THE FULL RENORMALIZATION HORSESHOE FOR UNIMODAL MAPS OF HIGHER DEGREE: EXPONENTIAL CONTRACTION ALONG HYBRID CLASSES

by Artur AVILA *and* Mikhail LYUBICH

ABSTRACT

We prove exponential contraction of renormalization along hybrid classes of infinitely renormalizable unimodal maps (with arbitrary combinatorics), in any even degree d. We then conclude that orbits of renormalization are asymptotic to the full renormalization horseshoe, which we construct. Our argument for exponential contraction is based on a precompactness property of the renormalization operator ("beau bounds"), which is leveraged in the abstract analysis of holomorphic iteration. Besides greater generality, it yields a unified approach to all combinatorics and degrees: there is no need to account for the varied geometric details of the dynamics, which were the typical source of contraction in previous restricted proofs.

CONTENTS

1. Introduction

1.1. *Renormalization Conjecture and Regular or Stochastic Theorem.* — The Renormalization Conjecture formulated in mid 1970's by Feigenbaum [F] and independently by Coullet and Tresser [TC] has been a focus of research ever since. Roughly speaking, it says that a certain "renormalization operator" is hyperbolic in an appropriate infinite-dimensional functional space. It explains remarkable universality properties on various families of dynamical systems (see [Cv] for a collection of early papers on the subject). More recently, it has played a central role in the measure-theoretical analysis of one-dimensional dynamical systems, particularly in the proofs of the Regular or Stochastic Dichotomy in the real quadratic family [L4] and more general spaces of quasiquadratic unimodal maps [ALM].

Here we will consider the renormalization operator R in the space $\mathcal{C}_d^{\mathbf{R}}$ of real unicritical polynomial-like maps of an arbitrary even degree $d \geq 2$. Hyperbolicity of R was proven for bounded combinatorics by Sullivan, McMullen and one of the authors in

DOI 10.1007/s10240-011-0034-2

[S, McM2, L3], and then for all combinatorics in the quadratic case [L4]. Our goal is to generalize the latter result to an arbitrary even degree d.

In this paper we will prove that the renormalization operator R has an invariant horseshoe $\mathcal{A}$ and is exponentially contracting on the corresponding hybrid lamination. In the forthcoming paper we will deal with the transverse unstable direction. Together with the previous analysis of non-renormalizable unimodal maps [ALS], this will prove the Regular or Stochastic Dichotomy in any unicritical family $p_c : x \mapsto x^d + c$ (with $d \geq 2$ even): For almost any real c (for which p_c has an invariant interval), the map p_c is either regular (i.e., it has an attracting cycle) or stochastic (i.e., it has an absolutely continuous invariant measure).

Besides supplying a more general version of the Renormalization Theorem, our goal is to address the issue of exponential contraction along hybrid leaves in a novel unified way, which does not involve fine geometric considerations (highly dependent on the combinatorics and degree). Our approach simplifies the previously known proofs in the quadratic-like case, even for the renormalization with bounded combinatorics. Namely, we will derive the desired result from the previously known *beau bounds* for real maps [S, LvS, LY], and basic facts of functional analysis and topology.

1.2. *Statement of the result.* — Let us now formulate our main result more precisely. To this end we need a few basic definitions that we now outline; a more detailed background will be supplied in the main body of the paper.

A *unicritical polynomial-like map* of degree d is a degree d branched covering $f : U \to V$ between two topological disks $U \Subset V$ that has a single critical point. We normalize f so that $f(z) = z^d + c + O(z^{d+1})$ at the origin. Note that this normalization survives rotations through $2\pi k/(d-1)$, $k \in \mathbf{Z}/(d-1)\mathbf{Z}$: conjugating a normalized map by such a rotation, we obtain a normalized map with rotated c.

The (filled) *Julia set* $K(f)$ is the set of non-escaping points. It is either connected or a Cantor set depending on whether $0 \in K(f)$ or not. If f is a polynomial-like map with connected Julia set then the corresponding polynomial-like *germ* is defined as the class of polynomial-like maps $\tilde{f}$ with the same Julia set and such that $\tilde{f} \mid K(\tilde{f}) = f \mid K(f)$.

