcal{O}} k)^H$ is finite dimensional for every open subgroup H of G . We note that it is enough to check this for a single open pro- p subgroup of G , see for example [54, 6.3.2]. Our definition of admissibility does not depend on the choice of Θ . Moreover, it is equivalent to that of [61], see [33, 6.5.7], which requires $\Theta^d := \text{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})$ to be a finitely generated module over $\mathcal{O}[[H]]$. We say that an L -Banach space representation Π is *irreducible*, if it does not contain a proper closed G -invariant subspace. We say that Π is *absolutely irreducible* if $\Pi \otimes_L L'$ is irreducible for every finite extension L' of L .

Lemma 4.1. — *Let Π be an absolutely irreducible and admissible unitary L -Banach space representation of G and let $\phi \in \text{End}_{L[G]}^{\text{cont}}(\Pi)$. If the algebra $L[\phi]$ is finite dimensional over L then $\phi \in L$.*

Proof. — Let $f \in L[X]$ be the minimal polynomial of ϕ over L , and let L' be the splitting field of f . If M is a finitely generated $L[[H]] := L \otimes \mathcal{O}[[H]]$ module, then $M_{L'}$ is a finitely generated $L'[[H]]$ -module. Thus, it follows from [61, Thm. 3.5] that $\Pi_{L'}$ is an admissible unitary L' -Banach space representation of G . Since by assumption $\Pi_{L'}$ is irreducible, it follows from the proof of [61, Cor. 3.7] that any non-zero continuous linear G -equivariant map $\psi : \Pi_{L'} \rightarrow \Pi_{L'}$ is an isomorphism. Since $f(\phi) = 0$ using this we may find $\lambda \in L'$ such that $f(\lambda) = 0$ and $\phi \otimes \text{id} - \lambda$ kills $\Pi_{L'}$. Now $\text{Gal}(L'/L)$ acts on $\Pi_{L'}$ via $\sigma(v \otimes \mu) = v \otimes \sigma(\mu)$ for all $\mu \in L'$. Choose a non-zero $v \in \Pi$, then $\phi(v) \in \Pi$, and hence $\sigma(\lambda)v = \lambda v$ for all $\sigma \in \text{Gal}(L'/L)$. This implies $\lambda \in L$, and hence $\phi = \lambda$. \square

Lemma 4.2. — *Let Π be an irreducible admissible unitary L -Banach space representation of G . If $\text{End}_{L[G]}^{\text{cont}}(\Pi) = L$ then Π is absolutely irreducible.*

Proof. — Suppose that Π is not absolutely irreducible. Then there exists a finite Galois extension L' of L such that $\Pi_{L'}$ contains a closed proper G -invariant subspace Σ . Since L' is a finite extension of L we have isomorphisms:

$$\text{End}_{L'[G]}^{\text{cont}}(\Pi_{L'}) \cong \text{Hom}_{L'[G]}^{\text{cont}}(\Pi, \Pi_{L'}) \cong \text{End}_{L[G]}^{\text{cont}}(\Pi)_{L'} \cong L'.$$

Hence, it is enough to show that $\text{End}_{L'[G]}^{\text{cont}}(\Pi_{L'})$ contains a non-trivial idempotent.

As observed in the proof of Lemma 4.1, $\Pi_{L'}$ is admissible. This implies that any descending chain of closed G -invariant subspaces must become constant. Hence we may assume that Σ is irreducible (and admissible). The group $\Gamma := \text{Gal}(L'/L)$ acts on $\Pi_{L'}$ by G -equivariant, L -linear isometries

$$r_{\gamma} : \Pi_{L'} \rightarrow \Pi_{L'}, \quad v \otimes \lambda \mapsto v \otimes \gamma(\lambda), \quad \forall \gamma \in \Gamma.$$

In particular, r_γ is continuous and $r_\gamma(\Sigma)$ is a closed G -invariant L' -subspace of $\Pi_{L'}$. Since Σ is an irreducible admissible unitary L' -Banach space representation of G , so are $r_\gamma(\Sigma)$ for all $\gamma \in \Gamma$. Let Υ be the image of the natural map

$$(56) \quad \bigoplus_{\gamma \in \Gamma} r_\gamma(\Sigma) \rightarrow \Pi_{L'}.$$

Since both representations are admissible Υ is a closed G -invariant subspace of $\Pi_{L'}$. Now Υ is Γ -invariant and $\Upsilon^\Gamma = \Upsilon \cap \bigcap_{\gamma \in \Gamma} \text{Ker}(r_\gamma - 1)$ is a closed G -invariant L -subspace of $\Pi_{L'}^\Gamma = \Pi$. Linear independence of characters implies that if $v \in \Upsilon$ is non-zero then there exists $\lambda \in L'$ such that $\sum_{\gamma \in \Gamma} r_\gamma(\lambda v) \neq 0$. Hence, Υ^Γ is non-zero. Since Π is irreducible we deduce that $\Upsilon^\Gamma = \Pi$ and hence (56) is surjective.

Now any non-zero continuous G -equivariant L -linear map between two admissible irreducible unitary L -Banach space representations of G is an isomorphism. Using this fact and arguing by induction on n one may show that any quotient of $\bigoplus_{i=1}^n \Pi_i$, where Π_i are admissible and irreducible, is semi-simple. Hence, $\Pi_{L'}$ is semi-simple. As we have assumed that $\Pi_{L'}$ is not irreducible $\text{End}_{L'[G]}^{\text{cont}}(\Pi_{L'})$ contains a non-trivial idempotent. \square

Lemma 4.3. — *Let Π be a unitary L -Banach space representation of G , let Θ and Ξ be open bounded G -invariant lattices in Π , and let π be an irreducible smooth k -representation of G . Then π is a subquotient of $\Theta \otimes_{\mathcal{O}} k$ if and only if it is a subquotient of $\Xi \otimes_{\mathcal{O}} k$. Moreover, if $\Theta \otimes_{\mathcal{O}} k$ is a G -representation of finite length then so is $\Xi \otimes_{\mathcal{O}} k$, and their semi-simplifications are isomorphic.*

Proof. — Let $\pi \hookrightarrow J$ be an injective envelope of π in $\text{Mod}_G^{\text{sm}}(k)$ the category of smooth k -representations of G . Since J is injective, $\text{Hom}_G(*, J)$ is exact, thus if π occurs as a subquotient of some smooth k -representation κ , then $\text{Hom}_G(\kappa, J) \neq 0$. Conversely, if there exists some non-zero $\varphi : \kappa \rightarrow J$, then the image of φ must contain π , as $\pi \hookrightarrow J$ is essential. Further, if κ is of finite length the same argument shows that π occurs in κ with multiplicity $\dim \text{Hom}_G(\kappa, J)$. Since $\Theta \otimes_{\mathcal{O}} k$ and $\Xi \otimes_{\mathcal{O}} k$ are smooth representations of G , the assertion of the lemma is equivalent to $\text{Hom}_G(\Theta \otimes_{\mathcal{O}} k, J) \neq 0$ if and only if $\text{Hom}_G(\Xi \otimes_{\mathcal{O}} k, J) \neq 0$; $\Theta \otimes_{\mathcal{O}} k$ is of finite length if and only if $\Xi \otimes_{\mathcal{O}} k$ is of finite length, in which case

$$\dim \text{Hom}_G(\Xi \otimes_{\mathcal{O}} k, J) = \dim \text{Hom}_G(\Theta \otimes_{\mathcal{O}} k, J).$$

Since any two open bounded lattices in Π are commensurable, one can show this by adapting the proof of analogous statement for finite groups, see the proof of Theorem 32 in Section 15.1 of [63] and use the exactness of $\text{Hom}_G(*, J)$. \square

Let Π be a unitary L -Banach space representation of G and Θ an open bounded G -invariant lattice in Π . We denote by Θ^d its Schikhof dual

$$\Theta^d := \text{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})$$

equipped with the topology of pointwise convergence. If $\Theta \otimes_{\mathcal{O}} k$ is a G -representation of finite length, then we denote by $\overline{\Pi}$ its semi-simplification

$$\overline{\Pi} := (\Theta \otimes_{\mathcal{O}} k)^{ss}.$$

Lemma 4.3 shows that $\overline{\Pi}$ does not depend on the choice of Θ .

Lemma 4.4. — Θ^d is an object of $\text{Mod}_G^{\text{pro aug}}(\mathcal{O})$.

Proof. — For every $n \geq 1$, $\Theta/\varpi^n\Theta$ is a smooth representation of G on an \mathcal{O} -torsion module, thus $(\Theta/\varpi^n\Theta)^\vee$ is an object of $\text{Mod}_G^{\text{pro aug}}(\mathcal{O})$. It follows from the proof of [57, Lem. 5.4] that we have a topological isomorphism:

$$(57) \quad \Theta^d \otimes_{\mathcal{O}} \mathcal{O}/\varpi^n\mathcal{O} \cong (\Theta/\varpi^n\Theta)^\vee.$$

Thus $\Theta^d \cong \varprojlim \Theta^d/\varpi^n\Theta^d \cong \varprojlim (\Theta/\varpi^n\Theta)^\vee$ is an object of $\text{Mod}_G^{\text{pro aug}}(\mathcal{O})$. \square

Lemma 4.5. — Suppose that Π is irreducible and admissible and let $\phi : \mathbf{M} \rightarrow \Theta^d$ be a non-zero morphism in $\text{Mod}_G^{\text{pro aug}}(\mathcal{O})$, then there exists an open bounded G -invariant lattice Ξ in Π such that $\Xi^d = \phi(\mathbf{M})$.

Proof. — Let H be an open p -adic analytic pro- p subgroup of G . The completed group algebra $\mathcal{O}[[H]]$ is noetherian. The admissibility of Π is equivalent to Θ^d being a finitely generated $\mathcal{O}[[H]]$ -module. Hence, $\phi(\mathbf{M})$ is a finitely generated $\mathcal{O}[[H]]$ -submodule of Θ^d and is \mathcal{O} -torsion free. Hence, there exist a unique Hausdorff topology on $\phi(\mathbf{M})$ such that $\mathcal{O}[[H]] \times \phi(\mathbf{M}) \rightarrow \phi(\mathbf{M})$ is continuous, [61, Prop. 3.1(i)], and $\phi(\mathbf{M})$ is a closed submodule of Θ^d with respect to this topology, [61, Prop. 3.1(ii)]. The uniqueness of the topology on $\phi(\mathbf{M})$ implies that the submodule topology coincides with the quotient topology. Since $\phi(\mathbf{M})$ is G -invariant and non-zero and Π is irreducible it follows from [61, Thm. 3.5], that Π is naturally isomorphic to the Banach space representation $\text{Hom}_{\mathcal{O}}^{\text{cont}}(\phi(\mathbf{M}), L)$ with the topology induced by the supremum norm. If we let $\Xi := \text{Hom}_{\mathcal{O}}^{\text{cont}}(\phi(\mathbf{M}), \mathcal{O})$ then Ξ will be an open bounded G -invariant lattice in Π and it follows from the proof of [61, Thm. 1.2] that $\Xi^d = \phi(\mathbf{M})$. \square

Let $\text{Mod}_G^2(\mathcal{O})$ be a full subcategory of $\text{Mod}_G^{\text{lfin}}(\mathcal{O})$ closed under subquotients and arbitrary direct sums in $\text{Mod}_G^{\text{lfin}}(\mathcal{O})$. Let $\mathfrak{C}(\mathcal{O})$ be a full subcategory of $\text{Mod}_G^{\text{pro aug}}(\mathcal{O})$ anti-equivalent to $\text{Mod}_G^2(\mathcal{O})$ via Pontryagin duality. We note that $\text{Mod}_G^2(\mathcal{O})$ has injective envelopes and so $\mathfrak{C}(\mathcal{O})$ has projective envelopes, see Section 2.

Lemma 4.6. — For an admissible unitary L -Banach space representation Π of G the following are equivalent:

- (i) there exists an open bounded G -invariant lattice Θ in Π such that Θ^d is an object of $\mathfrak{C}(\mathcal{O})$;

(ii) Θ^d is an object of $\mathfrak{C}(\mathcal{O})$ for every open bounded G -invariant lattice Θ in Π .

Proof. — Clearly (ii) implies (i). The converse holds because any two open bounded lattices are commensurable and $\mathfrak{C}(\mathcal{O})$ is closed under subquotients. \square

Definition 4.7. — Let $\text{Ban}_G^{\text{adm}}(\mathbf{L})$ be the category of admissible unitary \mathbf{L} -Banach space representations of G with morphisms continuous G -equivariant \mathbf{L} -linear homomorphisms. Let $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ be the full subcategory of $\text{Ban}_G^{\text{adm}}(\mathbf{L})$ with objects admissible unitary \mathbf{L} -Banach space representations of G satisfying the conditions of Lemma 4.6.

Lemma 4.8. — $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ is closed under subquotients in $\text{Ban}_G^{\text{adm}}(\mathbf{L})$. In particular, it is abelian.

Proof. — We note that it follows from [61] and [33, 6.2.16] that $\text{Ban}_G^{\text{adm}}(\mathbf{L})$ is an abelian category. Let Π be an object of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ and let Θ be an open bounded G -invariant lattice in Π . Then Θ^d is an object of $\mathfrak{C}(\mathcal{O})$ and any subquotient of Θ^d in $\text{Mod}_G^{\text{proaug}}(\mathcal{O})$ lies in $\mathfrak{C}(\mathcal{O})$, since $\mathfrak{C}(\mathcal{O})$ is a full subcategory of $\text{Mod}_G^{\text{proaug}}(\mathcal{O})$ closed under subquotients. Dually this implies that any subquotient of Π in $\text{Ban}_G^{\text{adm}}(\mathbf{L})$ lies in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$. Hence $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ is abelian. \square

Lemma 4.9. — Let $\tilde{\mathbb{P}}$ be a projective object in $\mathfrak{C}(\mathcal{O})$ and let $\tilde{\mathbb{E}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}})$. Let Π be in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$, choose an open bounded G -invariant lattice Θ in Π and put $m(\Pi) := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d) \otimes_{\mathcal{O}} \mathbf{L}$. Then $\Pi \mapsto m(\Pi)$ defines an exact functor from $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ to the category of right $\tilde{\mathbb{E}}[1/p]$ -modules.

Proof. — We note that since any two open bounded lattices in Π are commensurable the definition of $m(\Pi)$ does not depend on the choice of Θ . Let $0 \rightarrow \Pi_1 \rightarrow \Pi_2 \rightarrow \Pi_3 \rightarrow 0$ be an exact sequence in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$. Let Θ be an open bounded G -invariant lattice in Π_2 . Since all the Banach space representations are admissible, $\Pi_1 \cap \Theta$ is an open bounded G -invariant lattice in Π_1 and the image of Θ in Π_3 is an open bounded G -invariant lattice in Π_3 . So we have an exact sequence $0 \rightarrow \Theta_1 \rightarrow \Theta_2 \rightarrow \Theta_3 \rightarrow 0$ with Θ_i an open bounded G -invariant lattice in Π_i . Dually this gives an exact sequence $0 \rightarrow \Theta_3^d \rightarrow \Theta_2^d \rightarrow \Theta_1^d \rightarrow 0$ in $\mathfrak{C}(\mathcal{O})$. Since $\tilde{\mathbb{P}}$ is projective in $\mathfrak{C}(\mathcal{O})$ we obtain an exact sequence of right $\tilde{\mathbb{E}}$ -modules:

$$0 \rightarrow \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta_3^d) \rightarrow \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta_2^d) \rightarrow \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta_1^d) \rightarrow 0.$$

The sequence remains exact after tensoring with \mathbf{L} . \square

Corollary 4.10. — Let $\tilde{\mathbb{P}}$ be a projective object in $\mathfrak{C}(\mathcal{O})$ and let Π be in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ then there exists a smallest closed G -invariant subspace of Π_1 of Π such that $m(\Pi/\Pi_1)$ is zero.

Proof. — Since Π is admissible any descending chain of closed G -invariant subspaces must become stationary, [57, Lemma 5.8]. The assertion follows from the exactness of m . \square

In the application we will be in the following situation.

Lemma 4.11. — *Let $G = \mathrm{GL}_2(\mathbf{Q}_p)$, $\zeta : Z \rightarrow \mathcal{O}^\times$ be a continuous character of the centre of G and let $\mathfrak{C}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$ anti-equivalent to $\mathrm{Mod}_{G,\zeta}^{\mathrm{lin}}(\mathcal{O})$ by Pontryagin duality, see Section 2. Let Π be an admissible L -Banach space representation of G with a central character ζ and let Θ be an open bounded G -invariant lattice in Π . Then Θ^d is an object of $\mathfrak{C}(\mathcal{O})$. In particular, $\mathrm{Ban}_{\mathfrak{C}(\mathcal{O})}^{\mathrm{adm}} = \mathrm{Ban}_{G,\zeta}^{\mathrm{adm}}(L)$ the category of admissible unitary L -Banach space representations of G on which Z acts by the character ζ .*

Proof. — Recall that an object M of $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$ is an object of $\mathfrak{C}(\mathcal{O})$ if and only if $M \cong \varprojlim M_i$ where the limit is taken over all the quotients in $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(\mathcal{O})$ of finite length and Z acts on M via ζ^{-1} .

Since Π is admissible $\Theta/\varpi^n\Theta$ is an admissible smooth representation of G for all $n \geq 1$. Since Z acts on $\Theta/\varpi^n\Theta$ by a character ζ [30, Thm 2.3.8] says that any finitely generated subrepresentation of $\Theta/\varpi^n\Theta$ is of finite length. Hence by definition $(\Theta/\varpi^n\Theta)^\vee$ is an object of $\mathfrak{C}(\mathcal{O})$. The assertion follows from (57). \square

Remark 4.12. — If the Conjecture formulated by Emerton in [30, 2.3.7] holds then the proof of Lemma 4.11 goes through unchanged for a p -adic reductive group G .

Lemma 4.13. — *Let \tilde{P} be a projective envelope of an irreducible object S in $\mathfrak{C}(\mathcal{O})$, $\pi := S^\vee$ a smooth irreducible k -representation of G , Π an object of $\mathrm{Ban}_{\mathfrak{C}(\mathcal{O})}^{\mathrm{adm}}$ and Θ an open bounded G -invariant lattice in Π . Then the following are equivalent:*

- (i) π is a subquotient of $\Theta \otimes_{\mathcal{O}} k$;
- (ii) S is a subquotient of $\Theta^d \otimes_{\mathcal{O}} k$;
- (iii) $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d \otimes_{\mathcal{O}} k) \neq 0$;
- (iv) $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \neq 0$.

Proof. — It follows from (57) that (i) is equivalent to (ii). Since $\tilde{P} \twoheadrightarrow S$ is essential (iii) implies (ii). Since $\mathfrak{C}(\mathcal{O})$ is closed under subquotients and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, *)$ is exact (ii) implies (iii). We have isomorphisms:

$$(58) \quad \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \varprojlim \Theta^d/\varpi^n\Theta^d) \cong \varprojlim \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d/\varpi^n\Theta^d).$$

The transition maps are surjective since \tilde{P} is projective. Hence (iii) implies (iv). Since Θ^d is \mathcal{O} -torsion free multiplication by ϖ^n induces isomorphism $\Theta^d/\varpi^n\Theta^d \cong \varpi^n\Theta^d/\varpi^{n+1}\Theta^d$. If $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d/\varpi^n\Theta^d) = 0$ then by considering short exact sequences we obtain $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d/\varpi^n\Theta^d) = 0$ for all $n \geq 1$ and so (iv) implies (iii). \square

Lemma 4.14. — *Let \tilde{P} , S , π and Θ be as in Lemma 4.13. If $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \neq 0$ then π is an admissible representation of G . In particular, $\text{End}_{\mathfrak{C}(\mathcal{O})}(S) \cong \text{End}_G(\pi)$ is a finite field extension of k .*

Proof. — Let H be an open p -adic analytic pro- p subgroup of G . Since Π is admissible Θ^d is finitely generated over $\mathcal{O}[[H]]$. Since $\mathcal{O}[[H]]$ is noetherian it follows from Lemma 4.13 that S is a finitely generated $\mathcal{O}[[H]]$ -module. Since H is pro- p , dually this implies that π^H is finite dimensional. Since π is irreducible, $\text{End}_G(\pi)$ is a skew field over k contained in $\text{End}_k(\pi^H)$. Since π^H is finite dimensional $\text{End}_G(\pi)$ is finite dimensional. Since k is a finite field $\text{End}_G(\pi)$ is a finite field extension of k . \square

Lemma 4.15. — *Let \tilde{P} be a projective envelope of an irreducible object S in $\mathfrak{C}(\mathcal{O})$ with $d := \dim_k \text{End}_{\mathfrak{C}(\mathcal{O})}(S)$ finite. Let M be in $\mathfrak{C}(\mathcal{O})$, \mathcal{O} -torsion free and such that $M_k := M \otimes_{\mathcal{O}} k$ is of finite length in $\mathfrak{C}(\mathcal{O})$. Then $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M)$ is a free \mathcal{O} -module of rank equal to the multiplicity with which S occurs as a subquotient of M_k multiplied by d .*

Proof. — Since M is \mathcal{O} -torsion free so is $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M)$. Let m be the multiplicity with which S occurs as a subquotient of M_k . It follows from Lemma 3.3 that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M)_k \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M_k)$ is an md -dimensional k -vector space. Since $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M)$ is a compact \tilde{E} , and hence \mathcal{O} -module, the assertion follows from Nakayama's lemma. \square

From now on we assume (unless it is stated otherwise) the following setup. Let S_1, \dots, S_n be irreducible pairwise non-isomorphic objects of $\mathfrak{C}(\mathcal{O})$ such that $\text{End}_{\mathfrak{C}(\mathcal{O})}(S_i)$ is finite dimensional over k for $1 \leq i \leq n$. Let \tilde{P} be a projective envelope of $S := \bigoplus_{i=1}^n S_i$ and let $\tilde{E} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P})$. Recall from Section 2, that \tilde{E} is a compact ring and $\tilde{E}/\text{rad } \tilde{E} \cong \prod_{i=1}^n \text{End}_{\mathfrak{C}(\mathcal{O})}(S_i)$, where $\text{rad } \tilde{E}$ is the Jacobson radical of \tilde{E} . Moreover, uniqueness of projective envelopes implies that $\tilde{P} \cong \bigoplus_{i=1}^n \tilde{P}_i$, where \tilde{P}_i is a projective envelope of S_i in $\mathfrak{C}(\mathcal{O})$. For $1 \leq i \leq n$ let $\pi_i := S_i^\vee$, so that π_i is a smooth irreducible k -representation of G and $\pi := \bigoplus_{i=1}^n \pi_i \cong S^\vee$.

Remark 4.16. — The assumption on the finite dimensionality of $\text{End}_{\mathfrak{C}(\mathcal{O})}(S_i)$ holds if π_i is a subquotient of the reduction modulo ϖ of admissible Banach space representations, see Lemma 4.14.

Proposition 4.17. — *Let Π be in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ and let Θ be an open bounded G -invariant lattice in Π . Then $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)$ is a finitely generated module over \tilde{E} .*

Proof. — Let $m = \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)$ and let $M \subseteq \Theta^d$ be the image of the natural map $m \hat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Theta^d$. We may assume that $M \neq 0$, since otherwise $m = 0$ is finitely generated. We apply $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, *)$ to $m \hat{\otimes}_{\tilde{E}} \tilde{P} \twoheadrightarrow M \hookrightarrow \Theta^d$ and use Lemma 2.9 to obtain $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M) \cong m$. Since Π is admissible, Θ^d is a finitely generated $\mathcal{O}[[H]]$ -module,

which implies that $((\Theta)^d)^\vee$ is admissible-smooth. Since the quotients of admissible representations are admissible, we deduce that M^\vee is admissible-smooth. The G -socle of M^\vee is a finite direct sum of irreducible representations, because every summand contributes to invariants by a pro- p subgroup of G . Hence, $\mathrm{Hom}_G(\pi_i, M^\vee)$ is a finite dimensional k -vector space, for $1 \leq i \leq n$. Dually, we obtain that $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(M, S_i)$ is a finite dimensional k -vector space of dimension d_i (say), for all $1 \leq i \leq n$. Since M is a quotient of $m \widehat{\otimes}_{\tilde{E}} \tilde{P}$, all the irreducible summands appearing in its cosocle are isomorphic to S_i for some $1 \leq i \leq n$. Hence, $\mathrm{cosoc} M \cong \bigoplus_{i=1}^n S_i^{\oplus n_i}$, with n_i equal to d_i divided by the dimension of $\mathrm{End}_{\mathcal{E}(\mathcal{O})}(S_i)$. We may choose a surjection $a : \tilde{P}^{\oplus m} \rightarrow \mathrm{cosoc} M$ for some integer m . Since \tilde{P} is projective, a factors through $b : \tilde{P}^{\oplus m} \rightarrow M$. Since $M \rightarrow \mathrm{cosoc} M$ is an essential epimorphism, b is surjective. We apply $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, *)$ to $b : \tilde{P}^{\oplus m} \rightarrow M$ to obtain a surjection $\tilde{E}^{\oplus m} \rightarrow m$. \square

Proposition 4.18. — *Let Π be in $\mathrm{Ban}_{\mathcal{E}(\mathcal{O})}^{\mathrm{adm}}$ and let Θ be an open bounded G -invariant lattice in Π . Suppose that Π is irreducible and $\Theta \otimes_{\mathcal{O}} k$ contains π_i as a subquotient for some i . Let $\phi \in \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Theta^d)$ be non-zero and let $\mathfrak{a} := \{a \in \tilde{E} : \phi \circ a = 0\}$. There exists an open bounded G -invariant \mathcal{O} -lattice Ξ in Π such that $\phi(\tilde{P}) = \Xi^d$. Moreover,*

- (i) $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \cong \tilde{E}/\mathfrak{a}$ as a right \tilde{E} -module;
- (ii) $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d)_L$ is an irreducible right \tilde{E}_L -module;
- (iii) the natural map $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Xi^d$ is surjective.

Proof. — Since by assumption $\Theta \otimes_{\mathcal{O}} k$ contains π_i as a subquotient, Lemma 4.13 implies that $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Theta^d)$ is non-zero. Let $\phi, \psi \in \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Theta^d)$ be non-zero then by Lemma 4.5 there exists an open bounded G -invariant lattice Ξ in Π such that $\phi(\tilde{P}) = \Xi^d$. By applying $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, *)$ to the exact sequence $0 \rightarrow \mathrm{Ker} \phi \rightarrow \tilde{P} \rightarrow \phi(\tilde{P}) \rightarrow 0$ we obtain that

$$(59) \quad \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \phi(\tilde{P})) = \phi \tilde{E} \cong \tilde{E}/\mathfrak{a}.$$

Lemma 4.5 implies that $\psi(\tilde{P})$ is commensurable with $\phi(\tilde{P})$. Thus for some $n \geq 0$, $\varpi^n \psi(\tilde{P}) \subseteq \phi(\tilde{P})$, and hence $\varpi^n \psi \in \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \phi(\tilde{P}))$. It follows from (59) that $\varpi^n \psi = \phi \circ a$ for some $a \in \tilde{E}$. Hence, $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \otimes_{\mathcal{O}} L$ is an irreducible $\tilde{E} \otimes_{\mathcal{O}} L$ -module. The image of the natural map $\mathrm{ev} : \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Xi^d$ will contain $\phi(\tilde{P})$, and hence ev is surjective. \square

Proposition 4.19. — *Let Ξ be as in Proposition 4.18 then we have natural isomorphisms of rings:*

$$\mathrm{End}_{\mathcal{E}(\mathcal{O})}(\Xi^d) \cong \mathrm{End}_{\tilde{E}}(m) \cong \mathrm{End}_{\mathcal{E}(\mathcal{O})}(m \widehat{\otimes}_{\tilde{E}} \tilde{P}),$$

where $m := \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d)$.

Proof. — We note that m is a compact right \tilde{E} -module and $\text{End}_{\tilde{E}}(m)$ denotes continuous \tilde{E} -linear endomorphisms of m . We have natural maps

$$\text{End}_{\mathfrak{C}(\mathcal{O})}(\Xi^d) \rightarrow \text{End}_{\tilde{E}}(m) \rightarrow \text{End}_{\mathfrak{C}(\mathcal{O})}(m \widehat{\otimes}_{\tilde{E}} \tilde{P}),$$

where the first one sends ϕ to $\psi \mapsto \phi \circ \psi$, the second one sends ϕ to $\psi \widehat{\otimes} v \mapsto \phi(\psi) \widehat{\otimes} v$. The natural map $\text{ev} : m \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Xi^d$ is surjective by Proposition 4.18(iii), let K be its kernel. By applying $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, *)$ to the exact sequence $0 \rightarrow K \rightarrow m \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Xi^d \rightarrow 0$ and using Lemma 2.9 we deduce that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, K) = 0$.

We claim that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(K, \Xi^d) = 0$. Suppose we have a non-zero morphism $\phi : K \rightarrow \Xi^d$ in $\mathfrak{C}(\mathcal{O})$. It follows from Lemma 4.5 that $\phi(K)$ contains $\varpi^n \Xi^d$ for some $n \geq 1$. This implies that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \phi(K)) \neq 0$. Since \tilde{P} is projective we get $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, K) \neq 0$, which is a contradiction. The claim implies that every $\phi \in \text{End}_{\mathfrak{C}(\mathcal{O})}(m \widehat{\otimes}_{\tilde{E}} \tilde{P})$ maps K to itself. Hence we obtain a well defined map $\text{End}_{\mathfrak{C}(\mathcal{O})}(m \widehat{\otimes}_{\tilde{E}} \tilde{P}) \rightarrow \text{End}_{\mathfrak{C}(\mathcal{O})}(\Xi^d)$, which sends ϕ to $\psi \widehat{\otimes} v + K \mapsto \phi(\psi \widehat{\otimes} v) + K$. The composition of any three consecutive arrows is the identity map, hence all the maps are isomorphisms. \square

Proposition 4.20. — *Let $\Pi \in \text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ be irreducible and let Θ be an open bounded G -invariant lattice in Π . Suppose that $\Theta \otimes_{\mathcal{O}} k$ contains π_i as a subquotient for some i . If the centre \mathcal{Z} of \tilde{E} is noetherian and \tilde{E} is a finitely generated \mathcal{Z} -module then $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)_L$ is finite dimensional over L .*

Proof. — Let Ξ be an open bounded G -invariant lattice constructed in Proposition 4.18. Since Θ and Ξ are commensurable, $\Theta \otimes_{\mathcal{O}} k$ and $\Xi \otimes_{\mathcal{O}} k$ have the same irreducible subquotients by Lemma 4.3 and $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)_L \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Xi^d)_L$ as $\tilde{E}[1/p]$ -modules.

Since Π is admissible and irreducible it follows from [61, Thm. 3.5] that the ring $D := \text{End}_{L[G]}^{\text{cont}}(\Pi)$ is a skew field. Since Ξ is an open bounded G -invariant lattice in Π , [59, Prop. 3.1] implies that $\text{End}_{\mathcal{O}[G]}(\Xi)$ is an \mathcal{O} -order in D . It follows from the anti-equivalence of categories established in [61, Thm. 3.5] that sending f to its Schikhof dual f^d induces an isomorphism $B := \text{End}_{\mathfrak{C}(\mathcal{O})}(\Xi^d) \cong \text{End}_{\mathcal{O}[G]}(\Xi)^{\text{op}}$ and $B[1/p] \cong D^{\text{op}}$. Hence $B[1/p]$ is a skew field and since Ξ is \mathcal{O} -torsion free so is B and we have an injection $B \hookrightarrow B[1/p]$.

Let R be the centre of B . Since R is contained in a skew field $B[1/p]$ it is an integral domain and $B[1/p]$ contains the quotient field K of R . Let $s \in K \cap B$ be non-zero, then we may find non-zero $a, b \in R$ such that $as = b$. For all $t \in B$ we have $(st - ts)a = bt - tb = 0$ as a and b are central. Since B is contained in a skew field we deduce that $st = ts$ for all t and hence $B \cap K = R$. Since K is contained in $B[1/p]$ we deduce that for every $x \in K$ there exists $n \geq 0$ such that $p^n x \in R$ and so $K = R[1/p]$.

Without loss of generality we may assume that $n = 1$, so that S is irreducible. This may be seen as follows: since $\tilde{P} \cong \bigoplus_{i=1}^n \tilde{P}_i$ and thus $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_i, \Xi^d) \neq 0$ for some i , and if $e \in \tilde{E}$ denotes the idempotent such that $e\tilde{P} = \tilde{P}_i$, then $\text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_i) = e\tilde{E}e$ is a finitely

generated $e\mathcal{Z}e$ -module and $e\mathcal{Z}e$ is contained in the centre of $e\tilde{E}e$. Moreover, since \mathcal{Z} is noetherian so is $e\mathcal{Z}e$ and this implies that the centre of $e\tilde{E}e$ is noetherian.

Let $\mathfrak{m} := \text{Hom}_{\mathcal{O}(\mathcal{O})}(\tilde{\mathcal{P}}, \Xi^d)$. We have a natural map $\mathcal{Z} \rightarrow \text{End}_{\tilde{E}}(\mathfrak{m})$, which sends z to $\psi \mapsto \psi \circ z$. Let ϕ and \mathfrak{a} be as in Proposition 4.18. It follows from Proposition 4.18(i) that for every $\alpha \in \text{End}_{\tilde{E}}(\mathfrak{m})$ there exists $\beta \in \tilde{E}$ such that $\alpha(\phi) = \phi \circ \beta$ and the map $\alpha \mapsto \beta + \mathfrak{a}$ is an injection of \mathcal{Z} -modules $\text{End}_{\tilde{E}}(\mathfrak{m}) \hookrightarrow \tilde{E}/\mathfrak{a}$. Since by assumption \tilde{E} is finitely generated over \mathcal{Z} and \mathcal{Z} is noetherian we deduce that $\text{End}_{\tilde{E}}(\mathfrak{m})$ is finitely generated over \mathcal{Z} . The image of \mathcal{Z} in $\text{End}_{\tilde{E}}(\mathfrak{m})$ is contained in the centre. We identify $\text{End}_{\tilde{E}}(\mathfrak{m})$ with \mathbf{B} using Proposition 4.19. Then the image of \mathcal{Z} in \mathbf{B} is contained in \mathbf{R} , hence \mathbf{R} is a \mathcal{Z} -submodule of a finitely generated \mathcal{Z} -module \mathbf{B} . We deduce that \mathbf{R} is a noetherian ring.

Since $\mathbf{R}[1/p]$ is a field and \mathbf{R} is a noetherian integral domain, Theorem 146 in [38] implies that $\mathbf{R}/p\mathbf{R}$ is artinian. Hence, $\mathbf{R}/p\mathbf{R} \cong \prod_{i=1}^n (A_i, \mathfrak{n}_i)$, where (A_i, \mathfrak{n}_i) are artinian local rings. Let $\tilde{\mathcal{Z}}_i$ be the image of \mathcal{Z} in A_i/\mathfrak{n}_i via $\mathbf{R}/p\mathbf{R} \rightarrow A_i \rightarrow A_i/\mathfrak{n}_i$. Since \mathbf{R} is a finitely generated \mathcal{Z} -module, A_i/\mathfrak{n}_i is a finitely generated $\tilde{\mathcal{Z}}_i$ -module. Since A/\mathfrak{n}_i is a field we deduce that $\tilde{\mathcal{Z}}_i$ is a field. Since \mathcal{Z} is a local ring with residue field a finite extension of k , Corollary 2.6, we deduce that $\tilde{\mathcal{Z}}_i$ and hence A_i/\mathfrak{n}_i is a finite extension of k . Since A_i is an artinian local ring, A_i is an A_i -module of finite length with irreducible subquotients isomorphic to A_i/\mathfrak{n}_i . Hence A_i is a finitely generated \mathcal{O} -module and so $\mathbf{R}/p\mathbf{R}$ is a finitely generated \mathcal{O} -module. As Ξ^d is p -adically complete, so is \mathbf{B} and hence so is \mathbf{R} . Thus \mathbf{R} is a finitely generated \mathcal{O} -module. Since by assumption \tilde{E} is a finitely generated \mathcal{Z} -module we deduce from Proposition 4.18 that \mathfrak{m} is a finitely generated \mathcal{Z} -module and hence \mathfrak{m} is a finitely generated \mathbf{R} -module and so a finitely generated \mathcal{O} -module. Thus $\mathfrak{m} \otimes_{\mathcal{O}} \mathbf{L}$ is finite dimensional over \mathbf{L} . \square

Corollary 4.21. — *Let Π be an irreducible admissible unitary \mathbf{L} -Banach space representation of \mathbf{G} , let Θ be an open bounded \mathbf{G} -invariant lattice in Π . If the conditions of Proposition 4.20 are satisfied then there exists a finite extension \mathbf{L}' of \mathbf{L} such that $\Pi_{\mathbf{L}'}$ is isomorphic to a finite direct sum of absolutely irreducible unitary \mathbf{L}' -representations.*

Proof. — It follows from Propositions 4.18, 4.19 and 4.20 that $\text{End}_{\mathbf{G}}^{\text{cont}}(\Pi)$ is a skew field, finite dimensional over \mathbf{L} . Let \mathbf{L}' be a finite Galois extension of \mathbf{L} splitting $\text{End}_{\mathbf{G}}^{\text{cont}}(\Pi)$. Then $\text{End}_{\mathbf{G}}^{\text{cont}}(\Pi_{\mathbf{L}'}) \cong \text{End}_{\mathbf{G}}^{\text{cont}}(\Pi)_{\mathbf{L}'}$ is a matrix algebra over \mathbf{L}' . Let $\{e_i\}_{1 \leq i \leq n}$ be the complete set of orthogonal idempotents and let $\Pi_i := e_i(\Pi_{\mathbf{L}'})$. Then $\Pi_{\mathbf{L}'} \cong \bigoplus_{i=1}^n \Pi_i$ and $\text{End}_{\mathbf{G}}^{\text{cont}}(\Pi_i) = e_i \text{End}_{\mathbf{G}}^{\text{cont}}(\Pi_{\mathbf{L}'}) e_i = \mathbf{L}'$. It follows from the proof of Lemma 4.2 that, after possibly enlarging \mathbf{L}' , we may assume that Π_i is semi-simple. Since $\text{End}_{\mathbf{G}}^{\text{cont}}(\Pi_i) \cong \mathbf{L}'$, we deduce that Π_i is irreducible and hence absolutely irreducible by Lemma 4.2. \square

We equip every finitely generated \mathcal{O} -module (resp. every finite dimensional \mathbf{L} -vector space) with the p -adic topology.

Lemma 4.22. — *If \tilde{E} is right noetherian then any \mathcal{O} -linear right action of \tilde{E} on a finite dimensional L -vector space is continuous.*

Proof. — Let m_L be a finite dimensional L -vector space with an \mathcal{O} -linear right \tilde{E} -action. Choose a basis $\{v_1, \dots, v_n\}$ of m_L and let $m := v_1\tilde{E} + \dots + v_n\tilde{E}$. Since \tilde{E} is right noetherian the kernel of $\tilde{E}^{\oplus n} \rightarrow m$ is finitely generated as a right \tilde{E} -module and, since \tilde{E} is compact, the kernel is a closed submodule of $\tilde{E}^{\oplus n}$. Thus the quotient topology on m is Hausdorff and it has a system of open neighborhoods of 0 consisting of \tilde{E} -modules, and in particular of \mathcal{O} -modules. The action of \mathcal{O} on m via $\mathcal{O} \rightarrow \tilde{E}$ on m is continuous for the quotient topology. Any compact linear-topological \mathcal{O} -torsion free \mathcal{O} -module is isomorphic to $\prod_{i \in I} \mathcal{O}$ for some set I , see Remark 1.1 in [61]. Since m is contained in a finite dimensional L -vector space we deduce that m is an \mathcal{O} -module of finite rank. Thus, for each $n \geq 1$ the quotient topology on $m/\varpi^n m$ is discrete, as it is Hausdorff and the underlying set is finite. In particular, the sets $\varpi^n m$ are open in the quotient topology on m , for all $n \geq 0$. Since m is ϖ -adically complete, we deduce that the sets $\varpi^n m$ for $n \geq 0$ build a system of open neighborhoods of 0 in the quotient topology on m . In particular, the quotient topology and the p -adic topology on m coincide. Let $n \geq 0$, and let $v \in m_L$. The same argument as above shows that $(v\tilde{E} + \varpi^n m)/\varpi^n m$ with the discrete topology is a topological \tilde{E} -module. This implies that the set $\mathfrak{a}(v, n) := \{a \in \tilde{E} : va \in \varpi^n m\}$ is open in \tilde{E} . Let U be the preimage in $\tilde{E} \times m_L$ of $v + \varpi^n m$. If $(a, w) \in U$, then $(a + \mathfrak{a}(w, n), w + \varpi^n m)$ is open in $\tilde{E} \times m_L$, and is a subset of U containing (a, w) . In particular, U is open and hence the action of \tilde{E} on m_L is continuous. \square

Proposition 4.23. — *Let m be a compact right \tilde{E} -module, free of finite rank over \mathcal{O} . Assume that $(\tilde{E}/\text{rad } \tilde{E}) \hat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is a finitely generated $\mathcal{O}[[H]]$ -module, where $\text{rad } \tilde{E}$ is the Jacobson radical of \tilde{E} . Then $m \hat{\otimes}_{\tilde{E}} \tilde{P}$ is finitely generated over $\mathcal{O}[[H]]$ and $(m \hat{\otimes}_{\tilde{E}} \tilde{P}) \otimes_{\mathcal{O}} k$ is of finite length in $\mathfrak{C}(\mathcal{O})$.*

Proof. — Let n be a finite dimensional k -vector space with a continuous \tilde{E} -action. If n is an irreducible \tilde{E} -module then it is killed by $\text{rad } \tilde{E}$, and hence

$$n \hat{\otimes}_{\tilde{E}} \tilde{P} \cong n \hat{\otimes}_{\tilde{E}/\text{rad } \tilde{E}} ((\tilde{E}/\text{rad } \tilde{E}) \hat{\otimes}_{\tilde{E}} \tilde{P}).$$

Thus it follows from our assumptions that $n \hat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is a finitely generated $\mathcal{O}[[H]]$ -module. In general, arguing inductively on the dimension of n we deduce that $n \hat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is a finitely generated $\mathcal{O}[[H]]$ -module. Applying $\hat{\otimes}_{\tilde{E}} \tilde{P}$ to the exact sequence $m \xrightarrow{\varpi} m \rightarrow m_k \rightarrow 0$ we get

$$(m \hat{\otimes}_{\tilde{E}} \tilde{P}) \otimes_{\mathcal{O}} k \cong m_k \hat{\otimes}_{\tilde{E}} \tilde{P}.$$

Nakayama's lemma for compact $\mathcal{O}[[H]]$ -modules implies that $m \hat{\otimes}_{\tilde{E}} \tilde{P}$ is a finitely generated $\mathcal{O}[[H]]$ -module, see [21, Cor. 1.5]. \square

Remark 4.24. — If S is an irreducible object in $\mathfrak{C}(\mathcal{O})$ and $\kappa : \tilde{P} \twoheadrightarrow S$ is its projective envelope, then $\text{rad } \tilde{E} = \{\phi \in \tilde{E} : \kappa \circ \phi = 0\}$, which is the ideal \mathfrak{m} defined in Definition 3.6, and $(\tilde{E}/\text{rad } \tilde{E}) \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is the object Q considered in Section 3, see Remark 1.13 and Lemma 3.7.

Lemma 4.25. — *Under the hypotheses of Proposition 4.23 the maximal \mathcal{O} -torsion free quotient $(\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}$ of $\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is an object of $\mathfrak{C}(\mathcal{O})$. Moreover,*

$$\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, (\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}) \cong \mathfrak{m}.$$

Proof. — Since $\mathcal{O}[[H]]$ is noetherian and $\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is finitely generated, the torsion submodule $(\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tors}}$ is finitely generated, and hence is equal to the kernel of multiplication by ϖ^n for n large enough. So $(\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tors}}$ and $(\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}$ are both objects of $\mathfrak{C}(\mathcal{O})$. Now $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P}) \cong \mathfrak{m}$, see Lemma 2.9, is \mathcal{O} -torsion free. Hence, $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, (\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tors}}) = 0$. Since \tilde{P} is projective we obtain an isomorphism:

$$\mathfrak{m} \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P}) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, (\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}). \quad \square$$

Definition 4.26. — *Under the hypotheses of Proposition 4.23 to a right \tilde{E} -module \mathfrak{m} free of finite rank over \mathcal{O} we associate an admissible unitary L -Banach space representation of G :*

$$\Pi(\mathfrak{m}) := \text{Hom}_{\mathcal{O}}^{\text{cont}}((\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}, L)$$

with the topology induced by the supremum norm.

Remark 4.27. — We define $\Pi(\mathfrak{m})$ in terms of the maximal torsion free quotient of $\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P}$, as this allows us to appeal to the results of [61]. Since any \mathcal{O} -linear homomorphism to L kills off the \mathcal{O} -torsion, we have $\Pi(\mathfrak{m}) \cong \text{Hom}_{\mathcal{O}}^{\text{cont}}(\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P}, L)$.

A continuous homomorphism of compact \tilde{E} -modules $\mathfrak{m}_1 \rightarrow \mathfrak{m}_2$ induces a morphism $\mathfrak{m}_1 \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \mathfrak{m}_2 \widehat{\otimes}_{\tilde{E}} \tilde{P}$ in $\mathfrak{C}(\mathcal{O})$ and hence $\mathfrak{m} \mapsto \Pi(\mathfrak{m})$ defines a contravariant functor from the category of compact right \tilde{E} -modules, free of finite rank over \mathcal{O} to $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$. Since ϖ is invertible in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ the functor factors through the category of finite dimensional L -vector spaces with continuous \tilde{E} -action (note that \tilde{E} is compact).

Lemma 4.28. — *Let \mathfrak{m}_L be a finite dimensional L -vector space with continuous \tilde{E} -action. Assume that $(\tilde{E}/\text{rad } \tilde{E}) \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is a finitely generated $\mathcal{O}[[H]]$ -module. Then*

$$\mathfrak{m}(\Pi(\mathfrak{m}_L)) \cong \mathfrak{m}_L$$

where \mathfrak{m} is the functor defined in Lemma 4.9.

Proof. — Since \tilde{E} is compact and the action is continuous there exists an open bounded \mathcal{O} -lattice \mathfrak{m} in \mathfrak{m}_L which is \tilde{E} -stable. Then

$$\Pi(\mathfrak{m}_L) = \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}((\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\mathrm{tf}}, L)$$

and let $\Pi(\mathfrak{m}_L)^0$ be the unit ball in $\Pi(\mathfrak{m}_L)$ with respect to the supremum norm, so that $\Pi(\mathfrak{m}_L)^0 = \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}((\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\mathrm{tf}}, \mathcal{O})$. Then $(\Pi(\mathfrak{m}_L)^0)^d \cong (\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\mathrm{tf}}$ and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, (\Pi(\mathfrak{m}_L)^0)^d) \cong \mathfrak{m}$ by Lemma 4.25. Since \mathfrak{m} is an open \mathcal{O} -lattice in \mathfrak{m}_L we get $\mathfrak{m}(\Pi(\mathfrak{m}_L)) \cong \mathfrak{m}_L$. \square

Lemma 4.29. — *Let $\Pi \in \mathrm{Ban}_{\mathfrak{C}(\mathcal{O})}^{\mathrm{adm}}$ be irreducible, and let $\mathfrak{m}_L := \mathfrak{m}(\Pi)$, where \mathfrak{m} is the functor defined in Lemma 4.9. If \mathfrak{m}_L is a non-zero finite dimensional L -vector space then Π is isomorphic to a closed subspace of $\Pi(\mathfrak{m}_L)$.*

Proof. — Let Θ be an open bounded G -invariant lattice in Π . The evaluation map $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Theta^d$ induces a non-zero continuous, G -equivariant map $\Pi \rightarrow \Pi(\mathfrak{m}_L)$. Since Π is irreducible, and both representations are admissible, the map induces an isomorphism between Π and a closed subspace of $\Pi(\mathfrak{m}_L)$. \square

Lemma 4.30. — *Assume that $(\tilde{E}/\mathrm{rad}\tilde{E}) \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is a finitely generated $\mathcal{O}[[H]]$ -module. The functor $\mathfrak{m}_L \mapsto \Pi(\mathfrak{m}_L)$ is left exact.*

Proof. — This follows from the right exactness of $\widehat{\otimes}_{\tilde{E}} \tilde{P}$, left exactness of $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(*, L)$ and Remark 4.27. \square

Lemma 4.31. — *Assume that $(\tilde{E}/\mathrm{rad}\tilde{E}) \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is a finitely generated $\mathcal{O}[[H]]$ -module. Let \mathfrak{m}_L be a finite dimensional L -vector space with a continuous \tilde{E} -action. Then $\overline{\Pi}(\mathfrak{m}_L)$ is an admissible smooth, finite length representation of G .*

Proof. — The assertion follows from Proposition 4.23 together with (57). \square

Proposition 4.32. — *Assume that $(\tilde{E}/\mathrm{rad}\tilde{E}) \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is a finitely generated $\mathcal{O}[[H]]$ -module. Let \mathfrak{m}_L be a finite dimensional L -vector space with continuous \tilde{E} -action and let Π be a closed non-zero G -invariant subspace of $\Pi(\mathfrak{m}_L)$. Suppose that \mathfrak{m}_L is an irreducible right \tilde{E}_L -module, then $\mathfrak{m}(\Pi(\mathfrak{m}_L)/\Pi) = 0$ and $\mathfrak{m}(\Pi) \cong \mathfrak{m}(\Pi(\mathfrak{m}_L)) \cong \mathfrak{m}_L$. In particular, each π_i occurs in $\overline{\Pi}$ with the same (finite) multiplicity as in $\overline{\Pi}(\mathfrak{m}_L)$. Further, if S is irreducible and $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(S) = k$ then $\pi = S^\vee$ occurs in $\overline{\Pi}$ with multiplicity $\dim_L \mathfrak{m}_L$.*

Proof. — Let $\Pi(\mathfrak{m}_L)^0$ be the unit ball in $\Pi(\mathfrak{m}_L)$ with respect to the supremum norm and let $\Theta := \Pi \cap \Pi(\mathfrak{m}_L)^0$. Then Θ is an open bounded G -invariant lattice in Π . By [61, Prop. 1.3.iii] we have a surjection $\sigma : (\mathfrak{m} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\mathrm{tf}} \cong (\Pi(\mathfrak{m}_L)^0)^d \twoheadrightarrow \Theta^d$. As σ is non-zero, there exists $n \in \mathfrak{m}$ and $v \in \tilde{P}$ such that $\sigma(n \widehat{\otimes} v)$ is non-zero. Then $\psi : \tilde{P} \rightarrow \Theta^d$, $v \mapsto \sigma(n \widehat{\otimes} v)$ is a non-zero element of $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)$. Thus $\mathfrak{m}(\Pi) \neq 0$. Since the

functor m is exact and contravariant and m_L is an irreducible \tilde{E} -module we deduce that $m(\Pi(m_L)) \cong m(\Pi)$ and $m(\Pi(m_L)/\Pi) = 0$, which is a contradiction. The rest follows from Lemma 4.15 and (57). \square

Corollary 4.33. — *Assume the setup of Proposition 4.32 then $\Pi(m_L)$ contains a unique irreducible non-zero closed G -invariant subspace Π . Moreover, for any $\phi : \Pi(m_L) \rightarrow \Pi(m_L)$ continuous and G -equivariant we have $\phi(\Pi) \subseteq \Pi$.*

Proof. — This is immediate from Corollary 4.10 and Proposition 4.32. \square

Let $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ be the full subcategory of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ consisting of objects of finite length. Let $\text{Ker } m$ be the full subcategory of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ consisting of those Π such that $m(\Pi) = 0$. Since m is an exact functor, $\text{Ker } m$ is a thick subcategory of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ and hence we may build a quotient category $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m$, see [35, §III.1].

Theorem 4.34. — *Let \tilde{P} and \tilde{E} be as in the setup described before Proposition 4.18. Assume that*

- (i) $(\tilde{E}/\text{rad } \tilde{E}) \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is a finitely generated $\mathcal{O}[[H]]$ -module and is of finite length in $\mathfrak{C}(\mathcal{O})$;
- (ii) For every irreducible Π in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$, $m(\Pi)$ is finite dimensional.

Then the functors $m_L \mapsto \Pi(m_L)$ and $\Pi \mapsto m(\Pi)$ induce an anti-equivalence of categories between $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m$ and the category of finite dimensional L -vector spaces with continuous right \tilde{E} -action.

Proof. — Let $\mathcal{T} : \text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} \rightarrow \text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m$ be the natural functor. Recall that a morphism $\phi : \Pi_1 \rightarrow \Pi_2$ in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ induces an isomorphism $\mathcal{T}(\phi)$ in the quotient category if and only if $\text{Ker } \phi$ and $\text{Coker } \phi$ lie in $\text{Ker } m$ (that is $m(\text{Ker } \phi) = 0$ and $m(\text{Coker } \phi) = 0$), see Lemme 4 in [35, §III.1].

Since m is exact assumption (ii) implies that $m(\Pi)$ is finite dimensional for all Π in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$. Let Π be in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ and Θ be an open bounded G -invariant lattice in Π and let $m := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)$. Evaluation induces a morphism $m \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Theta^d$ in $\mathfrak{C}(\mathcal{O})$ and dually we obtain a morphism L -Banach spaces $\Pi \rightarrow \Pi(m(\Pi))$. We claim that the map $\mathcal{T}(\Pi) \rightarrow \mathcal{T}(\Pi(m(\Pi)))$ is an isomorphism. It is enough to prove the claim for irreducible Π , since then we get the rest by induction on the length of Π . The diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{T}(\Pi_1) & \longrightarrow & \mathcal{T}(\Pi_2) & \longrightarrow & \mathcal{T}(\Pi_3) \longrightarrow 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow \cong \\
 0 & \longrightarrow & \mathcal{T}(\Pi(m_1)) & \longrightarrow & \mathcal{T}(\Pi(m_2)) & \longrightarrow & \mathcal{T}(\Pi(m_3))
 \end{array}$$

where $m_i := m(\Pi_i)$ gives the induction step. We note that \mathcal{T} is exact by Proposition 1 in [35, §III.1] and hence the rows are exact.

Suppose that Π in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ is irreducible. If $\mathfrak{m}(\Pi) = 0$ then $\Pi \cong 0$ in the quotient category $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m}$ and hence $\mathcal{T}(\Pi) \cong \mathcal{T}(\Pi(\mathfrak{m}(\Pi)))$. Suppose that $\mathfrak{m}(\Pi) \neq 0$ then $\mathfrak{m}(\Pi)$ is an irreducible right $\tilde{\mathbb{E}}$ -module by Proposition 4.18(ii). By dualizing Proposition 4.18(iii) we obtain an injection $\iota : \Pi \hookrightarrow \Pi(\mathfrak{m}(\Pi))$ and it follows from Proposition 4.32 that $\mathfrak{m}(\Pi(\mathfrak{m}(\Pi)) / \Pi) = 0$. Hence, $\mathcal{T}(\iota)$ is an isomorphism between $\mathcal{T}(\Pi)$ and $\mathcal{T}(\Pi(\mathfrak{m}(\Pi)))$. For the other composition we observe that \mathfrak{m} factors through the quotient category, see [35, §III.1 Cor. 2], so $\mathfrak{m}(\mathcal{T}(\Pi(\mathfrak{m}_L))) \cong \mathfrak{m}(\Pi(\mathfrak{m}_L)) \cong \mathfrak{m}_L$, where the last assertion is given by Lemma 4.28. \square

Remark 4.35. — We note that since we assume that $(\tilde{\mathbb{E}} / \text{rad } \tilde{\mathbb{E}}) \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}$ is of finite length in $\mathfrak{C}(\mathcal{O})$, Lemma 4.31 implies that $\overline{\Pi(\mathfrak{m}_L)}$ is of finite length. The statement of Theorem 4.34 holds if instead of making the assumption (ii) we replace $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}}$ by a smaller category. Namely a full subcategory of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ with objects Π such that $\Theta \otimes_{\mathcal{O}} k$ is of finite length where Θ is an open bounded G -invariant lattice in Π . Such Π are of finite length and it follows from Lemma 4.15 that $\mathfrak{m}(\Pi)$ is finite dimensional. However, in the application to $\text{GL}_2(\mathbf{Q}_p)$ -representations we will verify that the assumption (ii) is satisfied using Proposition 4.20.

Theorem 4.36. — *Let $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}$ be as in the setup described before Proposition 4.18. Assume that*

- (i) $(\tilde{\mathbb{E}} / \text{rad } \tilde{\mathbb{E}}) \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}$ is a finitely generated $\mathcal{O}[[H]]$ -module and is of finite length in $\mathfrak{C}(\mathcal{O})$;
- (ii) the centre \mathcal{Z} of $\tilde{\mathbb{E}}$ is noetherian and $\tilde{\mathbb{E}}$ is a finitely generated \mathcal{Z} -module.

Then

$$\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m} \cong \bigoplus_{\mathfrak{n} \in \text{MaxSpec } \mathcal{Z}[1/p]} (\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m})_{\mathfrak{n}},$$

where the direct sum is taken over all the maximal ideals of $\mathcal{Z}[1/p]$, and for a maximal ideal \mathfrak{n} of $\mathcal{Z}[1/p]$, $(\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m})_{\mathfrak{n}}$ is the full subcategory of $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m}$, consisting of all Banach spaces which are killed by a power of \mathfrak{n} .

Further, the functor $\mathfrak{m} \mapsto \Pi(\mathfrak{m})$ induces an anti-equivalence of categories between the category of modules of finite length of the \mathfrak{n} -adic completion of $\tilde{\mathbb{E}}[1/p]$ and $(\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } \mathfrak{m})_{\mathfrak{n}}$.

Proof. — We claim that $\mathcal{Z}[1/p]/\mathfrak{n}$ is a finite extension of L for every maximal ideal \mathfrak{n} . For $1 \leq i \leq n$ let $e_i \in \tilde{\mathbb{E}}$ be orthogonal idempotents such that $e_i \tilde{\mathbb{P}} = \tilde{\mathbb{P}}_i$ and let \mathcal{Z}_i be the centre of $\text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_i)$. Since $\mathcal{Z} \subset \prod_{i=1}^n e_i \mathcal{Z} e_i \subset \prod_{i=1}^n \mathcal{Z}_i$, $\mathcal{Z}[1/p]/\mathfrak{n}$ will be a subfield of $\mathcal{Z}_i[1/p]/\mathfrak{n}_i$ for some $1 \leq i \leq n$ and some maximal ideal \mathfrak{n}_i of $\mathcal{Z}_i[1/p]$. It follows from the proof of Proposition 4.20 that $\mathcal{Z}_i[1/p]/\mathfrak{n}_i$ is a finite extension of L . Since $\tilde{\mathbb{E}}$ is a finitely generated \mathcal{Z} -module the claim implies that every irreducible $\tilde{\mathbb{E}}[1/p]$ -module is finite dimensional over L , see the proof of Proposition 4.20.

Proposition 4.20 says that the assumption (ii) in Theorem 4.34 is satisfied. Moreover, since \mathcal{Z} is noetherian and \tilde{E} is a finitely generated \mathcal{Z} -module we deduce that \tilde{E} is left and right noetherian and hence any \mathcal{O} -linear action of \tilde{E} on a finite dimensional L -vector space is automatically continuous by Lemma 4.22. Thus it follows from Theorem 4.34 and the claim that the functor $m \mapsto \Pi(m)$ induces an anti-equivalence of categories between the category of $\tilde{E}[1/p]$ -modules of finite length and $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m$.

Let \mathfrak{n} be a maximal ideal of $\mathcal{Z}[1/p]$ and let m be an \tilde{E} -module of finite length. It follows from the anti-equivalence that $\Pi(m)$ is an object of $(\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m)_{\mathfrak{n}}$ if and only if m is annihilated by a power of \mathfrak{n} , and, since m is of finite length and \mathfrak{n} is maximal, this is equivalent to $m = m_{\mathfrak{n}}$, the localization of m at \mathfrak{n} . As already observed, m is a finite dimensional L -vector space. Hence, the image of $\mathcal{Z}[1/p]$ in $\text{End}_L(m)$ is a finite dimensional L -algebra, which implies via the Chinese remainder theorem, that $m \cong \bigoplus_{\mathfrak{n}} m_{\mathfrak{n}}$, where the sum is taken over all the maximal ideals of $\mathcal{Z}[1/p]$, and $m_{\mathfrak{n}} = 0$ for almost all \mathfrak{n} . Applying the functor Π we deduce the last assertion. \square

Proposition 4.37. — *We assume the hypotheses of Theorem 4.36 and let \mathfrak{n} be a maximal ideal of $\mathcal{Z}[1/p]$ and $\mathfrak{n}_0 := \varphi^{-1}(\mathfrak{n})$, where $\varphi : \mathcal{Z} \rightarrow \mathcal{Z}[1/p]$. The irreducible objects of $(\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m)_{\mathfrak{n}}$ are precisely the irreducible Banach subrepresentations of $\text{Hom}_{\mathcal{O}}^{\text{cont}}((\tilde{P}/\mathfrak{n}_0\tilde{P})_{\text{tf}}, L)$.*

Proof. — Since \mathcal{Z} is noetherian \mathfrak{n}_0 is finitely generated and hence $\mathfrak{n}_0\tilde{P}$ is closed in \tilde{P} . Thus $\tilde{E}/\mathfrak{n}_0\tilde{E} \widehat{\otimes}_{\tilde{E}} \tilde{P} \cong \tilde{P}/\mathfrak{n}_0\tilde{P}$. Since \tilde{E} is a finitely generated \mathcal{Z} -module $\tilde{E}/\mathfrak{n}_0\tilde{E}$ is a finitely generated $\mathcal{Z}/\mathfrak{n}_0$ -module and so $(\tilde{E}/\mathfrak{n}_0\tilde{E})_{\text{tf}}$ is a finitely generated $(\mathcal{Z}/\mathfrak{n}_0)_{\text{tf}}$ -module. Now, $(\mathcal{Z}/\mathfrak{n}_0)_{\text{tf}}$ is equal to the image of \mathcal{Z} in $\mathcal{Z}[1/p]/\mathfrak{n}$ and hence is a finitely generated \mathcal{O} -module. We deduce that $(\tilde{E}/\mathfrak{n}_0\tilde{E})_{\text{tf}}$ is a free \mathcal{O} -module of finite rank. It follows from Lemma 4.25 that $((\tilde{E}/\mathfrak{n}_0\tilde{E})_{\text{tf}} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}$ is an \mathcal{O} -torsion free object of $\mathfrak{C}(\mathcal{O})$ and from Proposition 4.23 that it is finitely generated over $\mathcal{O}[[H]]$. It is immediate that any \mathcal{O} -linear homomorphism from $\tilde{E}/\mathfrak{n}_0\tilde{E} \widehat{\otimes}_{\tilde{E}} \tilde{P}$ to a torsion free \mathcal{O} -module must factor through $(\tilde{E}/\mathfrak{n}_0\tilde{E})_{\text{tf}} \widehat{\otimes}_{\tilde{E}} \tilde{P}$ and then through $((\tilde{E}/\mathfrak{n}_0\tilde{E})_{\text{tf}} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}$. We deduce that $(\tilde{P}/\mathfrak{n}_0\tilde{P})_{\text{tf}} \cong ((\tilde{E}/\mathfrak{n}_0\tilde{E})_{\text{tf}} \widehat{\otimes}_{\tilde{E}} \tilde{P})_{\text{tf}}$ is a finitely generated $\mathcal{O}[[H]]$ -module and is \mathcal{O} -torsion free, and so the Banach space representation $\Pi := \text{Hom}_{\mathcal{O}}^{\text{cont}}((\tilde{P}/\mathfrak{n}_0\tilde{P})_{\text{tf}}, L)$ is admissible.

Let Π_1 be a closed non-zero subspace of Π , irreducible as a Banach space representation of G . Let Π^0 be the unit ball in Π with respect to the supremum norm and let $\Pi_1^0 := \Pi_1 \cap \Pi^0$. Dually we obtain a surjection $\psi : (\tilde{P}/\mathfrak{n}_0\tilde{P})_{\text{tf}} \cong (\Pi^0)^d \twoheadrightarrow (\Pi_1^0)^d$. Composing ψ with the natural map $\tilde{P} \twoheadrightarrow (\tilde{P}/\mathfrak{n}_0\tilde{P})_{\text{tf}}$ we deduce that $m(\Pi_1) \neq 0$ and hence Π_1 is non-zero in the quotient category. Since \tilde{P} is projective we get a surjection of \tilde{E} -modules:

$$(\tilde{E}/\mathfrak{n}_0\tilde{E})_{\text{tf}} \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, (\tilde{P}/\mathfrak{n}_0\tilde{P})_{\text{tf}}) \twoheadrightarrow \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, (\Pi_1^0)^d),$$

where the first isomorphism follows from Lemma 4.25. Hence, \mathfrak{n} kills $m(\Pi_1)$ and so Π_1 is an object of $(\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m)_{\mathfrak{n}}$.

Conversely, let $\Pi_1 \in \text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ be irreducible with $m(\Pi_1) \neq 0$. Then Π_1 is non-zero in the quotient category. Suppose Π_1 is an object of $(\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm.fl}} / \text{Ker } m)_{\mathfrak{n}}$. Since $m(\Pi_1)$

is an irreducible $\tilde{\mathbb{E}}[1/p]$ -module by Proposition 4.18, \mathfrak{n} kills $\mathfrak{m}(\Pi_1)$. Let Θ be an open bounded G -invariant lattice in Π_1 , every $\psi \in \mathfrak{n}_0$ induces a map $\psi^* : \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d) \rightarrow \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d)$, which is zero after inverting p . Since Θ^d is \mathcal{O} -torsion free, so is $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d)$ and hence ψ^* is zero. We deduce that

$$\text{Hom}_{\mathfrak{C}(\mathcal{O})}((\tilde{\mathbb{P}}/\mathfrak{n}_0\tilde{\mathbb{P}})_{\text{tf}}, \Theta^d) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}/\mathfrak{n}_0\tilde{\mathbb{P}}, \Theta^d) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d).$$

Since $\mathfrak{m}(\Pi_1) \neq 0$, $\text{Hom}_{\mathfrak{C}(\mathcal{O})}((\tilde{\mathbb{P}}/\mathfrak{n}_0\tilde{\mathbb{P}})_{\text{tf}}, \Theta^d) \neq 0$ and dually $\text{Hom}_G^{\text{cont}}(\Pi_1, \Pi) \neq 0$. As both spaces are admissible and Π_1 is irreducible any such non-zero homomorphism induces an isomorphism between Π_1 and a closed subspace of Π . \square

Remark 4.38. — If we assume that \mathcal{Z} is noetherian, $\tilde{\mathbb{E}}$ is \mathcal{O} -torsion free and is a free module of finite rank over \mathcal{Z} then \mathcal{Z} is \mathcal{O} -torsion free, thus $\mathfrak{n}_0 = \mathcal{Z} \cap \mathfrak{n}$ and so $\mathcal{Z}/\mathfrak{n}_0$ is a free \mathcal{O} -module of finite rank, which implies $\tilde{\mathbb{E}}/\mathfrak{n}_0\tilde{\mathbb{E}}$ is a free \mathcal{O} -module of finite rank. If additionally we assume that $\tilde{\mathbb{P}}$ is flat over $\tilde{\mathbb{E}}$ then Corollary 3.14 implies that $\tilde{\mathbb{P}}/\mathfrak{n}_0\tilde{\mathbb{P}}$ is \mathcal{O} -torsion free. This situation will arise in the applications to $\text{GL}_2(\mathbf{Q}_p)$.

4.1. Relation to the deformation theory. — In this subsection we assume a more restrictive setup which will be used in the applications. Let $\tilde{\mathbb{P}}$ be a projective envelope of an irreducible object S in $\mathfrak{C}(\mathcal{O})$ such that $\text{End}_{\mathfrak{C}(\mathcal{O})}(S) = k$. Let $\tilde{\mathbb{E}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}})$ and $\pi := S^\vee$. Assume that there exists Q in $\mathfrak{C}(k)$ of finite length in $\mathfrak{C}(k)$, a finitely generated $\mathcal{O}[[H]]$ -module satisfying hypotheses (H1)–(H4) made in Section 3, (we do not assume (H5)). Then it follows from Lemma 3.7 that $(\tilde{\mathbb{E}}/\text{rad}\tilde{\mathbb{E}}) \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} \cong \tilde{\mathbb{P}}/(\text{rad}\tilde{\mathbb{E}})\tilde{\mathbb{P}} \cong Q$ and hence the hypothesis (i) in Theorem 4.34 is satisfied.

Remark 4.39. — If $G = \text{GL}_2(\mathbf{Q}_p)$ then it follows from the classification in [1] and [16] that every smooth irreducible k -representation of G with a central character is admissible and hence any smooth finite length k -representation of G with a central character is admissible. So the assumption that Q is finitely generated over $\mathcal{O}[[H]]$ will be automatically satisfied.

Theorem 4.40. — *There exists a natural bijection between isomorphism classes of*

- (i) *irreducible topological right $\tilde{\mathbb{E}}_L$ -modules, finite dimensional over L , and*
- (ii) *irreducible admissible unitary L -Banach space representations Π of G containing an open bounded G -invariant lattice Θ such that*
 - (a) $\Theta \otimes_{\mathcal{O}} k$ *is of finite length;*
 - (b) $\Theta \otimes_{\mathcal{O}} k$ *contains π as a subquotient;*
 - (c) Θ^d *is an object of $\mathfrak{C}(\mathcal{O})$.*

Proof. — We recall if the conditions are satisfied for one open bounded G -invariant lattice Θ then by Lemmas 4.3 and 4.6 they are satisfied for all such lattices inside Π .

Suppose we are given Π , containing such Θ , then $m(\Pi) := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d)_L$ does not depend on the choice of Θ and it follows from Proposition 4.18 that it is an irreducible $\tilde{\mathbb{E}}_L$ -module and from Lemma 4.15 that it is finite dimensional.

Given an irreducible $\tilde{\mathbb{E}}_L$ -module, finite dimensional over L , we may choose an $\tilde{\mathbb{E}}$ -invariant \mathcal{O} -lattice m inside it as $\tilde{\mathbb{E}}$ is compact. Let $\Pi(m)$ be the admissible unitary L -Banach space representation of G defined in 4.26. By Corollary 4.33 $\Pi(m)$ contains a unique closed irreducible G -invariant subspace of Π . Lemma 4.31 implies that $\bar{\Pi}$ is a G -representation of finite length.

It is shown at the end of the proof of Theorem 4.34 that we have a natural injection $\Pi \hookrightarrow \Pi(m(\Pi))$. This fact together with Propositions 4.32 implies that the two maps are mutually inverse. \square

Corollary 4.41. — *Let Π be an irreducible admissible unitary L -Banach space representation of G containing an open bounded G -invariant lattice Θ such that Θ^d is an object of $\mathfrak{C}(\mathcal{O})$ and π is a subquotient of $\Theta \otimes_{\mathcal{O}} k$. If the centre \mathcal{Z} of $\tilde{\mathbb{E}}$ is noetherian and $\tilde{\mathbb{E}}$ is a finitely generated \mathcal{Z} -module then $\Theta \otimes_{\mathcal{O}} k$ is of finite length as a G -representation.*

Proof. — Proposition 4.20 implies that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d)_L$ is finite dimensional over L and the assertion follows from Theorem 4.40. \square

Corollary 4.42. — *Let Π be as in Theorem 4.40 and $m := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d)$, where Θ is an open bounded G -invariant lattice in Π , then the following are equivalent:*

- (i) $\text{End}_{L[G]}^{\text{cont}}(\Pi) = L$;
- (ii) $\text{End}_{\tilde{\mathbb{E}}_L}(m_L) = L$;
- (iii) m_L is an absolutely irreducible right $\tilde{\mathbb{E}}_L$ -module;
- (iv) Π is an absolutely irreducible L -Banach space representation of G .

Proof. — It follows from Proposition 4.19 that $\text{End}_{L[G]}^{\text{cont}}(\Pi) \cong \text{End}_{\tilde{\mathbb{E}}_L}(m_L)^{\text{op}}$. Hence (i) is equivalent to (ii). The assumptions on Π made in Theorem 4.40 imply that m_L is finite dimensional. Hence (ii) is equivalent to (iii), see [11, Cor. 12.4]. Moreover, we deduce that $\text{End}_{L[G]}^{\text{cont}}(\Pi)$ is finite dimensional over L and so we deduce from Lemma 4.1 that (iv) implies (i). Finally Lemma 4.2 says that (i) implies (iv). \square

Corollary 4.43. — *Let Π and Θ be as in Theorem 4.40. If Π is absolutely irreducible then the image of the centre of $\tilde{\mathbb{E}}$ in $\text{End}_{\mathcal{O}}(\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d))$ is equal to \mathcal{O} .*

Proof. — The image of \mathcal{Z} contains \mathcal{O} and is contained in $\text{End}_{\tilde{\mathbb{E}}}(\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d))$, which is isomorphic to \mathcal{O} by Corollary 4.42. \square

Corollary 4.44. — *Let Π be an absolutely irreducible admissible L -Banach space representation of G containing an open bounded G -invariant lattice Θ such that Θ^d is an object of $\mathfrak{C}(\mathcal{O})$ and π is a subquotient of $\Theta \otimes_{\mathcal{O}} k$. If $\tilde{\mathbb{E}}$ is commutative then $\bar{\Pi} \subseteq (Q^\vee)^{\text{ss}}$.*

Proof. — As a consequence of the hypotheses (H1)–(H4) we know that the maximal ideal of \tilde{E} is generated by at most $1 + \dim_k \text{Ext}_{\mathfrak{C}(k)}^1(\mathcal{Q}, \mathcal{S})$ elements, see Lemma 3.7 and Proposition 3.8(iii), which implies that \tilde{E} is noetherian. We note that the proof of Lemma 3.7 and Proposition 3.8(iii) for $n = 1$ does not use (H5). Let Ξ and \mathfrak{a} be as in Proposition 4.18 then $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Xi^d)_L$ is finite dimensional by Proposition 4.20. Since \tilde{E} is commutative and Π is absolutely irreducible it follows from Corollary 4.43 that $\tilde{E}/\mathfrak{a} \cong \mathcal{O}$. Tensoring the surjection $\mathcal{O} \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Xi^d$ with k we obtain a surjection $\mathcal{Q} \cong k \widehat{\otimes}_{\tilde{E}} \tilde{P} \rightarrow \Xi^d \otimes_{\mathcal{O}} k$. Thus $\overline{\Pi} \cong (\Xi \otimes_{\mathcal{O}} k)^{ss} \cong ((\Xi^d \otimes_{\mathcal{O}} k)^\vee)^{ss} \subseteq (\mathcal{Q}^\vee)^{ss}$. \square

4.2. Extensions of Banach space representations. — Let $\text{Mod}_{\mathbb{G}}^2(\mathcal{O})$ be a full subcategory of $\text{Mod}_{\mathbb{G}}^{\text{fin}}(\mathcal{O})$ closed under subquotients and arbitrary direct sums in $\text{Mod}_{\mathbb{G}}^{\text{fin}}(\mathcal{O})$. Let $\mathfrak{C}(\mathcal{O})$ be a full subcategory of $\text{Mod}_{\mathbb{G}}^{\text{proaug}}(\mathcal{O})$ anti-equivalent to $\text{Mod}_{\mathbb{G}}^2(\mathcal{O})$ via Pontryagin duality. Assume that $\text{Mod}_{\mathbb{G}}^2(\mathcal{O})$ has only finitely many irreducible objects π_1, \dots, π_n , which are admissible. Let \tilde{P} be a projective envelope of $\pi_1^\vee \oplus \dots \oplus \pi_n^\vee$ in $\mathfrak{C}(\mathcal{O})$, and let $\tilde{E} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P})$. It follows from [35, §IV.4, Cor. 1] that the functor $M \mapsto \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, M)$ induces an equivalence of categories between $\mathfrak{C}(\mathcal{O})$ and the category of compact right \tilde{E} -modules, with the inverse functor given by $m \mapsto m \widehat{\otimes}_{\tilde{E}} \tilde{P}$. This implies that $\tilde{E}/\text{rad } \tilde{E} \widehat{\otimes}_{\tilde{E}} \tilde{P} \cong \pi_1^\vee \oplus \dots \oplus \pi_n^\vee$, which is a finitely generated $\mathcal{O}[[\mathbb{H}]]$ -module, as π_i are assumed to be admissible. We further assume that the centre \mathcal{Z} of \tilde{E} is noetherian, and \tilde{E} is a finitely generated module over \mathcal{Z} . Let $\text{Mod}_{\tilde{E}[1/\rho]}^{\text{fg}}$ be the category of finitely generated right $\tilde{E}[1/\rho]$ -modules.

Lemma 4.45. — *The functor $m : \text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}} \rightarrow \text{Mod}_{\tilde{E}[1/\rho]}^{\text{fg}}$ is fully faithful.*

Proof. — Lemma 4.9 and Proposition 4.17 show that m is well defined. It remains to show that it is fully faithful. Let Π_1, Π_2 be in $\text{Ban}_{\mathfrak{C}(\mathcal{O})}^{\text{adm}}$ and let Θ_1 and Θ_2 be open bounded \mathbb{G} -invariant lattices in Π_1 and Π_2 , respectively. Then Θ_1^d and Θ_2^d are objects of $\mathfrak{C}(\mathcal{O})$ by Lemma 4.6. For $i = 1$ and $i = 2$ let $m_i := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta_i^d)$, then, because of equivalence of categories explained above, we have $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\Theta_2^d, \Theta_1^d) \cong \text{Hom}_{\tilde{E}}(m_2, m_1)$. Since $\text{Hom}_{\mathbb{G}}(\Pi_1, \Pi_2) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\Theta_2^d, \Theta_1^d) \otimes_{\mathcal{O}} \mathbb{L}$ by [61], and $m(\Pi_i) = m_i \otimes_{\mathcal{O}} \mathbb{L}$, we deduce the result. \square

Proposition 4.46. — *Let \mathbb{R} be a ring, \mathcal{A} the category of finitely generated (right) modules of \mathbb{R} , and let \mathcal{B} be a full subcategory of \mathcal{A} containing all the modules of finite length and closed under extensions and subquotients in \mathcal{A} . Let \mathcal{Z} be the centre of \mathbb{R} . If \mathcal{Z} is noetherian and \mathbb{R} is a finitely generated \mathcal{Z} -module, then for every $A, B \in \mathcal{B}$ with B a module of finite length, the natural map between the Yoneda-Ext groups:*

$$\varphi^n : \text{Ext}_{\mathcal{B}}^n(A, B) \rightarrow \text{Ext}_{\mathcal{A}}^n(A, B)$$

is an isomorphism, for all $n \geq 0$.

Proof. — Since Z is noetherian, and R is a finitely generated Z -module, R is left and right noetherian. Hence \mathcal{A} is an abelian category. Since \mathcal{B} is a full subcategory, closed under subquotients in \mathcal{A} , \mathcal{B} is also an abelian category.

If φ^n is bijective for a given n and all $A, B \in \mathcal{B}$ then φ^{n+1} is injective for all $A, B \in \mathcal{B}$, see [52, Prop. 3.3]. Moreover, φ^0 and φ^1 are bijective by assumption. So it is enough to show that φ^n is surjective for $n \geq 2$. Let $0 \rightarrow B \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow A \rightarrow 0$ be an extension representing $\xi \in \text{Ext}_{\mathcal{A}}^n(A, B)$. Let I be the Z -annihilator of B , then by Artin-Rees lemma, there exists a positive integer c , such that $B \cap I^c X_1 = 0$. Since B is of finite length, Z/I^c is a Z -module of finite length, and hence $X_1/I^c X_1$ is an R -module of finite length. We thus may represent ξ with the extension $0 \rightarrow B \rightarrow X_1/I^c X_1 \rightarrow X_2/I^c X_1 \rightarrow \cdots \rightarrow A \rightarrow 0$. Arguing inductively, we deduce that ξ can be represented by an extension in \mathcal{B} , and so φ^n is surjective for $n \geq 2$. \square

Remark 4.47. — The upshot of Proposition 4.46 is that \mathcal{A} has enough projectives and the Yoneda Ext-groups can be calculated using projective resolutions.

Corollary 4.48. — *Let Π_1 and $\Pi_2 \in \text{Ban}_{\mathfrak{e}(\mathcal{O})}^{\text{adm}}$, with Π_1 of finite length. The functor m induces an isomorphism*

$$\text{Ext}_{\mathbb{G}}^i(\Pi_1, \Pi_2) \cong \text{Ext}_{\mathbb{E}[1/\rho]}^i(m(\Pi_2), m(\Pi_1))$$

between the Yoneda Ext-groups computed in $\text{Ban}_{\mathfrak{e}(\mathcal{O})}^{\text{adm}}$ and in $\text{Mod}_{\mathbb{E}[1/\rho]}^{\text{fg}}$, respectively.

Proof. — We apply Proposition 4.46 with $R = \tilde{\mathbb{E}}[1/\rho]$, $\mathcal{A} = \text{Mod}_{\mathbb{E}[1/\rho]}^{\text{fg}}$, and \mathcal{B} the full subcategory with objects all the finitely generated $\tilde{\mathbb{E}}[1/\rho]$ -modules, which are isomorphic to $m(\Pi)$, with $\Pi \in \text{Ban}_{\mathfrak{e}(\mathcal{O})}^{\text{adm}}$. Theorem 4.36 implies that \mathcal{B} contains all the modules of finite length. Let m be a finitely generated $\tilde{\mathbb{E}}[1/\rho]$ -module, and let m^0 be a finitely generated $\tilde{\mathbb{E}}$ -submodule, which is an \mathcal{O} -lattice in m . If $m^0 \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{P}$ is finitely generated over $\mathcal{O}[[H]]$, then $\Pi(m) := \text{Hom}_{\mathcal{O}}^{\text{cont}}(m^0 \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{P}, L)$ is an admissible Banach space representation of G , and $m(\Pi(m)) \cong m$. Since $\mathcal{O}[[H]]$ is noetherian, this implies that \mathcal{B} is closed under extensions and subquotients in \mathcal{A} . Lemma 4.45 implies that m induces an equivalence of categories between $\text{Ban}_{\mathfrak{e}(\mathcal{O})}^{\text{adm}}$ and \mathcal{B} . \square

Remark 4.49. — The assumptions made in this subsection are satisfied if $G = \text{GL}_2(\mathbf{Q}_p)$, $p \geq 5$ and $\text{Mod}_{\mathbb{G}}^{\text{fg}}(\mathcal{O})$ is a block in the category of smooth locally finite representations of G with a fixed central character, see Section 5.5. Further, for each block we will compute the ring $\tilde{\mathbb{E}}$ and show that it satisfies the assumptions made in this subsection. Since the decomposition into blocks is functorial, there are no extensions between Banach space representations lying in different blocks, so Corollary 4.48 will enable us to compute the Ext-groups in the category of admissible unitary Banach space representations of $\text{GL}_2(\mathbf{Q}_p)$ with a fixed central character.

5. Representations of $\mathrm{GL}_2(\mathbf{Q}_p)$

5.1. Notation. — Let $G := \mathrm{GL}_2(\mathbf{Q}_p)$, let P be the subgroup of upper-triangular matrices, T the subgroup of diagonal matrices, U be the unipotent upper triangular matrices and $K := \mathrm{GL}_2(\mathbf{Z}_p)$. Let $\mathfrak{p} := \mathfrak{p}\mathbf{Z}_p$ and

$$I := \begin{pmatrix} \mathbf{Z}_p^\times & \mathbf{Z}_p \\ \mathfrak{p} & \mathbf{Z}_p^\times \end{pmatrix}, \quad I_1 := \begin{pmatrix} 1 + \mathfrak{p} & \mathbf{Z}_p \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}, \quad K_1 := \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}.$$

For $\lambda \in \mathbf{F}_p$ we denote the Teichmüller lift of λ to \mathbf{Z}_p by $[\lambda]$. Set

$$H := \left\{ \begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} : \lambda, \mu \in \mathbf{F}_p^\times \right\}.$$

Let $\varepsilon : \mathbf{Q}_p \rightarrow \mathbf{L}$, $x \mapsto x|x|$, $\omega : \mathbf{Q}_p \rightarrow k$, $x \mapsto x|x| \pmod{\mathfrak{p}_L}$, where $|\cdot|$ is a norm on \mathbf{Q}_p with $|\mathfrak{p}| = \frac{1}{p}$, and $\alpha : T \rightarrow k^\times$ be the character

$$\alpha \left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \right) := \omega(\lambda\mu^{-1}).$$

Further, define

$$\Pi := \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

For $\lambda \in k^\times$ we define an unramified character $\mu_\lambda : \mathbf{Q}_p^\times \rightarrow k^\times$, by $x \mapsto \lambda^{\mathrm{val}(x)}$. Given two characters $\chi_1, \chi_2 : \mathbf{Q}_p^\times \rightarrow k^\times$ we consider $\chi_1 \otimes \chi_2$ as a character of P , which sends $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ to $\chi_1(a)\chi_2(d)$.

Let Z be the centre of G , and set $Z_1 := Z \cap I_1$. Let $G^0 := \{g \in G : \det g \in \mathbf{Z}_p^\times\}$ and set $G^+ := ZG^0$.

Let \mathcal{G} be a topological group. We denote by $\mathrm{Hom}(\mathcal{G}, k)$ the continuous group homomorphism from \mathcal{G} to $(k, +)$. If \mathcal{V} is a representation of \mathcal{G} and S is a subset of \mathcal{V} we denote by $\langle \mathcal{G} \cdot S \rangle$ the smallest subspace of \mathcal{V} containing S and stable under the action of \mathcal{G} . The socle $\mathrm{soc}_{\mathcal{G}} \mathcal{V}$ is the maximal semi-simple \mathcal{G} -subrepresentation of \mathcal{V} . The socle filtration $\mathrm{soc}_{\mathcal{G}}^i \mathcal{V} \subseteq \mathcal{V}$ is defined by an exact sequence $0 \rightarrow \mathrm{soc}_{\mathcal{G}}^i \mathcal{V} \rightarrow \mathrm{soc}_{\mathcal{G}}^{i+1} \mathcal{V} \rightarrow \mathrm{soc}_{\mathcal{G}}(\mathcal{V}/\mathrm{soc}_{\mathcal{G}}^i \mathcal{V}) \rightarrow 0$, for $i \geq 0$ and $\mathrm{soc}_{\mathcal{G}}^0 \mathcal{V} := 0$.

We make the same conventions as in [23] regarding local class field theory: if Λ is a topological ring let $\widehat{T}(\Lambda)$ be the set of continuous characters $\delta : \mathbf{Q}_p^\times \rightarrow \Lambda^\times$. Local class field theory gives us an isomorphism of topological groups between the abelianisation $W_{\mathbf{Q}_p}^{ab}$ of the Weil group $W_{\mathbf{Q}_p}$ of \mathbf{Q}_p and \mathbf{Q}_p^\times . This enables us to consider an element

$\delta \in \widehat{\mathcal{T}}(\Lambda)$ as a continuous character of $W_{\mathbf{Q}_p}$ by the formula:

$$(60) \quad \delta(g) = \delta(p)^{-\deg(g)} \delta(\varepsilon(g)), \quad \forall g \in W_{\mathbf{Q}_p},$$

where $\deg(g)$ is an integer defined by $g(x) = x^{p^{\deg(g)}}$, for all $x \in \overline{\mathbf{F}}_p$, and ε is the cyclotomic character. Since $\mathcal{G}_{\mathbf{Q}_p}$ is isomorphic to the profinite completion of $W_{\mathbf{Q}_p}$, the character δ defined by (60) extends to a continuous character $\delta : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \Lambda^\times$ if and only if $n \mapsto \delta(p^n)$ extends continuously to $\widehat{\mathbf{Z}}$. This is the case if $\Lambda = k$ or $\Lambda = \mathbf{L}$ and δ is unitary. The formula (60) identifies the cyclotomic character with the character $\mathbf{Q}_p^\times \rightarrow \mathbf{Z}_p^\times$, $x \mapsto x|x|$, which is also denoted by ε above.

5.2. Rationality.

Lemma 5.1. — *Let G be a group, K a field and L a field extension of K . Let V and W be $K[G]$ -modules. If either V is finitely generated over $K[G]$ or L is finite over K then the natural injection*

$$(61) \quad \mathrm{Hom}_{K[G]}(V, W) \otimes_K L \hookrightarrow \mathrm{Hom}_{L[G]}(V \otimes_K L, W \otimes_K L)$$

is an isomorphism. In particular, $W^G \otimes_K L \cong (W \otimes_K L)^G$.

Proof. — If V is finitely generated over $K[G]$ we have an exact sequence of $K[G]$ -modules $0 \rightarrow U \rightarrow K[G]^{\oplus n} \rightarrow V \rightarrow 0$. We obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{K[G]}(V, W)_L & \longrightarrow & (W^{\oplus n})_L & \longrightarrow & \mathrm{Hom}_{K[G]}(U, W)_L \\ & & \downarrow & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_{L[G]}(V_L, W_L) & \longrightarrow & (W_L)^{\oplus n} & \longrightarrow & \mathrm{Hom}_{L[G]}(U_L, W_L). \end{array}$$

Since the third vertical arrow is injective and the second is an isomorphism we deduce that the first is also an isomorphism. Since $W^G \cong \mathrm{Hom}_{K[G]}(\mathbf{1}, W)$ we deduce $W^G \otimes_K L \cong (W \otimes_K L)^G$.