Let $\mathcal{C} = \mathcal{C}_d$ stand for the space of normalized polynomial-like germs of degree d with connected Julia set. It intersects the polynomial family $p_c : z \mapsto z^d + c$, $c \in \mathbf{C}$, by the *Multibrot set* $\mathcal{M} = \mathcal{M}_d$ (defined as the set of c for which the Julia set $K(p_c)$ is connected).

Two polynomial-like germs are called *hybrid equivalent* if they have representatives $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ that are conjugate by a quasiconformal homeomorphism $h : V \to \tilde{V}$ such that $\bar{\partial} h = 0$ almost everywhere on $K(f)$. The corresponding equivalence classes are called *hybrid classes*. According to the Douady-Hubbard Straightening Theorem [DH], any hybrid class in $\mathcal{C}$ intersects the Multibrot set $\mathcal{M}$ by an orbit of the rotation group $\mathbf{Z}/(d-1)\mathbf{Z}$.

A unicritical polynomial-like germ f is called *renormalizable* if there is a disk $\Omega \ni 0$ and a $p \geq 2$ such that the map $f^p \mid \Omega$ is unicritical polynomial-like with connected Julia set (subject of a few extra technical requirements—see Section 2.10). Appropriately normalizing this polynomial-like germ, we obtain the *renormalization* of f. If p is the smallest period for which f is renormalizable, then the corresponding renormalization is denoted Rf.

We can now naturally define *infinitely renormalizable polynomial-like germs*. Let $\mathcal{I}^{\mathbf{R}}$ stand for the space of *real* infinitely renormalizable polynomial-like germs (that is, the germs preserving the real line), and let $\mathcal{I}^{(\mathbf{R})}$ stand for the space of polynomial-like germs that are hybrid equivalent to the real ones. The renormalization operator R naturally acts in both spaces preserving the hybrid partition. In what follows, this partition will serve as the *stable lamination*:

Main Theorem. — *There is an* R*-invariant precompact set* $\mathcal{A} \subset \mathcal{I}^{\mathbf{R}}$ (*the renormalization horseshoe*) *such that* $R\,|\,\mathcal{A}$ *is topologically conjugate to the two-sided shift in infinitely many symbols, and any germ* $f \in \mathcal{I}^{(\mathbf{R})}$ *is attracted to some orbit of* $\mathcal{A}$ *at a uniformly exponential rate, in a suitable "Carathéodory metric".*

See Theorem 9.2 for a slightly more detailed formulation.

Remark **1.1.** — Our approach to exponential contraction also applies to certain non-real renormalization combinatorics (for which the appropriate beau bounds have been established, see [K, KL1, KL2]). See Theorem 5.1.

1.3. *Outline of the proof.* — We start with the argument for exponential contraction along hybrid classes. To fix ideas, let us consider first the case of a *fixed hybrid leaf* $\mathcal{H}_c$ (the connected component of the hybrid class of p_c) so that every time we iterate the same renormalization operator R.

Hybrid lamination. — In Section 4 we endow hybrid leaves with a *path holomorphic structure* and show that all of them are bi-holomorphically equivalent. The path holomorphic structure endows these spaces with *Carathéodory pseudo-metrics* and we prove that they are *Carathéodory hyperbolic*, i.e., these pseudo-metrics are, in fact, metrics.

The Schwarz Lemma. — The renormalization operator, as a map from one hybrid leaf to another, is holomorphic with respect to their path holomorphic structure. This puts us in a position to apply the Schwarz Lemma to the analysis of its iterates: its weak form (see Section 3) already implies that renormalization is *weakly contracting* with respect to the Carathéodory metric.

Beau bounds in $\mathcal{H}_c$ mean by definition that there exists a compact set $\mathcal{K} \subset \mathcal{C}$ such that for every $f \in \mathcal{H}_c$, $R^n f \in \mathcal{K}$ for any n sufficiently large (depending only on the quality of the analytic extension of f), see Section 5.