Let V be arbitrary. The group G acts naturally on $\mathrm{Hom}_K(V, W)$ by conjugation. If L is finite over K then we have $\mathrm{Hom}_K(V, W)_L \cong \mathrm{Hom}_L(V_L, W_L)$, see for example Proposition 16(i) in §II.7.7 of [14]. Since by the previous part taking G -invariants commutes with the tensor product with L we deduce that (61) is an isomorphism. \square

Remark 5.2. — In the foundational papers [1], [16], [17], [67] the authors study representation theory over an algebraically closed field. Using the Lemma one may show that their results also hold over an extension of \mathbf{F}_p , provided the extension is “large enough”, see Lemma 5.10. Lemma 5.1 will allow us to deduce various results on Ext-groups between irreducible representations over k from the corresponding results over algebraically closed fields, which have already appeared in the literature, see Remark 5.15, Lemma 5.32, Proposition 5.33.

5.3. Irreducible representations. — We recall the classification of the (absolutely) irreducible smooth k -representations of G with a central character⁵ due to Barthel-Livné [1] and Breuil [16]. We then show that the category $\text{Mod}_{G,\zeta}^{\text{lfm}}(k)$ behaves well when we replace k by an extension. We let k be an arbitrary field of characteristic p until Proposition 5.11, from then onwards k is a finite field, which is the situation we are most interested in. This assumption is made for the sake of simplicity, one has to work harder if k is not a perfect field, see Remark 5.4. We assume from Section 5.4 onwards that k contains a square root of $\zeta(p)$, where ζ is the fixed central character.

Let σ be an irreducible smooth representation of K . Since K_1 is a normal pro- p subgroup of K , σ^{K_1} is non-zero and since σ is irreducible we deduce that K_1 acts trivially. Hence σ is an irreducible representation of $K/K_1 \cong \text{GL}_2(\mathbf{F}_p)$ and so $\sigma \cong \text{Sym}^r k^2 \otimes \det^a$ for uniquely determined integers $0 \leq r \leq p-1$ and $0 \leq a \leq p-2$. We also note that this implies that σ is absolutely irreducible and can be defined over \mathbf{F}_p .

Let $\zeta : Z \rightarrow k^\times$ be a smooth character extending the central character of σ . We extend the action of K on σ to the action of KZ by making p act by a scalar $\zeta(p)$. It is shown in [1, Prop. 8] that there exists an isomorphism of algebras:

$$(62) \quad \text{End}_G(\text{c-Ind}_{KZ}^G \sigma) \cong k[T]$$

for a certain Hecke operator $T \in \text{End}_G(\text{c-Ind}_{KZ}^G \sigma)$ defined in [1, §3].

Proposition 5.3. — *Let π be a smooth irreducible k -representation of G with a central character ζ . There exists a finite extension l of k such that $\pi \otimes_k l$ is of finite length and all the irreducible subquotients are absolutely irreducible in the sense of Remark 5.12(iii).*

Proof. — Following the proof of Proposition 32 in [1] we deduce that π is a quotient of $\text{c-Ind}_{KZ}^G \sigma / P(T) \text{c-Ind}_{KZ}^G \sigma$, where $P \in k[T]$ is a non-zero polynomial, irreducible over k and T is as in (62). We know that the assertion holds if $P(T) = T - \lambda$, for some $\lambda \in k$, by [1] if $\lambda \neq 0$ and [16] if $\lambda = 0$. We may take l to be the splitting field of P . \square

Remark 5.4. — If k is perfect then the same proof shows that for every smooth irreducible k -representation of G with a central character ζ there exist a finite extension l of k such that $\pi \otimes_k l$ is isomorphic to a finite direct sum of absolutely irreducible representations.

Corollary 5.5. — *If π is an object of $\text{Mod}_{G,\zeta}^{\text{lfm}}(k)$ then $\pi \otimes_k l$ is an object of $\text{Mod}_{G,\zeta}^{\text{lfm}}(l)$.*

Proof. — Given $v \in \pi \otimes_k l$ we may express $v = \sum_{i=1}^n \lambda_i v_i$ with $v_i \in \pi$ and $\lambda_i \in l$. Hence, we may assume that π is of finite length. Proposition 5.3 implies that $\pi \otimes_k l$ is of finite length. \square

⁵ Laurent Berger has shown recently in [6], that every smooth irreducible representation of G over an algebraically closed field of characteristic p admits a central character.

Corollary 5.6. — *Every smooth finite length k -representation of G with a central character is admissible.*

Proof. — It follows from the classification, see [1] and [16], that every absolutely irreducible representation is admissible. The assertion follows from Proposition 5.3 and Lemma 5.1. \square

Lemma 5.7. — *Let π and τ be objects of $\text{Mod}_{G,\zeta}^{\text{fin}}(k)$ (resp. $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$) and let l be a field extension of k . If π is finitely generated over G then the natural map*

$$(63) \quad \text{Ext}_{k[G],\zeta}^1(\pi, \tau) \otimes_k l \rightarrow \text{Ext}_{l[G],\zeta}^1(\pi \otimes_k l, \tau \otimes_k l),$$

is injective, where Ext^1 are computed in the corresponding categories.

Proof. — In terms of Yoneda Ext the map is given by sending an extension $0 \rightarrow \tau \rightarrow \kappa \rightarrow \pi \rightarrow 0$ to $0 \rightarrow \tau_l \rightarrow \kappa_l \rightarrow \pi_l \rightarrow 0$. Since π is assumed to be finitely generated over $k[G]$, it follows from Lemma 5.1 that $\text{Hom}_G(\pi_l, \kappa_l) = \text{Hom}_G(\pi, \kappa)_l$, hence any splitting of $0 \rightarrow \tau_l \rightarrow \kappa_l \rightarrow \pi_l \rightarrow 0$ is already defined over k . Thus the map is an injective. \square

Let \bar{k} be the algebraic closure of k . It follows from [1, Thm. 33] and [16, Thm. 1.1] that the irreducible smooth \bar{k} -representations of G with a central character fall into four disjoint classes:

- (i) characters, $\eta \circ \det$;
- (ii) special series, $\text{Sp} \otimes \eta \circ \det$;
- (iii) principal series $\text{Ind}_p^G \chi_1 \otimes \chi_2$, with $\chi_1 \neq \chi_2$;
- (iv) supersingular $c\text{-Ind}_{KZ}^G \sigma / (T)$.

The Steinberg representation Sp is defined by the exact sequence:

$$(64) \quad 0 \rightarrow \mathbf{1} \rightarrow \text{Ind}_p^G \mathbf{1} \rightarrow \text{Sp} \rightarrow 0.$$

Definition 5.8. — *Let π be a \bar{k} -representation of a group G and l a subfield of \bar{k} . We say that π can be defined over l if there exists an l -representation τ of G such that $\tau \otimes_l \bar{k} \cong \pi$. We say that l is a field of definition of π if it is the smallest subfield of \bar{k} over which π can be defined.*

Lemma 5.9. — *Let $\chi : \mathbf{Q}_p^\times \rightarrow \bar{k}^\times$ be a smooth character then the field of definition of χ is $\mathbf{F}_p[\chi(p)]$.*

Proof. — Since χ is smooth it is trivial on $1 + p\mathbf{Z}_p$ and hence $\chi(\mathbf{Z}_p^\times) \subseteq \mathbf{F}_p^\times$, the group of $(p-1)$ -st roots of unity in \bar{k} . Since $\mathbf{Q}_p^\times \cong \mathbf{Z}_p^\times \times p^{\mathbf{Z}}$ the assertion follows. \square

Lemma 5.10. — *Let π be a smooth irreducible \bar{k} -representation of G with a central character ζ . Then there exists a smallest subfield l of \bar{k} over which π can be defined. Moreover,*

- (i) *if $\pi \cong \eta \circ \det$ then $l = \mathbf{F}_p[\eta(p)]$;*
- (ii) *if $\pi \cong \mathrm{Sp} \otimes \eta \circ \det$ then $l = \mathbf{F}_p[\eta(p)]$;*
- (iii) *if $\pi \cong \mathrm{Ind}_p^G \chi_1 \otimes \chi_2$ then $l = \mathbf{F}_p[\chi_1(p), \chi_2(p)]$;*
- (iv) *if π is supersingular then $l = \mathbf{F}_p[\zeta(p)]$.*

Proof. — Let l be a subfield of \bar{k} and τ an l -representation of G such that $\tau \otimes_l \bar{k} \cong \pi$. Since π is irreducible τ is irreducible and hence it follows from Lemma 5.1 that τ is uniquely determined up to an isomorphism over l . As already mentioned π^{I_1} is finite dimensional (and non-zero), this implies $\mathrm{End}_G(\pi) \cong \bar{k}$. We deduce from Lemma 5.1 that $\mathrm{End}_G(\tau) \cong l$. Thus Z acts on τ by a central character. We deduce that $\zeta(p) \in l$.

If π is supersingular then we are done since $\sigma|_{\mathbf{K}}$ can be defined over \mathbf{F}_p , σ can be defined over $\mathbf{F}_p[\zeta(p)]$ by using $\mathbf{KZ} \cong \mathbf{K} \times p^{\mathbf{Z}}$ and the endomorphism \mathbf{T} can also be defined over $\mathbf{F}_p[\zeta(p)]$, as is immediate from [1, §3].

If π is a character or special series then π^{I_1} is 1-dimensional. Lemma 5.1 implies that τ^{I_1} is 1-dimensional. Since $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ acts on π^{I_1} by a scalar $\pm\eta(p)$, we deduce that $\eta(p) \in l$ and hence $\mathbf{F}_p[\eta(p)]$ is a field of definition of π . We note that it is immediate from (64) that Sp can be defined over \mathbf{F}_p .

If π is principal series then π^{I_1} is 2-dimensional with basis $\{\varphi_1, \varphi_2\}$, where $\mathrm{Supp} \varphi_1 = \mathrm{PsI}_1$, $\varphi_1(s) = 1$, $\mathrm{Supp} \varphi_2 = \mathrm{PI}_1$, $\varphi_2(1) = 1$. The \mathbf{K} -representation $\sigma := \langle \mathbf{K} \cdot \varphi_1 \rangle \subset \pi$ is irreducible, $\sigma^{I_1} = k\varphi_1$ and $\mathrm{Hom}_{\mathbf{K}}(\sigma, \pi)$ is 1-dimensional. Now σ can be realized over \mathbf{F}_p , so in particular over k . It follows from Lemma 5.1 that $\mathrm{Hom}_{\mathbf{K}}(\sigma, \tau)$ is 1-dimensional. Choose a non-zero $\phi \in \mathrm{Hom}_{\mathbf{K}}(\sigma, \tau)$ and let $v \in \phi(\sigma)^{I_1}$ be non-zero. The 1-dimensionality of the spaces involved implies that any G -equivariant isomorphism $\pi \cong \tau \otimes_l \bar{k}$ must map φ_1 to $v \otimes \lambda$ for some $\lambda \in \bar{k}$. A direct calculation shows that $\sum_{\lambda \in \mathbf{F}_p} \begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix} \varphi_1 = \chi_2(p)\varphi_1$, hence $\sum_{\lambda \in \mathbf{F}_p} \begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix} v = \chi_2(p)v$ and so $\chi_2(p) \in l$. Since $\zeta(p) = \chi_1(p)\chi_2(p)$ we deduce that $\chi_1(p) \in l$. Hence, $\mathbf{F}_p[\chi_1(p), \chi_2(p)]$ is the field of definition of π . Further, since both χ_1 and χ_2 maybe defined over $\mathbf{F}_p[\chi_1(p), \chi_2(p)]$ by Lemma 5.9 we deduce that τ is a principal series representation. \square

Let $\chi : \mathbf{T} \rightarrow \bar{k}^\times$ be a smooth character and let \mathbf{X} be the orbit of χ under the action of $\Gamma := \mathrm{Gal}(\bar{k}/k)$. It follows from Lemma 5.9 that χ can be defined over a finite extension of k and so \mathbf{X} is finite. Let

$$V_\chi := \left(\bigoplus_{\psi \in \mathbf{X}} \psi \right)^\Gamma,$$

where the action of Γ on $\bigoplus_{\psi \in \mathbf{X}} \psi$ is given by $\gamma \cdot (\lambda_\psi)_\psi := (\gamma(\lambda_\psi))_{\gamma(\psi)}$. Then V_χ is the unique irreducible k -representation of \mathbf{T} such that $V_\chi \otimes_k \bar{k}$ contains χ . We note that if χ factors through the determinant, then we may consider both χ and V_χ as representations of G .

Proposition 5.11. — *Let π be an irreducible smooth k -representation of G with a central character. Then π is isomorphic to one of the following:*

- (i) $V_{\eta_{\text{odet}}}, \eta : \mathbf{Q}_p^\times \rightarrow \bar{k}^\times$;
- (ii) $\text{Sp} \otimes V_{\eta_{\text{odet}}}, \eta : \mathbf{Q}_p^\times \rightarrow \bar{k}^\times$;
- (iii) $\text{Ind}_P^G V_\chi, \chi : T \rightarrow \bar{k}^\times$ with $\chi \neq \chi^s$;
- (iv) *supersingular $c\text{-Ind}_{\text{KZ}}^G \sigma / (T)$.*

Proof. — Since k is perfect it follows from the proof of Proposition 5.3 that there exists a finite Galois extension l of k such that

$$\pi \otimes_k l \cong \pi_1 \oplus \cdots \oplus \pi_n$$

with π_i absolutely irreducible and distinct. If π_1 is supersingular then, since the central character of π_1 is k -rational, π_1 can be realized over k by Lemma 5.10 and since π is irreducible Lemma 5.1 implies that $\pi_l \cong \pi_1$ and so π is absolutely irreducible supersingular.

The proof in the cases π_1 is a character, special series or principal series is the same. We only treat the principal series case so $\pi_1 \cong \text{Ind}_P^G \chi$ with $\chi : T \rightarrow \bar{k}^\times$ a smooth character with $\chi \neq \chi^s$. Let $\tau := \text{Ind}_P^G V_\chi$, then $\tau_l \cong \bigoplus_{\psi \in X} \text{Ind}_P^G \psi$. Since $\chi \neq \chi^s$ we have $\psi \neq \psi^s$ for every $\psi \in X$. Hence, all the principal series are irreducible and distinct. Since the Γ -action on $\tau \otimes_k l$ permutes the irreducible subspaces transitively we deduce that τ_l does not contain a proper G -invariant subspace, which is stable under the action of Γ . Hence τ is an irreducible G -representation. Since $\text{Hom}_G(\tau_l, \pi_l) \neq 0$ Lemma 5.1 implies that $\text{Hom}_G(\tau, \pi) \neq 0$. Since both π and τ are irreducible they must be isomorphic. \square

Remark 5.12. — It follows from the Proposition 5.11 that for an irreducible π with a central character the following are equivalent:

- (i) $\pi \otimes_k l$ is irreducible for all l/k finite;
- (ii) $\pi \otimes_k l$ is irreducible for all l/k ;
- (iii) $\pi \otimes_k l$ irreducible for some l/k with l algebraically closed.

In this case, we will say that π is *absolutely irreducible*.

Suppose $p \in Z$ acts trivially on σ and $\sigma|_K \cong \text{Sym}^r k^2$. Let $\varphi \in c\text{-Ind}_{\text{KZ}}^G \text{Sym}^r k^2$ be such that $\text{Supp } \varphi = ZK$ and $\varphi(1)$ is non-zero and I_1 -invariant. If we identify $\text{Sym}^r k^2$ with the space of homogeneous polynomials in two variables x and y of degree r , then we may take $\varphi(1) = x^r$. Since φ generates $c\text{-Ind}_{\text{KZ}}^G \text{Sym}^r k^2$ as a G -representation T is determined by $T\varphi$.

Lemma 5.13.

- (i) *If $r = 0$ then $T\varphi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \varphi + \sum_{\lambda \in \mathbf{F}_p} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t\varphi$.*
- (ii) *Otherwise, $T\varphi = \sum_{\lambda \in \mathbf{F}_p} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t\varphi$.*

Proof. — In the notation of [1] this is a calculation of $T([1, e_0])$. The claim follows from the formula (19) in the proof of [1] Theorem 19. \square

Theorem 19 in [1] says that $\text{c-Ind}_{\text{KZ}}^{\text{G}}\sigma$ is a free $k[T]$ -module. Hence, the map $T - \lambda$ is injective, for all $\lambda \in k$.

Definition 5.14. — Let $\pi(r, \lambda)$ be a representation of G defined by the exact sequence:

$$(65) \quad 0 \longrightarrow \text{c-Ind}_{\text{ZK}}^{\text{G}} \text{Sym}^r k^2 \xrightarrow{T-\lambda} \text{c-Ind}_{\text{ZK}}^{\text{G}} \text{Sym}^r k^2 \longrightarrow \pi(r, \lambda) \longrightarrow 0.$$

If $\eta : \mathbf{Q}_p^\times \rightarrow k^\times$ is a smooth character then let $\pi(r, \lambda, \eta) := \pi(r, \lambda) \otimes \eta \circ \det$.

It follows from [1, Thm. 30] and [16, Thm. 1.1] that $\pi(r, \lambda)$ is absolutely irreducible unless $(r, \lambda) = (0, \pm 1)$ or $(r, \lambda) = (p-1, \pm 1)$. Moreover, one has non-split exact sequences:

$$(66) \quad 0 \rightarrow \mu_{\pm 1} \circ \det \rightarrow \pi(p-1, \pm 1) \rightarrow \text{Sp} \otimes \mu_{\pm 1} \circ \det \rightarrow 0,$$

$$(67) \quad 0 \rightarrow \text{Sp} \otimes \mu_{\pm 1} \circ \det \rightarrow \pi(0, \pm 1) \rightarrow \mu_{\pm 1} \circ \det \rightarrow 0,$$

where $\mu_\lambda : \mathbf{Q}_p^\times \rightarrow k^\times, x \mapsto \lambda^{\text{val}(x)}$. Further, if $\lambda \neq 0$ and $(r, \lambda) \neq (0, \pm 1)$ then [1, Thm. 30] asserts that

$$(68) \quad \pi(r, \lambda) \cong \text{Ind}_p^{\text{G}} \mu_{\lambda^{-1}} \otimes \mu_\lambda \omega^r.$$

If π is an absolutely irreducible k -representation of G with a central character ζ and $\zeta(p)$ is a square in k then π is a quotient of $\pi(r, \lambda, \eta)$ for some $\lambda \in k$ and $\eta : \mathbf{Q}_p^\times \rightarrow k^\times$. The supersingular representations are isomorphic (over $k[\sqrt{\zeta(p)}]$) to $\pi(r, 0, \eta)$. All the isomorphism between supersingular representations corresponding to different r and η are given by

$$(69) \quad \pi(r, 0, \eta) \cong \pi(r, 0, \eta \mu_{-1}) \cong \pi(p-1-r, 0, \eta \omega^r) \cong \pi(p-1-r, 0, \eta \omega^r \mu_{-1})$$

see [16, Thm. 1.3]. We refer to the *regular* case if $\pi \cong \pi(r, 0, \eta)$ with $0 < r < p-1$, and *Iwahori* case if $\pi \cong \pi(0, 0, \eta) \cong \pi(p-1, 0, \eta)$.

5.4. Hecke algebra and extensions. — Let $\mathcal{H} := \text{End}_{\text{G}}(\text{c-Ind}_{\text{Z1}}^{\text{G}} \zeta)$ and let $\text{Mod}_{\mathcal{H}}$ be the category of right \mathcal{H} -modules. Let $\mathcal{I} : \text{Mod}_{\text{G}, \zeta}^{\text{sm}}(k) \rightarrow \text{Mod}_{\mathcal{H}}$ be the functor:

$$\mathcal{I}(\pi) := \pi^{I_1} \cong \text{Hom}_{\text{G}}(\text{c-Ind}_{\text{Z1}}^{\text{G}} \zeta, \pi).$$

Let $\mathcal{T} : \text{Mod}_{\mathcal{H}} \rightarrow \text{Mod}_{\text{G}, \zeta}^{\text{sm}}(k)$ be the functor:

$$\mathcal{T}(M) := M \otimes_{\mathcal{H}} \text{c-Ind}_{\text{Z1}}^{\text{G}} \zeta.$$

One has $\mathrm{Hom}_{\mathcal{H}}(\mathbf{M}, \mathcal{I}(\pi)) \cong \mathrm{Hom}_G(\mathcal{T}(\mathbf{M}), \pi)$. Moreover, Vignéras in [67, Thm. 5.4] shows that \mathcal{I} induces a bijection between irreducible objects in $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$ and $\mathrm{Mod}_{\mathcal{H}}$. Let $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)^{I_1}$ be the full subcategory of $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$ consisting of representations generated by their I_1 -invariants. Ollivier has shown⁶ in [51] that

$$(70) \quad \mathcal{I} : \mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)^{I_1} \rightarrow \mathrm{Mod}_{\mathcal{H}}, \quad \mathcal{T} : \mathrm{Mod}_{\mathcal{H}} \rightarrow \mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)^{I_1}$$

are quasi-inverse to each other and so $\mathrm{Mod}_{\mathcal{H}}$ is equivalent to $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)^{I_1}$.

Remark 5.15. — We note that $(\mathrm{c}\text{-Ind}_{Z I_1}^G \zeta) \otimes_k l \cong \mathrm{c}\text{-Ind}_{Z I_1}^G (\zeta \otimes_k l)$, and since it is finitely generated we have $\mathcal{H} \otimes_k l \cong \mathrm{End}_G(\mathrm{c}\text{-Ind}_{Z I_1}^G \zeta \otimes_k l)$. Moreover, $\mathcal{I}(\pi) \otimes_k l \cong \mathcal{I}(\pi \otimes_k l)$ by Lemma 5.1 and $\mathcal{T}(\mathbf{M}) \otimes_k l \cong \mathcal{T}(\mathbf{M} \otimes_k l)$. Hence, if we show that the functors in (70) induce an equivalence of categories over some extension of k then the same also holds over k .

In particular, if $\tau = \langle G \cdot \tau^{I_1} \rangle$ and π is in $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$ then one has:

$$(71) \quad \mathrm{Hom}_G(\tau, \pi) \cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\pi))$$

and the natural map $\mathcal{T}\mathcal{I}(\tau) \rightarrow \tau$ is an isomorphism. We have shown in [56, §9] that (71) gives an E_2 -spectral sequence:

$$(72) \quad \mathrm{Ext}_{\mathcal{H}}^i(\mathcal{I}(\tau), \mathbf{R}^j \mathcal{I}(\pi)) \implies \mathrm{Ext}_{G,\zeta}^{i+j}(\tau, \pi)$$

where $\mathrm{Ext}_{G,\zeta}^n(\tau, *)$ is the n -th right derived functor of $\mathrm{Hom}_G(\tau, *)$ on $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$. The 5-term sequence associated to (72) gives us:

$$(73) \quad \begin{aligned} 0 \rightarrow \mathrm{Ext}_{\mathcal{H}}^1(\mathcal{I}(\tau), \mathcal{I}(\pi)) &\rightarrow \mathrm{Ext}_{G,\zeta}^1(\tau, \pi) \rightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbf{R}^1 \mathcal{I}(\pi)) \\ &\rightarrow \mathrm{Ext}_{\mathcal{H}}^2(\mathcal{I}(\tau), \mathcal{I}(\pi)) \rightarrow \mathrm{Ext}_{G,\zeta}^2(\tau, \pi). \end{aligned}$$

Let $\mathrm{Mod}_{G,\zeta}^{\mathrm{ladm}}(\mathcal{O})$ (resp. $\mathrm{Mod}_{G,\zeta}^{\mathrm{ladm}}(k)$) be the full subcategory of $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(\mathcal{O})$ (resp. $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$) consisting of all locally admissible representations, see Section 2. As already explained in Section 2, it follows from [30, Thm. 2.3.8] that a smooth representation of G with a central character is locally admissible if and only if it is locally of finite length, so that $\mathrm{Mod}_{G,\zeta}^{\mathrm{ladm}}(\mathcal{O}) = \mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})$ and $\mathrm{Mod}_{G,\zeta}^{\mathrm{ladm}}(k) = \mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(k)$. The inclusion $\iota : \mathrm{Mod}_{G,\zeta}^{\mathrm{ladm}}(\mathcal{O}) \rightarrow \mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$ has a right adjoint functor $V \mapsto V_{\mathrm{ladm}}$, which associates to V the subset of all locally admissible elements. Taking locally admissible elements is a left

⁶ In fact, both Vignéras and Ollivier work with the full Hecke algebra $\mathrm{End}_G(\mathrm{c}\text{-Ind}_{I_1}^G \mathbf{1})$. Our Hecke algebra is the quotient of the full Hecke algebra by the ideal generated by all the elements of the form $T_z - \zeta(z)^{-1}$, where T_z is the Hecke operator corresponding to the (double) coset $z I_1$, see [54, §2], for all $z \in Z$. In particular, if π is a smooth representation of G , the action of the full Hecke algebra on π^{I_1} factors through the action of \mathcal{H} if and only if Z acts on π^{I_1} by the character ζ , or equivalently the subrepresentation of π generated by π^{I_1} has a central character equal to ζ . The results of [51] imply that (70) induces an equivalence of categories.

exact functor, see [30, 2.2.19], which is the identity functor on locally admissible representations. Let $\mathfrak{C}(\mathcal{O})$ (resp. $\mathfrak{C}(k)$) be the full subcategory of $\text{Mod}_G^{\text{proaug}}(\mathcal{O})$ anti-equivalent to $\text{Mod}_{G,\zeta}^{\text{ladm}}(\mathcal{O})$ (resp. $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$) via the Pontryagin duality.

Proposition 5.16. — *The functor $\iota : \text{Mod}_{G,\zeta}^{\text{ladm}}(k) \rightarrow \text{Mod}_{G,\zeta}^{\text{sm}}(k)$ maps injectives to injectives.*

Proof. — Let J be an injective object in $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$ and let $\iota(J) \hookrightarrow J_1$ be an injective envelope of $\iota(J)$ in $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$. If $\iota(J) \neq J_1$ then $(J_1/\iota(J))^{I_1} \neq 0$ and thus there exists $v \in (J_1/\iota(J))^{I_1}$ such that $\sigma := \langle \mathbf{K} \cdot v \rangle$ is an irreducible representation of \mathbf{K} . Let $A := \langle G \cdot v \rangle \subseteq J_1/\iota(J)$ then by pulling back we obtain $B \subseteq J_1$ and an exact sequence:

$$(74) \quad 0 \rightarrow \iota(J) \rightarrow B \rightarrow A \rightarrow 0.$$

Since $\iota(J) \hookrightarrow J_1$ is essential, the class of the sequence (74) is a non-zero element in $\text{Ext}_{G,\zeta}^1(A, \iota(J))$. We will show that $\text{Ext}_{G,\zeta}^1(A, \iota(J)) = 0$ and thus obtain a contradiction to $\iota(J) \neq J_1$.

Since J_1 has a central character ζ , Z acts on σ by ζ . Let $\varphi \in \text{c-Ind}_{\mathbf{KZ}}^G \sigma$ be such that $\text{Supp } \varphi = Z\mathbf{K}$ and $\varphi(1)$ spans σ^{I_1} . By Frobenius reciprocity we obtain a map $\psi : \text{c-Ind}_{\mathbf{KZ}}^G \sigma \rightarrow A$, which sends φ to v . Since v generates A as a G -representation, ψ is surjective. It is shown in [34, Cor. 3.8] that the restriction functor $\text{Mod}_{G,\zeta}^{\text{ladm}}(k) \rightarrow \text{Mod}_{\mathbf{K},\zeta}^{\text{sm}}(k)$, $\pi \mapsto \pi|_{\mathbf{K}}$ sends injectives to injectives. Hence,

$$(75) \quad \text{Ext}_{G,\zeta}^1(\text{c-Ind}_{\mathbf{KZ}}^G \sigma, \iota(J)) \cong \text{Ext}_{\mathbf{K},\zeta}^1(\sigma, \iota(J)) = 0$$

and so ψ cannot be injective. Thus $\text{Ker } \psi$ is non-zero, and [1, Prop. 18] asserts that $(\text{Ker } \psi)^{I_1}$ is of finite codimension in $(\text{c-Ind}_{\mathbf{KZ}}^G \sigma)^{I_1}$. In particular, the set $\{\psi(T^n \varphi) : n \geq 0\}$, where T is the Hecke operator defined in (62), is linearly dependent and so there exists a non-zero polynomial P such that $\psi(P(T)\varphi) = 0$. Hence, ψ factors through

$$(76) \quad \text{c-Ind}_{\mathbf{KZ}}^G \sigma \twoheadrightarrow \text{c-Ind}_{\mathbf{KZ}}^G \sigma / (P(T)) \twoheadrightarrow A.$$

Since $\text{c-Ind}_{\mathbf{KZ}}^G \sigma / (T - \lambda)$ is of finite length, for all λ , by base changing to the splitting field of $P(T)$, we see that $\text{c-Ind}_{\mathbf{KZ}}^G \sigma / (P(T))$ is of finite length and hence is admissible and thus A is admissible.

We claim that there exists a finite length subrepresentation κ of B such the $B = \kappa + \iota(J)$. The claim implies that B is locally finite (or equivalently locally admissible). Since J is injective in $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$, the claim implies that (74) is split. To prove the claim, we proceed as follows. Choose $w \in B$, which maps to v in A . Let τ be the \mathbf{KZ} -subrepresentation of B generated by v . Since Z acts by the central character and the action of \mathbf{K} is smooth, τ is finite dimensional. By Frobenius reciprocity we obtain a map $\theta : \text{c-Ind}_{\mathbf{KZ}}^G \tau \rightarrow B$, such that the composition $\text{c-Ind}_{\mathbf{KZ}}^G \tau \xrightarrow{\theta} B \rightarrow A$ is surjective. Let τ' be the kernel of the surjection $\tau \twoheadrightarrow \sigma$. Since compact induction is an exact functor we obtain an exact sequence

$$(77) \quad 0 \rightarrow \text{c-Ind}_{\mathbf{KZ}}^G \tau' \rightarrow \text{c-Ind}_{\mathbf{KZ}}^G \tau \rightarrow \text{c-Ind}_{\mathbf{KZ}}^G \sigma \rightarrow 0.$$

Let Υ be the subrepresentation of $\text{c-Ind}_{\text{KZ}}^G \tau$ fitting into the exact sequence

$$(78) \quad 0 \rightarrow \text{c-Ind}_{\text{KZ}}^G \tau' \rightarrow \Upsilon \rightarrow \text{P}(\text{T})(\text{c-Ind}_{\text{KZ}}^G \sigma) \rightarrow 0,$$

where $\text{P}(\text{T})$ is the endomorphism of $\text{c-Ind}_{\text{KZ}}^G \sigma$ constructed in (76). It follows from (76) that $\text{c-Ind}_{\text{KZ}}^G \tau / \Upsilon$ is of finite length, and the composition $\Upsilon \xrightarrow{\theta} \text{B} \rightarrow \text{A}$ is zero. Hence, $\theta(\Upsilon)$ is contained in $\iota(\text{J})$. It follows from (78) that Υ is a finitely generated G -representation, and since J is locally finite, we deduce that $\theta(\Upsilon)$ is of finite length. Thus $\kappa := \theta(\text{c-Ind}_{\text{KZ}}^G \tau)$ is of finite length, and the composition $\kappa \hookrightarrow \text{B} \twoheadrightarrow \text{A}$ is surjective, which implies the claim. \square

If τ is in $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$ then we denote by $\text{Ext}_{G,\zeta}^{\text{ladm},n}(\tau, *)$ (resp. $\text{Ext}_{G,\zeta}^n(\tau, *)$) the n -th right derived functor of $\text{Hom}_G(\tau, *)$ in $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$ (resp. $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$). It follows from Corollary 2.3 or [31, 2.1.1] that $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$ and $\text{Mod}_{G,\zeta}^{\text{ladm}}(\mathcal{O})$ have enough injectives.

Corollary 5.17. — *Let τ and π be in $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$ then ι induces an isomorphism*

$$(79) \quad \text{Ext}_{G,\zeta}^n(\iota(\tau), \iota(\pi)) \cong \text{Ext}_{G,\zeta}^{\text{ladm},n}(\tau, \pi) \cong \text{Ext}_{\mathfrak{C}(k)}^n(\pi^\vee, \tau^\vee), \quad \forall n \geq 0.$$

Corollary 5.18. — *The functor $\iota : \text{Mod}_{G,\zeta}^{\text{ladm}}(\mathcal{O}) \rightarrow \text{Mod}_{G,\zeta}^{\text{sm}}(\mathcal{O})$ maps injectives to injectives.*

Proof. — Let J be an injective object in $\text{Mod}_{G,\zeta}^{\text{ladm}}(\mathcal{O})$ and $\iota(\text{J}) \hookrightarrow \text{J}_1$ be an injective envelope of $\iota(\text{J})$ in $\text{Mod}_{G,\zeta}^{\text{sm}}(\mathcal{O})$. Now $\text{J}[\varpi]$ is injective in $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)$ and $\iota(\text{J}[\varpi]) \hookrightarrow \text{J}_1[\varpi]$ is an injective envelope of $\iota(\text{J}[\varpi])$ in $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$. It follows from Proposition 5.16 that $\text{J}_1[\varpi] \cong \iota(\text{J}[\varpi]) = \iota(\text{J})[\varpi]$. Hence, we obtain an injection $(\text{J}_1/\iota(\text{J}))[\varpi] \hookrightarrow \iota(\text{J})/\varpi\iota(\text{J})$. This implies that $(\text{J}_1/\iota(\text{J}))[\varpi]$ is an object of $\text{Mod}_{G,\zeta}^{\text{ladm}}(\mathcal{O})$ and so the extension

$$0 \rightarrow \iota(\text{J}) \rightarrow \text{A} \rightarrow (\text{J}_1/\iota(\text{J}))[\varpi] \rightarrow 0$$

splits, where $\text{A} \subset \text{J}_1$. Since $\iota(\text{J}) \hookrightarrow \text{J}_1$ is essential, we get that $(\text{J}_1/\iota(\text{J}))[\varpi] = 0$, which implies that $\text{J}_1 = \iota(\text{J})$. \square

Corollary 5.19. — *Projective objects in $\mathfrak{C}(\mathcal{O})$ are \mathcal{O} -torsion free. In particular, the hypothesis (H0) of Section 3.1 is satisfied.*

Proof. — Let P be a projective object in $\mathfrak{C}(\mathcal{O})$ then P^\vee is an injective object in $\text{Mod}_{G,\zeta}^{\text{ladm}}(\mathcal{O})$ and also in $\text{Mod}_{G,\zeta}^{\text{sm}}(\mathcal{O})$ by Corollary 5.18 and it is enough to show that P^\vee is ϖ -divisible. We claim that any V in $\text{Mod}_{G,\zeta}^{\text{sm}}(\mathcal{O})$ may be embedded into an object which is ϖ -divisible. The claim gives the result, since injectivity of P^\vee implies that the embedding must split. Since direct summands of ϖ -divisible modules are ϖ -divisible, we are done. We may embed $j : \text{V} \hookrightarrow \text{W}$ into a ϖ -divisible \mathcal{O} -torsion module, since the category of \mathcal{O} -torsion modules has enough injectives and these are ϖ -divisible. The

embedding $V \hookrightarrow C^u(G, W)$, $v \mapsto [g \mapsto j(gv)]$ where the target is the space of uniformly continuous functions with discrete topology on W solves the problem. \square

Corollary 5.20. — *Let M be an \mathcal{O} -torsion free object of $\mathfrak{C}(\mathcal{O})$ then M is projective in $\mathfrak{C}(\mathcal{O})$ if and only if $M \otimes_{\mathcal{O}} k$ is projective in $\mathfrak{C}(k)$.*

Proof. — Since every A in $\mathfrak{C}(k)$ is killed by ϖ we have

$$\mathrm{Hom}_{\mathfrak{C}(k)}(M \otimes_{\mathcal{O}} k, A) \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M, A).$$

Hence, if M is projective in $\mathfrak{C}(\mathcal{O})$ then the functor $\mathrm{Hom}_{\mathfrak{C}(k)}(M \otimes_{\mathcal{O}} k, *)$ is exact and so $M \otimes_{\mathcal{O}} k$ is projective in $\mathfrak{C}(k)$.

Let $\phi : P \twoheadrightarrow M \otimes_{\mathcal{O}} k$ be a projective envelope of $M \otimes_{\mathcal{O}} k$ in $\mathfrak{C}(\mathcal{O})$. Since $M \rightarrow M \otimes_{\mathcal{O}} k$ is essential and P is projective there exists a surjection $\psi : P \twoheadrightarrow M$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\psi} & M \\ & \searrow \phi & \downarrow \\ & & M \otimes_{\mathcal{O}} k \end{array}$$

commutes. Since $M \otimes_{\mathcal{O}} k$ is projective it is its own projective envelope in $\mathfrak{C}(k)$. Thus it follows from Lemma 2.11 that ϕ induces an isomorphism $P \otimes_{\mathcal{O}} k \cong M \otimes_{\mathcal{O}} k$. Since M is \mathcal{O} -torsion free we get $(\mathrm{Ker} \psi) \otimes_{\mathcal{O}} k = 0$. Nakayama's lemma implies that $\mathrm{Ker} \psi = 0$ and hence $M \cong P$ is projective. \square

Corollary 5.21. — *Let $P_{\bullet} \twoheadrightarrow M$ be a projective resolution of M in $\mathfrak{C}(k)$. Let \tilde{M} be an \mathcal{O} -torsion free object of $\mathfrak{C}(\mathcal{O})$ such that $\tilde{M} \otimes_{\mathcal{O}} k \cong M$. Then there exists a projective resolution $\tilde{P}_{\bullet} \twoheadrightarrow \tilde{M}$ of \tilde{M} in $\mathfrak{C}(\mathcal{O})$ lifting the resolution of M .*

Proof. — Let $\phi : P \twoheadrightarrow M$ be an epimorphism in $\mathfrak{C}(k)$ with P projective and let \tilde{P} be a projective envelope of P in $\mathfrak{C}(\mathcal{O})$. Lemma 2.11 says that $P \cong \tilde{P} \otimes_{\mathcal{O}} k$. Since $\tilde{M} \otimes_{\mathcal{O}} k \cong \tilde{M}/\varpi\tilde{M} \cong M$, the epimorphism $\tilde{M} \twoheadrightarrow M$ is essential by an application of Nakayama's lemma and since \tilde{P} is projective there exists $\tilde{\phi} : \tilde{P} \twoheadrightarrow \tilde{M}$ such that the diagram

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{\phi}} & \tilde{M} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\phi} & M \end{array}$$

commutes. Since \tilde{M} is \mathcal{O} -torsion free it is \mathcal{O} -flat and hence $(\mathrm{Ker} \tilde{\phi}) \otimes_{\mathcal{O}} k \cong \mathrm{Ker} \phi$. Moreover, \tilde{P} is \mathcal{O} -torsion free by Corollary 5.19, and hence $\mathrm{Ker} \tilde{\phi}$ is \mathcal{O} -torsion free. We may then continue to lift the whole resolution. \square

Lemma 5.22. — *Let π be a smooth k -representation of G with a central character ζ . Forgetting the \mathcal{H} -action induces an isomorphism $\mathbf{R}^i\mathcal{I}(\pi) \cong H^i(I_1/Z_1, \pi)$ for all $i \geq 0$.*

Proof. — Let $\text{Res}_{I_1} : \text{Mod}_{G,\zeta}^{\text{sm}}(k) \rightarrow \text{Mod}_{I_1,\zeta}^{\text{sm}}(k)$ be the restriction to I_1 . Since Res_{I_1} is right adjoint to an exact functor $\text{c-Ind}_{Z_1}^G$, Res_{I_1} maps injective objects to injective objects. Since ζ is smooth and I_1 is pro- p , ζ is trivial on $Z_1 := I_1 \cap Z$, hence we may identify $\text{Mod}_{I_1,\zeta}^{\text{sm}}(k)$ with $\text{Mod}_{I_1/Z_1}^{\text{sm}}(k)$. Choose an injective resolution $\pi \hookrightarrow \mathbf{J}^\bullet$ of π in $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$. Then $\pi|_{I_1} \hookrightarrow (\mathbf{J}|_{I_1})^\bullet$ is an injective resolution of $\pi|_{I_1}$ in $\text{Mod}_{I_1/Z_1}^{\text{sm}}(k)$. Hence, for all $i \geq 0$ we get an isomorphism of k -vector spaces $\mathbf{R}^i\mathcal{I}(\pi) \cong H^i(I_1/Z_1, \pi)$. \square

The results proved in the rest of the subsection will only be used in Section 10.

Lemma 5.23. — *If $p \geq 5$ then $\mathbf{R}^i\mathcal{I} = 0$ for $i \geq 4$.*

Proof. — We have an isomorphism

$$I_1/Z_1 \cong (I_1 \cap U^s) \times (I_1 \cap T)/Z_1 \times (I_1 \cap U) \cong \mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p.$$

Hence, I_1/Z_1 is a compact p -adic analytic group of dimension 3. Since we assume $p \geq 5$ it is p -saturable [45, III.3.2.7.5], and hence torsion free. Thus I_1/Z_1 is a Poincaré group of dimension 3, [45, V.2.5.8] and [62]. Hence $H^i(I_1/Z_1, *) = 0$ for all $i > 3$ and the assertion follows from Lemma 5.22. \square

Lemma 5.24. — *Let τ be a smooth irreducible representation of G with a central character ζ , such that $\tau \not\cong \pi(r, 0, \eta)$ with $0 < r < p - 1$. Then $\text{Ext}_{\mathcal{H}}^i(\mathcal{I}(\tau), *) = 0$ for $i \geq 2$.*

Proof. — Using Lemma 5.32 we may reduce to the case where τ is absolutely irreducible, which we now assume. It is enough to produce an exact sequence of \mathcal{H} -modules:

$$(80) \quad 0 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathcal{I}(\tau) \oplus M \rightarrow 0$$

with P_0 and P_1 projective and M arbitrary. We observe that if A is a direct summand of $\text{c-Ind}_{I_1 Z}^G \zeta$, then $\mathcal{I}(A)$ is a direct summand of $\mathcal{I}(\text{c-Ind}_{I_1 Z}^G \zeta) \cong \mathcal{H}$ and hence $\mathcal{I}(A)$ is projective. If $\tau \cong \pi(r, \lambda, \eta)$, with $\lambda \neq 0$ and $0 < r < p - 1$, then such sequence is constructed in [20, Cor. 6.6, Eq. (12)]. If $\tau \cong \pi(r, \lambda, \eta)$ with $r = 0$ or $r = p - 1$ then one may obtain (80) by applying \mathcal{I} to (65). The sequence remains exact by [1, 2.9, 2.8] in the non-, and by [16, 3.2.4, 3.2.5] in the supersingular case. If $\tau = \eta \circ \det$ or $\tau = \text{Sp} \otimes \eta \circ \det$, then τ may be realized as an H_0 of the diagram $\tau^{I_1} \hookrightarrow \tau^{K_1}$, see [20, Thm. 10.1]. This means an exact sequence:

$$(81) \quad 0 \rightarrow \text{c-Ind}_{\mathfrak{R}}^G \tau^{I_1} \otimes \delta \rightarrow \text{c-Ind}_{K_1}^G \tau^{K_1} \rightarrow \tau \rightarrow 0,$$

where $\delta(g) = (-1)^{\text{val}(\det g)}$, where \mathfrak{R} is the G -normalizer of I . Again applying \mathcal{I} we get (80). \square

Corollary 5.25. — *Let \mathbf{M} be a finite dimensional \mathcal{H} -module, such that the irreducible subquotients are isomorphic to $\mathcal{I}(\tau)$, where τ is as above, then $\mathrm{Ext}_{\mathcal{H}}^i(\mathbf{M}, *) = 0$, for $i \geq 2$.*

Proposition 5.26. — *Let π and τ be in $\mathrm{Mod}_{\mathbf{G}, \zeta}^{\mathrm{sm}}(k)$. Suppose that τ is admissible, generated by τ^{1_1} , and the irreducible subquotients of $\mathcal{I}(\tau)$ are not isomorphic to $\mathbf{M}(r, 0, \eta)$ with $0 < r < p - 1$. Then for $i \geq 1$ there exists an exact sequence:*

$$(82) \quad \mathrm{Ext}_{\mathcal{H}}^1(\mathcal{I}(\tau), \mathbf{R}^{i-1}\mathcal{I}(\pi)) \hookrightarrow \mathrm{Ext}_{\mathbf{G}, \zeta}^i(\tau, \pi) \rightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbf{R}^i\mathcal{I}(\pi)).$$

If $p \geq 5$ then $\mathrm{Ext}_{\mathbf{G}, \zeta}^4(\tau, \pi) \cong \mathrm{Ext}_{\mathcal{H}}^1(\mathcal{I}(\tau), \mathbf{R}^3\mathcal{I}(\pi))$ and $\mathrm{Ext}_{\mathbf{G}, \zeta}^i(\tau, \pi) = 0$ for $i \geq 5$.

Proof. — This follows from a calculation with the spectral sequence (72). Let $E_2^{pq} = \mathrm{Ext}_{\mathcal{H}}^p(\mathcal{I}(\tau), \mathbf{R}^q\mathcal{I}(\pi))$. Then $E_2^{pq} = 0$ for $p > 1$, by Corollary 5.25. Thus $E_{\infty}^{pq} = E_2^{pq}$ and for all $n \geq 0$ we obtain an exact sequence

$$(83) \quad 0 \rightarrow E_2^{0,n} \rightarrow E^n \rightarrow E_2^{1,n-1} \rightarrow 0.$$

If $p \geq 5$ then $E_2^{pq} = 0$ for $q > 3$ by Lemma 5.23, which implies the assertion. \square

Lemma 5.27. — *Let \mathbf{M}, \mathbf{N} be absolutely irreducible \mathcal{H} -modules and let d be the dimension of $\mathrm{Ext}_{\mathcal{H}}^1(\mathbf{M}, \mathbf{N})$. If $p > 2$ and $d \neq 0$ then one of the following holds:*

- (i) $\mathbf{M} \cong \mathbf{N} \cong \mathcal{I}(\pi(r, 0, \eta))$ with $0 < r < p - 1$ and $d = 2$;
- (ii) $\mathbf{M} \cong \mathbf{N}$ and $\mathbf{M} \not\cong \mathcal{I}(\mathrm{Sp} \otimes \eta)$, $\mathbf{M} \not\cong \mathcal{I}(\eta)$, $\mathbf{M} \not\cong \mathcal{I}(\pi(r, 0, \eta))$ with $0 < r < p - 1$ and $d = 1$;
- (iii) either $\mathbf{M} \cong \mathcal{I}(\eta)$ and $\mathbf{N} \cong \mathcal{I}(\mathrm{Sp} \otimes \eta)$ or $\mathbf{N} \cong \mathcal{I}(\eta)$ and $\mathbf{M} \cong \mathcal{I}(\mathrm{Sp} \otimes \eta)$ and $d = 1$,

where $\eta : \mathbf{G} \rightarrow k^\times$ is a smooth character.

Proof. — If $\mathbf{N} \cong \mathcal{I}(\mathrm{Sp} \otimes \eta)$ or $\mathbf{N} \cong \mathcal{I}(\eta)$ the assertion follows from [56, 11.3]. Otherwise, $\mathbf{N} \cong \mathcal{I}(\pi(r, \lambda, \eta))$ and the assertion follows from Corollaries 6.5, 6.6 and 6.7 [20]. We note that when the module denoted by \mathbf{M} in [20, Cor. 6.7] is irreducible, which is the case of interest here, it is isomorphic to the module denoted by \mathbf{M}' in [20, Cor. 6.7], as they are both isomorphic to $\mathcal{I}(\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}} \mu_\lambda \otimes \mu_{\lambda^{-1}})$. \square

Remark 5.28. — Let T_p be the Hecke operator in the full Hecke algebra $\mathrm{End}_{\mathbf{G}}(\mathrm{c}\text{-}\mathrm{Ind}_{\Gamma_1}^{\mathbf{G}} \mathbf{1})$ corresponding to the (double) coset $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \Gamma_1$, and let $\lambda = \zeta \left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \right)$. In [20] we work with the algebra $\mathcal{H}_{p=\lambda}$, which is the quotient of the full Hecke algebra by the ideal generated by $T_p - \lambda^{-1}$. Let $e_\zeta := |Z \cap \mathbf{K} / Z \cap \mathbf{K}_1|^{-1} \sum_{z \in Z \cap \mathbf{K} / Z \cap \mathbf{K}_1} \zeta(z) T_z$, where T_z is the Hecke operator corresponding to the (double) coset $z \Gamma_1$, see [54, §2]. Then e_ζ is a central idempotent in $\mathcal{H}_{p=\lambda}$ and $\mathcal{H} = e_\zeta \mathcal{H}_{p=\lambda} e_\zeta$. Since e_ζ is a central idempotent we may calculate the Ext-groups of \mathcal{H} -modules in the category of $\mathcal{H}_{p=\lambda}$ -modules, which allows us to use the results of [20].

5.5. Blocks. — We show that the category $\text{Mod}_{\mathcal{G}, \zeta}^{\text{lfm}}(\mathcal{O})$ naturally decomposes into a direct product of subcategories.

Lemma 5.29. — *Let \mathbf{J} be an injective object of $\text{Mod}_{\mathcal{G}}^{\text{sm}}(k)$, where \mathcal{G} is a profinite group. Let l be a field extension of k then $\mathbf{J} \otimes_k l$ is an injective object of $\text{Mod}_{\mathcal{G}}^{\text{sm}}(l)$.*

Proof. — Let V be a k -vector space and let $C(\mathcal{G}, V)$ be the space of continuous functions $f : \mathcal{G} \rightarrow V$. For every smooth k -representation π of \mathcal{G} the map $\phi \mapsto [v \mapsto \phi(v)(1)]$ induces an isomorphism

$$\text{Hom}_{\mathcal{G}}(\pi, C(\mathcal{G}, V)) \cong \text{Hom}_k(\pi, V).$$

The inverse is given by $\ell \mapsto [v \mapsto [g \mapsto \ell(gv)]]$. The functor $\text{Hom}_k(*, V)$ is exact and so $C(\mathcal{G}, V)$ is an injective object of $\text{Mod}_{\mathcal{G}}^{\text{sm}}(k)$. The natural injection $C(\mathcal{G}, V) \otimes_k l \hookrightarrow C(\mathcal{G}, V \otimes_k l)$ is also a surjection, since for every open subgroup \mathcal{P} of \mathcal{G} we have

$$(C(\mathcal{G}, V) \otimes_k l)^{\mathcal{P}} \cong (k[\mathcal{G}/\mathcal{P}] \otimes_k V) \otimes_k l \cong l[\mathcal{G}/\mathcal{P}] \otimes_l V_l \cong C(\mathcal{G}, V_l)^{\mathcal{P}},$$

as \mathcal{P} is of finite index in \mathcal{G} . This gives us the lemma for $\mathbf{J} = C(\mathcal{G}, V)$. In general, one can embed \mathbf{J} into $C(\mathcal{G}, V)$ by taking V to be the underlying vector space of \mathbf{J} . Since \mathbf{J} is injective the embedding splits. Thus $\mathbf{J} \otimes_k l$ is a direct summand of an injective object of $\text{Mod}_{\mathcal{G}}^{\text{sm}}(l)$ and hence it is itself injective. \square

Corollary 5.30. — *Let \mathcal{G} , π and l be as above then $H^i(\mathcal{G}, \pi) \otimes_k l \cong H^i(\mathcal{G}, \pi \otimes_k l)$ for all $i \geq 0$.*

Proof. — Choose an injective resolution $\pi \hookrightarrow \mathbf{J}^\bullet$ of π in $\text{Mod}_{\mathcal{G}}^{\text{sm}}(k)$. Lemma 5.29 says that $\pi_l \hookrightarrow \mathbf{J}_l^\bullet$ is an injective resolution of π_l in $\text{Mod}_{\mathcal{G}}^{\text{sm}}(l)$. Since taking \mathcal{G} -invariants commutes with $\otimes_k l$ by Lemma 5.1 we get the assertion. \square

Corollary 5.31. — *Let π be a smooth representation of \mathcal{G} with a central character ζ then*

$$(84) \quad \mathbf{R}^i \mathcal{I}(\pi) \otimes_k l \cong \mathbf{R}^i \mathcal{I}(\pi \otimes_k l)$$

for all field extensions l of k and all $i \geq 0$.

Proof. — Lemma 5.22, Corollary 5.30. \square

Lemma 5.32. — *Let \mathbf{M} and \mathbf{N} be \mathcal{H} -modules. If \mathbf{M} is finitely generated over \mathcal{H} then*

$$(85) \quad \text{Ext}_{\mathcal{H}}^i(\mathbf{M}, \mathbf{N}) \otimes_k l \cong \text{Ext}_{\mathcal{H}_l}^i(\mathbf{M} \otimes_k l, \mathbf{N} \otimes_k l)$$

for all field extensions l of k and all $i \geq 0$.

Proof. — It follows from the explicit description of \mathcal{H} given by Vignéras in [67] that the centre of \mathcal{H} is noetherian and \mathcal{H} is a finitely generated module over its centre, see [67, §1.2, 2.1.1, Cor. 2.3]. Hence \mathcal{H} is noetherian and since M is finitely generated we may find a resolution $P^\bullet \rightarrow M$ by free \mathcal{H} -modules of finite rank. Since $\mathrm{Hom}_{\mathcal{H}}(\mathcal{H}^{\oplus n}, N)_l \cong N_l^{\oplus n} \cong \mathrm{Hom}_{\mathcal{H}_l}(\mathcal{H}_l^{\oplus n}, N_l)$ we get the assertion. \square

Proposition 5.33. — *Let τ and π be in $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$ and suppose that τ is of finite length. Then*

$$(86) \quad \mathrm{Ext}_{k[G],\zeta}^i(\tau, \pi) \otimes_k l \cong \mathrm{Ext}_{l[G],\zeta}^i(\tau \otimes_k l, \pi \otimes_k l)$$

for all field extensions l of k and all $i \geq 0$.

Proof. — We will first prove the result when τ is irreducible. Then τ^{I_1} is finite dimensional and τ is generated by as a G -representation by the I_1 -invariants. By Lemma 5.1, $\mathcal{H} \otimes_k l \cong \mathrm{End}_{l[G]}(\mathrm{c}\text{-Ind}_{Z_{I_1}}^G \zeta \otimes_k l)$. Since τ^{I_1} is finite dimensional, it is a finitely generated \mathcal{H} -module. Combining (84) and (85) we get an isomorphism of spectral sequences:

$$\mathrm{Ext}_{\mathcal{H}}^i(\mathcal{I}(\tau), \mathbf{R}^j \mathcal{I}(\pi)) \otimes_k l \cong \mathrm{Ext}_{\mathcal{H}_l}^i(\mathcal{I}(\tau_l), \mathbf{R}^j \mathcal{I}(\pi_l)).$$

Since k is a field, l is k -flat and so it follows from (72) that $\mathrm{Ext}_{\mathcal{H}}^i(\mathcal{I}(\tau), \mathbf{R}^j \mathcal{I}(\pi))_l$ converges to $\mathrm{Ext}_{k[G],\zeta}^{i+j}(\tau, \pi)_l$. We use (72) again to deduce the assertion.

We will finish the proof by induction on the length of τ . We have already proved the result when τ is irreducible and (86) is an isomorphism for $i = 0$ by Lemma 5.1. If τ is not irreducible, then we may consider a short exact sequence $0 \rightarrow \tau_1 \rightarrow \tau \rightarrow \tau_2 \rightarrow 0$, with both τ_1 and τ_2 of length strictly less than the length of τ . The induction step is given by comparing the two long exact sequences induced by the short exact sequence and the 5-Lemma. \square

Let $\mathrm{Irr}_{G,\zeta}(k)$ be the set of equivalence classes of smooth irreducible k -representations of G with central character ζ . We write $\pi \leftrightarrow \tau$ if $\pi \cong \tau$ or $\mathrm{Ext}_{G,\zeta}^1(\pi, \tau) \neq 0$ or $\mathrm{Ext}_{G,\zeta}^1(\tau, \pi) \neq 0$. We write $\pi \sim \tau$ if there exists $\pi_1, \dots, \pi_n \in \mathrm{Irr}_{G,\zeta}(k)$, such that $\pi \cong \pi_1$, $\tau \cong \pi_n$ and $\pi_i \leftrightarrow \pi_{i+1}$ for $1 \leq i \leq n-1$. The relation \sim is an equivalence relation on $\mathrm{Irr}_{G,\zeta}(k)$. A block is an equivalence class of \sim .

Proposition 5.34. — *The category $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})$ decomposes into a direct product of subcategories*

$$(87) \quad \mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})^{\mathfrak{B}}$$

where the product is taken over all the blocks \mathfrak{B} and the objects of $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})^{\mathfrak{B}}$ are representations with all the irreducible subquotients lying in \mathfrak{B} . The equivalence in (87) is induced by sending $(\pi^{\mathfrak{B}})_{\mathfrak{B}}$, where each $\pi^{\mathfrak{B}}$ is an object of $\mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(\mathcal{O})^{\mathfrak{B}}$, to the direct sum $\bigoplus_{\mathfrak{B}} \pi^{\mathfrak{B}}$.

Proof. — This is standard, see [35, §IV.2], especially the Corollary after Theorem 2. Let us note that every irreducible object in $\text{Mod}_{G,\zeta}^{\text{fin}}(\mathcal{O})$ is killed by ϖ and so $\text{Irr}_{G,\zeta}(k) = \text{Irr}_{G,\zeta}(\mathcal{O})$. Moreover, if τ and π are irreducible then $\text{Ext}_{\mathcal{O}[G],\zeta}^1(\tau, \pi) \neq 0$ implies that either $\pi \cong \tau$ or $\text{Ext}_{k[G],\zeta}^1(\tau, \pi) \neq 0$, see the proof of Lemma 3.16. So we could have defined \sim by considering the extensions in $\text{Mod}_{G,\zeta}^{\text{fin}}(\mathcal{O})$. Let J_π and J_τ be injective envelopes of π and τ in $\text{Mod}_{G,\zeta}^{\text{fin}}(\mathcal{O})$. Then the following are equivalent: (1) $\text{Hom}_G(J_\pi, J_\tau) \neq 0$; (2) τ is a subquotient of J_π ; (3) there exists a representation κ of finite length which contains τ as a subquotient and $\text{soc}_G \kappa \cong \pi$. Using this one can show that our definition of a subcategory cut out by a block coincides with the one used in [35]. \square

Dually we obtain:

Corollary 5.35. — *The category $\mathfrak{C}(\mathcal{O})$ decomposes into a direct product of subcategories*

$$(88) \quad \mathfrak{C}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \mathfrak{C}(\mathcal{O})^{\mathfrak{B}},$$

where the product is taken over all the blocks \mathfrak{B} and the objects of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ are those M in $\mathfrak{C}(\mathcal{O})$ such that for every irreducible subquotient S of M , S^\vee lies in \mathfrak{B} . The equivalence in (88) is induced by sending $(M^{\mathfrak{B}})_{\mathfrak{B}}$, where each $M^{\mathfrak{B}}$ is an object of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$, to the direct product $\prod_{\mathfrak{B}} M^{\mathfrak{B}}$.

Let $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})$ be the category of admissible unitary L-Banach space representations of G with a central character ζ . We note that it follows from [61] and [33, 6.2.16] that $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})$ is an abelian category.

Proposition 5.36. — *The category $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})$ decomposes into a direct sum of categories:*

$$\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L}) \cong \bigoplus_{\mathfrak{B}} \text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}},$$

where the objects of $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$ are those Π in $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})$ such that for every open bounded G -invariant lattice Θ in Π the irreducible subquotients of $\Theta \otimes_{\mathcal{O}} k$ lie in \mathfrak{B} .

Proof. — Recall that we have showed in Lemma 4.3 that the reductions mod p of any two open bounded G -invariant lattices in Π have the same irreducible subquotients. Let Θ be an open bounded G -invariant lattice in Π , π an irreducible subquotient of $\Theta \otimes_{\mathcal{O}} k$ and \mathfrak{B} the block of π . By Lemma 4.11 the Schikhof dual Θ^d is an object of $\mathfrak{C}(\mathcal{O})$. Let $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ be the full subcategory of $\mathfrak{C}(\mathcal{O})$ as in Corollary 5.35 and let $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ be the full subcategory of $\mathfrak{C}(\mathcal{O})$, such that M is an object if and only if for all irreducible subquotients S of M , the Pontryagin dual S^\vee does not lie in \mathfrak{B} . It follows from Corollary 5.35 that we may canonically decompose $\Theta^d \cong (\Theta^d)_{\mathfrak{B}} \oplus (\Theta^d)^{\mathfrak{B}}$, where $(\Theta^d)_{\mathfrak{B}}$ is an object of $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ and $(\Theta^d)^{\mathfrak{B}}$ is an object of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$.

Let $\Pi_{\mathfrak{B}} := \text{Hom}_{\mathcal{O}}^{\text{cont}}((\Theta^d)_{\mathfrak{B}}, \mathbb{L})$ and $\Pi^{\mathfrak{B}} := \text{Hom}_{\mathcal{O}}^{\text{cont}}((\Theta^d)^{\mathfrak{B}}, \mathbb{L})$ with the supremum norm. Then it follows from the anti-equivalence of categories established in [61] that $\Pi \cong \Pi_{\mathfrak{B}} \oplus \Pi^{\mathfrak{B}}$. Further, since the decomposition in Corollary 5.35 is a decomposition of categories we have no non-zero morphisms in $\mathfrak{C}(\mathcal{O})$ between $(\Theta^d)_{\mathfrak{B}}$ and $(\Theta^d)^{\mathfrak{B}}$. Dually this implies that there are no non-zero morphisms between $\Pi_{\mathfrak{B}}$ and $\Pi^{\mathfrak{B}}$ in $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})$. Using $(\Theta \otimes_{\mathcal{O}} k)^{\vee} \cong \Theta^d \otimes_{\mathcal{O}} k$, see [57, Lem. 5.4], we deduce that $\Pi^{\mathfrak{B}}$ is a non-zero object of $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$ and none of the irreducible representations in \mathfrak{B} appear as subquotients of the reduction modulo \mathfrak{p} of any open bounded lattice in $\Pi_{\mathfrak{B}}$. Inductively we obtain a sequence of closed G -invariant subspaces Π_i of Π such that $\Pi_i \cong \Pi^{\mathfrak{B}_i} \oplus \Pi_{i+1}$ for some block \mathfrak{B}_i with $\Pi^{\mathfrak{B}_i} \neq 0$ if $\Pi_i \neq 0$. Since Π is admissible such sequence must become stationary, see [57, Lem. 5.8]. Hence, there exist finitely many blocks $\mathfrak{B}_1, \dots, \mathfrak{B}_m$ such that $\Pi^{\mathfrak{B}_i} \neq 0$ and so $\Pi \cong \bigoplus_{i=1}^m \Pi^{\mathfrak{B}_i}$. \square

Corollary 5.37. — *Let Π be an irreducible admissible \mathbb{L} -Banach space representation of G with a central character and let Θ be an open bounded G -invariant lattice in Π . Then $\Theta \otimes_{\mathcal{O}} k$ contains an irreducible subquotient π and all other irreducible subquotients lie in the block of π .*

Proposition 5.38. — *Let l be a field extension of k . Let π in $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$ be absolutely irreducible and let τ in $\text{Mod}_{G,\zeta}^{\text{sm}}(l)$ be irreducible. If $\pi \otimes_k l \leftrightarrow \tau$ then there exists an absolutely irreducible σ in $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$ such that $\tau \cong \sigma \otimes_k l$. Moreover, σ is unique up to isomorphism.*

Proof. — It follows from Lemma 5.1 that if such σ exists then it is unique. It follows from the proof of Proposition 5.3 that $\tau \otimes_l \bar{l} \cong \bigoplus_{i=1}^n \tau_i$, where each τ_i is of finite length and $\tau_i^{ss} \cong \kappa_i^{\oplus m_i}$ with κ_i absolutely irreducible. (Note that we do not require l to be a perfect field, and hence $\tau \otimes_l \bar{l}$ need not be semisimple.) Thus using Proposition 5.33 we may reduce the problem to the case when l is algebraically closed. In this case in [56] we have determined all possible τ such that $\tau \leftrightarrow \pi_l$. It follows from the explicit description (recalled in Proposition 5.42 below) and Lemma 5.10 that every such τ can be defined over the field of definition of π . \square

Corollary 5.39. — *Let $\pi \in \text{Irr}_{G,\zeta}(k)$ be absolutely irreducible and let $\pi \hookrightarrow J$ be an injective envelope of π in $\text{Mod}_{G,\zeta}^{\text{lfm}}(k)$. Then $\pi \otimes_k l \hookrightarrow J \otimes_k l$ is an injective envelope of $\pi \otimes_k l$ in $\text{Mod}_{G,\zeta}^{\text{lfm}}(l)$.*

Proof. — Let $\iota : J_l \hookrightarrow J'$ be an injective envelope of J_l in $\text{Mod}_{G,\zeta}^{\text{lfm}}(l)$. We claim that the quotient is zero. Otherwise, there exists $\tau \in \text{Irr}_{G,\zeta}(l)$ such that $\text{Hom}_G(\tau, J'/J_l) \neq 0$. Since ι is essential we have $\text{Hom}_G(\tau, J_l) \cong \text{Hom}_G(\tau, J')$ and so $\text{Ext}_{G,\zeta}^1(\tau, J_l) \neq 0$. Since we are working in the category of locally finite representations, every representation is equal to the union of its subrepresentations of finite length. Since τ is irreducible, and hence finitely generated as a G -representation, we deduce that $\text{Ext}_{G,\zeta}^1(\tau, J_l) \cong \varinjlim \text{Ext}_{G,\zeta}^1(\tau, \kappa_l)$, where the limit is taken over all the finite length subrepresentations κ of J . This implies the existence of a subobject κ of J of finite length, such that $\text{Ext}_{G,\zeta}^1(\tau, \kappa_l) \neq 0$. Further

passing to short exact sequences we may assume that κ is irreducible and lies in the block of π . Proposition 5.38 implies that κ is absolutely irreducible, and applying it again we deduce that there exists $\sigma \in \text{Irr}_{G,\zeta}(k)$ such that $\tau \cong \sigma \otimes_k l$. As J is injective in $\text{Mod}_{G,\zeta}^{\text{fin}}(k)$ we have $\text{Ext}_{G,\zeta}^1(\sigma, J) = 0$ and so Proposition 5.33 implies that $\text{Ext}_{G,\zeta}^1(\tau, J_l) = 0$. This is a contradiction and so $J_l \cong J'$ is injective.

Since all the irreducible subquotients σ of J are absolutely irreducible, all the irreducible subquotients of J_l can be defined over k . Since $\text{Hom}_G(\sigma_l, J_l) \cong \text{Hom}_G(\sigma, J)_l$ by Lemma 5.1, we deduce that $\pi_l \hookrightarrow J_l$ is essential. \square

Let L' be a finite extension of L with the ring of integers \mathcal{O}' and residue field k' . Let π be an absolutely irreducible k -representation of G with a central character ζ and let \tilde{P} be a projective envelope of $S := \pi^\vee$ in $\mathfrak{C}(\mathcal{O})$.

Corollary 5.40. — $\tilde{P} \otimes_{\mathcal{O}} \mathcal{O}'$ is a projective envelope of $S \otimes_k k'$ in $\mathfrak{C}(\mathcal{O}')$. Moreover, $\text{End}_{\mathfrak{C}(\mathcal{O}')}(\tilde{P} \otimes_{\mathcal{O}} \mathcal{O}') \cong \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}) \otimes_{\mathcal{O}} \mathcal{O}'$.

Proof. — It is enough to show that $\tilde{P} \otimes_{\mathcal{O}} k'$ is a projective envelope of $S \otimes_k k'$ in $\mathfrak{C}(k')$, see Corollary 5.20, as \tilde{P} is \mathcal{O} -torsion free by Corollary 5.19. Now,

$$\tilde{P} \otimes_{\mathcal{O}} k' \cong J_\pi^\vee \otimes_k k' \cong (J_\pi \otimes_k k')^\vee,$$

where J_π is an injective envelope of π in $\text{Mod}_{G,\zeta}^{\text{fin}}(k)$ and the last isomorphism follows from Lemma 5.1. Since $J_\pi \otimes_k k'$ is an injective envelope of $\pi \otimes_k k'$ in $\text{Mod}_{G,\zeta}^{\text{fin}}(k')$ by Corollary 5.39 we get the first assertion. The second assertion follows since \mathcal{O}' is free of finite rank over \mathcal{O} . \square

Corollary 5.41. — Let S be as above and suppose there exists Q in $\mathfrak{C}(k)$ satisfying the hypotheses (H1)–(H5) of Section 3. Then $Q \otimes_k k'$ and $S \otimes_k k'$ satisfy the hypotheses (H1)–(H5) in $\mathfrak{C}(k')$.

Proof. — Propositions 5.38, 5.33. \square

Proposition 5.42. — Let $\pi \in \text{Irr}_{G,\zeta}(k)$ be absolutely irreducible and let \mathfrak{B} be the equivalence class of π in $\text{Irr}_{G,\zeta}(k)$ under \sim . If $p \geq 5$ then one of the following holds:

- (i) if π is supersingular then $\mathfrak{B} = \{\pi\}$;
- (ii) if $\pi \cong \text{Ind}_P^G \chi_1 \otimes \chi_2 \omega^{-1}$ with $\chi_1 \chi_2^{-1} \neq \mathbf{1}$, $\omega^{\pm 1}$ then

$$\mathfrak{B} = \{\text{Ind}_P^G \chi_1 \otimes \chi_2 \omega^{-1}, \text{Ind}_P^G \chi_2 \otimes \chi_1 \omega^{-1}\};$$

- (iii) if $\pi \cong \text{Ind}_P^G \chi \otimes \chi \omega^{-1}$ then $\mathfrak{B} = \{\pi\}$;
- (iv) otherwise, $\mathfrak{B} = \{\eta, \text{Sp} \otimes \eta, (\text{Ind}_P^G \alpha) \otimes \eta\}$;

where $\eta : G \rightarrow k^\times$ is a smooth character.

Proof. — If k is algebraically closed this is the main result of [56]. It follows from Lemma 5.10, Propositions 5.38 and 5.33 that the same statement is true over k . \square

Remark 5.43. — In fact, [56] also computes the blocks for $p = 3$, the only difference is that the block in case (iv) contains 4 distinct irreducible representations, because $\text{Ind}_p^G \alpha$ is reducible if $p = 3$, and its semi-simplification is isomorphic to $\omega \circ \det \oplus \text{Sp} \otimes \omega \circ \det$. In [58] we have found a new method to compute the blocks, which also works for $p = 2$. If $p = 2$ then the cases (iii) and (iv) collapse to one: a block with two irreducible representations $\mathfrak{B} = \{\eta, \text{Sp} \otimes \eta\}$.

Corollary 5.44. — *Let Π be an absolutely irreducible admissible unitary L -Banach space representation of G with a central character ζ , let Θ be an open bounded G -invariant lattice in Π and let π be an irreducible subquotient of $\Theta \otimes_{\mathcal{O}} k$. Then either π is absolutely irreducible or there exists a smooth character $\chi : \mathbf{Q}_p^\times \rightarrow k^\times$ such that $l := k[\chi(p)]$ is a quadratic extension of k and*

$$\pi \otimes_k l \cong \text{Ind}_p^G \chi \otimes \chi^\sigma \omega^{-1} \oplus \text{Ind}_p^G \chi^\sigma \otimes \chi \omega^{-1},$$

where χ^σ is a conjugate of χ by the non-trivial element in $\text{Gal}(l/k)$.

Proof. — We observe that for every finite extension l of k the irreducible subquotients of π_l are contained in the same block. Since otherwise Proposition 5.36 implies that $\Pi_{L'}$ is not irreducible, where L' is a finite extension of L with residue field l . The assertion follows from the description of irreducible k -representations of G in Proposition 5.11 and Proposition 5.42. \square

It follows from Propositions 5.42, 5.11 and 5.33 that a block \mathfrak{B} contains only finitely many isomorphism classes of irreducible k -representations of G . Fix a representative π_i for each isomorphism class in \mathfrak{B} and let $\tilde{P}_{\mathfrak{B}}$ be a projective envelope of $(\bigoplus_{i=1}^n \pi_i)^\vee$ then $\tilde{E}_{\mathfrak{B}} := \text{End}_{\mathcal{C}(\mathcal{O})}(\tilde{P}_{\mathfrak{B}})$ is a compact ring, see Section 2.

Proposition 5.45. — *The category $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is anti-equivalent to the category of compact right $\tilde{E}_{\mathfrak{B}}$ -modules. The centre of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is isomorphic to the centre of $\tilde{E}_{\mathfrak{B}}$.*

Proof. — See [35, §IV.4 Thm. 4, Cor. 1, Cor. 5]. \square

5.6. Representations of the torus. — Let T be the subgroup of diagonal matrices in G , $T_0 := T \cap I$, $T_1 := T \cap I_1$.

Proposition 5.46. — *Every smooth irreducible k -representation of T is finite dimensional and hence admissible. The absolutely irreducible representations are 1-dimensional.*

Proof. — Let τ be an irreducible smooth k -representation of T . Since T_1 is a pro- p group we have $\tau^{T_1} \neq 0$, and since T_1 is normal in T and τ is irreducible we obtain

$\tau^{T_1} = \tau$. Since T_0/T_1 is a finite group of prime to p order and with all its absolutely irreducible representations defined over \mathbf{F}_p , we may find a smooth character $\chi : T_0 \rightarrow k^\times$, such that χ is a direct summand of $\tau|_{T_0}$. Since T is commutative and τ is irreducible we deduce that $\tau|_{T_0}$ is isomorphic to a direct sum of χ 's, in particular any k -subspace of τ is T_0 -invariant. Choose $t_1, t_2 \in T$ such that their images generate T/T_0 as a group. Let $R = k[t_1^{\pm 1}, t_2^{\pm 1}] \subset k[T]$ then any R -invariant subspace of τ is T_0 -invariant and hence T -invariant. Thus τ is an irreducible R -module and hence is isomorphic to R/\mathfrak{m} , where \mathfrak{m} is a maximal ideal of R . Since R is just the ring of Laurent polynomials in 2 variables, R/\mathfrak{m} is a finite extension of k . Thus, τ is finite dimensional and R/\mathfrak{m} is an absolutely irreducible R -module if and only if $R/\mathfrak{m} \cong k$. \square

Corollary 5.47. — *Let τ_1, τ_2 be smooth irreducible k -representations of T . If $\text{Ind}_p^G \tau_1$ and $\text{Ind}_p^G \tau_2$ have an irreducible subquotient in common then $\tau_1 \cong \tau_2$.*

Proof. — If τ_1 and τ_2 are absolutely irreducible then they are characters and the assertion follows from [1, §7]. In general, since τ_1 and τ_2 are finite dimensional, we may find a finite extension l of k such that

$$\tau_1 \otimes_k l \cong \bigoplus_{\sigma \in \text{Gal}(l/k)} \chi_1^\sigma, \quad \tau_2 \otimes_k l \cong \bigoplus_{\sigma \in \text{Gal}(l/k)} \chi_2^\sigma,$$

where χ_1 and χ_2 are smooth characters $T \rightarrow l^\times$. From the absolutely irreducible case, we deduce that χ_1 is Galois conjugate to χ_2 and thus $\tau_1 \otimes_k l \cong \tau_2 \otimes_k l$. Lemma 5.1 implies $\tau_1 \cong \tau_2$. \square

Lemma 5.48. — *Let $\psi : T \rightarrow k^\times$ be a smooth character such that $\psi|_Z = \zeta$ and let $0 \rightarrow \psi \rightarrow \epsilon \rightarrow \psi \rightarrow 0$ be a non-split extension in $\text{Mod}_{T, \zeta}^{\text{sm}}(k)$. If $p > 2$ then $\dim \text{Ext}_{T, \zeta}^1(\psi, \epsilon) = 2$.*

Proof. — After twisting we may assume that $\psi = \zeta = \mathbf{1}$. As we have seen in Proposition 3.34 the hypotheses (H0)–(H5) are satisfied for $T/Z \cong \mathbf{Q}_p^\times$ and $S = Q = \mathbf{1}^\vee$. Moreover, the endomorphism ring \tilde{E} of the projective envelope of $\mathbf{1}^\vee$ is isomorphic to $\mathcal{O}[[x, y]]$ and $E = \tilde{E} \otimes_{\mathcal{O}} k \cong k[[x, y]]$ with the maximal ideal $\mathfrak{m} = (x, y)$. The non-split extension ϵ^\vee , defines a 1-dimensional subspace W of $\mathfrak{m}/\mathfrak{m}^2$, see Section 3.3. Without loss of generality we may assume that the image of x is a basis of W . Then the image of $\{x, y^2\}$ in $(W + \mathfrak{m}^2)/(W\mathfrak{m} + \mathfrak{m}^3)$ is a basis and the assertion follows from Lemma 3.37. \square

5.7. Colmez's Montreal functor. — Let $\text{Mod}_{G, Z}^{\text{fin}}(\mathcal{O})$ be the full subcategory of $\text{Mod}_G^{\text{sm}}(\mathcal{O})$ consisting of representations of finite length with a central character. Let $\text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(\mathcal{O})$ be the category of continuous $\mathcal{G}_{\mathbf{Q}_p}$ -representations on \mathcal{O} -modules of finite length with the discrete topology, where $\mathcal{G}_{\mathbf{Q}_p}$ is the absolute Galois group of \mathbf{Q}_p . Colmez in [23] has defined an exact covariant functor $\mathbf{V} : \text{Mod}_{G, Z}^{\text{fin}}(\mathcal{O}) \rightarrow \text{Mod}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{fin}}(\mathcal{O})$.

If $\psi : \mathbf{Q}_p^\times \rightarrow \mathcal{O}^\times$ is a continuous character, then we may also consider it as a continuous character $\psi : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathcal{O}^\times$ via (60) and for all $\pi \in \text{Mod}_{G,Z}^{\text{fin}}(\mathcal{O})$ we have $\mathbf{V}(\pi \otimes \psi \circ \det) \cong \mathbf{V}(\pi) \otimes \psi$. Moreover, $\mathbf{V}(\mathbf{1}) = 0$, $\mathbf{V}(\text{Sp}) = \omega$, $\mathbf{V}(\text{Ind}_p^G(\chi_1 \otimes \chi_2 \omega^{-1})) \cong \chi_2$, $\mathbf{V}(\pi(r, 0)) \cong \text{ind}_{\mathcal{G}_{\mathbf{Q}_{p^2}}}^{\mathcal{G}_{\mathbf{Q}_p}} \omega_2^{r+1}$, where ω is the reduction of the cyclotomic character modulo p , $\omega_2 : \mathcal{G}_{\mathbf{Q}_{p^2}} \rightarrow k^\times$ is a character of the absolute Galois group of an unramified quadratic extension of \mathbf{Q}_p , which via local class field theory corresponds to the character $\mathbf{Q}_{p^2}^\times \rightarrow k^\times$, $x \mapsto x|x| \pmod{\varpi}$, see [23, §VII.4]. In particular, the representation $\mathbf{V}(\pi(r, 0))$ is absolutely irreducible.

Let $\zeta : Z \rightarrow \mathcal{O}^\times$ be a continuous character and let $\mathfrak{C}(\mathcal{O})$ be the category dual to $\text{Mod}_{G,\zeta}^{\text{fin}}(\mathcal{O})$. Let $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ be the category of continuous $\mathcal{G}_{\mathbf{Q}_p}$ -representations on compact \mathcal{O} -modules. We define a functor $\check{\mathbf{V}} : \mathfrak{C}(\mathcal{O}) \rightarrow \text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ as follows. Let M be in $\mathfrak{C}(\mathcal{O})$, if it is of finite length then $\check{\mathbf{V}}(M) := \mathbf{V}(M^\vee)^\vee(\varepsilon\zeta)$, where \vee denotes the Pontryagin dual, ε the cyclotomic character and we consider ζ as a character of $\mathcal{G}_{\mathbf{Q}_p}$ via the class field theory. In general, we may write $M \cong \varprojlim M_i$, where the limit is taken over all quotients in $\mathfrak{C}(\mathcal{O})$ of finite length, we define $\check{\mathbf{V}}(M) := \varprojlim \check{\mathbf{V}}(M_i)$. Since we have dualized twice, the functor $\check{\mathbf{V}}$ is covariant. Moreover, it preserves exactness of short exact sequences of objects of finite length. Since all the maps $M_i \rightarrow M_j$ in the projective system are surjective with M_i and M_j of finite length, we deduce that the maps $\check{\mathbf{V}}(M_i) \rightarrow \check{\mathbf{V}}(M_j)$ are surjective. The exactness of projective limits in $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ implies that the functor $\check{\mathbf{V}}$ is exact. Let us note that with our normalization of $\check{\mathbf{V}}$ we have:

$$\check{\mathbf{V}}(\pi^\vee) \cong \mathbf{V}(\pi), \quad \check{\mathbf{V}}((\text{Ind}_p^G \chi_1 \otimes \chi_2 \omega^{-1})^\vee) = \chi_1, \quad \check{\mathbf{V}}((\text{Sp} \otimes \eta \circ \det)^\vee) = \eta,$$

where π is a supersingular representation.

Let Π be an admissible unitary L-Banach space representation of G with central character ζ , and let Θ be an open bounded G -invariant lattice in Π . Then $\Theta/\varpi^n\Theta$ is admissible-smooth representation of G for all $n \geq 1$, and hence locally finite. It follows from Section 4 that Θ^d is an object of $\mathfrak{C}(\mathcal{O})$. Since Θ^d is \mathcal{O} -torsion free and $\check{\mathbf{V}}$ is covariant and exact, $\check{\mathbf{V}}(\Theta^d)$ is \mathcal{O} -torsion free. We let $\check{\mathbf{V}}(\Pi) := \check{\mathbf{V}}(\Theta^d) \otimes_{\mathcal{O}} L$. Since different open lattices in Π are commensurable, $\check{\mathbf{V}}(\Pi)$ does not depend on the choice of Θ .

Lemma 5.49. — *Let Π and Θ be as above. If $\Theta/\varpi\Theta$ is of finite length as a G -representation, then let $\mathbf{V}(\Theta) := \varprojlim \mathbf{V}(\Theta/\varpi^n\Theta)$, and $\mathbf{V}(\Pi) := \mathbf{V}(\Theta) \otimes_{\mathcal{O}} L$. Then $\check{\mathbf{V}}(\Pi) \cong \mathbf{V}(\Pi)^*(\varepsilon\zeta)$, where $*$ denotes L-linear dual. If $\check{\mathbf{V}}(\Pi)$ is 2-dimensional and $\det \check{\mathbf{V}}(\Pi) = \varepsilon\zeta$, then $\check{\mathbf{V}}(\Pi) \cong \mathbf{V}(\Pi)$.*

Proof. — Since \mathbf{V} sends irreducible representations of G to finite dimensional Galois representations, and $\Theta/\varpi\Theta$ is of finite length by assumption, we deduce

that $\mathbf{V}(\Theta/\varpi\Theta)$ is a finite dimensional k -vector space. It follows from [41, 2.2.2] that $\mathbf{V}(\Theta/\varpi^n\Theta)$ is a flat $\mathcal{O}/(\varpi^n)$ -module for all $n \geq 1$ and hence $\mathbf{V}(\Theta)$ is a free \mathcal{O} -module of finite rank, such that $\mathbf{V}(\Theta)/\varpi^n\mathbf{V}(\Theta) \cong \mathbf{V}(\Theta/\varpi^n\Theta)$ for all $n \geq 1$. For every $n \geq 1$ we have $\Theta^d/\varpi^n\Theta^d \cong (\Theta/\varpi^n\Theta)^\vee$, and hence $\check{\mathbf{V}}(\Theta^d)/\varpi^n\check{\mathbf{V}}(\Theta^d) \cong (\mathbf{V}(\Theta)/\varpi^n\mathbf{V}(\Theta))^\vee(\varepsilon\zeta) \cong \mathrm{Hom}_{\mathcal{O}}(\mathbf{V}(\Theta), \mathcal{O}/(\varpi^n))(\varepsilon\zeta)$, for all $n \geq 1$, which implies that $\check{\mathbf{V}}(\Theta^d) \cong \mathrm{Hom}_{\mathcal{O}}(\mathbf{V}(\Theta), \mathcal{O})(\varepsilon\zeta)$, and $\check{\mathbf{V}}(\Pi) \cong \mathbf{V}(\Pi)^*(\varepsilon\zeta)$. The last assertion follows from the fact that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ conjugates $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $(\det A)A^{t-1}$, where t denotes the transpose. \square

We are going to adapt an argument of Kisin [41, §2], which uses Colmez's functor to relate the deformation theory on the $\mathrm{GL}_2(\mathbf{Q}_p)$ -side to the deformation theory on the Galois side.

Lemma 5.50. — *Let M be in $\mathfrak{C}(\mathcal{O})$, let A be a noetherian \mathcal{O} -subalgebra of $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M)$ and let \mathfrak{m} be a finitely generated A -module, then there exists a natural isomorphism $\check{\mathbf{V}}(\mathfrak{m} \otimes_A M) \cong \mathfrak{m} \otimes_A \check{\mathbf{V}}(M)$.*

Proof. — This is identical to [41, 2.2.2], via [39, 1.2.7]. We recall the argument for the sake of completeness. Since $A \subset \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M)$ and $\check{\mathbf{V}}$ is a covariant functor, $\check{\mathbf{V}}(M)$ is naturally an A -module. Since \mathbf{V} is exact and additive, we have an isomorphism $\check{\mathbf{V}}(A^n \otimes_A M) \cong \check{\mathbf{V}}(M^{\oplus n}) \cong \check{\mathbf{V}}(M)^{\oplus n} \cong A^n \otimes_A \check{\mathbf{V}}(M)$. The isomorphism, exactness of $\check{\mathbf{V}}$ and the snake lemma imply that for any finitely presented A -module \mathfrak{m} we have an isomorphism $\mathfrak{m} \otimes_A \check{\mathbf{V}}(M) \cong \check{\mathbf{V}}(\mathfrak{m} \otimes_A M)$. We leave it as an exercise to the reader to check that the isomorphism does not depend on the presentation of \mathfrak{m} and is functorial. Since A is noetherian any finitely generated A -module \mathfrak{m} is also finitely presented. \square

Lemma 5.51. — *Let Q be an object of finite length in $\mathfrak{C}(k)$. Let $A \rightarrow A'$ be a morphism in \mathfrak{A} , let Q_A be a deformation of Q to A in the sense of Definition 3.21. Then $\check{\mathbf{V}}(Q_A)$ is a flat A -module and*

$$A' \otimes_A \check{\mathbf{V}}(Q_A) \cong \check{\mathbf{V}}(A' \otimes_A Q_A).$$

In particular, $k \otimes_A \check{\mathbf{V}}(Q_A) \cong \check{\mathbf{V}}(Q)$ and $\check{\mathbf{V}}(Q_A)$ is a finite free A -module of rank $\dim_k \check{\mathbf{V}}(Q)$.

Proof. — By definition of a deformation, Q_A is A -flat. Hence the functor $\mathfrak{m} \mapsto \mathfrak{m} \otimes_A Q_A$ is exact. Since $\check{\mathbf{V}}$ is exact, using Lemma 5.50 we deduce that the functor $\mathfrak{m} \mapsto \mathfrak{m} \otimes_A \check{\mathbf{V}}(Q_A)$ is exact, so that $\check{\mathbf{V}}(Q_A)$ is A -flat. The A -linear map $Q_A \rightarrow A' \otimes_A Q_A$, $v \mapsto 1 \otimes v$ induces an A -linear map $\check{\mathbf{V}}(Q_A) \rightarrow \check{\mathbf{V}}(A' \otimes_A Q_A)$. Since $\check{\mathbf{V}}$ is a functor, A' acts on $\check{\mathbf{V}}(A' \otimes_A Q_A)$ and by the universality of the tensor product we obtain an A' -linear map $A' \otimes_A \check{\mathbf{V}}(Q_A) \rightarrow \check{\mathbf{V}}(A' \otimes_A Q_A)$. This map is an isomorphism, since it follows from Lemma 5.50 that it is an isomorphism of A -modules. Since Q is of finite length and irreducible subquotients are mapped to finite dimensional k -vector spaces we deduce

that $\check{\mathbf{V}}(\mathcal{Q})$ is finite dimensional, as $k \otimes_A \check{\mathbf{V}}(\mathcal{Q}_A) \cong \check{\mathbf{V}}(k \otimes_A \mathcal{Q}_A) \cong \check{\mathbf{V}}(\mathcal{Q})$, by Lemma 5.50, we deduce the last assertion from Nakayama's lemma for A . \square

Corollary 5.52. — *Let \mathcal{Q} be an object of finite length in $\mathfrak{C}(k)$. The functor $\check{\mathbf{V}}$ induces natural transformations between the deformation functors $\text{Def}_{\mathcal{Q}} \rightarrow \text{Def}_{\check{\mathbf{V}}(\mathcal{Q})}$, $\text{Def}_{\mathcal{Q}}^{\text{cb}} \rightarrow \text{Def}_{\check{\mathbf{V}}(\mathcal{Q})}^{\text{cb}}$, $\mathcal{Q}_A \mapsto \check{\mathbf{V}}(\mathcal{Q}_A)$.*

Lemma 5.53. — *Let $\tilde{\mathcal{P}} \twoheadrightarrow S$ be a projective envelope of an absolutely irreducible object S in $\mathfrak{C}(\mathcal{O})$ and let $\tilde{\mathbb{E}} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathcal{P}})$. For every compact right $\tilde{\mathbb{E}}$ -module \mathfrak{m} there exists a natural isomorphism $\check{\mathbf{V}}(\mathfrak{m} \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}}) \cong \mathfrak{m} \widehat{\otimes}_{\tilde{\mathbb{E}}} \check{\mathbf{V}}(\tilde{\mathcal{P}})$.*

Proof. — Let $\{\mathfrak{m}_i\}_{i \in I}$ be a basis of open neighbourhoods of 0 in \mathfrak{m} , consisting of right $\tilde{\mathbb{E}}$ -modules, and let \mathfrak{a}_i be the $\tilde{\mathbb{E}}$ -annihilator of $\mathfrak{m}/\mathfrak{m}_i$. Since \mathfrak{m}_i is open in \mathfrak{m} , the quotient is an $\tilde{\mathbb{E}}$ -module of finite length, and the quotient topology on $\mathfrak{m}/\mathfrak{m}_i$ is discrete. In particular, \mathfrak{a}_i is open. Moreover, since $\tilde{\mathbb{E}}$ is a local ring with residue field k , we deduce that $\tilde{\mathbb{E}}/\mathfrak{a}_i$ is a finite \mathcal{O} -module, which implies that it is noetherian. Let $\tilde{\mathcal{P}}_i$ be the closure of $\mathfrak{a}_i \tilde{\mathcal{P}}$ in $\tilde{\mathcal{P}}$, so that $\tilde{\mathcal{P}}/\tilde{\mathcal{P}}_i \cong \tilde{\mathbb{E}}/\mathfrak{a}_i \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}}$. Then $\check{\mathbf{V}}(\mathfrak{m}/\mathfrak{m}_i \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}}) \cong \check{\mathbf{V}}(\mathfrak{m}/\mathfrak{m}_i \otimes_{\tilde{\mathbb{E}}/\mathfrak{a}_i} \tilde{\mathcal{P}}/\tilde{\mathcal{P}}_i) \cong \mathfrak{m}/\mathfrak{m}_i \otimes_{\tilde{\mathbb{E}}/\mathfrak{a}_i} \check{\mathbf{V}}(\tilde{\mathcal{P}}/\tilde{\mathcal{P}}_i) \cong \mathfrak{m}/\mathfrak{m}_i \widehat{\otimes}_{\tilde{\mathbb{E}}} \check{\mathbf{V}}(\tilde{\mathcal{P}})$, where the second isomorphism is given by Lemma 5.50. Since $\check{\mathbf{V}}$ and $\widehat{\otimes}$ commute with projective limits and $\mathfrak{m} \cong \varprojlim \mathfrak{m}/\mathfrak{m}_i$, by passing to the limit we obtain $\check{\mathbf{V}}(\mathfrak{m} \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}}) \cong \mathfrak{m} \widehat{\otimes}_{\tilde{\mathbb{E}}} \check{\mathbf{V}}(\tilde{\mathcal{P}})$. \square

Corollary 5.54. — *Let $\tilde{\mathcal{P}} \twoheadrightarrow S$ be a projective envelope of an absolutely irreducible object S in $\mathfrak{C}(\mathcal{O})$ and let $\tilde{\mathbb{E}} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathcal{P}})$. Let Π be an irreducible admissible unitary L -Banach space representation of G with a central character ζ and the reduction $\overline{\Pi}$ of finite length. Suppose that S^\vee is a subquotient of $\overline{\Pi}$ and let Ξ be an open bounded G -invariant lattice in Π such that the natural map $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathcal{P}}, \Xi^d) \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}} \rightarrow \Xi^d$ is surjective, see Proposition 4.18(iii). Then we have a surjection*

$$\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathcal{P}}, \Xi^d) \widehat{\otimes}_{\tilde{\mathbb{E}}} \check{\mathbf{V}}(\tilde{\mathcal{P}}) \cong \check{\mathbf{V}}(\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathcal{P}}, \Xi^d) \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}}) \twoheadrightarrow \check{\mathbf{V}}(\Xi^d).$$

Proof. — We note that since $\overline{\Pi}$ is of finite length, $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathcal{P}}, \Xi^d)$ is a finitely generated \mathcal{O} -module by Lemma 4.15. The isomorphism follows from Lemma 5.53, the surjection from the exactness of $\check{\mathbf{V}}$. \square

Corollary 5.55. — *Let $\tilde{\mathcal{P}} \twoheadrightarrow S$ be a projective envelope of an absolutely irreducible object S in $\mathfrak{C}(\mathcal{O})$, let $\tilde{\mathbb{E}} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathcal{P}})$ and let $\widehat{\text{Tor}}_{\tilde{\mathbb{E}}}^i(*, \tilde{\mathcal{P}})$ be the i -th derived functor of $\widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}}$ in the category of compact right $\tilde{\mathbb{E}}$ -modules. If $\text{SL}_2(\mathbf{Q}_p)$ acts trivially on $\widehat{\text{Tor}}_{\tilde{\mathbb{E}}}^1(k, \tilde{\mathcal{P}})$ then the functor $\mathfrak{m} \mapsto \mathfrak{m} \widehat{\otimes}_{\tilde{\mathbb{E}}} \check{\mathbf{V}}(\tilde{\mathcal{P}})$ is exact. Moreover, if $k \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}}$ is of finite length then $\check{\mathbf{V}}(\tilde{\mathcal{P}})$ is a free $\tilde{\mathbb{E}}$ -module of rank equal to $\dim_k \check{\mathbf{V}}(k \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathcal{P}})$.*

Proof. — Since every compact $\tilde{\mathbb{E}}$ -module can be written as a projective limit of $\tilde{\mathbb{E}}$ -modules of finite length, and $\widehat{\otimes}$ commutes with projective limits, which are exact in the

category of continuous $\mathcal{G}_{\mathbf{Q}_p}$ -representations on compact \mathcal{O} -modules, it is enough to show that the functor $m \mapsto m \widehat{\otimes}_{\tilde{E}} \check{\mathbf{V}}(\tilde{P})$ maps short exact sequences of continuous \tilde{E} -modules of finite length to short exact sequences.

If m is a continuous \tilde{E} -module of finite length then by devissage we deduce that $\mathrm{SL}_2(\mathbf{Q}_p)$ acts trivially on $\widehat{\mathrm{Tor}}_{\tilde{E}}^1(m, \tilde{P})$. Since $\check{\mathbf{V}}$ kills every irreducible on which $\mathrm{SL}_2(\mathbf{Q}_p)$ acts trivially, we obtain $\check{\mathbf{V}}(\widehat{\mathrm{Tor}}_{\tilde{E}}^1(m, \tilde{P})) = 0$ for all finite length modules m . Using Lemma 5.50 we deduce that the functor $m \mapsto m \widehat{\otimes}_{\tilde{E}} \check{\mathbf{V}}(\tilde{P})$ maps short exact sequence of continuous \tilde{E} -modules of finite length to short exact sequences.

If $k \widehat{\otimes}_{\tilde{E}} \tilde{P}$ is of finite length then $\check{\mathbf{V}}(k \widehat{\otimes}_{\tilde{E}} \tilde{P})$ is a finite dimensional k -vector space. Since $k \widehat{\otimes}_{\tilde{E}} \check{\mathbf{V}}(\tilde{P}) \cong \check{\mathbf{V}}(k \widehat{\otimes}_{\tilde{E}} \tilde{P})$ by Lemma 5.50, we deduce from Nakayama's lemma that $\check{\mathbf{V}}(\tilde{P})$ is a free \tilde{E} -module of rank equal to $\dim_k \check{\mathbf{V}}(k \widehat{\otimes}_{\tilde{E}} \tilde{P})$. \square

Let $\tilde{P} \twoheadrightarrow S$ be a projective envelope of an absolutely irreducible object S in $\mathfrak{C}(\mathcal{O})$, let $\tilde{E} = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P})$. From now on we assume the existence of Q in $\mathfrak{C}(k)$ of finite length, satisfying the hypothesis (H1)–(H5) in $\mathfrak{C}(k)$. Since (H0) holds in $\mathfrak{C}(\mathcal{O})$ by Corollary 5.19, the hypotheses (H1)–(H5) hold in $\mathfrak{C}(\mathcal{O})$ by Proposition 3.17, and so \tilde{P} is a flat \tilde{E} -module by Corollary 3.12.

Since Q is of finite length, $\check{\mathbf{V}}(Q)$ is a continuous representation of $\mathcal{G}_{\mathbf{Q}_p}$ on a finite dimensional k -vector space. Let $\mathrm{Def}_{\check{\mathbf{V}}(Q)}^{ab, \psi}$ be a subfunctor of $\mathrm{Def}_{\check{\mathbf{V}}(Q)}^{ab}$ parameterising deformations with determinant equal to $\psi := \varepsilon \zeta$, where ε is the cyclotomic character. If $\mathrm{Def}_{\check{\mathbf{V}}(Q)}^{ab}$ is pro-representable then so is $\mathrm{Def}_{\check{\mathbf{V}}(Q)}^{ab, \psi}$, see [48, §24], and we denote the corresponding ring by R^ψ , and the universal deformation of $\check{\mathbf{V}}(Q)$ with determinant equal to ψ by $\rho^{un, \psi}$. Let \mathfrak{m} be a maximal ideal of $R^\psi[1/p]$, then the residue field $\kappa(\mathfrak{m})$ is a finite totally ramified extension of L , and the image of R^ψ in $\kappa(\mathfrak{m})$ is equal to the ring of integers $\mathcal{O}_{\kappa(\mathfrak{m})}$. Let \mathfrak{a} be the intersection of those maximal ideals \mathfrak{m} of $R^\psi[1/p]$ for which $\kappa(\mathfrak{m}) \otimes_{R^\psi} \rho^{un, \psi}$ is an absolutely irreducible $\kappa(\mathfrak{m})[\mathcal{G}_{\mathbf{Q}_p}]$ -module and let R' be the image of R^ψ in $R^\psi[1/p]/\mathfrak{a}$.

Proposition 5.56. — *Assume that S^\vee is not a character, so that $\check{\mathbf{V}}(S) \neq 0$. Suppose that the following hold:*

- (i) $\mathrm{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\check{\mathbf{V}}(Q)) = k$;
- (ii) \mathbf{V} induces an injection,

$$\mathrm{Ext}_{G, \zeta}^1(Q^\vee, Q^\vee) \hookrightarrow \mathrm{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\mathbf{V}(Q^\vee), \mathbf{V}(Q^\vee));$$

- (iii) *for every irreducible representation ρ of $\mathcal{G}_{\mathbf{Q}_p}$ defined over some finite totally ramified extension L' of L and satisfying $\det \rho = \psi$ and $\bar{\rho} \cong \check{\mathbf{V}}(Q)^{ss}$ there exists an open bounded G -invariant lattice Ξ in a unitary admissible L' -Banach space representation Π of G such that the following hold:*

- (a) ζ is the central character of Π ;
- (b) $\overline{\Pi}$ contains S^\vee with multiplicity 1;
- (c) $\rho \cong \check{\mathbf{V}}(\Xi^d) \otimes_{\mathcal{O}} L$.

Then there exists a natural surjection $\tilde{\mathbf{E}}^{ab} \rightarrow \mathbf{R}'$.

Proof. — We note that in this Proposition we allow only commutative coefficients for our deformations. In particular, all the rings representing different functors are commutative. Corollary 5.52 gives a natural transformation of functors $\text{Def}_{\mathbf{Q}}^{ab} \rightarrow \text{Def}_{\check{\mathbf{V}}(\mathbf{Q})}^{ab}$. Since both functors are pro-representable we obtain a map $\varphi : \mathbf{R} \rightarrow \tilde{\mathbf{E}}^{ab}$, where \mathbf{R} is the ring pro-representing $\text{Def}_{\check{\mathbf{V}}(\mathbf{Q})}^{ab}$. Now (ii) is equivalent to the assertion that $\check{\mathbf{V}}$ induces an injection

$$\text{Def}_{\mathbf{Q}}^{ab}(k[\epsilon]) \cong \text{Ext}_{\mathfrak{e}(k)}^1(\mathbf{Q}, \mathbf{Q}) \hookrightarrow \text{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\check{\mathbf{V}}(\mathbf{Q}), \check{\mathbf{V}}(\mathbf{Q})) \cong \text{Def}_{\check{\mathbf{V}}(\mathbf{Q})}^{ab}(k[\epsilon]),$$

which is equivalent to the assertion that φ induces a surjection

$$\mathbf{m}_{\mathbf{R}}/(\mathbf{m}_{\mathbf{R}}^2 + \varpi \mathbf{R}) \rightarrow \tilde{\mathbf{m}}_{ab}/(\tilde{\mathbf{m}}_{ab}^2 + \varpi \tilde{\mathbf{E}}^{ab}).$$

Since both rings are complete we deduce that $\varphi : \mathbf{R} \rightarrow \tilde{\mathbf{E}}^{ab}$ is surjective.

Let \mathfrak{m} be a maximal ideal of $\mathbf{R}^\psi[1/p]$, such that $\kappa(\mathfrak{m}) \otimes_{\mathbf{R}^\psi} \rho^{um, \psi}$ is absolutely irreducible. We claim that there exists a map of \mathcal{O} -algebras $x : \tilde{\mathbf{E}} \rightarrow \mathcal{O}_{\kappa(\mathfrak{m})}$, such that $\kappa(\mathfrak{m}) \otimes_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}})$ is isomorphic to $\kappa(\mathfrak{m}) \otimes_{\mathbf{R}^\psi} \rho^{um, \psi}$ as a $\kappa(\mathfrak{m})$ -representation of $\mathcal{G}_{\mathbf{Q}_p}$. Since $\check{\mathbf{V}}(\mathbf{Q})$ has only scalar endomorphisms by (i), there exists a unique $\mathcal{G}_{\mathbf{Q}_p}$ -invariant $\mathcal{O}_{\kappa(\mathfrak{m})}$ -lattice in $\kappa(\mathfrak{m}) \otimes_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}})$, which reduces to $\check{\mathbf{V}}(\mathbf{Q})$ modulo the maximal ideal of $\mathcal{O}_{\kappa(\mathfrak{m})}$. Hence, the claim implies that $\mathcal{O}_{\kappa(\mathfrak{m})} \otimes_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}})$ and $\mathcal{O}_{\kappa(\mathfrak{m})} \otimes_{\mathbf{R}^\psi} \rho^{um, \psi}$ define the same deformation of $\check{\mathbf{V}}(\mathbf{Q})$ with determinant ψ . Thus the natural map $\mathbf{R}^\psi \rightarrow \kappa(\mathfrak{m})$ factors through $x \circ \varphi$, which implies that the surjection $\mathbf{R}^\psi \rightarrow \mathbf{R}'$ factors through $\varphi : \mathbf{R}^\psi \rightarrow \tilde{\mathbf{E}}^{ab}$.

We will deduce the claim from (iii). Let Ξ and Π be as in (iii) with $L' = \kappa(\mathfrak{m})$ and $\rho = \kappa(\mathfrak{m}) \otimes_{\mathbf{R}^\psi} \rho^{um, \psi}$, so that $\rho \cong \check{\mathbf{V}}(\Xi^d) \otimes_{\mathcal{O}} L$. As $\check{\mathbf{V}}$ is exact, and Ξ^d is \mathcal{O} -torsion free, we deduce that $\check{\mathbf{V}}(\Xi^d)$ is \mathcal{O} -torsion free and it follows from (iii)(c) that $\check{\mathbf{V}}(\Xi^d)$ is an $\mathcal{O}_{\kappa(\mathfrak{m})}$ -lattice in ρ . Part (iii)(b) implies that S occurs as a subquotient of $k \otimes_{\mathcal{O}_{\kappa(\mathfrak{m})}} \Xi^d$ with multiplicity one. It follows from Lemma 4.15 and Corollary 5.40 that $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbf{P}}, \Xi^d)$ is a free $\mathcal{O}_{\kappa(\mathfrak{m})}$ -module of rank 1. The action of $\tilde{\mathbf{E}}$ gives us an \mathcal{O} -linear map $x : \tilde{\mathbf{E}} \rightarrow \text{End}_{\mathcal{O}_{\kappa(\mathfrak{m})}}(\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbf{P}}, \Xi^d)) \cong \mathcal{O}_{\kappa(\mathfrak{m})}$. Let C be cokernel of the natural map $\mathcal{O}_{\kappa(\mathfrak{m})} \widehat{\otimes}_{\tilde{\mathbf{E}}, x} \tilde{\mathbf{P}} \rightarrow \Xi^d$. It follows from Lemma 2.9 that $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbf{P}}, C) = 0$ thus S is not a subquotient of C by Lemma 4.13. Since $\check{\mathbf{V}}(S) \neq 0$ by assumption, and $\check{\mathbf{V}}$ maps distinct irreducibles to distinct irreducibles, we deduce that $\check{\mathbf{V}}(S)$ is not a subquotient of $\check{\mathbf{V}}(C)$. Hence, the map $\check{\mathbf{V}}(\mathcal{O}_{\kappa(\mathfrak{m})} \widehat{\otimes}_{\tilde{\mathbf{E}}, x} \tilde{\mathbf{P}}) \rightarrow \check{\mathbf{V}}(\Xi^d)$ is non-zero. Lemma 5.53 and the irreducibility of ρ implies that the induced map $\kappa(\mathfrak{m}) \otimes_{\tilde{\mathbf{E}}, x} \check{\mathbf{V}}(\tilde{\mathbf{P}}) \rightarrow \rho$ is surjective. The map is an isomorphism as both $\kappa(\mathfrak{m})$ -vector spaces have dimension equal to $\dim_k \check{\mathbf{V}}(\mathbf{Q})$. \square

5.8. *The strategy in the generic case.* — We are now in a position to explain how in the generic case the proof of the main theorem reduces to a computation of dimensions of some Ext groups in the category of smooth k -representations of G with a central character, when $p \geq 5$. By the generic case we mean that Q^\vee is an *atome atomorphe* in the sense of Colmez, which is either irreducible supersingular, so that $S = Q$, or Q^\vee is a non-split extension of $\text{Ind}_p^G \chi_1 \otimes \chi_2 \omega^{-1}$ by $\text{Ind}_p^G \chi_2 \otimes \chi_1 \omega^{-1}$ with $\chi_1 \chi_2^{-1} \neq \omega^{\pm 1}$, $\mathbf{1}$ and S^\vee is a principal series representation.

We know that the hypothesis (H0) is satisfied by Proposition 5.19 and to verify (H1)–(H5) we only need to compute the dimensions of some Ext groups. Suppose that we can do this and (H1)–(H5) hold. Now $\check{\mathbf{V}}(Q)$ is 2-dimensional and is either irreducible or a non-split extension of two characters χ_2 by χ_1 . Since $p \geq 5$ and $\chi_1 \chi_2^{-1} \neq \omega^{\pm 1}$, $\mathbf{1}$ the universal deformation ring $\text{Def}_{\check{\mathbf{V}}(Q)}^{ab}$ is representable by $R \cong \mathcal{O}[[x_1, \dots, x_5]]$ and the deformation ring with the determinant condition R^ψ is isomorphic to $\mathcal{O}[[x_1, x_2, x_3]]$. Moreover, one may show that the irreducible locus is dense, hence the ring R' introduced before Proposition 5.56 is isomorphic to R^ψ . The condition (ii) in Proposition 5.56 in this case is a result of Colmez [23, VII.5.2], and the condition (iii) is a result of Kisin [41, 2.3.8]. Hence, Proposition 5.56 gives us a surjection $\tilde{E}^{ab} \twoheadrightarrow R^\psi \cong \mathcal{O}[[x_1, x_2, x_3]]$. One may calculate that $\dim \text{Ext}_{G, \zeta}^1(Q^\vee, Q^\vee) = 3$ and hence $\dim \tilde{m}/(\tilde{m}^2 + \varpi \tilde{E}) = 3$. If we can show that for every non-split extension $0 \rightarrow Q^\vee \rightarrow \tau \rightarrow Q^\vee \rightarrow 0$ in $\text{Mod}_{G, \zeta}^{\text{sm}}(k)$ the dimension of $\text{Ext}_{G, \zeta}^1(S^\vee, \tau)$ is at most 3 then using Theorem 3.39 we may deduce that $\tilde{E} \cong R^\psi$. In particular, \tilde{E} is commutative and hence Corollary 4.44 says that every absolutely irreducible admissible unitary L-Banach space representation Π of G with the central character ζ and such that $\overline{\Pi}$ contains S^\vee satisfies $\overline{\Pi} \subseteq (Q^\vee)^{\text{ss}}$.

6. Supersingular representations

In this section we carry out the strategy described in Section 5.8 in the supersingular case. The main result is Theorem 6.4 and its Corollaries. In Section 6.1 we carry out some Ext calculations, we suggest to skip them at first reading. We assume throughout this section that $p \geq 5$. Let $\pi \cong \pi(r, 0, \eta)$ be a supersingular representation with a central character congruent to ζ .

Proposition 6.1. — *The hypotheses (H1)–(H5) of Section 3.1 hold with $Q = S = \pi^\vee$. Moreover, $\dim \text{Ext}_{\mathfrak{C}(k)}^1(S, S) = \dim \text{Ext}_{G, \zeta}^1(\pi, \pi) = 3$.*

Proof. — Let τ be an irreducible representation in $\text{Mod}_{G, \zeta}^{\text{sm}}(k)$ not isomorphic to π , then $\text{Ext}_{G, \zeta}^1(\tau, \pi) = 0$, [56, 10.7]. Moreover, $\dim \text{Ext}_{G, \zeta}^1(\pi, \pi) = 3$, [56, 10.13]. This implies (H3) and (H4) via Corollary 5.17, the rest holds trivially. \square

Since (H0) holds vacuously in the supersingular case, we may apply the results of Section 3.1 and Section 4. Let $\tilde{P} \twoheadrightarrow S$ be a projective envelope of S in $\mathfrak{C}(\mathcal{O})$, let $\tilde{E} =$

$\text{End}_{\mathcal{O}}(\tilde{\mathbb{P}})$, $\tilde{\mathfrak{m}}$ the maximal ideal of $\tilde{\mathbb{E}}$ and let \mathfrak{m} be the maximal ideal of $\tilde{\mathbb{E}} \otimes_{\mathcal{O}} k$. We note that Proposition 6.1 and Lemma 3.29 imply that $d := \dim \mathfrak{m}/\mathfrak{m}^2 = 3$. Let $\rho := \mathbf{V}(\pi)$ then $\rho \cong \text{ind } \omega_2^{r+1} \otimes \eta$ is absolutely irreducible. We note that $\det \rho$ is congruent to $\zeta \varepsilon$, where ε is the cyclotomic character. Let \mathbf{R}_ρ be the universal deformation ring of ρ and $\mathbf{R}_\rho^{\zeta \varepsilon}$ be the deformation ring of ρ pro-representing a deformation problem with a fixed determinant equal to $\zeta \varepsilon$.

Proposition 6.2. — *The functor $\check{\mathbf{V}}$ induces a surjection*

$$\tilde{\mathbb{E}} \twoheadrightarrow \mathbf{R}_\rho^{\zeta \varepsilon} \cong \mathcal{O}[[x_1, x_2, x_3]].$$

Proof. — This is a consequence of Proposition 5.56. We note that $\check{\mathbf{V}}(\mathbf{S}) \cong \mathbf{V}(\pi) = \rho$. Since $p \geq 5$ using local Tate duality and Euler characteristic, we may calculate that $\mathbf{H}^2(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho) = 0$ and $\mathbf{H}^1(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho)$ is 5-dimensional. This implies, see [47, §1.6], [48, §24], that the universal deformation problem $\text{Def}_{\check{\mathbf{V}}(\mathbf{S})}^{ab}$ is represented by $\mathbf{R}_\rho \cong \mathcal{O}[[x_1, \dots, x_5]]$ and the deformation problem with the fixed determinant is represented by $\mathbf{R}_\rho^{\zeta \varepsilon} \cong \mathcal{O}[[x_1, x_2, x_3]]$. Since the residual representation is irreducible, the ring \mathbf{R}' in Proposition 5.56 is isomorphic to $\mathbf{R}_\rho^{\zeta \varepsilon}$. Part (ii) of Proposition 5.56 is satisfied by [23, VII.5.2], and (iii) is satisfied by [41, 2.3.8]. \square

Proposition 6.3. — *The functor $\check{\mathbf{V}}$ induces an isomorphism $\tilde{\mathbb{E}} \cong \mathbf{R}_\rho^{\zeta \varepsilon}$. In particular, $\tilde{\mathbb{E}}$ is commutative and $\check{\mathbf{V}}(\tilde{\mathbb{P}})$ is the universal deformation of ρ with determinant $\zeta \varepsilon$.*

Proof. — Since $\dim \mathfrak{m}/\mathfrak{m}^2 = 3$ we deduce from the map in Proposition 6.2 induces an isomorphism $\tilde{\mathbb{E}}^{ab} \cong \mathbf{R}_\rho^{\zeta \varepsilon} \cong \mathcal{O}[[x_1, x_2, x_3]]$. It follows then from Lemma 3.41 that the natural map $\text{Hom}(\mathbf{E}, k[x]/(x^3)) \rightarrow \text{Hom}(\mathbf{E}, k[x]/(x^2))$ is surjective. In view of Theorem 3.39 and Lemma 3.43, it is enough to find a 2-dimensional subspace \mathbf{V} of $\text{Ext}_{\mathbf{G}, \zeta}^1(\pi, \pi)$ such that for every non-zero $\xi \in \mathbf{V}$, representing an extension $0 \rightarrow \pi \rightarrow \mathbf{E}_\xi \rightarrow \pi \rightarrow 0$ we have $\dim \text{Ext}_{\mathbf{G}, \zeta}^1(\pi, \mathbf{E}_\xi) \leq 3$ or equivalently $\dim \text{Ext}_{\mathbf{G}, \zeta}^1(\mathbf{E}_\xi, \pi) \leq 3$.

We have shown in [56, 10.14] that for any non-zero ξ lying in the image of $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\pi), \mathcal{I}(\pi))$ in $\text{Ext}_{\mathbf{G}, \zeta}^1(\pi, \pi)$ via (73), we have $\dim \text{Ext}_{\mathbf{G}, \zeta}^1(\mathbf{E}_\xi, \pi) \leq 3$. In the regular case, we have $\dim \text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\pi), \mathcal{I}(\pi)) = 2$, [20, Cor. 6.6], and so we are done. In the Iwahori case, $\dim \text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\pi), \mathcal{I}(\pi)) = 1$, but in Proposition 6.23 below, we find a two dimensional subspace \mathbf{V} in $\text{Ext}_{\mathbf{G}, \zeta}^1(\pi, \pi)$ such that for any non-zero $\xi \in \mathbf{V}$ we have $\dim \text{Ext}_{\mathbf{G}, \zeta}^1(\pi, \mathbf{E}_\xi) \leq 3$. Hence, $\check{\mathbf{V}}$ induces an isomorphism of deformation functors, Corollary 5.52, and so $\check{\mathbf{V}}(\tilde{\mathbb{P}})$ is the universal deformation of ρ with determinant $\zeta \varepsilon$. \square

Theorem 6.4. — *Let Π be a unitary absolutely irreducible L -Banach space representation with a central character ζ . Suppose that the reduction of some open \mathbf{G} -invariant lattice in Π contains π as a subquotient then $\overline{\Pi} \cong \pi$.*

Proof. — Since $\tilde{\mathbb{E}}$ is commutative the assertion follows from Corollary 4.44. \square

Recall that the block \mathfrak{B} of π consists of only one isomorphism class, Proposition 5.42. Then $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is the full subcategory of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$ consisting of representations with every irreducible subquotient isomorphic to π .

Corollary 6.5. — *The category $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is anti-equivalent to the category of compact $R_{\rho}^{\zeta\varepsilon}$ -modules. The centre of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is naturally isomorphic to $R_{\rho}^{\zeta\varepsilon}$.*

Proof. — The assertion follows from Proposition 5.45 and Proposition 6.3. \square

Remark 6.6. — Since ρ is absolutely irreducible and $p > 2 = \dim \rho$, sending a deformation to its trace induces an isomorphism between $R_{\rho}^{\zeta\varepsilon}$ and $R_{\text{tr}\rho}^{\text{ps},\zeta\varepsilon}$, the deformation ring parameterizing 2-dimensional pseudocharacters with determinant $\zeta\varepsilon$ lifting $\text{tr}\rho$, see [50].

Corollary 6.7. — *Let $T : \mathcal{G}_{\mathbf{Q}_p} \rightarrow R_{\text{tr}\rho}^{\text{ps},\zeta\varepsilon}$ be the universal 2-dimensional pseudocharacter with determinant $\zeta\varepsilon$ lifting $\text{tr}\rho$. For every N in $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$, $\check{\mathbf{V}}(N)$ is killed by $g^2 - T(g)g + \zeta\varepsilon(g)$, for all $g \in \mathcal{G}_{\mathbf{Q}_p}$.*

Proof. — Proposition 6.3 and Remark 6.6 imply that the assertion is true if $N = \tilde{P}$. In general, $\check{\mathbf{V}}(N) \cong \check{\mathbf{V}}(\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, N) \hat{\otimes}_{\tilde{\mathbb{E}}} \tilde{P}) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, N) \hat{\otimes}_{\tilde{\mathbb{E}}} \check{\mathbf{V}}(\tilde{P})$, by Lemma 5.53. \square

Let $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})^{\mathfrak{B}}$ be as in Proposition 5.36 and let $\text{Ban}_{G,\zeta}^{\text{adm,fl}}(\mathbf{L})^{\mathfrak{B}}$ be the full subcategory consisting of objects of finite length.

Corollary 6.8. — *We have an equivalence of categories*

$$\text{Ban}_{G,\zeta}^{\text{adm,fl}}(\mathbf{L})^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{n} \in \text{MaxSpec } R_{\rho}^{\zeta\varepsilon}[1/p]} \text{Ban}_{G,\zeta}^{\text{adm,fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}.$$

The category $\text{Ban}_{G,\zeta}^{\text{adm,fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $R_{\rho}^{\zeta\varepsilon}[1/p]$. In particular, $\text{Ban}_{G,\zeta}^{\text{adm,fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ contains only one irreducible object.

Proof. — Apply Theorem 4.36 with $\mathfrak{C}(\mathcal{O}) = \mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. \square

6.1. Iwahori case. — Let $\pi \cong \pi(0, 0, \eta) \cong \pi(p-1, 0, \eta)$. In this section we identify a two dimensional subspace V of $\text{Ext}_{G,\zeta}^1(\pi, \pi)$ such that for any non-zero $\xi \in V$, the equivalence class of an extension $0 \rightarrow \pi \rightarrow E_{\xi} \rightarrow \pi \rightarrow 0$, we have either $\dim \text{Ext}_{G,\zeta}^1(E_{\xi}, \pi) \leq 3$ or $\dim \text{Ext}_{G,\zeta}^1(\pi, E_{\xi}) \leq 3$, thus completing the proof of Proposition 6.3. The proof involves tracking down the dimension of various Ext groups. Essentially the same argument should also work for $p = 3$, but we have not checked the details.

After twisting we may assume that η is the trivial character, and so Z acts trivially on π . We will write $\text{Mod}_{G/Z}^{\text{sm}}(k)$ instead of $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$ and $\text{Ext}_{G/Z}^1$ instead of $\text{Ext}_{G,\zeta}^1$. To ease the notation in this section we will also write Rep to mean smooth representations on k -vector spaces. It follows from [16, 3.2.4, 4.1.4] that π^{I_1} is 2-dimensional. Moreover, [16, 4.1.5] implies that there exists a basis $\{v_{\mathbf{1}}, v_{\text{st}}\}$ of π^{I_1} , such that $\Pi v_{\mathbf{1}} = v_{\text{st}}$, $\Pi v_{\text{st}} = v_{\mathbf{1}}$ and there exists an isomorphism of \mathbf{K} -representations:

$$(89) \quad \langle \mathbf{K} \cdot v_{\mathbf{1}} \rangle \cong \mathbf{1}, \quad \langle \mathbf{K} \cdot v_{\text{st}} \rangle \cong \text{st},$$

where st is the inflation of the Steinberg representation of $\text{GL}_2(\mathbf{F}_p)$. In particular, \mathbf{H} acts trivially on $v_{\mathbf{1}}$ and v_{st} . We recall the results of [56, §4]. Let

$$(90) \quad \begin{aligned} \mathbf{M}_{\mathbf{1}} &:= \left\langle \begin{pmatrix} p^{2n} & b \\ 0 & 1 \end{pmatrix} v_{\mathbf{1}} : n \geq 0, b \in \mathbf{Z}_p \right\rangle, \\ \mathbf{M}_{\text{st}} &:= \left\langle \begin{pmatrix} p^{2n} & b \\ 0 & 1 \end{pmatrix} v_{\text{st}} : n \geq 0, b \in \mathbf{Z}_p \right\rangle. \end{aligned}$$

Then $\mathbf{M}_{\mathbf{1}}, \mathbf{M}_{\text{st}}$ are stable under the action of \mathbf{I} , [56, 4.6], $\mathbf{M}_{\mathbf{1}}^{I_1} = kv_{\mathbf{1}}$ and $\mathbf{M}_{\text{st}}^{I_1} = kv_{\text{st}}$. We set

$$(91) \quad \pi_{\mathbf{1}} := \mathbf{M}_{\mathbf{1}} + \Pi \cdot \mathbf{M}_{\text{st}}, \quad \pi_{\text{st}} := \mathbf{M}_{\text{st}} + \Pi \cdot \mathbf{M}_{\mathbf{1}}.$$

The subspaces $\pi_{\mathbf{1}}$ and π_{st} are stable under the action of G^+ , [56, 4.12]. Moreover, we have

$$(92) \quad \pi|_{G^+} \cong \pi_{\mathbf{1}} \oplus \pi_{\text{st}}.$$

This implies

$$(93) \quad \pi \cong \text{Ind}_{G^+}^G \pi_{\mathbf{1}} \cong \text{Ind}_{G^+}^G \pi_{\text{st}}.$$

Further, [56, 6.4] says that

$$(94) \quad \pi_{\mathbf{1}}^{I_1} = \mathbf{M}_{\mathbf{1}} \cap \Pi \mathbf{M}_{\text{st}} = kv_{\mathbf{1}}, \quad \pi_{\text{st}}^{I_1} = \mathbf{M}_{\text{st}} \cap \Pi \mathbf{M}_{\mathbf{1}} = kv_{\text{st}}.$$

The key observation that goes into the proof of this result is that the restrictions of $\mathbf{M}_{\mathbf{1}}$ and \mathbf{M}_{st} to $\mathbf{H}(\mathbf{I} \cap \mathbf{U})$ are injective envelopes of the trivial representation in $\text{Rep}_{\mathbf{H}(\mathbf{I} \cap \mathbf{U})}$.

Lemma 6.9. — *Let \mathbf{N} be a representation of \mathbf{I}/Z_1 such that $\mathbf{N}|_{\mathbf{I}_1 \cap \mathbf{U}}$ is an injective envelope of the trivial representation in $\text{Rep}_{\mathbf{I}_1 \cap \mathbf{U}}$. Let $v \in \mathbf{N}$ such that \mathbf{H} acts on v by a character χ and let $\kappa := \langle \mathbf{I} \cdot v \rangle$, then*

- (i) $\dim(\mathbf{N}/\kappa)^{I_1} = 1$;
- (ii) \mathbf{H} acts on $(\mathbf{N}/\kappa)^{I_1}$ by a character $\chi\alpha^{-1}$;
- (iii) $\chi\alpha^{-1} \hookrightarrow (\mathbf{N}/\kappa)|_{\mathbf{H}(\mathbf{I}_1 \cap \mathbf{U})}$ is an injective envelope of $\chi\alpha^{-1}$ in $\text{Rep}_{\mathbf{H}(\mathbf{I}_1 \cap \mathbf{U})}$.

Proof. — Since N is smooth and κ is finitely generated, κ is of finite length. We argue by induction on the length ℓ of κ . Suppose the length of κ is 1, then $\kappa = N^{I_1} = N^{I_1 \cap U}$ and the assertion follows from [56, Prop. 5.9]. In general, let $N_1 := N/N^{I_1}$ and let κ_1 denote the image of κ in N_1 . Now [56, Prop. 5.9] says that $N_1|_{I_1 \cap U}$ is an injective envelope of the trivial representation in $\text{Rep}_{I_1 \cap U}$. Since $N^{I_1} = N^{I_1 \cap U}$ is 1-dimensional and $\kappa \cap N^{I_1} = \kappa^{I_1} \neq 0$ we have $\ell(\kappa_1) = \ell(\kappa) - 1$, and hence we get the assertion by induction. \square

Proposition 6.10. — *Let*

$$(95) \quad w_{\mathbf{1}} := \sum_{\lambda, \mu \in \mathbf{F}_p} \lambda \begin{pmatrix} 1 & [\mu] + \rho[\lambda] \\ 0 & 1 \end{pmatrix} t^2 v_{\mathbf{1}}, \quad w_{\text{st}} := \sum_{\lambda \in \mathbf{F}_p} \lambda \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v_{\mathbf{1}},$$

and set $\tau := \langle \mathbf{K} \cdot w_{\mathbf{1}} \rangle + \langle \mathbf{K} \cdot (\Pi w_{\text{st}}) \rangle \subset \pi_{\mathbf{1}}$. *There exist an exact non-split sequence of \mathbf{K} -representations*

$$(96) \quad 0 \rightarrow \mathbf{1} \rightarrow \tau \rightarrow \text{Ind}_1^{\mathbf{K}} \alpha \rightarrow 0.$$

Moreover, $\text{Ext}_{\mathbf{1}/\mathbf{Z}_1}^1(\mathbf{1}, \pi_{\mathbf{1}}/\tau) = 0$ and $(\pi_{\mathbf{1}}/\tau)^{I_1} \cong \alpha^{-2} \oplus \alpha^2$.

Proof. — From (95) we get

$$(97) \quad \sum_{\mu \in \mathbf{F}_p} \begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} s(\Pi w_{\text{st}}) = w_{\mathbf{1}}.$$

Let $\bar{\tau}$, u_1 and u_2 be the images of τ , $w_{\mathbf{1}}$ and Πw_{st} in $\pi_{\mathbf{1}}/\mathbf{1}$, respectively. It follows from [56, Lem. 6.1] that u_1 and u_2 are I_1 -invariant. Moreover, they are linearly independent, since H acts on u_1 by α^{-1} and on u_2 by α , and these characters are distinct, as $p \geq 5$. Now (94) implies $\pi_{\mathbf{1}}/k v_{\mathbf{1}} \cong M_{\mathbf{1}}/k v_{\mathbf{1}} \oplus \Pi(M_{\text{st}}/k v_{\text{st}})$. Moreover, since the restrictions of $M_{\mathbf{1}}$ and M_{st} to $H(I \cap U)$ are injective envelopes of the trivial representation in $\text{Rep}_{H(I \cap U)}$, Lemma 6.9 implies that the space of I_1 -invariants of $M_{\mathbf{1}}/k v_{\mathbf{1}} \oplus \Pi(M_{\text{st}}/k v_{\text{st}})$ is two dimensional. Hence, $\{u_1, u_2\}$ is a basis for $(\pi_{\mathbf{1}}/\mathbf{1})^{I_1}$ and $\Pi w_{\text{st}} \in \Pi M_{\text{st}}$. Moreover, (97) implies that the natural surjection $\text{Ind}_1^{\mathbf{K}} \alpha \twoheadrightarrow \langle \mathbf{K} \cdot u_2 \rangle$ is injective, since it induces an injection on $(\text{Ind}_1^{\mathbf{K}} \alpha)^{I_1}$. Thus $\bar{\tau} \cong \text{Ind}_1^{\mathbf{K}} \alpha$ and the extension $0 \rightarrow \mathbf{1} \rightarrow \tau \rightarrow \bar{\tau} \rightarrow 0$ is non-split, since $\text{soc}_{\mathbf{K}} \tau \subseteq \text{soc}_{\mathbf{K}} \pi_{\mathbf{1}} \cong \mathbf{1}$. Now su_2 is the image of

$$s(\Pi w_{\text{st}}) = t w_{\text{st}} = \sum_{\lambda \in \mathbf{F}_p} \lambda \begin{pmatrix} 1 & \rho[\lambda] \\ 0 & 1 \end{pmatrix} t^2 v_{\mathbf{1}},$$

which lies in $M_{\mathbf{1}}$. Since $\bar{\tau} = k u_2 \oplus \langle I_1 \cdot (s u_2) \rangle$ we obtain

$$\pi_{\mathbf{1}}/\tau \cong M_{\mathbf{1}}/\langle I_1 \cdot (t w_{\text{st}}) \rangle \oplus \Pi(M_{\text{st}}/\langle I_1 \cdot w_{\text{st}} \rangle).$$

Let $N_1 := M_1 / \langle I \cdot (tw_{st}) \rangle$. Lemma 6.9 gives that $N_1^{I_1}$ is 1-dimensional, H acts on $N_1^{I_1}$ by the character α^{-2} and $\alpha^{-2} \hookrightarrow N_1|_{H(I_1 \cap U)}$ is an injective envelope of α^{-2} in $\text{Rep}_{H(I_1 \cap U)}$. Let $\psi : I \rightarrow k^\times$ be a smooth character, [56, Prop. 7.2, Cor. 7.4] say that $\text{Ext}_{I/Z_1}^1(\psi, N_1) \neq 0$ if and only if $\psi = \alpha^{-1}$ or $\psi = \alpha^{-2}$. Since $p \geq 5$ we get $\text{Ext}_{I/Z_1}^1(\mathbf{1}, N_1) = 0$. Similarly, one gets $\text{Ext}_{I/Z_1}^1(\psi, \Pi(M_{st}/\langle I \cdot w_{st} \rangle)) \neq 0$ if and only if $\psi = (\alpha^{-1})^\Pi = \alpha$ or $\psi = (\alpha^{-2})^\Pi = \alpha^2$. Again we obtain, $\text{Ext}_{I/Z_1}^1(\mathbf{1}, \Pi(M_{st}/\langle I \cdot w_{st} \rangle)) = 0$ and hence $\text{Ext}_{I/Z_1}^1(\mathbf{1}, \pi_1/\tau) = 0$. \square

Lemma 6.11. — *We have*

- (i) $\text{Ext}_{K/Z_1}^1(st, \mathbf{1}) = \text{Ext}_{K/Z_1}^1(\mathbf{1}, st) = 0$;
- (ii) $\dim \text{Ext}_{K/Z_1}^1(st, st) = 1$;
- (iii) $\dim H^1(I/Z_1, st) = 1$.

Proof. — Since $\mathbf{1}$ and st are self dual, sending an extension to its dual induces an isomorphism $\text{Ext}_{K/Z_1}^1(st, \mathbf{1}) \cong \text{Ext}_{K/Z_1}^1(\mathbf{1}, st)$. So for (i) it is enough to prove $\text{Ext}_{K/Z_1}^1(st, \mathbf{1}) = 0$. Let κ be a smooth representation of K/Z_1 , then

$$(98) \quad \text{Hom}_{K/Z_1}(st, \kappa) \cong \text{Hom}_{K/K_1}(st, \kappa^{K_1}),$$

since K_1 acts trivially on st . Now st is a projective object in Rep_{K/K_1} , [63, §16.4]. Thus, $\text{Hom}_{K/K_1}(st, *)$ is exact and we get:

$$(99) \quad \text{Ext}_{K/Z_1}^1(st, \kappa) \cong \text{Hom}_{K/K_1}(st, H^1(K_1/Z_1, \kappa)).$$

If K_1 acts trivially on κ we have an isomorphism of K -representations:

$$(100) \quad H^1(K_1/Z_1, \kappa) \cong \text{Hom}(K_1/Z_1, k) \otimes \kappa \cong (\text{Sym}^2 k^2 \otimes \det^{-1}) \otimes \kappa,$$

see [20, Prop 5.1]. Now $\dim \text{Hom}_K(st, st \otimes \text{Sym}^2 k^2 \otimes \det^{-1}) = 1$, by [20, Prop 5.4 (ii)] and $\text{Hom}_K(st, \text{Sym}^2 k^2 \otimes \det^{-1}) = 0$ as $p \geq 5$. So we get the assertions (i) and (ii). For (iii) we observe that

$$H^1(I/Z_1, st) \cong \text{Ext}_{K/Z_1}^1(\text{Ind}_I^K \mathbf{1}, st) \cong \text{Ext}_{K/Z_1}^1(\mathbf{1} \oplus st, st). \quad \square$$

Lemma 6.12. — *We have $\dim \text{Ext}_{I/Z_1}^1(st, \alpha) = 1$. The natural map*

$$(101) \quad \text{Ext}_{I/Z_1}^1(st, \alpha) \rightarrow \text{Ext}_{(I \cap P)/Z_1}^1(st^I, \alpha)$$

is an isomorphism.

Proof. — Since $\text{Ind}_I^K \mathbf{1} \cong \mathbf{1} \oplus st$, we have an isomorphism $st|_I \cong \text{Ind}_{HK_1}^I \mathbf{1}$ and hence

$$(102) \quad \text{Ext}_{I/Z_1}^1(st, \alpha) \cong \text{Ext}_{HK_1/Z_1}^1(\mathbf{1}, \alpha) \cong H^1(K_1/Z_1, \alpha)^H,$$

which is one dimensional, see the proof of [56, Prop. 5.4]. We identify $H^1(\mathbf{K}_1/Z_1, \alpha)$ with $\text{Hom}(\mathbf{K}_1/Z_1, k)$, then $\text{Ext}_{I/Z_1}^1(\text{st}, \alpha)$ is identified with the subspace generated by κ , where

$$\kappa(g) = (bp^{-1}) \pmod{p}, \quad \forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{K}_1.$$

Let $0 \rightarrow \alpha \rightarrow \mathbf{E} \rightarrow \text{st} \rightarrow 0$ be the unique non-split extension and let v be the basis vector of α . For each coset $c \in I/\text{HK}_1$ fix a coset representative \bar{c} of the form $\begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix}$. We note that given $g \in I_1$ the element $\bar{g}\bar{c}^{-1}g\bar{c}$ lies in \mathbf{K}_1 . The isomorphism $\text{st}|_I \cong \text{Ind}_{\text{HK}_1}^1 \mathbf{1}$ and (102) imply that there exists $w \in \mathbf{E}$ such that the image of $\{\bar{c}w : c \in I/\text{HK}_1\}$ is a basis of st and for all $g \in \mathbf{K}_1$ we have $gw = w + \kappa(g)v$. Let $w_1 = \sum_c \bar{c}w$, then the image of w_1 in st spans st^1 . We have

$$\begin{aligned} (103) \quad gw_1 &= g \sum_c \bar{c}w = \sum_c \bar{g}\bar{c}(\bar{g}\bar{c}^{-1}g\bar{c})w = w_1 + \sum_c \kappa(\bar{g}\bar{c}^{-1}g\bar{c})v \\ &= w_1 + \kappa\left(\prod_c (\bar{g}\bar{c}^{-1}g\bar{c})\right)v. \end{aligned}$$

If $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then g commutes with \bar{c} , and so $\prod_c (\bar{g}\bar{c}^{-1}g\bar{c}) = g^{\sharp}$, thus $gw_1 = w_1 + v$. This implies that the map (101) is non-zero. Moreover, the target is 1-dimensional by [56, Prop. 5.4], hence (101) is an isomorphism. \square

Corollary 6.13. — $\dim \text{Ext}_{\mathbf{K}/Z_1}^1(\sigma, \pi_1) = 1$ and $\text{Ext}_{\mathbf{K}/Z_1}^1(\sigma, \pi_1/\tau) = 0$, where $\sigma = \mathbf{1}$ or $\sigma = \text{st}$.

Proof. — It follows from 6.10 that

$$\text{Ext}_{\mathbf{K}/Z_1}^1(\text{Ind}_I^{\mathbf{K}} \mathbf{1}, \pi_1/\tau) \cong \text{Ext}_{I/Z_1}^1(\mathbf{1}, \pi_1/\tau) = 0,$$

$$\text{Hom}_{\mathbf{K}/Z_1}(\text{Ind}_I^{\mathbf{K}} \mathbf{1}, \pi_1/\tau) \cong \text{Hom}_{I/Z_1}(\mathbf{1}, \pi_1/\tau) = 0.$$

Since $\text{Ind}_I^{\mathbf{K}} \mathbf{1} \cong \mathbf{1} \oplus \text{st}$, we get $\text{Ext}_{\mathbf{K}/Z_1}^1(\sigma, \pi_1/\tau) = 0$ and $\text{Hom}_{\mathbf{K}}(\sigma, \pi_1/\tau) = 0$. Thus, applying $\text{Hom}_{\mathbf{K}}(\sigma, *)$ to the exact sequence

$$0 \rightarrow \text{Ind}_I^{\mathbf{K}} \alpha \rightarrow \pi_1/\mathbf{1} \rightarrow \pi_1/\tau \rightarrow 0$$

we obtain

$$\text{Ext}_{\mathbf{K}/Z_1}^1(\sigma, \pi_1/\mathbf{1}) \cong \text{Ext}_{\mathbf{K}/Z_1}^1(\sigma, \text{Ind}_I^{\mathbf{K}} \alpha) \cong \text{Ext}_{I/Z_1}^1(\sigma, \alpha).$$

Now $\text{Ext}_{I/Z_1}^1(\text{st}, \alpha)$ is 1-dimensional by Lemma 6.12 and $\text{Ext}_{I/Z_1}^1(\mathbf{1}, \alpha)$ is 1-dimensional by [56, Prop. 5.4]. Hence,

$$\text{Ext}_{I/Z_1}^1(\mathbf{1}, \pi_1/\mathbf{1}) \cong \text{Ext}_{\mathbf{K}/Z_1}^1(\mathbf{1} \oplus \text{st}, \pi_1/\mathbf{1})$$

is 2-dimensional. We know that $\text{Ext}_{I/Z_1}^1(\mathbf{1}, \mathbf{1}) = 0$, [56, Prop. 5.4], so

$$(104) \quad \text{Ext}_{I/Z_1}^1(\mathbf{1}, \pi_1) \rightarrow \text{Ext}_{I/Z_1}^1(\mathbf{1}, \pi_1/\mathbf{1})$$

is an injection. The source has dimension 2 by [56, Thm. 7.9]. Hence, (104) is an isomorphism. Using $\text{Ind}_1^K \mathbf{1} \cong \mathbf{1} \oplus \text{st}$ again we get that $\text{Ext}_{K/Z_1}^1(\text{st}, \pi_1)$ and $\text{Ext}_{K/Z_1}^1(\mathbf{1}, \pi_1)$ are both 1-dimensional. \square

Lemma 6.14. — *We have exact sequences of G^+ -representations:*

$$(105) \quad 0 \rightarrow \text{c-Ind}_{K^{\Pi Z}}^{G^+} \mathbf{1} \rightarrow \text{c-Ind}_{KZ}^{G^+} \mathbf{1} \rightarrow \pi_1 \rightarrow 0,$$

$$(106) \quad 0 \rightarrow \text{c-Ind}_{K^{\Pi Z}}^{G^+} \text{st}^{\Pi} \rightarrow \text{c-Ind}_{KZ}^{G^+} \text{st} \rightarrow \pi_{\text{st}} \rightarrow 0.$$

Proof. — Below we let $(\sigma = \text{st}$ and $\check{\sigma} = \mathbf{1})$ or $(\check{\sigma} = \text{st}$ and $\sigma = \mathbf{1})$. Let

$$\mathcal{F}^+ := \{f \in \text{c-Ind}_{KZ}^G \sigma : \text{Supp} f \subseteq G^+\} \cong \text{c-Ind}_{KZ}^{G^+} \sigma,$$

$$\mathcal{F}^- := \{f \in \text{c-Ind}_{KZ}^G \sigma : \text{Supp} f \subseteq \Pi G^+\} \cong \text{c-Ind}_{K^{\Pi Z}}^{G^+} \sigma^{\Pi}.$$

We have $\text{c-Ind}_{KZ}^G \sigma|_{G^+} \cong \mathcal{F}^+ \oplus \mathcal{F}^-$. Let $\varphi \in \text{c-Ind}_{KZ}^G \sigma$, such that $\text{Supp} \varphi = KZ$ and $\varphi(1)$ spans σ^{1_1} . Then $\mathcal{F}^+ = \langle G^+ \cdot \varphi \rangle$ and $\mathcal{F}^- = \langle G^+ \cdot \Pi \varphi \rangle$. It follows from Lemma 5.13 that $T\varphi \in \mathcal{F}^-$ and $T\Pi\varphi = \Pi T\varphi \in \mathcal{F}^+$. Hence, $T(\mathcal{F}^+) \subset \mathcal{F}^-$ and $T(\mathcal{F}^-) \subset \mathcal{F}^+$. Hence,

$$(107) \quad \pi|_{G^+} \cong \mathcal{F}^+/T(\mathcal{F}^-) \oplus \mathcal{F}^-/T(\mathcal{F}^+).$$

Now (94) implies that $\text{Hom}_K(\check{\sigma}, \pi_{\sigma}) = 0$, and thus (92) and (107) give $\pi_{\sigma} \cong \mathcal{F}^+/T(\mathcal{F}^-)$, $\pi_{\check{\sigma}} \cong \mathcal{F}^-/T(\mathcal{F}^+)$. Since T is an injection we obtain the result. \square

Proposition 6.15. — *We have*

- (i) $\dim \text{Ext}_{G^+/Z}^1(\pi_{\text{st}}, \pi_1) = \dim \text{Ext}_{G^+/Z}^1(\pi_1, \pi_{\text{st}}) = 2$;
- (ii) $\dim \text{Ext}_{G^+/Z}^1(\pi_{\text{st}}, \pi_{\text{st}}) = \dim \text{Ext}_{G^+/Z}^1(\pi_1, \pi_1) = 1$.

Proof. — By applying $\text{Hom}_{G^+/Z}(*, \pi_1)$ to (105) we get an exact sequence

$$\text{Hom}_{K^{\Pi}}(\mathbf{1}, \pi_1) \rightarrow \text{Ext}_{G^+/Z}^1(\pi_1, \pi_1) \rightarrow \text{Ext}_{K/Z_1}^1(\mathbf{1}, \pi_1).$$

It follows from (90) that $\pi_1^{\Pi} \cong \pi_{\text{st}}$, hence

$$\text{Hom}_{K^{\Pi}}(\mathbf{1}, \pi_1) \cong \text{Hom}_K(\mathbf{1}, \pi_{\text{st}}) = 0$$

and so $\dim \text{Ext}_{G^+/Z}^1(\pi_1, \pi_1) \leq 1$. Similarly, by applying $\text{Hom}_{G^+/Z}(*, \pi_1)$ to (106) we obtain $\dim \text{Ext}_{G^+/Z}^1(\pi_{\text{st}}, \pi_1) \leq 2$. On the other hand $\pi \cong \text{Ind}_{G^+}^G \pi_1$ and we know that

$$\text{Ext}_{G^+/Z}^1(\pi, \pi) \cong \text{Ext}_{G^+/Z}^1(\pi_1 \oplus \pi_{\text{st}}, \pi_1)$$

is 3-dimensional, [56, 10.13]. Hence, both inequalities are in fact equalities. We obtain the rest by using $\pi_{\text{st}} = \pi_{\mathbf{1}}^{\Pi}$ and $\pi_{\mathbf{1}} = \pi_{\text{st}}^{\Pi}$. \square

Corollary 6.16. — We have $\dim \text{Ext}_{\mathbf{K}/Z_1}^1(\text{st}, \pi_{\text{st}}) = \dim \text{Ext}_{\mathbf{K}/Z_1}^1(\mathbf{1}, \pi_{\text{st}}) = 1$.

Proof. — Applying $\text{Hom}_{G^+/Z}(*, \pi_{\text{st}})$ to (105) we get an exact sequence:

$$\text{Hom}_{\mathbf{K}^{\Pi}}(\mathbf{1}, \pi_{\text{st}}) \hookrightarrow \text{Ext}_{G^+/Z}^1(\pi_{\mathbf{1}}, \pi_{\text{st}}) \rightarrow \text{Ext}_{\mathbf{K}/Z_1}^1(\mathbf{1}, \pi_{\text{st}}).$$

Proposition 6.15 implies that $\dim \text{Ext}_{\mathbf{K}/Z_1}^1(\mathbf{1}, \pi_{\text{st}}) \geq 1$. We apply $\text{Hom}_{G^+/Z}(*, \pi_{\text{st}})$ to (106) we get an injection

$$\text{Ext}_{G^+/Z}^1(\pi_{\text{st}}, \pi_{\text{st}}) \hookrightarrow \text{Ext}_{\mathbf{K}/Z_1}^1(\text{st}, \pi_{\text{st}}).$$

Proposition 6.15 implies that $\dim \text{Ext}_{\mathbf{K}/Z_1}^1(\text{st}, \pi_{\text{st}}) \geq 1$. We know [56, 7.9] that

$$\text{Ext}_{\mathbf{1}/Z_1}^1(\mathbf{1}, \pi_{\text{st}}) \cong \text{Ext}_{\mathbf{K}/Z_1}^1(\text{Ind}_{\mathbf{1}}^{\mathbf{K}} \mathbf{1}, \pi_{\text{st}}) \cong \text{Ext}_{\mathbf{K}/Z_1}^1(\mathbf{1} \oplus \text{st}, \pi_{\text{st}})$$

is 2-dimensional. This implies that both inequalities must be equalities. \square

Lemma 6.17. — Let E be the unique non-split extension $0 \rightarrow \mathbf{1} \rightarrow E \rightarrow \mathbf{1} \rightarrow 0$ of $(\mathbf{I} \cap \mathbf{P})/Z_1$ -representations. Then the natural map

$$(108) \quad \text{Ext}_{(\mathbf{I} \cap \mathbf{P})/Z_1}^1(E, \alpha) \rightarrow \text{Ext}_{(\mathbf{I} \cap \mathbf{P})/Z_1}^1(\mathbf{1}, \alpha)$$

is zero.

Proof. — We know that $\text{Ext}_{(\mathbf{I} \cap \mathbf{P})/Z_1}^1(\mathbf{1}, \mathbf{1}) \cong \text{Hom}((\mathbf{I} \cap \mathbf{P})/Z_1, k)$ is one dimensional and we may choose a basis $\{w_1, w_2\}$ of E such that w_2 is fixed by $\mathbf{I} \cap \mathbf{P}$ and $dw_1 = w_2 + w_1$, $uw_1 = w_1$, where $d = \begin{pmatrix} 1+\rho & 0 \\ 0 & 1 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, [56, 5.7]. Suppose that the map is non-zero, then we have an extension $0 \rightarrow \alpha \rightarrow E' \rightarrow E \rightarrow 0$. Since $\text{Ext}_{(\mathbf{I} \cap \mathbf{P})/Z_1}^1(\mathbf{1}, \alpha)$ is one dimensional, [56, 5.7], we may choose a basis $\{v_1, v_2, v_3\}$ of E' such that \mathbf{I} acts by α on v_3 , v_2 maps to w_2 , $dv_2 = v_2$, $uv_2 = v_2 + v_3$, v_1 maps to w_1 and \mathbf{H} act trivially on v_1 and v_2 . Now \mathbf{H} act trivially on $(d-1)v_1$, hence $(d-1)v_1 = \lambda v_1 + \mu v_2$. By considering the image in E we get $dv_1 = v_1 + v_2$. The image of $(u-1)v_1$ is zero in E . Hence $(u-1)v_1 = \lambda v_3$, for some λ , $(u^\rho - 1)v_1 = (u-1)^\rho v_1 = 0$ and so $u^{\rho+1}v_1 = uv_1 = v_1 + \lambda v_3$. Now

$$\begin{aligned} dw_1 &= d(v_1 + \lambda v_2) = v_1 + v_2 + \lambda v_3, \\ u^{\rho+1}dv_1 &= u^{\rho+1}(v_1 + v_2) = v_1 + \lambda v_3 + v_2 + v_3. \end{aligned}$$

Since $du = u^{\rho+1}d$ in $\mathbf{I} \cap \mathbf{P}$ we deduce that E' cannot exist. \square

Lemma 6.18. — Let e be the unique non-split extension $0 \rightarrow \text{st} \rightarrow e \rightarrow \text{st} \rightarrow 0$ of \mathbf{K}/Z_1 -representations. Then $e^{\mathbf{I} \cap \mathbf{U}}$ is the unique non-split extension $0 \rightarrow \mathbf{1} \rightarrow e^{\mathbf{I} \cap \mathbf{U}} \rightarrow \mathbf{1} \rightarrow 0$ of $(\mathbf{I} \cap \mathbf{P})/Z_1$ -representations.

Proof. — By taking I-invariants we obtain an exact sequence:

$$\mathrm{st}^I \hookrightarrow e^I \rightarrow \mathrm{st}^I \xrightarrow{\partial} H^1(I/Z_1, \mathrm{st}).$$

Now $\dim e^I = 1$, since $\mathrm{Hom}_I(\mathbf{1}, e) \cong \mathrm{Hom}_K(\mathrm{st}, e)$ by Frobenius reciprocity. Hence, ∂ is an injection. Since by Lemma 6.11 $\dim H^1(I/Z_1, \mathrm{st}) = 1$ we get that ∂ is an isomorphism. Fix a non-zero $v \in \mathrm{st}^I = e^I$. To prove the assertion it is enough to give a 1-cocycle $f : I/Z_1 \rightarrow \mathrm{st}$, such that for some scalar $\lambda \neq 0$ we have

$$(109) \quad f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \frac{a-d}{p}\lambda v, \quad \forall \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in I_1 \cap P.$$

Since then there exists $w \in e$ such that $(g-1)w = f(g)$, for all $g \in I$, as ∂ is an isomorphism. Then (109) implies that $w \in e^{I \cap U}$ and $w \notin e^{I \cap P}$. Thus v and w are linearly independent, and so $\dim e^{I \cap U} \geq 2$. Since $\mathrm{st}^I = \mathrm{st}^{I \cap U}$ is 1-dimensional we get that $e^{I \cap U} = \langle v, w \rangle$, thus we have an exact sequence $0 \rightarrow \mathrm{st}^I \rightarrow e^{I \cap U} \rightarrow \mathrm{st}^I \rightarrow 0$ and (109) implies that this sequence is non-split.

We will construct a cocycle f satisfying (109). We have $\mathbf{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbf{Z}_p)$, let $\mathrm{pr} : \mathbf{Z}_p^\times \rightarrow 1 + p\mathbf{Z}_p$ denote the projection and let $\delta : K \rightarrow 1 + p\mathbf{Z}_p$ be the character $\delta(g) = \mathrm{pr}(\det(g))$. Let $M := \mathrm{Sym}^{p-1} \mathbf{Z}_p^2 \otimes \delta^{\frac{1-p}{2}}$, then K acts on M , Z_1 acts trivially and $M/pM \cong \mathrm{st}$. We have an exact sequence of $\mathbf{Z}_p/p^2\mathbf{Z}_p[K]$ -modules

$$0 \rightarrow M/pM \xrightarrow{p} M/p^2M \rightarrow M/pM \rightarrow 0.$$

Let $w := x^{p-1} + p^2M \in M/p^2M$, then the image of w in M/pM is I-invariant. Thus $f(g) := (g-1)w$ takes values in $pM/p^2M \cong \mathrm{st}$ for all $g \in I$. Moreover, it is immediate that f satisfies (109) with $v = x^{p-1} + pM$ and $\lambda = \frac{p-1}{2}$. \square

Lemma 6.19. — *Let e be the unique non-split extension $0 \rightarrow \mathrm{st} \rightarrow e \rightarrow \mathrm{st} \rightarrow 0$ of K/Z_1 -representations. Then the natural map*

$$(110) \quad \mathrm{Ext}_{I/Z_1}^1(e, \alpha) \rightarrow \mathrm{Ext}_{I/Z_1}^1(\mathrm{st}, \alpha)$$

is zero.

Proof. — Lemma 6.18 gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{st}^I & \longrightarrow & e^{I \cap U} & \longrightarrow & \mathrm{st}^I \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{st} & \longrightarrow & e & \longrightarrow & \mathrm{st} \longrightarrow 0 \end{array}$$

with exact rows, and the top row a non-split extension of $I \cap P$ -representations, with the middle vertical arrow $I \cap P$ -equivariant. Applying $\mathrm{Hom}_{I/Z_1}(*, \alpha)$ to the bottom row, and $\mathrm{Hom}_{(I \cap P)/Z_1}(*, \alpha)$ to the top row we obtain a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}_{I/Z_1}^1(e, \alpha) & \xrightarrow{\beta} & \mathrm{Ext}_{I/Z_1}^1(st, \alpha) \\ \downarrow & & \downarrow \cong \\ \mathrm{Ext}_{(I \cap P)/Z_1}^1(e^{I \cap U}, \alpha) & \xrightarrow[0]{(108)} & \mathrm{Ext}_{(I \cap P)/Z_1}^1(st^I, \alpha). \end{array} \quad (101)$$

It follows from the diagram that β is the zero map. \square

Lemma 6.20. — *Let e be the unique non-split extension $0 \rightarrow st \rightarrow e \rightarrow st \rightarrow 0$ of K/Z_1 -representations, then $\dim \mathrm{Ext}_{K/Z_1}^1(e, \pi_1) = 1$. Moreover, let*

$$0 \rightarrow \pi_1 \rightarrow E'_1 \rightarrow e \rightarrow 0$$

be an exact sequence of K/Z_1 -representations, then $\mathrm{Hom}_K(st, E'_1) \neq 0$.

Proof. — Applying $\mathrm{Hom}_{K/Z_1}(*, \pi_1)$ to $0 \rightarrow st \rightarrow e \rightarrow st \rightarrow 0$ we get an exact sequence

$$\mathrm{Ext}_{K/Z_1}^1(st, \pi_1) \hookrightarrow \mathrm{Ext}_{K/Z_1}^1(e, \pi_1) \xrightarrow{\beta} \mathrm{Ext}_{K/Z_1}^1(st, \pi_1).$$

We claim that β is zero. The claim and Lemma 6.12 gives $\dim \mathrm{Ext}_{K/Z_1}^1(e, \pi_1) = 1$. The Yoneda interpretation of the claim gives the second assertion. Let $\tau \subset \pi_1$ be the representation considered in Proposition 6.10, then $\mathrm{Hom}_K(st, \pi_1/\tau) = 0$, hence $\mathrm{Hom}_K(e, \pi_1/\tau) = 0$. Moreover, Corollary 6.13 says that $\mathrm{Ext}_{K/Z_1}^1(st, \pi_1/\tau) = 0$, this implies $\mathrm{Ext}_{K/Z_1}^1(e, \pi_1/\tau) = 0$. Hence, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}_{K/Z_1}^1(e, \tau) & \xrightarrow{\gamma} & \mathrm{Ext}_{K/Z_1}^1(st, \tau) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Ext}_{K/Z_1}^1(e, \pi_1) & \xrightarrow{\beta} & \mathrm{Ext}_{K/Z_1}^1(st, \pi_1). \end{array}$$

Recall that (96) is an exact sequence $0 \rightarrow \mathbf{1} \rightarrow \tau \rightarrow \mathrm{Ind}_I^K \alpha \rightarrow 0$ of K/Z_1 -representations. Lemma 6.11 says that $\mathrm{Ext}_{K/Z_1}^1(st, \mathbf{1}) = 0$. This implies $\mathrm{Ext}_{K/Z_1}^1(e, \mathbf{1}) = 0$. Thus applying $\mathrm{Hom}_{K/Z_1}(e, *)$ and $\mathrm{Hom}_{K/Z_1}(st, *)$ to (96) we get a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}_{K/Z_1}^1(e, \tau) & \xrightarrow{\gamma} & \mathrm{Ext}_{K/Z_1}^1(st, \tau) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{K/Z_1}^1(e, \mathrm{Ind}_I^K \alpha) & \xrightarrow{\delta} & \mathrm{Ext}_{K/Z_1}^1(st, \mathrm{Ind}_I^K \alpha). \end{array}$$

Now δ is zero by Shapiro's lemma and Lemma 6.19, hence $\gamma = \beta = 0$. \square

Proposition 6.21. — *Let*

$$(111) \quad 0 \rightarrow \pi_{\mathbf{1}} \rightarrow E_1 \rightarrow \pi_{st} \rightarrow 0$$

be a non-split extension of G^+/Z representations. Suppose that $E_1^{\mathbf{1}}$ is 1-dimensional, then $\text{Ext}_{\mathbf{K}/Z_1}^1(st, E_1) = 0$.

Proof. — We note that the assumption $\dim E_1^{\mathbf{1}} = 1$, implies that $E_1^{\mathbf{1}} = \pi_{\mathbf{1}}^{\mathbf{1}}$ and hence $\text{Hom}_{\mathbf{K}}(st, E_1) = 0$. Let $0 \rightarrow st \rightarrow e \rightarrow st \rightarrow 0$ be the unique non-split extension of \mathbf{K}/Z_1 -representations. Now, e cannot be a subrepresentation of π_{st} , since in that case by pulling back we would obtain a subrepresentation $E'_1 \subset E_1$ such that we have an exact sequence $0 \rightarrow \pi_{\mathbf{1}} \rightarrow E'_1 \rightarrow e \rightarrow 0$ of \mathbf{K}/Z_1 -representations with $\text{Hom}_{\mathbf{K}}(st, E'_1) = \text{Hom}_{\mathbf{K}}(st, E_1) = 0$, which would contradict Lemma 6.20. Hence, $\text{Hom}_{\mathbf{K}}(st, \pi_{st}/st) = 0$ and so we obtain an injection

$$(112) \quad \text{Ext}_{\mathbf{K}/Z_1}^1(st, st) \hookrightarrow \text{Ext}_{\mathbf{K}/Z_1}^1(st, \pi_{st}).$$

Corollary 6.16 asserts $\dim \text{Ext}_{\mathbf{K}/Z_1}^1(st, \pi_{st}) = 1$, so the map of (112) is an isomorphism. The Yoneda interpretation of this says if we let

$$(113) \quad 0 \rightarrow \pi_{st} \rightarrow E_2 \rightarrow st \rightarrow 0$$

be the unique non-split extension of \mathbf{K}/Z_1 -representations, then E_2 contains e as a subrepresentation. Corollary 6.13 says that $\dim \text{Ext}_{\mathbf{K}/Z_1}^1(st, \pi_{\mathbf{1}}) = 1$ so applying $\text{Hom}_{\mathbf{K}/Z_1}(st, *)$ to (111) gives us an injection

$$(114) \quad \text{Ext}_{\mathbf{K}/Z_1}^1(st, E_1) \hookrightarrow \text{Ext}_{\mathbf{K}/Z_1}^1(st, \pi_{st}).$$

Suppose that $\text{Ext}_{\mathbf{K}/Z_1}^1(st, E_1) \neq 0$ then (114) would give a non-split extension of \mathbf{K}/Z_1 -representations

$$(115) \quad 0 \rightarrow E_1 \rightarrow E_3 \rightarrow st \rightarrow 0$$

such that $E_3/\pi_{\mathbf{1}} \cong E_2$. Now, $\text{Hom}_{\mathbf{K}}(st, E_1) = 0$ and so $\text{Hom}_{\mathbf{K}}(st, E_3) = 0$, as otherwise we would obtain a splitting of (115). As e is a subrepresentation of E_2 , by pulling back we obtain a subrepresentation $E'_3 \subset E_3$, which sits in an exact sequence:

$$0 \rightarrow \pi_{\mathbf{1}} \rightarrow E'_3 \rightarrow e \rightarrow 0.$$

Since $\text{Hom}_{\mathbf{K}}(st, E_3) = 0$, we have $\text{Hom}_{\mathbf{K}}(st, E'_3) = 0$. Hence, we obtain a contradiction to Lemma 6.20. \square

Corollary 6.22. — *Let E_1 be as above, then $\dim H^1(\mathbf{I}_1/Z_1, E_1) \leq 2$.*

Proof. — Taking I_1/Z_1 -invariants of (111) gives us an exact sequence:

$$(116) \quad \pi_{\text{st}}^{I_1} \hookrightarrow H^1(I_1/Z_1, \pi_{\mathbf{1}}) \rightarrow H^1(I_1/Z_1, E_1) \rightarrow H^1(I_1/Z_1, \pi_{\text{st}}).$$

By [56, 7.9] H acts trivially on $H^1(I_1/Z_1, \pi_{\mathbf{1}})$ and $H^1(I_1/Z_1, \pi_{\text{st}})$. Hence,

$$(117) \quad \begin{aligned} H^1(I_1/Z_1, E_1) &\cong H^1(I_1/Z_1, E_1)^H \cong \text{Ext}_{I_1/Z_1}^1(\mathbf{1}, E_1) \\ &\cong \text{Ext}_{K/Z_1}^1(\text{Ind}_1^K \mathbf{1}, E_1) \cong \text{Ext}_{K/Z_1}^1(\mathbf{1}, E_1) \oplus \text{Ext}_{K/Z_1}^1(\text{st}, E_1). \end{aligned}$$

Application of $\text{Hom}_{K/Z_1}(\mathbf{1}, *)$ to (111) gives an exact sequence:

$$(118) \quad \text{Ext}_{K/Z_1}^1(\mathbf{1}, \pi_{\mathbf{1}}) \hookrightarrow \text{Ext}_{K/Z_1}^1(\mathbf{1}, E_1) \rightarrow \text{Ext}_{K/Z_1}^1(\mathbf{1}, \pi_{\text{st}}).$$

It follows from (118) and Corollaries 6.13, 6.16 that $\dim \text{Ext}_{K/Z_1}^1(\mathbf{1}, E_1) \leq 2$. Proposition 6.21 says $\text{Ext}_{K/Z_1}^1(\text{st}, E_1) = 0$, hence (117) gives us the assertion. \square

Recall that $\pi \cong \text{Ind}_{G^+}^G \pi_{\mathbf{1}} \cong \text{Ind}_{G^+}^G \pi_{\text{st}}$. Thus we have an injection

$$(119) \quad \text{Ext}_{G^+/Z}^1(\pi_{\text{st}}, \pi_{\mathbf{1}}) \hookrightarrow \text{Ext}_{G^+/Z}^1(\pi_{\text{st}} \oplus \pi_{\mathbf{1}}, \pi_{\mathbf{1}}) \cong \text{Ext}_{G^+/Z}^1(\pi, \pi).$$

Proposition 6.23. — *Let ξ lie in the image of (119). Suppose that $\xi \neq 0$ then either $\dim \text{Ext}_{G^+/Z}^1(\pi, E_\xi) \leq 3$ or $\dim \text{Ext}_{G^+/Z}^1(E_\xi, \pi) \leq 3$, where E_ξ is the corresponding extension of π by π .*

Proof. — Since ξ lies in the image of (119) we have $E_\xi \cong \text{Ind}_{G^+}^G E_1$, where E_1 is an extension of G^+/Z -representations: $0 \rightarrow \pi_{\mathbf{1}} \rightarrow E_1 \rightarrow \pi_{\text{st}} \rightarrow 0$. Moreover, $\xi \neq 0$ implies that E_1 is non-split. If $\dim E_1^{I_1} = 2$ then $\dim E_\xi^{I_1} = 4$ and hence ξ lies in the image of $\text{Ext}_{\mathcal{H}}^1(\pi^{I_1}, \pi^{I_1}) \hookrightarrow \text{Ext}_{G^+/Z}^1(\pi, \pi)$ via (73) and we know that the assertion is true for such ξ by [56, 10.14]. Suppose that $\dim E_1^{I_1} = 1$ then, since $E_\xi|_{I_1} \cong E_1 \oplus E_1^\Pi$, we get $\dim E_\xi^{I_1} = 2$, hence $E_\xi^{I_1} = \pi^{I_1}$ and [20, 6.7], or Lemma 5.27 gives $\dim \text{Ext}_{\mathcal{H}}^1(\pi^{I_1}, E_\xi^{I_1}) = 1$. Moreover, Corollary 6.22 implies that $\dim H^1(I_1/Z_1, E_\xi) = 2 \dim H^1(I_1/Z_1, E_1) \leq 4$. Since π^{I_1} is an irreducible \mathcal{H} -module and its underlying vector space is 2-dimensional, we deduce from Lemma 5.22 that $\dim \text{Hom}_{\mathcal{H}}(\pi^{I_1}, \mathbf{R}^1\mathcal{I}(E_\xi)) \leq 2$. Now (73) implies $\dim \text{Ext}_{G^+/Z}^1(\pi, E_\xi) \leq 3$. \square

7. Non-supersingular representations

We recall the properties of Emerton's functor of ordinary parts. This functor is an extremely useful tool for calculating Ext groups, when some principal series are involved. In Section 7.2 we discuss Banach space representations of G obtained by parabolic induction of admissible unitary Banach space representations of the torus T . We assume throughout this section $p \geq 3$.

7.1. Ordinary parts. — Let A be a complete local noetherian commutative \mathcal{O} -algebra with a finite residue field. Emerton in [30] has defined a functor $\text{Ord}_{\mathbb{P}} : \text{Mod}_{G,\zeta}^{\text{ladm}}(A) \rightarrow \text{Mod}_{T,\zeta}^{\text{ladm}}(A)$, satisfying

$$(120) \quad \text{Hom}_{A[G]}(\text{Ind}_{\bar{P}}^G U, V) \cong \text{Hom}_{A[M]}(U, \text{Ord}_{\mathbb{P}}(V)),$$

where \bar{P} is the parabolic subgroup of G opposite to P with respect to T , see Theorem 4.4.6 in [30] if U is admissible, the general statement follows from the fact that $\text{Ord}_{\mathbb{P}}$ and $\text{Ind}_{\bar{P}}^G$ commute with inductive limits, see Lemmas 3.2.2 and 4.1.4 in [30]. Since induction is an exact functor, [30, 4.1.5], the functor $\text{Ord}_{\mathbb{P}}$ is left exact. It follows from [30, Prop. 4.3.4] that for every U in $\text{Mod}_{T,\zeta}^{\text{ladm}}(A)$ we have:

$$(121) \quad \text{Ord}_{\mathbb{P}}(\text{Ind}_{\bar{P}}^G U) \cong U.$$

From now on we suppose that A is artinian. It is shown in [31, §3.7] that (120) induces an E_2 -spectral sequence:

$$(122) \quad \text{Ext}_{T,\zeta}^i(U, \mathbf{R}^j \text{Ord}_{\mathbb{P}} V) \implies \text{Ext}_{G,\zeta}^{i+j}(\text{Ind}_{\bar{P}}^G U, V).$$

The Ext groups in [31] are computed in the category of locally admissible representations. This category coincides with the category of locally finite representations by Proposition 5.46, Corollary 5.6 and [30, 2.3.8]. However, we have shown in Corollary 5.17 that for $G = \text{GL}_2(\mathbf{Q}_p)$ and $A = k$ these groups coincide with the Ext groups computed in $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$. This answers a question raised in [31, 3.7.8]. It follows from [34] that $\mathbf{R}^j \text{Ord}_{\mathbb{P}} = 0$ for $j \geq 2$. Moreover, it follows from Proposition 3.34 that each block of the category $\text{Mod}_{T,\zeta}^{\text{ladm}}(k)$ is anti-equivalent to the category of compact $k[[x,y]]$ -modules. Hence, $\text{Ext}_{T,\zeta}^i = 0$ for $i \geq 3$. Thus (122) yields an exact sequence:

$$(123) \quad \begin{aligned} \text{Ext}_{T,\zeta}^1(U, \text{Ord}_{\mathbb{P}} V) &\hookrightarrow \text{Ext}_{G,\zeta}^1(\text{Ind}_{\bar{P}}^G U, V) \rightarrow \text{Hom}_T(U, \mathbf{R}^1 \text{Ord}_{\mathbb{P}} V) \\ &\rightarrow \text{Ext}_{T,\zeta}^2(U, \text{Ord}_{\mathbb{P}} V) \rightarrow \text{Ext}_{G,\zeta}^2(\text{Ind}_{\bar{P}}^G U, V) \rightarrow \text{Ext}_{T,\zeta}^1(U, \mathbf{R}^1 \text{Ord}_{\mathbb{P}} V) \end{aligned}$$

and an isomorphism

$$(124) \quad \text{Ext}_{G,\zeta}^3(\text{Ind}_{\bar{P}}^G U, V) \cong \text{Ext}_{T,\zeta}^2(U, \mathbf{R}^1 \text{Ord}_{\mathbb{P}} V).$$

Moreover, we have $\text{Ext}_{G,\zeta}^i(\text{Ind}_{\bar{P}}^G U, V) = 0$ for $i \geq 4$. Since, we prefer working with P instead of \bar{P} we note that the map $f \mapsto [g \mapsto f(sg)]$ induces an isomorphism:

$$(125) \quad \text{Ind}_{\mathbb{P}}^G U \cong \text{Ind}_{\bar{P}}^G U^s.$$

It follows from [31, 4.2.10] that

$$(126) \quad \text{Ord}_{\mathbb{P}}(\text{Ind}_{\mathbb{P}}^G U) \cong U^s, \quad \mathbf{R}^1 \text{Ord}_{\mathbb{P}}(\text{Ind}_{\mathbb{P}}^G U) \cong U \otimes \alpha^{-1}.$$

Proposition 7.1. — *Let $\chi : \mathbb{T} \rightarrow k^\times$ be a smooth character such that $\chi|_Z = \zeta$. Let $\iota : \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \hookrightarrow \mathbb{J}$ be an injective envelope of $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi$ in $\text{Mod}_{\mathbb{G}, \zeta}^{\text{ladm}}(\mathbb{A})$. Then the following hold*

- (i) $\text{Ord}_{\mathbb{P}}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi) \hookrightarrow \text{Ord}_{\mathbb{P}} \mathbb{J}$ is an injective envelope of χ in $\text{Mod}_{\mathbb{T}, \zeta}^{\text{ladm}}(\mathbb{A})$;
- (ii) the adjoint map $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \text{Ord}_{\mathbb{P}} \mathbb{J} \rightarrow \mathbb{J}$ is injective;
- (iii) There exists a natural surjective ring homomorphism

$$\text{End}_{\mathbb{A}[\mathbb{G}]}(\mathbb{J}) \twoheadrightarrow \text{End}_{\mathbb{A}[\mathbb{G}]}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \text{Ord}_{\mathbb{P}} \mathbb{J}) \cong \text{End}_{\mathbb{A}[\mathbb{T}]}(\text{Ord}_{\mathbb{P}} \mathbb{J}).$$

Proof. — Since $\text{Ord}_{\mathbb{P}}$ is right adjoint to the exact functor $\text{Ind}_{\mathbb{P}}^{\mathbb{G}}$, $\text{Ord}_{\mathbb{P}} \mathbb{J}$ is injective in $\text{Mod}_{\mathbb{T}, \zeta}^{\text{ladm}}(\mathbb{A})$ and we obtain an injection $\text{Ord}_{\mathbb{P}} \iota : \chi \hookrightarrow \text{Ord}_{\mathbb{P}} \mathbb{J}$. For every τ in $\text{Mod}_{\mathbb{T}, \zeta}^{\text{ladm}}(\mathbb{A})$ we have a commutative diagram:

$$(127) \quad \begin{array}{ccc} \text{Hom}_{\mathbb{A}[\mathbb{T}]}(\tau, \chi) & \hookrightarrow & \text{Hom}_{\mathbb{A}[\mathbb{T}]}(\tau, \text{Ord}_{\mathbb{P}} \mathbb{J}) \\ \downarrow \cong & & \downarrow \cong \\ \text{Hom}_{\mathbb{A}[\mathbb{G}]}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \tau, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi) & \hookrightarrow & \text{Hom}_{\mathbb{A}[\mathbb{G}]}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \tau, \mathbb{J}). \end{array}$$

We claim that if τ is irreducible then the bottom arrow is an isomorphism. Suppose that $\text{Hom}_{\mathbb{A}[\mathbb{G}]}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \tau, \mathbb{J})$ is non-zero, then as $\iota : \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \hookrightarrow \mathbb{J}$ is essential, the representations $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \tau$ and $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi$ have an irreducible subquotient in common. In this case it follows from Corollary 5.47 that $\tau \cong \chi$. If $\chi \neq \chi^s$ then $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi$ is irreducible and the claim follows from the essentiality of ι . If $\chi = \chi^s$ then χ factors through the determinant and thus extends to a character $\chi : \mathbb{G} \rightarrow k^\times$ and we have an exact non-split sequence $0 \rightarrow \chi \rightarrow \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \rightarrow \text{Sp} \otimes \chi \rightarrow 0$. Since the sequence is non-split, \mathbb{J} is also an injective envelope of χ and any non-zero map $\mathbb{A}[\mathbb{G}]$ -equivariant map $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \rightarrow \mathbb{J}$ is an injection. Thus the claim in this case is equivalent to the assertion that $\text{Ext}_{\mathbb{G}/Z}^1(\text{Sp}, \mathbf{1})$ is one dimensional. This is shown in [56, Thm 11.4]. The claim implies that the top arrow in (127) is an isomorphism and hence $\text{Ord}_{\mathbb{P}} \iota$ is essential, which proves (i).

We claim that the map $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \text{Ord}_{\mathbb{P}} \iota : \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \rightarrow \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \text{Ord}_{\mathbb{P}} \mathbb{J}$ is essential. It is enough to show that the natural map

$$(128) \quad \text{Hom}_{\mathbb{G}}(\pi, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi) \rightarrow \text{Hom}_{\mathbb{G}}(\pi, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \text{Ord}_{\mathbb{P}} \mathbb{J})$$

is an isomorphism for all irreducible representations π . By adjointness this is equivalent to showing that the natural map

$$(129) \quad \text{Hom}_{\mathbb{T}}(\pi_{\overline{\mathbb{U}}}, \chi) \rightarrow \text{Hom}_{\mathbb{T}}(\pi_{\overline{\mathbb{U}}}, \text{Ord}_{\mathbb{P}} \mathbb{J})$$

is an isomorphism, where $\pi_{\overline{\mathbb{U}}}$ denotes the coinvariants by the subgroup of lower-triangular unipotent matrices. Since π is an irreducible representation the coinvariants are either zero or an irreducible representation of \mathbb{T} . Since $\chi \hookrightarrow \text{Ord}_{\mathbb{P}} \mathbb{J}$ is essential by Part (i), in both cases we obtain that (129) is an isomorphism.

Applying $\mathrm{Hom}_{A[G]}(\mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J, *)$ to the injection $\mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J \hookrightarrow J$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{A[T]}(\mathrm{Ord}_{\mathbb{P}} J, \mathrm{Ord}_{\mathbb{P}} J) & \xrightarrow{\mathrm{id}} & \mathrm{Hom}_{A[T]}(\mathrm{Ord}_{\mathbb{P}} J, \mathrm{Ord}_{\mathbb{P}} J) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{A[G]}(\mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J, \mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J) & \longrightarrow & \mathrm{Hom}_{A[G]}(\mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J, J). \end{array}$$

Hence, the bottom arrow is an isomorphism. Applying $\mathrm{Hom}_{A[G]}(*, J)$ and using injectivity of J we get a surjection

$$\mathrm{Hom}_{A[G]}(J, J) \twoheadrightarrow \mathrm{Hom}_{A[G]}(\mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J, J).$$

This implies that every endomorphism of J maps $\mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J$ to itself and every endomorphism of $\mathrm{Ind}_{\mathbb{P}}^G \mathrm{Ord}_{\mathbb{P}} J$ extends to an endomorphism of J , which implies (iii). \square

Corollary 7.2. — *Let $\chi : T \rightarrow k^\times$ be a smooth character such that $\chi|_Z = \zeta$. Let $\tilde{\mathbb{P}}$ be a projective envelope of $(\mathrm{Ind}_{\mathbb{P}}^G \chi)^\vee$ in $\mathfrak{C}_{G, \zeta}(\mathcal{O})$, let $\tilde{\mathbb{P}}_{\chi^\vee}$ be a projective envelope of χ^\vee in $\mathfrak{C}_{T, \zeta}(\mathcal{O})$ and let $\tilde{\mathbb{M}} = (\mathrm{Ind}_{\mathbb{P}}^G (\tilde{\mathbb{P}}_{\chi^\vee})^\vee)^\vee$ then there exists a continuous surjection of rings:*

$$(130) \quad \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}) \twoheadrightarrow \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{M}}) \cong \mathcal{O}[[x, y]].$$

Proof. — It follows from Proposition 7.1 that $\mathrm{Ord}_{\mathbb{P}}(\tilde{\mathbb{P}}^\vee)$ is an injective envelope of χ in $\mathrm{Mod}_{T, \zeta}^{\mathrm{ladm}}(\mathcal{O})$. Since injective envelopes are unique up to an isomorphism we deduce that $\mathrm{Ord}_{\mathbb{P}}(\tilde{\mathbb{P}}^\vee) \cong (\tilde{\mathbb{P}}_{\chi^\vee})^\vee$. Duality induces an isomorphism between the endomorphism ring of an object and the opposite of the endomorphism ring of its dual. Thus it follows from Proposition 7.1(iii) that we have natural maps:

$$\mathrm{End}_{\mathfrak{C}_G(\mathcal{O})}(\tilde{\mathbb{P}}) \twoheadrightarrow \mathrm{End}_{\mathfrak{C}_G(\mathcal{O})}(\tilde{\mathbb{M}}) \cong \mathrm{End}_{\mathfrak{C}_T(\mathcal{O})}(\tilde{\mathbb{P}}_{\chi^\vee}).$$

The last ring is isomorphic to $\mathcal{O}[[x, y]]$ by Proposition 3.34. \square

Corollary 7.3. — *We keep notations of Corollary 7.2. Let $\mathbb{R} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{M}})$ and let \mathfrak{m} be a compact \mathbb{R} -module then there exists a natural isomorphism:*

$$\mathfrak{m} \hat{\otimes}_{\mathbb{R}} \tilde{\mathbb{M}} \cong (\mathrm{Ind}_{\mathbb{P}}^G (\mathfrak{m} \hat{\otimes}_{\mathbb{R}} \tilde{\mathbb{P}}_{\chi^\vee})^\vee)^\vee.$$

In particular, $\mathfrak{m} \mapsto \mathfrak{m} \hat{\otimes}_{\mathbb{R}} \tilde{\mathbb{M}}$ defines an exact functor from the category of compact \mathbb{R} -modules to $\mathfrak{C}(\mathcal{O})$. Moreover, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \mathfrak{m} \hat{\otimes}_{\mathbb{R}} \tilde{\mathbb{M}}) \cong \mathfrak{m}$.

Proof. — The assertion is true by definition if $\mathfrak{m} = \mathbb{R}$. If $\mathfrak{m} = \prod_{i \in I} \mathbb{R}$ for some set I then

$$(\mathfrak{m} \hat{\otimes}_{\mathbb{R}} \tilde{\mathbb{M}})^\vee \cong \bigoplus_{i \in I} \tilde{\mathbb{M}}^\vee \cong \mathrm{Ind}_{\mathbb{P}}^G \left(\bigoplus_{i \in I} \tilde{\mathbb{P}}_{\chi^\vee}^\vee \right) \cong \mathrm{Ind}_{\mathbb{P}}^G (\mathfrak{m} \hat{\otimes}_{\mathbb{R}} \tilde{\mathbb{P}}_{\chi^\vee})^\vee.$$

In general, we may present \mathfrak{m} as $\prod_{j \in J} \mathbf{R} \rightarrow \prod_{i \in I} \mathbf{R} \rightarrow \mathfrak{m} \rightarrow 0$ and argue as in Lemma 2.9. Since $\tilde{\mathbf{P}}_{\chi^\vee}$ is a free \mathbf{R} -module of rank 1 by Proposition 3.34, the functor $\mathfrak{m} \mapsto \mathfrak{m} \widehat{\otimes}_{\mathbf{R}} \tilde{\mathbf{P}}_{\chi^\vee}$ is exact and since induction and Pontryagin dual are exact functors $\mathfrak{m} \mapsto \mathfrak{m} \widehat{\otimes}_{\mathbf{R}} \tilde{\mathbf{M}}$ is exact.

The last assertion is proved similarly. It follows from Proposition 7.1 that we have an isomorphism $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}, \tilde{\mathbf{M}}) \cong \mathrm{Hom}_{\mathcal{O}[\Gamma]}(\mathrm{Ord}_{\mathfrak{p}} \mathbf{J}, \mathrm{Ord}_{\mathfrak{p}} \mathbf{J}) \cong \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{M}})$. Hence the assertion is true when $\mathfrak{m} = \mathbf{R}$ and thus it is also true when $\mathfrak{m} \cong \prod_{i \in I} \mathbf{R}$ for some set I . In general, we may present \mathfrak{m} as $\prod_{j \in J} \mathbf{R} \rightarrow \prod_{i \in I} \mathbf{R} \rightarrow \mathfrak{m} \rightarrow 0$ and argue as in Lemma 2.9. \square

Lemma 7.4. — *Let \mathbf{U} and \mathbf{J} be in $\mathrm{Mod}_{\Gamma, \zeta}^{\mathrm{ladm}}(k)$ and suppose that \mathbf{J} is injective. Then*

$$(131) \quad \mathrm{Ext}_{G, \zeta}^1(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{U}, \mathrm{Ind}_{\mathfrak{p}}^G \mathbf{J}) \cong \mathrm{Hom}_{\Gamma}(\mathbf{U}^s, \mathbf{J} \otimes \alpha^{-1})$$

and $\mathrm{Ext}_{G, \zeta}^i(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{U}, \mathrm{Ind}_{\mathfrak{p}}^G \mathbf{J}) = 0$, for $i \geq 2$.

Proof. — It follows from (126) that $\mathrm{Ord}_{\mathfrak{p}}(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{J})$ and $\mathbf{R}^1 \mathrm{Ord}_{\mathfrak{p}}(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{J})$ are both injective objects. Thus the terms $\mathrm{Ext}_{\Gamma, \zeta}^i$ in (123) and (124) vanish and we get the assertion. \square

Proposition 7.5. — *Let $\chi : \Gamma \rightarrow k^\times$ be a smooth character, such that $\chi \neq \chi^s$ and $\chi \neq \chi^s \alpha^2$. Then there exists an exact sequence in $\mathfrak{C}_{G, \zeta}(k)$:*

$$(132) \quad 0 \rightarrow \mathbf{P}_{S'} \rightarrow \mathbf{P}_S \rightarrow \mathbf{M}_{\chi^\vee} \rightarrow 0$$

where $\mathbf{S} = (\mathrm{Ind}_{\mathfrak{p}}^G \chi)^\vee$, $\mathbf{S}' = (\mathrm{Ind}_{\mathfrak{p}}^G \chi^s \alpha)^\vee$, \mathbf{P}_S a projective envelope of \mathbf{S} in $\mathfrak{C}_{G, \zeta}(k)$ and

$$\mathbf{M}_{\chi^\vee} := (\mathrm{Ind}_{\mathfrak{p}}^G (\mathbf{P}_{\chi^\vee})^\vee)^\vee,$$

where \mathbf{P}_{χ^\vee} is a projective envelope of χ^\vee in $\mathfrak{C}_{\Gamma, \zeta}(k)$.

Remark 7.6. — If we write $\chi = \chi_1 \otimes \chi_2 \omega^{-1}$ then $\chi^s \alpha = \chi_2 \otimes \chi_1 \omega^{-1}$ and we exclude the case $\chi_1 \chi_2^{-1} = \omega^{\pm 1}$. In particular, both principal series representations are irreducible. For analogous sequences, when $\chi_1 \chi_2^{-1} = \omega^{\pm 1}$ see Proposition 10.17 and (234), (235).

Proof. — We show the existence of the dual sequence in $\mathrm{Mod}_{G, \zeta}(k)$. Let \mathbf{J}_χ be an injective envelope of χ in $\mathrm{Mod}_{\Gamma, \zeta}^{\mathrm{ladm}}(k)$ and let \mathbf{J}_{π_χ} be an injective envelope of $\pi_\chi := \mathrm{Ind}_{\mathfrak{p}}^G \chi$ in $\mathrm{Mod}_{G, \zeta}^{\mathrm{ladm}}(k)$. Then Proposition 7.1 gives an injection $\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{J}_\chi \hookrightarrow \mathbf{J}_{\pi_\chi}$ and we denote the quotient by κ_1 . Let π be an irreducible smooth representation of G , then by applying $\mathrm{Hom}_G(\pi, *)$ we get an isomorphism

$$(133) \quad \mathrm{Hom}_G(\pi, \kappa_1) \cong \mathrm{Ext}_{G, \zeta}^1(\pi, \mathrm{Ind}_{\mathfrak{p}}^G \mathbf{J}_\chi).$$

If $\mathrm{Hom}_G(\pi, \kappa_1) \neq 0$ then π is a subquotient of \mathbf{J}_{π_χ} and hence lies in the block of $\mathrm{Ind}_{\mathfrak{p}}^G \chi$, see the proof of Proposition 5.34. Hence, $\pi \cong \mathrm{Ind}_{\mathfrak{p}}^G \chi$ or $\pi \cong \mathrm{Ind}_{\mathfrak{p}}^G \chi^s \alpha$, see Proposition 5.42. It follows from Lemma 7.4 and (133) that $\pi \cong \mathrm{Ind}_{\mathfrak{p}}^G \chi^s \alpha$ and $\mathrm{Hom}_G(\pi, \kappa_1)$ is

one dimensional. Thus we may embed $\kappa_1 \hookrightarrow J_\pi$, where J_π is an injective envelope of π . Let κ_2 be the quotient. Then for every irreducible τ we have isomorphisms

$$(134) \quad \mathrm{Hom}_G(\tau, \kappa_2) \cong \mathrm{Ext}_{G,\zeta}^1(\tau, \kappa_1) \cong \mathrm{Ext}_{G,\zeta}^2(\tau, \mathrm{Ind}_P^G J_\chi).$$

If $\mathrm{Hom}_G(\tau, \kappa_2) \neq 0$ then τ lies in the block of $\mathrm{Ind}_P^G \chi$. Hence, $\tau \cong \mathrm{Ind}_P^G \chi$ or $\tau \cong \mathrm{Ind}_P^G \chi^s \alpha$. It follows from Lemma 7.4 and (133) that Ext^2 term vanishes. Since every non-zero object of $\mathrm{Mod}_{G,\zeta}^{\mathrm{adm}}(k) = \mathrm{Mod}_{G,\zeta}^{\mathrm{lfm}}(k)$ has a non-zero socle, we deduce that $\kappa_2 = 0$. \square

Corollary 7.7. — *Let $\chi : T \rightarrow k^\times$ be a smooth character, such that $\chi \neq \chi^s$ and $\chi \neq \chi^s \alpha^2$. Then there exists an exact sequence in $\mathfrak{C}_{G,\zeta}(\mathcal{O})$:*

$$(135) \quad 0 \rightarrow \tilde{P}_{S'} \rightarrow \tilde{P}_S \rightarrow \tilde{M}_{\chi^\vee} \rightarrow 0$$

where $S = (\mathrm{Ind}_P^G \chi)^\vee$, $S' = (\mathrm{Ind}_P^G \chi^s \alpha)^\vee$, \tilde{P}_S a projective envelope of S in $\mathfrak{C}_{G,\zeta}(\mathcal{O})$ and

$$\tilde{M}_{\chi^\vee} := (\mathrm{Ind}_P^G (\tilde{P}_{\chi^\vee})^\vee)^\vee,$$

where \tilde{P}_{χ^\vee} is a projective envelope of χ^\vee in $\mathfrak{C}_{T,\zeta}(\mathcal{O})$.

Proof. — Recall that if A is an object in $\mathfrak{C}(k)$ and $\tilde{P} \twoheadrightarrow A$ is a projective envelope of A in $\mathfrak{C}(\mathcal{O})$ then $\tilde{P}/\varpi \tilde{P} \rightarrow A$ is a projective envelope of A in $\mathfrak{C}(k)$, see Lemma 2.11. From this and Corollary 7.3 we deduce that $\tilde{M}_{\chi^\vee} \otimes_{\mathcal{O}} k \cong M_{\chi^\vee}$. Proposition 3.34 says that \tilde{P}_{χ^\vee} is \mathcal{O} -torsion free, hence its dual is ϖ -divisible, hence $\mathrm{Ind}_P^G \tilde{P}_{\chi^\vee}^\vee$ is ϖ -divisible and so \tilde{M}_{χ^\vee} is \mathcal{O} -torsion free. The assertion follows from Corollary 5.21 and Proposition 7.5. \square

7.2. Parabolic induction of unitary characters. — Let $\mathrm{Mod}_{T,\zeta}^{\mathrm{lfm}}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_T^{\mathrm{sm}}(\mathcal{O})$ with objects locally finite representations on which Z acts by ζ . The irreducible objects correspond to the maximal ideals of $k[T]/(z - \zeta(z) : z \in Z)$, or alternatively $\mathrm{Gal}(k/k)$ -orbits of smooth characters $\chi : T \rightarrow \bar{k}^\times$, such that the restriction of χ to Z is congruent to ζ , and are of the form V_χ , see Proposition 5.11. It follows from the proof of Proposition 3.34 that there are no extensions between distinct irreducible representations. Hence, each block consists of only one irreducible representation and so if $\mathfrak{B} = \{\tau\}$ then $\mathrm{Mod}_{T,\zeta}^{\mathrm{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is a full subcategory of $\mathrm{Mod}_{T,\zeta}^{\mathrm{lfm}}(\mathcal{O})$ with objects locally finite representations with all the irreducible subquotients isomorphic to τ . It follows from [35, §IV.2] that we have a decomposition of categories:

$$(136) \quad \mathrm{Mod}_{T,\zeta}^{\mathrm{lfm}}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \mathrm{Mod}_{T,\zeta}^{\mathrm{lfm}}(\mathcal{O})^{\mathfrak{B}},$$

where the product is taken over all the blocks \mathfrak{B} . Using (136) and arguing as in Proposition 5.36, we obtain a decomposition of the category of admissible unitary L-Banach space representations of T on which Z acts by ζ into a direct sum of subcategories:

$$(137) \quad \mathrm{Ban}_{T,\zeta}^{\mathrm{adm}}(\mathbb{L}) \cong \bigoplus_{\mathfrak{B}} \mathrm{Ban}_{T,\zeta}^{\mathrm{adm}}(\mathbb{L})^{\mathfrak{B}},$$

where Π is an object of $\text{Ban}_{T,\zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$ if and only if all the irreducible subquotients of $\Theta/\varpi\Theta$ lie in \mathfrak{B} , where Θ is an open bounded T -invariant lattice in Π . By Proposition 5.46, an irreducible τ is absolutely irreducible if and only if it is a character.

Let $\chi : T \rightarrow k^\times$ be a smooth character with $\chi|_Z \equiv \zeta$, let $\tilde{\mathbb{P}} \rightarrow \chi^\vee$ be a projective envelope of χ^\vee in $\mathfrak{C}_{T,\zeta}(\mathcal{O})$, the category anti-equivalent to $\text{Mod}_{T,\zeta}^{\text{lfm}}(\mathcal{O})$ by Pontryagin duality, and let $\tilde{\mathbb{E}}$ be the endomorphism ring of $\tilde{\mathbb{P}}$ in $\mathfrak{C}_{T,\zeta}(\mathcal{O})$. We have showed in Proposition 3.34 that $\tilde{\mathbb{E}} \cong \mathcal{O}[[x,y]]$. In particular, $\tilde{\mathbb{E}}$ is commutative and noetherian. Let \mathfrak{B} be the block of χ and let $\text{Ban}_{T,\zeta}^{\text{adm,fl}}(\mathbb{L})^{\mathfrak{B}}$ be the full subcategory of $\text{Ban}_{T,\zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$ consisting of all objects of finite length.

Lemma 7.8. — *We have an equivalence of categories*

$$\text{Ban}_{T,\zeta}^{\text{adm,fl}}(\mathbb{L})^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{n} \in \text{MaxSpec } \tilde{\mathbb{E}}[1/p]} \text{Ban}_{T,\zeta}^{\text{adm,fl}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}.$$

The category $\text{Ban}_{T,\zeta}^{\text{adm,fl}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\tilde{\mathbb{E}}[1/p]$. In particular, $\text{Ban}_{T,\zeta}^{\text{adm,fl}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}$ contains only one irreducible object $\Pi_{\mathfrak{n}}$.

Proof. — Apply Theorem 4.36 with $\mathfrak{C}(\mathcal{O}) = \mathfrak{C}_{T,\zeta}(\mathcal{O})^{\mathfrak{B}}$. □

Lemma 7.9. — *Let \mathfrak{n} be a maximal ideal of $\tilde{\mathbb{E}}[1/p]$ and let $\Pi_{\mathfrak{n}}$ be as above then $\Pi_{\mathfrak{n}}$ is finite dimensional over \mathbb{L} , with $\dim_{\mathbb{L}} \Pi_{\mathfrak{n}} = [\tilde{\mathbb{E}}[1/p]/\mathfrak{n} : \mathbb{L}]$.*

Proof. — Let Θ be an open bounded T -invariant lattice in $\Pi_{\mathfrak{n}}$. It follows from Theorem 4.36 that $\mathfrak{m}(\Pi_{\mathfrak{n}}) := \text{Hom}_{\mathfrak{C}_{T,\zeta}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d)_{\mathbb{L}}$ is an irreducible $\tilde{\mathbb{E}}[1/p]$ -module killed by \mathfrak{n} . Since $\tilde{\mathbb{E}}[1/p]/\mathfrak{n}$ is a field we have $\mathfrak{m}(\Pi_{\mathfrak{n}}) \cong \tilde{\mathbb{E}}[1/p]/\mathfrak{n}$. Corollary 4.41 implies that $\Theta \otimes_{\mathcal{O}} k$ is of finite length and the irreducible subquotients are isomorphic to χ . Lemma 4.15 says that χ occurs in $\Theta/\varpi\Theta$ with multiplicity $[\tilde{\mathbb{E}}[1/p]/\mathfrak{n} : \mathbb{L}]$. Hence, $[\tilde{\mathbb{E}}[1/p]/\mathfrak{n} : \mathbb{L}] = \dim_k \Theta \otimes_{\mathcal{O}} k = \dim_{\mathbb{L}} \Pi_{\mathfrak{n}}$. □

Let Π be in $\text{Ban}_{T,\zeta}^{\text{adm}}(\mathbb{L})$ and let $|\cdot|$ be a T -invariant norm defining the topology on Π . We may consider Π as a representation of \mathbb{P} by letting U act trivially. We let $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Pi)_{\text{cont}}$ be the space of continuous functions $f : \mathbb{G} \rightarrow \Pi$ such that $f(bg) = bf(g)$ for all $b \in \mathbb{P}$ and $g \in \mathbb{G}$. The function $g \mapsto |f(g)|$ is continuous and constant on the cosets $\mathbb{P}g$ since the norm on Π is T -invariant. Since $\mathbb{P} \backslash \mathbb{G}$ is compact, the function $f \mapsto \|f\| := \sup_{g \in \mathbb{G}} |f(g)|$ defines a \mathbb{G} -invariant norm on $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Pi)_{\text{cont}}$ with respect to which it is complete. If Θ is an open bounded T -invariant lattice in Π , then $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Theta)_{\text{cont}}$ is an open bounded \mathbb{G} -invariant lattice in $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Pi)_{\text{cont}}$ and we have

$$(138) \quad (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Theta)_{\text{cont}} \otimes_{\mathcal{O}} \mathcal{O}/(\varpi^n) \cong \text{Ind}_{\mathbb{P}}^{\mathbb{G}} (\Theta/\varpi^n \Theta), \quad \forall n \geq 1.$$

Using (138) one may show that the admissibility of Π implies the admissibility of $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Pi)_{\text{cont}}$. Let $\tilde{\mathbb{M}} := (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \tilde{\mathbb{P}}^{\vee})^{\vee}$ and recall that $\text{End}_{\mathfrak{C}_{G,\zeta}(\mathcal{O})}(\tilde{\mathbb{M}})$ is naturally isomorphic to $\tilde{\mathbb{E}} = \text{End}_{\mathfrak{C}_{T,\zeta}(\mathcal{O})}(\tilde{\mathbb{P}})$ by Corollary 7.2.

Lemma 7.10. — *Let Π be in $\widehat{\text{Ban}}_{\mathbb{T}, \zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$, let Θ be an open bounded \mathbb{T} -invariant lattice in Π and let $\mathfrak{m} := \text{Hom}_{\mathfrak{C}_{\mathbb{T}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{P}}, \Theta^d)$. Then*

$$(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Theta)_{\text{cont}}^d \cong \mathfrak{m} \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{M}}, \quad (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Pi)_{\text{cont}} \cong \text{Hom}_{\mathcal{O}}^{\text{cont}}(\mathfrak{m} \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{M}}, \mathbb{L}).$$

Proof. — Since χ is the only irreducible object of $\text{Mod}_{\mathbb{T}, \zeta}^{\text{lin}}(\mathcal{O})^{\mathfrak{B}}$, for every object \mathbb{N} of $\mathfrak{C}_{\mathbb{T}, \zeta}(\mathcal{O})^{\mathfrak{B}}$, $\text{Hom}_{\mathfrak{C}_{\mathbb{T}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{P}}, \mathbb{N}) = 0$ is equivalent to $\mathbb{N} = 0$. Thus, it follows from Lemma 2.9 that the map $\text{Hom}_{\mathfrak{C}_{\mathbb{T}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{P}}, \mathbb{N}) \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{P}} \rightarrow \mathbb{N}$ is an isomorphism. In particular, $\mathfrak{m} \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{P}} \cong \Theta^d$, $(\mathfrak{m}/\varpi^n \mathfrak{m}) \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{P}} \cong \Theta^d/\varpi^n \Theta^d$, for all $n \geq 1$. It follows from (57) that $\Theta/\varpi^n \Theta \cong ((\mathfrak{m}/\varpi^n \mathfrak{m}) \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{P}})^{\vee}$, for all $n \geq 1$. Hence,

$$(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \Theta)_{\text{cont}}^d \otimes_{\mathcal{O}} \mathcal{O}/(\varpi^n) \cong (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} (\Theta/\varpi^n \Theta))^{\vee} \cong (\mathfrak{m} \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{M}}) \otimes_{\mathcal{O}} \mathcal{O}/(\varpi^n),$$

where the first isomorphism is (57) and (138), the second is given by Corollary 7.3. We get the first assertion by passing to the limit. The second assertion follows from [61]. \square

Proposition 7.11. — *Let Π be an absolutely irreducible admissible unitary \mathbb{L} -Banach space representation of \mathbb{G} with a central character ζ . If $\text{Hom}_{\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{M}}, \Xi^d) \neq 0$ for some open \mathbb{G} -invariant lattice Ξ in Π , then either $\Pi \cong \eta \circ \det$ or $\Pi \cong (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi)_{\text{cont}}$ for some continuous unitary character $\psi : \mathbb{T} \rightarrow \mathbb{L}^{\times}$ lifting χ with $\psi \neq \psi^s$.*

Proof. — Let $\mathbb{S} := (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi)^{\vee}$ and let $\widetilde{\mathbb{P}}_{\mathbb{S}} \twoheadrightarrow \mathbb{S}$ be a projective envelope of \mathbb{S} in $\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})$. We note that if $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi$ is reducible then it is a non-split extension of two irreducible representations, hence $\widetilde{\mathbb{P}}_{\mathbb{S}}$ is a projective envelope of an irreducible object in $\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})$, namely the cosocle of \mathbb{S} . Let $\widetilde{\mathbb{E}}_{\mathbb{S}} := \text{End}_{\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{P}}_{\mathbb{S}})$ and let $\widetilde{\mathbb{E}} = \text{End}_{\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{M}})$ as above. Recall that in the Corollary 7.2 we have shown that the natural map $\widetilde{\mathbb{P}}_{\mathbb{S}} \twoheadrightarrow \widetilde{\mathbb{M}}$ induces a surjection of rings $\varphi : \widetilde{\mathbb{E}}_{\mathbb{S}} \twoheadrightarrow \widetilde{\mathbb{E}}$.

Lemma 4.5 allows us to assume that there exists a surjection $\phi : \widetilde{\mathbb{M}} \twoheadrightarrow \Xi^d$ in $\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})$. Let γ be the composition $\widetilde{\mathbb{P}}_{\mathbb{S}} \rightarrow \widetilde{\mathbb{M}} \xrightarrow{\phi} \Xi^d$ and let $\mathfrak{m} := \text{Hom}_{\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{P}}_{\mathbb{S}}, \Xi^d)$. It follows from Proposition 4.18 that $\mathfrak{m} = \gamma \circ \widetilde{\mathbb{E}}_{\mathbb{S}}$. Since γ factors through $\widetilde{\mathbb{M}}$ it will be killed by any $\phi_1 \in \text{Ker } \phi$. Hence, $\mathfrak{m} \cong \phi \circ \widetilde{\mathbb{E}} \cong \text{Hom}_{\mathfrak{C}_{\mathbb{G}, \zeta}(\mathcal{O})}(\widetilde{\mathbb{M}}, \Xi^d)$ and the action of $\widetilde{\mathbb{E}}_{\mathbb{S}}$ on \mathfrak{m} factors through the action of $\widetilde{\mathbb{E}}$. By Proposition 4.18, $\mathfrak{m}_{\mathbb{L}}$ is an irreducible $\widetilde{\mathbb{E}}[1/p]$ -module and, since $\widetilde{\mathbb{E}} \cong \mathcal{O}[[x, y]]$, $\mathfrak{m}_{\mathbb{L}}$ is a finite dimensional \mathbb{L} -vector space. Moreover, $\text{End}_{\widetilde{\mathbb{E}}_{\mathbb{S}}}(\mathfrak{m}) \cong \text{End}_{\widetilde{\mathbb{E}}}(\mathfrak{m})$. Since $\mathfrak{m}_{\mathbb{L}}$ is finite dimensional, $\widetilde{\mathbb{E}}$ is commutative and Π absolutely irreducible we deduce from Proposition 4.19 and Lemma 4.1 that $\mathfrak{m}_{\mathbb{L}}$ is one dimensional and so \mathfrak{m} is a free \mathcal{O} -module of rank 1. Since $\phi \in \mathfrak{m}$ the map $\text{ev} : \mathfrak{m} \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{M}} \rightarrow \Xi^d$ is surjective. Dually this means that we have an injection $\Pi \hookrightarrow \text{Hom}_{\mathcal{O}}^{\text{cont}}(\mathcal{O} \widehat{\otimes}_{\widetilde{\mathbb{E}}} \widetilde{\mathbb{M}}, \mathbb{L}) \cong (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi)_{\text{cont}}$ and the character ψ comes from Lemma 7.9, where the last isomorphism is given by Lemma 7.10, which identifies Π with a closed \mathbb{G} -invariant subspace of $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi)_{\text{cont}}$. (We note that both Banach space representations are admissible.) If $\psi \neq \psi^s$ then $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi)_{\text{cont}}$ is topologically irreducible and if $\psi = \psi^s$ then

it has a unique closed G -invariant subspace isomorphic to a character, [29, 5.3.4]. This implies the assertion. \square

8. Generic residually reducible case

In this section we deal with the case where in Colmez's terminology the *atome automorphe* consists of two distinct irreducible representations. More precisely, let $\chi_1, \chi_2 : \mathbf{Q}_p^\times \rightarrow k^\times$ be smooth characters and assume that $\chi_1 \chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$. We assume throughout this section that $p \geq 3$. Let $\chi : \mathbf{T} \rightarrow k^\times$ be the character $\chi = \chi_1 \otimes \chi_2 \omega^{-1}$ then $\chi^s \alpha = \chi_2 \otimes \chi_1 \omega^{-1}$. Let

$$\pi_1 := \text{Ind}_p^G \chi, \quad \pi_2 := \text{Ind}_p^G \chi^s \alpha.$$

We note that the assumption on χ_1 and χ_2 implies that both representations are irreducible and distinct. Let π be an irreducible smooth representation of G with a central character. It is well known, see for example [56, 11.5], that if $\text{Ext}_{G,\zeta}^1(\pi, \pi_1) \neq 0$ then $\pi \cong \pi_1$ or $\pi \cong \pi_2$. Moreover,

$$\dim \text{Ext}_{G,\zeta}^1(\pi_1, \pi_1) = 2, \quad \dim \text{Ext}_{G,\zeta}^1(\pi_2, \pi_1) = 1.$$

Let

$$(139) \quad 0 \rightarrow \pi_1 \rightarrow \kappa \rightarrow \pi_2 \rightarrow 0$$

be the unique non-split extension.

Lemma 8.1. — $\text{Ord}_p \kappa \cong \text{Ord}_p \pi_1 \cong \chi^s, \mathbf{R}^1 \text{Ord}_p \kappa \cong \mathbf{R}^1 \text{Ord}_p \pi_2 \cong \chi^s$.

Proof. — Since $\mathbf{R}^i \text{Ord}_p = 0$ for $i \geq 2$, we apply Ord_p to (139) to get an exact sequence:

$$(140) \quad \begin{aligned} 0 &\rightarrow \text{Ord}_p \pi_1 \rightarrow \text{Ord}_p \kappa \rightarrow \text{Ord}_p \pi_2 \\ &\rightarrow \mathbf{R}^1 \text{Ord}_p \pi_1 \rightarrow \mathbf{R}^1 \text{Ord}_p \kappa \rightarrow \mathbf{R}^1 \text{Ord}_p \pi_2 \rightarrow 0. \end{aligned}$$

It follows from (126) that $\text{Ord}_p \pi_1 \cong \chi^s$ and $\text{Ord}_p \pi_2 \cong \chi \alpha^{-1}$. Hence, if the map $\text{Ord}_p \kappa \rightarrow \text{Ord}_p \pi_2$ is non-zero then it must be surjective. Hence, we have an exact sequence of \mathbf{T} -representations $0 \rightarrow \chi^s \rightarrow \text{Ord}_p \kappa \rightarrow \chi \alpha^{-1} \rightarrow 0$. Since $\chi^s \neq \chi \alpha^{-1}$ this sequence must split, see Corollary 3.35. But then using adjointness (120) we would obtain a splitting of (139). Hence, the map $\text{Ord}_p \pi_2 \rightarrow \mathbf{R}^1 \text{Ord}_p \pi_1$ is non-zero, and since $\mathbf{R}^1 \text{Ord}_p \pi_1 \cong \chi \alpha^{-1}$ the map is an isomorphism. Thus we obtain the claim. \square

Lemma 8.2. — *Let π be irreducible and suppose that $\text{Ext}_{G,\zeta}^1(\pi, \kappa) \neq 0$ then $\pi \cong \pi_1$. Moreover, $\dim \text{Ext}_{G,\zeta}^1(\pi_1, \kappa) \leq 3$ and $\text{Ext}_{G,\zeta}^i(\pi_2, \kappa) = 0$ for all $i \geq 0$.*

Proof. — If $\text{Ext}_G^1(\pi, \kappa) \neq 0$ then $\text{Ext}_G^1(\pi, \pi_1) \neq 0$ or $\text{Ext}_G^1(\pi, \pi_2) \neq 0$ and hence $\pi \cong \pi_1$ or $\pi \cong \pi_2$. The assertion follows from the degeneration of spectral sequence (123), Lemma 8.1 and the fact that for distinct characters $\chi, \psi : \mathbb{T} \rightarrow k^\times$ we have $\text{Ext}_{\mathbb{T}, \zeta}^i(\chi, \psi) = 0$ for all $i \geq 0$, see Corollary 3.35. \square

Proposition 8.3. — *Let $S := \pi_1^\vee$ and $Q := \kappa^\vee$ then the hypotheses (H1)–(H5) of Section 3.1 are satisfied.*

Proof. — (H1) holds because (139) is non-split, (H2) holds as $\pi_1 \not\cong \pi_2$, (H3), (H4) and (H5) follow from Lemma 8.2. \square

Since (H0) holds for G by Corollary 5.19, we may apply the results of Section 3.1 and Section 4. Let $\tilde{\mathbb{P}} \twoheadrightarrow S$ be a projective envelope of S in $\mathfrak{C}(\mathcal{O})$, let $\tilde{\mathbb{E}} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}})$ and let \mathfrak{m} be the maximal ideal of $\tilde{\mathbb{E}} \otimes_{\mathcal{O}} k$. Let $\rho := \check{\mathbf{V}}(Q)$ then since $\check{\mathbf{V}}$ is exact we get an exact sequence of Galois representations

$$0 \rightarrow \chi_2 \rightarrow \rho \rightarrow \chi_1 \rightarrow 0.$$

This sequence is non-split by [23, VII.4.13]. We note that $\det \rho$ is congruent to $\varepsilon \zeta$, where ε is the cyclotomic character.

Proposition 8.4. — *The functor $\check{\mathbf{V}}$ induces a surjection*

$$\tilde{\mathbb{E}} \twoheadrightarrow \mathbf{R}_\rho^{\varepsilon \zeta} \cong \mathcal{O}[[x, y, z]],$$

where $\mathbf{R}_\rho^{\varepsilon \zeta}$ pro-represents the deformation functor of ρ with determinant $\varepsilon \zeta$.

Proof. — Since $\chi_1 \neq \chi_2$ and the sequence is non-split, we get that $\text{End}_{k[\mathcal{G}_{\mathbf{Q}_p}]}(\rho) = k$ and hence the universal deformation functor Def_ρ^{ab} is representable. Since $\chi_1 \chi_2^{-1} \neq \omega^{\pm 1}$ a standard calculation with local Tate duality and Euler characteristic gives $H^2(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho) = 0$ and $H^1(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho)$ is 5-dimensional. This implies, see [47, §1.6], [48, §24], that Def_ρ^{ab} is represented by $\mathbf{R} \cong \mathcal{O}[[x_1, \dots, x_5]]$ and the deformation problem with the fixed determinant is represented by $\mathbf{R}^{\varepsilon \zeta} \cong \mathcal{O}[[x_1, x_2, x_3]]$. It follows from [41, 2.3.4] that $\text{Spec } \tilde{\mathbb{E}}^{ab}$ is a closed subset of $\text{Spec } \mathbf{R}$ and contains $\text{Spec } \mathbf{R}^{\varepsilon \zeta}$, which is stronger than (iii) in Proposition 5.56. Since $\mathbf{R}^{\varepsilon \zeta}$ is reduced we obtain a surjection $\tilde{\mathbb{E}}^{ab} \twoheadrightarrow \mathbf{R}^{\varepsilon \zeta} \cong \mathcal{O}[[x_1, x_2, x_3]]$. \square

Corollary 8.5. — *We have*

$$\dim \text{Ext}_{G, \zeta}^1(\pi_1, \kappa) = \dim \text{Ext}_{\mathfrak{C}(k)}^1(Q, S) = \dim \text{Ext}_{\mathfrak{C}(k)}^1(Q, Q) = 3.$$

Proof. — We note that all three Ext^1 groups are isomorphic, the first two by anti-equivalence of categories, the last two by Lemma 3.1. Now $\text{Ext}_{\mathfrak{C}(k)}^1(Q, Q)$ is isomorphic to

$(\mathfrak{m}/\mathfrak{m}^2)^*$ by Lemma 3.29 and the surjection of Proposition 8.4 implies that $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq 3$. Since $\dim \text{Ext}_{G,\zeta}^1(\pi_1, \kappa) \leq 3$ by Lemma 8.2 we are done. \square

Proposition 8.6. — $\dim \text{Ext}_{G,\zeta}^1(\pi_1, \tau) \leq 3$ for all non-split extensions $0 \rightarrow \kappa \rightarrow \tau \rightarrow \kappa \rightarrow 0$ in $\text{Mod}_{G,\zeta}^{\text{sm}}(k)$.

Proof. — Proposition 8.4, Corollary 8.5 and Lemma 3.41 imply that the equivalent conditions of Lemma 3.40 are satisfied and thus by Lemma 3.43 it is enough to check the statement for every non-split extension in some 2-dimensional subspace of $\text{Ext}_{G,\zeta}^1(\kappa, \kappa)$. Let Υ be the image of:

$$\text{Ext}_{T,\zeta}^1(\chi^s, \chi^s) \cong \text{Ext}_{G,\zeta}^1(\pi_1, \pi_1) \hookrightarrow \text{Ext}_{G,\zeta}^1(\pi_1, \kappa) \cong \text{Ext}_{G,\zeta}^1(\kappa, \kappa).$$

The extension class of $0 \rightarrow \kappa \rightarrow \tau \rightarrow \kappa \rightarrow 0$ lies in Υ if and only if there exists an extension $0 \rightarrow \chi \rightarrow \epsilon \rightarrow \chi \rightarrow 0$ in $\text{Mod}_{T,\zeta}^{\text{sm}}(k)$ and an injection $\text{Ind}_{\mathfrak{p}}^G \epsilon \hookrightarrow \tau$. We denote the quotient by κ_1 . Since the semi-simplification $\tau^{ss} \cong \pi_1^{\oplus 2} \oplus \pi_2^{\oplus 2}$ we have $\kappa_1^{ss} \cong \pi_2^{\oplus 2}$. As $\chi \neq \chi^s \alpha$, the 5-term sequence (123) implies that $\text{Ext}_{G,\zeta}^1(\pi_2, \text{Ind}_{\mathfrak{p}}^G \epsilon)$ is 1-dimensional. Since $\text{Hom}_G(\pi_2, \tau) = 0$ we deduce that κ_1 cannot be semisimple. We use (123) again to obtain $\text{Ext}_{G,\zeta}^1(\pi_2, \pi_2) \cong \text{Ext}_{T,\zeta}^1(\chi^s \alpha, \chi^s \alpha)$. Hence, $\kappa_1 \cong \text{Ind}_{\mathfrak{p}}^G \delta$, where $0 \rightarrow \chi^s \alpha \rightarrow \delta \rightarrow \chi^s \alpha \rightarrow 0$ is an extension in $\text{Mod}_{T,\zeta}^{\text{sm}}(k)$. Applying $\text{Ord}_{\mathfrak{p}}$ to $0 \rightarrow \text{Ind}_{\mathfrak{p}}^G \epsilon \rightarrow \tau \rightarrow \text{Ind}_{\mathfrak{p}}^G \delta \rightarrow 0$ gives an exact sequence:

$$0 \rightarrow \epsilon^s \rightarrow \text{Ord}_{\mathfrak{p}} \tau \rightarrow \delta^s \xrightarrow{\partial} \epsilon \alpha^{-1} \rightarrow \mathbf{R}^1 \text{Ord}_{\mathfrak{p}} \tau \rightarrow \delta \alpha^{-1} \rightarrow 0.$$

Since $\text{Hom}_G(\pi_2, \kappa) = 0$ we have $\text{Hom}_T(\chi \alpha^{-1}, \text{Ord}_{\mathfrak{p}} \tau) \cong \text{Hom}_G(\pi_2, \tau) = 0$. Since $\chi^s \neq \chi \alpha^{-1}$ we have $\text{Ext}_{T,\zeta}^1(\chi \alpha^{-1}, \chi^s) = 0$ and hence ∂ is injective. Since the source and the target are 2-dimensional, ∂ is an isomorphism and hence $\text{Ord}_{\mathfrak{p}} \tau \cong \mathbf{R}^1 \text{Ord}_{\mathfrak{p}} \tau \cong \epsilon^s$ and we have an exact sequence

$$0 \rightarrow \text{Ext}_{T,\zeta}^1(\chi^s, \epsilon^s) \rightarrow \text{Ext}_{G,\zeta}^1(\pi_1, \tau) \rightarrow \text{Hom}_T(\chi^s, \epsilon^s).$$

Since the first term is 2-dimensional by Lemma 5.48 and the last term is 1-dimensional as ϵ is non-split, we deduce that $\dim \text{Ext}_{G,\zeta}^1(\pi_1, \tau) \leq 3$. \square

Corollary 8.7. — The functor $\check{\mathbf{V}}$ induces an isomorphism $\check{\mathbf{E}} \cong \mathbf{R}_{\rho}^{\epsilon \zeta}$. In particular, $\check{\mathbf{V}}(\check{\mathbf{P}})$ is the universal deformation of ρ with determinant $\zeta \epsilon$.

Proof. — The first assertion follows from Theorem 3.39. We then deduce that $\check{\mathbf{V}}$ induces an isomorphism of deformation functors, Corollary 5.52, and thus $\check{\mathbf{V}}(\check{\mathbf{P}})$ is the universal deformation of ρ with determinant $\zeta \epsilon$. \square

Theorem 8.8. — Let Π be an admissible unitary absolutely irreducible L -Banach space representation of G with a central character ζ . Suppose that the reduction of some open bounded G -invariant lattice in Π contains π_1 as a subquotient then $\overline{\Pi} \subseteq \pi_1 \oplus \pi_2$.

Proof. — The Schikhof dual of an open bounded G -invariant lattice in Π is an object of $\mathfrak{C}(\mathcal{O})$ by Lemma 4.11. Since \tilde{E} is commutative the assertion follows from Corollary 4.44. \square

Corollary 8.9. — Let Π be as in Theorem 8.8 and suppose that $\bar{\Pi}$ does not contain π_2 then $\Pi \cong (\text{Ind}_p^G \psi)_{\text{cont}}$ for some continuous unitary character $\psi : \Gamma \rightarrow L^\times$ lifting χ and satisfying $\psi|_Z = \zeta$.

Proof. — Let \tilde{P}_2 be a projective envelope of π_2^\vee in $\mathfrak{C}_{G,\zeta}(\mathcal{O})$ and let Θ be an open bounded G -invariant lattice in Π . Theorem 8.8 implies that $\bar{\Pi} \cong \pi_1$. Hence Lemma 4.15 says that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_2, \Theta^d) = 0$ and $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \neq 0$. We deduce from Corollary 7.7 that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{M}, \Theta^d) \neq 0$, where $\tilde{M} = (\text{Ind}_p^G \tilde{P}_{\chi^\vee})^\vee$ and \tilde{P}_{χ^\vee} is a projective envelope of χ^\vee in $\mathfrak{C}_{\Gamma,\zeta}(\mathcal{O})$. The assertion follows from Proposition 7.11. \square

8.1. The centre. — Recall that the block \mathfrak{B} of π_1 contains only two irreducible representations π_1 and π_2 , Proposition 5.42, and so $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is the full subcategory of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})$ consisting of representations with every irreducible subquotient isomorphic to either π_1 or π_2 . Let $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ be the full subcategory of $\mathfrak{C}(\mathcal{O})$ anti-equivalent to $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$, as in Proposition 5.35. Let \tilde{P}_1 and \tilde{P}_2 be projective envelopes of $S_1 := \pi_1^\vee$ and $S_2 := \pi_2^\vee$ in $\mathfrak{C}(\mathcal{O})$, respectively. Let $\tilde{P}_{\mathfrak{B}} := \tilde{P}_1 \oplus \tilde{P}_2$ and $\tilde{E}_{\mathfrak{B}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\mathfrak{B}})$. The aim of this subsection is to compute the ring $\tilde{E}_{\mathfrak{B}}$ and determine its centre.

Lemma 8.10. — Let M and N be objects of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ then \check{V} induces an isomorphism $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(M, N) \cong \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\check{V}(M), \check{V}(N))$.

Proof. — Since \check{V} commutes with projective limits it is enough to show the statement for objects of finite length. Now $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ has only two irreducible objects S_1, S_2 . For A and B isomorphic to S_1 or S_2 we have $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(A, B) \cong \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\check{V}(A), \check{V}(B))$, since both sides are equal either to k or to 0 and an injection $\text{Ext}_{\mathfrak{C}(\mathcal{O})}^1(A, B) \hookrightarrow \text{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\check{V}(A), \check{V}(B))$ by [23, §VII.5]. We then may argue by induction on $\ell(M) + \ell(N)$, where ℓ denotes the length, see the proof of Lemma A.1 in [56]. \square

Let ρ_1 and ρ_2 be 2-dimensional k -representations of $\mathcal{G}_{\mathbf{Q}_p}$ such that we have exact non-split sequences of Galois representations:

$$0 \rightarrow \chi_2 \rightarrow \rho_1 \rightarrow \chi_1 \rightarrow 0, \quad 0 \rightarrow \chi_1 \rightarrow \rho_2 \rightarrow \chi_2 \rightarrow 0.$$

Since $\text{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\chi_1, \chi_2)$ and $\text{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\chi_2, \chi_1)$ are one dimensional such representations exist and are uniquely determined up to isomorphism. We note that $\det \rho_1 = \det \rho_2$ is congruent to $\zeta \varepsilon$. Let ρ_1^{un} and ρ_2^{un} be the universal deformations of ρ_1 and ρ_2 respectively with determinant $\zeta \varepsilon$. Let $\chi := \text{tr } \rho_1 = \text{tr } \rho_2$ and let $R_\chi^{\text{ps}, \varepsilon \zeta}$ be the universal deformation ring parameterizing 2-dimensional pseudocharacters with determinant $\zeta \varepsilon$ lifting χ .

Corollary 8.11. — *The category $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is anti-equivalent to the category of compact $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{um} \oplus \rho_2^{um})$ -modules. The centre of $\text{Mod}_{G,\zeta}^{\text{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is naturally isomorphic to $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}$.*

Proof. — Corollary 8.7 and Lemma 8.10 imply that

$$\tilde{\mathbf{E}}_{\mathfrak{B}} \cong \text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\check{\mathbf{V}}(\tilde{\mathbf{P}}_1) \oplus \check{\mathbf{V}}(\tilde{\mathbf{P}}_2)) \cong \text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{um} \oplus \rho_2^{um}).$$

In Proposition B.26 we have showed that $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{um} \oplus \rho_2^{um})$ is a free $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}$ -module of rank 4 and its centre is isomorphic to $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}$. The assertion follows from Proposition 5.45. \square

Corollary 8.12. — *Let $T : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}$ be the universal 2-dimensional pseudocharacter with determinant $\zeta\varepsilon$ lifting χ . For every N in $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$, $\check{\mathbf{V}}(N)$ is killed by $g^2 - T(g)g + \zeta\varepsilon(g)$, for all $g \in \mathcal{G}_{\mathbf{Q}_p}$.*

Proof. — Corollary 8.7 and Proposition B.17 imply that the assertion is true if $N = \tilde{\mathbf{P}}_1$ or $N = \tilde{\mathbf{P}}_2$. Hence, the assertion holds for $N = \tilde{\mathbf{P}}_{\mathfrak{B}}$. The general case follows from the isomorphism:

$$\check{\mathbf{V}}(N) \cong \check{\mathbf{V}}(\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathfrak{B}}, N) \hat{\otimes}_{\tilde{\mathbf{E}}_{\mathfrak{B}}} \tilde{\mathbf{P}}_{\mathfrak{B}}) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathfrak{B}}, N) \hat{\otimes}_{\tilde{\mathbf{E}}_{\mathfrak{B}}} \check{\mathbf{V}}(\tilde{\mathbf{P}}_{\mathfrak{B}}),$$

which is proved in the same way as Lemma 5.53. \square

Let $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})^{\mathfrak{B}}$ be as in Proposition 5.36 and let $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}}$ be the full subcategory consisting of objects of finite length.

Corollary 8.13. — *We have an equivalence of categories*

$$\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{n} \in \text{MaxSpec } \mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}[1/\rho]} \text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}.$$

The category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{um} \oplus \rho_2^{um})[1/\rho]$.

Proof. — Apply Theorem 4.36 with $\mathfrak{C}(\mathcal{O}) = \mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. \square

Corollary 8.14. — *Suppose that the pseudo-character corresponding to a maximal ideal \mathfrak{n} of $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}[1/\rho]$ is irreducible over the residue field of \mathfrak{n} then the category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}[1/\rho]$. In particular, it contains only one irreducible object.*

Proof. — Since the pseudocharacter corresponding to \mathfrak{n} is irreducible, \mathfrak{n} cannot contain the reducibility ideal of $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}[1/\rho]$, see Section B.1. It follows from Corollary B.27 that for such \mathfrak{n} the \mathfrak{n} -adic completion of $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{um} \oplus \rho_2^{um})[1/\rho]$ is isomorphic to the ring of two by two matrices over the \mathfrak{n} -adic completion of $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}[1/\rho]$. \square

Let \mathfrak{n} be a maximal ideal of $\mathbf{R}_\chi^{\text{ps}, \zeta \varepsilon}[1/p]$ with residue field L , let $T_{\mathfrak{n}} : \mathcal{G}_{\mathbf{Q}_p} \rightarrow L$ be the pseudocharacter corresponding to \mathfrak{n} and let $\text{Irr}(\mathfrak{n})$ denote the set (of equivalence classes of) irreducible objects in $\text{Ban}_{G, \zeta}^{\text{adm. fl}}(L)_{\mathfrak{n}}^{\text{B}}$.

Corollary 8.15. — *If $T_{\mathfrak{n}} = \psi_1 + \psi_2$ with $\psi_1, \psi_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow L^\times$ continuous homomorphisms then*

$$\text{Irr}(\mathfrak{n}) = \left\{ \left(\text{Ind}_{\mathbf{P}}^G \psi_1 \otimes \psi_2 \varepsilon^{-1} \right)_{\text{cont}}, \left(\text{Ind}_{\mathbf{P}}^G \psi_2 \otimes \psi_1 \varepsilon^{-1} \right)_{\text{cont}} \right\}.$$

Proof. — Corollary 8.12 implies that, since

$$\mathbf{V}\left(\left(\text{Ind}_{\mathbf{P}}^G \psi_1 \otimes \psi_2 \varepsilon^{-1}\right)_{\text{cont}}\right) = \psi_2, \quad \mathbf{V}\left(\left(\text{Ind}_{\mathbf{P}}^G \psi_2 \otimes \psi_1 \varepsilon^{-1}\right)_{\text{cont}}\right) = \psi_1,$$

both Banach space representations lie in $\text{Irr}(\mathfrak{n})$. Since $\chi_1 \chi_2^{-1} \neq \omega^{\pm 1}, \mathbf{1}$ we also have $\psi_1 \psi_2^{-1} \neq \varepsilon^{\pm 1}, \mathbf{1}$. Thus the Banach space representations are irreducible and distinct. It follows from the explicit description of $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{\text{un}} \oplus \rho_2^{\text{un}})$ in Proposition B.26 that the ring $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_1^{\text{un}} \oplus \rho_2^{\text{un}})[1/p]/\mathfrak{n}$ has two non-isomorphic irreducible modules. \square

9. Non-generic case I

In this section we deal with the case where in Colmez's terminology the *atome automorphe* consists of two isomorphic irreducible representations. We assume throughout this section that $p \geq 3$. Let $\pi := \text{Ind}_{\mathbf{P}}^G \chi$, where $\chi : T \rightarrow k^\times$ is the character $\chi = \chi_1 \otimes \chi_1 \omega^{-1}$, for some smooth character $\chi_1 : \mathbf{Q}_p^\times \rightarrow k^\times$. We note that $\chi^s \alpha = \chi$. We show that the formalism of Section 3 applies with $Q = S = \pi^\vee$. Hence, the projective envelope $\tilde{\mathbf{P}}$ of S is the universal deformation of S , and its endomorphism ring $\tilde{\mathbf{E}}$ is the universal deformation ring in the sense of Theorem 3.26. The new feature in this case is that the ring $\tilde{\mathbf{E}}$ is non-commutative. Indeed, if $\tilde{\mathbf{E}}$ were commutative, then by arguing as in the proof of Theorem 6.4, we would deduce that if π is a subquotient of a reduction modulo ϖ of an open bounded G -invariant lattice in an absolutely irreducible L -Banach space representation Π with central character ζ , then the reduction is isomorphic to π . However, the Banach space representations corresponding to 2-dimensional crystalline Galois representations of small weight provide a counterexample to this, see [17, 5.3.3.1] with $a_p = 2p$. By applying the functor $\check{\mathbf{V}}$ we deduce that $\check{\mathbf{V}}(\tilde{\mathbf{P}})$ is a deformation to $\tilde{\mathbf{E}}$ of one dimensional Galois representation $\check{\mathbf{V}}(S) = \mathbf{V}(\pi) = \chi_1$. Since we allow the coefficients in our deformation theory be non-commutative, Lemma 3.32 implies that the ring $\mathcal{O}[[\mathcal{G}_{\mathbf{Q}_p}(p)]]^{\text{op}}$ solves the universal deformation problem of χ_1 , where $\mathcal{G}_{\mathbf{Q}_p}(p)$ is the maximal pro- p quotient of $\mathcal{G}_{\mathbf{Q}_p}$. Hence we obtain a map $\varphi_{\check{\mathbf{V}}} : \mathcal{O}[[\mathcal{G}_{\mathbf{Q}_p}(p)]]^{\text{op}} \rightarrow \tilde{\mathbf{E}}$ uniquely determined up to $\tilde{\mathbf{E}}$ -conjugation. We show that $\varphi_{\check{\mathbf{V}}}$ is surjective by looking at the tangent spaces.

Let $R_{2\chi_1}^{\text{ps}, \zeta \varepsilon}$ be the universal deformation ring parameterising 2-dimensional pseudocharacters of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant $\zeta \varepsilon$ lifting $2\mathbf{V}(\pi) = 2\chi_1$ and let $T : \mathcal{G}_{\mathbf{Q}_p} \rightarrow R_{2\chi_1}^{\text{ps}, \zeta \varepsilon}$ be the universal pseudocharacter. Kisin has shown that every two dimensional Galois representation, with reduction modulo ϖ equal to $\chi_1 \oplus \chi_1$, lies in the image of \mathbf{V} . This result combined with a “non-commutative Zariski closure” argument, see Corollary 9.6, shows that $\varphi_{\mathbf{V}}$ induces a surjection $\tilde{\mathbf{E}} \twoheadrightarrow (R_{2\chi_1}^{\text{ps}, \zeta \varepsilon} [[\mathcal{G}_{\mathbf{Q}_p}]]/J)^{\text{op}}$, where J is a closed two-sided ideal generated by $g^2 - T(g)g + \zeta \varepsilon(g)$ for all $g \in \mathcal{G}_{\mathbf{Q}_p}$. We show that this map is an isomorphism, Corollary 9.27, by proving structure theorems about both rings, see Lemma 9.3 and Proposition 9.23. We also show that $R_{2\chi_1}^{\text{ps}, \zeta \varepsilon} [[\mathcal{G}_{\mathbf{Q}_p}]]/J$ is a free module of rank 4 over its center, which is isomorphic to $R_{2\chi_1}^{\text{ps}, \zeta \varepsilon}$. We record the consequences for Banach space representations in Section 9.3.

The idea to try and show that $\tilde{\mathbf{E}}$ is isomorphic to a Cayley-Hamilton quotient was inspired by [10].

9.1. Deformation theory.

Proposition 9.1. — *Let $S = \mathbf{Q} = \pi^\vee$ then the hypotheses (H1)–(H5) of Section 3.1 are satisfied. Moreover, $d := \dim \text{Ext}_{\mathfrak{C}(k)}^1(S, S) = 2$.*

Proof. — Let τ be irreducible in $\text{Mod}_{G, \zeta}^{\text{sm}}(k)$. It is well known, see for example [56, Thm 11.5], that if $\text{Ext}_{G, \zeta}^1(\tau, \pi) \neq 0$ then $\pi \cong \tau$ and $\dim \text{Ext}_{G, \zeta}^1(\pi, \pi) = 2$. Dually this implies (H3) and (H4) and all the other hypotheses hold trivially. \square

Since (H0) holds for G by Corollary 5.19, we may apply the results of Section 3.1 and Section 4. Let $\tilde{\mathbf{P}} \twoheadrightarrow S$ be a projective envelope of S in $\mathfrak{C}(\mathcal{O})$, let $\tilde{\mathbf{E}} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}})$, $\tilde{\mathfrak{m}}$ the maximal ideal of $\tilde{\mathbf{E}}$ and let \mathfrak{m} be the maximal ideal of $\tilde{\mathbf{E}} \otimes_{\mathcal{O}} k$. We note that the last part of Proposition 9.1 and Lemma 3.29 gives $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$.

Let $\tilde{\mathbf{P}}_{\chi^\vee}$ be a projective envelope of χ^\vee in $\mathfrak{C}_{T, \zeta}(\mathcal{O})$ and let $\tilde{\mathbf{M}} := (\text{Ind}_{\mathbf{P}}^G \tilde{\mathbf{P}}_{\chi^\vee})^\vee$. Corollary 7.2 gives us a surjection

$$(141) \quad \tilde{\mathbf{E}} \twoheadrightarrow \text{End}_{\mathfrak{C}_{G, \zeta}(\mathcal{O})}(\tilde{\mathbf{M}}) \cong \text{End}_{\mathfrak{C}_{T, \zeta}(\mathcal{O})}(\tilde{\mathbf{P}}_{\chi^\vee}) \cong \mathcal{O}[[x, y]].$$

Let \mathfrak{a} be the kernel of (141). Since $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$ we deduce from (141) that $\tilde{\mathbf{E}}/\mathfrak{a} \cong \tilde{\mathbf{E}}^{\text{ab}}$.

Lemma 9.2. — *There exists $t \in \tilde{\mathfrak{m}}^2$ such that $\mathfrak{a} = \tilde{\mathbf{E}}t$ and $\phi t \neq 0$ for all non-zero $\phi \in \tilde{\mathbf{E}}$.*

Proof. — Since $\chi = \chi^s \alpha$ Corollary 7.7 gives us an exact sequence

$$(142) \quad 0 \rightarrow \tilde{\mathbf{P}} \xrightarrow{t} \tilde{\mathbf{P}} \rightarrow \tilde{\mathbf{M}} \rightarrow 0.$$

Applying the exact functor $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}, *)$ to (142) we get an exact sequence

$$(143) \quad 0 \rightarrow \tilde{\mathbf{E}} \xrightarrow{t_*} \tilde{\mathbf{E}} \rightarrow \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}, \tilde{\mathbf{M}}) \rightarrow 0.$$

The last term is isomorphic to $\text{End}_{\mathcal{O}(\mathcal{O})}(\tilde{\mathbf{M}})$ by Proposition 7.1(iii), Corollary 7.2. Hence, $\mathfrak{a} = t_*(\tilde{\mathbf{E}}) = \tilde{\mathbf{E}}t$ and since t_* is injective we get that $\phi t = 0$ implies $\phi = 0$. As the image of t in $\tilde{\mathbf{E}}^{ab}$ is zero, the image of t in the commutative ring $\tilde{\mathbf{E}}/\tilde{\mathfrak{m}}^2$ will also be zero. Hence, $t \in \tilde{\mathfrak{m}}^2$. \square

Lemma 9.3. — *Let $\varphi : \tilde{\mathbf{E}} \rightarrow \mathbf{R}$ be a quotient such that $\mathbf{R}^{ab} \cong \mathcal{O}[[x, y]]$ and there exists an element $t' \in \mathbf{R}$ such that $\text{Ker}(\mathbf{R} \rightarrow \mathbf{R}^{ab}) = \mathbf{R}t'$ and $at' = 0$ implies that $a = 0$ for all $a \in \mathbf{R}$. Then φ is an isomorphism.*

Proof. — The composition $\tilde{\mathbf{E}} \rightarrow \mathbf{R} \rightarrow \mathbf{R}^{ab}$ factors through $\tilde{\mathbf{E}}^{ab}$ and since both rings are formally smooth of the same dimension we deduce that $\varphi^{ab} : \tilde{\mathbf{E}}^{ab} \rightarrow \mathbf{R}^{ab}$ is an isomorphism. Thus $\text{Ker}(\mathbf{R} \rightarrow \mathbf{R}^{ab}) = \mathbf{R}\varphi(t)$. Hence, we may write $\varphi(t) = at'$ and $t' = b\varphi(t)$ for some $a, b \in \mathbf{R}$. Hence, $(1 - ba)t' = 0$ and so $ba = 1$ and this implies that b and a are units in \mathbf{R} . (Note that any element of $1 + \mathfrak{m}_{\mathbf{R}}$ is a unit and hence if the image of a in \mathbf{R}^{ab} is a unit then a is a unit in \mathbf{R} .) So we may assume that $t' = \varphi(t)$.

Since \mathfrak{a} is a two-sided ideal and $\mathfrak{a} = \tilde{\mathbf{E}}t$, for every $b \in \tilde{\mathbf{E}}$ there exists $a \in \tilde{\mathbf{E}}$ such that $tb = at$. This implies that for $n \geq 1$ we have $\mathfrak{a}^n = \tilde{\mathbf{E}}t^n$. Moreover, since the right multiplication by t is injective, multiplication by t^n induces an isomorphism $\tilde{\mathbf{E}}/\mathfrak{a} \cong \mathfrak{a}^n/\mathfrak{a}^{n+1}$. Since the multiplication by $\varphi(t)$ is injective in \mathbf{R} , multiplication by $\varphi(t)^n$ induces an isomorphism $\mathbf{R}/\varphi(\mathfrak{a}) \cong \varphi(\mathfrak{a})^n/\varphi(\mathfrak{a})^{n+1}$. Hence, φ induces an isomorphism $\mathfrak{a}^n/\mathfrak{a}^{n+1} \cong \varphi(\mathfrak{a})^n/\varphi(\mathfrak{a})^{n+1}$, for all $n \geq 1$. Thus an isomorphism $\tilde{\mathbf{E}}/\mathfrak{a}^n \cong \mathbf{R}/\varphi(\mathfrak{a})^n$ for all n . Passing to the limit we get $\tilde{\mathbf{E}} \cong \mathbf{R}$. \square

Now $\check{\mathbf{V}}(\mathbf{S})$ is a 1-dimensional k -representation of $\mathcal{G}_{\mathbf{Q}_p}$, the absolute Galois group of \mathbf{Q}_p . Let \mathfrak{A} be the category of local finite artinian augmented (possibly non-commutative) \mathcal{O} -algebras defined in Definition 3.19 and let $\text{Def}_{\check{\mathbf{V}}(\mathbf{S})} : \mathfrak{A} \rightarrow \text{Sets}$ be the functor, such that $\text{Def}_{\check{\mathbf{V}}(\mathbf{S})}(\mathbf{A})$ is the set of isomorphism classes of deformations of $\check{\mathbf{V}}(\mathbf{S})$ to \mathbf{A} , see Definition 3.21. Lemma 3.32 says that the functor $\text{Def}_{\check{\mathbf{V}}(\mathbf{S})} : \mathfrak{A} \rightarrow \text{Sets}$ is pro-represented (in the sense of Theorem 3.26) by the ring $\mathcal{O}[[\mathcal{G}]]^{op} \cong \text{End}_{\mathcal{O}[[\mathcal{G}]]}(\mathcal{O}[[\mathcal{G}]])$, where $\mathcal{G} := \mathcal{G}_{\mathbf{Q}_p}(p)$ is the maximal pro- p quotient of $\mathcal{G}_{\mathbf{Q}_p}$ and $\mathcal{O}[[\mathcal{G}]]$ is the universal deformation.

It follows from Corollary 5.52 that $\check{\mathbf{V}}$ induces a natural transformation $\check{\mathbf{V}} : \text{Def}_{\mathbf{S}} \rightarrow \text{Def}_{\check{\mathbf{V}}(\mathbf{S})}$. Since $\text{Def}_{\mathbf{S}}$ is pro-represented by $\tilde{\mathbf{E}}$ by Theorem 3.26 we deduce from Yoneda's Lemma in this non-commutative context, see Lemma 3.30, that the natural transformation of functors is induced by

$$\varphi_{\check{\mathbf{V}}} : \mathcal{O}[[\mathcal{G}]]^{op} \rightarrow \tilde{\mathbf{E}},$$

where the morphism $\varphi_{\check{\mathbf{V}}}$ is uniquely determined up to conjugation by $\tilde{\mathbf{E}}^\times$. Since by a result of Colmez, [23, VII.4.15], we know that $\check{\mathbf{V}}$ induces an injection

$$\text{Ext}_{\mathfrak{e}_{G, \zeta}(k)}^1(\mathbf{S}, \mathbf{S}) \hookrightarrow \text{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\check{\mathbf{V}}(\mathbf{S}), \check{\mathbf{V}}(\mathbf{S})),$$

we deduce via Lemma 3.29 and the proof of Proposition 5.56 that $\varphi_{\check{\mathbf{V}}}$ is surjective.

Remark 9.4. — A note on actions: our groups always act on the left, $(g, v) \mapsto gv$, hence a representation (ρ, V) of \mathcal{G} gives rise to a left $\mathcal{O}[[\mathcal{G}]]$ -module, which we may write down as a homomorphism $\rho : \mathcal{O}[[\mathcal{G}]] \rightarrow \text{End}_{\mathcal{O}}(V)$. In our context, it is also natural to consider ρ as a right $\mathcal{O}[[\mathcal{G}]]^{\text{op}}$ -module, via the isomorphisms $\text{Hom}_{\mathcal{O}[[\mathcal{G}]]}(\mathcal{O}[[\mathcal{G}]], \rho) \cong \rho$, $\phi \mapsto \phi(1)$, and $\mathcal{O}[[\mathcal{G}]]^{\text{op}} \cong \text{End}_{\mathcal{O}[[\mathcal{G}]]}(\mathcal{O}[[\mathcal{G}]])$. Having made this point we will not distinguish between left $\mathcal{O}[[\mathcal{G}]]$ -modules and right $\mathcal{O}[[\mathcal{G}]]^{\text{op}}$ -modules.

Proposition 9.5. — *Let M be a finite extension of L and let $\rho : \mathcal{O}[[\mathcal{G}]] \rightarrow \text{End}_M(W)$ be a continuous absolutely irreducible representation of \mathcal{G} with $\dim_M W \leq 2$. Then $\text{Ker } \varphi_{\check{V}} \subset \text{Ker } \rho$.*

Proof. — If $\dim W = 1$ then ρ factors through $\mathcal{O}[[\mathcal{G}]]^{\text{ab}}$. Since $p > 2$, \mathcal{G} is a free pro- p group on 2-generators, [49, 7.5.8]. It follows from (141) that φ induces an isomorphism $\mathcal{O}[[\mathcal{G}]]^{\text{ab}} \cong \tilde{E}^{\text{ab}}$ and we are done.

Suppose that $\dim W = 2$ by base change we may assume that $M = L$. It follows from [41, 2.3.8] that there exists an open bounded G -invariant lattice Ξ in a unitary admissible L -Banach space representation Π of G such that $L \otimes_{\mathcal{O}} \check{V}(\Xi^d) \cong \rho$ and $\overline{\Pi} \cong \pi^{\oplus 2}$. Since all open bounded lattices are commensurable, $L \otimes_{\mathcal{O}} \check{V}(\Xi^d)$ does not depend on the choice of Ξ . Thus we may choose Ξ so that we have a surjection

$$\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \widehat{\otimes}_{\tilde{E}} \tilde{P} \twoheadrightarrow \Xi^d,$$

see Proposition 4.18. Corollary 5.55 says that $\check{V}(\tilde{P})$ is a free \tilde{E} -module of rank 1 and Corollary 5.54 gives us a surjection

$$\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \widehat{\otimes}_{\tilde{E}} \check{V}(\tilde{P}) \twoheadrightarrow \check{V}(\Xi^d).$$

Choose a basis element of $\check{V}(\tilde{P})$ over \tilde{E} , then this gives us an isomorphism of \tilde{E} -modules, $\tilde{P} \cong \tilde{E}$ and hence a map $\mathcal{O}[[\mathcal{G}]] \twoheadrightarrow \tilde{P}$ compatible with $\varphi_{\check{V}}$. (We note that all such choices differ by a unit of \tilde{E} , and in the non-commutative setting $\varphi_{\check{V}}$ is uniquely determined up to conjugation by \tilde{E}^{\times} .) And thus we have a surjection of \mathcal{G} -representations

$$\begin{aligned} \text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) &\cong \text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \widehat{\otimes}_{\mathcal{O}[[\mathcal{G}]]^{\text{op}}} \mathcal{O}[[\mathcal{G}]] \\ &\twoheadrightarrow \text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \widehat{\otimes}_{\tilde{E}} \check{V}(\tilde{P}) \twoheadrightarrow \check{V}(\Xi^d), \end{aligned}$$

where the first isomorphism is given by $\phi \mapsto \phi \widehat{\otimes} 1$. The \mathcal{G} -action on $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d)$ is given by

$$(144) \quad g \cdot \phi = g \cdot (\phi \widehat{\otimes} 1) = \phi \widehat{\otimes} g = \phi \widehat{\otimes} (1 \cdot g) = (\phi \circ \varphi_{\check{V}}(g)) \widehat{\otimes} 1 = \phi \circ \varphi_{\check{V}}(g).$$

Since $\overline{\Pi} \cong \pi^{\oplus 2}$ Lemma 4.15 says that $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d)$ is a free \mathcal{O} -module of rank 2, and hence $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{P}, \Xi^d) \cong \check{V}(\Xi^d)$. Since $\check{V}(\Xi^d)$ is a lattice in ρ we deduce that $\text{Ker } \varphi_{\check{V}} \subset \text{Ker } \rho$. \square

Corollary 9.6. — Let $\varphi : \mathcal{O}[[\mathcal{G}]] \rightarrow \mathbf{R}$ be a quotient such that $\bigcap_{\rho} \text{Ker } \rho = 0$, where the intersection is taken over all continuous representations $\rho : \mathbf{R} \rightarrow \text{End}_{\mathbf{M}}(\mathbf{W})$, where \mathbf{M} is a finite extension of \mathbf{L} , $\dim_{\mathbf{M}} \mathbf{W} \leq 2$ and (ρ, \mathbf{W}) is absolutely irreducible. Then $\text{Ker } \varphi_{\mathbf{V}} \subseteq \text{Ker } \varphi$.

9.2. Cayley-Hamilton quotient. — We will construct a quotient $\mathcal{O}[[\mathcal{G}]] \twoheadrightarrow \mathbf{R}$, satisfying the conditions of Corollary 9.6, and such that \mathbf{R}^{op} satisfies the conditions of Lemma 9.3. This will imply that $\tilde{\mathbf{E}} \cong \mathbf{R}^{op}$. After twisting we may assume that χ_1 is trivial and $\zeta = \varepsilon^{-1}$. The ring \mathbf{R} will turn out to be isomorphic to $\mathbf{R}^{\text{ps},1}[[\mathcal{G}]]/\mathbf{J}$, where $\mathbf{R}^{\text{ps},1}$ is a (commutative) deformation ring parameterizing all 2-dimensional pseudocharacters lifting the trace of the trivial 2-dimensional k -representation of \mathcal{G} with determinant equal to 1, see the conditions (o)–(iii) in Proposition 9.12 below, and \mathbf{J} is a closed two-sided ideal generated by $g^2 - \text{T}(g)g + 1$, for all $g \in \mathcal{G}$, where $\text{T} : \mathcal{G} \rightarrow \mathbf{R}^{\text{ps},1}$ is the universal pseudocharacter with determinant 1. Using this we will show that for a finite extension \mathbf{M} of \mathbf{L} an absolutely irreducible \mathbf{M} -representation of $\tilde{\mathbf{E}}_{\mathbf{M}}$ can be at most 2-dimensional.

Recall that the maximal pro- p quotient \mathcal{G} of $\mathcal{G}_{\mathbf{Q}_p}$ is a free pro- p group generated by 2 elements, which we denote by γ and δ . We let

$$(145) \quad \mathbf{R} := \frac{\mathcal{O}[[t_1, t_2, t_3]] \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[\mathcal{G}]]}{\mathbf{J}},$$

where \mathbf{J} is a closed two-sided ideal generated by

$$(146) \quad \gamma^2 - 2(1 + t_1)\gamma + 1, \quad \delta^2 - 2(1 + t_2)\delta + 1,$$

$$(147) \quad (\gamma\delta)^2 - 2(1 + t_3)\gamma\delta + 1, \quad (\delta\gamma)^2 - 2(1 + t_3)\delta\gamma + 1.$$

Sending $x \mapsto \gamma - 1, y \mapsto \delta - 1$ induces an isomorphism between $\mathcal{O}[[\mathcal{G}]]$ and $\mathcal{O}[[x, y]]^{\text{nc}}$, the ring of non-commutative formal power series. We denote the images of t_1, t_2, t_3, x, y in \mathbf{R} by the same letters. We note that the elements t_1, t_2 and t_3 are central in \mathbf{R} .

Substituting $\gamma = 1 + x$ and $\delta = 1 + y$ in the relations defining the ideal \mathbf{J} we get that the following relations hold in \mathbf{R} :

$$(148) \quad x^2 = 2t_1(1 + x), \quad y^2 = 2t_2(1 + y),$$

$$(149) \quad (x + y + xy)^2 = 2t_3(1 + x + y + xy), \quad (x + y + yx)^2 = 2t_3(1 + x + y + yx).$$

Since 2 is invertible in \mathbf{R} , and every $a \in 1 + \mathfrak{m}_{\mathbf{R}}$ is a unit, we get that $t_1, t_2, t_3 \in \mathfrak{m}_{\mathbf{R}}^2$. Thus the natural map $\mathcal{O}[[\mathcal{G}]] \rightarrow \mathbf{R}$ is surjective on tangent spaces and, since both rings are complete,

$$(150) \quad \mathcal{O}[[\mathcal{G}]] \twoheadrightarrow \mathbf{R}$$

is surjective and $\dim \mathfrak{m}_R/(\mathfrak{m}_R^2 + \varpi_L R) \leq 2$. Let J^{ab} be the ideal generated by the relations (148), (149) in the commutative ring $\mathcal{O}[[t_1, t_2, t_3, x, y]]$. Then we have a natural surjection

$$(151) \quad R \twoheadrightarrow \frac{\mathcal{O}[[t_1, t_2, t_3, x, y]]}{J^{ab}} \cong \mathcal{O}[[x, y]].$$

Since the target is commutative, (151) factors through $R^{ab} \twoheadrightarrow \mathcal{O}[[x, y]]$. Since $\dim \mathfrak{m}_R/(\mathfrak{m}_R^2 + \varpi_L R) \leq 2$ we obtain $R^{ab} \cong \mathcal{O}[[x, y]]$ and $\dim \mathfrak{m}_R/(\mathfrak{m}_R^2 + \varpi_L R) = 2$.

Definition 9.7. — Let C be the commutative ring

$$\frac{\mathcal{O}[[t_1, t_2, t_3]][a_1, a_2, b_1, b_2]}{(a_1 + a_2 - 2t_1, a_1 a_2 - 2t_1, b_1 + b_2 - 2t_2, b_1 b_2 - 2t_2)}$$

let \mathfrak{m}_C be the maximal ideal of C and let $\mathfrak{A} := \begin{pmatrix} C & C \\ \mathfrak{m}_C & C \end{pmatrix}$, $\mathfrak{P} = \begin{pmatrix} \mathfrak{m}_C & C \\ \mathfrak{m}_C & \mathfrak{m}_C \end{pmatrix}$.

Proposition 9.8. — There exists a continuous representation of \mathcal{G} on a free rank 2 module over C , which induces a homomorphism of $\mathcal{O}[[t_1, t_2, t_3]]$ -algebras $\rho : R \rightarrow \mathfrak{A}$. In particular, $\text{tr } \rho(\gamma) = 2(1 + t_1)$, $\text{tr } \rho(\delta) = 2(1 + t_2)$, $\text{tr } \rho(\gamma\delta) = 2(1 + t_3)$ and $\det(\rho(g)) = 1$, for all $g \in \mathcal{G}$.

Proof. — We note that \mathfrak{P} is a two sided ideal of \mathfrak{A} and \mathfrak{A} is \mathfrak{P} -adically complete. Let

$$\alpha = \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} b_1 & 0 \\ b & b_2 \end{pmatrix}$$

with $b = 2 + 2t_3 - (1 + a_1)(1 + b_1) - (1 + a_2)(1 + b_2) \in \mathfrak{m}_C$. Sending $g \mapsto g - 1$ induces an isomorphism $(1 + \mathfrak{P}^i)/(1 + \mathfrak{P}^{i+1}) \cong \mathfrak{P}^i/\mathfrak{P}^{i+1}$, where the right hand side is a group with respect to addition. Since $\mathfrak{P}^i/\mathfrak{P}^{i+1}$ is a finite dimensional $\mathcal{O}/\varpi_L \mathcal{O}$ -vector space, we deduce that $1 + \mathfrak{P}$ is a pro- p group. Hence, $\gamma \mapsto 1 + \alpha$ and $\delta \mapsto 1 + \beta$ induces a group homomorphism $\mathcal{G} \rightarrow 1 + \mathfrak{P}$ and hence an algebra homomorphism

$$(152) \quad \mathcal{O}[[t_1, t_2, t_3]][[\mathcal{G}]] \rightarrow \mathfrak{A}.$$

By construction of C we have $\det(1 + \alpha) = \det(1 + \beta) = 1$, $\text{tr}(1 + \alpha) = 2(1 + t_1)$ and $\text{tr}(1 + \beta) = 2(1 + t_2)$. Hence, $\det((1 + \alpha)(1 + \beta)) = 1$ and a direct calculation shows that $\text{tr}((1 + \alpha)(1 + \beta)) = 2(1 + t_3)$. Hence, (152) factors through $\rho : R \rightarrow \mathfrak{A}$. \square

Corollary 9.9. — The natural map $\mathcal{O}[[t_1, t_2, t_3]] \rightarrow R$ is injective.

Proof. — Since the composition $\mathcal{O}[[t_1, t_2, t_3]] \rightarrow R \xrightarrow{\rho} \mathfrak{A}$ is injective, where ρ is the representation constructed in the Proposition 9.8, we obtain the claim. \square

Let H be the subgroup of \mathcal{G} generated as an abstract group by γ and δ . There is a natural length function $\ell : H \rightarrow \mathbf{Z}_{\geq 0}$, $\ell(h) = \min(\sum_{i \geq 1} |m_i|)$, where the minimum is taken over all finite expressions $h = \gamma^{m_1} \delta^{m_2} \dots$, with $m_i \in \mathbf{Z}$. We let

$$\Gamma := \{1, \gamma, \delta, \gamma\delta, \delta\gamma\}$$

and given an integer $m \geq 0$ we define

$$S_m := \{g \in H : \ell(g) \leq m\},$$

and for a subset S of H we define

$$\Sigma(S) := \{g_1 g_2 : g_1, g_2, g_1^{-1} g_2 \in S\} \cup \{g_1^{-1} g_2 : g_1, g_2, g_1 g_2 \in S\}.$$

We note that if $1 \in S$ then by taking $g_1 = 1$ we obtain that $\Sigma(S)$ contains S and by taking $g_2 = 1$ we get that $\Sigma(S)$ contains S^{-1} , the set of inverses of the elements of S .

Lemma 9.10.

- (i) $S_2 \subset \Sigma(\Sigma(\Gamma))$;
- (ii) $S_m \subset \Sigma(\Sigma(\Sigma(S_{m-1})))$, for $m \geq 3$.

Proof. — Since $1 \in \Gamma$, $\Sigma(\Gamma)$ will contain $\Gamma \cup \Gamma^{-1}$ and also $\gamma^2, \delta^2, \gamma^{-1}\delta, \delta^{-1}\gamma$. Thus all the elements of S_2 , except for $\gamma\delta^{-1}$ and $\delta\gamma^{-1}$, are contained in $\Sigma(\Gamma) \cup \Sigma(\Gamma)^{-1}$, which is a subset of $\Sigma(\Sigma(\Gamma))$. To finish the proof of (i), we observe that since $\gamma, \delta^{-1}, \gamma^{-1}\delta^{-1} \in \Sigma(\Gamma)$, $\gamma\delta^{-1} \in \Sigma(\Sigma(\Gamma))$ and since $\delta, \gamma^{-1}, \delta^{-1}\gamma^{-1} \in \Sigma(\Gamma)$, $\delta\gamma^{-1} \in \Sigma(\Sigma(\Gamma))$.

Let $g = \gamma^{m_1} \delta^{m_2} \dots$ be an expression of g such that $\ell(g) = \sum_{i \geq 1} |m_i| \geq 3$. Without loss of generality we may assume $m_1 \neq 0$. If $|m_1| > 1$ then $g \in \Sigma(S_{m-1})$ as we may take $g_1 = \gamma^\varepsilon, g_2 = \gamma^{-\varepsilon} g$ where $\varepsilon = m_1/|m_1|$, so that $g = g_1 g_2$ and $g_1^{-1} g_2 \in S_{m-1}$. Hence, if $|m_j| > 1$ for some j , then $g \in \Sigma(\Sigma(S_{m-1}))$, for we may take (for odd j) $g_1 = \gamma^{m_1} \dots \delta^{m_{j-1}} \gamma^{m_j}$ and $g_2 = \delta^{m_{j+1}} \dots$ and so $g = g_1 g_2$ and $g_1^{-1} g_2 \in \Sigma(S_{m-1})$ by the previous calculation. Thus we may assume $|m_i| = 1$ for all i and since $\ell(g) \geq 3$ this implies $m_3 \neq 0$. Let $g_1 = \gamma^{m_1} \delta^{m_2}$, $g_2 = \gamma^{m_3} \dots$, if m_1 and m_3 have the same sign then $g_1^{-1} g_2 \in S_{m-1}$, if not then $g_1^{-1} g_2 = \delta^{-m_2} \gamma^{m_3 - m_1} \dots \in \Sigma(\Sigma(S_{m-1}))$. \square

Corollary 9.11. — *Let B be a topological ring and let $f : \mathcal{G} \rightarrow B$ be a continuous function such that*

$$(153) \quad f(g^{-1}h) - f(g)f(h) + f(gh) = 0, \quad \forall g, h \in \mathcal{G}.$$

Then f is uniquely determined by its values at the elements of Γ . Moreover, the image of f is contained in the closure of the subring of B generated by $f(g), g \in \Gamma$.

Proof. — Since f is continuous and H is dense in \mathcal{G} , f is uniquely determined by its restriction to H . Using (153) and Lemma 9.10 we deduce that $f|_H$ is uniquely determined by $f(g)$, $g \in \Gamma$ and $f(H)$ is contained in the subring of B generated by $f(g)$, $g \in \Gamma$. Since H is dense in G , $f(G)$ is contained in the closure of this ring. \square

Proposition 9.12. — *Let $T(g) := g + g^{-1} \in R$ then $T(g) \in \mathcal{O}[[t_1, t_2, t_3]]$ for each $g \in \mathcal{G}$. Moreover, T is the unique continuous function $T : \mathcal{G} \rightarrow \mathcal{O}[[t_1, t_2, t_3]]$ such that*

- (o) $T(1) = 2$;
- (i) $\frac{T(g)^2 - T(g^2)}{2} = 1$;
- (ii) $T(gh) = T(hg)$;
- (iii) $T(g^{-1}h) - T(g)T(h) + T(gh) = 0$;
- (iv) $T(\gamma) = 2(1 + t_1)$, $T(\delta) = 2(1 + t_1)$, $T(\gamma\delta) = T(\delta\gamma) = 2(1 + t_3)$.

Proof. — We note that $T : \mathcal{G} \rightarrow R$ is continuous and satisfies (o), (i) and (iv). Now

$$(154) \quad T(g^{-1}h) - T(g)T(h) + T(gh) = h^{-1}T(g) - T(g)h^{-1}.$$

So (iii) holds for all $g, h \in G$ such that $T(g)$ is central in R . Since $T(g)$ is central in R for every $g \in \Gamma$, using Lemma 9.10 we deduce that (iii) holds for every $g, h \in H$ and by continuity of T and density of H , we get that (iii) holds. It follows from Lemma 9.9 and (iv) that the closure of the subring of R generated by $T(g)$, $g \in \Gamma$ is $\mathcal{O}[[t_1, t_2, t_3]]$. It follows from Corollary 9.11 that $T(g) \in \mathcal{O}[[t_1, t_2, t_3]]$ and is uniquely determined. It remains to show that T satisfies (ii). Let $\rho : R \rightarrow \mathfrak{A}$ be the homomorphism constructed in Proposition 9.8. Recall that \mathfrak{A} is a subring of the ring of 2×2 matrices over a commutative ring C . Hence for every $a \in \mathfrak{A}$ we have

$$(155) \quad a^2 - \text{tr}(a)a + \det a = 0,$$

where $\text{tr} : \mathfrak{A} \rightarrow C$ and $\det : \mathfrak{A} \rightarrow C$ are the usual trace and determinant. Since by construction $\det \rho(\gamma) = \det \rho(\delta) = 1$ we get that $\det \rho(g) = 1$ for all $g \in \mathcal{G}$, and so we may rewrite (155) to get

$$(156) \quad \rho(T(g)) = \rho(g) + \rho(g)^{-1} = \text{tr}(\rho(g))$$

and hence $\rho(T(gh) - T(hg)) = 0$. The restriction of ρ to $\mathcal{O}[[t_1, t_2, t_3]]$ is injective and this implies (ii). \square

Corollary 9.13. — *Let $\eta : \mathcal{G}_{\mathbf{Q}_p} \rightarrow k^\times$ and $\psi : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathcal{O}^\times$ be continuous characters such that $\psi \equiv \eta^2 \pmod{\mathfrak{p}_L}$. Then the universal deformation ring $R^{\text{ps}, \psi}$ parameterizing 2-dimensional pseudocharacters of $\mathcal{G}_{\mathbf{Q}_p}$ lifting 2η with determinant ψ is isomorphic to $\mathcal{O}[[x_1, x_2, x_3]]$ and the universal pseudocharacter is equal to the trace of the representation constructed in Proposition 9.8 twisted with $\sqrt{\psi}$.*

Proof. — Since ψ modulo \mathfrak{p}_L is a square and $p \neq 2$ there exists a continuous character $\psi_1 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathcal{O}^\times$ such that $\psi_1^2 = \psi$. Corollary A.3 implies that it is enough to show that the assertion after replacing $\mathcal{G}_{\mathbf{Q}_p}$ with its maximal pro- p quotient \mathcal{G} and this follows Proposition 9.12. \square

Following [10] we introduce an involution $*$ on \mathbf{R} , by letting $g^* := g^{-1}$, extending it linearly on $\mathcal{O}[[t_1, t_2, t_3]][[\mathcal{G}]]$ and observing that $\mathbf{J}^* = \mathbf{J}$.

Lemma 9.14. — $\mathcal{O}[[t_1, t_2, t_3]] = \{a \in \mathbf{R} : a = a^*\}$.

Proof. — Every $a \in \mathcal{O}[[t_1, t_2, t_3]]$ is fixed by $*$ by construction. The map $\mathbf{R} \rightarrow \mathbf{R}$, $a \mapsto \frac{a+a^*}{2}$ is continuous and maps the subring $\mathcal{O}[[t_1, t_2, t_3]][[\mathcal{G}]] + \mathbf{J}$ into $\mathcal{O}[[t_1, t_2, t_3]]$ by Proposition 9.12. Since the subring is dense in \mathbf{R} , we conclude that the fixed points of $*$ are contained in $\mathcal{O}[[t_1, t_2, t_3]]$. \square

Corollary 9.15. — Let $a \in \mathbf{R}$ then $a + a^*$ and a^*a are in $\mathcal{O}[[t_1, t_2, t_3]]$.

Proof. — This follows from $(ab)^* = b^*a^*$ and $(a^*)^* = a$ and Lemma 9.14. \square

Corollary 9.16. — Let $\rho : \mathbf{R} \rightarrow \mathfrak{A}$ be the representation constructed in Proposition 9.8. Then

$$(157) \quad \rho(a + a^*) = \mathrm{tr}(\rho(a)), \quad \rho(a^*a) = \det(\rho(a)), \quad \forall a \in \mathbf{R}.$$

Proof. — The function $\mathbf{R} \rightarrow \mathfrak{A}$, $a \mapsto \rho(a + a^*) - \mathrm{tr}(\rho(a))$ is $\mathcal{O}[[t_1, t_2, t_3]]$ -linear, continuous, and zero on \mathcal{G} by (156). Hence, it is zero on $\mathcal{O}[[t_1, t_2, t_3]][[\mathcal{G}]] + \mathbf{J}$ and since it is dense the function is zero on \mathbf{R} . Now $a^2 - (a + a^*)a + a^*a = 0$ in \mathbf{R} . Hence,

$$0 = \rho(a)^2 - \rho(a + a^*)\rho(a) + \rho(a^*a) = \rho(a)^2 - \mathrm{tr}(\rho(a))\rho(a) + \rho(a^*a).$$

Since, $\rho(a)^2 - \mathrm{tr}(\rho(a))\rho(a) + \det(\rho(a)) = 0$ in \mathfrak{A} we get $\rho(a^*a) = \det \rho(a)$. \square

Corollary 9.17. — Let $g \in \mathcal{G}$ and \mathbf{T} as in Proposition 9.12 then $g^2 - \mathbf{T}(g)g + 1 = 0$ in \mathbf{R} .

Proof. — We have $\mathbf{T}(g) = g + g^{-1} = g + g^*$ and the assertion follows from (157) and the identity $g^2 - (g + g^*)g + g^*g = 0$. \square

To ease the calculations we set

$$(158) \quad u := x - t_1 = \gamma - 1 - t_1, \quad v := y - t_2 = \delta - 1 - t_2.$$

We note that the images of u and v form a k -basis of $\mathfrak{m}_{\mathbf{R}}/(\mathfrak{m}_{\mathbf{R}}^2 + \varpi_L \mathbf{R})$ and hence it follows from (150) that u and v generate \mathbf{R} topologically over \mathcal{O} . Then (148) reads

$$(159) \quad u^2 = 2t_1 - t_1^2, \quad v^2 = 2t_2 - t_2^2.$$

In particular, u^2 and v^2 are central in \mathbf{R} . Hence,

$$(160) \quad u(uv - vu) = -(uv - vu)u, \quad v(uv - vu) = -(uv - vu)v.$$

We also note that substituting $t_1 = \frac{\gamma+\gamma^{-1}}{2} - 1$ and $t_2 = \frac{\delta+\delta^{-1}}{2} - 1$ in (158) gives $u = \frac{\gamma-\gamma^{-1}}{2}$, $v = \frac{\delta-\delta^{-1}}{2}$. Hence,

$$(161) \quad u^* = -u, \quad v^* = -v, \quad (uv - vu)^* = -(uv - vu), \quad (uv + vu)^* = uv + vu.$$

Lemma 9.18. — *Every element $a \in \mathbf{R}$ maybe written as*

$$(162) \quad a = \lambda_1 + \lambda_2 u + \lambda_3 v + \lambda_4 (uv - vu)$$

with $\lambda_i \in \mathcal{O}[[t_1, t_2, t_3]]$.

Proof. — It follows from (150) that a may be written as a formal power series with coefficients in \mathcal{O} in (non-commuting) variables u, v . It follows from (159) that $u^2, v^2 \in \mathcal{O}[[t_1, t_2, t_3]]$ so we only need to deal with monomials of the form $(uv)^n, (uv)^n u, (vu)^n, (vu)^n v$. Lemma 9.14 and (161) give that $(uv - vu)^2, uv + vu \in \mathcal{O}[[t_1, t_2, t_3]]$ and since 2 is invertible in \mathbf{R} , we may substitute $uv = \frac{(uv-vu)+(uv+vu)}{2}$ and $vu = \frac{(uv+vu)-(uv-vu)}{2}$. Thus $(uv)^2 = \lambda uv - \lambda vu + \mu$ and $(vu)^2 = \lambda vu - \lambda uv + \mu$ with $\lambda, \mu \in \mathcal{O}[[t_1, t_2, t_3]]$, which leaves us to deal with uvu and vuv . Since $uvu = (uv + vu)u - u^2v$ and $vuv = (uv + vu)v - v^2u$ we are done. \square

Corollary 9.19. — *Let L' be a finite extension of L and W a finite dimensional L' -vector space with continuous \mathbf{R} -action $\tau : \mathbf{R} \otimes_{\mathcal{O}} L' \rightarrow \text{End}_{L'}(W)$. Suppose that the representation W is absolutely irreducible, then $\dim_{L'} W \leq 2$.*

Proof. — Since W is absolutely irreducible and finite dimensional over L' we have $\text{End}_{\mathcal{G}}(W) = L'$ and thus τ induces a continuous homomorphism of \mathcal{O} -algebras $\mathcal{O}[[t_1, t_2, t_3]] \rightarrow L'$. Since W is absolutely irreducible τ is surjective. It follows from Lemma 9.18 that $(\dim_{L'} W)^2 = \dim_{L'} \text{End}_{L'} W = \dim_{L'}(\tau(\mathbf{R}) \otimes_{\mathcal{O}} L') \leq 4$. \square

Lemma 9.20. — $(uv - vu)^*(uv - vu) \neq 0$ in \mathbf{R} .

Proof. — It follows from (158) that $uv - vu = \gamma\delta - \delta\gamma$. Using Corollary 9.16 it is enough to show that $\det(\rho(\gamma\delta - \delta\gamma)) \neq 0$. If we specialize $t_1 = t_2 = 0$ this means it is enough to show that the determinant of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2t_3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2t_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2t_3 & 0 \\ 0 & -2t_3 \end{pmatrix}$$

is non-zero in $\mathcal{O}[[t_3]]$, which is clear. \square

Lemma 9.21. — *Let \mathfrak{m} be a maximal ideal of $\mathbb{C}[1/\rho]$ and let $\rho_{\mathfrak{m}} : \mathbb{R} \rightarrow \mathfrak{A} \otimes_{\mathbb{C}} \kappa(\mathfrak{m})$ be the specialization at \mathfrak{m} of the representation ρ constructed in Proposition 9.8. Then the following are equivalent:*

- (i) $\rho_{\mathfrak{m}}$ is absolutely irreducible;
- (ii) $\rho_{\mathfrak{m}}(uv - vu)$ is invertible;
- (iii) $(uv - vu)(uv - vu)^* \notin \mathfrak{m}$.

Proof. — (i) implies (ii). The kernel of $\rho_{\mathfrak{m}}(uv - vu)$ is stable under u and v , see (160). If $\rho_{\mathfrak{m}}(uv - vu) = 0$ then the action of \mathcal{G} factors throughout its abelian quotient, as $\gamma\delta - \delta\gamma = uv - vu$. Since $\rho_{\mathfrak{m}}$ is absolutely irreducible, this would force the dimension of $\rho_{\mathfrak{m}}$ over $\kappa(\mathfrak{m})$ to be 1. Since the dimension is 2, we deduce that the kernel of $\rho_{\mathfrak{m}}(uv - vu)$ is zero, and hence it is invertible.

(ii) implies (i). Suppose that $\rho_{\mathfrak{m}}$ is not absolutely irreducible. Then after replacing $\kappa(\mathfrak{m})$ by a finite extension, we may choose a basis such that the matrices of $\rho_{\mathfrak{m}}(\gamma)$ and $\rho_{\mathfrak{m}}(\delta)$ are both upper-triangular. Since, $uv - vu = \gamma\delta - \delta\gamma$ we deduce that $\rho_{\mathfrak{m}}(uv - vu)$ is nilpotent.

(ii) is equivalent to (iii). It follows from (157) that the image of $(uv - vu)(uv - vu)^*$ in $\kappa(\mathfrak{m})$ is equal to the determinant of $\rho_{\mathfrak{m}}(uv - vu)$. \square

Lemma 9.22. — *Let $f \in \mathcal{O}[[x_1, \dots, x_n]]$ be non-zero then there exists $a_i \in \mathfrak{p}_L$, $1 \leq i \leq n$ such that $f(a_1, \dots, a_n) \neq 0$.*

Proof. — Since $\mathcal{O}[[x_1, \dots, x_n]]$ is a unique factorisation domain, [46, 20.3], f is divisible by only finitely many prime elements. Hence, we may find $a_n \in \mathfrak{p}_L$ such that $x_n - a_n$ does not divide f . Let f_1 be the image of f in $\mathcal{O}[[x_1, \dots, x_n]]/(x_n - a_n) \cong \mathcal{O}[[x_1, \dots, x_{n-1}]]$. By construction f_1 is non-zero and we proceed as before. \square

Proposition 9.23. — *Let $a = \lambda_1 + \lambda_2 u + \lambda_3 v + \lambda_4 (uv - vu) \in \mathbb{R}$ with $\lambda_i \in \mathcal{O}[[t_1, t_2, t_3]]$ and not all λ_i equal to zero. Then there exists a finite extension L' of L and a 2-dimensional L' -vector space W , with a continuous action $\tau : \mathbb{R} \otimes_{\mathcal{O}} L' \rightarrow \text{End}_{L'} W$ such that τ is absolutely irreducible and $\tau(a) \neq 0$.*

Proof. — Let $\lambda = (uv - vu)^*(uv - vu) \in \mathbb{R}$, we note that λ is non-zero in \mathbb{R} by Lemma 9.20. Let $f \in \mathcal{O}[[t_1, t_2, t_3]]$ be the product of λ and non-zero λ_i 's. By Lemma 9.22 we may find $a_1, a_2, a_3 \in \mathfrak{p}_L$ such that $f(a_1, a_2, a_3) \neq 0$. Let \mathbb{C} be the ring defined in Definition 9.7 and let \mathfrak{m} be any maximal ideal of $\mathbb{C}[1/\rho]$ containing $(t_1 - a_1, t_2 - a_2, t_3 - a_3)$. Then the residue field $\kappa(\mathfrak{m})$ of \mathfrak{m} is a finite extension of L . Moreover, the image of f in $\kappa(\mathfrak{m})$ is equal to $f(a_1, a_2, a_3)$ and hence is non-zero. So the image of λ in $\kappa(\mathfrak{m})$ is non-zero, and not all λ_i map to 0 in $\kappa(\mathfrak{m})$. Let $\tau = \rho_{\mathfrak{m}} : \mathbb{R} \otimes_{\mathcal{O}} \kappa(\mathfrak{m}) \rightarrow \mathfrak{A} \otimes_{\mathbb{C}} \kappa(\mathfrak{m})$ as in Lemma 9.21. Since the image of λ in $\kappa(\mathfrak{m})$ is non-zero by construction, Lemma 9.21 implies that τ is absolutely irreducible. Thus, τ is surjective. Since $\dim_{\kappa(\mathfrak{m})} \mathfrak{A} \otimes_{\mathbb{C}} \kappa(\mathfrak{m}) = 4$,

we deduce from Lemma 9.18 that $1, \tau(u), \tau(v)$ and $\tau(uv - vu)$ are linearly independent. Hence, $\tau(a) \neq 0$, as $\kappa(\mathfrak{m})$ was constructed so that the images of non-zero λ_i are non-zero. \square

Corollary 9.24. — *The centre of \mathbf{R} is equal to $\mathcal{O}[[t_1, t_2, t_3]]$.*

Proof. — Suppose there exists a non-zero element z in the centre of \mathbf{R} such that $z^* = -z$. Let (τ, W) and L' be as in Proposition 9.23 with $\tau(z) \neq 0$ then $\tau(z)$ is a scalar matrix in $\text{End}_{L'}(W)$. It follows from Corollary 9.16 that $\text{tr } \tau(z) = \tau(z^* + z) = 0$ and thus $\tau(z) = 0$. We obtain a contradiction. Since 2 is invertible in \mathbf{R} , Lemma 9.14 implies that the centre is contained in $\mathcal{O}[[t_1, t_2, t_3]]$. The other inclusion holds by construction. \square

Corollary 9.25. — *\mathbf{R} is a free $\mathcal{O}[[t_1, t_2, t_3]]$ -module of rank 4.*

Proof. — If $0 = \lambda_1 + \lambda_2 u + \lambda_3 v + \lambda_4 (uv - vu)$ then it follows from Proposition 9.23 that all $\lambda_i = 0$. The result then follows from Lemma 9.18. \square

Corollary 9.26. — *Let $a \in \mathbf{R}$ and suppose that $a(uv - vu) = 0$ or $(uv - vu)a = 0$ then $a = 0$.*

Proof. — Since $(uv - vu)^2 = -(uv - vu)(uv - vu)^*$ is in $\mathcal{O}[[t_1, t_2, t_3]]$ and is non-zero by Lemma 9.20, the assertion follows from Corollary 9.25. \square

Corollary 9.27. — *$\tilde{\mathbf{E}} \cong \mathbf{R}^{op}$. In particular, the functor $\check{\mathbf{V}}$ induces an equivalence of categories between $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ and the category of compact \mathbf{R}^{op} -modules.*

Proof. — Proposition 9.23 says that $\varphi : \mathcal{O}[[\mathcal{G}]] \rightarrow \mathbf{R}$ satisfies the conditions of Corollary 9.6 and thus we have $\text{Ker } \varphi_{\check{\mathbf{V}}} \subseteq \text{Ker } \varphi$ and hence a surjection $\tilde{\mathbf{E}} \rightarrow \mathbf{R}^{op}$. Corollary 9.26 implies that \mathbf{R}^{op} satisfies the conditions of Lemma 9.3 with $t' = uv - vu$, hence the surjection is an isomorphism. The last assertion follows from Proposition 5.45. \square

Corollary 9.28. — *Let \mathcal{Z} be the centre of \mathbf{R} , let \mathfrak{n} be a maximal ideal of $\mathcal{Z}[1/p]$ with residue field $\kappa(\mathfrak{n})$ and let $\Gamma_{\mathfrak{n}} : \mathcal{G} \rightarrow \kappa(\mathfrak{n})$ be the specialization at \mathfrak{n} of the universal pseudocharacter Γ , see Proposition 9.12. If the image of $(uv - vu)(uv - vu)^*$ in $\kappa(\mathfrak{n})$ is non-zero then $\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{n})$ is a central simple $\kappa(\mathfrak{n})$ -algebra of dimension 4. Moreover, $\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{n})$ is a matrix algebra over $\kappa(\mathfrak{n})$ if and only if $\Gamma_{\mathfrak{n}}$ is the trace of a 2-dimensional representation of \mathcal{G} defined over $\kappa(\mathfrak{n})$.*

Proof. — It follows from Corollary 9.25 that $\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{n})$ is a 4-dimensional $\kappa(\mathfrak{n})$ algebra. Let \mathfrak{m} be any maximal ideal of $\mathbb{C}[1/p]$ containing \mathfrak{n} and let $\kappa(\mathfrak{m})$ be its residue field. The representation $\rho_{\mathfrak{m}}$ is absolutely irreducible, as part (iii) of Lemma 9.21 is satisfied. Hence, $\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{m}) \cong M_2(\kappa(\mathfrak{m}))$ the algebra of 2×2 matrices over $\kappa(\mathfrak{n})$. Thus the centre of $\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{n})$ is a one dimensional $\kappa(\mathfrak{n})$ -vector space, which implies that

$\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{n})$ is a central simple $\kappa(\mathfrak{n})$ -algebra. If $\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{n}) \cong M_2(\kappa(\mathfrak{n}))$ then letting τ be the standard module, we obtain that $\tau \otimes_{\kappa(\mathfrak{n})} \kappa(\mathfrak{m}) \cong \rho_{\mathfrak{m}}$ and hence $\mathrm{tr} \tau = \mathrm{tr} \rho_{\mathfrak{m}} = T_{\mathfrak{n}}$ by Proposition 9.8. Conversely, if there exists a representation $\tau : \mathcal{G} \rightarrow \mathrm{GL}_2(\kappa(\mathfrak{n}))$ such that $\mathrm{tr} \tau = T_{\mathfrak{n}}$ then $\mathrm{tr} \tau = \mathrm{tr} \rho_{\mathfrak{m}}$ and so τ is absolutely irreducible and the surjection $\mathbf{R} \twoheadrightarrow \mathrm{End}_{\kappa(\mathfrak{n})}(\tau)$ factors through $\mathbf{R} \otimes_{\mathcal{Z}} \kappa(\mathfrak{n})$ and is then an isomorphism, since both the source and the target are 4-dimensional. \square

Corollary 9.29. — *Let \mathcal{Z} be the centre of \mathbf{R} , let \mathfrak{n} be a maximal ideal of $\mathcal{Z}[1/p]$ with residue field $\kappa(\mathfrak{n})$ and let $T_{\mathfrak{n}} : \mathcal{G} \rightarrow \kappa(\mathfrak{n})$ be the specialization at \mathfrak{n} of the universal pseudocharacter T , see Proposition 9.12. If the image of $(uv - vu)(uv - vu)^*$ in $\kappa(\mathfrak{n})$ is non-zero then the \mathfrak{n} -adic completion of $\mathbf{R}[1/p]$ is an Azumaya algebra of rank 4 over the \mathfrak{n} -adic completion of $\mathcal{Z}[1/p]$. Moreover, it is a matrix algebra over the \mathfrak{n} -adic completion of \mathcal{Z} if and only if $T_{\mathfrak{n}}$ is the trace of a 2-dimensional, absolutely irreducible representation of \mathcal{G} defined over $\kappa(\mathfrak{n})$.*

Proof. — It follows from Corollaries 9.24, 9.25 that the \mathfrak{n} -adic completion of \mathbf{R} is a free, rank 4 module over the \mathfrak{n} -adic completion of \mathcal{Z} . The assertion follows from Corollary 9.28 and the idempotent lifting Lemma. \square

Corollary 9.30. — *Let \mathcal{Z} be the centre of \mathbf{R} and let \mathfrak{n} be a maximal ideal of $\mathcal{Z}[1/p]$. If the image of $(uv - vu)(uv - vu)^*$ in $\mathcal{Z}[1/p]/\mathfrak{n}$ is zero then $\mathbf{R}[1/p]/\mathfrak{n}\mathbf{R}[1/p]$ has at most 2 non-isomorphic irreducible modules.*

Proof. — Let $\mathbf{R}_1 := \mathbf{R}[1/p]/\mathfrak{n}\mathbf{R}[1/p]$, $L' := \mathcal{Z}[1/p]/\mathfrak{n}$ and let θ be the image of $uv - vu$ in \mathbf{R}_1 . Let V be an irreducible right \mathbf{R}_1 -module. It follows from (160) that $V\theta$ is an \mathbf{R}_1 -submodule of V . Since the image of $(uv - vu)^2 = -(uv - vu)(uv - vu)^*$ in L' is zero, we deduce that $\theta^2 = 0$ and since V is irreducible we get $V\theta = 0$. Thus $\mathrm{Hom}_{\mathbf{R}_1}(\mathbf{R}_1/\mathbf{R}_1\theta, V) \neq 0$ and it is enough to show that $\dim_{L'} \mathbf{R}_1/\mathbf{R}_1\theta \leq 2$. It follows from Corollary 9.25 that \mathbf{R}_1 is a 4-dimensional L' vector space. Let $m_{\theta} : \mathbf{R}_1 \rightarrow \mathbf{R}_1$, $a \mapsto a\theta$, then $\dim \mathrm{Ker} m_{\theta} + \dim \mathrm{Im} m_{\theta} = 4$ and since $\theta^2 = 0$ we have $\dim \mathrm{Im} m_{\theta} \leq \dim \mathrm{Ker} m_{\theta}$. Thus $\dim_{L'} \mathbf{R}_1/\mathbf{R}_1\theta \leq 2$. \square

9.3. The centre and Banach space representations.

Theorem 9.31. — *Let Π be a unitary absolutely irreducible admissible L -Banach space representation with the central character ζ . Suppose that the reduction of some open bounded G -invariant lattice in Π contains π as a subquotient then $\overline{\Pi} \subseteq \pi \oplus \pi$.*

Proof. — By Proposition 4.18 we may choose an open bounded G -invariant lattice Ξ in Π such that the natural map $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbf{P}}, \Xi^d) \widehat{\otimes}_{\tilde{\mathbf{E}}} \tilde{\mathbf{P}} \rightarrow \Xi^d$ is surjective. It follows from Corollaries 9.25 and 9.27 that the centre of $\tilde{\mathbf{E}}$ is noetherian and $\tilde{\mathbf{E}}$ is a finite module over its centre. Hence $\Xi \otimes_{\mathcal{O}} k$ is of finite length by Corollary 4.41 and $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbf{P}}, \Xi^d)_L$

is finite dimensional over L . Since the block of π consists only of π itself we deduce that

$$\overline{\Pi} \cong ((\Xi^d \otimes_{\mathcal{O}} k)^{ss})^{\vee} \cong \pi^{\oplus m},$$

where m is equal to the dimension of $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}, \Xi^d)_L$ by Lemma 4.15. Since $\overline{\Pi}$ is absolutely irreducible, $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}, \Xi^d)_L$ is an absolutely irreducible right \tilde{E}_L -module by Proposition 4.42. Since $\tilde{E} \cong R^{op}$ we deduce from Corollary 9.19 that the dimension of $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}, \Xi^d)_L$ is at most 2. \square

Corollary 9.32. — *Let $\overline{\Pi}$ be as in Theorem 9.31 and suppose that $\overline{\Pi} \cong \pi$ then $\Pi \cong (\mathrm{Ind}_{\mathbb{P}}^G \psi)_{cont}$ for some continuous unitary character $\psi : \Gamma \rightarrow L^{\times}$ lifting χ and satisfying $\psi|_Z = \zeta$.*

Proof. — Let Ξ be as in the proof of Theorem 9.31. Since $\overline{\Pi} \cong \pi$ we deduce from Lemma 4.15 that $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}, \Xi^d)$ is a free \mathcal{O} -module of rank 1. Hence, the action of \tilde{E} on it factors through the action of \tilde{E}^{ab} . In particular, the element $t \in \tilde{E}$ defined in (142) kills $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}, \Xi^d)$, and hence it follows from (142) that we have an isomorphism $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{M}}, \Xi^d) \cong \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}, \Xi^d)$. The assertion follows from Proposition 7.11. \square

Let $\chi_1 : \mathbf{Q}_p^{\times} \rightarrow k^{\times}$ be a continuous character. Recall that the block \mathfrak{B} of $\pi := \mathrm{Ind}_{\mathbb{P}}^G \chi_1 \otimes \chi_1 \omega^{-1}$ consists of only one isomorphism class, Proposition 5.42. So $\mathrm{Mod}_{G, \zeta}^{\mathrm{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is the full subcategory of $\mathrm{Mod}_{G, \zeta}^{\mathrm{lfm}}(\mathcal{O})$ consisting of representations with every irreducible subquotient isomorphic to π . Let $R_{\chi}^{\mathrm{ps}, \varepsilon \zeta}$ be the universal deformation ring parameterizing 2-dimensional pseudocharacters of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant $\zeta \varepsilon$ lifting $\chi := 2\chi_1$ and let $\Gamma : \mathcal{G}_{\mathbf{Q}_p} \rightarrow R_{\chi}^{\mathrm{ps}, \varepsilon \zeta}$ be the universal deformation of χ .

Corollary 9.33. — *The category $\mathrm{Mod}_{G, \zeta}^{\mathrm{lfm}}(\mathcal{O})^{\mathfrak{B}}$ is anti-equivalent to the category of right compact $R_{\chi}^{\mathrm{ps}, \zeta \varepsilon}[[\mathcal{G}_{\mathbf{Q}_p}]]/J$ -modules, where J is a closed two-sided ideal generated by $g^2 - \Gamma(g)g + \varepsilon \zeta(g)$ for all $g \in \mathcal{G}_{\mathbf{Q}_p}(p)$.*

Proof. — By twisting we may assume that χ_1 is trivial and $\zeta = \varepsilon^{-1}$, see the proof of Corollary 9.13. We have shown in Corollary 9.13 that Γ factors through \mathcal{G} , the maximal pro- p quotient of $\mathcal{G}_{\mathbf{Q}_p}$. Corollary A.4 says that $R_{\chi}^{\mathrm{ps}, \zeta \varepsilon}[[\mathcal{G}_{\mathbf{Q}_p}]]/J \cong R_{\chi}^{\mathrm{ps}, \zeta \varepsilon}[[\mathcal{G}]]/J'$, where the ideal J' is a closed two-sided ideal of $R_{\chi}^{\mathrm{ps}, \zeta \varepsilon}[[\mathcal{G}]]$ defined by the same relations. It follows from Proposition 9.12 and Corollary 9.17 that $R_{\chi}^{\mathrm{ps}, 1}[[\mathcal{G}]]/J'$ is the ring R considered above. The assertion follows from Corollaries 9.24, 9.27 and Proposition 5.45. We also note that the involution $*$ induces an isomorphism between R and R^{op} , so the category of right compact R is equivalent to the category of left compact R -modules. \square

Corollary 9.34. — *The centre of the category $\mathrm{Mod}_{G, \zeta}^{\mathrm{ladm}}(\mathcal{O})^{\mathfrak{B}}$ is naturally isomorphic to $R_{\chi}^{\mathrm{ps}, \zeta \varepsilon}$.*

Proof. — Corollary 9.28, Corollary 9.13, Proposition 5.45. \square

Let $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})^{\mathfrak{B}}$ be as in Proposition 5.36 and let $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}}$ be the full subcategory consisting of objects of finite length. Let Π be in $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}}$, and let $m(\Pi) := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \Theta^d) \otimes_{\mathcal{O}} \mathbf{L}$, where Θ is an open bounded G -invariant lattice in Π . It follows from Proposition 4.20 that $m(\Pi)$ is a finite dimensional \mathbf{L} -vector space with continuous $\tilde{\mathbb{E}}$ -action. Let \mathfrak{n} be a maximal ideal in $\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[1/p]$, recall that $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is the full subcategory of $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}}$ consisting of those Π such that $m(\Pi)$ is killed by a power of \mathfrak{n} .

Corollary 9.35. — *We have an equivalence of categories*

$$\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{n} \in \text{MaxSpec } \mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[1/p]} \text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}.$$

The category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $(\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[[\mathcal{G}_{\mathbf{Q}_p}]]/J)[1/p]$.

Proof. — Apply Theorem 4.36 with $\mathfrak{C}(\mathcal{O}) = \mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. □

Corollary 9.36. — *Suppose that the pseudo-character corresponding to a maximal ideal \mathfrak{n} of $\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[1/p]$ is the trace of an absolutely irreducible representation of $\mathcal{G}_{\mathbf{Q}_p}$ defined over the residue field of \mathfrak{n} then the category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[1/p]$. In particular, it contains only one irreducible object.*

Proof. — Corollaries 9.29 and 9.35. The last assertion follows from the fact that the only irreducible module is $\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[1/p]/\mathfrak{n}$. □

Let \mathfrak{n} be a maximal ideal of $\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[1/p]$ with residue field \mathbf{L} , let $T_{\mathfrak{n}} : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{L}$ be the pseudocharacter corresponding to \mathfrak{n} and let $\text{Irr}(\mathfrak{n})$ denote the set (of equivalence classes of) irreducible objects in $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$.

Corollary 9.37. — *If $T_{\mathfrak{n}} = \psi_1 + \psi_2$ with $\psi_1, \psi_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{L}^{\times}$ continuous homomorphisms then*

$$\text{Irr}(\mathfrak{n}) = \left\{ (\text{Ind}_{\mathbb{P}}^G \psi_1 \otimes \psi_2 \varepsilon^{-1})_{\text{cont}}, (\text{Ind}_{\mathbb{P}}^G \psi_2 \otimes \psi_1 \varepsilon^{-1})_{\text{cont}} \right\}.$$

Proof. — Let \mathcal{Z} be the centre of $\tilde{\mathbb{E}}$. We may identify $\tilde{\mathbb{E}}$ with \mathbf{R} and \mathcal{Z} with $\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}$. Corollary 9.13 says that the universal pseudocharacter is equal to the trace of the representation constructed in the proof of Lemma 9.9. In particular, if the image of $(uv - vu)(uv - vu)^*$ in $\mathcal{Z}[1/p]/\mathfrak{n}$ is non-zero, then $T_{\mathfrak{n}}$ is the trace of an absolutely irreducible 2-dimensional representation, see the proof of Proposition 9.23. Since $T_{\mathfrak{n}} = \psi_1 + \psi_2$ we deduce that the image of $(uv - vu)(uv - vu)^*$ in $\mathcal{Z}[1/p]/\mathfrak{n}$ is zero.

Corollary 9.33 implies that for every N in $\mathfrak{C}(\mathcal{O})$, $\check{\mathbf{V}}(N)$ is killed by $g^2 - T(g)g + \varepsilon\zeta(g)$, for all $g \in \mathcal{G}_{\mathbf{Q}_p}$. Since

$$\mathbf{V}((\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi_1 \otimes \psi_2 \varepsilon^{-1})_{\mathrm{cont}}) = \psi_2, \quad \mathbf{V}((\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi_2 \otimes \psi_1 \varepsilon^{-1})_{\mathrm{cont}}) = \psi_1,$$

both Banach space representations lie in $\mathrm{Irr}(\mathfrak{n})$. If $\psi_1 \neq \psi_2$ then the representations are non-isomorphic and we are done, since Corollary 9.30 says that $\mathrm{Ban}_{\mathbb{G}, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}$ has at most 2 irreducible objects. Suppose that $\psi_1 = \psi_2$ and $\mathrm{Irr}(\mathfrak{n})$ contains an irreducible object $\Pi \not\cong (\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi_1 \otimes \psi_1 \varepsilon^{-1})_{\mathrm{cont}}$. Then it follows from the proof of Corollary 9.30 that $\mathrm{m}(\Pi)$ is one dimensional. By Corollary 9.32, Π is isomorphic to the parabolic induction of a unitary character, and thus must be contained in one of the components that we have handled already. Hence, if $\psi_1 = \psi_2$ then $|\mathrm{Irr}(\mathfrak{n})| = 1$. \square

10. Non-generic case II

In this section we deal with the case, where in Colmez's terminology the *atome automorphe* consists of three distinct irreducible representations. We assume throughout this section that $p \geq 5$. After twisting we may assume that our fixed central character is trivial and the block \mathfrak{B} consists of $\mathbf{1}$, Sp and $\pi_{\alpha} := \mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha$. The formalism developed in Section 3 does not work in the category $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. However, Colmez's functor kills off all the representations on which $\mathrm{SL}_2(\mathbf{Q}_p)$ acts trivially and so it is natural to work in the quotient category. We show in Section 10.3 that the category of compact \mathcal{O} -modules with the trivial \mathbb{G} -action is a thick subcategory of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ and the formalism of Section 3 applies in the quotient category $\mathfrak{Q}(\mathcal{O})^{\mathfrak{B}}$ to a projective envelope $\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}$ of π_{α}^{\vee} . Using Proposition 5.56 we show that $\check{\mathbf{V}}$ induces a surjection $\varphi : \tilde{\mathbb{E}} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}) \twoheadrightarrow \mathbb{R}_{\rho}^{\psi}$, where ρ is the non-split extension $0 \rightarrow \mathbf{1} \rightarrow \rho \rightarrow \omega \rightarrow 0$ and \mathbb{R}_{ρ}^{ψ} is the universal deformation ring of ρ with a fixed determinant. The proof requires all kinds of Ext calculations, which are carried out in Sections 10.1, 10.2. (We suggest to skip them on first reading.)

The second difficulty is that \mathbb{R}_{ρ}^{ψ} is not formally smooth and hence we cannot use the same argument as in the generic case. The functor $\mathrm{Ind}_{\mathbb{P}}^{\mathbb{G}} \mathrm{Ord}_{\mathbb{P}}$ is left exact and we have a natural transformation to the identity functor. This induces a functorial filtration on every object of $\mathrm{Mod}_{\mathbb{G}, \zeta}^{\mathrm{ladm}}(\mathcal{O})$ and dually on every object of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ and by functoriality on $\tilde{\mathbb{E}}$. In Section 10.4 we compare this filtration to the filtration on \mathbb{R}_{ρ}^{ψ} induced by powers of the ideal defined by the intersection of \mathbb{R}_{ρ}^{ψ} and the reducible locus in $\mathbb{R}_{\rho}^{\psi}[1/p]$. We show in Theorem 10.71 that φ is an isomorphism and $\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}})$ is the universal deformation of ρ with the fixed determinant. In order to do this we need a good knowledge of the ring \mathbb{R}_{ρ}^{ψ} . This is provided by the Appendix B using results of Böckle [9].

In Section 10.5 we compute the endomorphism ring of $\tilde{\mathbb{P}}_{\mathbf{1}^{\vee}} \oplus \tilde{\mathbb{P}}_{\mathrm{Sp}^{\vee}} \oplus \tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}$ and show that its centre is naturally isomorphic to \mathbb{R}_{ρ}^{ψ} and it is a finitely generated module over its centre. As a consequence we may describe $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ as a module category over an explicit ring.

In Section 10.6 we apply the theory of Section 4 to describe the category of admissible unitary L-Banach space representations of G of finite length whose reduction mod ϖ lies in $\text{Mod}_{G,\zeta}^{\text{ladm}}(k)^{\text{B}}$.

If π and τ are smooth k -representations of G on which Z acts trivially, in order to simplify the notation we will write:

$$e_{G/Z}^i(\pi, \tau) := \dim_k \text{Ext}_{G/Z}^i(\pi, \tau).$$

If it is clear from the context that we are working with G -representations, then we will drop the index G/Z and write $e^i(\pi, \tau)$ instead. Similarly, if π and τ are representations of T on which Z acts trivially, we will let $e_{T/Z}^i(\pi, \tau) := \dim_k \text{Ext}_{T/Z}^i(\pi, \tau)$.

We assume all the way till Section 10.6 that our fixed central character ζ is trivial. This is harmless since we may always twist to achieve this, see Lemma 10.103. We recall that the representation $\pi(0, 1)$, defined in (67), is the unique non-split extension of Sp by $\mathbf{1}$ with 2-dimensional I_1 -invariants.

10.1. Higher Ext-groups. — The dimensions of $\text{Ext}_{G/Z}^1$ groups between irreducible representations in the block of the trivial representation, are given by:

$$(163) \quad e^1(\mathbf{1}, \mathbf{1}) = 0, \quad e^1(\text{Sp}, \mathbf{1}) = 1, \quad e^1(\text{Ind}_p^G \alpha, \mathbf{1}) = 1,$$

$$(164) \quad e^1(\mathbf{1}, \text{Sp}) = 2, \quad e^1(\text{Sp}, \text{Sp}) = 0, \quad e^1(\text{Ind}_p^G \alpha, \text{Sp}) = 0,$$

$$(165) \quad e^1(\mathbf{1}, \text{Ind}_p^G \alpha) = 0, \quad e^1(\text{Sp}, \text{Ind}_p^G \alpha) = 1, \quad e^1(\text{Ind}_p^G \alpha, \text{Ind}_p^G \alpha) = 2,$$

see Theorems 11.4 and 11.5(ii) in [56]. We are going to determine the dimensions of higher Ext-groups. It is shown in [31, 4.1.3] that

$$(166) \quad \text{Ord}_p \mathbf{1} = 0, \quad \mathbf{R}^1 \text{Ord}_p \mathbf{1} = \alpha^{-1},$$

$$(167) \quad \text{Ord}_p \text{Sp} = \mathbf{1}, \quad \mathbf{R}^1 \text{Ord}_p \text{Sp} = 0.$$

It follows directly from (123), (124) and (166) that

$$(168) \quad e_{G/Z}^i(\text{Ind}_p^G \mathbf{1}, \mathbf{1}) = 0, \quad i \geq 0$$

and from (123), (124), (167) and Corollary 3.35 that

$$(169) \quad e_{G/Z}^1(\text{Ind}_p^G \mathbf{1}, \text{Sp}) = e_{T/Z}^1(\mathbf{1}, \mathbf{1}) = 2, \quad e_{G/Z}^2(\text{Ind}_p^G \mathbf{1}, \text{Sp}) = e_{T/Z}^2(\mathbf{1}, \mathbf{1}) = 1$$

and $e^i(\text{Ind}_p^G \mathbf{1}, \text{Sp}) = 0$ for $i \geq 3$.

Proposition 10.1. — $\mathbf{R}^1 \mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\text{Ind}_p^G \alpha)$, $\mathbf{R}^2 \mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\text{Ind}_p^G \alpha)$, $\mathbf{R}^3 \mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\mathbf{1})$ and $\mathbf{R}^i \mathcal{I}(\mathbf{1}) = 0$ for $i \geq 4$.

Proof. — The first assertion is given by [56, 11.2]. Since I_1/Z_1 is a Poincaré group of dimension 3, see the proof of Corollary 5.23, $H^3(I_1/Z_1, \mathbf{1})$ is one dimensional and $H^i(I_1/Z_1, \mathbf{1}) = 0$ for $i \geq 4$. We deduce that $\mathbf{R}^3\mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\pi \otimes \mu)$, where $\pi = \mathbf{1}$ or $\pi = \text{Sp}$ and $\mu : G \rightarrow k^\times$ is a smooth character, since all the 1-dimensional modules of the Hecke algebra \mathcal{H} are of this form. It follows from Proposition 5.26 that $\text{Ext}_{G/Z}^3(\pi \otimes \mu, \mathbf{1}) \neq 0$. Hence, $\pi \otimes \mu$ is in the block of $\mathbf{1}$ and so μ is trivial. If $\pi \cong \text{Sp}$ then the same argument implies $\text{Ext}_{G/Z}^3(\text{Ind}_p^G \mathbf{1}, \mathbf{1}) \neq 0$, thus contradicting (168). It follows from (123), (124) and (166) that $e^i(\text{Ind}_p^G \alpha, \mathbf{1}) = 0$ for $i \geq 4$ and

$$(170) \quad e^3(\text{Ind}_p^G \alpha, \mathbf{1}) = 1, \quad e^2(\text{Ind}_p^G \alpha, \mathbf{1}) = 2.$$

Since $\mathbf{R}^1\mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\text{Ind}_p^G \alpha)$, Lemma 5.27 (i) implies that $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\text{Ind}_p^G \alpha), \mathbf{R}^1\mathcal{I}(\mathbf{1}))$ is one dimensional. Proposition 5.26 and (170) imply that $\text{Hom}_{\mathcal{H}}(\mathcal{I}(\text{Ind}_p^G \alpha), \mathbf{R}^2\mathcal{I}(\mathbf{1}))$ is non-zero. Since I_1/Z_1 is a Poincaré group of dimension 3 we have

$$\dim H^2(I_1/Z_1, \mathbf{1}) = \dim H^1(I_1/Z_1, \mathbf{1}) = 2.$$

As $\mathcal{I}(\text{Ind}_p^G \alpha)$ is irreducible and 2-dimensional we obtain $\mathbf{R}^2\mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\text{Ind}_p^G \alpha)$. \square

Corollary 10.2. — For $i \geq 2$, $e^i(\mathbf{1}, \mathbf{1}) = 0$ and $e^i(\text{Sp}, \mathbf{1}) = 0$, except $e^3(\mathbf{1}, \mathbf{1}) = 1$ and $e^4(\text{Sp}, \mathbf{1}) = 1$.

Proof. — The only non-zero $\text{Ext}_{\mathcal{H}}^i$ groups for $i \geq 1$ between $\mathcal{I}(\mathbf{1})$ and $\mathcal{I}(\text{Sp})$ are $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\mathbf{1}), \mathcal{I}(\text{Sp}))$ and $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\text{Sp}), \mathcal{I}(\mathbf{1}))$, see Lemmas 5.24 and 5.27, which are 1-dimensional. The assertion follows from Proposition 10.1 and Proposition 5.26. \square

Lemma 10.3. — Let $\chi : T \rightarrow k^\times$ be a smooth character, then $H^2(I_1/Z_1, \text{Ind}_p^G \chi)$ is 2-dimensional and $H^i(I_1/Z_1, \text{Ind}_p^G \chi) = 0$, for $i \geq 3$.

Proof. — By restricting to I_1 we obtain

$$(171) \quad (\text{Ind}_p^G \chi)|_{I_1} \cong \text{Ind}_{I_1 \cap P}^{I_1} \mathbf{1} \oplus \text{Ind}_{I_1 \cap P^s}^{I_1} \mathbf{1}.$$

Shapiro's lemma gives

$$(172) \quad H^i(I_1/Z_1, \text{Ind}_p^G \chi) \cong H^i((I_1 \cap P)/Z_1, \mathbf{1}) \oplus H^i((I_1 \cap P^s)/Z_1, \mathbf{1}).$$

Since $(I_1 \cap P)/Z_1 \cong (I_1 \cap P^s)/Z_1 \cong \mathbf{Z}_p \rtimes \mathbf{Z}_p$ is a compact torsion-free p -adic analytic group of dimension 2, the assertion follows from [45, V.2.5.8] and [62]. \square

Corollary 10.4. — Let $\chi : T \rightarrow k^\times$ be a smooth character, then $\mathbf{R}^2\mathcal{I}(\text{Ind}_p^G \chi)$ is a 2-dimensional k -vector space and $\mathbf{R}^i\mathcal{I}(\text{Ind}_p^G \chi) = 0$, for $i \geq 3$.

Proof. — Lemma 5.22 provides a natural isomorphism of k -vector spaces between $H^i(I_1/Z_1, \text{Ind}_p^G \chi)$ and $\mathbf{R}^i\mathcal{I}(\text{Ind}_p^G \chi)$ and the assertion follows from Lemma 10.3. \square

Proposition 10.5. — $\mathbf{R}^1\mathcal{I}(\mathrm{Sp}) \cong \mathcal{I}(\mathrm{Ind}_p^G \mathbf{1})$, $\mathbf{R}^2\mathcal{I}(\mathrm{Sp}) \cong \mathcal{I}(\mathbf{1})$ and $\mathbf{R}^i\mathcal{I}(\mathrm{Sp}) = 0$ for $i \geq 3$.

Proof. — Proposition 11.2 of [56] says that the natural maps induce isomorphisms $\mathbf{R}^1\mathcal{I}(\mathrm{Ind}_p^G \mathbf{1}) \cong \mathbf{R}^1\mathcal{I}(\mathbf{1}) \oplus \mathbf{R}^1\mathcal{I}(\mathrm{Sp})$ and $\mathbf{R}^1\mathcal{I}(\mathrm{Sp}) \cong \mathcal{I}(\mathrm{Ind}_p^G \mathbf{1})$. Hence, applying \mathcal{I} to the exact sequence $0 \rightarrow \mathbf{1} \rightarrow \mathrm{Ind}_p^G \mathbf{1} \rightarrow \mathrm{Sp} \rightarrow 0$, and observing that $\mathbf{R}^3(\mathrm{Ind}_p^G \mathbf{1})$ vanishes by Corollary 10.4, we get

$$(173) \quad \mathbf{R}^2\mathcal{I}(\mathbf{1}) \hookrightarrow \mathbf{R}^2\mathcal{I}(\mathrm{Ind}_p^G \mathbf{1}) \rightarrow \mathbf{R}^2\mathcal{I}(\mathrm{Sp}) \twoheadrightarrow \mathbf{R}^3\mathcal{I}(\mathbf{1}).$$

The first arrow in (173) is a surjection, since both the source and the target are 2-dimensional, see Proposition 10.1 and Corollary 10.4 respectively. This implies the last arrow is an isomorphism. Further, we deduce from Corollary 10.4 and Proposition 10.1 that $\mathbf{R}^i\mathcal{I}(\mathrm{Sp}) \cong \mathbf{R}^{i+1}\mathcal{I}(\mathbf{1}) = 0$ for $i \geq 3$. \square

Proposition 10.6. — Let U be in $\mathrm{Mod}_{T/Z}^{\mathrm{ladm}}(k)$ then for all $i \geq 0$ we have an exact sequence

$$(174) \quad \mathrm{Ext}_{G/Z}^i(\mathrm{Sp}, \mathrm{Ind}_p^G U) \hookrightarrow \mathrm{Ext}_{G/Z}^i(\mathrm{Ind}_p^G \mathbf{1}, \mathrm{Ind}_p^G U) \twoheadrightarrow \mathrm{Ext}_{G/Z}^i(\mathbf{1}, \mathrm{Ind}_p^G U).$$

Proof. — Recall that by Corollary 5.17 it does not matter whether we compute the Ext groups in $\mathrm{Mod}_{G/Z}^{\mathrm{sm}}(k)$ or in $\mathrm{Mod}_{G/Z}^{\mathrm{ladm}}(k)$. If V is in $\mathrm{Mod}_{G/Z}^{\mathrm{sm}}(k)$ then

$$(175) \quad \mathrm{Ext}_{G/Z}^i(V, \mathrm{Ind}_p^G U) \cong \mathrm{Ext}_{P/Z}^i(V, U),$$

see [31, 4.2.1]. Since the sequence $0 \rightarrow \mathbf{1} \rightarrow \mathrm{Ind}_p^G \mathbf{1} \rightarrow \mathrm{Sp} \rightarrow 0$ splits, when restricted to P , we obtain the result. \square

Corollary 10.7. — Let κ be in $\mathrm{Mod}_{T/Z}^{\mathrm{ladm}}(k)$ such that $\mathrm{Hom}_T(\chi, \kappa) = 0$ for all $\chi \in \mathrm{Irr}_{T/Z}(k)$, $\chi \neq \mathbf{1}_T$. Then $e^i(\mathrm{Sp}, \mathrm{Ind}_p^G \kappa) = 0$ and $\mathrm{Ext}_{G/Z}^i(\mathbf{1}, \mathrm{Ind}_p^G \kappa) \cong \mathrm{Ext}_{T/Z}^i(\mathbf{1}, \kappa)$ for all $i \geq 0$.

Proof. — Suppose that $\kappa = J$ is injective in $\mathrm{Mod}_{T/Z}^{\mathrm{ladm}}(k)$. Then it follows from (123) and (124) that $\mathrm{Ext}_{G/Z}^i(\mathrm{Ind}_p^G \mathbf{1}, \mathrm{Ind}_p^G J) = 0$ for all $i \geq 1$. Proposition 10.6 implies that $\mathrm{Ind}_p^G J$ is acyclic for $\mathrm{Hom}_{G/Z}(\mathbf{1}, *)$ and $\mathrm{Hom}_{G/Z}(\mathrm{Sp}, *)$. Moreover, since $\mathbf{R}^1 \mathrm{Ord}_p \mathrm{Sp} = 0$, the U -coinvariants Sp_U are zero by [31, 3.6.2]. Hence, $\mathrm{Hom}_G(\mathrm{Sp}, \mathrm{Ind}_p^G J) = 0$ and $\mathrm{Hom}_G(\mathbf{1}_G, \mathrm{Ind}_p^G J) \cong \mathrm{Hom}_T(\mathbf{1}_T, J)$.

In general, let $\kappa \hookrightarrow J^\bullet$ be an injective resolution of κ in $\mathrm{Mod}_{T/Z}^{\mathrm{ladm}}(k)$. Since the block of $\mathbf{1}_T$ contains only $\mathbf{1}_T$ itself, see Corollary 3.35, we may assume that for each $i \geq 1$ all $\mathrm{Hom}_T(\chi, J^i) = 0$ for all $\chi \in \mathrm{Irr}_{T/Z}(k)$, $\chi \neq \mathbf{1}_T$. By inducing we obtain a resolution $\mathrm{Ind}_p^G \kappa \hookrightarrow \mathrm{Ind}_p^G J^\bullet$ by acyclic objects for functors $\mathrm{Hom}_G(\mathbf{1}, *)$, $\mathrm{Hom}_G(\mathrm{Sp}, *)$. Since $\mathrm{Hom}_G(\mathrm{Sp}, \mathrm{Ind}_p^G J^i) = 0$ and $\mathrm{Hom}_G(\mathbf{1}, \mathrm{Ind}_p^G J^i) \cong \mathrm{Hom}_T(\mathbf{1}, J^i)$ we obtain the assertion. \square

Corollary 10.8. — *Let κ be in $\text{Mod}_{\Gamma/Z}^{\text{ladm}}(k)$ such that $\text{Hom}_{\Gamma}(\chi, \kappa) = 0$ for all $\chi \in \text{Irr}_{\Gamma/Z}(k)$, $\chi \neq \alpha$, then $e^i(\mathbf{1}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \kappa) = 0$ for all $i \geq 0$.*

Proof. — Frobenius reciprocity and the assumption on κ imply the assertion for $i = 0$. It is enough to show the assertion when κ is injective in $\text{Mod}_{\Gamma/Z}^{\text{ladm}}(k)$, since then we may deduce the general case as in the proof of Corollary 10.7. Suppose that $\kappa = \mathbf{J}$ is injective, it follows from Proposition 10.6 and Lemma 7.4 that Ext^i vanishes for $i \geq 2$. It is enough to show the statement for $i = 1$. We know that $\text{Ext}_{G/Z}^1(\mathbf{1}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) = 0$, [56, 11.5]. Hence, if U is any representation of finite length with irreducible subquotients isomorphic to $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha$ then $\text{Ext}_{G/Z}^1(\mathbf{1}, U) = 0$. Since $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \mathbf{J}$ is a union of subobjects of finite length with the irreducible subquotients isomorphic to $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha$ we deduce the assertion. \square

Corollary 10.9. — *We have $\text{Ext}_{G/Z}^i(\mathbf{1}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) = 0$ for all $i \geq 0$. Moreover,*

$$(176) \quad e_{G/Z}^2(\text{Sp}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) = 2, \quad e_{G/Z}^3(\text{Sp}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) = 1,$$

and $e_{G/Z}^i(\text{Sp}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) = 0$ for $i \geq 4$.

Proof. — The first assertion follows from Corollary 10.8. It follows from Proposition 10.6 that $\text{Ext}_{G/Z}^i(\text{Sp}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) \cong \text{Ext}_{G/Z}^i(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \mathbf{1}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha)$, for all $i \geq 0$. The last assertion follows from (122). \square

Corollary 10.10. — $\mathbf{R}^1 \mathcal{I}(\pi_{\alpha}) \cong \mathcal{I}(\pi_{\alpha}) \oplus \mathcal{I}(\pi(0, 1))$, $\mathbf{R}^2 \mathcal{I}(\pi_{\alpha}) \cong \mathcal{I}(\pi(0, 1))$, $\mathbf{R}^i \mathcal{I}(\pi_{\alpha}) = 0$ for $i \geq 3$.

Proof. — The first assertion is [20, Thm. 7.16]. Since $\text{Ext}_{G/Z}^3(\text{Sp}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) \neq 0$ by Corollary 10.9 and $\mathbf{R}^3 \mathcal{I}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) = 0$ by Corollary 10.4, Proposition 5.26 implies that $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\text{Sp}), \mathbf{R}^2 \mathcal{I}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha)) \neq 0$. If M is irreducible then $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\text{Sp}), M) \neq 0$ implies that $M \cong \mathcal{I}(\mathbf{1})$, see [56, 11.3]. Thus $\mathcal{I}(\mathbf{1})$ is an irreducible subquotient of $\mathbf{R}^2 \mathcal{I}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha)$. Since $\text{Ext}_{G/Z}^2(\mathbf{1}, \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) = 0$, it follows from Proposition 5.26 that $\mathcal{I}(\mathbf{1})$ cannot be a submodule of $\mathbf{R}^2 \mathcal{I}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha)$. If M is irreducible then $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\mathbf{1}), M) \neq 0$ implies that $M \cong \mathcal{I}(\text{Sp})$, [56, 11.3]. Since by Corollary 10.4 the underlying vector space of $\mathbf{R}^2 \mathcal{I}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha)$ is 2-dimensional, we deduce that there exists a non-split sequence:

$$(177) \quad 0 \rightarrow \mathcal{I}(\text{Sp}) \rightarrow \mathbf{R}^2 \mathcal{I}(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha) \rightarrow \mathcal{I}(\mathbf{1}) \rightarrow 0.$$

Now $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\mathbf{1}), \mathcal{I}(\text{Sp}))$ is one dimensional, [56, 11.3], and the only non-split extension is obtained by applying \mathcal{I} to (67). \square

We record below the dimensions of $\text{Ext}_{G/Z}^i(\pi, \tau)$, where τ, π are $\mathbf{1}$, Sp or $\text{Ind}_p^G \alpha$. All the other Ext-groups vanish.

$\tau = \mathbf{1}$					$\tau = \text{Sp}$				$\tau = \text{Ind}_p^G \alpha$			
i	1	2	3	4	i	1	2	3	i	1	2	3
$\mathbf{1}$	0	0	1	0	$\mathbf{1}$	2	2	0	$\mathbf{1}$	0	0	0
Sp	1	0	0	1	Sp	0	0	1	Sp	1	2	1
$\text{Ind}_p^G \alpha$	1	2	1	0	$\text{Ind}_p^G \alpha$	0	0	0	$\text{Ind}_p^G \alpha$	2	1	0

Using the table one can construct minimal injective resolutions of $\mathbf{1}$, Sp and $\text{Ind}_p^G \alpha$:

Remark 10.11. — Let κ be an object of $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ and $\iota : \text{soc}_G \kappa \hookrightarrow \mathbf{J}$ an injective envelope of $\text{soc}_G \kappa$ in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$. Since \mathbf{J} is injective there exists $\phi : \kappa \rightarrow \mathbf{J}$ such that the composition $\text{soc}_G \kappa \rightarrow \kappa \rightarrow \mathbf{J}$ is equal to ι . Since ι is an injection, we deduce that $\text{soc}_G \text{Ker } \phi = 0$ and since $\text{Ker } \phi$ is an object of $\text{Mod}_{G/Z}^{\text{ladm}}(k)$, we deduce that ϕ is injective. Since ι is essential, so is ϕ and hence for every irreducible object π of $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ we have $\text{Hom}_{G/Z}(\pi, \kappa) \cong \text{Hom}_{G/Z}(\pi, \mathbf{J})$ and thus $\text{Hom}_{G/Z}(\pi, \mathbf{J}/\kappa) \cong \text{Ext}_{G/Z}^1(\pi, \kappa)$. Hence, if we know the dimensions of $\text{Ext}_{G/Z}^1(\pi, \kappa)$ for all irreducible π then we may determine $\text{soc}_G(\mathbf{J}/\kappa)$ and thus construct the next step in the injective resolution. This way we obtain an injective resolution $\kappa \hookrightarrow \mathbf{J}^\bullet$ such that for all irreducible π in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ the complex $\text{Hom}_G(\pi, \mathbf{J}^\bullet)$ has zero differentials. In particular, $\text{Ext}_{G/Z}^i(\pi, \kappa) \cong \text{Hom}_G(\pi, \mathbf{J}^i)$ for all $i \geq 0$. Since in a locally finite category every injective object is determined by its socle up to isomorphism, the knowledge of $\text{Ext}_{G/Z}^i(\pi, \kappa)$ for all irreducible π determines \mathbf{J}^i up to isomorphism. It should be pointed out that these kind of arguments are standard in commutative algebra, see for example [46, Thm. 18.5].

Using the table and Remark 10.11 we get:

$$(178) \quad 0 \rightarrow \mathbf{1} \rightarrow \mathbf{J}_1 \rightarrow \mathbf{J}_{\text{Sp}} \oplus \mathbf{J}_{\pi_\alpha} \rightarrow \mathbf{J}_{\pi_\alpha}^{\oplus 2} \rightarrow \mathbf{J}_{\pi_\alpha} \oplus \mathbf{J}_1 \rightarrow \mathbf{J}_{\text{Sp}} \rightarrow 0,$$

$$(179) \quad 0 \rightarrow \text{Sp} \rightarrow \mathbf{J}_{\text{Sp}} \rightarrow \mathbf{J}_1^{\oplus 2} \rightarrow \mathbf{J}_1^{\oplus 2} \rightarrow \mathbf{J}_{\text{Sp}} \rightarrow 0,$$

$$(180) \quad 0 \rightarrow \pi_\alpha \rightarrow \mathbf{J}_{\pi_\alpha} \rightarrow \mathbf{J}_{\text{Sp}} \oplus \mathbf{J}_{\pi_\alpha}^{\oplus 2} \rightarrow \mathbf{J}_{\text{Sp}}^{\oplus 2} \oplus \mathbf{J}_{\pi_\alpha} \rightarrow \mathbf{J}_{\text{Sp}} \rightarrow 0$$

where $\pi_\alpha = \text{Ind}_p^G \alpha$ and \mathbf{J}_π denotes an injective envelope of π .

10.2. Preparation. — Since $e^1(\mathbf{1}, \text{Sp}) = 2$ there exists a unique smooth k -representation τ_1 of G/Z such that $\text{Hom}_G(\mathbf{1}, \tau_1) = 0$ and we have an exact sequence:

$$(181) \quad 0 \rightarrow \text{Sp} \rightarrow \tau_1 \rightarrow \mathbf{1} \oplus \mathbf{1} \rightarrow 0.$$

Applying Ord_p to (181) and using (166), (167) we get

$$(182) \quad \text{Ord}_p \tau_1 \cong \text{Ord}_p \text{Sp} \cong \mathbf{1}, \quad \mathbf{R}^1 \text{Ord}_p \tau_1 \cong (\mathbf{R}^1 \text{Ord}_p \mathbf{1})^{\oplus 2} \cong (\alpha^{-1})^{\oplus 2}.$$

Lemma 10.12. — $e^1(\mathbf{1}, \tau_1) = 0$, $e^1(\text{Sp}, \tau_1) = 2$, $e^1(\text{Ind}_p^G \alpha, \tau_1) = 2$.

Proof. — Since $e^1(\mathbf{1}, \mathbf{1}) = 0$, we get the first claim by applying $\text{Hom}_{G/Z}(\mathbf{1}, *)$ to (181). From (182) and the 5-term sequence for Ord_p , see (123), we get that

$$(183) \quad \text{Ext}_{G/Z}^1(\text{Ind}_p^G \mathbf{1}, \tau_1) \cong \text{Ext}_{T/Z}^1(\mathbf{1}, \mathbf{1}),$$

$$(184) \quad \text{Ext}_{G/Z}^1(\text{Ind}_p^G \alpha, \tau_1) \cong \text{Hom}_{T/Z}(\alpha^{-1}, (\alpha^{-1})^{\oplus 2})$$

are both 2-dimensional. Since $e^1(\mathbf{1}, \tau_1) = 0$, by applying $\text{Hom}_{G/Z}(*, \tau_1)$ to the exact sequence $0 \rightarrow \mathbf{1} \rightarrow \text{Ind}_p^G \mathbf{1} \rightarrow \text{Sp} \rightarrow 0$ we deduce that $\text{Ext}_{G/Z}^1(\text{Sp}, \tau_1) \cong \text{Ext}_{G/Z}^1(\text{Ind}_p^G \mathbf{1}, \tau_1)$. \square

Proposition 10.13. — $\mathcal{I}(\tau_1) \cong \mathcal{I}(\pi(0, 1))$, $\mathbf{R}^1 \mathcal{I}(\tau_1) \cong \mathcal{I}(\text{Sp}) \oplus \mathcal{I}(\text{Ind}_p^G \alpha)^{\oplus 2}$, $\mathbf{R}^2 \mathcal{I}(\tau_1) \cong \mathcal{I}(\mathbf{1}) \oplus \mathcal{I}(\text{Ind}_p^G \alpha)^{\oplus 2}$.

Proof. — We apply \mathcal{I} to (181). Suppose that the connecting homomorphism $\partial : \mathcal{I}(\mathbf{1})^{\oplus 2} \rightarrow \mathbf{R}^1 \mathcal{I}(\text{Sp})$ is zero. Then we would have an exact sequence of \mathcal{H} -modules $0 \rightarrow \mathcal{I}(\text{Sp}) \rightarrow \mathcal{I}(\tau_1) \rightarrow \mathcal{I}(\mathbf{1})^{\oplus 2} \rightarrow 0$. Since $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\mathbf{1}), \mathcal{I}(\text{Sp})) = 1$ by [56, 11.3], we would obtain $\text{Hom}_G(\mathbf{1}, \tau_1) \cong \text{Hom}_{\mathcal{H}}(\mathcal{I}(\mathbf{1}), \mathcal{I}(\tau_1)) \neq 0$ contradicting the construction of τ_1 . Hence, ∂ is non-zero. Since $\mathbf{R}^1 \mathcal{I}(\text{Sp}) \cong \mathcal{I}(\text{Ind}_p^G \mathbf{1})$ by Proposition 10.5, the image of ∂ is 1-dimensional. Hence, we obtain a non-split extension $0 \rightarrow \mathcal{I}(\text{Sp}) \rightarrow \mathcal{I}(\tau_1) \rightarrow \mathcal{I}(\mathbf{1}) \rightarrow 0$. Since the only non-split extension between $\mathcal{I}(\text{Sp})$ and $\mathcal{I}(\mathbf{1})$ is realized by applying \mathcal{I} to (67) we deduce that $\mathcal{I}(\tau_1) \cong \mathcal{I}(\pi(0, 1))$. The cokernel of ∂ is isomorphic to $\mathcal{I}(\text{Sp})$. Hence, we obtain an exact sequence

$$(185) \quad 0 \rightarrow \mathcal{I}(\text{Sp}) \rightarrow \mathbf{R}^1 \mathcal{I}(\tau_1) \rightarrow \mathbf{R}^1 \mathcal{I}(\mathbf{1})^{\oplus 2}.$$

As $e^1(\text{Ind}_p^G \alpha, \tau_1) = 2$ by Lemma 10.12 and $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\text{Ind}_p^G \alpha), \mathcal{I}(\tau_1)) = 0$ by Lemma 5.27, Proposition 5.26 implies that $\dim \text{Hom}_{\mathcal{H}}(\mathcal{I}(\text{Ind}_p^G \alpha), \mathbf{R}^1 \mathcal{I}(\tau_1)) = 2$. Since $\mathbf{R}^1 \mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\text{Ind}_p^G \alpha)$ by Proposition 10.1, we deduce that the last arrow in (185) is surjective. Since $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\text{Ind}_p^G \alpha), \mathcal{I}(\text{Sp})) = 0$ by Lemma 5.27, we get $\mathbf{R}^1 \mathcal{I}(\tau_1) \cong \mathcal{I}(\text{Sp}) \oplus \mathcal{I}(\text{Ind}_p^G \alpha)^{\oplus 2}$. As $\mathbf{R}^3 \mathcal{I}(\text{Sp}) = 0$ by Proposition 10.5, we have an exact sequence:

$$(186) \quad 0 \rightarrow \mathbf{R}^2 \mathcal{I}(\text{Sp}) \rightarrow \mathbf{R}^2 \mathcal{I}(\tau_1) \rightarrow \mathbf{R}^2 \mathcal{I}(\mathbf{1})^{\oplus 2} \rightarrow 0.$$

Propositions 10.5 and 10.1 give $\mathbf{R}^2 \mathcal{I}(\text{Sp}) \cong \mathcal{I}(\mathbf{1})$ and $\mathbf{R}^2 \mathcal{I}(\mathbf{1}) \cong \mathcal{I}(\text{Ind}_p^G \alpha)$. Lemma 5.27 implies that the sequence (186) is split. This gives the last assertion. \square

Since $e^i(\mathbf{1}, \text{Ind}_p^G \alpha) = 0$ for $i \geq 0$, Corollary 10.9, by applying $\text{Hom}_G(*, \text{Ind}_p^G \alpha)$ to (181), we deduce that $\text{Ext}_{G/Z}^i(\tau_1, \text{Ind}_p^G \alpha) \cong \text{Ext}_{G/Z}^i(\text{Sp}, \text{Ind}_p^G \alpha)$ for all $i \geq 0$. In particular, $e^1(\tau_1, \text{Ind}_p^G \alpha) = 1$ and hence there exists a unique smooth k -representation τ_2 of G/Z such that $\text{Hom}_G(\text{Sp}, \tau_2) = 0$ and there exists an exact sequence:

$$(187) \quad 0 \rightarrow \text{Ind}_p^G \alpha \rightarrow \tau_2 \rightarrow \tau_1 \rightarrow 0.$$

Lemma 10.14. — $\text{Ord}_p \tau_2 \cong \alpha^{-1}$, $\mathbf{R}^1 \text{Ord}_p \tau_2 \cong \alpha^{-1} \oplus \alpha^{-1}$.

Proof. — We apply Ord_p to (187). Since $\text{Hom}_G(\text{Sp}, \tau_2) = \text{Hom}_G(\mathbf{1}, \tau_2) = 0$ we have $\text{Hom}_T(\mathbf{1}, \text{Ord}_p \tau_2) \cong \text{Hom}_G(\text{Ind}_p^G \mathbf{1}, \tau_2) = 0$. Since $\text{Ord}_p \tau_1 = \mathbf{1}$ and there are no extensions between α^{-1} and $\mathbf{1}$, we deduce that the connecting homomorphism $\partial : \text{Ord}_p \tau_1 \rightarrow \mathbf{R}^1 \text{Ord}_p(\text{Ind}_p^G \alpha)$ is injective. Since both the source and the target are 1-dimensional we deduce that ∂ is an isomorphism. Hence, $\text{Ord}_p \tau_2 \cong \text{Ord}_p(\text{Ind}_p^G \alpha)$ and $\mathbf{R}^1 \text{Ord}_p \tau_2 \cong \mathbf{R}^1 \text{Ord}_p \tau_1$. \square

Corollary 10.15. — $e^i(\text{Ind}_p^G \mathbf{1}, \tau_2) = 0$, for $i \geq 0$.

Proof. — Lemma 10.14, (123). \square

Lemma 10.16. — $e^1(\mathbf{1}, \tau_2) = 0$, $e^1(\text{Sp}, \tau_2) = e^2(\text{Sp}, \tau_2) = 0$, $e^1(\text{Ind}_p^G \alpha, \tau_2) \leq 4$.

Proof. — Since $e^1(\mathbf{1}, \text{Ind}_p^G \alpha) = e^1(\mathbf{1}, \tau_1) = 0$ we deduce that $e^1(\mathbf{1}, \tau_2) = 0$. By applying $\text{Hom}_{G/Z}(*, \tau_2)$ to the exact sequence $0 \rightarrow \mathbf{1} \rightarrow \text{Ind}_p^G \mathbf{1} \rightarrow \text{Sp} \rightarrow 0$ and using Corollary 10.15 we obtain $\text{Ext}_{G/Z}^i(\mathbf{1}, \tau_2) \cong \text{Ext}_{G/Z}^{i+1}(\text{Sp}, \tau_2)$ for $i \geq 0$. Hence, $e^1(\text{Sp}, \tau_2) = e^2(\text{Sp}, \tau_2) = 0$. Since $e^1(\text{Ind}_p^G \alpha, \text{Ind}_p^G \alpha) = e^1(\text{Ind}_p^G \alpha, \tau_1) = 2$ the last assertion follows after applying $\text{Hom}_{G/Z}(\text{Ind}_p^G \alpha, *)$ to (187). \square

For a smooth character $\chi : T/Z \rightarrow k^\times$ we denote by J_χ its injective envelope in $\text{Mod}_{T/Z}^{\text{ladm}}(k)$. We note that uniqueness of injective envelopes implies that $(J_\chi)^s \cong J_{\chi^s}$ and $J_\chi \cong J_{\mathbf{1}} \otimes \chi$. Let $J_{\mathbf{1}_G}$, J_{Sp} and J_{π_α} be injective envelopes of the trivial representation, Sp and $\pi_\alpha := \text{Ind}_p^G \alpha$ in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$, respectively.

Proposition 10.17. — *There exist exact sequences:*

$$(188) \quad 0 \rightarrow \text{Ind}_p^G J_{\mathbf{1}_T} \rightarrow J_{\mathbf{1}_G} \rightarrow J_{\pi_\alpha} \rightarrow 0,$$

$$(189) \quad 0 \rightarrow \text{Ind}_p^G J_\alpha \rightarrow J_{\pi_\alpha} \rightarrow J_{\text{Sp}} \rightarrow 0,$$

$$(190) \quad 0 \rightarrow (\text{Ind}_p^G J_{\mathbf{1}_T})/\mathbf{1}_G \rightarrow J_{\text{Sp}} \rightarrow J_{\pi_\alpha}^{\oplus 2}.$$

Proof. — The injections in (188) and (189) follow from Proposition 7.1. Lemma 7.4 gives $e^i(\mathrm{Ind}_p^G \mathbf{1}, \mathrm{Ind}_p^G \mathbf{J}_1) = 0$ for $i \geq 1$. Proposition 10.6 gives $e^i(\mathbf{1}, \mathrm{Ind}_p^G \mathbf{J}_1) = 0$ and $e^i(\mathrm{Sp}, \mathrm{Ind}_p^G \mathbf{J}_1) = 0$ for $i \geq 1$. Lemma 7.4 implies that $e^1(\mathrm{Ind}_p^G \alpha, \mathrm{Ind}_p^G \mathbf{J}_1) = 1$, and $e^i(\mathrm{Ind}_p^G \alpha, \mathrm{Ind}_p^G \mathbf{J}_1) = 0$ for all $i \geq 2$. This gives (188), see Remark 10.11. Similarly we obtain (189), noting that $e^i(\mathbf{1}, \mathrm{Ind}_p^G \mathbf{J}_\alpha) = 0$ for all $i \geq 0$, see Corollary 10.8.

Applying $\mathrm{Hom}_{G/Z}(\mathbf{1}, *)$ to the exact sequence:

$$(191) \quad 0 \rightarrow \mathbf{1}_G \rightarrow \mathrm{Ind}_p^G \mathbf{J}_1 \rightarrow \kappa \rightarrow 0$$

we get $e^i(\mathbf{1}, \kappa) = 0$ for $i = 0$ and $i = 1$. (Here we are using the fact that $e^1(\mathbf{1}, \mathbf{1}) = e^2(\mathbf{1}, \mathbf{1}) = e^1(\mathbf{1}, \mathrm{Ind}_p^G \mathbf{J}_1) = 0$.) As $e^1(\mathrm{Sp}, \mathbf{1}) = 1$ and $e^1(\mathrm{Sp}, \mathrm{Ind}_p^G \mathbf{J}_1) = e^2(\mathrm{Sp}, \mathbf{1}) = 0$ we get $\dim \mathrm{Hom}_{G/Z}(\mathrm{Sp}, \kappa) = 1$ and $\mathrm{Ext}_{G/Z}^1(\mathrm{Sp}, \kappa) = 0$. Since all the irreducible subquotients of $\mathrm{Ind}_p^G \mathbf{J}_1$ are either $\mathbf{1}_G$ or Sp we have $\mathrm{Hom}_{G/Z}(\mathrm{Ind}_p^G \alpha, \kappa) = 0$. Moreover, $e^1(\mathrm{Ind}_p^G \alpha, \mathbf{1}) = e^1(\mathrm{Ind}_p^G \alpha, \mathrm{Ind}_p^G \mathbf{J}_1) = 1$, $e^2(\mathrm{Ind}_p^G \alpha, \mathrm{Ind}_p^G \mathbf{J}_1) = 0$ and thus $\mathrm{Ext}_{G/Z}^1(\mathrm{Ind}_p^G \alpha, \kappa) \cong \mathrm{Ext}_{G/Z}^2(\mathrm{Ind}_p^G \alpha, \mathbf{1})$ is 2-dimensional. Hence, we deduce the existence of (190). \square

Let

$$(192) \quad 0 \rightarrow \mathbf{1} \rightarrow \kappa \rightarrow \pi_\alpha \rightarrow 0$$

be a non-split extension. Since $\mathrm{Ext}_{G/Z}^1(\pi_\alpha, \mathbf{1})$ is one dimensional κ is uniquely determined up to isomorphism. Applying Ord_p to (192) we obtain:

$$(193) \quad \mathrm{Ord}_p \kappa = 0, \quad \mathbf{R}^1 \mathrm{Ord}_p \kappa \cong \mathbf{R}^1 \mathrm{Ord}_p \pi_\alpha \cong \mathbf{1}.$$

It follows from (123) that

$$(194) \quad e^1(\pi_\alpha, \kappa) = 0, \quad e^1(\mathrm{Ind}_p^G \mathbf{1}, \kappa) = 1.$$

Lemma 10.18. — $e^1(\kappa, \kappa) = 0$, $e^1(\mathrm{Sp}, \kappa) = 2$, $e^1(\kappa, \mathrm{Sp}) = 2$.

Proof. — The first assertion follows since $e^1(\mathbf{1}, \mathbf{1}) = 0$, $e^1(\mathbf{1}, \pi_\alpha) = 0$ thus $e^1(\mathbf{1}, \kappa) = 0$ and $e^1(\pi_\alpha, \kappa) = 0$ by (194). For the second apply $\mathrm{Hom}_{G/Z}(*, \kappa)$ to $0 \rightarrow \mathbf{1} \rightarrow \mathrm{Ind}_p^G \mathbf{1} \rightarrow \mathrm{Sp} \rightarrow 0$ and use (194). Since $e^1(\pi_\alpha, \mathrm{Sp}) = e^2(\pi_\alpha, \mathrm{Sp}) = 0$ we have $e^1(\kappa, \mathrm{Sp}) = e^1(\mathbf{1}, \mathrm{Sp}) = 2$. \square

Lemma 10.19. — Let β in $\mathrm{Mod}_{G/Z}^{\mathrm{sm}}(k)$ be such that $\mathrm{soc}_G \beta \cong \mathrm{Sp}$ and the semisimplification is isomorphic to $\mathrm{Sp} \oplus \mathbf{1} \oplus \pi_\alpha$ then $e^1(\beta, \beta) \leq 3$.

Proof. — Since $e^1(\pi_\alpha, \mathrm{Sp}) = e^2(\pi_\alpha, \mathrm{Sp}) = 0$ there exists an exact sequence $0 \rightarrow \mathrm{Sp} \rightarrow \beta \rightarrow \kappa \rightarrow 0$. Since $e^1(\mathrm{Sp}, \mathrm{Sp}) = e^2(\mathrm{Sp}, \mathrm{Sp}) = 0$ we get $e^1(\mathrm{Sp}, \beta) = e^1(\mathrm{Sp}, \kappa) = 2$. Since $e^1(\kappa, \kappa) = 0$ we get $e^1(\kappa, \beta) = e^1(\kappa, \mathrm{Sp}) - e^0(\kappa, \kappa) = 1$. Thus $e^1(\beta, \beta) \leq e^1(\kappa, \beta) + e^1(\mathrm{Sp}, \beta) = 3$. \square

Remark 10.20. — Using the bound of Lemma 10.19 and the results of Kisin [41] one may show that \mathbf{V} induces an isomorphism between the deformation functors of β with a fixed central character and $\mathbf{V}(\beta)$ with a fixed determinant.

Lemma 10.21. — *Let G be a compact torsion-free p -adic analytic pro- p group of dimension d and let τ be in $\text{Mod}_G^{\text{sm}}(k)$ then there exists a natural isomorphism between $\text{Ext}_G^d(\mathbf{1}, \tau)$ and the G -coinvariants τ_G .*

Proof. — Since $H^0(G, *) \cong \text{Hom}_G(\mathbf{1}, *)$ and $H^i(G, *)$ is the i -th derived functor of $H^0(G, *)$, [64, §2.2], for all $i \geq 0$ we have a natural isomorphism of functors $\text{Ext}_G^i(\mathbf{1}, *) \cong H^i(G, *)$. Since G is compact torsion-free and p -adic analytic, it is a Poincaré group of dimension d , [45, 2.5.8], [62]. Since G is pro- p , it acts trivially on the dualizing module. If τ is finite then Poincaré duality induces an isomorphism $H^d(G, \tau) \cong H^0(G, \tau^*)^* \cong \tau_G$, [64, I.4.5], where $*$ denotes k -linear dual. In general, we may write τ as a union of finite subrepresentations $\tau = \varinjlim \tau_i$. We have

$$H^d(G, \tau) \cong \varinjlim H^d(G, \tau_i) \cong \varinjlim (\tau_i)_G \cong \tau_G,$$

where the first isomorphism is given by [64, I.2.2 Cor. 2]. □

In Lemmas below κ is the representation defined in (192).

Lemma 10.22. — $\mathcal{I}(\kappa) \cong \mathcal{I}(\mathbf{1})$, $\mathbf{R}^3\mathcal{I}(\kappa) = 0$.

Proof. — Since $\text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\pi_\alpha), \mathcal{I}(\mathbf{1})) = 0$, Lemma 5.27, we have $\mathcal{I}(\kappa) \cong \mathcal{I}(\mathbf{1})$. Lemma 10.3 and Lemma 10.21 imply that the I_1/Z_1 -coinvariants of π_α are zero. Hence, I_1/Z_1 -coinvariants of κ are also zero, since otherwise we would obtain a I_1 -equivariant splitting of (192), which would contradict $\mathcal{I}(\kappa) \cong \mathcal{I}(\mathbf{1})$. Lemma 10.21 implies that $H^3(I_1/Z_1, \kappa) = 0$ and it follows from Lemma 5.22 that $\mathbf{R}^3\mathcal{I}(\kappa) = 0$. □

Lemma 10.23. — $\mathbf{R}^1\mathcal{I}(\kappa) \cong \mathcal{I}(\pi(0, 1))$, $\mathbf{R}^2\mathcal{I}(\kappa) \cong \mathcal{I}(\text{Sp})$, $\mathbf{R}^i\mathcal{I}(\kappa) = 0$ for $i \geq 3$.

Proof. — Lemma 10.22 implies that $\mathbf{R}^3\mathcal{I}(\kappa) = 0$. It follows from Lemma 5.23 that $\mathbf{R}^i\mathcal{I}(\kappa) = 0$ for $i \geq 4$. Applying \mathcal{I} to (192) and using Proposition 10.1, Lemma 10.22 and Corollary 10.10 we obtain an exact sequence:

$$\begin{aligned} \mathbf{R}^1\mathcal{I}(\kappa) &\hookrightarrow \mathcal{I}(\pi_\alpha) \oplus \mathcal{I}(\pi(0, 1)) \rightarrow \mathcal{I}(\pi_\alpha) \rightarrow \mathbf{R}^2\mathcal{I}(\kappa) \\ &\rightarrow \mathcal{I}(\pi(0, 1)) \twoheadrightarrow \mathcal{I}(\mathbf{1}). \end{aligned}$$

It follows from Proposition 5.26 and (194) that $\text{Hom}_{\mathcal{H}}(\mathcal{I}(\pi_\alpha), \mathbf{R}^1\mathcal{I}(\kappa)) = 0$, which implies the assertion. □

Lemma 10.24. — $e^1(\pi(0, 1), \kappa) = 2$, $e^2(\pi(0, 1), \kappa) = 1$.

Proof. — Using Lemmas 5.24, 5.27 one obtains $\text{Ext}_{\mathcal{H}}^i(\mathcal{I}(\pi(0, 1)), \mathcal{I}(\pi(0, 1)))$ is 1-dimensional and $\text{Ext}_{\mathcal{H}}^i(\mathcal{I}(\pi(0, 1)), \mathcal{I}(\text{Sp})) = 0$ for $i = 0, i = 1$. The assertion follows from Proposition 5.26 and Lemma 10.23. \square

10.3. Quotient category.

Lemma 10.25. — *Let $0 \rightarrow \pi_1 \rightarrow \pi_2 \rightarrow \pi_3 \rightarrow 0$ be an extension in $\text{Mod}_{G/Z}^{\text{sm}}(\mathcal{O})$ then G acts trivially on π_1 and π_3 if and only if it acts trivially on π_2 .*

Proof. — Choose v in π_2 then the map $g \mapsto (g - 1)v$ defines a group homomorphism $\psi : G \rightarrow (\pi_1, +)$. Since Z acts trivially on π_2 , ψ will factor through $G/Z \text{SL}_2(\mathbf{Q}_p)$. The order of $G/Z \text{SL}_2(\mathbf{Q}_p)$ is prime to p , as $p > 2$. Since every element of π_2 is killed by a power of p , we deduce that ψ is zero. Hence, G acts trivially on π_2 . The other implication is trivial. \square

Let $\mathfrak{T}(\mathcal{O})$ be the category of compact \mathcal{O} -modules with the trivial G -action. It follows from Lemma 10.25 that $\mathfrak{T}(\mathcal{O})$ is a thick subcategory of $\mathfrak{C}(\mathcal{O})$ and hence we may build a quotient category $\mathfrak{Q}(\mathcal{O}) := \mathfrak{C}(\mathcal{O})/\mathfrak{T}(\mathcal{O})$. Recall, [35, §III.1], that the objects of $\mathfrak{Q}(\mathcal{O})$ are the same as the objects of $\mathfrak{C}(\mathcal{O})$, the morphisms are given by

$$(195) \quad \text{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathbf{M}, \mathbf{N}) := \varinjlim \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathbf{M}', \mathbf{N}/\mathbf{N}'),$$

where the limit is taken over all subobjects \mathbf{M}' of \mathbf{M} and \mathbf{N}' of \mathbf{N} such that G acts trivially on \mathbf{M}/\mathbf{M}' and \mathbf{N}' . Let $\mathcal{T} : \mathfrak{C}(\mathcal{O}) \rightarrow \mathfrak{Q}(\mathcal{O})$ be the functor $\mathcal{T}\mathbf{M} = \mathbf{M}$ for every object of $\mathfrak{C}(\mathcal{O})$ and $\mathcal{T}f : \mathcal{T}\mathbf{M} \rightarrow \mathcal{T}\mathbf{N}$ is the image of $f : \mathbf{M} \rightarrow \mathbf{N}$ in $\varinjlim \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathbf{M}', \mathbf{N}/\mathbf{N}')$ under the natural map. The category $\mathfrak{Q}(\mathcal{O})$ is abelian and \mathcal{T} is an exact functor, [35, Prop 1, §III.1]. In our situation it is easy to describe the homomorphisms in the quotient category explicitly. For an object \mathbf{M} of $\mathfrak{C}(\mathcal{O})$, we denote by $\mathbf{I}_G(\mathbf{M}) := (\mathbf{M}^\vee / (\mathbf{M}^\vee)^G)^\vee \subseteq \mathbf{M}$.

Lemma 10.26. — *Let \mathbf{M} and \mathbf{N} be objects of $\mathfrak{C}(\mathcal{O})$, then $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathbf{I}_G(\mathbf{M}), \mathbf{1}) = 0$ and $(\mathbf{N}/\mathbf{N}^G)^G = 0$. In particular,*

$$(196) \quad \text{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T}\mathbf{M}, \mathcal{T}\mathbf{N}) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathbf{I}_G(\mathbf{M}), \mathbf{N}/\mathbf{N}^G).$$

Proof. — The first two assertions follow from Lemma 10.25. Hence, it follows from the definition that $\text{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T}(\mathbf{I}_G(\mathbf{M})), \mathcal{T}(\mathbf{N}/\mathbf{N}^G)) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathbf{I}_G(\mathbf{M}), \mathbf{N}/\mathbf{N}^G)$. Moreover, Lemme 4 in [35, §III.1] implies that the natural maps induce isomorphisms $\mathcal{T}\mathbf{I}_G(\mathbf{M}) \cong \mathcal{T}(\mathbf{M})$, $\mathcal{T}\mathbf{N} \cong \mathcal{T}(\mathbf{N}/\mathbf{N}^G)$. \square

Lemma 10.27. — *If P is a projective object of $\mathfrak{C}(\mathcal{O})$ with $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \mathbf{1}) = 0$ then $\mathcal{T}P$ is a projective object of $\mathfrak{Q}(\mathcal{O})$ and*

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, N) \cong \mathrm{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T}P, \mathcal{T}N)$$

for all N .

Proof. — Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \mathbf{1}) = 0$ we get $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, N^G) = 0$. Since P is projective we deduce $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, N) \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, N/N^G)$. The second assertion follows from Lemma 10.26. The exactness of $\mathrm{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T}P, *)$ follows from [35, Cor 1, §III.1], which says that every exact sequence of $\mathfrak{Q}(\mathcal{O})$ is isomorphic to an exact sequence of the form $0 \rightarrow \mathcal{T}M_1 \rightarrow \mathcal{T}M_2 \rightarrow \mathcal{T}M_3 \rightarrow 0$, where $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence in $\mathfrak{C}(\mathcal{O})$. \square

Lemma 10.28. — *The category $\mathfrak{Q}(\mathcal{O})$ has enough projectives.*

Proof. — Let M be in $\mathfrak{C}(\mathcal{O})$ and let $P \rightarrow I_G(M)$ be a projective envelope of $I_G(M)$ in $\mathfrak{C}(\mathcal{O})$. Since $\mathrm{Hom}_{\mathfrak{Q}(\mathcal{O})}(I_G(M), \mathbf{1}) = 0$ by Lemma 10.26 we also have $\mathrm{Hom}_{\mathfrak{Q}(\mathcal{O})}(P, \mathbf{1}) = 0$. Thus $\mathcal{T}P$ is projective in $\mathfrak{Q}(\mathcal{O})$ by Lemma 10.27 and since \mathcal{T} is exact we have $\mathcal{T}P \rightarrow \mathcal{T}I_G(M) \cong \mathcal{T}M$. \square

Lemma 10.29. — *If $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(N, \mathbf{1}) = 0$ then for every essential epimorphism $q : M \rightarrow N$, $\mathcal{T}q : \mathcal{T}M \rightarrow \mathcal{T}N$ is an essential epimorphism in $\mathfrak{Q}(\mathcal{O})$.*

Proof. — Let $a : T \rightarrow \mathcal{T}M$ be a morphism in $\mathfrak{Q}(\mathcal{O})$ such that the composition $\mathcal{T}q \circ a : T \rightarrow \mathcal{T}N$ is an epimorphism. We claim that a is an epimorphism. After replacing T with the image of a we may assume that a is a monomorphism. It follows from [35, Prop 1, §III.1] that there exists a monomorphism $u : M' \rightarrow M$ in $\mathfrak{C}(\mathcal{O})$ such that $a : T \rightarrow \mathcal{T}M$ is isomorphic to $\mathcal{T}u : \mathcal{T}M' \rightarrow \mathcal{T}M$. Now $\mathcal{T}q \circ \mathcal{T}u = \mathcal{T}(q \circ u) : \mathcal{T}M' \rightarrow \mathcal{T}N$ is an epimorphism, and hence G acts trivially on the cokernel of $q \circ u$ in $\mathfrak{C}(\mathcal{O})$, see Lemme 3 in [35, §III.1]. As $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(N, \mathbf{1}) = 0$, we get that $q \circ u$ is an epimorphism, and since q is essential, $u : M' \rightarrow M$ is an epimorphism, which implies that $\mathcal{T}u$ (and hence a) is an epimorphism. \square

We note that the category $\mathfrak{Q}(\mathcal{O})$ is \mathcal{O} -linear. Since \mathcal{T} is exact we have $(\mathcal{T}M)[\varpi] \cong \mathcal{T}(M[\varpi])$ and $\mathcal{T}M/\varpi\mathcal{T}M \cong \mathcal{T}(M/\varpi M)$. The composition $\mathfrak{C}(k) \rightarrow \mathfrak{C}(\mathcal{O}) \xrightarrow{\mathcal{T}} \mathfrak{Q}(\mathcal{O})$ factors through the quotient category $\mathfrak{Q}(k) := \mathfrak{C}(k)/\mathfrak{T}(k)$ and induces an equivalence of categories between $\mathfrak{Q}(k)$ and the full subcategory of $\mathfrak{Q}(\mathcal{O})$ consisting of the objects killed by ϖ . We denote by $T_{\mathbf{1}}$ and T_{α} the following objects of $\mathfrak{Q}(k)$:

$$(197) \quad T_{\mathbf{1}} := \mathcal{T}(\mathrm{Ind}_p^G \mathbf{1})^{\vee}, \quad T_{\alpha} := \mathcal{T}(\mathrm{Ind}_p^G \alpha)^{\vee}.$$

We note that since $\mathcal{T}(\mathbf{1}) \cong 0$ in $\mathfrak{Q}(k)$ and since \mathcal{T} is exact we have

$$(198) \quad T_1 \cong \mathcal{T} \text{Sp}^\vee \cong \mathcal{T} \tau_1^\vee,$$

where τ_1 is the representation defined by (181).

Lemma 10.30. — $\text{Hom}_{\mathfrak{Q}(k)}(\mathcal{T}M, T_1) \cong \text{Hom}_G(\text{Ind}_p^G \mathbf{1}, M^\vee)$, for all M in $\mathfrak{C}(k)$.

Proof. — Since $\text{Ext}_{G/Z}^i(\text{Ind}_p^G \mathbf{1}, \mathbf{1}) = 0$ for $i \geq 0$ by (168), we have

$$\text{Hom}_G(\text{Ind}_p^G \mathbf{1}, M^\vee) \cong \text{Hom}_G(\text{Ind}_p^G \mathbf{1}, M^\vee / (M^\vee)^G) \cong \text{Hom}_{\mathfrak{Q}(k)}(\mathcal{T}M, T_1).$$

The last isomorphism follows from Lemma 10.26. \square

Proposition 10.31. — The hypotheses (H1)–(H5) hold in $\mathfrak{Q}(k)$ with $S = T_\alpha$ and $Q = \mathcal{T} \tau_2^\vee$, where τ_2 is the representation defined by (187).

Remark 10.32. — The hypotheses (H1)–(H5) are stated in Section 3 assuming that \mathfrak{C} is a full subcategory of $\text{Mod}_G^{\text{pro-}a\text{ug}}(\mathcal{O})$, where G is a locally pro- p group, but the statements make sense in any k -linear abelian category, such as $\mathfrak{Q}(k)$.

Proof of Proposition 10.31. — If π_1, π_2 are irreducible non-trivial in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ then it follows from Lemma 10.26 that $\mathcal{T} \pi_1^\vee$ and $\mathcal{T} \pi_2^\vee$ are irreducible in $\mathfrak{Q}(k)$ and $\mathcal{T} \pi_1^\vee \cong \mathcal{T} \pi_2^\vee$ implies $\pi_1 \cong \pi_2$. In particular, T_1 and T_α are irreducible, non-zero and non-isomorphic in $\mathfrak{Q}(k)$. Conversely, it follows from Lemma 10.26 that every irreducible non-zero object of $\mathfrak{Q}(k)$ is isomorphic to $\mathcal{T} \pi^\vee$, where π is an irreducible non-trivial representation in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$. Let J_{π_α} be an injective envelope of $\text{Ind}_p^G \alpha$ in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$, then $P := J_{\pi_\alpha}^\vee$ is a projective envelope of $(\text{Ind}_p^G \alpha)^\vee$ in $\mathfrak{C}(k)$ and it follows from Lemmas 10.27 and 10.29 that $\mathcal{T}P$ is a projective envelope of T_α in $\mathfrak{Q}(k)$. It follows from Lemma 10.16 and Remark 10.11 that we have an exact sequence in $\text{Mod}_{G/Z}^{\text{ladm}}$:

$$(199) \quad 0 \rightarrow \tau_2 \rightarrow J_{\pi_\alpha} \rightarrow J_{\pi_\alpha}^{\oplus d} \rightarrow \kappa \rightarrow 0,$$

where $d = \dim \text{Ext}_{G/Z}^1(\text{Ind}_p^G \alpha, \tau_2) \leq 4$ and $\text{Hom}_G(\text{Ind}_p^G \mathbf{1}, \kappa) = 0$ by Corollary 10.15. By dualizing (199) and applying \mathcal{T} we get an exact sequence:

$$(200) \quad 0 \rightarrow \mathcal{T} \kappa^\vee \rightarrow \mathcal{T}P^{\oplus d} \rightarrow \mathcal{T}P \rightarrow \mathcal{T} \tau_2^\vee \rightarrow 0.$$

Let π be an irreducible representation of $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ with $\pi \not\cong \mathbf{1}$ and $\pi \not\cong \text{Ind}_p^G \alpha$. Since $\mathcal{T} \pi^\vee$ is irreducible in $\mathfrak{Q}(k)$, is not isomorphic to T_α and $\mathcal{T}P$ is a projective envelope of T_α in $\mathfrak{Q}(k)$, we deduce that $\text{Hom}_{\mathfrak{Q}(k)}(\mathcal{T}P, \mathcal{T} \pi^\vee) = 0$. Applying $\text{Hom}_{\mathfrak{Q}(k)}(*, \mathcal{T} \pi^\vee)$ to (200) we get that

$$(201) \quad \text{Hom}_{\mathfrak{Q}(k)}(\mathcal{T} \tau_2^\vee, \mathcal{T} \pi^\vee) = 0, \quad \text{Ext}_{\mathfrak{Q}(k)}^1(\mathcal{T} \tau_2^\vee, \mathcal{T} \pi^\vee) = 0,$$

$$(202) \quad \mathrm{Ext}_{\Omega(k)}^2(\mathcal{T}\tau_2^\vee, \mathcal{T}\pi^\vee) \cong \mathrm{Hom}_{\Omega(k)}(\mathcal{T}\kappa^\vee, \mathcal{T}\pi^\vee).$$

It follows from (201) that (H1) and (H3) hold. Dualizing (187) and applying \mathcal{T} we get an exact sequence $0 \rightarrow \mathbf{T}_1 \rightarrow \mathcal{T}\tau_2^\vee \rightarrow \mathbf{T}_\alpha \rightarrow 0$. Since $\mathbf{T}_\alpha \not\cong \mathbf{T}_1$, (H2) holds. Further, applying $\mathrm{Hom}_{\Omega(k)}(*, \mathbf{T}_\alpha)$ to (200) we deduce that

$$(203) \quad \dim \mathrm{Ext}_{\Omega(k)}^1(\mathcal{T}\tau_2^\vee, \mathbf{T}_\alpha) \leq d \leq 4,$$

hence (H4) holds. Since $\mathrm{Hom}_G(\mathrm{Ind}_p^G \mathbf{1}, \kappa) = 0$ we deduce from Lemma 10.30 that $\mathrm{Hom}_{\Omega(k)}(\mathcal{T}\kappa^\vee, \mathbf{T}_1) = 0$. Since \mathbf{T}_1 is the maximal proper subobject of $\mathcal{T}\tau_2^\vee$, it follows from (202) that (H5) is satisfied. \square

Remark 10.33. — It follows from (199) that the hypotheses (H1)–(H4) hold in $\mathfrak{C}(k)$ with $S = \pi_\alpha^\vee$ and $Q = \tau_2^\vee$. The problem is that (H5) does not hold in $\mathfrak{C}(k)$: one may calculate using the results of Section 10.1 that $\mathrm{Ext}_{G/Z}^2(\mathbf{1}, \tau_2) \cong \mathrm{Ext}_{G/Z}^2(\mathbf{1}, \tau_1) \cong \mathrm{Ext}_{G/Z}^2(\mathbf{1}, \mathrm{Sp}) \neq 0$. This implies that $\mathrm{Ext}_{G/Z}^2(\tau_1, \tau_2) \neq 0$ since $e^1(\mathrm{Sp}, \tau_2) = 0$ by Lemma 10.16. Dually we obtain that $\mathrm{Ext}_{\mathfrak{C}(k)}^2(\tau_2^\vee, \tau_1^\vee) \neq 0$.

Lemma 10.34. — $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_1, \mathbf{T}_1)$, $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_1, \mathbf{T}_\alpha)$, $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_\alpha, \mathbf{T}_\alpha)$ are 2-dimensional and $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_\alpha, \mathbf{T}_1)$ is 1-dimensional.

Proof. — Let $J_{\mathrm{Sp}}, J_{\pi_\alpha}$ be injective envelopes of Sp and $\pi_\alpha := \mathrm{Ind}_p^G \alpha$ in $\mathrm{Mod}_{G/Z}^{\mathrm{ladm}}(k)$. It follows from Lemma 10.12 that we have an exact sequence:

$$(204) \quad 0 \rightarrow \tau_1 \rightarrow J_{\mathrm{Sp}} \rightarrow J_{\mathrm{Sp}}^{\oplus 2} \oplus J_{\pi_\alpha}^{\oplus 2}.$$

Moreover, if we let κ be the cokernel of the second arrow then the monomorphism $\kappa \hookrightarrow J_{\mathrm{Sp}}^{\oplus 2} \oplus J_{\pi_\alpha}^{\oplus 2}$ induced by the third arrow is essential. Let π be Sp or π_α then we know from Lemmas 10.27 and 10.29 that $\mathcal{T}J_\pi^\vee$ is a projective envelope of $\mathcal{T}\pi^\vee$ in $\Omega(k)$. By dualizing (204), applying \mathcal{T} and then $\mathrm{Hom}_{\Omega(k)}(*, \mathcal{T}\pi^\vee)$ we obtain

$$\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_1, \mathcal{T}\pi^\vee) \cong \mathrm{Hom}_{\Omega(k)}(\mathcal{T}\kappa^\vee, \mathcal{T}\pi^\vee) \cong \mathrm{Hom}_{\Omega(k)}(\mathcal{T}J^\vee, \mathcal{T}\pi^\vee),$$

where $J = J_{\mathrm{Sp}}^{\oplus 2} \oplus J_{\pi_\alpha}^{\oplus 2}$. The last isomorphism follows from the fact that $\mathcal{T}\pi^\vee$ is irreducible, and $\mathcal{T}J^\vee \twoheadrightarrow \mathcal{T}\kappa^\vee$ is essential by Lemma 10.29. Hence $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_1, \mathbf{T}_1)$ and $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_1, \mathbf{T}_\alpha)$ are 2-dimensional. To calculate dimensions of $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_\alpha, \mathbf{T}_\alpha)$ and $\mathrm{Ext}_{\Omega(k)}^1(\mathbf{T}_\alpha, \mathbf{T}_1)$ the same argument may be applied to (180). \square

The functor $\check{\mathbf{V}} : \mathfrak{C}(\mathcal{O}) \rightarrow \mathrm{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ kills the trivial representation and hence every object in $\mathfrak{T}(\mathcal{O})$. It follows from Corollaire 2 in [35, §III.1] that $\check{\mathbf{V}}$ factors through $\mathcal{T} : \mathfrak{C}(\mathcal{O}) \rightarrow \Omega(\mathcal{O})$. We denote $\check{\mathbf{V}} : \Omega(\mathcal{O}) \rightarrow \mathrm{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ by the same letter. We have

$$(205) \quad \check{\mathbf{V}}(\mathbf{T}_1) \cong \mathbf{V}(\mathrm{Sp})^\vee(\varepsilon) \cong \mathbf{1}, \quad \check{\mathbf{V}}(\mathbf{T}_\alpha) \cong \mathbf{V}(\mathrm{Ind}_p^G \alpha)^\vee(\varepsilon) \cong \omega.$$

Lemma 10.35. — *The functor $\check{\mathbf{V}}$ induces an injection*

$$\check{\mathbf{V}} : \mathrm{Ext}_{\check{\Omega}(\mathcal{O})}^1(S_1, S_2) \hookrightarrow \mathrm{Ext}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}^1(\check{\mathbf{V}}(S_1), \check{\mathbf{V}}(S_2)),$$

for $S_1, S_2 \in \{\mathbf{T}_1, \mathbf{T}_\alpha\}$.

Proof. — We interpret Ext^1 as Yoneda Ext and the extension $0 \rightarrow S_2 \rightarrow E \rightarrow S_1 \rightarrow 0$ is mapped to $0 \rightarrow \check{\mathbf{V}}(S_2) \rightarrow \check{\mathbf{V}}(E) \rightarrow \check{\mathbf{V}}(S_1) \rightarrow 0$. If this extension splits, then $\check{\mathbf{V}}(E) \cong \check{\mathbf{V}}(S_1) \oplus \check{\mathbf{V}}(S_2)$ is killed by ϖ . Since $\check{\mathbf{V}}(S_1)$ and $\check{\mathbf{V}}(S_2)$ are non-zero by (205) the exactness of $\check{\mathbf{V}}$ implies that E is killed by ϖ . Thus it is enough to show that $\check{\mathbf{V}}$ induces an injection

$$\mathrm{Ext}_{\check{\Omega}(k)}^1(S_1, S_2) \hookrightarrow \mathrm{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\check{\mathbf{V}}(S_1), \check{\mathbf{V}}(S_2)).$$

This assertion follows from the work of Colmez. We first treat the case $S_1 \cong S_2$. Let $\chi : \mathbf{T}/Z \rightarrow k^\times$ be a smooth character. Since $\mathbf{T}/Z \cong \mathbf{Q}_p^\times$ the space $\mathrm{Ext}_{\mathbf{T}/Z}^1(\mathbf{1}, \mathbf{1}) \cong \mathrm{Hom}(\mathbf{Q}_p^\times, k)$ is 2-dimensional. Fix $\tau \in \mathrm{Hom}(\mathbf{Q}_p^\times, k)$ and let Y_τ be the corresponding extension of $\mathbf{1}$ by itself. Since parabolic induction is exact we have an exact sequence

$$0 \rightarrow \mathrm{Ind}_p^G \chi \rightarrow \mathrm{Ind}_p^G Y_\tau \otimes \chi \rightarrow \mathrm{Ind}_p^G \chi \rightarrow 0.$$

We denote $\pi_\chi := \mathrm{Ind}_p^G \chi$. Since χ is trivial on Z we may write it as $\chi = \chi_1^{-1} \otimes \chi_1$, then $\mathbf{V}(\pi_\chi) \cong \chi_1 \omega$. It is shown in the proof of [23, VII.4.14] that the composition of

$$\mathrm{Hom}(\mathbf{Q}_p^\times, k) \rightarrow \mathrm{Ext}_{G/Z}^1(\pi_\chi, \pi_\chi) \xrightarrow{\mathbf{V}} \mathrm{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\chi_1 \omega, \chi_1 \omega) \cong \mathrm{Hom}(\mathbf{Q}_p^\times, k)$$

is the identity map. Using the anti-equivalence of categories, we obtain a surjection

$$\mathrm{Ext}_{\mathfrak{C}(k)}^1(\pi_\chi^\vee, \pi_\chi^\vee) \xrightarrow{\check{\mathbf{V}}} \mathrm{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\chi_1^{-1}, \chi_1^{-1}) \cong \mathrm{Hom}(\mathbf{Q}_p^\times, k).$$

Since $\check{\mathbf{V}}$ factors through \mathcal{T} , we obtain a surjection

$$\mathrm{Ext}_{\check{\Omega}(k)}^1(\mathcal{T} \pi_\chi^\vee, \mathcal{T} \pi_\chi^\vee) \xrightarrow{\check{\mathbf{V}}} \mathrm{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\chi_1^{-1}, \chi_1^{-1}) \cong \mathrm{Hom}(\mathbf{Q}_p^\times, k).$$

When $\chi = \mathbf{1}$ or $\chi = \alpha$ we know by Lemma 10.34 that the source is 2-dimensional. Since the target is 2-dimensional, the map is an isomorphism. We deal with the case $S_1 \not\cong S_2$ similarly.

We claim that the map $\check{\mathbf{V}} : \mathrm{Ext}_{\check{\Omega}(k)}^1(\mathbf{T}_1, \mathbf{T}_\alpha) \rightarrow \mathrm{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\mathbf{1}, \omega)$ is surjective. For every non-zero smooth homomorphism $\tau : \mathbf{Q}_p^\times \rightarrow k$, Colmez constructs an extension $0 \rightarrow \mathrm{Sp} \rightarrow E_\tau \rightarrow \mathbf{1} \rightarrow 0$, see [23, VII.4.19], and shows that $\mathrm{Ext}_{G/Z}^1(\mathrm{Ind}_p^G \alpha, E_\tau)$ is 1-dimensional, [23, VII.4.26], see also Lemma 10.18. If we let ϵ_τ be a non-split extension $0 \rightarrow E_\tau \rightarrow \Pi \rightarrow \mathrm{Ind}_p^G \alpha \rightarrow 0$, then $\mathbf{V}(\epsilon_\tau)$ defines an element of $\mathrm{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\mathbf{1}, \omega)$. It follows from [23, VII.4.25], that the $\mathbf{V}(\epsilon_\tau)$ for different τ span the 2-dimensional

space $\text{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\mathbf{1}, \omega)$. Since $\mathcal{T}\mathcal{E}_\tau^\vee \cong \mathbf{T}_1$ we get our claim by applying $\check{\mathbf{V}}$ to the extension $0 \rightarrow \mathbf{T}_\alpha \rightarrow \mathcal{T}\Pi^\vee \rightarrow \mathcal{T}\mathcal{E}_\tau^\vee \rightarrow 0$. Since $\text{Ext}_{\Omega(k)}^1(\mathbf{T}_1, \mathbf{T}_\alpha)$ is 2-dimensional by Lemma 10.34 we deduce that $\check{\mathbf{V}}$ induces an isomorphism.

Finally, since $\text{Ext}_{\Omega(k)}^1(\mathbf{T}_\alpha, \mathbf{T}_1)$ is 1-dimensional, it is enough to produce an extension $0 \rightarrow \text{Ind}_p^G \alpha \rightarrow \Pi \rightarrow \text{Sp} \rightarrow 0$, such that $0 \rightarrow \mathbf{1} \rightarrow \mathbf{V}(\Pi) \rightarrow \omega \rightarrow 0$ is non-split. We know that $\text{Ext}_{G/\mathbb{Z}}^1(\text{Sp}, \text{Ind}_p^G \alpha)$ is 1-dimensional, see [56, 11.5 (ii)]. Let $0 \rightarrow \text{Ind}_p^G \alpha \rightarrow \Pi \rightarrow \text{Sp} \rightarrow 0$ be a non-split extension. Applying Ord_p to it gives an isomorphism $\mathbf{R}^1 \text{Ord}_p \Pi \cong \mathbf{R}^1 \text{Ord}_p \text{Sp} = 0$. It follows from [31, 3.3.1] that the space of U-coinvariants of Π is zero. Since the space of U-coinvariants of $\text{Ind}_p^G \alpha$ is 1-dimensional, we deduce from [23, VII.1.8] that the space of $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p^{ab})$ -invariants of $\mathbf{V}(\Pi)$ is 1-dimensional, where \mathbf{Q}_p^{ab} is the maximal abelian extension of \mathbf{Q}_p . Hence, $\mathbf{V}(\Pi)$ can not be split. \square

Let $\mathfrak{B} = \{\mathbf{1}, \text{Sp}^\vee, (\text{Ind}_p^G \alpha)^\vee\}$ be the block of the trivial representation. Let $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ be the full subcategory of $\mathfrak{C}(\mathcal{O})$ consisting of all M whose irreducible subquotients lie in \mathfrak{B} . It follows from 5.5 that $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ is abelian and $\mathfrak{C}(\mathcal{O}) \cong \mathfrak{C}(\mathcal{O})^{\mathfrak{B}} \oplus \mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, where $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ is the full subcategory of $\mathfrak{C}(\mathcal{O})$ consisting of those M which no irreducible subquotient lies in \mathfrak{B} . Since $\mathfrak{T}(\mathcal{O})$ is contained in $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ we may build a quotient category $\Omega(\mathcal{O})^{\mathfrak{B}} := \mathfrak{C}(\mathcal{O})^{\mathfrak{B}}/\mathfrak{T}(\mathcal{O})$ and we have an isomorphism of categories $\Omega(\mathcal{O}) \cong \Omega(\mathcal{O})^{\mathfrak{B}} \oplus \mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$. Recall that $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ is the category of continuous representations of $\mathcal{G}_{\mathbf{Q}_p}$ on compact \mathcal{O} -modules. Let $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$ be the full subcategory of $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ with objects τ such that there exists M in $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$, such that $\tau \cong \check{\mathbf{V}}(M)$.

Proposition 10.36. — *The functor $\check{\mathbf{V}}$ induces an equivalence of categories between $\Omega(\mathcal{O})^{\mathfrak{B}}$ and $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$.*

Proof. — We note that since $\Omega(\mathcal{O})^{\mathfrak{B}}$ is a direct summand of $\Omega(\mathcal{O})$, for every object M of $\Omega(\mathcal{O})^{\mathfrak{B}}$ a projective envelope of M in $\Omega(\mathcal{O})$ lies in $\Omega(\mathcal{O})^{\mathfrak{B}}$. This implies that if M and N are objects of $\Omega(\mathcal{O})^{\mathfrak{B}}$ then $\text{Ext}_{\Omega(\mathcal{O})^{\mathfrak{B}}}^i(M, N) = \text{Ext}_{\Omega(\mathcal{O})}^i(M, N)$ for all $i \geq 0$. It is enough to show that for M and N objects of $\Omega(\mathcal{O})^{\mathfrak{B}}$, $\check{\mathbf{V}}$ induces a bijection

$$(206) \quad \text{Hom}_{\Omega(\mathcal{O})}(M, N) \xrightarrow{\cong} \text{Hom}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}(\check{\mathbf{V}}(M), \check{\mathbf{V}}(N)),$$

where $\text{Hom}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}$ means morphisms in the category $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$. We may write $M \cong \varprojlim M_i$ and $N \cong \varprojlim N_j$, where the limit is taken over all the quotients of finite length. Then $\check{\mathbf{V}}(M) \cong \varprojlim \check{\mathbf{V}}(M_i)$ and $\check{\mathbf{V}}(N) \cong \varprojlim \check{\mathbf{V}}(N_j)$, where $\check{\mathbf{V}}(M_i)$ and $\check{\mathbf{V}}(N_j)$ are of finite length. Now

$$(207) \quad \text{Hom}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}(\check{\mathbf{V}}(M), \check{\mathbf{V}}(N)) \cong \varprojlim \text{Hom}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}(\check{\mathbf{V}}(M_i), \check{\mathbf{V}}(N_j)).$$

The kernels of $\check{\mathbf{V}}(\mathbf{M}) \rightarrow \check{\mathbf{V}}(\mathbf{M}_i) \rightarrow \check{\mathbf{V}}(\mathbf{N}_j)$ form a basis of open neighbourhoods of 0 in $\check{\mathbf{V}}(\mathbf{M})$. Since $\check{\mathbf{V}}(\mathbf{N}_j)$ is of finite length it carries the discrete topology and hence every $\phi : \check{\mathbf{V}}(\mathbf{M}) \rightarrow \check{\mathbf{V}}(\mathbf{N}_j)$ in $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}(\mathcal{O})$ factors through $\check{\mathbf{V}}(\mathbf{M}_i) \rightarrow \check{\mathbf{V}}(\mathbf{N}_j)$ for some i . We obtain:

$$(208) \quad \text{Hom}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}(\check{\mathbf{V}}(\mathbf{M}), \check{\mathbf{V}}(\mathbf{N}_j)) \cong \varinjlim \text{Hom}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}(\check{\mathbf{V}}(\mathbf{M}_i), \check{\mathbf{V}}(\mathbf{N}_j)).$$

Since (207) and (208) also hold for \mathbf{M} and \mathbf{N} in $\Omega(\mathcal{O})$, it is enough to verify (206) when \mathbf{M} and \mathbf{N} are of finite length.

One may show that (206) holds, when \mathbf{M} and \mathbf{N} are of finite length, by proving a stronger statement: (206) holds and $\check{\mathbf{V}}$ induces an injection

$$(209) \quad \text{Ext}_{\Omega(\mathcal{O})}^1(\mathbf{M}, \mathbf{N}) \hookrightarrow \text{Ext}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}^1(\check{\mathbf{V}}(\mathbf{M}), \check{\mathbf{V}}(\mathbf{N})).$$

The proof is by induction on $\ell(\mathbf{M}) + \ell(\mathbf{N})$, where ℓ denotes the number of irreducible subquotients, see the proof of Lemma A.1 in [56]. Since the only irreducible objects in $\Omega(\mathcal{O})^{\mathfrak{B}}$ are T_α and T_1 the initial induction step follows from Lemma 10.35. \square

Corollary 10.37. — *The category $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$ is abelian.*

Lemma 10.38. — *If \mathbf{M} is an object of $\Omega(\mathcal{O})^{\mathfrak{B}}$ then the action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\check{\mathbf{V}}(\mathbf{M})$ factors through $\text{Gal}(\mathbf{F}(p)|\mathbf{Q}_p)$, where $\mathbf{F} = \mathbf{Q}_p(\mu_p)$ and $\mathbf{F}(p)$ denotes the maximal pro- p extension of \mathbf{F} .*

Proof. — If \mathbf{M} is irreducible then $\mathbf{M} \cong T_1$ or $\mathbf{M} \cong T_\alpha$, and it follows from (205) that $\text{Gal}(\overline{\mathbf{Q}_p}|\mathbf{F})$ acts trivially on $\check{\mathbf{V}}(\mathbf{M})$. If \mathbf{M} is of finite length then the cosocle filtration on \mathbf{M} induces a filtration of $\check{\mathbf{V}}(\mathbf{M})$ such that $\text{Gal}(\overline{\mathbf{Q}_p}|\mathbf{F})$ acts trivially on the graded pieces. This implies that the image of $\text{Gal}(\overline{\mathbf{Q}_p}|\mathbf{F})$ in $\text{Aut}_{\mathcal{O}}(\check{\mathbf{V}}(\mathbf{M}))$ is a p -group, and hence $\text{Gal}(\overline{\mathbf{Q}_p}|\mathbf{F}(p))$ acts trivially on $\check{\mathbf{V}}(\mathbf{M})$. The general case may be deduced from this by taking projective limits, as in the proof of Proposition 10.36. \square

Remark 10.39. — Lemma 10.38 allows us to consider $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$ as the full subcategory of $\text{Mod}_{\text{Gal}(\mathbf{F}(p)|\mathbf{Q}_p)}^{\text{pro-} \text{aug}}(\mathcal{O})$. Since $\text{Gal}(\mathbf{F}(p)|\mathbf{F})$ is an open pro- p subgroup of $\text{Gal}(\mathbf{F}(p)|\mathbf{Q}_p)$, this enables us to apply the results of Section 3 with $\mathfrak{C} = \text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$.

Let $\tilde{\mathbf{P}} \rightarrow (\text{Ind}_{\mathbf{p}}^{\mathbf{G}} \alpha)^{\vee}$ be a projective envelope of $(\text{Ind}_{\mathbf{p}}^{\mathbf{G}} \alpha)^{\vee}$ in $\mathfrak{C}(\mathcal{O})$. Then $\mathcal{T}\tilde{\mathbf{P}} \rightarrow T_\alpha$ is a projective envelope of T_α in $\Omega(\mathcal{O})$ by Lemma 10.27 and hence $\check{\mathbf{V}}(\tilde{\mathbf{P}}) \rightarrow \omega$ is a projective envelope of ω in $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$. Let

$$\tilde{\mathbf{E}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}) \xrightarrow{\mathcal{T}} \text{End}_{\Omega(\mathcal{O})}(\mathcal{T}\tilde{\mathbf{P}}) \xrightarrow{\check{\mathbf{V}}} \text{End}_{\mathcal{O}[\mathcal{G}_{\mathbf{Q}_p}]}^{\text{cont}}(\check{\mathbf{V}}(\tilde{\mathbf{P}})),$$

where the first isomorphism is given by Lemma 10.27 and the second by Proposition 10.36.

Corollary 10.40. — *The hypotheses (H0)–(H5) hold in $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$ with $\mathbf{S} = \check{\mathbf{V}}(\mathbf{T}_\alpha) \cong \omega$ and $\mathbf{Q} = \check{\mathbf{V}}(\mathcal{T}\tau_2^\vee) \cong \check{\mathbf{V}}(\tau_2^\vee)$, which is uniquely determined up to isomorphism by the non-split extension*

$$(210) \quad 0 \rightarrow \mathbf{1} \rightarrow \check{\mathbf{V}}(\tau_2^\vee) \rightarrow \omega \rightarrow 0.$$

Proof. — Since $\tilde{\mathbf{P}}$ is \mathcal{O} -torsion free by Corollary 5.19 and $\check{\mathbf{V}}$ is exact and \mathcal{O} -linear, we deduce that the sequence $0 \rightarrow \check{\mathbf{V}}(\tilde{\mathbf{P}}) \xrightarrow{\varpi} \check{\mathbf{V}}(\tilde{\mathbf{P}}) \rightarrow \check{\mathbf{V}}(\tilde{\mathbf{P}}/\varpi\tilde{\mathbf{P}}) \rightarrow 0$ is exact. Hence, $\check{\mathbf{V}}(\tilde{\mathbf{P}})$ is \mathcal{O} -torsion free and so (H0) holds in $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$. The equivalence of categories established in Proposition 10.36 and the Ext- calculations made in Proposition 10.31 show that (H1)–(H5) hold in $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(k)$, and hence in $\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathfrak{B}}(\mathcal{O})$ by Proposition 3.17. \square

Corollary 10.41. — *The functor $m \mapsto m \widehat{\otimes}_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}})$ is exact.*

Proof. — Since the hypotheses are satisfied by Corollary 10.40, the assertion follows from the Corollary 3.12. \square

Lemma 10.42. — *For a compact right $\tilde{\mathbf{E}}$ -module m , we let $m \widehat{\otimes}_{\tilde{\mathbf{E}}} \mathcal{T}\tilde{\mathbf{P}}$ be an object of $\mathfrak{Q}(\mathcal{O})^{\mathfrak{B}}$ corresponding to $m \widehat{\otimes}_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}})$ under the equivalence of categories induced by $\check{\mathbf{V}}$, see Proposition 10.36. Then $\mathcal{T}(m \widehat{\otimes}_{\tilde{\mathbf{E}}} \tilde{\mathbf{P}}) \cong m \widehat{\otimes}_{\tilde{\mathbf{E}}} \mathcal{T}\tilde{\mathbf{P}}$.*

Proof. — This follows from Lemma 5.53 and the fact that $\check{\mathbf{V}}$ factors through \mathcal{T} . \square

Corollary 10.43. — *$\text{SL}_2(\mathbf{Q}_p)$ acts trivially on $\widehat{\text{Tor}}_{\tilde{\mathbf{E}}}^1(k, \tilde{\mathbf{P}})$.*

Proof. — It follows from Corollary 10.41 and Lemma 10.42 that $\mathcal{T}(\widehat{\text{Tor}}_{\tilde{\mathbf{E}}}^1(k, \tilde{\mathbf{P}})) = 0$, which implies the assertion. \square

Definition 10.44. — *Let \mathbf{R} be the universal deformation ring of $\check{\mathbf{V}}(\tau_2^\vee)$ and let \mathbf{R}^ψ be the deformation ring parameterizing deformations of $\check{\mathbf{V}}(\tau_2^\vee)$ with determinant equal to the cyclotomic character. Here we consider the usual deformations with commutative coefficients.*

In the Appendix B we have recalled a construction of an explicit presentation of \mathbf{R} and \mathbf{R}^ψ due to Böckle, [9].

Proposition 10.45. — *The functor $\check{\mathbf{V}}$ induces a surjection $\varphi : \tilde{\mathbf{E}} \twoheadrightarrow \mathbf{R}^\psi$.*

Proof. — The intersection of maximal ideals of $\mathbf{R}^\psi[1/p]$ corresponding to the irreducible representations is zero by Lemma B.10. Moreover, it follows from Corollary B.5 that \mathbf{R}^ψ is \mathcal{O} -torsion free. Hence, the ring denoted by \mathbf{R}' in the statement of the Proposition 5.56 is equal to \mathbf{R}^ψ . We will prove the assertion by modifying the proof of Proposition 5.56.

We note that $\check{\mathbf{V}}(\tau_2^\vee) \cong k \widehat{\otimes}_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}}) \cong \check{\mathbf{V}}(k \widehat{\otimes}_{\tilde{\mathbf{E}}} \tilde{\mathbf{P}})$, and has only scalar endomorphisms. Since $\check{\mathbf{V}}(\tilde{\mathbf{P}})$ is $\tilde{\mathbf{E}}$ -flat by Corollary 10.41, $\tilde{\mathbf{E}}^{ab} \widehat{\otimes}_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}})$ is a deformation of $\check{\mathbf{V}}(\tau_2^\vee)$ to $\tilde{\mathbf{E}}^{ab}$. Thus we obtain a natural map $\varphi : \mathbf{R} \rightarrow \tilde{\mathbf{E}}^{ab}$, where \mathbf{R} is the universal deformation ring of $\check{\mathbf{V}}(\tau_2^\vee)$. To show the surjectivity of φ it is enough to show that it induces a surjection on tangent spaces, which is equivalent to showing that the natural map $\text{Ext}_{\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}^{\mathfrak{B}}}(k)}^1(\check{\mathbf{V}}(\tau_2^\vee), \check{\mathbf{V}}(\tau_2^\vee)) \rightarrow \text{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\check{\mathbf{V}}(\tau_2^\vee), \check{\mathbf{V}}(\tau_2^\vee))$ is injective. This assertion follows from (209). Hence, $\varphi : \mathbf{R} \rightarrow \tilde{\mathbf{E}}^{ab}$ is surjective. We proceed as in the proof of Proposition 5.56 to show that every closed point in $\mathfrak{m} \in \text{Spec } \mathbf{R}^\psi[1/\rho]$, corresponding to an irreducible representation lies in $\text{Spec } \tilde{\mathbf{E}}^{ab}$. This implies that $\mathbf{R} \rightarrow \mathbf{R}^\psi$ factors through φ . As explained in the proof of Proposition 5.56 it is enough to produce a map of \mathcal{O} -algebras $x : \tilde{\mathbf{E}} \rightarrow \kappa(\mathfrak{m})$, such that $\kappa(\mathfrak{m}) \otimes_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}})$ is isomorphic to $\kappa(\mathfrak{m}) \otimes_{\tilde{\mathbf{E}}} \rho^{un,\psi}$, where $\rho^{un,\psi}$ is the universal deformation with determinant ψ . It follows from [41, 2.3.8] that part (iii) of Proposition 5.56 holds, and then the argument in the proof of Proposition 5.56 allows us to conclude. \square

Corollary 10.46. — *Let $\tilde{\mathfrak{m}}$ be the maximal ideal of $\tilde{\mathbf{E}}$ then $\dim \tilde{\mathfrak{m}}/(\tilde{\mathfrak{m}}^2 + \varpi \tilde{\mathbf{E}}) = 4$.*

Proof. — It follows from Corollary B.5 that the tangent space of \mathbf{R}^ψ is 4-dimensional. Hence, Proposition 10.45 implies that the tangent space of $\tilde{\mathbf{E}}$ is at least 4-dimensional. By Proposition 10.36 and since (H1) and (H3) hold in $\Omega(k)$ we have:

$$\text{Ext}_{\text{Rep}_{\mathcal{G}_{\mathbf{Q}_p}^{\mathfrak{B}}}(k)}^1(\check{\mathbf{V}}(\tau_2^\vee), \check{\mathbf{V}}(\tau_2^\vee)) \cong \text{Ext}_{\Omega(k)}^1(\mathcal{T}\tau_2^\vee, \mathcal{T}\tau_2^\vee) \cong \text{Ext}_{\Omega(k)}^1(\mathcal{T}\tau_2^\vee, \mathbf{T}\alpha)$$

and is of dimension at most 4 by (203). Hence it follows from Lemma 3.29 that the tangent space of $\tilde{\mathbf{E}}$ is at most 4-dimensional. \square

Let $\tilde{\mathbf{P}}_{\alpha^\vee}$ be a projective envelope of α^\vee in $\mathfrak{C}_{\mathbf{T}/Z}(\mathcal{O})$ and $\tilde{\mathbf{M}} = (\text{Ind}_{\mathbf{P}}^{\mathbf{G}}(\tilde{\mathbf{P}}_{\alpha^\vee})^\vee)^\vee$. All the irreducible subquotients of $\tilde{\mathbf{M}}$ are isomorphic to $(\text{Ind}_{\mathbf{P}}^{\mathbf{G}} \alpha)^\vee$ and hence \mathcal{T} induces an isomorphism $\text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{M}}) \cong \text{End}_{\Omega(\mathcal{O})}(\mathcal{T}\tilde{\mathbf{M}})$ by Lemma 10.26. Thus it follows from Proposition 7.2 that we have a natural surjection

$$(211) \quad \tilde{\mathbf{E}} \cong \text{End}_{\Omega(\mathcal{O})}(\mathcal{T}\tilde{\mathbf{P}}) \twoheadrightarrow \text{End}_{\Omega(\mathcal{O})}(\mathcal{T}\tilde{\mathbf{M}}) \cong \mathcal{O}[[x, y]].$$

We let $\tilde{\mathfrak{a}}$ be the kernel of (211), then $\tilde{\mathfrak{a}}$ is also the kernel of $\text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}) \twoheadrightarrow \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{M}})$.

Proposition 10.47. — *The image of $\tilde{\mathfrak{a}}$ in \mathbf{R}^ψ is equal to $\mathfrak{r} := \mathbf{R}^\psi \cap \bigcap_x \mathfrak{m}_x$ where the intersection is taken over all maximal ideals of $\mathbf{R}^\psi[1/\rho]$ such that the corresponding representation ρ_x is reducible. Moreover, $\tilde{\mathbf{E}}/\tilde{\mathfrak{a}} \cong \mathbf{R}^\psi/\varphi(\tilde{\mathfrak{a}})$.*

Proof. — Since we know that $\mathbf{R}^\psi/\mathfrak{r} \cong \mathcal{O}[[x, y]]$ by Corollary B.6, $\tilde{\mathbf{E}}/\tilde{\mathfrak{a}} \cong \mathcal{O}[[x, y]]$ by (211), and φ is surjective, it is enough to show that \mathfrak{r} contains $\varphi(\tilde{\mathfrak{a}})$.

Let x be a maximal ideal of $\mathbf{R}^\psi[1/p]$ with residue field L and let ρ_x be the corresponding representation. Suppose that ρ_x is reducible then since $\det \rho_x = \varepsilon$ we have an exact sequence $0 \rightarrow \delta^{-1} \rightarrow \rho_x \rightarrow \delta\varepsilon \rightarrow 0$, where $\delta : \mathcal{G}_{\mathbf{Q}_p} \rightarrow L^\times$ is a continuous character, lifting the trivial character $\mathbf{1} : \mathcal{G}_{\mathbf{Q}_p} \rightarrow k^\times$.

Let $\chi : T \rightarrow L^\times$ be the character $\chi := \delta\varepsilon \otimes \delta^{-1}\varepsilon^{-1}$. Then χ is trivial on Z and is a deformation of $\alpha : T \rightarrow k^\times$ and hence defines a maximal ideal $y : \tilde{E} \rightarrow \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{M}) \rightarrow L$, such that $\text{Hom}_{\mathcal{O}}^{\text{cont}}(\mathcal{O} \otimes_{\tilde{E},y} \tilde{M}, L) \cong (\text{Ind}_p^G \chi)_{\text{cont}}$. It follows from the construction of Colmez's functor that $\mathbf{V}((\text{Ind}_p^G \chi)_{\text{cont}}) \cong \delta^{-1}$ and hence $L \otimes_{\tilde{E},y} \check{\mathbf{V}}(\tilde{M}) \cong \check{\mathbf{V}}(\mathcal{O} \otimes_{\tilde{E},y} \tilde{M})_L \cong \delta\varepsilon$. Since $\check{\mathbf{V}}(\tilde{P})$ is a free \tilde{E} -module of rank 2 by Corollary 5.55, $L \otimes_{\tilde{E},y} \check{\mathbf{V}}(\tilde{P})$ is a 2-dimensional L -representation of $\mathcal{G}_{\mathbf{Q}_p}$ lifting $k \otimes_{\tilde{E}} \check{\mathbf{V}}(\tilde{P}) \cong \check{\mathbf{V}}(\tau_2^\vee)$. Moreover, we know that $L \otimes_{\tilde{E},y} \check{\mathbf{V}}(\tilde{P})$ admits $L \otimes_{\tilde{E},y} \check{\mathbf{V}}(\tilde{M}) \cong \delta\varepsilon$ as a quotient. Lemma B.9 implies that $\rho_x \cong L \otimes_{\tilde{E},y} \check{\mathbf{V}}(\tilde{P})$, which implies that $y = \varphi^{-1}(x)$. Since by construction y contains $\tilde{\mathfrak{a}}$, we deduce that x contains $\varphi(\tilde{\mathfrak{a}})$. Hence, $\varphi(\tilde{\mathfrak{a}})$ is contained in $\mathbf{R}^\psi \cap \bigcap_x \mathfrak{m}_x$ where the intersection is taken over all maximal ideals of $\mathbf{R}^\psi[1/p]$ with residue field L such that the corresponding representation ρ_x is reducible. Remark B.8 implies that this ideal is equal to \mathfrak{r} . \square

10.4. Filtration by ordinary parts. — Let P be a projective envelope of $(\text{Ind}_p^G \alpha)^\vee$ in $\mathfrak{C}(k)$ and let $E = \text{End}_{\mathfrak{C}(k)}(P)$. Recall that uniqueness of projective envelopes implies the existence of an isomorphism $P \cong \tilde{P} \otimes_{\mathcal{O}} k$, and hence $E \cong \tilde{E} \otimes_{\mathcal{O}} k$. Moreover, \mathcal{TP} is projective in $\mathfrak{Q}(k)$ and $\text{End}_{\mathfrak{Q}(k)}(\mathcal{TP}) \cong E$ by Proposition 10.27. Since $\tilde{E}/\tilde{\mathfrak{a}} \cong \mathcal{O}[[x, y]]$ is \mathcal{O} -torsion free, we have an injection $\tilde{\mathfrak{a}} \otimes_{\mathcal{O}} k \hookrightarrow E$, and we denote the ideal $\tilde{\mathfrak{a}} \otimes_{\mathcal{O}} k$ by \mathfrak{a} . We are going to show that φ , defined in Proposition 10.45, induces an isomorphism $\mathfrak{a}^n/\mathfrak{a}^{n+1} \xrightarrow{\cong} \varphi(\mathfrak{a})^n/\varphi(\mathfrak{a})^{n+1}$, for all $n \geq 1$. Using this we will show in Theorem 10.71 that φ is an isomorphism.

Lemma 10.48. — *Let κ be an object of $\text{Mod}_{\mathbf{T}/Z}^{\text{ladm}}(k)$, let θ be a subspace of $(\text{Ind}_p^G \kappa)^G$ and let τ be the quotient:*

$$(212) \quad 0 \rightarrow \theta \rightarrow \text{Ind}_p^G \kappa \rightarrow \tau \rightarrow 0.$$

Then $\text{Ord}_p \tau \cong \kappa^s$, $\mathbf{R}^1 \text{Ord}_p \tau \cong (\kappa/\theta) \otimes \alpha^{-1}$, where we have identified θ with the subspace of κ^T by evaluating at 1.

Proof. — We note that evaluation at 1 induces an isomorphism $(\text{Ind}_p^G \kappa)^G \cong \kappa^T$, which allows us to identify θ with a subspace of κ^T . It follows from [31, 4.1.1] that $\text{Ord}_p \theta = 0$ and from the proof of [31, 4.1.2] that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{R}^1 \text{Ord}_p \theta & \longrightarrow & \mathbf{R}^1 \text{Ord}_p \text{Ind}_p^G \kappa \\ \downarrow \cong & & \downarrow \cong \\ \theta \otimes \alpha^{-1} & \hookrightarrow & \kappa \otimes \alpha^{-1} \end{array}$$

Hence, $\text{Ord}_p \tau \cong \text{Ord}_p \text{Ind}_p^G \kappa \cong \kappa^s$ and $\mathbf{R}^1 \text{Ord}_p \tau \cong (\kappa/\theta) \otimes \alpha^{-1}$. \square

Lemma 10.49. — *Let τ be in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ such that $\text{Hom}_G(\pi', \tau) = 0$ for all irreducible π' not isomorphic to $\mathbf{1}$, Sp or $\text{Ind}_p^G \alpha$. Then G acts trivially on the kernel of the natural map $\text{Ind}_p^G \text{Ord}_p \tau \rightarrow \tau$.*

Proof. — We denote the kernel by \mathbf{K} . By construction we have $\text{Ord}_p \mathbf{K} = 0$. If χ is irreducible in $\text{Mod}_{T/Z}^{\text{ladm}}(k)$ and $\chi \neq \mathbf{1}$, $\chi \neq \alpha^{-1}$, then $\text{Hom}_T(\chi, \text{Ord}_p \tau) \cong \text{Hom}_G(\text{Ind}_p^G \chi, \tau) = 0$, the first quality holding by the adjointness property of Ord_p , and the second by our assumption on τ together with Corollary 5.47. Since there are no extensions between α^{-1} and $\mathbf{1}$ in $\text{Mod}_{T/Z}^{\text{ladm}}(k)$, we deduce that $\text{Ord}_p \tau \cong \kappa_{\mathbf{1}} \oplus \kappa_{\alpha^{-1}}$, where all the irreducible subquotients of $\kappa_{\mathbf{1}}$ are isomorphic to $\mathbf{1}$ and all the irreducible subquotients of $\kappa_{\alpha^{-1}}$ are isomorphic to α^{-1} . Hence, there are no non-zero homomorphisms between $\text{Ind}_p^G \kappa_{\mathbf{1}}$ and $\text{Ind}_p^G \kappa_{\alpha^{-1}}$ and so we may write $\mathbf{K} = \mathbf{K}_{\mathbf{1}} \oplus \mathbf{K}_{\alpha^{-1}}$ where all the irreducible subquotients of $\mathbf{K}_{\mathbf{1}}$ are $\mathbf{1}$ or Sp and all the irreducible subquotients of $\mathbf{K}_{\alpha^{-1}}$ are $\text{Ind}_p^G \alpha^{-1}$. Since $\text{Ord}_p \mathbf{K} = 0$ we get that $\mathbf{K}_{\alpha^{-1}} = 0$. Now $(\mathbf{K}/\mathbf{K}^G)^G = 0$ by Lemma 10.25. Hence, if $\mathbf{K} \neq \mathbf{K}^G$ then we must have $\text{Hom}_G(\text{Sp}, \mathbf{K}/\mathbf{K}^G) \neq 0$. However, this implies that \mathbf{K} contains Sp or $\text{Ind}_p^G \mathbf{1}$ as a subobject, which contradicts $\text{Ord}_p \mathbf{K} = 0$. \square

Lemma 10.50. — *Let \mathbf{J} be an injective object in $\text{Mod}_{T/Z}^{\text{ladm}}(k)$ and τ an object of $\text{Mod}_{G/Z}^{\text{ladm}}(k)$. If $\text{Hom}_T(\text{Ord}_p \tau, \mathbf{J} \otimes \alpha^{-1}) = 0$ then $\text{Ext}_{G/Z}^1(\tau, \text{Ind}_p^G \mathbf{J}) = 0$.*

Proof. — Since by (126) and assumption $\text{Hom}_T(\text{Ord}_p \tau, \mathbf{R}^1 \text{Ord}_p \text{Ind}_p^G \mathbf{J}) = 0$, by applying Ord_p to the extension $0 \rightarrow \text{Ind}_p^G \mathbf{J} \rightarrow \kappa \rightarrow \tau \rightarrow 0$ we obtain an injection $\mathbf{J} \otimes \alpha^{-1} \hookrightarrow \mathbf{R}^1 \text{Ord}_p \kappa$. Since \mathbf{J} is injective the injection splits. As $G = \text{GL}_2(\mathbf{Q}_p)$ we have $\mathbf{R}^1 \text{Ord}_p \kappa \cong \kappa_U \otimes \alpha$, where subscript U denotes the coinvariants by the unipotent radical of P , [31, 3.6.2]. Thus $\kappa_U \cong (\text{Ind}_p^G \mathbf{J})_U \oplus \tau_U \cong \mathbf{J} \oplus \tau_U$. Since $\text{Hom}_G(\kappa, \text{Ind}_p^G \mathbf{J}) \cong \text{Hom}_T(\kappa_U, \mathbf{J})$ we obtain a splitting. \square

On every τ in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ we define an increasing filtration τ^\bullet by subobjects uniquely determined by (1) $\tau^0 = 0$ and (2) $\text{gr}^{i+1} \tau := \tau^{i+1}/\tau^i$ is the image of $\text{Ind}_p^G \text{Ord}_p(\tau/\tau^i) \rightarrow \tau/\tau^i$. Dually on every M in $\mathfrak{C}(k)$ we define a decreasing filtration M^\bullet by subobjects $M^0 = M$ and M^i be the kernel of $M \rightarrow ((M^\vee)^i)^\vee$ and let $\text{gr}^i M := M^i/M^{i+1}$.

Lemma 10.51. — *The filtration is functorial: for every $\phi : \tau \rightarrow \kappa$ in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ we have $\phi(\tau^i) \subseteq \kappa^i$ and for every $\psi : M \rightarrow N$ in $\mathfrak{C}(k)$ we have $\psi(M^i) \subseteq N^i$, for all $i \geq 0$.*

Proof. — Trivially $\phi(\tau^0) \subseteq \kappa^0$. Suppose $\phi(\tau^i) \subseteq \kappa^i$ then we get a map $\phi : \tau/\tau^i \rightarrow \kappa/\kappa^i$. The natural transformation $\text{Ind}_p^G \text{Ord}_p \rightarrow \text{id}$ induces a map $\text{gr}^{i+1} \tau \rightarrow \text{gr}^{i+1} \kappa$ and hence $\phi(\tau^{i+1}) \subseteq \kappa^{i+1}$. \square

Lemma 10.52. — Let \mathbf{J} be an injective object in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$. Then for $i \geq 1$ we have:

$$(213) \quad \text{Ord}_p \text{gr}^{i+1} \mathbf{J} \cong \text{Ord}_p \mathbf{J}/\mathbf{J}^i \cong \mathbf{R}^1 \text{Ord}_p \text{gr}^i \mathbf{J}.$$

Proof. — From $\text{Ind}_p^G \text{Ord}_p(\mathbf{J}/\mathbf{J}^{i-1}) \rightarrow \text{gr}^i \mathbf{J} \hookrightarrow \mathbf{J}/\mathbf{J}^{i-1}$ and left exactness of Ord_p we deduce that $\text{Ord}_p \text{gr}^i \mathbf{J} \xrightarrow{\cong} \text{Ord}_p \mathbf{J}/\mathbf{J}^{i-1}$, for all $i \geq 1$. This gives the first isomorphism in (213). Since \mathbf{J} is injective we have $\mathbf{R}^1 \text{Ord}_p \mathbf{J} = 0$ and since $\mathbf{R}^2 \text{Ord}_p = 0$, we get $\mathbf{R}^1 \text{Ord}_p \mathbf{J}/\mathbf{J}^i = 0$, for all $i \geq 1$. Thus applying Ord_p to $0 \rightarrow \text{gr}^i \mathbf{J} \rightarrow \mathbf{J}/\mathbf{J}^{i-1} \rightarrow \mathbf{J}/\mathbf{J}^i \rightarrow 0$ we get the isomorphism $\text{Ord}_p \mathbf{J}/\mathbf{J}^i \cong \mathbf{R}^1 \text{Ord}_p \text{gr}^i \mathbf{J}$, for all $i \geq 1$. \square

Lemma 10.53. — $\mathbf{J}_{\mathbf{1}_G}^1 \cong \text{Ind}_p^G \mathbf{J}_{\mathbf{1}_T}, \mathbf{J}_{\pi_\alpha}^1 \cong \text{Ind}_p^G \mathbf{J}_\alpha, \mathbf{J}_{\text{Sp}}^1 \cong (\text{Ind}_p^G \mathbf{J}_{\mathbf{1}_T})/\mathbf{1}_G$, where $\mathbf{J}_{\mathbf{1}_T}$ and \mathbf{J}_α denote injective envelopes of $\mathbf{1}_T$ and α in $\text{Mod}_{T/Z}^{\text{ladm}}(k)$.

Proof. — The first two isomorphisms follow from the Propositions 10.17 and 7.1(i). Applying Ord_p to (189) gives us an isomorphism $\text{Ord}_p(\mathbf{J}_{\text{Sp}}) \cong \mathbf{R}^1 \text{Ord}_p(\text{Ind}_p^G \mathbf{J}_\alpha) \cong \mathbf{J}_\alpha \otimes \alpha^{-1} \cong \mathbf{J}_{\mathbf{1}_T}$. The last isomorphism follows from (190). \square

Lemma 10.54. — For all $i \geq 0$ we have exact sequences $0 \rightarrow \mathbf{J}_{\mathbf{1}_G}^1 \rightarrow \mathbf{J}_{\mathbf{1}_G}^{i+1} \rightarrow \mathbf{J}_{\pi_\alpha}^i \rightarrow 0$, $0 \rightarrow \mathbf{J}_{\pi_\alpha}^1 \rightarrow \mathbf{J}_{\pi_\alpha}^{i+1} \rightarrow \mathbf{J}_{\text{Sp}}^i \rightarrow 0$, $0 \rightarrow \mathbf{J}_{\mathbf{1}_G}^2 \rightarrow \mathbf{J}_{\mathbf{1}_G}^{i+2} \rightarrow \mathbf{J}_{\text{Sp}}^i \rightarrow 0$.

Proof. — By construction of the filtration for each τ in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$ and $i, j \geq 0$ we have an isomorphism $(\tau/\tau^i)^j \cong \tau^{i+j}/\tau^i$. We apply this observation to (188) and (189). \square

Lemma 10.55. — $\text{gr}^2 \mathbf{J}_{\text{Sp}} \cong \text{Ind}_p^G(\mathbf{J}_\alpha/\alpha), e^1(\mathbf{1}_G, \mathbf{J}_{\text{Sp}}^2) = 0$.

Proof. — It follows from (213) and Lemma 10.48 that $\text{Ord}_p \text{gr}^2 \mathbf{J}_{\text{Sp}} \cong \mathbf{J}_{\alpha^{-1}}/\alpha^{-1}$ and hence there is a surjection $\text{Ind}_p^G(\mathbf{J}_\alpha/\alpha) \rightarrow \text{gr}^2 \mathbf{J}_{\text{Sp}}$. The injectivity of this map follows from Lemma 10.49 and (190). Corollary 10.8 implies that $e^1(\mathbf{1}_G, \text{gr}^2 \mathbf{J}_{\text{Sp}}) = 0$. It follows from (190) that $e^1(\mathbf{1}_G, \mathbf{J}_{\text{Sp}}^1) = 0$. Hence, $e^1(\mathbf{1}_G, \mathbf{J}_{\text{Sp}}^2) = 0$. \square

Lemma 10.56. — Let $0 \rightarrow \mathbf{1} \rightarrow \kappa \rightarrow \pi_\alpha \rightarrow 0$ be a non-split extension. There exists an exact sequence $0 \rightarrow \kappa \rightarrow \mathbf{J}_{\mathbf{1}_G}^2 \rightarrow \mathbf{J}_{\text{Sp}}^2 \rightarrow 0$.

Proof. — We recall from Section 10.2 that κ is uniquely determined up to isomorphism, since $e^1(\pi_\alpha, \mathbf{1}) = 1$. Since $\text{soc}_G \kappa \cong \mathbf{1}$ there exists an injection $\iota : \kappa \hookrightarrow \mathbf{J}_{\mathbf{1}_G}$. Since $\mathbf{1}$ occurs as a subquotient of κ with multiplicity one, $\text{Hom}_G(\kappa, \mathbf{J}_{\mathbf{1}_G})$ is 1-dimensional and so the image of ι does not depend on the choice of ι . Since $\mathbf{1} \hookrightarrow \mathbf{J}_{\mathbf{1}_G}$ is essential $\mathbf{J}_{\mathbf{1}_G}^1 \cap \kappa \neq 0$. Since $\mathbf{J}_{\mathbf{1}_G}^1 \cong \text{Ind}_p^G \mathbf{J}_{\mathbf{1}_T}$ by Lemma 10.53, it does not contain π_α as a subquotient and we deduce that $\mathbf{J}_{\mathbf{1}_G}^1 \cap \kappa = \mathbf{1}$ and hence we have an injection $\pi_\alpha \cong \kappa/\mathbf{1} \hookrightarrow \mathbf{J}_{\mathbf{1}_G}/\mathbf{J}_{\mathbf{1}_G}^1 \cong \mathbf{J}_{\pi_\alpha}$. Hence, κ is contained in $\mathbf{J}_{\mathbf{1}_G}^2$ and since $\mathbf{J}_{\mathbf{1}_G}^2/\mathbf{J}_{\mathbf{1}_G}^1 \cong \mathbf{J}_{\pi_\alpha}^1 \cong \text{Ind}_p^G \mathbf{J}_\alpha$ by Lemmas 10.54 and 10.53, we obtain an exact sequence

$$(214) \quad 0 \rightarrow \text{Ind}_p^G \mathbf{J}_{\mathbf{1}_T}/\mathbf{1}_G \rightarrow \mathbf{J}_{\mathbf{1}_G}^2/\kappa \rightarrow \text{Ind}_p^G(\mathbf{J}_\alpha/\alpha) \rightarrow 0.$$

Since J_{Sp} is injective there exist a map $J_{\mathbf{1}_G}^2/\kappa \rightarrow J_{\text{Sp}}$ extending the injection $\text{Ind}_p^G J_{\mathbf{1}_T}/\mathbf{1}_G \cong J_{\text{Sp}}^1 \hookrightarrow J_{\text{Sp}}$. Since $e^1(\pi_\alpha, \kappa) = 0$, see (194), and $\text{Hom}_G(\pi_\alpha, J_{\text{Sp}}^2) = 0$ we obtain $\text{Hom}_G(\pi_\alpha, J_{\mathbf{1}_G}^2/\kappa) = 0$ and it follows from (214) that $\text{soc}_G J_{\mathbf{1}_G}^2/\kappa \cong \text{soc}_G \text{Ind}_p^G J_{\mathbf{1}_T}/\mathbf{1}_G$, which implies that the map $J_{\mathbf{1}_G}^2/\kappa \rightarrow J_{\text{Sp}}$, constructed above, is an embedding, as it induces an isomorphism on G -socles. By applying Ord_p to (214) and using (213) we obtain isomorphisms

$$\text{Ord}_p((J_{\mathbf{1}_G}^2/\kappa)/J_{\text{Sp}}^1) \cong \mathbf{R}^1 \text{Ord}_p J_{\text{Sp}}^1 \cong \text{Ord}_p(J_{\text{Sp}}/J_{\text{Sp}}^1),$$

and so $J_{\mathbf{1}_G}^2/\kappa \cong J_{\text{Sp}}^2$. Thus (214) coincides with the tautological exact sequence $0 \rightarrow J_{\text{Sp}}^1 \rightarrow J_{\text{Sp}}^2 \rightarrow \text{gr}^2 J_{\text{Sp}} \rightarrow 0$. \square

Lemma 10.57. — $e^1(\pi(0, 1), J_{\text{Sp}}^2) = 1$.

Proof. — Combining (188) and (189) we obtain an exact sequence $0 \rightarrow J_{\mathbf{1}_G}^2 \rightarrow J_{\mathbf{1}_G} \rightarrow J_{\text{Sp}} \rightarrow 0$. Hence, $e^1(\pi(0, 1), J_{\mathbf{1}_G}^2) = 1$ and $e^2(\pi(0, 1), J_{\mathbf{1}_G}^2) = 0$. Since Sp occurs only once as a subquotient of $\pi(0, 1)$ we have $e^0(\pi(0, 1), J_{\text{Sp}}) = 1$. It follows from (190) and Lemma 10.53 that any map $\pi(0, 1) \rightarrow J_{\text{Sp}}$ has image lying in J_{Sp}^1 , and hence in J_{Sp}^2 . Thus $e^0(\pi(0, 1), J_{\text{Sp}}^2) = 1$. We apply $\text{Hom}_G(\pi(0, 1), *)$ to the exact sequence of Lemma 10.56. Since $e^1(\pi(0, 1), \kappa) = 2$, Lemma 10.24, we obtain an isomorphism $\text{Ext}_{G/Z}^1(\pi(0, 1), J_{\text{Sp}}^2) \cong \text{Ext}_{G/Z}^2(\pi(0, 1), \kappa)$. The assertion follows from Lemma 10.24. \square

It follows from Lemma 10.55 and (213) that $\text{Ord}_p \text{gr}^3 J_{\text{Sp}} \cong J_{\mathbf{1}_T}/\mathbf{1}_T$. Moreover, since $e^1(\mathbf{1}_G, J_{\text{Sp}}^2) = 0$ by Lemma 10.55, we have an exact sequence

$$(215) \quad 0 \rightarrow (\text{Ind}_p^G (J_{\mathbf{1}_T}/\mathbf{1}_T))^G \rightarrow \text{Ind}_p^G (J_{\mathbf{1}_T}/\mathbf{1}_T) \rightarrow \text{gr}^3 J_{\text{Sp}} \rightarrow 0.$$

In particular, $\text{Hom}_G(\mathbf{1}, \text{gr}^3 J_{\text{Sp}}) = \text{Hom}_G(\pi_\alpha, \text{gr}^3 J_{\text{Sp}}) = 0$ and

$$\text{Hom}_G(\text{Sp}, \text{gr}^3 J_{\text{Sp}}) \cong \text{Hom}_G(\text{Ind}_p^G \mathbf{1}, \text{gr}^3 J_{\text{Sp}}) \cong \text{Hom}_T(\mathbf{1}, J_{\mathbf{1}_T}/\mathbf{1}_T),$$

where the last isomorphism follows from Lemma 10.48. Hence, $\text{soc}_G \text{gr}^3 J_{\text{Sp}} \cong \text{Sp}^{\oplus 2}$ and we have an isomorphism

$$(216) \quad \text{gr}^3 J_{\text{Sp}} / \text{soc}_G \text{gr}^3 J_{\text{Sp}} \cong \text{Ind}_p^G (J_{\mathbf{1}_T} / \text{soc}_T^2 J_{\mathbf{1}_T}).$$

Since $e^1(\mathbf{1}, \mathbf{1}) = e^2(\mathbf{1}, \mathbf{1}) = 0$ we have isomorphisms:

$$\text{Ext}_{G/Z}^1(\mathbf{1}, \text{gr}^3 J_{\text{Sp}}) \cong \text{Ext}_{G/Z}^1(\mathbf{1}, \text{Ind}_p^G (J_{\mathbf{1}_T}/\mathbf{1}_T)) \stackrel{10.7}{\cong} \text{Ext}_{T/Z}^1(\mathbf{1}, J_{\mathbf{1}_T}/\mathbf{1}_T).$$

In particular, $e_{G/Z}^1(\mathbf{1}, \text{gr}^3 J_{\text{Sp}}) = e_{T/Z}^2(\mathbf{1}, \mathbf{1}) = 1$. Using (216) we deduce that

$$\text{Hom}_G(\mathbf{1}, \text{gr}^3 J_{\text{Sp}} / \text{soc}_G \text{gr}^3 J_{\text{Sp}}) \cong \text{Hom}_T(\mathbf{1}, J_{\mathbf{1}_T} / \text{soc}_T^2 J_{\mathbf{1}_T})$$

is 3-dimensional. Since $e^1(\mathbf{1}, \text{soc}_G \text{gr}^3 J_{\text{Sp}}) = e^1(\mathbf{1}, \text{Sp}^{\oplus 2}) = 4$ we deduce that the natural map

$$(217) \quad \text{Ext}_{G/Z}^1(\mathbf{1}, \text{gr}^3 J_{\text{Sp}}) \rightarrow \text{Ext}_{G/Z}^1(\mathbf{1}, \text{gr}^3 J_{\text{Sp}} / \text{soc}_G \text{gr}^3 J_{\text{Sp}})$$

is zero.

Proposition 10.58. — $e^1(\mathbf{1}, J_{\text{Sp}}^3) = 0$.

Proof. — Let τ be the subrepresentation of J_{Sp}^3 such that τ contains J_{Sp}^2 and $\tau/J_{\text{Sp}}^2 \cong \text{soc}_G \text{gr}^3 J_{\text{Sp}}$. In particular, $J_{\text{Sp}}^3/\tau = \text{gr}^3 J_{\text{Sp}} / \text{soc}_G \text{gr}^3 J_{\text{Sp}}$. We have a commutative diagram:

$$\begin{array}{ccc} \text{Ext}_{G/Z}^1(\mathbf{1}, J_{\text{Sp}}^3) & \longrightarrow & \text{Ext}_{G/Z}^1(\mathbf{1}, J_{\text{Sp}}^3/\tau) \\ \downarrow \text{10.55} & & \downarrow = \\ \text{Ext}_{G/Z}^1(\mathbf{1}, \text{gr}^3 J_{\text{Sp}}) & \xrightarrow[0]{(217)} & \text{Ext}_{G/Z}^1(\mathbf{1}, J_{\text{Sp}}^3/\tau). \end{array}$$

Hence, the top horizontal arrow is zero and we obtain an exact sequence:

$$(218) \quad 0 \rightarrow \text{Hom}_G(\mathbf{1}, J_{\text{Sp}}^3/\tau) \rightarrow \text{Ext}_{G/Z}^1(\mathbf{1}, \tau) \rightarrow \text{Ext}_{G/Z}^1(\mathbf{1}, J_{\text{Sp}}^3) \rightarrow 0.$$

As $(J_{\text{Sp}}^3/\tau)^G \cong (J_{\mathbf{1}_T} / \text{soc}_T^2 J_{\mathbf{1}_T})^T$ is 3-dimensional, if $e^1(\mathbf{1}, J_{\text{Sp}}^3) \neq 0$ then $e^1(\mathbf{1}, \tau) \geq 4$. As $e^1(\mathbf{1}, J_{\text{Sp}}^2) = 0$ we have an injection $\text{Ext}_{G/Z}^1(\mathbf{1}, \tau) \hookrightarrow \text{Ext}_{G/Z}^1(\mathbf{1}, \tau/J_{\text{Sp}}^2)$. Since $\tau/J_{\text{Sp}}^2 \cong \text{Sp}^{\oplus 2}$, $e^1(\mathbf{1}, \tau/J_{\text{Sp}}^2) = 4$ and the injection must be an isomorphism. This implies the existence of an exact sequence

$$(219) \quad 0 \rightarrow J_{\text{Sp}}^2 \rightarrow \tau' \rightarrow \tau_1^{\oplus 2} \rightarrow 0,$$

with $\text{soc}_G \tau' \cong \text{Sp}$, where τ_1 is the representation defined by (181). Since $\text{soc}_G \tau' \cong \text{Sp}$, we have $e^0(\pi(0, 1), J_{\text{Sp}}^2) = e^0(\pi(0, 1), \tau') = 1$. Applying $\text{Hom}_{G/Z}(\pi(0, 1), *)$ to (219) we deduce that $e^1(\pi(0, 1), J_{\text{Sp}}^2) \geq e^0(\pi(0, 1), \tau_1^{\oplus 2}) \geq 2$. This contradicts Lemma 10.57. \square

Corollary 10.59. — $e^1(\mathbf{1}, J_{\pi_\alpha}^4) = 0$.

Proof. — Lemma 10.54 gives an exact sequence $0 \rightarrow J_{\pi_\alpha}^1 \rightarrow J_{\pi_\alpha}^4 \rightarrow J_{\text{Sp}}^3 \rightarrow 0$. Corollary 10.8 says that $e^i(\mathbf{1}, J_{\pi_\alpha}^1) = 0$ for all $i \geq 0$. Hence, $e^1(\mathbf{1}, J_{\pi_\alpha}^4) = e^1(\mathbf{1}, J_{\text{Sp}}^3) = 0$, where the last equality follows from Proposition 10.58. \square

The following technical result will be useful later, in the arguments of Section 10.5.

Lemma 10.60. — *Let J be an injective envelope of $\mathbf{1}_G$ or π_α in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$. The exact sequence $0 \rightarrow J^1 \rightarrow J \rightarrow J/J^1 \rightarrow 0$ induces isomorphisms $\text{End}_G(J) \cong \text{End}_G(J/J^1)$, $\text{End}_G(\text{gr}^1 J) \cong \text{End}_G(\text{gr}^1(J/J^1))$.*

Remark 10.61. — Proposition 10.17 and Lemma 10.53 imply $J_{\mathbf{1}_G}/J_{\mathbf{1}_G}^1 \cong J_{\pi_\alpha}$ and $J_{\pi_\alpha}/J_{\pi_\alpha}^1 \cong J_{\text{Sp}}$.

Proof. — Let J_χ be an injective envelope of χ in $\text{Mod}_{G/Z}^{\text{ladm}}(k)$, where χ is either $\mathbf{1}_T$ or α . Lemma 10.53 says that $J^1 \cong \text{Ind}_p^G J_\chi$. It follows from (213) that $\text{Ord}_p J \cong \text{Ord}_p \text{Ind}_p^G J_\chi \cong J_{\chi^s}$. Since $\chi^s \neq \chi\alpha^{-1}$ we get $\text{Hom}_T(\text{Ord}_p J, J_\chi \otimes \alpha^{-1}) = 0$ and thus $\text{Ext}_{G/Z}^1(J, \text{Ind}_p^G J_\chi) = 0$ by Lemma 10.50. Hence we obtain an exact sequence $0 \rightarrow \text{Hom}_G(J, J^1) \rightarrow \text{Hom}_G(J, J) \rightarrow \text{Hom}_G(J, J/J^1) \rightarrow 0$. Since J is injective $\mathbf{R}^1 \text{Ord}_p J = 0$ thus the U -coinvariants J_U are zero, [31, 3.6.2], and so $\text{Hom}_G(J, J^1) = 0$. As $\text{Ord}_p J/J^1 \cong \mathbf{R}^1 \text{Ord}_p J^1 \cong J_\chi \otimes \alpha^{-1}$ we get $\text{Hom}_G(J^1, J/J^1) = 0$ and so $\text{Hom}_G(J, J/J^1) \cong \text{Hom}_G(J/J^1, J/J^1)$.

The second assertion follows by the same argument with J^2 instead of J . Note that $J^1 = \text{gr}^1 J \cong \text{Ind}_p^G J_\chi$ and $\text{gr}^2 J = \text{gr}^1(J/J^1)$. Now $\text{Ord}_p J^2 \cong \text{Ord}_p J$, hence $\text{Ext}_{G/Z}^1(J^2, \text{Ind}_p^G J_\chi) = 0$ by Lemma 10.50, and $\mathbf{R}^1 \text{Ord}_p J^2 \cong \text{Ord}_p J/J^2 \cong \mathbf{R}^1 \text{Ord}_p \text{gr}^2 J$ by (213). Hence, $J_U^2 \cong (\text{gr}^2 J)_U$ and so

$$\text{Hom}_G(J^2, \text{gr}^1 J) \cong \text{Hom}_G(\text{gr}^2 J, \text{gr}^1 J) = 0.$$

The last equality follows from the fact that $\text{gr}^1 J$ and $\text{gr}^2 J$ do not have a common irreducible subquotient, as $\text{gr}^1 J \cong \text{Ind}_p^G J_\chi$ and $\text{gr}^2 J$ is a quotient of $\text{Ind}_p^G J_{\chi'\alpha}$ by (213) and $\chi \neq \chi'\alpha$ since $p \geq 5$. We obtain an isomorphism $\text{Hom}_G(J^2, J^2) \cong \text{Hom}_G(J^2, \text{gr}^2 J) \cong \text{Hom}_G(\text{gr}^2 J, \text{gr}^2 J)$. On the other hand from the exact sequence $0 \rightarrow \text{Hom}_G(J^1, J) \rightarrow \text{Hom}_G(J^2, J) \rightarrow \text{Hom}_G(\text{gr}^2 J, J) \rightarrow 0$ we obtain an exact sequence $0 \rightarrow \text{Hom}_G(J^1, J^1) \rightarrow \text{Hom}_G(J^2, J^2) \rightarrow \text{Hom}_G(\text{gr}^2 J, J^1) \rightarrow 0$ from the functoriality of the filtration. Hence, an isomorphism $\text{Hom}_G(J^1, J^1) \cong \text{Hom}_G(J^2, J^2)$. \square

We let $d\text{Ind} : \mathfrak{C}_T(\mathcal{O}) \rightarrow \mathfrak{C}_G(\mathcal{O})$ be the functor $d\text{Ind}_p^G N := (\text{Ind}_p^G N^\vee)^\vee$. With this notation we have $\tilde{M} \cong d\text{Ind}_p^G \tilde{P}_{\alpha^\vee}$, and hence $M := \tilde{M} \otimes_{\mathcal{O}} k \cong d\text{Ind}_p^G P_{\alpha^\vee}$, where \tilde{P}_{α^\vee} (resp. P_{α^\vee}) is a projective envelope of α^\vee in $\mathfrak{C}_T(\mathcal{O})$ (resp. $\mathfrak{C}_T(k)$). Moreover, we have

$$E/\mathfrak{a} \cong \text{End}_{\mathfrak{C}_T(k)}(P_{\alpha^\vee}) \cong \text{End}_{\mathfrak{C}_G(k)}(M) \cong \text{End}_{\Omega(k)}(\mathcal{T}M),$$

see Corollary 7.2, (211). We let $\check{\mathcal{J}} : \mathfrak{C}_G(k) \rightarrow \mathfrak{C}_G(k)$ be the functor $\check{\mathcal{J}}(N) = (\text{Ind}_p^G(\text{Ord}_p N^\vee))^\vee$.

Proposition 10.62. — Let P_{α^\vee} be a projective envelope of α^\vee in $\mathfrak{C}_{T/Z}(k)$. There exists a decreasing filtration $P_{\alpha^\vee}^\bullet$ of P_{α^\vee} by subobjects, such that

- (i) $P_{\alpha^\vee}^0 = P_{\alpha^\vee}$;
- (ii) $\text{rad} P_{\alpha^\vee}^i \subseteq P_{\alpha^\vee}^{i+1} \subseteq P_{\alpha^\vee}^i$, for all $i \geq 0$;
- (iii) for $i \geq 0$ we have

$$(220) \quad \mathcal{T}P_{\alpha^\vee}^{2i} / \mathcal{T}P_{\alpha^\vee}^{2i+1} \cong \mathcal{T} d\text{Ind}_p^G P_{\alpha^\vee}^i,$$

$$(221) \quad \mathcal{TP}^{2i+1}/\mathcal{TP}^{2i+2} \cong \mathcal{T} \operatorname{dInd}_{\mathbb{P}}^{\mathbb{G}} \left((\mathbb{P}_{\alpha^{\vee}}^i)^s \otimes \alpha^{\vee} \right),$$

where $\mathbb{P}^i = (\mathbb{J}/\mathbb{J}^i)^{\vee}$ and \mathbb{J}^{\bullet} is the filtration of \mathbb{J} by ordinary parts. Moreover, $\operatorname{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathbb{T}_{\mathbf{1}}) = 0$.

Proof. — Let \mathbb{J} be an injective envelope of $\operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \alpha$ in $\operatorname{Mod}_{\mathbb{G}/Z}^{\operatorname{ladm}}(k)$. It follows from Lemma 10.49 that for $i \geq 1$ we have an exact sequence

$$(222) \quad 0 \rightarrow \theta_i \rightarrow \operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \operatorname{Ord}_{\mathbb{P}}(\operatorname{gr}^i \mathbb{J}) \rightarrow \operatorname{gr}^i \mathbb{J} \rightarrow 0,$$

where \mathbb{G} acts trivially on θ_i . By evaluating at the identity, we may identify θ_i with a subspace of $(\operatorname{Ord}_{\mathbb{P}} \operatorname{gr}^i \mathbb{J})^{\Gamma}$. Lemma 10.48 says that

$$(223) \quad \mathbf{R}^1 \operatorname{Ord}_{\mathbb{P}} \operatorname{gr}^i \mathbb{J} \cong \left((\operatorname{Ord}_{\mathbb{P}} \operatorname{gr}^i \mathbb{J}) / \theta_i \right)^s \otimes \alpha^{-1}.$$

For $i \geq 1$ let $\kappa_i := \operatorname{Ord}_{\mathbb{P}} \operatorname{gr}^i \mathbb{J}$. We deduce from Proposition 7.1 that we have an injection $\operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \mathbb{J}_{\alpha^{-1}} \hookrightarrow \mathbb{J}$. Hence, $\kappa_1 \cong \mathbb{J}_{\alpha^{-1}} \cong \mathbb{J}_{\mathbf{1}} \otimes \alpha^{-1}$ and $\theta_1 = 0$. It follows from (213) and (223) that $\kappa_{2i-1}^{\Gamma} = 0$ and hence $\theta_{2i-1} = 0$, for all $i \geq 1$. We deduce from (213) and (223) that $\kappa_{2i} \cong (\kappa_{2i-1})^s \otimes \alpha^{-1}$ and we have an exact sequence

$$(224) \quad 0 \rightarrow \theta_{2i} \rightarrow \kappa_{2i} \rightarrow \kappa_{2i+2} \rightarrow 0$$

of \mathbb{T} -representations, where \mathbb{T} acts trivially on θ_{2i} . In particular, κ_i is a successive extension of copies of α^{-1} when i is odd, and a successive extension of copies of $\mathbf{1}_{\mathbb{T}}$ when i is even.

Since $\mathcal{T}\theta_i^{\vee} = 0$ we deduce from (222) that

$$(225) \quad \mathcal{T}(\operatorname{gr}^i \mathbb{J})^{\vee} \cong \mathcal{T}(\operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \kappa_i)^{\vee} \cong \mathcal{T}(\operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \kappa_i^s)^{\vee}.$$

We let $\mathbb{P}_{\alpha^{\vee}}^i := (\kappa_{2i+1}^s)^{\vee} \cong (\kappa_{2i+2} \otimes \alpha)^{\vee}$ then $\mathbb{P}_{\alpha^{\vee}}^0 \cong (\mathbb{J}_{\alpha^{-1}}^s)^{\vee} \cong \mathbb{P}_{\alpha^{\vee}}$. Moreover, by twisting (224) by α and dualizing we obtain injections $\mathbb{P}_{\alpha^{\vee}}^{i+1} \hookrightarrow \mathbb{P}_{\alpha^{\vee}}^i$ with semi-simple cokernel. Hence, $\operatorname{rad} \mathbb{P}_{\alpha^{\vee}}^i \subseteq \mathbb{P}_{\alpha^{\vee}}^{i+1}$. Part (iii) follows from (225). Moreover, we deduce from (213) that

$$\operatorname{Hom}_{\mathbb{G}}(\operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \mathbf{1}, \mathbb{J}/\mathbb{J}^{2i}) \cong \operatorname{Hom}_{\mathbb{T}}(\mathbf{1}, \kappa_{2i+1}) = 0.$$

Since $\mathbb{P}^i \cong (\mathbb{J}/\mathbb{J}^i)^{\vee}$, Lemma 10.30 implies that

$$\begin{aligned} \operatorname{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathbb{T}_{\mathbf{1}}) &\stackrel{(10.30)}{\cong} \operatorname{Hom}_{\mathbb{G}}(\operatorname{Ind}_{\mathbb{P}}^{\mathbb{G}} \mathbf{1}, \mathbb{J}/\mathbb{J}^{2i}) \cong \operatorname{Hom}_{\mathbb{T}}(\mathbf{1}, \operatorname{Ord}_{\mathbb{P}}(\mathbb{J}/\mathbb{J}^{2i})) \\ &\stackrel{(213)}{\cong} \operatorname{Hom}_{\mathbb{T}}(\mathbf{1}, \operatorname{Ord}_{\mathbb{P}} \operatorname{gr}^{2i+1} \mathbb{J}) \cong \operatorname{Hom}_{\mathbb{T}}(\mathbf{1}, \kappa_{2i+1}) = 0. \end{aligned}$$

□

Lemma 10.63. — $\operatorname{Hom}_{\mathcal{E}(k)}(\mathbb{P}^2, \mathbf{1}_{\mathbb{G}}^{\vee}) = 0$, $\operatorname{Hom}_{\mathcal{E}(k)}(\mathbb{P}^4, \mathbf{1}_{\mathbb{G}}^{\vee}) = 0$.

Proof. — The assertion is equivalent to $\operatorname{Hom}_{\mathbb{G}}(\mathbf{1}, \mathbb{J}_{\pi_{\alpha}}/\mathbb{J}_{\pi_{\alpha}}^{2i}) = \operatorname{Ext}_{\mathbb{G}/Z}^1(\mathbf{1}, \mathbb{J}_{\pi_{\alpha}}^{2i}) = 0$ for $i = 1, 2$. If $i = 2$ this follows from Corollary 10.59. Proposition 10.17 gives an exact sequence $0 \rightarrow \mathbb{J}_{\pi_{\alpha}}^2 \rightarrow \mathbb{J}_{\pi_{\alpha}} \rightarrow \mathbb{J}_{\pi_{\alpha}}^{\oplus 2}$, which proves the assertion for $i = 1$. □

Lemma 10.64. — *The ideal \mathfrak{a} is a finitely generated right E -module and $E/\mathfrak{a} \widehat{\otimes}_E P \cong P/P^2$, $\mathfrak{a}P \cong P^2$, $\mathfrak{a} \widehat{\otimes}_E \mathcal{TP} \cong \mathfrak{a} \mathcal{TP} \cong \mathcal{TP}^2$.*

Proof. — Since $\widetilde{M} \otimes_{\mathcal{O}} k \cong P/P^1$ and \widetilde{M} is \mathcal{O} -flat, we deduce from the definition of $\widetilde{\mathfrak{a}}$ in (211) that

$$\mathfrak{a} = \{\phi \in E : \phi(P) \subseteq P^1\} = \{\phi \in E : \phi(P) \subseteq P^2\},$$

where the second equality follows from the fact that $\mathrm{Hom}_{\mathcal{E}(k)}(P, P^1/P^2) = 0$ as Lemma 10.65 implies that π_{α^\vee} is not a subquotient of P^1/P^2 . Hence, $\mathfrak{a}P \subseteq P^2$ and $\mathfrak{a} \mathcal{TP} \subseteq \mathcal{TP}^2$. On the other hand using (189) and (190) we get a surjection $P \oplus P \twoheadrightarrow P^2$. For $i = 1$ and $i = 2$ let $\phi_i \in E$ be the composition

$$P \rightarrow P \oplus P \twoheadrightarrow P^2 \hookrightarrow P,$$

where the first arrow is $(\mathrm{id}, 0)$ if $i = 1$ and $(0, \mathrm{id})$ if $i = 2$. Then $\phi_1, \phi_2 \in \mathfrak{a}$ and $P^2 = \phi_1(P) + \phi_2(P)$. Hence, $P^2 \subseteq \mathfrak{a}P$ and so $\mathfrak{a}P = P^2$ is closed in P , which implies $E/\mathfrak{a} \widehat{\otimes}_E P \cong P/\mathfrak{a}P \cong P/P^2$. Using Lemma 2.9 and exactness of $\mathrm{Hom}_{\mathcal{E}(k)}(P, *)$ we get

$$\mathfrak{a} \cong \mathrm{Hom}_{\mathcal{E}(k)}(P, \mathfrak{a}P) \cong \mathrm{Hom}_{\mathcal{E}(k)}(P, P^2).$$

Hence, $\mathfrak{a} = \phi_1 E + \phi_2 E$ is a finitely generated right E -module. In particular, $\mathfrak{a} \mathcal{TP} = \phi_1(\mathcal{TP}) + \phi_2(\mathcal{TP}) = \mathcal{TP}^2$ is an object of $\Omega(k)$. Since \mathcal{TP} is E -flat by Lemma 10.41, we obtain $\mathfrak{a} \widehat{\otimes}_E \mathcal{TP} \cong \mathfrak{a} \mathcal{TP}$. \square

Lemma 10.65. — *$\mathrm{Hom}_{\Omega(k)}(\mathcal{TP}, \mathcal{TP}^{2i-1}/\mathcal{TP}^{2i}) = 0$, for all $i \geq 1$.*

Proof. — All the irreducible subquotients of $\mathcal{TP}^{2i-1}/\mathcal{TP}^{2i}$ are isomorphic to T_1 , see (221). Since \mathcal{TP} is a projective envelope of T_α in $\Omega(k)$, see Lemmas 10.27 and 10.29, there are no non-zero homomorphisms. \square

Lemma 10.66. — *We have an isomorphism of E -modules:*

$$(226) \quad \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}, \mathcal{TP}^{2i}/\mathcal{TP}^{2i+2}) \cong \mathrm{Hom}_{\mathcal{E}_T(k)}(P_{\alpha^\vee}, P_{\alpha^\vee}^i),$$

for all $i \geq 0$.

Proof. — The E -module structure on the left hand side is given by the action of $E = \mathrm{End}_{\Omega(k)}(\mathcal{TP})$ on \mathcal{TP} and on the right hand side by the action of $E/\mathfrak{a} \cong \mathrm{End}_{\mathcal{E}_T(k)}(P_{\alpha^\vee})$ on P_{α^\vee} . Since \mathcal{TP} is projective, $\mathrm{Hom}_{\Omega(k)}(\mathcal{TP}, *)$ is exact and so we get:

$$\begin{aligned} \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}, \mathcal{TP}^{2i}/\mathcal{TP}^{2i+2}) &\stackrel{10.65}{\cong} \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}, \mathcal{TP}^{2i}/\mathcal{TP}^{2i+1}) \\ &\stackrel{(220)}{\cong} \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}, \mathcal{T} \mathrm{dInd}_P^G P_{\alpha^\vee}^i) \end{aligned}$$

$$\begin{aligned}
& \stackrel{10.27}{\cong} \mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}, \mathrm{dInd}_{\mathbf{P}}^{\mathbf{G}} \mathbf{P}_{\alpha^{\vee}}^i) \\
& \stackrel{10.51}{\cong} \mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}/\mathbf{P}^1, \mathrm{dInd}_{\mathbf{P}}^{\mathbf{G}} \mathbf{P}_{\alpha^{\vee}}^i) \\
& \stackrel{(220)}{\cong} \mathrm{Hom}_{\mathfrak{C}(k)}(\mathrm{dInd}_{\mathbf{P}}^{\mathbf{G}} \mathbf{P}_{\alpha^{\vee}}, \mathrm{dInd}_{\mathbf{P}}^{\mathbf{G}} \mathbf{P}_{\alpha^{\vee}}^i) \\
& \cong \mathrm{Hom}_{\mathfrak{C}_{\mathbf{T}}(k)}(\mathbf{P}_{\alpha^{\vee}}, \mathbf{P}_{\alpha^{\vee}}^i). \quad \square
\end{aligned}$$

Lemma 10.67. — *Let \mathfrak{m} be a compact \mathbf{E}/\mathfrak{a} -module. If $\mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}^{2i}, \mathbf{1}_{\mathbf{G}}^{\vee}) = 0$ for a fixed i then*

$$(227) \quad \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}/\mathcal{TP}^{2i+2}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}) \cong \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}).$$

Proof. — Since \mathfrak{a} acts trivially on \mathfrak{m} we have $\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP} \cong \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}/\mathfrak{a}\mathcal{TP}$. It follows from Lemma 10.51 that the filtration on \mathcal{TP} is \mathbf{E} -invariant. Lemma 10.64 gives us an exact sequence:

$$(228) \quad \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}^1/\mathcal{TP}^2 \rightarrow \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP} \rightarrow \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}/\mathcal{TP}^1 \rightarrow 0.$$

We may find an exact sequence of compact \mathbf{E} -modules:

$$(229) \quad \prod_{i \in \mathbf{I}} \mathbf{E}/\mathfrak{a} \rightarrow \prod_{j \in \mathbf{J}} \mathbf{E}/\mathfrak{a} \rightarrow \mathfrak{m} \rightarrow 0$$

for some index sets \mathbf{I} and \mathbf{J} . Applying $\widehat{\otimes}_{\mathbf{E}} \mathcal{TP}^1/\mathcal{TP}^2$ to (229) we deduce that $\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}^1/\mathcal{TP}^2$ is a quotient of $\prod_{j \in \mathbf{J}} \mathcal{TP}^1/\mathcal{TP}^2$. Hence, it follows from (221) that all the irreducible subquotients of $\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}^1/\mathcal{TP}^2$ are isomorphic to \mathbf{T}_1 . Since $\mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathbf{T}_1) = 0$ by Proposition 10.62, we get an injection:

$$(230) \quad \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}) \hookrightarrow \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}/\mathcal{TP}^1).$$

Hence, we obtain a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}) & \hookrightarrow & \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}/\mathcal{TP}^1) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i+2}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}) & \hookrightarrow & \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i+2}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathcal{TP}/\mathcal{TP}^1).
\end{array}$$

It is enough to show that the right vertical arrow is zero. As $\mathbf{P}/\mathbf{P}^1 \cong \mathrm{dInd}_{\mathbf{P}}^{\mathbf{G}} \mathbf{P}_{\alpha^{\vee}}$, Corollary 7.3 says $\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathbf{P}/\mathbf{P}^1 \cong \mathrm{dInd}_{\mathbf{P}}^{\mathbf{G}}(\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathbf{P}_{\alpha^{\vee}})$. In particular, $\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathbf{P}/\mathbf{P}^1 \cong \check{\mathcal{J}}(\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathbf{P}/\mathbf{P}^1)$ and so the second step of filtration on $\mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathbf{P}/\mathbf{P}^1$ is zero. Hence, Lemma 10.51 implies that

$$\mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}^{2i}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathbf{P}/\mathbf{P}^1) \cong \mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}^{2i}/\mathbf{P}^{2i+1}, \mathfrak{m} \widehat{\otimes}_{\mathbf{E}} \mathbf{P}/\mathbf{P}^1).$$

Since $\mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}^{2i}, \mathbf{1}_G^\vee) = 0$ by assumption, Lemma 10.26 implies that

$$(231) \quad \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathcal{T}(\mathfrak{m} \widehat{\otimes}_E \mathbf{P}/\mathbf{P}^1)) \cong \mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}/\mathcal{TP}^{2i+1}, \mathcal{T}(\mathfrak{m} \widehat{\otimes}_E \mathbf{P}/\mathbf{P}^1)).$$

Since $\mathcal{T}(\mathfrak{m} \widehat{\otimes}_E \mathbf{P}/\mathbf{P}^1) \cong \mathfrak{m} \widehat{\otimes}_E \mathcal{TP}/\mathcal{TP}^1$ by Lemma 10.42, the right vertical arrow in the diagram above is zero. \square

Let $\varphi : \widetilde{E} \rightarrow \mathbf{R}^\psi$ be the homomorphism defined in Proposition 10.45, let \mathfrak{r} be the ideal of \mathbf{R}^ψ defined in Proposition 10.47 and let $\mathfrak{r}_k = \mathfrak{r} \otimes_{\mathcal{O}} k$.

Lemma 10.68. — *If $\mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}^{2i}, \mathbf{1}_G^\vee) = 0$ and $\mathfrak{a}^i \widehat{\otimes}_E \mathcal{TP} \cong \mathcal{TP}^{2i}$ for a fixed i then the map φ induces an isomorphism $\mathfrak{a}^i/\mathfrak{a}^{i+1} \xrightarrow{\cong} \mathfrak{r}_k^i/\mathfrak{r}_k^{i+1} \cong \mathfrak{n}^i$, where \mathfrak{n} is the maximal ideal of E/\mathfrak{a} . Moreover, $\mathfrak{a}^{i+1} \widehat{\otimes}_E \mathcal{TP} \cong \mathcal{TP}^{2i+2}$.*

Proof. — Recall that $E/\mathfrak{a} \cong k[[x, y]]$, let \mathfrak{n} be the maximal ideal of E/\mathfrak{a} and let K be the quotient field of E/\mathfrak{a} . We have a surjection

$$\mathcal{TP}^{2i} \cong \mathfrak{a}^i \widehat{\otimes}_E \mathcal{TP} \rightarrow \mathfrak{a}^i/\mathfrak{a}^{i+1} \widehat{\otimes}_E \mathcal{TP},$$

We note that since \mathfrak{a} is a finitely generated right E -module, \mathfrak{a}^j is a closed submodule of \mathfrak{a}^{j-1} for all $j \geq 1$ and hence $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is a compact E/\mathfrak{a} -module. It follows from Lemma 10.67 that the surjection factors through $\mathcal{TP}^{2i}/\mathcal{TP}^{2i+2} \rightarrow \mathfrak{a}^i/\mathfrak{a}^{i+1} \widehat{\otimes}_E \mathcal{TP}$. We apply $\mathrm{Hom}_{\Omega(k)}(\mathcal{TP}, *)$ and use Lemmas 2.9 and 10.66 to get a surjection of (right) E -modules: $\mathrm{Hom}_{\mathfrak{C}_T(k)}(\mathbf{P}_{\alpha^\vee}, \mathbf{P}_{\alpha^\vee}^i) \rightarrow \mathfrak{a}^i/\mathfrak{a}^{i+1}$, where E acts on $\mathfrak{m} := \mathrm{Hom}_{\mathfrak{C}_T(k)}(\mathbf{P}_{\alpha^\vee}, \mathbf{P}_{\alpha^\vee}^i)$ via $E/\mathfrak{a} \cong \mathrm{End}_{\mathfrak{C}_T(k)}(\mathbf{P}_{\alpha^\vee})$. It follows from Proposition 10.62(ii) that $\mathrm{rad}^i \mathbf{P}_{\alpha^\vee} \subseteq \mathbf{P}_{\alpha^\vee}^i$. Since \mathbf{P}_{α^\vee} is flat over E/\mathfrak{a} , see the proof of Proposition 3.34, and $k \widehat{\otimes}_{E/\mathfrak{a}} \mathbf{P}_{\alpha^\vee} \cong \alpha^\vee$ is irreducible, we get that $\mathrm{rad}^i \mathbf{P}_{\alpha^\vee} \cong \mathfrak{n}^i \widehat{\otimes}_E \mathbf{P}_{\alpha^\vee} \cong \mathfrak{n}^i \mathbf{P}_{\alpha^\vee}$. Since $\mathbf{P}_{\alpha^\vee}^i \subseteq \mathbf{P}_{\alpha^\vee}$ we have $\mathfrak{n}^i \subseteq \mathfrak{m} \subseteq E/\mathfrak{a}$. Hence, $\dim_K \mathfrak{m} \otimes_E K = 1$ and we have an injection $\mathfrak{m} \hookrightarrow \mathfrak{m} \otimes_E K$. Proposition 10.47 and Corollary B.6(ii) give a surjection $\mathfrak{m} \twoheadrightarrow \varphi(\mathfrak{a})^i/\varphi(\mathfrak{a})^{i+1} \rightarrow \mathfrak{n}^i$. Since $\mathfrak{n}^i \otimes_E K$ is 1-dimensional, the map induces an isomorphism $\mathfrak{m} \otimes_E K \cong \mathfrak{n}^i \otimes_E K$. Hence, the composition $\mathfrak{m} \rightarrow \mathfrak{n}^i$ is injective, and so $\varphi : \mathfrak{a}^i/\mathfrak{a}^{i+1} \rightarrow \varphi(\mathfrak{a})^i/\varphi(\mathfrak{a})^{i+1}$ is injective, and thus an isomorphism. Since $\mathrm{Hom}_{\Omega(k)}(\mathcal{TP}^{2i}, \mathbf{T}_1) = 0$ by Proposition 10.62, Lemma 2.10 implies that the evaluation map $\mathfrak{m} \widehat{\otimes}_E \mathcal{TP} \rightarrow \mathcal{TP}^{2i}/\mathcal{TP}^{2i+2}$ is surjective. Since the composition $\mathfrak{m} \widehat{\otimes}_E \mathcal{TP} \rightarrow \mathcal{TP}^{2i}/\mathcal{TP}^{2i+2} \rightarrow \mathfrak{a}^i/\mathfrak{a}^{i+1} \widehat{\otimes}_E \mathcal{TP}$ is an isomorphism, we deduce that $\mathcal{TP}^{2i}/\mathcal{TP}^{2i+2} \cong \mathfrak{a}^i/\mathfrak{a}^{i+1} \widehat{\otimes}_E \mathcal{TP}$. Since \mathcal{TP} is E -flat and $\mathfrak{a}^i \widehat{\otimes}_E \mathcal{TP} \cong \mathcal{TP}^{2i}$ by assumption, we deduce that $\mathcal{TP}^{2i+2} \cong \mathfrak{a}^{i+1} \widehat{\otimes}_E \mathcal{TP}$. \square

Lemma 10.69. — *The map φ induces isomorphisms $\mathfrak{a}/\mathfrak{a}^2 \cong \mathfrak{r}_k/\mathfrak{r}_k^2 \cong \mathfrak{n}$ and $\mathfrak{a}^2/\mathfrak{a}^3 \cong \mathfrak{r}_k^2/\mathfrak{r}_k^3 \cong \mathfrak{n}^2$, where \mathfrak{n} is the maximal ideal of E/\mathfrak{a} .*

Proof. — Since $\mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}^2, \mathbf{1}_G^\vee) = 0$ by Lemma 10.63 and $\mathfrak{a} \widehat{\otimes}_E \mathcal{TP} \cong \mathcal{TP}^2$ by Lemma 10.64, Lemma 10.68 implies that $\mathfrak{a}/\mathfrak{a}^2 \cong \mathfrak{r}_k/\mathfrak{r}_k^2 \cong \mathfrak{n}$ and $\mathfrak{a}^2 \widehat{\otimes}_E \mathcal{TP} \cong \mathcal{TP}^4$. Since $\mathrm{Hom}_{\mathfrak{C}(k)}(\mathbf{P}^4, \mathbf{1}_G^\vee) = 0$ by Lemma 10.63, Lemma 10.64 implies $\mathfrak{a}^2/\mathfrak{a}^3 \cong \mathfrak{r}_k^2/\mathfrak{r}_k^3 \cong \mathfrak{n}^2$. \square

Proposition 10.70. — *The surjection of graded rings $\varphi^\bullet : \text{gr}_\alpha^\bullet(\mathbf{E}) \rightarrow \text{gr}_{\mathfrak{r}_k}^\bullet(\mathbf{R}_k^\psi)$ is an isomorphism.*

Proof. — It follows from Lemma B.5 that $\mathbf{R}_k^\psi \cong k[[x, y, z, w]]/(xz - yw)$ and $\mathfrak{r}_k = (z, w)$. Thus $\mathbf{R}_k^\psi/\mathfrak{r}_k \cong k[[x, y]]$ and $\text{gr}_{\mathfrak{r}_k}^\bullet(\mathbf{R}_k^\psi) \cong (\mathbf{R}_k^\psi/\mathfrak{r}_k)[\bar{z}, \bar{w}]/(x\bar{z} - y\bar{w})$. It follows from Proposition 10.47 that φ^0 induces an isomorphism $\mathbf{E}/\mathfrak{a} \cong \mathbf{R}_k^\psi/\mathfrak{r}_k \cong k[[x, y]]$. Lemma 10.69 implies that φ^1 induces an isomorphism $\mathfrak{a}/\mathfrak{a}^2 \cong \mathfrak{r}_k/\mathfrak{r}_k^2$. In particular, $\text{gr}_\alpha^\bullet(\mathbf{E})/\text{gr}_\alpha^{>1}(\mathbf{E}) \cong \text{gr}_{\mathfrak{r}_k}^\bullet(\mathbf{R}_k^\psi)/\text{gr}_{\mathfrak{r}_k}^{>1}(\mathbf{R}_k^\psi)$ is a commutative ring. Hence, we have a surjection

$$(232) \quad \beta : (\mathbf{E}/\mathfrak{a})[\bar{z}, \bar{w}]^{\text{nc}} \rightarrow \text{gr}_\alpha^\bullet(\mathbf{E})$$

such that the image of $x\bar{z} - y\bar{w}$ is zero, where the source is a polynomial ring in two non-commutative variables with coefficients in $\tilde{\mathbf{E}}/\mathfrak{a}$. Lemma 10.69 implies that φ^2 induces an isomorphism $\mathfrak{a}^2/\mathfrak{a}^3 \cong \mathfrak{r}_k^2/\mathfrak{r}_k^3$. Hence, $\bar{z}\bar{w} - \bar{w}\bar{z}$ maps to zero in $\text{gr}_\alpha^\bullet(\mathbf{E})$. Thus $\text{gr}_\alpha^\bullet(\mathbf{E})$ is a commutative ring and (232) factors through

$$(\mathbf{E}/\mathfrak{a})[\bar{z}, \bar{w}]/(x\bar{z} - y\bar{w}) \rightarrow \text{gr}_\alpha^\bullet(\mathbf{E}) \rightarrow \text{gr}_{\mathfrak{r}_k}^\bullet(\mathbf{R}_k^\psi) \cong (\mathbf{E}/\mathfrak{a})[\bar{z}, \bar{w}]/(x\bar{z} - y\bar{w}).$$

Since any surjection of a noetherian ring onto itself is an isomorphism we deduce the assertion. \square

Theorem 10.71. — *The map φ induces an isomorphism $\tilde{\mathbf{E}} \cong \mathbf{R}^\psi$.*

Proof. — We deduce from Proposition 10.70 that φ induces an isomorphism $\mathbf{E}/\mathfrak{a}^i \xrightarrow{\cong} \mathbf{R}_k^\psi/\varphi(\mathfrak{a})^i$, for all $i \geq 1$. Passing to the limit we get an isomorphism $\mathbf{E} \cong \mathbf{R}_k^\psi$. Since \mathbf{R}^ψ is \mathcal{O} -flat by Corollary B.5, we get that $(\text{Ker } \varphi) \otimes_{\mathcal{O}} k = 0$. Hence, $\text{Ker } \varphi = 0$ by Nakayama's lemma. \square

Corollary 10.72. — $\check{\mathbf{V}}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee})$ is the universal deformation of ρ with determinant equal to $\zeta \varepsilon$.

Proof. — It follows from Theorem 10.71 that $\check{\mathbf{V}}$ induces an isomorphism between deformation functors and hence an isomorphism between the universal objects. \square

10.5. The centre. — Let $\mathfrak{B} = \{\mathbf{1}, \text{Sp}, \pi_\alpha\}$ and let $\tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}$, $\tilde{\mathbf{P}}_{\text{Sp}^\vee}$ and $\tilde{\mathbf{P}}_{\pi_\alpha^\vee}$ be projective envelopes of $\mathbf{1}_G^\vee$, Sp^\vee and π_α^\vee in $\mathfrak{C}(\mathcal{O})$. Let $\tilde{\mathbf{P}}_{\mathfrak{B}} := \tilde{\mathbf{P}}_{\pi_\alpha^\vee} \oplus \tilde{\mathbf{P}}_{\text{Sp}^\vee} \oplus \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}$ and $\tilde{\mathbf{E}}_{\mathfrak{B}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathfrak{B}})$. Recall that the functor $\mathbf{N} \mapsto \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathfrak{B}}, \mathbf{N})$ induces an equivalence of categories between $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ and the category of compact $\tilde{\mathbf{E}}_{\mathfrak{B}}$ -modules, Proposition 5.45. In this section we compute the ring $\tilde{\mathbf{E}}_{\mathfrak{B}}$ and show that it is a finitely generated module over its centre, and that the centre is naturally isomorphic to \mathbf{R}^ψ .

After twisting we may assume that our fixed central character ζ is trivial, see Lemma 10.103 below. For a character $\chi : T/Z \rightarrow k^\times$ we let $\tilde{\mathbf{P}}_{\chi^\vee}$ be a projective envelope of χ^\vee in $\mathfrak{C}_{T/Z}(\mathcal{O})$ and let $\tilde{\mathbf{M}}_{\chi^\vee} := (\text{Ind}_{\mathbf{P}}^{\mathbf{G}}(\tilde{\mathbf{P}}_{\chi^\vee})^\vee)^\vee$. Further we define $\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee, 0}$ by the exact sequence:

$$(233) \quad 0 \rightarrow \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee, 0} \xrightarrow{\xi_{32}} \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee} \xrightarrow{\theta} \mathcal{O} \rightarrow 0,$$

where \mathcal{O} is equipped with the trivial \mathbf{G} -action. Proposition 10.17 and Corollary 5.21 imply the existence of exact sequences:

$$(234) \quad 0 \rightarrow \tilde{\mathbf{P}}_{\pi_\alpha^\vee} \xrightarrow{\varphi_{31}} \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee} \xrightarrow{\psi_3} \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee} \rightarrow 0,$$

$$(235) \quad 0 \rightarrow \tilde{\mathbf{P}}_{\text{Sp}^\vee} \xrightarrow{\varphi_{12}} \tilde{\mathbf{P}}_{\pi_\alpha^\vee} \xrightarrow{\psi_1} \tilde{\mathbf{M}}_{\alpha^\vee} \rightarrow 0,$$

$$(236) \quad \tilde{\mathbf{P}}_{\pi_\alpha^\vee}^{\oplus 2} \rightarrow \tilde{\mathbf{P}}_{\text{Sp}^\vee} \xrightarrow{\psi_2} \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee, 0} \rightarrow 0.$$

Lemma 10.73. — $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}, \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}), \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}, \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee, 0}), \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathbf{M}}_{\alpha^\vee})$ and $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\text{Sp}^\vee}, \tilde{\mathbf{M}}_{\alpha^\vee})$, all vanish.

Proof. — The proof in all the cases is the same, so we prove only the vanishing of $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}, \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee})$. The irreducible subquotients of $\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}$ are isomorphic to $\mathbf{1}_G^\vee$ and Sp^\vee . In particular, π_α^\vee is not a subquotient. Since $\tilde{\mathbf{P}}_{\pi_\alpha^\vee}$ is a projective envelope of π_α^\vee , we deduce that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}, \tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}) = 0$. \square

We let $\varphi_{32} := \varphi_{31} \circ \varphi_{12} : \tilde{\mathbf{P}}_{\text{Sp}^\vee} \hookrightarrow \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}$ and denote:

$$\tilde{\mathbf{E}}_{11} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}), \quad \tilde{\mathbf{E}}_{22} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\text{Sp}^\vee}), \quad \tilde{\mathbf{E}}_{33} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}).$$

For $i = 1, 2, 3$ we let $\tilde{\mathbf{a}}_i := \{\phi \in \tilde{\mathbf{E}}_i : \psi_i \circ \phi = 0\}$, with ψ_i defined in (234), (235), (236). Let e_1, e_2 and e_3 be idempotents in $\tilde{\mathbf{E}}_{\mathfrak{B}}$ cutting out $\tilde{\mathbf{P}}_{\pi_\alpha^\vee}, \tilde{\mathbf{P}}_{\text{Sp}^\vee}$ and $\tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}$ respectively.

Lemma 10.74.

$$(237) \quad \tilde{\mathbf{E}}_{11} \xrightarrow{\cong} \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}, \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}), \quad z_{11} \mapsto \varphi_{31} \circ z_{11},$$

$$(238) \quad \tilde{\mathbf{E}}_{22} \xrightarrow{\cong} \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\text{Sp}^\vee}, \tilde{\mathbf{P}}_{\pi_\alpha^\vee}), \quad z_{22} \mapsto \varphi_{12} \circ z_{22},$$

$$(239) \quad \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathbf{P}}_{\text{Sp}^\vee}) \xrightarrow{\cong} \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathbf{P}}_{\pi_\alpha^\vee}), \quad z_{23} \mapsto \varphi_{12} \circ z_{23},$$

$$(240) \quad \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}, \tilde{\mathbf{P}}_{\text{Sp}^\vee}) \xrightarrow{\cong} \tilde{\mathbf{a}}_{11}, \quad z_{21} \mapsto \varphi_{12} \circ z_{21},$$

$$(241) \quad \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathbb{P}}_{\pi_\alpha^\vee}) \xrightarrow{\cong} \tilde{\mathfrak{a}}_{33}, \quad z_{13} \mapsto \varphi_{31} \circ z_{13},$$

$$(242) \quad \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}) \xrightarrow{\cong} \tilde{\mathfrak{a}}_{33}, \quad z_{23} \mapsto \varphi_{32} \circ z_{23}.$$

Proof. — The proof in all the cases is the same, one uses (234), (235) and (236) together with Lemma 10.73 and the left exactness of Hom . The assertion in (242) follows from (239) and (241). \square

Lemma 10.75. — *There exists $\beta : \tilde{\mathbb{P}}_{\mathrm{Sp}^\vee} \rightarrow \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}$ such that $\psi_3 \circ \beta = \xi_{32} \circ \psi_2$. Moreover, the following sequence:*

$$(243) \quad \tilde{\mathbb{P}}_{\pi_\alpha^\vee} \oplus \tilde{\mathbb{P}}_{\mathrm{Sp}^\vee} \xrightarrow{\varphi_{13} \oplus \beta} \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee} \rightarrow \mathcal{O} \rightarrow 0$$

is exact.

Proof. — Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}, \mathcal{O}) = 0$, we deduce from (233) that ξ_{32} induces an isomorphism $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}, \tilde{\mathbb{M}}_{\mathbf{1}_T^\vee, 0}) \xrightarrow{\xi_{32} \circ} \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}, \tilde{\mathbb{M}}_{\mathbf{1}_T^\vee})$. Since $\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}$ is projective we deduce from (234) that ψ_3 induces a surjection $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}, \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}) \xrightarrow{\psi_3 \circ} \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}, \tilde{\mathbb{M}}_{\mathbf{1}_T^\vee})$. Hence, there exists $\beta : \tilde{\mathbb{P}}_{\mathrm{Sp}^\vee} \rightarrow \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}$ such that $\psi_3 \circ \beta = \xi_{32} \circ \psi_2$. Combining (233) with (234) we obtain (243). \square

Proposition 10.76. — *Restriction to $\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$ in (234) and to $\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}$ in (235) induces isomorphisms: $\tilde{\mathbb{E}}_{33} \xrightarrow{\cong} \tilde{\mathbb{E}}_{11}$, $z_{33} \mapsto z_{33}|_{\tilde{\mathbb{P}}_{\pi_\alpha^\vee}}$ and $\tilde{\mathbb{E}}_{11} \xrightarrow{\cong} \tilde{\mathbb{E}}_{22}$, $z_{11} \mapsto z_{11}|_{\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}}$.*

Proof. — We only show the first claim, the second can be proved in an identical manner. Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee}, \tilde{\mathbb{M}}_{\mathbf{1}_T^\vee}) = 0$ by Lemma 10.73, every endomorphism of $\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}$ maps $\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$ to itself. Hence, we obtain a well defined map $r : \tilde{\mathbb{E}}_{33} \rightarrow \tilde{\mathbb{E}}_{11}$. Now both $\tilde{\mathbb{E}}_{33}$ and $\tilde{\mathbb{E}}_{11}$ are \mathcal{O} -torsion free, since $\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}$ and $\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$ are by Corollary 5.19. Nakayama's lemma for compact \mathcal{O} -modules applied to the cokernel and then to the kernel of r implies that it is enough to show that $r \otimes_{\mathcal{O}} k : \tilde{\mathbb{E}}_{33} \otimes_{\mathcal{O}} k \rightarrow \tilde{\mathbb{E}}_{11} \otimes_{\mathcal{O}} k$ is an isomorphism. Let J_π be an injective envelope of an irreducible representation π in $\mathrm{Mod}_{G/Z}^{\mathrm{ladm}}(k)$, \mathbb{P}_{π^\vee} projective envelope of π^\vee in $\mathfrak{C}(k)$ and $\tilde{\mathbb{P}}_{\pi^\vee}$ projective envelope of π^\vee in $\mathfrak{C}(\mathcal{O})$. Then

$$\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\pi^\vee}) \otimes_{\mathcal{O}} k \cong \mathrm{End}_{\mathfrak{C}(k)}(\mathbb{P}_{\pi^\vee}) \cong \mathrm{End}_G(J_\pi)^{\mathrm{op}},$$

where the first isomorphism follows from (28) in Section 3.1, the second since J_π^\vee is a projective envelope of π^\vee and thus is isomorphic to \mathbb{P}_{π^\vee} . Now the assertion of the Proposition follows from Lemma 10.60. \square

Corollary 10.77. — *Let z lie in the centre of $\tilde{\mathbb{E}}_{33}$. If the restriction of z to any of $\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$, $\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}$ or $\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}$ is equal to zero then $z = 0$.*

Proof. — Since $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ is equivalent to the category of compact $\tilde{\mathbb{E}}_{\mathfrak{B}}$ -modules, for every object M of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$, z defines a functorial homomorphism $z_M : M \rightarrow M$. It follows from the functoriality that for every subobject N of M , z_N is equal to the restriction of z_M to N . The assertion follows from Proposition 10.76 and this observation. \square

Corollary 10.78. — *The rings $\tilde{\mathbb{E}}_{11}$, $\tilde{\mathbb{E}}_{22}$ and $\tilde{\mathbb{E}}_{33}$ are naturally isomorphic to \mathbb{R}^{ψ} . In particular, they are commutative noetherian integral domains.*

Proof. — The isomorphism $\tilde{\mathbb{E}}_{11} \cong \mathbb{R}^{\psi}$ in Theorem 10.71 is natural since it is induced by a morphism of deformation functors. The sequences (234) and (235) are not canonical, but are minimal projective resolutions of $\tilde{M}_{1_G^{\vee}}$ and $\tilde{M}_{\alpha^{\vee}}$ respectively. Since any two minimal projective resolutions of the same object are isomorphic, a different choice of an exact sequence in (234) would conjugate the homomorphism $\tilde{\mathbb{E}}_{33} \rightarrow \tilde{\mathbb{E}}_{11}$ by an element of $\tilde{\mathbb{E}}_{33}$. Since as a consequence of Proposition 10.76 all the rings are isomorphic and hence are commutative, we deduce that the homomorphism $\tilde{\mathbb{E}}_{33} \rightarrow \tilde{\mathbb{E}}_{11}$ does not depend on the choice of (234). The last assertion follows from the explicit description of \mathbb{R}^{ψ} in Corollary B.5 below. \square

Corollary 10.79. — *For $i = 1, 2, 3$, $\tilde{\mathfrak{a}}_{ii}$ is the annihilator of $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_{\alpha^{\vee}}}, \tilde{M}_{\alpha^{\vee}})$, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\mathrm{Sp}^{\vee}}, \tilde{M}_{1_{\Gamma,0}^{\vee}})$ and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{1_G^{\vee}}, \tilde{M}_{1_{\Gamma}^{\vee}})$ respectively. Moreover, $\tilde{\mathbb{E}}_{ii}/\tilde{\mathfrak{a}}_{ii}$ is \mathcal{O} -torsion free.*

Proof. — The proof in all cases is the same. We deal with $i = 1$. By applying $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_{\alpha^{\vee}}}, *)$ to (234) we deduce that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_{\alpha^{\vee}}}, \tilde{M}_{\alpha^{\vee}}) = \psi_1 \circ \tilde{\mathbb{E}}_{11} \cong \tilde{\mathbb{E}}_{11}/\tilde{\mathfrak{a}}_{11}$. Since $\tilde{\mathbb{E}}_{11}$ is commutative the annihilator of ψ_1 coincides with the annihilator $\psi_1 \circ \tilde{\mathbb{E}}_{11}$. Further, since $\tilde{M}_{\alpha^{\vee}}$ is \mathcal{O} -torsion free so is $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_{\alpha^{\vee}}}, \tilde{M}_{\alpha^{\vee}})$ and hence $\tilde{\mathbb{E}}_{11}/\tilde{\mathfrak{a}}_{11}$. \square

It follows from Corollary B.5 that \mathbb{R}^{ψ} is \mathcal{O} -torsion free. Thus we have an injection $\mathbb{R}^{\psi} \hookrightarrow \mathbb{R}^{\psi}[1/p]$. Let \mathfrak{r} be the intersection of the reducible locus in $\mathbb{R}^{\psi}[1/p]$ with \mathbb{R}^{ψ} , see Corollary B.6.

Lemma 10.80. — *The image of \mathfrak{r} in $\tilde{\mathbb{E}}_{ii}$ via the natural isomorphism of Corollary 10.78 is equal to $\tilde{\mathfrak{a}}_{ii}$.*

Proof. — If $i = 1$ then the assertion follows from Proposition 10.47 and Theorem 10.71. We claim that the isomorphisms $\tilde{\mathbb{E}}_{33} \xrightarrow{\cong} \tilde{\mathbb{E}}_{11} \xrightarrow{\cong} \tilde{\mathbb{E}}_{22}$ of Proposition 10.76 identify $\tilde{\mathfrak{a}}_{33}$ with $\tilde{\mathfrak{a}}_{11}$ and $\tilde{\mathfrak{a}}_{11}$ with $\tilde{\mathfrak{a}}_{22}$. Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{1_G^{\vee}}, \tilde{M}_{\alpha^{\vee}})$ and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_{\alpha^{\vee}}}, \tilde{M}_{1_{\Gamma,0}^{\vee}})$ are zero by Lemma 10.73, using Corollary 10.79, we get that the image of $\tilde{\mathfrak{a}}_{33}$ is contained in $\tilde{\mathfrak{a}}_{11}$ and the image of $\tilde{\mathfrak{a}}_{11}$ is contained in $\tilde{\mathfrak{a}}_{22}$. Hence, we obtain surjections $\tilde{\mathbb{E}}_{33}/\tilde{\mathfrak{a}}_{33} \rightarrow \tilde{\mathbb{E}}_{11}/\tilde{\mathfrak{a}}_{11} \rightarrow \tilde{\mathbb{E}}_{22}/\tilde{\mathfrak{a}}_{22}$. Since $\tilde{\mathbb{E}}_{ii}/\tilde{\mathfrak{a}}_{ii}$ is \mathcal{O} -torsion free it is enough to show that the surjections are isomorphisms after tensoring with k . This assertion follows from the last assertion in Lemma 10.60. \square

We embed R^ψ into $\tilde{E}_{\mathfrak{B}}$ diagonally using the isomorphisms of Corollary 10.78:

$$(244) \quad R^\psi \hookrightarrow \tilde{E}_{11} \oplus \tilde{E}_{22} \oplus \tilde{E}_{33} \hookrightarrow \tilde{E}_{\mathfrak{B}}, \quad z \mapsto z_{11} \oplus z_{22} \oplus z_{33}.$$

Lemma 10.81. — *Let $\psi \in \tilde{E}_{\mathfrak{B}}$ such that $e_3 \circ \psi \circ e_2 = 0$ then $z \circ \psi = \psi \circ z$ for all $z \in R^\psi$.*

Proof. — It follows from the definition of the embedding that z commutes with φ_{31} and φ_{12} and hence with their composition φ_{32} . Since the rings \tilde{E}_{11} , \tilde{E}_{22} and \tilde{E}_{33} are commutative the assertion follows from Lemma 10.74. \square

Lemma 10.82. — $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\mathcal{O}, \tilde{P}_{1_G^\vee}) = 0$.

Proof. — If not then by composing $\tilde{P}_{1_G^\vee} \rightarrow \mathcal{O} \rightarrow \tilde{P}_{1_G^\vee}$ we would obtain a zero divisor in $\text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{1_G^\vee}) \cong R^\psi$. \square

Lemma 10.83. — *Let $\tilde{P}_{1_G^\vee,0}$ be the kernel of $\tilde{P}_{1_G^\vee} \rightarrow \mathcal{O}$. Then restriction induces an isomorphism $\text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{1_G^\vee}) \cong \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{1_G^\vee,0})$.*

Proof. — It follows from (243) that $\tilde{P}_{1_G^\vee,0}$ is a quotient of $\tilde{P}_{\text{Sp}^\vee} \oplus \tilde{P}_{\pi_\alpha^\vee}$, which implies that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{1_G^\vee,0}, \mathcal{O}) = 0$. Thus every endomorphism of $\tilde{P}_{1_G^\vee}$ maps $\tilde{P}_{1_G^\vee,0}$ to itself. Since $\tilde{P}_{1_G^\vee,0}$ contains the image of φ_{13} the assertion follows from Proposition 10.76. \square

Lemma 10.84. — *Let N be an object of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. Then G acts trivially on N if and only if $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_\alpha^\vee} \oplus \tilde{P}_{\text{Sp}^\vee}, N) = 0$.*

Proof. — Let N be an object of $\mathfrak{C}(\mathcal{O})$ then $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_\alpha^\vee} \oplus \tilde{P}_{\text{Sp}^\vee}, N) = 0$ is equivalent to the assertion that none of the irreducible subquotients of N are isomorphic to π_α^\vee or Sp^\vee . If N is an object of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ then the last condition is equivalent to the assertion that all the irreducible subquotients of N are isomorphic to $\mathbf{1}_G^\vee$, which is equivalent to G acting trivially on N by Lemma 10.25. \square

Corollary 10.85. — *Let $\tilde{P} := \tilde{P}_{\pi_\alpha^\vee} \oplus \tilde{P}_{\text{Sp}^\vee}$ and let $\tilde{E} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P})$. The functor $\mathcal{T}N \mapsto \text{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T}\tilde{P}, \mathcal{T}N)$ induces an equivalence of categories between $\mathfrak{Q}(\mathcal{O})^{\mathfrak{B}}$ and the category of compact \tilde{E} -modules.*

Proof. — It follows from Lemma 10.84 that the category $\mathfrak{I}(\mathcal{O})$, defined in Section 10.3, is precisely the kernel of the functor $N \mapsto \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, N)$. Given this, the assertion follows from [35, §IV.4, Thm. 4]. \square

Lemma 10.86. — *Let $z \in \tilde{E}_{33}$ and let z_{11} and z_{22} denote the restriction of z to $\tilde{P}_{\pi_\alpha^\vee}$ and $\tilde{P}_{\text{Sp}^\vee}$ respectively via (234) and (235). Then $\xi \circ (z_{11} \oplus z_{22}) = z \circ \xi$ for all $\xi \in \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi_\alpha^\vee} \oplus \tilde{P}_{\text{Sp}^\vee}, \tilde{P}_{1_G^\vee})$.*

Proof. — Let $\tilde{\mathbb{P}} := \tilde{\mathbb{P}}_{\pi_\alpha^\vee} \oplus \tilde{\mathbb{P}}_{\text{Sp}^\vee}$, $\tilde{\mathbb{E}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}})$, let $\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0}$ be the kernel of $\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee} \rightarrow \mathcal{O}$ and let

$$m := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0}) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0}).$$

Since $\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0}$ is a quotient of $\tilde{\mathbb{P}}$ by (243), Lemma 2.10 implies that the natural map $m \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} \rightarrow \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0}$ is surjective. Let \mathbf{K} be the kernel, Lemma 2.9 implies that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}, \mathbf{K}) = 0$. Thus G acts trivially on \mathbf{K} by Lemma 10.84. It follows from Lemma 10.82 that every endomorphism of $m \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}$ maps \mathbf{K} to itself. Thus we obtain well defined sequence of maps

$$\text{End}_{\tilde{\mathbb{E}}}(\mathfrak{m}) \rightarrow \text{End}_{\mathfrak{C}(\mathcal{O})}(m \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}}) \rightarrow \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0}) \rightarrow \text{End}_{\tilde{\mathbb{E}}}(\mathfrak{m}),$$

in which composition of any three consecutive one is an identity. The arrows are given by $\phi \mapsto [\xi \widehat{\otimes} v \mapsto \phi(\xi) \widehat{\otimes} v]$; $\phi \mapsto [\xi \widehat{\otimes} v + \mathbf{K} \mapsto \phi(\xi \widehat{\otimes} v) + \mathbf{K}]$; $\phi \mapsto [\xi \mapsto \phi \circ \xi]$ respectively, see also the proof of Proposition 4.19. Since $z_{11} \oplus z_{22}$ lies in the centre of $\tilde{\mathbb{E}}$ by Lemma 10.81, it defines an element of $\text{End}_{\tilde{\mathbb{E}}}(\mathfrak{m})$ by $\xi \mapsto \xi \circ (z_{11} \oplus z_{22})$. Let z' be the image of $z_{11} \oplus z_{22}$ in $\text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0})$ via the above maps. Tautologically we have $z' \circ \xi = \xi \circ (z_{11} \oplus z_{22})$. From (243) we obtain a commutative diagram

$$\begin{array}{ccccc} \tilde{\mathbb{P}}_{\pi_\alpha^\vee} \oplus \tilde{\mathbb{P}}_{\text{Sp}^\vee} & \xrightarrow{\varphi_{13} \oplus \beta} & m \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} & \longrightarrow & \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0} \\ \downarrow z_{11} \oplus z_{22} & & \downarrow z_{11} \oplus z_{22} & & \downarrow z' \\ \tilde{\mathbb{P}}_{\pi_\alpha^\vee} \oplus \tilde{\mathbb{P}}_{\text{Sp}^\vee} & \xrightarrow{\varphi_{13} \oplus \beta} & m \widehat{\otimes}_{\tilde{\mathbb{E}}} \tilde{\mathbb{P}} & \longrightarrow & \tilde{\mathbb{P}}_{\mathbf{1}_G^\vee,0} \end{array}$$

Thus the restriction of z' to $\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$ is equal to z_{11} , which is equal to the restriction of z to $\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$. It follows from Proposition 10.76 and Lemma 10.83 that $z = z'$. \square

Theorem 10.87. — *The centre of $\tilde{\mathbb{E}}_{\mathfrak{B}}$ (and hence the centre of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$) is naturally isomorphic to \mathbb{R}^ψ , defined in Definition 10.44.*

Proof. — It follows from Lemma 10.81 and Lemma 10.86 that the image of \mathbb{R}^ψ via (244) lies in the centre of $\tilde{\mathbb{E}}_{\mathfrak{B}}$. Conversely, suppose that z' lies in the centre of $\tilde{\mathbb{E}}$. Since the restriction map $z \mapsto z|_{\tilde{\mathbb{P}}_{\pi_\alpha^\vee}}$ induces an isomorphism $\mathbb{R}^\psi \cong \tilde{\mathbb{E}}_{11}$ there exists $z \in \mathbb{R}^\psi$ such that $(z - z')|_{\tilde{\mathbb{P}}_{\pi_\alpha^\vee}} = 0$. It follows from Corollary 10.77 that $z = z'$. \square

Remark 10.88. — We note that it is shown in Corollary B.16 below that sending deformation to its trace induces an isomorphism between the ring \mathbb{R}^ψ and $\mathbb{R}_{\text{tr}\rho}^{\text{ps},\psi}$ the deformation ring parameterizing 2-dimensional pseudocharacters lifting $\text{tr}\rho$ with determinant ψ .

Corollary 10.89. — Let $T : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{R}_{\text{tr}\rho}^{\text{ps}, \zeta\varepsilon}$ be the universal 2-dimensional pseudocharacter with determinant $\zeta\varepsilon$ lifting $\text{tr}\rho$. For every N in $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$, $\check{\mathbf{V}}(N)$ is killed by $g^2 - T(g)g + \zeta\varepsilon(g)$, for all $g \in \mathcal{G}_{\mathbf{Q}_p}$.

Proof. — If $N \cong \tilde{\mathbf{P}}_{\pi_\alpha^\vee}$ then the assertion follows from Theorem 10.71 and Corollary B.16. Since $\check{\mathbf{V}}$ is exact, $\check{\mathbf{V}}(\tilde{\mathbf{P}}_{\text{Sp}^\vee})$ is a $\mathcal{G}_{\mathbf{Q}_p}$ -subrepresentation $\check{\mathbf{V}}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee})$, see (235), thus the assertion also holds for $N \cong \tilde{\mathbf{P}}_{\text{Sp}^\vee}$ and hence for $\tilde{\mathbf{P}} := \tilde{\mathbf{P}}_{\pi_\alpha^\vee} \oplus \tilde{\mathbf{P}}_{\text{Sp}^\vee}$. Let $\tilde{\mathbf{E}} := \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}})$ then Proposition 10.36 and Lemma 5.53 imply that

$$\check{\mathbf{V}}(N) \cong \check{\mathbf{V}}(\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}, N) \hat{\otimes}_{\tilde{\mathbf{E}}} \tilde{\mathbf{P}}) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}, N) \hat{\otimes}_{\tilde{\mathbf{E}}} \check{\mathbf{V}}(\tilde{\mathbf{P}}),$$

which implies the claim. \square

Lemma 10.90. — $\tilde{\mathbf{E}}_{\mathfrak{B}}$ is a finitely generated torsion-free \mathbf{R}^ψ -module.

Proof. — It is enough to prove the statement for $e_i \tilde{\mathbf{E}}_{\mathfrak{B}} e_j$, $i, j = 1, 2, 3$. If $(i, j) \neq (3, 2)$ then the assertion follows from Lemma 10.74 as $\tilde{\mathbf{E}}_{ii} \cong \mathbf{R}^\psi$, $\tilde{\mathbf{a}}_{ii} \cong \mathfrak{t}$ and \mathbf{R}^ψ is an integral domain. Let $\mathfrak{m} := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\text{Sp}^\vee}, \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee})$. It follows from the proof of Lemma 10.86 that \mathfrak{m} is generated over \mathbf{R}^ψ by φ_{23} and β . It remains to show that \mathfrak{m} is torsion free. We may dualize (178) and using Proposition 5.21 lift it to an exact sequence:

$$(245) \quad 0 \rightarrow \tilde{\mathbf{P}}_{\text{Sp}^\vee} \rightarrow \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee} \oplus \tilde{\mathbf{P}}_{\pi_\alpha^\vee} \rightarrow \tilde{\mathbf{P}}_{\pi_\alpha^\vee}^{\oplus 2} \rightarrow \tilde{\mathbf{P}}_{\text{Sp}^\vee} \oplus \tilde{\mathbf{P}}_{\pi_\alpha^\vee} \rightarrow \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee} \rightarrow \mathcal{O} \rightarrow 0.$$

We apply $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\text{Sp}^\vee}, *)$ to (245) and use Lemma 10.74 to obtain an exact sequence $0 \rightarrow \mathbf{R}^\psi \rightarrow \mathfrak{m} \oplus \mathbf{R}^\psi \rightarrow \mathbf{R}^\psi \oplus \mathbf{R}^\psi$ of \mathbf{R}^ψ -modules. Since \mathbf{R}^ψ is an integral domain we deduce that \mathfrak{m} is torsion free. \square

For $\delta \in \tilde{\mathbf{E}}_{\mathfrak{B}}$ we let $\delta_{ij} = e_i \circ \delta \circ e_j$. This notation is consistent with (244).

Lemma 10.91. — Let $\delta, \gamma \in \tilde{\mathbf{E}}_{\mathfrak{B}}$ then the image of $\delta_{ij} \circ \gamma_{ji}$ under the isomorphism $\tilde{\mathbf{E}}_{ii} \cong \tilde{\mathbf{E}}_{jj}$ of Proposition 10.76 is equal to $\gamma_{ji} \circ \delta_{ij}$.

Proof. — There exists $z \in \mathbf{R}^\psi$ such that $z_{ii} = \delta_{ij} \circ \gamma_{ji}$. Since $(\gamma_{ji} \circ \delta_{ij} - z_{ii}) \circ \gamma_{ji} = \gamma_{ji} \circ z_{ii} - z_{ii} \circ \gamma_{ji} = 0$ and $\tilde{\mathbf{E}}_{\mathfrak{B}}$ is a torsion free \mathbf{R}^ψ -module we obtain the claim. \square

By Corollary B.6 the ideal \mathfrak{t} is generated by two elements e_0 and e_1 . For each pair $e_i \tilde{\mathbf{P}}_{\mathfrak{B}}$ and $e_j \tilde{\mathbf{P}}_{\mathfrak{B}}$ appearing in (240), (241), (242) we may choose $\varphi_{ji}^k \in \text{Hom}_{\mathfrak{C}(\mathcal{O})}(e_i \tilde{\mathbf{P}}_{\mathfrak{B}}, e_j \tilde{\mathbf{P}}_{\mathfrak{B}})$ such that $\varphi_{ij} \circ \varphi_{ji}^k = c_k e_i$ for $k = 0$ and $k = 1$. It follows from Lemma 10.74 that the elements φ_{ji}^0 and φ_{ji}^1 generate $e_j \tilde{\mathbf{E}}_{\mathfrak{B}} e_i$ as an \mathbf{R}^ψ -module, which is isomorphic to \mathfrak{t} . It follows from Lemma 10.91 that $\varphi_{ji}^k \circ \varphi_{ij} = c_k e_j$ for $k = 0, 1$. We record this below:

$$(246) \quad \varphi_{12} \circ \varphi_{21}^k = c_k e_1, \quad \varphi_{31} \circ \varphi_{13}^k = c_k e_3, \quad \varphi_{32} \circ \varphi_{23}^k = c_k e_3,$$

$$(247) \quad \varphi_{21}^k \circ \varphi_{12} = c_k \ell_2, \quad \varphi_{13}^k \circ \varphi_{31} = c_k \ell_1, \quad \varphi_{23}^k \circ \varphi_{32} = c_k \ell_2.$$

By definition of φ_{32} and (241) we have:

$$(248) \quad \varphi_{31} \circ \varphi_{12} = \varphi_{32}, \quad \varphi_{12} \circ \varphi_{23}^k = \varphi_{13}^k,$$

$$(249) \quad \varphi_{21}^k \circ \varphi_{13}^l = \varphi_{21}^k \circ \varphi_{12} \circ \varphi_{23}^l = c_k \varphi_{23}^l, \quad \varphi_{32} \circ \varphi_{21}^k = \varphi_{31} \circ \varphi_{12} \circ \varphi_{21}^k = c_k \varphi_{31}.$$

Since $\varphi_{12} \circ (\varphi_{23}^k \circ \varphi_{31} - \varphi_{21}^k) = \varphi_{13}^k \circ \varphi_{31} - c_k \ell_1 = 0$ and (240) is an isomorphism we obtain:

$$(250) \quad \varphi_{23}^k \circ \varphi_{31} = \varphi_{21}^k, \quad \varphi_{13}^k \circ \varphi_{32} = \varphi_{13}^k \circ \varphi_{31} \circ \varphi_{12} = c_k \varphi_{12}.$$

Lemma 10.92. — *Let $\beta \in \text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\text{Sp}^\vee}, \tilde{\mathcal{P}}_{\mathbf{1}_G^\vee})$ be the morphism constructed in Lemma 10.75. Then there exist unique $d_0, d_1 \in \mathbf{R}^\psi$ such that $c_0 \beta = d_0 \varphi_{23}$ and $c_1 \beta = d_1 \varphi_{23}$. Moreover,*

$$(251) \quad \beta \circ \varphi_{21}^k = d_k \varphi_{31}, \quad \varphi_{13}^k \circ \beta = d_k \varphi_{12}, \quad \beta \circ \varphi_{23}^k = d_k \ell_3, \quad \varphi_{23}^k \circ \beta = d_k \ell_2.$$

Proof. — The uniqueness follows from the fact that $\tilde{\mathcal{E}}_{\mathfrak{B}}$ is \mathbf{R}^ψ -torsion free, see Lemma 10.90. It follows from (237) that there exists $d_k \in \mathbf{R}^\psi$ such that $\beta \circ \varphi_{21}^k = d_k \varphi_{31}$. It follows from (247) that $c_k \beta = \beta \circ \varphi_{21}^k \circ \varphi_{12} = d_k \varphi_{31} \circ \varphi_{12} = d_k \varphi_{32}$. There exists $a_k \in \mathbf{R}^\psi$ such that $\beta \circ \varphi_{23}^k = a_k \ell_3$. We may multiply by c_k to get $c_k a_k \ell_3 = d_k \varphi_{32} \circ \varphi_{23}^k = d_k c_k \ell_3$. Since \mathbf{R}^ψ is an integral domain we obtain $a_k = d_k$. Lemma 10.91 implies $\varphi_{23}^k \circ \beta = d_k \ell_2$. Moreover, $\varphi_{13}^k \circ \beta = \varphi_{12} \circ \varphi_{23}^k \circ \beta = d_k \varphi_{12}$. \square

Lemma 10.93. — *Sending $x \mapsto c_0, y \mapsto c_1, z \mapsto d_0, w \mapsto d_1$ induces an isomorphism of rings $\mathcal{O}[[x, y, z, w]]/(xw - yz) \cong \mathbf{R}^\psi$.*

Proof. — Since $\tilde{\mathcal{E}}_{\mathfrak{B}}$ is a torsion free \mathbf{R}^ψ -module, $(c_0 d_1 - c_1 d_0) \beta = (d_0 d_1 - d_1 d_0) \varphi_{23} = 0$ implies $c_0 d_1 = c_1 d_0$. Thus the map is well defined. It is enough to show that it is surjective, since we know that \mathbf{R}^ψ can be presented as $\mathcal{O}[[x, y, z, w]]/(f)$, see Corollary B.5, and $xw - yz$ is a prime element in a factorial ring. Let $\mathfrak{b} := \{b \in \mathbf{R}^\psi : \theta \circ b = 0\}$, where $\theta : \tilde{\mathcal{M}}_{\mathbf{1}_T^\vee} \rightarrow \mathcal{O}$ is defined in (233). Applying $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, *)$ to (236) we obtain a surjection $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathcal{P}}_{\text{Sp}^\vee}) \rightarrow \text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathcal{M}}_{\mathbf{1}_T^\vee, 0})$, thus $\psi_2 \circ \varphi_{23}^0$ and $\psi_2 \circ \varphi_{23}^1$ generate $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathcal{M}}_{\mathbf{1}_T^\vee, 0})$ as an \mathbf{R}^ψ -module. Applying $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, *)$ to (233) we obtain an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathcal{M}}_{\mathbf{1}_T^\vee, 0}) \rightarrow \text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathcal{M}}_{\mathbf{1}_T^\vee}) \rightarrow \mathcal{O} \rightarrow 0.$$

As $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathcal{P}}_{\mathbf{1}_G^\vee}, \tilde{\mathcal{M}}_{\mathbf{1}_T^\vee}) \cong \tilde{\mathcal{E}}_{33}/\tilde{\mathfrak{a}}_{33} \cong \mathbf{R}^\psi/\mathfrak{r} \cong \mathcal{O}[[x, y]]$ we deduce that \mathfrak{b} contains \mathfrak{r} and the images of $\xi_{32} \circ \psi_2 \circ \varphi_{23}^0$ and $\xi_{32} \circ \psi_2 \circ \varphi_{23}^1$ generate $\mathfrak{b}/\mathfrak{r}$ as an \mathbf{R}^ψ -module. Since by definition, see the proof of Lemma 10.86, $\psi_3 \circ \beta = \xi_{32} \circ \psi_2$ and $\beta \circ \varphi_{23}^k = d_k \ell_3$ by (251), we deduce that the images of d_0 and d_1 generate $\mathfrak{b}/\mathfrak{r}$. Hence $\mathbf{R}^\psi/(d_0, d_1, c_0, c_1) \cong \mathcal{O}$ and so the map is surjective. \square

Corollary 10.94. — *The $\mathcal{G}_{\mathbf{Q}_p}$ -representation corresponding to the ideal (c_0, c_1, d_0, d_1) in $\mathbf{R}^\psi[1/p]$ is characterized as the unique non-split extension $0 \rightarrow \mathbf{1} \rightarrow \mathbf{V} \rightarrow \varepsilon \rightarrow 0$.*

Proof. — Let $\mathfrak{n}_0 = (c_0, c_1, d_0, d_1) \subset \mathbf{R}^\psi$. It follows from the proof of Lemma 10.93 that there exists a surjection $\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee} \rightarrow \mathcal{O}$ where G acts trivially on \mathcal{O} . It follows from Lemma 7.10 applied with $\mathfrak{m} = \mathbf{R}^\psi/\mathfrak{n}_0 \cong \mathcal{O}$ that $\Pi := \text{Hom}_{\mathcal{O}}^{\text{cont}}(\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}, \mathbf{L})$ is a parabolic induction of a unitary character, which reduces to the trivial character modulo ϖ . Since $\Pi^G \neq 0$ we deduce that $\Pi \cong (\text{Ind}_P^G \mathbf{1})_{\text{cont}}$ and thus $\mathbf{V}(\Pi) \cong \varepsilon$. Let \mathfrak{n}' be the maximal ideal of $\mathbf{R}^\psi[1/p]$ corresponding to \mathbf{V} and let $\mathfrak{n}'_0 := \mathbf{R}^\psi \cap \mathfrak{n}'$. Theorem 10.71 and Corollary 10.72 imply that $\mathbf{V} \cong \check{\mathbf{V}}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}/\mathfrak{n}'_0\tilde{\mathbf{P}}_{\pi_\alpha^\vee}) \otimes_{\mathcal{O}} \mathbf{L}$. It follows from Proposition 10.36 that

$$\begin{aligned} & \text{Hom}_{\Omega(\mathcal{O})}(\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}, \tilde{\mathbf{P}}_{\pi_\alpha^\vee}/\mathfrak{n}'_0\tilde{\mathbf{P}}_{\pi_\alpha^\vee}) \otimes_{\mathcal{O}} \mathbf{L} \\ & \cong \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\check{\mathbf{V}}(\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\mathbf{1}_T^\vee}), \check{\mathbf{V}}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee}/\mathfrak{n}'_0\tilde{\mathbf{P}}_{\pi_\alpha^\vee})) \otimes_{\mathcal{O}} \mathbf{L} \end{aligned}$$

is non-zero thus $\mathfrak{n}_0 = \mathfrak{n}'_0$. □

Since

$$\tilde{\mathbf{E}}_{\mathfrak{B}} = \begin{pmatrix} \mathbf{R}^\psi e_1 & \mathbf{R}^\psi \varphi_{12} & \mathbf{R}^\psi \varphi_{13}^0 + \mathbf{R}^\psi \varphi_{13}^1 \\ \mathbf{R}^\psi \varphi_{21}^0 + \mathbf{R}^\psi \varphi_{21}^1 & \mathbf{R}^\psi e_2 & \mathbf{R}^\psi \varphi_{23}^0 + \mathbf{R}^\psi \varphi_{23}^1 \\ \mathbf{R}^\psi \varphi_{31} & \mathbf{R}^\psi \varphi_{32} + \mathbf{R}^\psi \beta & \mathbf{R}^\psi e_3 \end{pmatrix}$$

the multiplication in $\tilde{\mathbf{E}}_{\mathfrak{B}}$ is determined by (246), (247), (250) and (251). One may check that the \mathbf{R}^ψ -module structure of $\text{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbf{P}}_{\text{Sp}^\vee}, \tilde{\mathbf{P}}_{\mathbf{1}_G^\vee})$ is completely determined by Lemmas 10.92, 10.93 and Corollary 10.94. We also point out that

$$(252) \quad \tilde{\mathbf{E}} := \text{End}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_\alpha^\vee} \oplus \tilde{\mathbf{P}}_{\text{Sp}^\vee}) = \begin{pmatrix} \mathbf{R}^\psi e_1 & \mathbf{R}^\psi \varphi_{12} \\ \mathbf{R}^\psi \varphi_{21}^0 + \mathbf{R}^\psi \varphi_{21}^1 & \mathbf{R}^\psi e_2 \end{pmatrix}$$

and the multiplication is given by $\varphi_{12} \circ \varphi_{21}^k = c_k e_1$, $\varphi_{21}^k \circ \varphi_{12} = c_k e_2$ for $k = 0, 1$, where c_0 and c_1 are generators of \mathfrak{r} , the intersection of \mathbf{R}^ψ and the reducible locus in $\mathbf{R}^\psi[1/p]$.

Lemma 10.95. — *Let $\tilde{\mathbf{E}}$ be the ring in (252) and let \mathfrak{n} be a maximal ideal of $\mathbf{R}^\psi[1/p]$ with residue field \mathbf{L} containing \mathfrak{r} . Then $\tilde{\mathbf{E}} \otimes_{\mathbf{R}^\psi} \mathbf{R}^\psi[1/p]/\mathfrak{n}$ has two non-isomorphic irreducible modules, both of them 1-dimensional.*

Proof. — Let \mathfrak{b} be the two sided ideal in $\tilde{\mathbf{E}} \otimes_{\mathbf{R}^\psi} \mathbf{R}^\psi[1/p]/\mathfrak{n}$ generated by the images of φ_{12} , φ_{21}^0 , φ_{21}^1 . Since \mathfrak{n} contains $\mathfrak{r} = (c_0, c_1)$ we have $\mathfrak{b}^2 = 0$ and the quotient by \mathfrak{b} of $\tilde{\mathbf{E}} \otimes_{\mathbf{R}^\psi} \mathbf{R}^\psi[1/p]/\mathfrak{n}$ is isomorphic to $\mathbf{L} \times \mathbf{L}$. This implies the assertion. □

Remark 10.96. — We note that the Galois side sees only the quotient category $\Omega(\mathcal{O})^{\mathfrak{B}}$, see Proposition 10.36, and this category is equivalent to the category of compact modules of the endomorphism ring of $\mathcal{T}\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee} \oplus \mathcal{T}\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$, which is isomorphic to the ring in (252) by Corollary 10.85. Moreover, it follows from Proposition 10.36 that the ring is isomorphic to $\mathrm{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}) \oplus \check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee}))$.

Remark 10.97. — We are going to describe $\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee})$ as a $\mathcal{G}_{\mathbf{Q}_p}$ -representation. Corollary 10.72 says that $\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee})$ is the universal deformation of ρ with determinant $\zeta\varepsilon$. Hence, $\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee})/\mathfrak{t}\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee})$ is the universal reducible deformation of ρ with the determinant $\zeta\varepsilon$. Thus we have an exact sequence $0 \rightarrow \mathbf{N}_1 \rightarrow \check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee})/\mathfrak{t}\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee}) \rightarrow \mathbf{N}_\omega \rightarrow 0$, where \mathbf{N}_1 is the deformation of the trivial representation and \mathbf{N}_ω is a deformation of ω to $\mathbf{R}_\rho^\psi/\mathfrak{t}$. One may deduce from Theorem B.16 and the proof of Proposition B.19 that these deformations are universal. We apply $\check{\mathbf{V}}$ to (235) to obtain an exact sequence $0 \rightarrow \check{\mathbf{V}}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}) \rightarrow \check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee}) \rightarrow \check{\mathbf{V}}(\tilde{\mathbb{M}}_{\alpha^\vee}) \rightarrow 0$. Proposition 10.47 implies that \mathfrak{t} acts trivially on $\check{\mathbf{V}}(\tilde{\mathbb{M}}_{\alpha^\vee})$, and since all the irreducible subquotients of $\check{\mathbf{V}}(\tilde{\mathbb{M}}_{\alpha^\vee})$ are isomorphic to ω , the surjection $\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee})/\mathfrak{t}\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee}) \twoheadrightarrow \check{\mathbf{V}}(\tilde{\mathbb{M}}_{\alpha^\vee})$ factors through the surjection $\mathbf{N}_\omega \twoheadrightarrow \check{\mathbf{V}}(\tilde{\mathbb{M}}_{\alpha^\vee})$. But both are free $\mathbf{R}_\rho^\psi/\mathfrak{t}$ -modules of rank 1. Hence, the surjection is an isomorphism. This implies that $\check{\mathbf{V}}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee})$ is the kernel of the map from the universal deformation of ρ with determinant $\zeta\varepsilon$ to \mathbf{N}_ω .

Let $\tilde{\mathbb{P}}_*$ be a direct summand of $\tilde{\mathbb{P}}_{\mathfrak{B}}$, let $\tilde{\mathbf{E}}_* := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_*)$. The rings $\tilde{\mathbf{E}}_*$ and $\tilde{\mathbf{E}}_{\mathfrak{B}}$ are finitely generated modules over a noetherian ring \mathbf{R}^ψ , thus they are right and left noetherian. Every finitely generated module carries a canonical topology, with respect to which the action is continuous. Since the rings are noetherian the canonical topology is Hausdorff. Let c be a non-zero element of \mathbf{R}^ψ and let $\mathrm{Mod}_{\tilde{\mathbf{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}]$ denote the full subcategory of finitely generated $\tilde{\mathbf{E}}_{\mathfrak{B}}[1/p]$ -modules consisting of those modules on which c acts invertibly. Define a functor

$$Q : \mathrm{Mod}_{\tilde{\mathbf{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}] \rightarrow \mathrm{Mod}_{\tilde{\mathbf{E}}_*[1/p]}^{\mathrm{fg}}[c^{-1}],$$

$$m \mapsto \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_*, m^0 \otimes_{\tilde{\mathbf{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}})_L,$$

where we have chosen a finitely generated $\tilde{\mathbf{E}}_{\mathfrak{B}}$ -submodule $m^0 \subset m$ such that $m = m^0[1/p]$ and equipped it with the canonical topology. Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_*, \tilde{\mathbb{P}}_{\mathfrak{B}})$ is a finitely generated \mathbf{R}^ψ -module, $Q(m)$ is a finitely generated $\tilde{\mathbf{E}}_*[1/p]$ -module. The definition of Q does not depend on the choice of m^0 , since any two are commensurable.

Lemma 10.98. — *If Q is faithful then it induces an equivalence of categories.*

Proof. — Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, *)$ induces an equivalence between $\mathfrak{C}(\mathcal{O})$ and the category of compact $\tilde{\mathbb{E}}_{\mathfrak{B}}$ -modules, Lemma 2.9 implies that the natural map

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, \mathbb{N}) \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}} \rightarrow \mathbb{N}$$

is an isomorphism. Thus the functor $m^0 \mapsto m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}}$ is exact and hence Q is exact. Define

$$\begin{aligned} R &: \mathrm{Mod}_{\tilde{\mathbb{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}] \rightarrow \mathrm{Mod}_{\tilde{\mathbb{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}], \\ m &\mapsto \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}})_L. \end{aligned}$$

We claim that $Q \circ R$ is equivalent to the identity functor. The claim implies that Q is fully faithful and surjective, hence an equivalence of categories. We may choose $R(m)^0$ to be the maximal \mathcal{O} -torsion free quotient of $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}})$. Then we have a surjection

$$m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}} \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}}) \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}} \twoheadrightarrow R(m)^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}}$$

with the kernel killed by a power of p . Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}}) \cong m^0$, see Lemma 2.9, is \mathcal{O} -torsion free, we get $m^0 \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, R(m)^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}})$. \square

Lemma 10.99. — *Let m be in $\mathrm{Mod}_{\tilde{\mathbb{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}]$ and choose $m^0 \subset m$ as above. Then the kernel of $c : m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}} \rightarrow m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}}$ is zero and the cokernel is killed by a power of p .*

Proof. — Let \mathbb{K} be the kernel and \mathbb{C} be the cokernel. Lemma 2.9 gives an exact sequence

$$0 \rightarrow \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, \mathbb{K}) \rightarrow m^0 \xrightarrow{c} m^0 \rightarrow \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, \mathbb{C}) \rightarrow 0.$$

Since m^0 is finitely generated and c is invertible on m we deduce that there exist p^n such that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, p^n \mathbb{C}) = 0$ and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, \mathbb{K}) = 0$. Since \mathbb{K} and $p^n \mathbb{C}$ are objects of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ this implies that they are 0. \square

Proposition 10.100. — *Let $c \in \mathfrak{r}$ be non-zero and $\tilde{\mathbb{P}}_*$ be either $\tilde{\mathbb{P}}_{\pi_\alpha^\vee}$, $\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}$ or $\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}$ then Q induces an equivalence of categories between $\mathrm{Mod}_{\tilde{\mathbb{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}]$ and $\mathrm{Mod}_{\mathbb{R}^\psi[1/p]}^{\mathrm{fg}}[c^{-1}]$.*

Proof. — Let m be in $\mathrm{Mod}_{\tilde{\mathbb{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}]$ and let $\mathbb{N} = m^0 \widehat{\otimes}_{\tilde{\mathbb{E}}_{\mathfrak{B}}} \tilde{\mathbb{P}}_{\mathfrak{B}}$. It follows from Lemma 10.80 that c kills $\tilde{\mathbb{M}}_{\mathbf{1}_T^\vee}$, $\tilde{\mathbb{M}}_{\alpha^\vee}$. Since c acts invertibly on m Lemma 10.99 implies that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{M}}_*, \mathbb{N}) = 0$ and $\mathrm{Ext}_{\mathfrak{C}(\mathcal{O})}^1(\tilde{\mathbb{M}}_*, \mathbb{N}) = 0$ is killed by a power of p , where $* = \mathbf{1}_T^\vee$ or $* = \alpha^\vee$. Thus (234) and (235) imply that we have an isomorphism of $\mathbb{R}^\psi[1/p]$ -modules:

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathbf{1}_G^\vee}, \mathbb{N})_L \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\pi_\alpha^\vee}, \mathbb{N})_L \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathrm{Sp}^\vee}, \mathbb{N})_L.$$

If $Q(\mathfrak{m}) = 0$ then $0 = \mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}_{\mathfrak{B}}, \mathbf{N})_{\mathbb{L}} \cong \mathfrak{m}$. Hence the functor $\mathfrak{m} \mapsto Q(\mathfrak{m})$ is faithful. The assertion follows from Lemma 10.98. \square

Proposition 10.101. — *Let \mathfrak{n} be the maximal ideal of $\mathbb{R}^{\psi}[1/p]$ corresponding to $0 \rightarrow \mathbf{1} \rightarrow \mathbf{V} \rightarrow \varepsilon \rightarrow 0$, let $c \in \mathbb{R}^{\psi} \cap \mathfrak{n}$ be non-zero and let $\tilde{\mathbb{P}}_* = \tilde{\mathbb{P}}_{\mathrm{Sp}^{\vee}} \oplus \tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}$ then Q induces an equivalence of categories between $\mathrm{Mod}_{\tilde{\mathbb{E}}_{\mathfrak{B}}[1/p]}^{\mathrm{fg}}[c^{-1}]$ and $\mathrm{Mod}_{\tilde{\mathbb{E}}[1/p]}^{\mathrm{fg}}[c^{-1}]$, where $\tilde{\mathbb{E}}$ is the ring described in (252).*

Proof. — We have an exact sequence $\tilde{\mathbb{P}}_* \rightarrow \tilde{\mathbb{P}}_{\mathbf{1}_G^{\vee}} \rightarrow \mathcal{O} \rightarrow 0$, see (243), and c kills \mathcal{O} , see Corollary 10.94. The proof is then the same as the proof of Proposition 10.100. \square

Let \mathfrak{n} be a maximal ideal of $\mathbb{R}^{\psi}[1/p]$ with residue field \mathbb{L} and let $\mathfrak{n}_0 := \mathfrak{n} \cap \mathbb{R}^{\psi}$. Suppose that \mathfrak{n}_0 contains \mathfrak{r} . Then the Galois representation corresponding to \mathfrak{n} is reducible. Thus it follows from Theorem 10.71 and Corollary 10.72 that we have a non-split sequence $0 \rightarrow \psi_1 \rightarrow \check{\mathbf{V}}(\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}) \rightarrow \psi_2 \rightarrow 0$, where $\psi_1, \psi_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathcal{O}^{\times}$ are continuous characters such that ψ_1 is congruent to $\check{\mathbf{V}}(\mathrm{Sp}^{\vee}) = \mathbf{1}$ and ψ_2 is congruent $\check{\mathbf{V}}(\pi_{\alpha}^{\vee}) = \omega$ modulo ϖ .

Proposition 10.102. — *Let \mathfrak{n} , ψ_1 and ψ_2 be as above then we have isomorphisms of Banach space representations of G :*

$$\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\tilde{\mathbb{M}}_{\mathbf{1}_T^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{M}}_{\mathbf{1}_T^{\vee}}, \mathbb{L}) \cong (\mathrm{Ind}_{\mathbb{P}}^G \psi_1 \otimes \psi_2 \varepsilon^{-1})_{\mathrm{cont}},$$

$$\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\tilde{\mathbb{M}}_{\alpha^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{M}}_{\alpha^{\vee}}, \mathbb{L}) \cong (\mathrm{Ind}_{\mathbb{P}}^G \psi_2 \otimes \psi_1 \varepsilon^{-1})_{\mathrm{cont}}.$$

Proof. — Lemma 10.80 identifies \mathfrak{r} with $\tilde{\mathfrak{a}}_{11}$, and by the definition of $\tilde{\mathfrak{a}}_{11}$, we have that $\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}/\tilde{\mathfrak{a}}_{11}\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}$ is the quotient of $\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}$ by the submodule generated by the images of all endomorphisms of $\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}$, whose image lies in the first term $\tilde{\mathbb{P}}_{\mathrm{Sp}^{\vee}}$ of (235). Now using the fact that $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}, \tilde{\mathbb{M}}_{\mathbf{1}_{T,0}^{\vee}}) = 0$, see Lemma 10.73, we deduce that this submodule is precisely the image of the first arrow in (236). Hence, we obtain an exact sequence:

$$(253) \quad 0 \rightarrow \tilde{\mathbb{M}}_{\mathbf{1}_{T,0}^{\vee}} \rightarrow \tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}/\mathfrak{r}\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}} \rightarrow \tilde{\mathbb{M}}_{\alpha^{\vee}} \rightarrow 0.$$

As $\tilde{\mathbb{M}}_{\alpha^{\vee}}$ is $\mathbb{R}^{\psi}/\mathfrak{r}$ -flat, see Corollary 7.3, and \mathfrak{n}_0 contains \mathfrak{r} by applying $\mathbb{R}^{\psi}/\mathfrak{n}_0 \widehat{\otimes}_{\mathbb{R}^{\psi}/\mathfrak{r}}$ we obtain an exact sequence:

$$(254) \quad 0 \rightarrow \tilde{\mathbb{M}}_{\mathbf{1}_{T,0}^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{M}}_{\mathbf{1}_{T,0}^{\vee}} \rightarrow \tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{P}}_{\pi_{\alpha}^{\vee}} \rightarrow \tilde{\mathbb{M}}_{\alpha^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{M}}_{\alpha^{\vee}} \rightarrow 0.$$

Applying $\mathbb{R}^{\psi}/\mathfrak{n}_0 \widehat{\otimes}_{\mathbb{R}^{\psi}/\mathfrak{r}}$ to (233) gives an exact sequence:

$$(255) \quad \tilde{\mathbb{M}}_{\mathbf{1}_{T,0}^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{M}}_{\mathbf{1}_{T,0}^{\vee}} \rightarrow \tilde{\mathbb{M}}_{\mathbf{1}_T^{\vee}}/\mathfrak{n}_0\tilde{\mathbb{M}}_{\mathbf{1}_T^{\vee}} \rightarrow \mathcal{O} \widehat{\otimes}_{\mathbb{R}^{\psi}} \mathbb{R}^{\psi}/\mathfrak{n}_0 \rightarrow 0.$$

Lemma 7.10 implies that $\check{\mathbf{V}}(\tilde{\mathbf{M}}_{\mathbf{T}^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\mathbf{T}^\vee}) \neq 0$. Since $\check{\mathbf{V}}$ is exact and it kills the representations on which \mathbf{G} acts trivially we deduce that

$$(256) \quad \check{\mathbf{V}}(\tilde{\mathbf{M}}_{\mathbf{T}^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\mathbf{T}^\vee}) \cong \psi_1, \quad \check{\mathbf{V}}(\tilde{\mathbf{M}}_{\alpha^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\alpha^\vee}) \cong \psi_2.$$

Lemma 7.10 says that $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\tilde{\mathbf{M}}_{\mathbf{T}^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\mathbf{T}^\vee}, \mathbf{L})$ and $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\tilde{\mathbf{M}}_{\alpha^\vee}/\mathfrak{n}_0\tilde{\mathbf{M}}_{\alpha^\vee}, \mathbf{L})$ are parabolic inductions of unitary characters. As $\mathbf{V}((\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \chi_1 \otimes \chi_2 \varepsilon^{-1})_{\mathrm{cont}}) \cong \chi_2$ and the central character is trivial we deduce the assertion. \square

10.6. Banach space representations. — Let $\psi : \mathbf{T} \rightarrow \mathbf{L}^\times$ be a unitary character. It is shown in [29, 5.3.4] that if $\psi \neq \psi^s$ then $(\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \psi)_{\mathrm{cont}}$ is irreducible and otherwise ψ factors through \det , and so extends to $\psi : \mathbf{G} \rightarrow \mathbf{L}^\times$ and we have a non-split exact sequence of admissible unitary \mathbf{L} -Banach space representations

$$(257) \quad 0 \rightarrow \psi \rightarrow (\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \psi)_{\mathrm{cont}} \rightarrow \widehat{\mathrm{Sp}} \otimes \psi \rightarrow 0,$$

where $\widehat{\mathrm{Sp}}$ is the universal unitary completion of the smooth Steinberg representation over \mathbf{L} . Moreover, $\widehat{\mathrm{Sp}}$ is irreducible, see [18, 4.5.1], [29, 5.1.8 (1)].

Let $\zeta : \mathbf{Z} \rightarrow \mathbf{L}^\times$ be a continuous unitary character and let Π be an admissible unitary \mathbf{L} -Banach space representation of \mathbf{G} with a central character ζ and let Θ be an open bounded \mathbf{G} -invariant lattice in Π . Let $\eta : \mathbf{Q}_p^\times \rightarrow k^\times$ be a smooth character.

Lemma 10.103. — *If $\Theta \otimes_{\mathcal{O}} k$ contains $\eta \circ \det$, $\mathrm{Sp} \otimes \eta \circ \det$ or $(\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \alpha) \otimes \eta \circ \det$ as a subquotient then there exist a unique continuous unitary character $\tilde{\eta} : \mathbf{Q}_p^\times \rightarrow \mathbf{L}^\times$ such that $\zeta = \tilde{\eta}^2$ and $\tilde{\eta} \equiv \eta \pmod{\mathfrak{p}_L}$.*

Proof. — Since Π is unitary, the central character ζ is unitary and \mathbf{Z} acts on $\Theta \otimes_{\mathcal{O}} k$ by the character ζ modulo \mathfrak{p}_L . Since the central character of $\eta \circ \det$, $\mathrm{Sp} \otimes \eta \circ \det$ and $(\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \alpha) \otimes \eta \circ \det$ is η^2 , we deduce that $\zeta \equiv \eta^2 \pmod{\mathfrak{p}_L}$. Let $[\eta] : \mathbf{Q}_p^\times \rightarrow \mathcal{O}^\times$ be the Teichmüller lift of η . Then $\zeta [\eta]^{-2}$ takes values in $1 + \mathfrak{p}_L$. Since $p \neq 2$ we may take a square root by the usual power series expansion. Let $\tilde{\eta} := [\eta] \sqrt{\zeta [\eta]^{-2}}$. This proves existence. For the uniqueness we may assume that both η and ζ are trivial, in which case the assertion follows since (as $p \neq 2$) the equation $X^2 - 1$ has a unique solution in \mathbf{L} , which is congruent to 1 modulo \mathfrak{p}_L . \square

Proposition 10.104. — *Suppose that Π is absolutely irreducible then:*

- (i) *if $\Theta \otimes_{\mathcal{O}} k$ contains $\eta \circ \det$ and does not contain $(\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \alpha) \otimes \eta \circ \det$ as subquotients and*
 - (a) *if $\Theta \otimes_{\mathcal{O}} k$ contains $\mathrm{Sp} \otimes \eta \circ \det$ as a subquotient then $\Pi \cong (\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \psi)_{\mathrm{cont}}$ and $\overline{\Pi} \cong (\mathbf{1} \oplus \mathrm{Sp}) \otimes \eta \circ \det$;*
 - (b) *if $\Theta \otimes_{\mathcal{O}} k$ does not contain $\mathrm{Sp} \otimes \eta \circ \det$ as a subquotient then $\Pi \cong \tilde{\eta} \circ \det$ and $\overline{\Pi} = \eta \circ \det$;*

- (ii) if $\Theta \otimes_{\mathcal{O}} k$ does not contain $\eta \circ \det$ and contains $\mathrm{Sp} \otimes \eta \circ \det$ as subquotients then $\Pi \cong \widehat{\mathrm{Sp}} \otimes \tilde{\eta} \circ \det$ and $\overline{\Pi} \cong \mathrm{Sp} \otimes \eta \circ \det$.
- (iii) if $\Theta \otimes_{\mathcal{O}} k$ contains $(\mathrm{Ind}_{\mathfrak{p}}^G \alpha) \otimes \eta \circ \det$ and does not contain $\mathrm{Sp} \otimes \eta \circ \det$ as subquotients then $\Pi \cong (\mathrm{Ind}_{\mathfrak{p}}^G \psi)_{\mathrm{cont}}$ and $\overline{\Pi} \cong (\mathrm{Ind}_{\mathfrak{p}}^G \alpha) \otimes \eta \circ \det$.

Proof. — After twisting by $\tilde{\eta}^{-1} \circ \det$, constructed in Lemma 10.103, we may assume that η and ζ are trivial. Let $\pi_{\alpha} = \mathrm{Ind}_{\mathfrak{p}}^G \alpha$ and let $\tilde{\mathbf{P}}_{\mathbf{1}_G^{\vee}}$, $\tilde{\mathbf{P}}_{\mathrm{Sp}^{\vee}}$ and $\tilde{\mathbf{P}}_{\pi_{\alpha}^{\vee}}$ be projective envelopes of $\mathbf{1}_G^{\vee}$, Sp^{\vee} and π_{α}^{\vee} in $\mathfrak{C}_{G/Z}(\mathcal{O})$. Let $\tilde{\mathbf{P}}_{\mathbf{1}_T^{\vee}}$ be a projective envelope of the trivial representation of T in $\mathfrak{C}_{T/Z}(\mathcal{O})$ and let $\tilde{\mathbf{M}}_{\mathbf{1}_T^{\vee}} := (\mathrm{Ind}_{\mathfrak{p}}^G (\tilde{\mathbf{P}}_{\mathbf{1}_T^{\vee}})^{\vee})^{\vee}$. Recall that (234) is an exact sequence:

$$(258) \quad 0 \rightarrow \tilde{\mathbf{P}}_{\pi_{\alpha}^{\vee}} \rightarrow \tilde{\mathbf{P}}_{\mathbf{1}_G^{\vee}} \rightarrow \tilde{\mathbf{M}}_{\mathbf{1}_T^{\vee}} \rightarrow 0.$$

Lemma 4.11 says that the Schikhof dual Θ^d is an object of $\mathfrak{C}(\mathcal{O})$. Suppose that $\Theta \otimes_{\mathcal{O}} k$ contains $\mathbf{1}$ and does not contain π_{α} as subquotients. Then Lemma 4.13 implies that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi_{\alpha}^{\vee}}, \Theta^d) = 0$ and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathbf{1}_G^{\vee}}, \Theta^d) \neq 0$. Using (258) we get $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{M}}_{\mathbf{1}_T^{\vee}}, \Theta^d) \neq 0$. The assertion in (i) follows from Proposition 7.11.

Let $(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{1})_{\mathrm{cont}}^0$ be a unit ball in $(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{1})_{\mathrm{cont}}$ with respect to the supremum norm. Let $(\widehat{\mathrm{Sp}})^0$ be the image of $(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{1})_{\mathrm{cont}}^0$ inside $\widehat{\mathrm{Sp}}$, then $(\widehat{\mathrm{Sp}})^0$ is an open bounded G -invariant lattice in $\widehat{\mathrm{Sp}}$. Since $(\mathrm{Ind}_{\mathfrak{p}}^G \mathbf{1})_{\mathrm{cont}}^0 \otimes_{\mathcal{O}} k \cong \mathrm{Ind}_{\mathfrak{p}}^G \mathbf{1}$ we deduce that $(\widehat{\mathrm{Sp}})^0 \otimes_{\mathcal{O}} k \cong \mathrm{Sp}$ and hence $((\widehat{\mathrm{Sp}})^0)^d \otimes_{\mathcal{O}} k \cong \mathrm{Sp}^{\vee}$. Now using (179) and Corollary 5.21 we get an exact sequence

$$\tilde{\mathbf{P}}_{\mathbf{1}_G^{\vee}}^{\oplus 2} \rightarrow \tilde{\mathbf{P}}_{\mathrm{Sp}^{\vee}} \rightarrow ((\widehat{\mathrm{Sp}})^0)^d \rightarrow 0.$$

If $\Theta \otimes_{\mathcal{O}} k$ contains Sp and does not contain $\mathbf{1}$ then $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathrm{Sp}^{\vee}}, \Theta^d) \neq 0$ and $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\mathbf{1}_G^{\vee}}, \Theta^d) = 0$. Hence, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(((\widehat{\mathrm{Sp}})^0)^d, \Theta^d) \neq 0$ and so dually $\mathrm{Hom}_{\mathfrak{L}[G]}^{\mathrm{cont}}(\Pi, \widehat{\mathrm{Sp}}) \neq 0$. As both representations are irreducible and admissible we deduce that $\Pi \cong \widehat{\mathrm{Sp}}$.

The proof of (iii) is identical to the proof of (i), using (235) instead of (234), and $\tilde{\mathbf{M}}_{\alpha^{\vee}}$ instead of $\tilde{\mathbf{M}}_{\mathbf{1}_T^{\vee}}$. \square

Theorem 10.105. — *Suppose that Π is absolutely irreducible and $\Theta \otimes_{\mathcal{O}} k$ contains $\eta \circ \det$, $\mathrm{Sp} \otimes \eta \circ \det$ or $(\mathrm{Ind}_{\mathfrak{p}}^G \alpha) \otimes \eta \circ \det$ as a subquotient then $\overline{\Pi}$ is contained in $(\mathbf{1} \oplus \mathrm{Sp} \oplus \mathrm{Ind}_{\mathfrak{p}}^G \alpha) \otimes \eta \circ \det$. Moreover, if the inclusion is not an isomorphism then we are in one of the cases of Proposition 10.104.*

Proof. — By twisting we may assume that ζ and η are both trivial. Let π be either $\mathbf{1}$, Sp or π_{α} and $\tilde{\mathbf{P}}_{\pi^{\vee}}$ a projective envelope of π^{\vee} in $\mathfrak{C}(\mathcal{O})$. If Π is not one of the representations described in Proposition 10.104 then $\overline{\Pi}$ contains $\mathbf{1}$, Sp and π_{α} . Thus it follows from Lemma 4.15 that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi^{\vee}}, \Theta^d)$ is non-zero. Since by Corollary 10.78, $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi^{\vee}}) \cong \mathbf{R}^{\psi}$ is commutative and Π is absolutely irreducible, we deduce from Theorem 4.36 that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{\mathbf{P}}_{\pi^{\vee}}, \Theta^d)_{\mathbf{L}}$ is an absolutely irreducible finite dimensional

$R^\psi[1/\rho]$ -module. Hence, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\pi^\vee}, \Theta^d)_L$ is one dimensional and Lemma 4.15 implies that π occurs in $\bar{\Pi}$ with multiplicity 1. \square

Let $\mathfrak{B} = \{\eta \circ \det, \mathrm{Sp} \otimes \eta \circ \det, (\mathrm{Ind}_p^G \alpha) \otimes \eta \circ \det\}$, $\pi_{\mathfrak{B}} := (\mathbf{1} \oplus \mathrm{Sp} \oplus \pi_\alpha) \otimes \eta \circ \det$, $\tilde{P}_{\mathfrak{B}}$ a projective envelope of $\pi_{\mathfrak{B}}^\vee$ in $\mathfrak{C}(\mathcal{O})$ and $\tilde{E}_{\mathfrak{B}} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\mathfrak{B}})$. The ring $\tilde{E}_{\mathfrak{B}}$ is a finitely generated module over its centre, and the centre is naturally isomorphic to $R_\chi^{\mathrm{ps}, \varepsilon \zeta}$, see Theorem 10.87 and Remark 10.88, where $R_\chi^{\mathrm{ps}, \varepsilon \zeta}$ is the universal deformation ring parameterizing 2-dimensional pseudocharacters with determinant $\zeta \varepsilon$ lifting $\chi := \eta + \omega \eta$. Let $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}}(\mathbf{L})^{\mathfrak{B}}$ be as in Proposition 5.36 and let $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})^{\mathfrak{B}}$ be the full subcategory consisting of objects of finite length. Let Π be in $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})^{\mathfrak{B}}$, choose an open bounded G -invariant lattice Θ and let $\mathfrak{m}(\Pi) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}_{\mathfrak{B}}, \Theta^d) \otimes_{\mathcal{O}} L$. It follows from Proposition 4.20 that $\mathfrak{m}(\Pi)$ is a finite dimensional L -vector space with continuous $\tilde{E}_{\mathfrak{B}}$ -action. Let \mathfrak{n} be a maximal ideal in $R_\chi^{\mathrm{ps}, \varepsilon \zeta}[1/\rho]$ and $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ the full subcategory of $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})^{\mathfrak{B}}$ consisting of those Π such that $\mathfrak{m}(\Pi)$ is killed by a power of \mathfrak{n} .

Corollary 10.106. — *We have an equivalence of categories*

$$\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})^{\mathfrak{B}} \cong \bigoplus_{\mathfrak{n} \in \mathrm{MaxSpec} R_\chi^{\mathrm{ps}, \varepsilon \zeta}[1/\rho]} \mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}.$$

The category $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\tilde{E}_{\mathfrak{B}}[1/\rho]$.

Proof. — Apply Theorem 4.36 with $\mathfrak{C}(\mathcal{O}) = \mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. \square

Let Π in $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})^{\mathfrak{B}}$ be absolutely irreducible, we say that Π is *ordinary* if it is isomorphic to one of the representations in Proposition 10.104, otherwise we say that Π is *non-ordinary*.

The ring $\tilde{E}_{\mathfrak{B}}$ is described explicitly in Section 10.5. However, in many cases one can give a simpler description of the category $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$. Let \mathfrak{n} be a maximal ideal of $R_\chi^{\mathrm{ps}, \varepsilon \zeta}[1/\rho]$ with residue field L , let $T_{\mathfrak{n}} : \mathcal{G}_{\mathbf{Q}_p} \rightarrow L$ be the pseudocharacter corresponding to \mathfrak{n} and let $\mathrm{Irr}(\mathfrak{n})$ denote the set (of equivalence classes of) irreducible objects in $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm}, \mathrm{fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$.

Proposition 10.107.

(i) if $T_{\mathfrak{n}} = \tilde{\eta} + \tilde{\eta} \varepsilon$ then

$$\mathrm{Irr}(\mathfrak{n}) = \left\{ \tilde{\eta} \circ \det, \widehat{\mathrm{Sp}}_L \otimes \tilde{\eta} \circ \det, \left(\mathrm{Ind}_p^G \tilde{\eta} \varepsilon \otimes \tilde{\eta} \varepsilon^{-1} \right)_{\mathrm{cont}} \right\}.$$

(ii) if $T_{\mathfrak{n}} = \psi_1 + \psi_2$ with $\psi_1, \psi_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow L^\times$ continuous homomorphisms and $T_{\mathfrak{n}} \neq \tilde{\eta} + \tilde{\eta} \varepsilon$ then

$$\mathrm{Irr}(\mathfrak{n}) = \left\{ \left(\mathrm{Ind}_p^G \psi_1 \otimes \psi_2 \varepsilon^{-1} \right)_{\mathrm{cont}}, \left(\mathrm{Ind}_p^G \psi_2 \otimes \psi_1 \varepsilon^{-1} \right)_{\mathrm{cont}} \right\}$$

and $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is naturally equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\tilde{\mathbb{E}}[1/p]$, see (252) for definition and description of $\tilde{\mathbb{E}}$.

- (iii) if $T_{\mathfrak{n}}$ is irreducible then $\text{Irr}(\mathfrak{n}) = \{\Pi_{\mathfrak{n}}\}$ with $\Pi_{\mathfrak{n}}$ absolutely irreducible non-ordinary. The category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}[1/p]$.

Proof. — Suppose that \mathfrak{n} contains the reducible locus in $\mathbf{R}_{\chi}^{\text{ps},\varepsilon\zeta}[1/p]$. Since $\mathbf{1} \neq \omega$ and the residue field of \mathfrak{n} is L , we get that $T_{\mathfrak{n}} = \psi_1 + \psi_2$ with $\psi_1, \psi_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow L^{\times}$ continuous homomorphisms. It follows from Proposition 10.102 that $\text{Irr}(\mathfrak{n})$ contains the semi-simplification of unitary principal series appearing in (ii). Recall that $(\text{Ind}_p^G \chi_1 \otimes \chi_2)_{\text{cont}}$ is irreducible if and only if $\chi_1 \neq \chi_2$. We get that $|\text{Irr}(\mathfrak{n})| \geq 3$ if $T_{\mathfrak{n}} = \tilde{\eta} + \tilde{\eta}\varepsilon$ and $|\text{Irr}(\mathfrak{n})| \geq 2$, otherwise. The representations in $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$, on which $\text{SL}_2(\mathbf{Q}_p)$ acts trivially, form a thick subcategory. The quotient category $\text{QBan}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is equivalent to the category of $\tilde{\mathbb{E}}[1/p]$ -modules of finite length, Theorem 4.36 and Lemma 10.84. Since we have fixed a central character and $p > 2$ any Π in $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ on which $\text{SL}_2(\mathbf{Q}_p)$ acts trivially is isomorphic to $\tilde{\eta}^{\oplus m}$. Hence, if \mathfrak{n} does not kill $m(\tilde{\eta})$ then

$$\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}} \cong \text{QBan}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$$

and the last category is equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\tilde{\mathbb{E}}[1/p]$ by Theorem 4.36. This category has 2-irreducible objects by Lemma 10.95. If \mathfrak{n} kills $m(\tilde{\eta})$ then $\text{QBan}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ has one irreducible object less than $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$. Again Lemma 10.95 allows us to conclude.

Suppose that \mathfrak{n} does not contain the reducible locus then it follows from Proposition 10.100 that $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is equivalent to the category of modules of finite length of the \mathfrak{n} -adic completion of $\mathbf{R}_{\chi}^{\text{ps},\zeta\varepsilon}[1/p]$. This category contains only one irreducible object and hence $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ contains only one irreducible object Π . Since all the ordinary representations have already appeared in (i) and (ii) and $|\text{Irr}(\mathfrak{n})| > 1$ in those cases, we deduce that Π cannot be ordinary. \square

11. p -adic Langlands correspondence

Let Π be a unitary irreducible admissible L -Banach space representation of G with a central character. We say that Π is *ordinary* if Π is either a unitary character $\Pi \cong \eta \circ \det$, a twist of the universal completion of the smooth Steinberg representation by a unitary character $\Pi \cong \widehat{\text{Sp}} \otimes \eta \circ \det$ or Π is a unitary parabolic induction of a unitary character. We assume throughout that $p \geq 5$.

Theorem 11.1. — *Let Π be a unitary admissible absolutely irreducible L -Banach space representation of G with a central character and let Θ be an open bounded G -invariant lattice in Π . Then $\Theta \otimes_{\mathcal{O}} k$ is of finite length. Moreover, one of the following holds:*

- (i) $\Theta \otimes_{\mathcal{O}} k$ is absolutely irreducible supersingular;
- (iii) $\Theta \otimes_{\mathcal{O}} k$ is irreducible and

$$(259) \quad \Theta \otimes_{\mathcal{O}} l \cong \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \otimes \chi^{\sigma} \omega^{-1} \oplus \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi^{\sigma} \otimes \chi \omega^{-1},$$

where l is a quadratic extension of k , $\chi : \mathbf{Q}_p^{\times} \rightarrow l^{\times}$ a smooth character and χ^{σ} is a conjugate of χ by the non-trivial element in $\text{Gal}(l/k)$;

- (iii) $(\Theta \otimes_{\mathcal{O}} k)^{ss} \subseteq (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi_1 \otimes \chi_2 \omega^{-1})^{ss} \oplus (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi_2 \otimes \chi_1 \omega^{-1})^{ss}$ for some smooth characters $\chi_1, \chi_2 : \mathbf{Q}_p^{\times} \rightarrow k^{\times}$.

Further, the inclusion in (iii) is not an isomorphism if and only if Π is ordinary.

Proof. — Let π be an irreducible subquotient of $\Theta \otimes_{\mathcal{O}} k$. Suppose that π is absolutely irreducible. Then it follows from Theorems 6.4, 8.8, 9.31 and 10.105 that either (i) or (iii) holds. Further, if the inclusion in (iii) is not an isomorphism then Π is ordinary, see Corollaries 8.9, 9.32 and Theorem 10.105. If π is not absolutely irreducible then it follows from Corollary 5.44 that $\pi \otimes_k l$ is isomorphic to $\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \otimes \chi^{\sigma} \omega^{-1} \oplus \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi^{\sigma} \otimes \chi \omega^{-1}$. The previous argument applied to $\Pi_{L'}$ where L' is a quadratic unramified extension of L shows that

$$(\Theta \otimes_{\mathcal{O}} l)^{ss} \subseteq \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi \otimes \chi^{\sigma} \omega^{-1} \oplus \text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi^{\sigma} \otimes \chi \omega^{-1}.$$

Since $\Theta \otimes_{\mathcal{O}} k$ contains π we deduce that $\Theta \otimes_{\mathcal{O}} k \cong \pi$. □

We refer the reader to Section 5.7 for the definition of the functors \mathbf{V} and $\check{\mathbf{V}}$.

Corollary 11.2. — *If Π is a unitary admissible absolutely irreducible L-Banach space representation of \mathbb{G} with a central character then $\dim_L \mathbf{V}(\Pi) \leq 2$. Moreover, $\dim_L \mathbf{V}(\Pi) < 2$ if and only if Π is ordinary.*

Proof. — Let Θ be an open bounded \mathbb{G} -invariant lattice in Π . It follows from Theorem 11.1 that $\dim_k \mathbf{V}(\Theta \otimes_{\mathcal{O}} k) \leq 2$ and the equality is strict if and only if the inclusion in Theorem 11.1(iii) is not an isomorphism. Hence $\mathbf{V}(\Theta)$ is a free \mathcal{O} -module of rank at most 2, see [41, 2.2.2] or the proof Lemma 5.51. Since $\mathbf{V}(\Theta)$ is an \mathcal{O} -lattice in $\mathbf{V}(\Pi)$ we get the assertion. □

Let Π be an absolutely irreducible non-ordinary unitary L-Banach space representation of \mathbb{G} with a central character ζ . Then Π is an object of $\text{Ban}_{\mathbb{G}, \zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$ for some block \mathfrak{B} , Proposition 5.36. Let $\overline{\Pi}$ be the semi-simplification of the reduction modulo ϖ of an open bounded \mathbb{G} -invariant lattice Θ in Π . Suppose that $\overline{\Pi}$ contains an absolutely irreducible representation; this can be achieved by replacing L with a quadratic unramified extension. Then there are essentially four possibilities for \mathfrak{B} , described in Proposition 5.42. Recall that $\text{Mod}_{\mathbb{G}, \zeta}^{\text{fin}}(\mathcal{O})^{\mathfrak{B}}$ is the full subcategory of $\text{Mod}_{\mathbb{G}, \zeta}^{\text{fin}}(\mathcal{O})$ consisting of representations with all the irreducible subquotients in \mathfrak{B} , and $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ is the

full subcategory of $\mathfrak{C}(\mathcal{O})$ anti-equivalent to $\text{Mod}_{G,\zeta}^{\text{fin}}(\mathcal{O})^{\mathfrak{B}}$ via Pontryagin duality. The centre of the category $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ is naturally isomorphic to $\mathbf{R}_{\text{tr}\mathbf{V}(\overline{\Pi})}^{\text{ps},\psi}$, the deformation ring parameterizing 2-dimensional pseudocharacters of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant $\psi = \varepsilon\zeta$ lifting $\text{tr}\mathbf{V}(\overline{\Pi})$, Corollaries 6.5, 8.11, 9.34, Theorem 10.87. Since Θ^d is an object of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ and $\Pi \cong \text{Hom}_{\mathcal{O}}^{\text{cont}}(\Theta^d, \mathbf{L})$ we obtain a ring homomorphism:

$$x : \mathbf{R}_{\text{tr}\mathbf{V}(\overline{\Pi})}^{\text{ps},\psi}[1/p] \rightarrow \text{End}_G^{\text{cont}}(\Pi) \cong \mathbf{L},$$

where the last isomorphism follows from Corollary 4.42.

Proposition 11.3. — *The representation $\mathbf{V}(\Pi)$ is absolutely irreducible with determinant $\varepsilon\zeta$. Moreover, $\text{tr}\mathbf{V}(\Pi)$ is equal to the pseudocharacter corresponding to $x \in \text{Spec}\mathbf{R}_{\text{tr}\mathbf{V}(\overline{\Pi})}^{\text{ps},\psi}[1/p]$.*

Proof. — Let $T_x : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{L}$ be the pseudocharacter corresponding to x . There exists a unique semi-simple continuous representation V_x of $\mathcal{G}_{\mathbf{Q}_p}$, defined over a finite extension of \mathbf{L} , such that $\text{tr}V_x = T_x$ and $\det V_x = \zeta\varepsilon$, [65, Thm. 1]. The representation V_x is absolutely irreducible, since otherwise Corollaries 8.15, 9.37 and Proposition 10.107 would imply that Π is ordinary. It follows from Corollaries 6.7, 8.12, 9.33 and 10.89 that $\mathbf{V}(\Pi)$ is killed by $g^2 - T_x(g)g + \varepsilon\zeta(g)$ for all $g \in \mathcal{G}_{\mathbf{Q}_p}$. Since $\mathbf{V}(\Pi)$ is 2-dimensional by Corollary 11.2, the main result of [10] implies that $V_x \cong \mathbf{V}(\Pi)$. \square

Theorem 11.4. — *Assume $p \geq 5$, the functor \mathbf{V} induces a bijection between isomorphism classes of*

- (i) *absolutely irreducible admissible unitary non-ordinary \mathbf{L} -Banach space representations of G with the central character ζ , and*
- (ii) *absolutely irreducible 2-dimensional continuous \mathbf{L} -representations of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant equal to $\zeta\varepsilon$,*

where ε is the cyclotomic character, and we view ζ as a character of $\mathcal{G}_{\mathbf{Q}_p}$ via the class field theory.

Proof. — It follows from Proposition 11.3 that \mathbf{V} maps one set to the other. The surjectivity follows from [41, 2.3.8]. We show injectivity: suppose that $\mathbf{V}(\Pi_1) \cong \mathbf{V}(\Pi_2)$. As $\mathbf{V}(\overline{\Pi}) \cong \overline{\mathbf{V}(\overline{\Pi})}$, Theorem 11.1 implies that Π_1 and Π_2 lie in $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})^{\mathfrak{B}}$ for the same block \mathfrak{B} . Let $x \in \text{Spec}\mathbf{R}_{\text{tr}\mathbf{V}(\overline{\Pi})}^{\text{ps},\psi}[1/p]$ be the maximal ideal corresponding to $\text{tr}\mathbf{V}(\Pi_1) = \text{tr}\mathbf{V}(\Pi_2)$. Proposition 11.3 implies that Π_1 and Π_2 are killed by x and hence are objects of $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})_x^{\mathfrak{B}}$. Since $\mathbf{V}(\Pi_1) \cong \mathbf{V}(\Pi_2)$ is absolutely irreducible this category contains only one irreducible object, see Corollaries 6.8, 8.14, 9.36 and Proposition 10.107. Hence, $\Pi_1 \cong \Pi_2$. \square

Let V be an absolutely irreducible 2-dimensional \mathbf{L} -representation of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant $\psi := \zeta\varepsilon$. Let \overline{V} be the semi-simplification of a reduction modulo ϖ of a

$\mathcal{G}_{\mathbf{Q}_p}$ -stable \mathcal{O} -lattice in V . We assume that \bar{V} is either absolutely irreducible or a direct sum of two one dimensional representations. This can always be achieved by replacing L by a quadratic unramified extension. Let R_V^ψ be the deformation ring representing the deformation problem of V with determinant $\zeta \varepsilon$ to local artinian L -algebras, and let V^u be the universal deformation of V with the determinant $\zeta \varepsilon$.

Lemma 11.5. — *Let m_1, m_2 be R_V^ψ -modules of finite length. Then the natural map $\mathrm{Hom}_{R_V^\psi}(m_1, m_2) \rightarrow \mathrm{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(m_1 \otimes_{R_V^\psi} V^u, m_2 \otimes_{R_V^\psi} V^u)$ is an isomorphism.*

Proof. — The assertion is true if both modules are of length one, since then both groups are isomorphic to $\mathrm{End}_{\mathcal{G}_{\mathbf{Q}_p}}(V) \cong L$. Moreover,

$$\mathrm{Ext}_{R_V^\psi}^1(L, L) \cong \mathrm{Hom}(R_V^\psi, L[\varepsilon]/(\varepsilon^2)) \cong \mathrm{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(V, V).$$

Given this we may finish the proof in the same way as in Proposition 10.36: we argue by induction on $\ell(m_1) + \ell(m_2)$ that the functor $m \mapsto m \otimes_{R_V^\psi} V^u$ induces an isomorphism between Hom -groups and an injection on Ext^1 -groups. \square

Corollary 11.6. — *Let m be a R_V^ψ -module of finite length. Then*

$$m \cong \mathrm{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathrm{cont}}(V^u, m \otimes_{R_V^{\zeta \varepsilon}} V^u).$$

Proof. — Let \mathfrak{m} be the maximal ideal of $R := R_V^\psi$, and let $V(\mathfrak{m}) := m \otimes_R V^u$. Then $\mathrm{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}^{\mathrm{cont}}(V^u, V(\mathfrak{m})) \cong \varinjlim \mathrm{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(V^u/\mathfrak{m}^n V^u, V(\mathfrak{m})) \cong \varinjlim \mathrm{Hom}_R(R/\mathfrak{m}^n, m) \cong m$, where the second isomorphism follows from Lemma 11.5. \square

Let Π be an absolutely irreducible admissible unitary L -Banach space representation of G with central character ζ , corresponding to V by Theorem 11.4, so that $\mathbf{V}(\Pi) \cong V$. Let $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm. fl}}(L)$ be the category of admissible unitary L -Banach space representations of G of finite length with the central character ζ and let $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm. fl}}(L)_\Pi$ be the full subcategory of consisting of the representations with all irreducible subquotients isomorphic to Π .

Theorem 11.7. — *Let $B \in \mathrm{Ban}_{G, \zeta}^{\mathrm{adm}}(L)$ be of finite length with all irreducible subquotients isomorphic to Π and let $m(B) = \mathrm{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(V^u, \check{\mathbf{V}}(B))$ then*

$$\check{\mathbf{V}}(B) \cong m(B) \otimes_{R_V^\psi} V^u.$$

Moreover, the functor $B \mapsto m(B)$ induces an anti-equivalence of categories between $\mathrm{Ban}_{G, \zeta}^{\mathrm{adm. fl}}(L)_\Pi$ and the category of R_V^ψ -modules of finite length.

Proof. — Let \mathfrak{B} be the block corresponding to \bar{V} , so that if \bar{V} is absolutely irreducible then $\mathfrak{B} = \{\pi\}$, where π is a supersingular representation of G , with $\mathbf{V}(\pi) \cong \bar{V}$, and if $\bar{V} \cong \chi_1 \oplus \chi_2$, then \mathfrak{B} consists of all the irreducible subquotients of the principal series representations $\text{Ind}_p^G \chi_1 \otimes \chi_2 \omega^{-1}$ and $\text{Ind}_p^G \chi_2 \otimes \chi_1 \omega^{-1}$, and let \mathcal{Z} be the centre of the category $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. Let Θ be an open bounded G -invariant lattice in Π , and let $\bar{\Pi}$ denote the semi-simplification of $\Theta/\varpi\Theta$. The isomorphism $\mathbf{V}(\Pi) \cong V$ implies that $\mathbf{V}(\bar{\Pi}) \cong \bar{V}$. This implies that Π is an object of $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$ and Θ^d is an object of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$. The action of \mathcal{Z} on Θ^d induces a ring homomorphism $x: \mathcal{Z}[1/p] \rightarrow \text{End}_G(\Pi) \cong \mathbb{L}$, and we let \mathfrak{n} be the kernel of x . Let $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}$ be the full subcategory of $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})^{\mathfrak{B}}$ consisting of all Banach space representations of finite length, which are killed by some power of \mathfrak{n} . We note that Π is in $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}$ by construction of \mathfrak{n} . Moreover, it follows from Corollaries 6.8, 8.14, 9.35 and 9.36 and Proposition 10.107(iii) that Π is the only irreducible object in the category. Hence, $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}} = \text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\Pi}$. The second part of the Corollaries referred to above says that $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is anti-equivalent to the category of modules of finite length over the \mathfrak{n} -adic completion of $\mathcal{Z}[1/p]$. To prove the theorem we need to write out how this equivalence is realized.

If \bar{V} is absolutely irreducible we let π be supersingular representation of G , with $\mathbf{V}(\pi) \cong \bar{V}$. If $\bar{V} \cong \chi_1 \oplus \chi_2$ then we let $\pi = \text{Ind}_p^G \chi_1 \otimes \chi_2 \omega^{-1}$. Since $p \geq 5$ we may assume without loss of generality that $\chi_1 \neq \chi_2 \omega^{-1}$, so that π is irreducible. Let \tilde{P} be a projective envelope of π^\vee in $\mathfrak{C}(\mathcal{O})$ and let $\tilde{E} = \text{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P})$. The action of \mathcal{Z} on \tilde{P} induces a homomorphism of rings $\mathcal{Z} \rightarrow \tilde{E}$. If π is supersingular, or π is a principal series with $\chi_1 \neq \chi_2$ then this map is an isomorphism, see Proposition 6.3, Corollary 8.11 and Proposition B.17, Corollary 10.78 and Theorem 10.87. If $\pi \cong \text{Ind}_p^G \chi \otimes \chi \omega^{-1}$ then $\mathfrak{B} = \{\pi\}$ and so \mathcal{Z} is the centre of \tilde{E} .

The functor $B \mapsto m(B) := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)_L$, where Θ is any open bounded G -invariant lattice in B , is faithful when restricted to $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\Pi}$: since π appears as a subquotient of $\bar{\Pi}$, Lemma 4.13 implies that $m(\Pi) \neq 0$, and the assertion follows from the exactness of m , see Lemma 4.9. Since $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\Pi} = \text{Ban}_{G,\zeta}^{\text{adm}}(\mathbb{L})_{\mathfrak{n}}^{\mathfrak{B}}$ and m is faithful, it follows from Theorem 4.36 that m induces an anti-equivalence of categories between $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\Pi}$ and the category of modules of finite length of the \mathfrak{n} -adic completion of $\tilde{E}[1/p]$. The inverse functor $m \mapsto \Pi(m)$ is defined in Definition 4.26. So that for $B \in \text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbb{L})_{\Pi}$ and Θ an open bounded G -invariant lattice in B , we have $\Theta^d \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \hat{\otimes}_{\tilde{E}} \tilde{P}$. Lemma 5.53 implies that $\check{\mathbf{V}}(\Theta^d) \cong \text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d) \hat{\otimes}_{\tilde{E}} \check{\mathbf{V}}(\tilde{P})$. Since π occurs in \bar{B} with finite multiplicity, Lemma 4.15 implies that $\text{Hom}_{\mathfrak{C}(\mathcal{O})}(\tilde{P}, \Theta^d)$ is a free \mathcal{O} -module of finite rank. In particular, it is finitely generated over \tilde{E} . Since \tilde{E} is noetherian, the module is finitely presented and hence we may replace $\hat{\otimes}$ with \otimes . Hence, $\check{\mathbf{V}}(B) \cong m(B) \otimes_{\tilde{E}} \check{\mathbf{V}}(\tilde{P})$. As $m(B)$ is killed by a power of \mathfrak{n} we may replace \tilde{E} with the \mathfrak{n} -adic completion of $\tilde{E}[1/p]$ and $\check{\mathbf{V}}(\tilde{P})$ with the \mathfrak{n} -adic completion of $\check{\mathbf{V}}(\tilde{P})_L$.

To finish the proof we only have to relate the \mathfrak{n} -adic completion of $\tilde{E}[1/\rho]$ to R_V^ψ and the \mathfrak{n} -adic completion of $\check{V}(\tilde{P})_L$ to V^u . Assume that $\pi \not\cong \text{Ind}_p^G \chi \otimes \chi \omega^{-1}$ for any character χ , so that \tilde{E} is commutative. In this case we know that $\check{V}(\tilde{P})$ is the universal deformation with determinant $\varepsilon\zeta$ of a 2-dimensional representation ρ , and $\tilde{E} \cong R_\rho^\psi$ is the deformation ring representing this deformation problem, where $\rho \cong \mathbf{V}(\pi)$ if π is supersingular and ρ is a non-split extension of χ_2 by χ_1 if $\pi \cong \text{Ind}_p^G \chi_1 \otimes \chi_2 \omega^{-1}$, see Proposition 6.3, Corollary 8.7, Theorem 10.71, Corollary 10.72. Since $V \cong \check{V}(\Pi) \cong L \otimes_{\tilde{E}, x} \check{V}(\tilde{P})$, [39, (2.3.5)] implies that the \mathfrak{n} -adic completion of $R_\rho^\psi[1/\rho]$ is isomorphic to R_V^ψ , and the \mathfrak{n} -adic completion of $\check{V}(\tilde{P})_L$ is isomorphic to V^u . Hence, $\check{V}(B) \cong \mathfrak{m}(B) \otimes_{R_V^\psi} V^u$ for all $B \in \text{Ban}_{G, \zeta}^{\text{adm.fl}}(L)_\Pi$, and Corollary 11.6 implies that $\mathfrak{m}(B) \cong \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(V^u, \check{V}(B))$.

We assume that $\pi \cong \text{Ind}_p^G \chi \otimes \chi \omega^{-1}$, for some character χ , so that $\bar{V} \cong \chi \oplus \chi$. Let $R_{\text{tr}\bar{V}}^{\text{ps}, \psi}$ be the deformation ring parameterizing 2-dimensional pseudocharacters of $\mathcal{G}_{\mathbf{Q}_p}$ with determinant $\psi = \varepsilon\zeta$ lifting $\text{tr}\bar{V}$, and let $T : \mathcal{G}_{\mathbf{Q}_p} \rightarrow R_{\text{tr}\bar{V}}^{\text{ps}, \psi}$ be the universal pseudocharacter with determinant ψ lifting $\text{tr}\bar{V}$. In this case \tilde{E} is isomorphic to the opposite ring of $R_{\text{tr}\bar{V}}^{\text{ps}, \psi}[[\mathcal{G}_{\mathbf{Q}_p}]]/J$, where J is a closed two-sided ideal generated by the elements $g^2 - T(g)g + \psi(g)$ for all $g \in \mathcal{G}_{\mathbf{Q}_p}$, and $\mathcal{Z} \cong R_{\text{tr}\bar{V}}^{\text{ps}, \psi}$, see Section 9.3. Moreover, $\check{V}(\tilde{P})$ is a free \tilde{E} -module of rank 1, see Section 9.1. It follows from (144) that if \mathfrak{m} is a compact right \tilde{E} -module then $\check{V}(\mathfrak{m} \otimes_{\tilde{E}} \tilde{P}) \cong \mathfrak{m} \otimes_{\tilde{E}} \check{V}(\tilde{P}) \cong \mathfrak{m}$, where the action of $\mathcal{G}_{\mathbf{Q}_p}$ on \mathfrak{m} is induced by the natural homomorphism $\mathcal{G}_{\mathbf{Q}_p} \rightarrow R_{\text{tr}\bar{V}}^{\text{ps}, \psi}[[\mathcal{G}_{\mathbf{Q}_p}]]/J$. In particular, $\check{V}(\tilde{P}) \cong R_{\text{tr}\bar{V}}^{\text{ps}, \psi}[[\mathcal{G}_{\mathbf{Q}_p}]]/J$ with $\mathcal{G}_{\mathbf{Q}_p}$ acting on the left. Since the specialization of T at \mathfrak{n} is the trace of V , the \mathfrak{n} -adic completion of $R_{\text{tr}\bar{V}}^{\text{ps}, \psi}$ is isomorphic to $R_{\text{tr}V}^{\text{ps}, \psi}$.

Let \mathcal{E} be the \mathfrak{n} -adic completion of $\tilde{E}[1/\rho]$. Corollary 9.29 implies that \mathcal{E} is isomorphic to the ring of 2×2 -matrices over $R_{\text{tr}V}^{\text{ps}, \psi}$. Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $e\mathcal{E}$ is a free $R_{\text{tr}V}^{\text{ps}, \psi}$ -module of rank 2 with a continuous $\mathcal{G}_{\mathbf{Q}_p}$ -action. Since for every invertible 2×2 -matrix A we have $A + (\det A)A^{-1} = (\text{tr} A)\text{Id}$, the trace of $\mathcal{G}_{\mathbf{Q}_p}$ -representation on $e\mathcal{E}$ is equal to T , and the trace of $\mathcal{G}_{\mathbf{Q}_p}$ -representation on $L \otimes_{R_{\text{tr}V}^{\text{ps}, \psi}} e\mathcal{E}$ is equal to $\text{tr} V$. Since V is irreducible, this implies $e\mathcal{E}$ is a deformation of V to $R_{\text{tr}V}^{\text{ps}, \psi}$, which induces a ring homomorphism $R_V^\psi \rightarrow R_{\text{tr}V}^{\text{ps}, \psi}$. Moreover, the composition $R_{\text{tr}V}^{\text{ps}, \psi} \rightarrow R_V^\psi \rightarrow R_{\text{tr}V}^{\text{ps}, \psi}$, where the first arrow is induced by taking a trace of a deformation of V , is the identity map. Since V is absolutely irreducible the first arrow is an isomorphism by [50], and hence the second arrow is an isomorphism, which implies that $e\mathcal{E} \cong V^u$. Since the same argument applies with $1 - e$ instead of e , we deduce that the \mathfrak{n} -adic completion of $\check{V}(\tilde{P})_L$ is isomorphic to $V^u \oplus V^u$ as a $\mathcal{G}_{\mathbf{Q}_p}$ -representation, and \mathcal{E} is the ring of 2×2 -matrices over R_V^ψ . Thus the rings R_V^ψ and \mathcal{E} are Morita equivalent, which implies that $\check{V}(B) \cong \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(V^u, \check{V}(B)) \otimes_{R_V^\psi} V^u$ for all $B \in \text{Ban}_{G, \zeta}^{\text{adm.fl}}(L)_\Pi$. \square

Remark 11.8. — Since $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}} = \text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\Pi}$, it follows from Proposition 5.36 and Theorem 4.36 that the category $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\Pi}$ is a direct summand of $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})$. Concretely this means that every admissible unitary L-Banach space representation B , which is of finite length and has a central character ζ decomposes as $B \cong B_1 \oplus B_2$, where all the irreducible subquotients of B_1 are isomorphic to Π and none of the irreducible subquotients of B_2 is isomorphic to Π .

Remark 11.9. — We note that one may also prove an analog of Theorem 11.7, when V is reducible. Let $\psi_1, \psi_2 : \mathbf{Q}_p^\times \rightarrow \mathbf{L}^\times$ be unitary characters satisfying $\psi_1 \psi_2 = \varepsilon \zeta$. Let $\chi_1, \chi_2 : \mathbf{Q}_p^\times \rightarrow k^\times$ be their reduction modulo ϖ . Let \mathfrak{B} be the block corresponding to $\chi_1 \oplus \chi_2$, \mathcal{Z} the centre of $\mathfrak{C}(\mathcal{O})^{\mathfrak{B}}$ and \mathfrak{n} the maximal ideal of $\mathcal{Z}[1/p]$ corresponding to the pseudocharacter $\psi_1 + \psi_2$. Then it follows from Corollaries 8.15, 9.37 and Proposition 10.107 that the irreducible representations of $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ are precisely the irreducible subquotients of $(\text{Ind}_p^G \psi_1 \otimes \psi_2 \varepsilon^{-1})_{\text{cont}}$, $(\text{Ind}_p^G \psi_2 \otimes \psi_1 \varepsilon^{-1})_{\text{cont}}$. Since $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ is closed under subquotients and extensions in $\text{Ban}_{G,\zeta}^{\text{adm}}(\mathbf{L})$, it is uniquely determined by its irreducible objects. One then can reinterpret the anti-equivalence of categories between $\text{Ban}_{G,\zeta}^{\text{adm.fl}}(\mathbf{L})_{\mathfrak{n}}^{\mathfrak{B}}$ and the category of modules of finite length over certain \mathfrak{n} -adic completions, see Corollaries 8.13, 9.35 and Proposition 10.107(i), (ii) and Remark 10.96 in terms of the Galois side.

For example, if $\psi_1 \psi_2^{-1} \neq \varepsilon^{\pm 1}, \mathbf{1}$, so that both unitary principal series representations are irreducible and distinct, then Theorem 11.7 holds if we replace V^u with $V_1^u \oplus V_2^u$, and R_V^ψ with $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}^{\text{cont}}(V_1^u \oplus V_2^u)$, where V_1^u is the universal deformation of the non-split extension ψ_1 by ψ_2 , and V_2^u is the universal deformation of the non-split extension ψ_2 by ψ_1 with determinant $\varepsilon \zeta$. Our assumptions imply that the extensions are unique up to isomorphism. If $\chi_1 \chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$ then the assertion follows from Proposition B.26, Corollary 8.13 and [39, (2.3.5)]. If $\chi_1 = \chi_2 \omega^{\pm 1}$ then one may show the assertion using Remarks 10.96 and 10.97 instead. If $\chi_1 = \chi_2$ then one has to do some work to show that the \mathfrak{n} -adic completion of $\check{\mathbf{V}}(\tilde{P})$ is isomorphic to $V_1^u \oplus V_2^u$. We leave the details to the interested reader.

12. Unitary completions

We determine all the absolutely irreducible admissible unitary completions of absolutely irreducible locally algebraic L-representations of G with $p \geq 5$. Such representations are of the form $\pi \otimes_L W_{l,k}$, where π is a smooth absolutely irreducible L-representation of G , that is a stabilizer of v is an open subgroup of G for all $v \in \pi$, and $W_{l,k} = \det^l \otimes \text{Sym}^{k-1} \mathbf{L}^2$, see [53]. The study of such completions was initiated by Breuil [17], [18] and our results confirm his philosophy, see [18, §1.3]. We deduce the main result of this section, Theorem 12.7, by combining Theorem 11.4 with some deep results of Colmez.

Lemma 12.1. — *If $\pi = \eta \circ \det$ is a character then $\pi \otimes_{\mathbb{L}} W_{l,k}$ admits a unitary completion if and only if $k = 1$ and $\text{val}(\eta(p)) = -l$.*

Proof. — This is well known, see for example [57, Lem. 7.3]. \square

Lemma 12.2. — *Let $\chi_1, \chi_2 : \mathbf{Q}_p^\times \rightarrow \mathbb{L}^\times$ be smooth characters. If the representation $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi_1 \otimes \chi_2 | \cdot |^{-1})_{\text{sm}} \otimes W_{l,k}$ admits a unitary completion then*

- (i) $-(k+l) \leq \text{val}(\chi_1(p)), \text{val}(\chi_2(p)) \leq -l$ and
- (ii) $\text{val}(\chi_1(p)) + \text{val}(\chi_2(p)) = -(k+2l)$.

Proof. — See [57, Lem 7.9], [28, Lem. 2.1]. \square

Theorem 12.3. — *Suppose that $\pi \cong (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi_1 \otimes \chi_2 | \cdot |^{-1})_{\text{sm}}$ satisfies the conditions of Lemma 12.2 then the universal unitary completion of $\pi \otimes W_{l,k}$ is an admissible absolutely irreducible \mathbb{L} -Banach space representation. Moreover, the universal completion is ordinary if and only if $\text{val}(\chi_1(p)) = -l$ or $\text{val}(\chi_2(p)) = -l$.*

Proof. — Since by assumption π is irreducible, $\chi_1 \chi_2^{-1} \neq | \cdot |^{\pm 1}$ and so

$$(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi_1 \otimes \chi_2 | \cdot |^{-1})_{\text{sm}} \cong (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi_2 \otimes \chi_1 | \cdot |^{-1})_{\text{sm}}.$$

We may assume that $\text{val}(\chi_1(p)) \leq \text{val}(\chi_2(p))$. Suppose that $\text{val}(\chi_2(p)) < -l$ then if $\chi_1 \neq \chi_2$ the assertion is a deep result of Berger-Breuil [7, 5.3.4], if $\chi_1 = \chi_2$ then the assertion follows from [55]. If $\text{val}(\chi_2(p)) = -l$ then it follows from [19, 2.2.1] that the universal unitary completion is isomorphic to $(\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi)_{\text{cont}}$, where $\psi \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \chi_2(a) a^l \chi_1(d) |d|^{-1} d^{k+l-1}$. \square

Lemma 12.4. — *Let $\psi : \mathbb{P} \rightarrow \mathbb{L}^\times$ be a continuous character and let \mathbb{P}_0 be a compact open subgroup of \mathbb{P} . Then $\text{Hom}_{\mathbb{P}_0}(W_{l,k}, \psi)$ is at most 1-dimensional and is non-zero if and only if $\psi \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = a^l d^{k+l-1}$ for all $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathbb{P}_0$.*

Proof. — The restriction of ψ to \mathbb{U} is trivial, since \mathbb{U} is contained in the derived subgroup of \mathbb{P} . We identify $W_{l,k}$ with the space of homogeneous polynomials in x and y of degree $k-1$ with \mathbb{G} -action given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbb{P}(x, y) = (ad - bc)^l \mathbb{P}(ax + cy, bx + dy)$. The space of $\mathbb{U} \cap \mathbb{P}_0$ -coinvariants of $W_{l,k}$ is 1-dimensional, spanned by the image of y^{k-1} . Since $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} y^{k-1} = d^{k-1} (ad)^l y^{k-1}$ we obtain the assertion. \square

Lemma 12.5. — *Let $\psi : \mathbb{P} \rightarrow \mathbb{L}^\times$ be a continuous unitary character and let $\Pi := (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \psi)_{\text{cont}}$. If $\Pi^{\text{alg}} \neq 0$ then $\Pi^{\text{alg}} \cong (\text{Ind}_{\mathbb{P}}^{\mathbb{G}} \chi_1 \otimes \chi_2)_{\text{sm}} \otimes W_{l,k}$ and $\psi \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \chi_1(a) a^l \chi_2(d) d^{k+l-1}$, for some smooth characters $\chi_1, \chi_2 : \mathbf{Q}_p^\times \rightarrow \mathbb{L}^\times$ and integers k, l with $k \geq 0$.*

Proof. — Let τ be a smooth L -representation of G and $W = W_{l,k}$ for some integers k, l , then

$$\begin{aligned} \mathrm{Hom}_G(\tau \otimes W, \Pi) &\cong \mathrm{Hom}_G(\tau, \mathrm{Hom}_L(W, \Pi)) \\ &\cong \mathrm{Hom}_G(\tau, \mathrm{Hom}_L(W, \Pi)^{\mathrm{sm}}), \end{aligned}$$

where $\mathrm{Hom}_L(W, \Pi)^{\mathrm{sm}}$ denotes smooth vectors for the action of G on $\mathrm{Hom}_L(W, \Pi)$ by conjugation; explicitly it is the union of $\mathrm{Hom}_H(W_{l,k}, \Pi)$ for all compact open subgroups H of G . If $\mathrm{Hom}_G(\tau \otimes W, \Pi) \neq 0$ there exists a compact open subgroup H_0 of G such that $\mathrm{Hom}_{H_0}(W, \Pi) \neq 0$. Frobenius reciprocity and Lemma 12.4 imply that $\psi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a^l d^{k+l-1}$ for all $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in H_0 \cap P$. Hence, $\psi = \psi_{\mathrm{sm}} \psi_{\mathrm{alg}}$, where $\psi_{\mathrm{sm}} : P \rightarrow L^\times$ is a smooth character, trivial on $H_0 \cap P$ and $\psi_{\mathrm{alg}}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a^l d^{k+l-1}$ for all $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P$. Lemma 12.4 implies that if $(l', k') \neq (l, k)$ then $\mathrm{Hom}_G(\tau \otimes W_{l',k'}, \Pi) = 0$ for all smooth representations τ . It follows from [53] that $\Pi^{\mathrm{alg}} \cong \mathrm{Hom}_L(W, \Pi)^{\mathrm{sm}} \otimes W$. We identify W with the space homogeneous polynomials in x and y of degree $k-1$. The map $f \otimes P \mapsto [g \mapsto f(g)P(c, d)]$, for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ induces an injection $(\mathrm{Ind}_P^G \psi_{\mathrm{sm}})_{\mathrm{sm}} \otimes W \hookrightarrow \Pi$, and hence an injection $(\mathrm{Ind}_P^G \psi_{\mathrm{sm}})_{\mathrm{sm}} \hookrightarrow \mathrm{Hom}_L(W, \Pi)^{\mathrm{sm}}$. It follows from Lemma 12.4 and Frobenius reciprocity that for all open subgroups $H \subseteq H_0$ the space of H -invariants in the source and the target have the same dimension equal to $|H \backslash G/P|$. Hence, the injection is an isomorphism. \square

Lemma 12.6. — *Let $\eta : \mathbf{Q}_p^\times \rightarrow L^\times$ be a continuous unitary character. If $(\widehat{\mathrm{Sp}} \otimes \eta \circ \det)^{\mathrm{alg}} \neq 0$ then η is locally algebraic and $(\widehat{\mathrm{Sp}} \otimes \eta \circ \det)^{\mathrm{alg}} \cong \mathrm{Sp} \otimes \eta \circ \det$.*

Proof. — Since the surjection $q : (\mathrm{Ind}_P^G \eta \otimes \eta)_{\mathrm{cont}} \twoheadrightarrow \widehat{\mathrm{Sp}} \otimes \eta \circ \det$ admits a P -equivariant splitting, [23, VI.2.3] implies that q induces a surjection on locally algebraic vectors. The assertion follows from Lemma 12.5. \square

Theorem 12.7. — *Suppose that the central character of $\pi \otimes W_{l,k}$ is unitary and either π is special series and $k > 1$ or π is supercuspidal. Then $\pi \otimes W_{l,k}$ admits precisely $\mathbf{P}^1(L)$ non-isomorphic absolutely irreducible admissible unitary completions.*

Proof. — Let Π be an absolutely irreducible admissible unitary L -Banach space representation of G containing $\pi \otimes W_{l,k}$ as a G -invariant dense subspace. Since $\pi \otimes W_{l,k}$ is dense in Π , the central character of Π is equal to the central character of $\pi \otimes W_{l,k}$. It follows from Lemmas 12.5 and 12.6 that Π is not ordinary. Hence, $V := \mathbf{V}(\Pi)$ is an absolutely irreducible 2-dimensional L -representation of $\mathcal{G}_{\mathbf{Q}_p}$ by Theorem 11.4. Since Π contains a locally algebraic representation $\pi \otimes W_{l,k}$, V is de Rham [23, VI.6.13], with Hodge-Tate weights $a < b$, [23, VI.5.1], where $b - a = k$ (the precise formula for a and b depends on the normalization of the correspondence). Since V is de Rham, it is potentially semistable and to V one may associate a 2-dimensional Weil-Deligne representation $\mathrm{WD}(V)$, see for example [36]. Colmez has shown that $\Pi^{\mathrm{alg}} \cong \mathrm{LL}(\mathrm{WD}(V)) \otimes W_{l,k}$,

[23, Thm. 0.21], where LL denotes the classical (modified) local Langlands correspondence [23, §VI.6.11]. In the supercuspidal case the proof was conditional on the results of Emerton, which have now appeared in [32, §7.4]. Thus determining all the isomorphism classes of the absolutely irreducible admissible unitary completions of $\pi \otimes W$ is equivalent to determining all the isomorphism classes of the absolutely irreducible 2-dimensional potentially semistable L-representations V of $\mathcal{G}_{\mathbf{Q}_p}$ with Hodge-Tate weights $a < b$, such that $\text{Hom}_G(\pi, \text{LL}(\text{WD}(V))) \neq 0$. If π is special series then (after twisting by a smooth unitary character) it follows from [23, VI.6.50] that the set of such V consists of a family of semi-stable non-crystalline representations indexed by the L-invariant $\mathcal{L} \in L$ and one crystalline representation. If π is supercuspidal then the last condition is equivalent to $\text{LL}(\text{WD}(V)) \cong \pi$ and the assertion follows from [36]. \square

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Appendix A: Two dimensional pseudocharacters

We recall some standard facts about 2-dimensional pseudocharacters. We refer the reader to [3, §1] for more information. Let \mathcal{G} be a profinite group and (A, \mathfrak{m}) a local artinian \mathcal{O} -algebra. We assume that $p > 2$. A 2-dimensional A -valued pseudocharacter is a continuous function $T : \mathcal{G} \rightarrow A$ satisfying: (1) $T(1) = 2$; (2) $T(gh) = T(hg)$ for all $g, h \in \mathcal{G}$; (3) the relation

$$T(g)T(h)T(k) = T(g)T(hk) + T(h)T(gk) + T(k)T(gh) - T(ghk) - T(gkh)$$

for all $g, h, k \in \mathcal{G}$. One may show that if $\rho : \mathcal{G} \rightarrow \text{GL}_2(A)$ is a continuous representation then $\text{tr } \rho$ is a 2-dimensional pseudocharacter. Given a 2-dimensional pseudocharacter $T : \mathcal{G} \rightarrow A$ one may show, [22, Prop. 1.29], that the function $D(g) := \frac{T(g)^2 - T(g^2)}{2}$

defines a continuous group homomorphism $\mathcal{G} \rightarrow A^\times$. It is shown in [22, 1.9, 1.29] that $T \mapsto (T, D)$ induces a bijection between 2-dimensional pseudocharacters and pairs of functions (T, D) , where $D : \mathcal{G} \rightarrow A^\times$ is a continuous group homomorphism and $T : \mathcal{G} \rightarrow A$ is a continuous function satisfying: $T(1) = 2$, $T(gh) = T(hg)$, $D(g)T(g^{-1}h) - T(g)T(h) + T(gh) = 0$ for all $g, h \in \mathcal{G}$.

Let $\rho : \mathcal{G} \rightarrow \mathrm{GL}_2(k)$ be a continuous representation and let D^{ps} be the functor from local artinian augmented \mathcal{O} -algebras with residue field k to the category of sets, such that $D^{\mathrm{ps}}(A)$ is the set of all 2-dimensional A -valued pseudocharacters T , such that $T \equiv \mathrm{tr} \rho \pmod{\mathfrak{m}_A}$. If for every open subgroup \mathcal{H} of \mathcal{G} , $\mathrm{Hom}^{\mathrm{cont}}(\mathcal{H}, \mathbf{F}_p)$ is a finite dimensional \mathbf{F}_p -vector space then the functor D^{ps} is pro-represented by a complete local noetherian \mathcal{O} -algebra. We note that this finiteness condition is satisfied if \mathcal{G} is the absolute Galois group of a local field. We usually work with a variant: fix a continuous character $\psi : \mathcal{G} \rightarrow \mathcal{O}^\times$ lifting $\det \rho$ and let $D^{\mathrm{ps}, \psi}$ be a subfunctor of D^{ps} such that $T \in D^{\mathrm{ps}, \psi}(A)$ if and only if $\frac{T(g)^2 - T(g^2)}{2}$ is equal to (the image of) $\psi(g)$ for all $g \in \mathcal{G}$. We will refer to $D^{\mathrm{ps}, \psi}$ as a deformation problem with a fixed determinant. One may show that if D^{ps} is pro-represented by R then $D^{\mathrm{ps}, \psi}$ is pro-represented by a quotient of R .

Lemma A.1. — *Let G be a finite group, let $S = k[G]/J$, where J is the two sided ideal in $k[G]$ generated by $g^2 - 2g + 1$ for all $g \in G$. Then the image of G in S^\times is a p -group.*

Proof. — Suppose not then there exists a prime $l \neq p$ and $g \in G$ such that the image of g in S^\times has order l . Since the greatest common divisor of $x^l - 1$ and $(x - 1)^2$ in $k[x]$ equal to $x - 1$ we may find polynomials $a(x), b(x) \in k[x]$ such that $(x^l - 1)a(x) + (x - 1)^2 b(x) = x - 1$. Since the images of $g^l - 1$ and $g^2 - 2g + 1$ are equal to 0 in S , we deduce that the image of g in S is trivial. \square

Let $\rho : \mathcal{G} \rightarrow \mathrm{GL}_2(k)$ be a continuous representation, \mathcal{K} be the kernel of ρ , $\mathcal{K}(p)$ the maximal pro- p quotient of \mathcal{K} and \mathcal{H} the kernel of $\mathcal{K} \twoheadrightarrow \mathcal{K}(p)$. We note that \mathcal{H} is a normal subgroup of \mathcal{G} .

Let (A, \mathfrak{m}) be a local artinian \mathcal{O} -algebra with residue field k . Let $T : \mathcal{G} \rightarrow A$ be a continuous 2-dimensional pseudocharacter lifting ρ . Since A is finite and T is continuous $\mathrm{Ker} T := \{h \in \mathcal{G} : T(gh) = T(g), \forall g \in \mathcal{G}\}$ is an open subgroup of \mathcal{G} .

Proposition A.2. — $\mathcal{H} \subseteq \mathrm{Ker} T$.

Proof. — Choose an open normal subgroup \mathcal{N} of \mathcal{G} contained in $\mathcal{K} \cap \mathrm{Ker} T$. Let $G := \mathcal{G}/\mathcal{N}$ and let J be the two sided ideal in $A[G]$ generated by elements $g^2 - T(g)g + \frac{T(g)^2 - T(g^2)}{2}$, for all $g \in G$ and let $S := A[G]/J$. We claim that the image of \mathcal{K} in S^\times is a p -group. Since the kernel of $S^\times \rightarrow (S/\mathfrak{m}S)^\times$ is a p -group, it is enough to show that the image of \mathcal{K} in $(S/\mathfrak{m}S)^\times$ is a p -group. Since $S/\mathfrak{m}S$ is a quotient of $k[G]/(g^2 - \mathrm{tr} \rho(g)g + \det \rho(g) : g \in G)$ the claim follows from Lemma A.1. For each $g \in \mathcal{G}$ we denote the image of g in S

by \bar{g} . It follows from the claim that $\bar{h} = 1$ for all $h \in \mathcal{H}$. We may extend $T : \mathcal{G} \rightarrow A$ linearly to $T : A[\mathcal{G}] \rightarrow A$, which factors through $T : A[G] \rightarrow A$ as $\mathcal{N} \subseteq \text{Ker } T$, and then factors through $T : S \rightarrow A$ and so we have $T(\bar{g}) = T(g)$ for all $g \in \mathcal{G}$. In particular, if $h \in \mathcal{H}$ then $T(gh) = T(\bar{g}\bar{h}) = T(\bar{g}) = T(g)$ for all $g \in \mathcal{G}$. \square

Corollary A.3. — *The inclusion $D_{\mathcal{G}/\mathcal{H}}^{\text{ps}} \subseteq D_{\mathcal{G}}^{\text{ps}}$ is an isomorphism of functors.*

Proof. — It follows from Proposition A.2 that for all artinian local \mathcal{O} -algebras (A, \mathfrak{m}) with residue field k we have $D_{\mathcal{G}/\mathcal{H}}^{\text{ps}}(A) = D_{\mathcal{G}}^{\text{ps}}(A)$. \square

Suppose that $D_{\mathcal{G}/\mathcal{H}}^{\text{ps}}$ is pro-represented by a complete local noetherian \mathcal{O} -algebra (R, \mathfrak{m}) , then $D_{\mathcal{G}}^{\text{ps}}$ is also pro-represented by (R, \mathfrak{m}) by Corollary A.3. Let $T : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H} \rightarrow R$ be the universal pseudocharacter lifting $\text{tr } \rho$. Let J (resp. J') be a closed two-sided ideal in $R[[\mathcal{G}]]$ (resp. $R[[\mathcal{G}/\mathcal{H}]]$) generated by the elements $g^2 - T(g)g + \frac{T(g)^2 - T(g^2)}{2}$ for all $g \in \mathcal{G}$ (resp. $g \in \mathcal{G}/\mathcal{H}$).

Corollary A.4. — *The natural map $R[[\mathcal{G}]]/J \rightarrow R[[\mathcal{G}/\mathcal{H}]]/J'$ is an isomorphism.*

Proof. — If \mathcal{N} is an open normal subgroup of \mathcal{G} and $n \geq 1$ let $\mathfrak{a}(\mathcal{N}, n)$ be the kernel of $R[[\mathcal{G}]] \rightarrow R/\mathfrak{m}^n[[\mathcal{G}/\mathcal{N}]]$. The ideals $\mathfrak{a}(\mathcal{N}, n)$ for all open normal subgroups \mathcal{N} and all $n \geq 1$ form a system of open neighbourhoods of 0 in $R[[\mathcal{G}]]$. It follows from the proof of Proposition A.2 that for each $n \geq 1$ we may choose an open normal subgroup \mathcal{N}_n of \mathcal{G} such that for all open normal subgroups \mathcal{N} of \mathcal{G} contained in \mathcal{N}_n the image of \mathcal{H} in $R[[\mathcal{G}]]/(J + \mathfrak{a}(\mathcal{N}, n))$ is trivial. Thus $R[[\mathcal{G}]]/(J + \mathfrak{m}^n R[[\mathcal{G}]]) \cong R[[\mathcal{G}/\mathcal{H}]]/(J' + \mathfrak{m}^n R[[\mathcal{G}/\mathcal{H}]])$ for all $n \geq 1$, which yields the claim. \square

Appendix B: Some deformation rings

Let $\omega : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{F}_p^\times \hookrightarrow k^\times$ be the cyclotomic character modulo p . It follows from local Tate duality and Euler characteristic that $\text{Ext}_{\mathbf{F}_p[[\mathcal{G}_{\mathbf{Q}_p}]]}^1(\omega, \mathbf{1})$ is one dimensional. Let $0 \rightarrow \mathbf{1} \rightarrow \rho \rightarrow \omega \rightarrow 0$ be a non-split extension. This determines ρ up to isomorphism. The purpose of this appendix is to describe explicitly various deformation rings of ρ , by spelling out what a general result of Böckle in [9], says in this particular case. We then show using results of Bellaïche [2] that the universal deformation ring of ρ is isomorphic to the universal deformation ring of $\text{tr } \rho$. In Section B.1 we consider the easier, generic reducible case. We assume $p \geq 5$ until Section B.1, and $p \geq 3$ in Section B.1.

We may think of ρ as a group homomorphism $\rho : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \text{GL}_2(\mathbf{F}_p) \hookrightarrow \text{GL}_2(k)$, $g \mapsto \begin{pmatrix} 1 & \kappa(g) \\ 0 & \omega(g) \end{pmatrix}$. Let H be the image of $\mathcal{G}_{\mathbf{Q}_p}$ in $\text{GL}_2(k)$ and let U be the p -Sylow subgroup of H . Since ρ is non-split U is non-trivial, hence $U \cong \mathbf{F}_p$, let G be the subgroup of diagonal matrices in H , then $G \cong \mathbf{F}_p^\times$ and $H \cong U \rtimes G$. Let L be the fixed field of $\text{Ker } \rho$ and let

$F = L^U$. Then F is the fixed field of $\text{Ker } \omega$ and hence is equal to $\mathbf{Q}_p(\mu_p)$, where μ_p is the group of p -th roots of unity. We identify $\text{Gal}(F/\mathbf{Q}_p)$ with G . Let \mathcal{G}_F be the absolute Galois group of F .

If $\rho_A : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \text{GL}_2(A)$ is a deformation of ρ to (A, \mathfrak{m}) then $\rho_A(\mathcal{G}_F)$ is contained in $\begin{pmatrix} 1+\mathfrak{m} & \mathfrak{A} \\ \mathfrak{m} & 1+\mathfrak{m} \end{pmatrix}$, and hence ρ_A factors through $\text{Gal}(F(p)/\mathbf{Q}_p)$, where $F(p)$ is the compositum of all finite extensions of p power order of F in $\overline{\mathbf{Q}_p}$. Now $\text{Gal}(F(p)/F) \cong \mathcal{G}_F(p)$ the maximal pro- p quotient of \mathcal{G}_F . Since the order of $G = \text{Gal}(F/\mathbf{Q}_p)$ is prime to p , we may choose a splitting of exact sequence $1 \rightarrow \mathcal{G}_F(p) \rightarrow \text{Gal}(F(p)/\mathbf{Q}_p) \rightarrow G \rightarrow 1$, so that $\text{Gal}(F(p)/\mathbf{Q}_p) \cong \mathcal{G}_F(p) \rtimes G$.

We will recall some facts about Demuškin groups, see for example [49, §III.9] for details. A finitely generated pro- p group P is a Demuškin group, if $H^2(P, \mathbf{F}_p)$ is one dimensional and the cup product $H^1(P, \mathbf{F}_p) \times H^1(P, \mathbf{F}_p) \xrightarrow{\cup} H^2(P, \mathbf{F}_p)$ is a non-degenerate bilinear form. If $p > 2$ a Demuškin group P is uniquely determined up to isomorphism by two parameters $n = n(P)$ the dimension of $H^1(P, \mathbf{F}_p)$ and $q = q(P)$ the number of elements in the torsion subgroup of P^{ab} , and is isomorphic to a pro- p group generated by n elements x_1, \dots, x_n and one relation $x_1^q(x_1, x_2)(x_3, x_4) \dots (x_{n-1}, x_n)$, where $(x, y) = x^{-1}y^{-1}xy$. We note that since $p > 2$ the non-degeneracy of bilinear form implies that n is even and it follows from the presentation of P that $P^{ab} \cong \mathbf{Z}_p^{n-1} \oplus \mathbf{Z}/q\mathbf{Z}$. It is well known, see for example [49, 7.5.8], that if F is a finite extension of \mathbf{Q}_p containing μ_p , then $\mathcal{G}_F(p)$ is a Demuškin group with $n = [F : \mathbf{Q}_p] + 2$ and q equal to the number of p power order roots of unity in F . In our situation $F = \mathbf{Q}_p(\mu_p)$ and so $n = p + 1$ and $q = p$.

Following [9] we are going to construct a universal deformation of ρ using the presentation of $\mathcal{G}_F(p)$. For a p -group P we define a filtration $P_1 = P$, $P_{i+1} = P_i^p(P_i, P)$, where (P_i, P) denotes a closed subgroup generated by the commutators, and let $\text{gr}_i P := P_i/P_{i+1}$. We let \mathcal{F} be a free pro- p group on $p + 1$ generators, and we choose a surjection $\varphi : \mathcal{F} \twoheadrightarrow \mathcal{G}_F(p)$. Since $\mathcal{G}_F(p)$ is a Demuškin group there exists an element $r \in \mathcal{F}$ such that $\text{Ker } \varphi$ is the smallest normal closed subgroup of \mathcal{F} containing r . Since the order of G is prime to p , we may let G act on \mathcal{F} so that φ is G -equivariant, see Lemma 3.1 in [9]. We denote by $\tilde{\omega} : G \rightarrow \mathbf{Z}_p^\times$ the Teichmüller lift of ω .

Lemma B.1. — *We may choose generators x_0, \dots, x_p of \mathcal{F} so that*

- (i) $gx_i g^{-1} = x_i^{\tilde{\omega}(g)^i}$, for $g \in G$ and $0 \leq i \leq p$;
- (ii) *the image of r in $\text{gr}_2 \mathcal{F}$ is equal to the image of*

$$r' := x_1^p(x_1, x_{p-1})(x_2, x_{p-2}) \dots (x_{\frac{p-1}{2}}, x_{\frac{p+1}{2}})(x_p, x_0).$$

Proof. — The assertion follows from [42, Prop. 3], where the cup product is described in terms of the image of r in $\text{gr}_2 \mathcal{F}$. We know that

$$\text{gr}_1 \mathcal{F} \cong \text{gr}_1 \mathcal{G}_F(p) \cong \mathbf{F}_p \oplus \mu_p \oplus \mathbf{F}_p[G]$$

as a representation of G , see Theorems 4.1 and 4.2 in [9]. Moreover, the summand μ_p is the image of the torsion subgroup of $\mathcal{G}_F(p)^{ab}$ under the natural map $\mathcal{G}_F(p)^{ab} \rightarrow \text{gr}_1 \mathcal{G}_F(p)$. We fix $\xi_1 \in \mu_p \subset \text{gr}_1 \mathcal{G}_F(p)$, which generates μ_p as $\mathbf{F}_p[G]$ -module. Now $H^1(\mathcal{G}_F(p), \mathbf{F}_p) \cong \text{Hom}^{\text{cont}}(\mathcal{G}_F(p), \mathbf{F}_p) \cong (\text{gr}_1 \mathcal{G}_F(p))^*$ as a G -representation. Hence, we may find an \mathbf{F}_p -basis χ_0, \dots, χ_p of $H^1(\mathcal{G}_F(p), \mathbf{F}_p)$ such that G acts on χ_i by ω^{-i} , $\chi_1(\xi_1) \neq 0$, $\chi_p(\xi_1) = 0$ and, since the cup product defines a non-degenerate bilinear pairing and G acts on $H^2(\mathcal{G}_F(p), \mathbf{F}_p)$ by ω^{-1} , we have $\chi_i \cup \chi_j = 0$ unless $i + j \equiv 1 \pmod{p-1}$. Further, by replacing χ_i by a scalar multiple $\lambda \chi_i$, with $\lambda \in \mathbf{F}_p^\times$, we may achieve that $\bar{r}(\chi_p \cup \chi_0) = 1$ and $\bar{r}(\chi_i \cup \chi_{p-i}) = 1$ for $1 \leq i \leq (p-1)/2$, where $\bar{r} : H^2(\mathcal{G}_F(p), \mathbf{F}_p) \xrightarrow{\cong} \mathbf{F}_p$ is the isomorphism defined in [42, Prop 3]. Let ξ_0, \dots, ξ_p be an \mathbf{F}_p -basis of $\text{gr}_1 \mathcal{G}_F(p)$ dual to χ_0, \dots, χ_p . Then G acts on ξ_i by the character ω^i . Since the order of G is prime to p , we may find $x_i \in \mathcal{F}$ satisfying (i) and mapping to ξ_i in $\text{gr}_1 \mathcal{F}$. Since the images of x_0, \dots, x_p form a basis of $\text{gr}_1 \mathcal{F}$, they generate \mathcal{F} . Part (ii) follows by construction from the Proposition 3 in [42]. \square

Let R be the ring

$$(B.1) \quad R := \frac{\mathcal{O}[[a_0, a_1, c_0, c_1, d_0, d_1]]}{(pc_0 + c_0d_1 + c_1d_0)}.$$

Let P be a pro- p subgroup of $GL_2(R)$ generated by the matrices m_i for $0 \leq i \leq p$, where $m_i = 1$ if $i \not\equiv 0, \pm 1 \pmod{p-1}$, and

$$m_{p-2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad m_{1+(p-1)j} = \begin{pmatrix} 1 & 0 \\ c_j & 1 \end{pmatrix},$$

$$m_{(p-1)j} = \begin{pmatrix} (1+a_j)^{\frac{1}{2}}(1+d_j)^{\frac{1}{2}} & 0 \\ 0 & (1+a_j)^{\frac{1}{2}}(1+d_j)^{-\frac{1}{2}} \end{pmatrix},$$

for $j = 0$ and $j = 1$. We embed $G \hookrightarrow GL_2(R)$, $g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\omega}(g) \end{pmatrix}$. One has $gm_i g^{-1} = m_i^{\tilde{\omega}(g)^i}$ for all $0 \leq i \leq p$ and hence $x_i \mapsto m_i$ defines a G -equivariant homomorphism $\alpha : \mathcal{F} \rightarrow P$ and hence a group homomorphism $\alpha : \mathcal{F} \rtimes G \rightarrow GL_2(R)$.

Proposition B.2. — *There exists a continuous group homomorphism*

$$\varphi' : \mathcal{F} \rtimes G \rightarrow \text{Gal}(\mathbf{F}(p)/\mathbf{Q}_p)$$

such that $\varphi'(g) \equiv \varphi(g) \pmod{\mathcal{G}_F(p)_3}$, and a commutative diagram:

$$\begin{array}{ccc} \mathcal{F} \rtimes G & \xrightarrow{\alpha} & GL_2(R) \\ & \searrow \varphi' & \uparrow \tilde{\rho} \\ & & \text{Gal}(\mathbf{F}(p)/\mathbf{Q}_p). \end{array}$$

Proof. — Let us observe that for $j = 0$ and $j = 1$ the commutator

$$(m_{1+(p-1)j}, m_{(p-1)(1-j)}) = \begin{pmatrix} 1 & 0 \\ c_j d_{1-j} & 1 \end{pmatrix}$$

and $(m_i, m_{p-i}) = 1$ if $i \not\equiv 1, 0 \pmod{p-1}$ hence

$$\mathbf{(B.2)} \quad m_1^p(m_1, m_{p-1})(m_2, m_{p-2}) \dots (m_{\frac{p-1}{2}}, m_{\frac{p+1}{2}})(m_p, m_0) = 1$$

as $pc_0 + c_0d_1 + c_1d_0 = 0$ in \mathbf{R} . Since $\alpha(x_i) = m_i$, we get that $\alpha(r') = 1$, where r' is defined in Lemma B.1. Since $r \equiv r' \pmod{\mathcal{F}_3}$ we deduce that $\alpha(r) \in \alpha(\mathcal{F}_3)$ and the assertion follows from Proposition 3.8 in [9]. Namely, it is shown there that there exists an element $r_1 \in \text{Ker } \alpha \cap \mathcal{F}^p(\mathcal{F}, \mathcal{F})$, such that $r_1 \equiv r \pmod{\mathcal{F}_3}$, and G acts on r_1 by a character. It follows Lemma B.1(ii) that the character is equal to $\tilde{\omega}$. Let \mathcal{R} be the smallest closed normal subgroup of \mathcal{F} containing r_1 and set $D := \mathcal{F}/\mathcal{R}$. Since $\alpha(r_1) = 1$ and G acts on r_1 by a character, we deduce that α factors through $\alpha : D \rtimes G \rightarrow \text{GL}_2(\mathbf{R})$.

We claim that $D \rtimes G \cong \text{Gal}(\mathbf{F}(p)/\mathbf{Q}_p)$. Since $r_1 \equiv r \equiv r' \pmod{\mathcal{F}_3}$, D is a Demuškin group with $n(D) = p + 1$ and $q(D) = p$, see [49, 3.9.17]. Hence, we know that $D \cong \text{Gal}(\mathbf{F}(p)/\mathbf{F})$. To see that we may choose this isomorphism G -equivariantly we observe that since $r \equiv r_1 \pmod{\mathcal{F}_3}$ Proposition 3 in [42] implies that the diagram:

$$\begin{array}{ccc} \mathrm{H}^1(D, \mathbf{F}_p) \times \mathrm{H}^1(D, \mathbf{F}_p) & \xrightarrow{\cup} & \mathrm{H}^2(D, \mathbf{F}_p) \\ \uparrow \cong & & \downarrow \cong \bar{r}_1 \\ \mathrm{H}^1(\mathcal{F}, \mathbf{F}_p) \times \mathrm{H}^1(\mathcal{F}, \mathbf{F}_p) & \xrightarrow{\quad} & \mathbf{F}_p \\ \cong \downarrow & & \uparrow \cong \bar{r} \\ \mathrm{H}^1(\mathcal{G}_{\mathbf{F}}(p), \mathbf{F}_p) \times \mathrm{H}^1(\mathcal{G}_{\mathbf{F}}(p), \mathbf{F}_p) & \xrightarrow{\cup} & \mathrm{H}^2(\mathcal{G}_{\mathbf{F}}(p), \mathbf{F}_p) \end{array}$$

commutes and is G -equivariant. The claim follows from Theorem 3.4 in [9]. \square

Theorem B.3 ([9]). — \mathbf{R} is the universal deformation ring of ρ and the equivalence class of $\tilde{\rho}$, defined in Proposition B.2, is the universal deformation.

Proof. — We note that since $\omega \neq \mathbf{1}$, $\text{End}_{k[\mathcal{G}_{\mathbf{Q}_p}]}(\rho) = k$ and hence the deformation functor Def_{ρ} is representable. Moreover, local Tate duality implies that

$$\mathrm{H}^2(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho) \cong \mathrm{H}^0(\mathcal{G}_{\mathbf{Q}_p}, \text{Hom}(\text{Ad } \rho, \omega)) \cong \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho, \rho \otimes \omega)$$

is 1-dimensional and hence $\mathrm{H}^1(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho)$ is 6-dimensional by local Euler-Poincaré characteristic. We have a natural transformation of functors $\eta : h_{\mathbf{R}} \rightarrow \text{Def}_{\rho}$, which maps a homomorphism $\psi : \mathbf{R} \rightarrow \mathbf{A}$ to the equivalence class of the representation $\rho_{\mathbf{A}} : \text{Gal}(\mathbf{F}(p)/\mathbf{Q}_p) \xrightarrow{\hat{\rho}} \text{GL}_2(\mathbf{R}) \xrightarrow{\psi} \text{GL}_2(\mathbf{A})$. Moreover, one may check directly that this

induces an isomorphism $h_{\mathbb{R}}(k[\epsilon]) \cong \text{Def}_{\rho}(k[\epsilon])$. Hence, we obtain a surjection $\mathbb{R}_{\rho} \twoheadrightarrow \mathbb{R}$, where \mathbb{R}_{ρ} is the ring representing Def_{ρ} . It is shown in Theorem 6.2 of [9] that this map is an isomorphism. \square

Corollary B.4. — *Let $x \in \text{Spec } \mathbb{R}[1/p]$ be a maximal ideal with residue field \mathbb{E} . The corresponding representation $\rho_x : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(\mathbb{E})$ is reducible if and only if c_0 and c_1 are 0 in \mathbb{E} .*

Proof. — Let \mathfrak{a} be the ideal of \mathbb{R} generated by c_0 and c_1 . It follows from the construction of the universal deformation $\tilde{\rho}$ that the image of $\mathcal{G}_{\mathbb{Q}_p} \xrightarrow{\tilde{\rho}} \text{GL}_2(\mathbb{R}) \rightarrow \text{GL}_2(\mathbb{R}/\mathfrak{a})$ is contained in the subgroup of upper triangular matrices. Hence, if the image of c_0 and c_1 in \mathbb{E} is zero then ρ_x is reducible. Conversely, suppose that ρ_x is reducible then for all $g, h \in \mathcal{G}_{\mathbb{Q}_p}$ the matrix $\rho_x(g)\rho_x(h) - \rho_x(h)\rho_x(g)$ is nilpotent. In particular, for $j = 0$ and $j = 1$ the matrix $\rho_x(\varphi'(x_{1+(p-1)j}))\rho_x(\varphi'(x_{p-2})) - \rho_x(\varphi'(x_{p-2}))\rho_x(\varphi'(x_{1+(p-1)j}))$ is nilpotent. Since it is equal to

$$\begin{pmatrix} 1 & 0 \\ \bar{c}_j & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{c}_j & 1 \end{pmatrix} = \begin{pmatrix} -\bar{c}_j & 0 \\ 0 & \bar{c}_j \end{pmatrix}$$

we deduce that \bar{c}_j the image of c_j in \mathbb{E} is zero. \square

Let $\psi : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathcal{O}^{\times}$ be a continuous character, lifting ω and let Def_{ρ}^{ψ} be subfunctor of Def_{ρ} parameterizing the deformations with determinant equal to ψ .

Corollary B.5. — *The functor Def_{ρ}^{ψ} is represented by*

$$\mathbb{R}^{\psi} \cong \frac{\mathcal{O}[[c_0, c_1, d_0, d_1]]}{(pc_0 + c_0d_1 + c_1d_0)}.$$

Proof. — Let $\lambda_0, \lambda_1 \in \varpi\mathcal{O}$ such that $\psi(\varphi'(x_{j(p-1)})) = 1 + \lambda_j$, for $j = 0$ and $j = 1$. By construction we have $\det m_i = 1$, if $i \not\equiv 0 \pmod{p-1}$, and $\det m_{j(p-1)} = 1 + \lambda_j$. We deduce that Def_{ρ}^{ψ} is represented by $\mathbb{R}/(a_0 - \lambda_0, a_1 - \lambda_1)$, which implies the claim. \square

Corollary B.6. — *Let $\mathfrak{r} = \mathbb{R}^{\psi} \cap \bigcap_x \mathfrak{m}_x$ where the intersection is taken over all maximal ideals of $\mathbb{R}^{\psi}[1/p]$ such that ρ_x is reducible. Then $\mathfrak{r} = (c_0, c_1)$. In particular,*

- (i) $\mathbb{R}^{\psi}/\mathfrak{r} \cong \mathcal{O}[[d_0, d_1]]$;
- (ii) let \mathfrak{n} be the maximal ideal of $\mathbb{R}_k^{\psi}/\mathfrak{r}_k$, then for all $i \geq 0$ there exists a surjection of \mathbb{R}^{ψ} -modules: $\mathfrak{r}_k^i/\mathfrak{r}_k^{i+1} \twoheadrightarrow \mathfrak{n}^i$.

Proof. — Corollary B.4 implies that (c_0, c_1) is contained in \mathfrak{r} and the image of \mathfrak{r} in $\mathbb{R}^{\psi}/(c_0, c_1)$ is equal to the intersection of all the maximal ideals of $\mathbb{R}^{\psi}/(c_0, c_1)[1/p]$. Since $\mathbb{R}^{\psi}/(c_0, c_1) \cong \mathcal{O}[[d_0, d_1]]$ by Corollary B.5, we deduce that $\mathfrak{r} = (c_0, c_1)$. Now $\mathbb{R}_k^{\psi} \cong k[[c_0, c_1, d_0, d_1]]/(c_0d_1 + d_0c_1)$. Let $S = k[[c_0, c_1, d_0, d_1]]$ and we denote by \mathfrak{b} the ideal of S

generated by c_0, c_1 . Then $\mathrm{gr}_{\mathfrak{b}}^{\bullet} \mathbf{S}$ is isomorphic to a polynomial ring in two variables over $k[[d_0, d_1]]$. The element $t = c_0 d_1 + d_0 c_1$ is pure of grade 1. Since $\mathbf{R}_k^{\psi} \cong \mathbf{S}/t\mathbf{S}$, and \mathfrak{r}_k is the image of \mathfrak{b} , we have an exact sequence $0 \rightarrow \mathrm{gr}_{\mathfrak{b}}^{i-1} \mathbf{S} \rightarrow \mathrm{gr}_{\mathfrak{b}}^i \mathbf{S} \rightarrow \mathfrak{r}_k^i / \mathfrak{r}_k^{i+1} \rightarrow 0$ for all $i \geq 1$, where the first non-trivial arrow is given by multiplication by t . Now $\mathrm{gr}_{\mathfrak{b}}^i \mathbf{S}$ is a free $k[[d_0, d_1]]$ -module with monomials in c_0 and c_1 of homogeneous degree i as a basis. Sending $c_0 \mapsto d_0, c_1 \mapsto -d_1$ induces a surjection of $k[[d_0, d_1]]$ -modules $\mathrm{gr}_{\mathfrak{b}}^i \mathbf{S} \twoheadrightarrow \mathfrak{n}^i$. Since this map kills t the surjection factors through $\mathfrak{r}_k^i / \mathfrak{r}_k^{i+1} \twoheadrightarrow \mathfrak{n}^i$. \square

Remark B.7. — We will deduce in the course of the proof of Proposition 10.70 that the map in (ii) is an isomorphism.

Remark B.8. — We note that in the definition of \mathfrak{r} it is enough to consider the ideals with residue field \mathbf{L} , since it follows from Lemma 9.22 that such ideals are Zariski dense in $\mathcal{O}[[d_0, d_1]][1/\rho]$.

Lemma B.9. — Suppose that the representations ρ_x and ρ_y corresponding to maximal ideals x and y of $\mathbf{R}^{\psi}[1/\rho]$ with residue field \mathbf{L} are reducible and have a common subquotient then $x = y$.

Proof. — Since the determinant is fixed we deduce that ρ_x and ρ_y have the same semisimplification $\delta \oplus \delta^{-1}\psi$, where $\delta : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathbf{L}^{\times}$ is a continuous character, lifting the trivial character $\mathbf{1} : \mathcal{G}_{\mathbf{Q}_p} \rightarrow k^{\times}$. If ρ_x is semisimple then the action of $\mathcal{G}_{\mathbf{Q}_p}$ on ρ_x factors through $\mathcal{G}_{\mathbf{Q}_p}^{ab}$, and hence the action of $\mathcal{G}_{\mathbf{Q}_p}$ on any stable \mathcal{O} -lattice of ρ_x factors through $\mathcal{G}_{\mathbf{Q}_p}^{ab}$, and hence the same holds for the reduction of any stable \mathcal{O} -lattice modulo \mathfrak{w} . Since the action of $\mathcal{G}_{\mathbf{Q}_p}$ on ρ does not factor through $\mathcal{G}_{\mathbf{Q}_p}^{ab}$ we deduce that both ρ_x and ρ_y are not semisimple. Since the reduction of $\delta^2\psi^{-1}$ modulo \mathfrak{w} is equal to ω^{-1} and $p \geq 5$, $\delta^{-2}\varepsilon^{-1}$ cannot be equal to the trivial or the cyclotomic character. This implies $\mathrm{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\varepsilon\delta^{-1}, \delta)$ is 1-dimensional. Hence, $\rho_x \cong \rho_y$ and so $x = y$. \square

Corollary B.10. — The intersection of all the maximal ideals of $\mathbf{R}^{\psi}[1/\rho]$ such that ρ_x is irreducible is zero.

Proof. — Let $\mathbf{S} = \mathcal{O}[[c_0, c_1, d_0, d_1]]$ and $g = pc_0 + c_1 d_0 + c_0 d_1$ and $f \in \mathbf{S}$, not divisible by g . It is enough to construct $\varphi : \mathbf{S} \rightarrow \overline{\mathbf{Q}}_p$, such that $\varphi(f) \neq 0$, $\varphi(g) = 0$ and $\varphi(c_0) \neq 0$, since the last condition implies that the representation associated to $\ker \varphi$ is irreducible via Corollary B.4.

Substituting $c'_1 := c_1 - d_0$ we get $g = d_0^2 + c'_1 d_0 + c_0 d_1 + pc_0$. Hence, we may write $f = qg + r$, where $r = d_0 f_1 + f_2$, with $f_1, f_2 \in \mathcal{O}[[c_0, c'_1, d_1]]$, see [44, IV§9]. The polynomial $\mathbf{X}^2 + c'_1 \mathbf{X} + c_0 d_1 + pc_0$ is irreducible over $\mathcal{O}[[c_0, c'_1, d_1]]$ and hence also over its quotient field. As $r \neq 0$ we deduce that $h := f_2^2 - c'_1 f_1 f_2 + (c_0 d_1 + pc_0) f_1^2 \neq 0$. We may choose $\varphi : \mathcal{O}[[c_0, c'_1, d_1]] \rightarrow \overline{\mathbf{Q}}_p$ such that $\varphi(c_0) \neq 0$ and $\varphi(h) \neq 0$, see Lemma 9.22. We may extend it to \mathbf{S} so that $\varphi(g) = 0$. If $\varphi(f) = 0$ then $\varphi(f_1)\varphi(d_0) + \varphi(f_0) = 0$, and since $\varphi(d_0)$ is a root of $\mathbf{X}^2 + \varphi(c'_1)\mathbf{X} + \varphi(c_0 d_1 + pc_0)$, we would obtain that $\varphi(h) = 0$. \square

Let $D_{\text{tr}\rho}^{\text{ps}}$ be the deformation functor parameterizing all the 2-dimensional pseudocharacters of $\mathcal{G}_{\mathbf{Q}_p}$ lifting $\text{tr}\rho$. We know that $D_{\text{tr}\rho}^{\text{ps}}$ is pro-represented by a complete local noetherian \mathcal{O} -algebra (S, \mathfrak{m}_S) . Trace induces a morphism of functors $\text{Def}_\rho \rightarrow D_{\text{tr}\rho}^{\text{ps}}$ and hence homomorphism of local \mathcal{O} -algebras $\theta : S \rightarrow R$. It follows from [40, 1.4.4] that θ is surjective. We note that this can also be deduced from Theorem B.3, since we have written down the universal deformation explicitly. We are going to show that θ is an isomorphism.

Lemma B.11. — *Trace induces a bijection $\text{Def}_\rho(k[x]/(x^2)) \xrightarrow{\cong} D_{\text{tr}\rho}^{\text{ps}}(k[x]/(x^2))$.*

Proof. — Since θ is surjective we already know that the map is an injection. Hence to show surjectivity it is enough to show that both spaces have the same dimension as k -vector spaces. It follows from [2, Thm 2] that $\dim_k D_{\text{tr}\rho}^{\text{ps}}(k[x]/(x^2)) = 6$, which is also the dimension of $\text{Def}_\rho(k[x]/(x^2))$. \square

Let $F := \mathcal{O}[[a_0, a_1, c_0, c_1, d_0, d_1]]$ it follows from Lemma B.11 that there exist surjections $F \xrightarrow{\kappa} S \xrightarrow{\theta} R$, which induce isomorphisms on the tangent spaces. We may assume that the composition $\kappa \circ \theta$ is the one used to present R in (B.1). Given a local \mathcal{O} -algebra (R, \mathfrak{m}) we denote $\bar{R} := R/\varpi R$, and let $\bar{R}_n := R/(\mathfrak{m}^n + \varpi R)$.

Let $\beta : F \rightarrow k[x]/(x^3)$ be a homomorphism of \mathcal{O} -algebras such that $a_0, a_1, c_0, d_1 \mapsto 0, c_1, d_0 \mapsto x$. Let $\rho_\beta : \mathcal{F} \rtimes G \rightarrow \text{GL}_2(k[x]/(x^3))$ be a representation defined by the same formulas used to define α in Proposition B.2.

Lemma B.12. — *Let $h \in \mathcal{F} \rtimes G$ be such that $\text{tr}\rho_\beta(hg) = \text{tr}\rho_\beta(g)$ for all $g \in \mathcal{F} \rtimes G$ then $\rho_\beta(h) = 1$.*

Proof. — Since $\mathbf{1} \neq \omega$, $\text{tr}\rho_\beta(g)$ determines the diagonal entries of $\rho_\beta(g)$ for all $g \in \mathcal{F} \rtimes G$, see (B.3) below. In particular, for all $g \in \mathcal{F} \rtimes G$ the diagonal entries of $\rho_\beta(gh)$ are equal to the diagonal entries of $\rho_\beta(g)$. Let $x_i \in \mathcal{F}$ be the generators defined in Lemma B.1. Applying the last observation to $g = x_0, g = x_{p-2}$ we deduce that $\rho_\beta(h)$ is unipotent upper-triangular, and to $g = x_p$ we deduce that $\rho_\beta(h) = 1$. \square

Lemma B.13. — *The surjection $\kappa : \bar{F}_3 \rightarrow \bar{S}_3$ is not an isomorphism.*

Proof. — Suppose κ is injective then $\bar{F}_3 \cong \bar{S}_3$. We may consider $\text{tr}\rho$ as a pseudocharacter of $\mathcal{F} \rtimes G$ and let D' be the deformation functor parameterizing 2-dimensional pseudocharacters of $\mathcal{F} \rtimes G$ lifting $\text{tr}\rho$. Corollary A.3 says that every 2-dimensional pseudocharacter of $\mathcal{G}_{\mathbf{Q}_p}$ lifting $\text{tr}\rho$ is a pseudocharacter of $\text{Gal}(F(p)/\mathbf{Q}_p)$ and thus using Proposition B.2 we may consider $D_{\text{tr}\rho}^{\text{ps}}$ as a subfunctor of D' . Using [2, Thm. 2] we deduce that $\dim_k D'(k[x]/(x^2)) = 6$. Thus if $\bar{S}_3 \cong \bar{F}_3$ then $D'(k[x]/(x^3)) = D_{\text{tr}\rho}^{\text{ps}}(k[x]/(x^3))$. This would mean that every 2-dimensional pseudocharacter of $\mathcal{F} \rtimes G$ lifting $\text{tr}\rho$ is

automatically a pseudocharacter of $\text{Gal}(\mathbb{F}(\rho)/\mathbf{Q}_p)$. This would mean that there exists $T \in D_{\text{tr } \rho}^{\text{ps}}(k[x]/(x^3))$, such that $T(\varphi'(g)) = \text{tr } \rho_\beta(g)$ for all $g \in \mathcal{F} \rtimes G$. The equality would imply that for all $h \in \text{Ker } \varphi'$ and $g \in \mathcal{F} \rtimes G$ we have $\text{tr } \rho_\beta(hg) = \text{tr } \rho_\beta(g)$. Lemma B.12 implies that $\text{Ker } \varphi'$ is contained in $\text{Ker } \rho_\beta$. However, as $\beta(\rho c_0 + c_1 d_0 + c_1 d_0) = x^2 \neq 0$ we obtain a contradiction to the universality of the representation constructed in Proposition B.2. \square

Lemma B.14. — *The map θ induces an isomorphism $\bar{S}_3 \cong \bar{R}_3$.*

Proof. — Since $\bar{S}_3 \rightarrow \bar{R}_3$ is surjective it is enough to show the equality of dimensions as k -vector spaces. We have $\dim_k \bar{R}_3 \leq \dim_k \bar{S}_3 < \dim_k \bar{F}_3 = \dim_k \bar{R}_3 + 1$, where the strict inequality follows from Lemma B.13 and the equality from (B.1). \square

Theorem B.15. — *The map $\theta : S \rightarrow R$ is an isomorphism.*

Proof. — Since R is \mathcal{O} -torsion free it is enough to show that $\theta : \bar{S} \rightarrow \bar{R}$ is an isomorphism. Let $f = c_0 d_1 + c_1 d_0 \in \bar{F}$, so that $\bar{R} = \bar{F}/(f)$ and let \mathfrak{m} be the maximal ideal of \bar{F} . It is enough to show that $\kappa(f) = 0$. It follows from Lemma B.14 that there exists $g \in \mathfrak{m}^3$, such that $\kappa(f) = \kappa(g)$. Thus $\theta(\kappa(g)) = 0$ and so there exists $h \in \bar{F}$ such that $g = fh$. Now h cannot be a unit as $g \in \mathfrak{m}^3$ and $f \notin \mathfrak{m}^3$. Hence $h \in \mathfrak{m}$ and so $1 - h$ is a unit. Since $\kappa(f(1 - h)) = \kappa(f) - \kappa(g) = 0$ and $1 - h$ is a unit we deduce that $\kappa(f) = 0$. \square

Corollary B.16. — *Let $\psi : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathcal{O}^\times$ be a continuous character lifting $\det \rho$ and let S^ψ and R^ψ be the rings pro-representing functors $D_{\text{tr } \rho}^{\text{ps}, \psi}$ and Def_ρ^ψ then trace induces an isomorphism $S^\psi \cong R^\psi$.*

B.1 Generic reducible case. — Let $\chi_1, \chi_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow k^\times$ be continuous characters, such that $\chi_1 \chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$. We assume that $p \geq 3$. This assumption and a standard calculation with local duality and local Euler-Poincaré characteristic imply that both subspaces $\text{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\chi_1, \chi_2)$ and $\text{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\chi_2, \chi_1)$ are 1-dimensional. Let

$$0 \rightarrow \chi_1 \rightarrow \rho_{12} \rightarrow \chi_2 \rightarrow 0, \quad 0 \rightarrow \chi_2 \rightarrow \rho_{21} \rightarrow \chi_1 \rightarrow 0$$

be non-split extensions. From now on the indices i, j will mean either $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$. Since $\text{Ext}_{k[\mathcal{G}_{\mathbf{Q}_p}]}^1(\chi_j, \chi_i)$ is 1-dimensional, such ρ_{ij} exists and is unique up to isomorphism. Since $\chi_1 \neq \chi_2$ we have $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\rho_{ij}) = k$ thus the universal deformation problem $D_{\rho_{ij}}$ for ρ_{ij} is (pro-)representable by a ring R_{ij} . Our assumptions $\chi_1 \chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$ imply that $H^2(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho_{ij}) = 0$ and $H^1(\mathcal{G}_{\mathbf{Q}_p}, \text{Ad } \rho_{ij})$ is 5-dimensional. Hence, R_{ij} is formally smooth of relative dimension 5 over \mathcal{O} . Let $\tilde{\rho}_{ij}$ be the universal deformation of ρ_{ij} .

Let G be the image of $\mathcal{G}_{\mathbf{Q}_p}$ in $k^\times \times k^\times$ under the map $g \mapsto (\chi_1(g), \chi_2(g))$ and let P be the maximal pro- p quotient of the kernel of this map. Since the order of G is prime

to p after choosing some splitting we may assume that the action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\tilde{\rho}_{ij}$ factors through $P \rtimes G$. Let $\tilde{\chi}_1$ and $\tilde{\chi}_2$ be the Teichmüller lifts of χ_1 and χ_2 respectively.

Let χ be the trace of ρ_{ij} , so that $\chi = \chi_1 + \chi_2$ and let D_χ^{ps} be the functor parameterizing all the 2-dimensional pseudo-characters lifting χ . The functor is represented by a ring R_χ^{ps} . Trace induces a morphism of functors $D_{\rho_{ij}} \rightarrow D_\chi^{\text{ps}}$ and hence a ring homomorphism $\theta : R_\chi^{\text{ps}} \rightarrow R_{ij}$.

Proposition B.17. — *Trace induces an isomorphism $\theta : R_\chi^{\text{ps}} \xrightarrow{\cong} R_{ij}$.*

Proof. — The map is surjective by [40, 1.4.4]. Now the tangent space of R_χ^{ps} is at most 5-dimensional by Theorem 2 in [2], alternatively the Proposition can be deduced from [2] Theorem 4 and Remark 3. Since R_{ij} is formally smooth of relative dimension 5 over \mathcal{O} , we deduce that the map is an isomorphism. \square

Corollary B.18. — *Let $T : \mathcal{G}_{\mathbf{Q}_p} \rightarrow R_\chi^{\text{ps}}$ be the universal 2-dimensional pseudocharacter lifting χ then $\text{tr } \tilde{\rho}_{12}(g) = T(g) = \text{tr } \tilde{\rho}_{21}(g)$, for all $g \in \mathcal{G}_{\mathbf{Q}_p}$.*

Let \mathfrak{t} be the residubility ideal in R_χ^{ps} in the sense of [3, Def. 1.5.2]. It is uniquely determined by the following universal property: an ideal J of R_χ^{ps} contains \mathfrak{t} if and only if $T \pmod{J} = T_1 + T_2$, where $T_1, T_2 : \mathcal{G}_{\mathbf{Q}_p} \rightarrow R_\chi^{\text{ps}}/J$, are deformations of χ_1 and χ_2 , respectively, to R_χ^{ps}/J , see [3, Prop. 1.5.1]. We note that since $\chi_1 \neq \chi_2$, T_1 and T_2 are determined uniquely by $T \pmod{J}$ by the formulas:

$$(B.3) \quad T_i(g) = \frac{1}{|G|} \sum_{h \in G} \tilde{\chi}_i^{-1}(h) T(hg) \pmod{J}.$$

Proposition B.19. — *$R_\chi^{\text{ps}}/\mathfrak{t}$ is formally smooth of relative dimension 4 over \mathcal{O} .*

Proof. — Let D_{χ_1} and D_{χ_2} be the universal deformation problems for χ_1 and χ_2 , respectively. A standard argument shows that they are represented by formally smooth \mathcal{O} -algebras R_{χ_1}, R_{χ_2} , which are of relative dimension 2 over \mathcal{O} . The functor $D_{\chi_1} \times D_{\chi_2}$ is represented by the ring $R_{\chi_1} \widehat{\otimes}_{\mathcal{O}} R_{\chi_2}$, which is formally smooth of relative dimension 4 over \mathcal{O} . By the definition of the reducibility ideal, the map $a : D_{\chi_1} \times D_{\chi_2} \rightarrow D_\chi$, $(T_1, T_2) \mapsto T_1 + T_2$ factors through the map $\bar{a} : D_{\chi_1} \times D_{\chi_2} \rightarrow \text{Hom}(R_\chi^{\text{ps}}/\mathfrak{t}, *)$. It is trivially injective, when the functors are evaluated, and it follows from (B.3), that is also surjective. Hence, \bar{a} is an isomorphism of functors and so $R_\chi^{\text{ps}}/\mathfrak{t} \cong R_{\chi_1} \widehat{\otimes}_{\mathcal{O}} R_{\chi_2}$. \square

Corollary B.20. — *The reducibility ideal \mathfrak{t} is a principal ideal.*

Proof. — In fact we prove a stronger statement. It follows from Proposition B.17 that R_χ^{ps} is formally smooth of relative dimension 5 over \mathcal{O} . Thus we may deduce from Proposition B.19 that \mathfrak{t} is generated by an element contained in the maximal ideal of R_χ^{ps} ,

but not contained in the square of the maximal ideal. Alternatively, one could use [3, Prop. 1.7.1]. \square

Since the order of G is prime to p , we may choose a basis $\{v_1^j, v_2^j\}$ of $\tilde{\rho}_{ij}$, such that G acts on v_1^j by the character $\tilde{\chi}_1$ and on v_2^j by the character $\tilde{\chi}_2$. Fixing a basis allows us to think about $\tilde{\rho}_{ij}$ as a continuous group homomorphism $\tilde{\rho}_{ij} : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathrm{GL}_2(\mathbf{R}_{ij})$, so that $\rho_{12} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ and $\rho_{21} = \begin{pmatrix} \chi_1 & 0 \\ * & \chi_2 \end{pmatrix}$.

Lemma B.21. — $\mathrm{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij}) = \mathbf{R}_{ij}$.

Proof. — Since the characters $\tilde{\chi}_1$ and $\tilde{\chi}_2$ are distinct $\mathrm{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij}) \subseteq \mathrm{End}_G(\tilde{\rho}_{ij}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} : \lambda, \mu \in \mathbf{R}_{ij} \right\}$. Since ρ_{ij} is non-split, there exist $g \in \mathcal{G}_{\mathbf{Q}_p}$ such that either the entry (1, 2) or the entry (2, 1) of $\tilde{\rho}_{ij}(g)$ is a unit in \mathbf{R}_{ij} . The only elements of $\mathrm{End}_G(\tilde{\rho}_{ij})$ commuting with $\tilde{\rho}_{ij}(g)$ are scalar matrices, which then commute with everything. \square

Definition B.22. — For $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$, let \mathfrak{r}_{ij} be the ideal of \mathbf{R}_{ij} generated by the (j, i) -entry of $\tilde{\rho}_{ij}(g)$, for all $g \in \mathcal{G}_{\mathbf{Q}_p}$.

Proposition B.23. — The isomorphism θ of Proposition B.17 maps \mathfrak{r} to \mathfrak{r}_{ij} .

Proof. — Since $\rho_{12} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ and $\rho_{21} = \begin{pmatrix} \chi_1 & 0 \\ * & \chi_2 \end{pmatrix}$, the ideal \mathfrak{r}_{ij} is contained in the maximal ideal of \mathbf{R}_{ij} . By construction of \mathfrak{r}_{ij} , the representation $\tilde{\rho}_{ij} \pmod{\mathfrak{r}_{ij}}$ is reducible. Hence, its trace is a direct sum of two characters, which are deformations of χ_1 and χ_2 to $\mathbf{R}_{ij}/\mathfrak{r}_{ij}$. Thus, \mathfrak{r} is contained in $\theta^{-1}(\mathfrak{r}_{ij})$, and so $\theta(\mathfrak{r}) \subset \mathfrak{r}_{ij}$.

Let \mathbf{K} be the quotient field of $\mathbf{R}_{ij}^{\mathrm{ps}}/\mathfrak{r}$. If for some $g \in \mathcal{G}_{\mathbf{Q}_p}$ the (j, i) -entry of $\tilde{\rho}_{ij}(g) \pmod{\theta(\mathfrak{r})}$ is non-zero, then the representation $\tilde{\rho}_{ij} \otimes_{\mathbf{R}_{ij}} \mathbf{K}$ is absolutely irreducible. However, this is impossible as the trace of $\tilde{\rho}_{ij} \otimes_{\mathbf{R}_{ij}} \mathbf{K}$ is a sum of two characters. This implies that $\mathfrak{r}_{ij} \subset \theta(\mathfrak{r})$. \square

We fix a generator c of the ideal \mathfrak{r} . It follows from Proposition B.23 that $\theta(c)$ is a generator of \mathfrak{r}_{12} . Let $\tilde{\rho}_{12}^c : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathrm{GL}_2(\mathbf{R}_{12})$ be the representation defined by

$$\tilde{\rho}_{12}^c(g) := \begin{pmatrix} \theta(c) & 0 \\ 0 & 1 \end{pmatrix} \tilde{\rho}_{12}(g) \begin{pmatrix} \theta(c)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

A priori the image of $\tilde{\rho}_{12}^c$ lands in the GL_2 of the quotient field of \mathbf{R}_{12} , but since $\mathbf{R}_{12} = \theta(c)^{-1}\mathfrak{r}_{12}$, the image is contained in $\mathrm{GL}_2(\mathbf{R}_{12})$.

Proposition B.24. — *The representation $\tilde{\rho}_{12}^c$ is a deformation of ρ_{21} to \mathbf{R}_{12} . The induced map $\alpha : \mathbf{R}_{21} \rightarrow \mathbf{R}_{12}$ is an isomorphism, making the following diagram:*

$$\begin{array}{ccc} \mathbf{R}_{21} & \xrightarrow{\alpha} & \mathbf{R}_{12} \\ \theta \uparrow & & \uparrow \theta \\ \mathbf{R}_\chi^{\text{ps}} & \xrightarrow{=} & \mathbf{R}_\chi^{\text{ps}} \end{array}$$

commute.

Proof. — The reduction of $\tilde{\rho}_{12}^c$ modulo the maximal ideal of \mathbf{R}_{12} is of the form $\begin{pmatrix} \chi_1 & 0 \\ * & \chi_2 \end{pmatrix}$. Since $\mathbf{R}_{12} = \theta(c)^{-1}\mathfrak{r}_{12}$, there exists $g \in \mathbf{P}$, such that $*(g) \neq 0$. This implies that the extension is non-split. Since $\text{Ext}_{\mathcal{G}_{\mathbf{Q}_p}}^1(\chi_1, \chi_2)$ is one dimensional, we deduce that the reduction of $\tilde{\rho}_{12}^c$ modulo the maximal ideal of \mathbf{R}_{12} is isomorphic to ρ_{21} . Hence, $\tilde{\rho}_{12}^c$ is a deformation of ρ_{21} to \mathbf{R}_{12} . This induces the map $\alpha : \mathbf{R}_{21} \rightarrow \mathbf{R}_{12}$. Since $\text{tr } \rho_{12}^c = \text{tr } \rho_{12} = \text{tr } \rho_{21}$ we obtain a commutative diagram as above. Since θ is an isomorphism, we deduce from the diagram that α is also an isomorphism. \square

Corollary B.25. — *$\text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij}^c, \tilde{\rho}_{ji}^c)$ is a free $\mathbf{R}_\chi^{\text{ps}}$ module of rank 1.*

Proof. — It follows from the Proposition B.24 that $\tilde{\rho}_{12}^c \cong \tilde{\rho}_{21} \otimes_{\mathbf{R}_{21}, \alpha} \mathbf{R}_{12}$. Hence, it is enough to show that $\text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{12}^c, \tilde{\rho}_{12}^c)$ is a free \mathbf{R}_{12} -module of rank 1. If we think of $\tilde{\rho}_{12}^c$, $\tilde{\rho}_{12}^c$ as representations of $\mathcal{G}_{\mathbf{Q}_p}$ on $\mathbf{R}_{12}v_1^{12} \oplus \mathbf{R}_{12}v_2^{12}$, then inside the ring of 2×2 -matrices over the quotient field of \mathbf{R}_{ij} , we have equalities of \mathbf{R}_{ij} -modules:

$$(B.4) \quad \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{12}^c, \tilde{\rho}_{12}^c) = \begin{pmatrix} \theta(c) & 0 \\ 0 & 1 \end{pmatrix} \text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{12}^c),$$

$$(B.5) \quad \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{12}^c, \tilde{\rho}_{12}^c) = \text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{12}^c) \begin{pmatrix} 1 & 0 \\ 0 & \theta(c) \end{pmatrix}.$$

The assertion follows from Lemma B.21. \square

Proposition B.26. — *The centre of the ring $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij} \oplus \tilde{\rho}_{ji})$ is isomorphic to $\mathbf{R}_\chi^{\text{ps}}$. Moreover, $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij} \oplus \tilde{\rho}_{ji})$ is a free $\mathbf{R}_\chi^{\text{ps}}$ -module of rank 4.*

Proof. — The ring $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij} \oplus \tilde{\rho}_{ji})$ is isomorphic to

$$(B.6) \quad \begin{pmatrix} \text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij}^c) & \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij}^c, \tilde{\rho}_{ji}^c) \\ \text{Hom}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ji}^c, \tilde{\rho}_{ij}^c) & \text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ji}^c) \end{pmatrix} \cong \begin{pmatrix} \mathbf{R}_\chi^{\text{ps}} \mathbf{1}_i & \mathbf{R}_\chi^{\text{ps}} \Phi_{ij} \\ \mathbf{R}_\chi^{\text{ps}} \Phi_{ji} & \mathbf{R}_\chi^{\text{ps}} \mathbf{1}_j \end{pmatrix}$$

where Φ_{ij} is described in Corollary B.25. It follows from Corollary B.25 that $\Phi_{ij} \circ \Phi_{ji} = c \mathbf{1}_i$ and $\Phi_{ji} \circ \Phi_{ij} = c \mathbf{1}_j$. Since $\mathbf{R}_\chi^{\text{ps}}$ is an integral domain we deduce that $\alpha \mathbf{1}_i + \beta \Phi_{ji} + \gamma \Phi_{ij} + \delta \mathbf{1}_j$ is central if and only if $\beta = \gamma = 0$ and $\alpha = \delta$. \square

Corollary B.27. — Let c be a generator of the reducibility ideal \mathfrak{r} in $\mathbf{R}_\chi^{\text{ps}}$. Then $\text{End}_{\mathcal{G}_{\mathbf{Q}_p}}(\tilde{\rho}_{ij} \oplus \tilde{\rho}_{ji})[1/c]$ is isomorphic to the ring of 2×2 matrices over $\mathbf{R}_\chi^{\text{ps}}[1/c]$.

Proof. — The isomorphism is induced by sending $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto \mathbf{1}_i$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \Phi_{ij}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto c^{-1}\Phi_{ji}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \mathbf{1}_j$. \square

Remark B.28. — Let $\psi : \mathcal{G}_{\mathbf{Q}_p} \rightarrow \mathcal{O}^\times$ be a continuous character, congruent to $\chi_1 \chi_2$ modulo \mathfrak{m} . The results of this section hold if instead of working with an unrestricted deformation problem, we consider only those deformations with determinate equal to ψ . The proofs carry over word for word, except that one has to subtract 2 from every dimension, and in the proof of Proposition B.19 one obtains an isomorphism $\mathbf{R}_\chi^{\text{ps}, \psi} / \mathfrak{r} \cong \mathbf{R}_{\chi_1}$, since the determinant condition imposes the relation $T_1 T_2 = \psi$, and hence T_2 is uniquely determined by T_1 .

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