For real maps with stationary combinatorics, beau bounds were proved by Sullivan (see [S, MvS]) in early 1990's, for complex maps with *primitive* stationary combinatorics, they have been recently established by Kahn [K]. (Note that this result covers all real stationary combinatorics except the period doubling.) Exponential contraction can be easily concluded from beau bounds through the entire complex hybrid leaf $\mathcal{H}_c$:

The Strong Schwarz Lemma shows that holomorphic endomorphisms are strongly contracting with respect to the Carathéodory metric, provided the image is "small" in the range, in the sense that the diameter is less than 1,[1] see Section 3. Beau bounds imply that for any compact set $\mathcal{Q} \subset \mathcal{H}_c$ there exists N such that $R^n(\mathcal{Q})$ contained in the (universal) compact set $\mathcal{K}$ for $n \geq N$. By selecting $\mathcal{Q} \supset \mathcal{K}$ sufficiently large, we fulfill the smallness condition of $\mathcal{K}$ inside $\mathcal{Q}$, and can conclude that $R^N \mid \mathcal{Q}$ is strongly contracting (Section 5).

We will now give a different argument for stationary real combinatorics that relies only on the beau bounds for *real* maps. It makes use of one general idea of functional analysis:

Almost periodicity. — The beau bounds for real maps (and even just for real polynomials) imply that the cyclic semigroup $\{R^n\}_{n=0}^{\infty}$ is precompact in the topology of uniform convergence on compact sets of $\mathcal{H}_c$. (Such semigroups are called almost periodic, see [Lju1, Lju2].) Then the ω-limit set of this semigroup is a *group*. Its unit element is a retraction $P : \mathcal{H}_c \to \mathcal{Z} = \operatorname{Fix} P$.

Remark **1.2.** — More directly, the precompactness of $\{R^n\}$ allows us to find nearby iterates R^n and R^m with $n > 2m$. It follows that R^{n-m} is close to the identity in $\operatorname{Im} R^m \supset \operatorname{Im} R^{n-m}$. Taking limits we get a map P which is exactly the identity in $\operatorname{Im} P$, so that P is a retraction.

Let $P^{\mathbf{R}} : \mathcal{H}_c^{\mathbf{R}} \to \mathcal{Z}^{\mathbf{R}}$ be the restriction of P to the real slice.

Topological argument and analytic continuation (Section 7). — The beau bounds for real maps imply that the real slice $\mathcal{Z}^{\mathbf{R}}$ is compact. By the Implicit Function Theorem, $\mathcal{Z}^{\mathbf{R}}$ is a finite-dimensional manifold. But one can show that the space $\mathcal{H}_c^{\mathbf{R}}$ is contractible (see Lemma 2.1 and Theorem 2.2), and hence the retract $\mathcal{Z}^{\mathbf{R}}$ is contractible as well. But the only contractible compact finite dimensional manifold (without boundary) is a single point. So, $P^{\mathbf{R}}$ collapses the real slice $\mathcal{H}_c^{\mathbf{R}}$ to a single point f_*. Since P is holomorphic, it collapses the whole space $\mathcal{H}_c$ to f_* as well.

Since P is constant it follows that $R^n \to P$ uniformly on compact subsets of $\mathcal{H}_c$, and we can conclude exponential contraction through the strong Schwarz Lemma as before.

This completes the argument for the case of stationary combinatorics.

[1] We gauge the Carathéodory metric so that any space has diameter at most 1. Condition that the diameter of $\mathcal{K}$ in $\mathcal{Q}$ is less than 1 means that "$\mathcal{K}$ is well inside of $\mathcal{Q}$".

Unbounded combinatorics. — Beau bounds for arbitrary real maps were established in [LvS, LY]. For complex maps, they have been recently established for a fairly big class of combinatorics in [KL1, KL2] (see also [L2] for earlier results). Our first argument that uses the beau bounds for complex maps extends to the unbounded combinatorics case in a straightforward way, using the leafwise Carathéodory metric on the hybrid lamination (Section 5). The second argument based on almost periodicity requires an extension of this idea from semigroups to *cocycles* (or *grupoids*). This is carried out in Section 8.1.

Horseshoe. — Once contraction is proved, the horseshoe is constructed in the familiar way (see [L4]) using *rigidity* of real maps [L2, GS, KSS]. It is automatically semiconjugate to the full shift. A further argument based on the analysis of the analytic continuation of anti-renormalizable maps (which becomes substantially more involved in the higher degree case, see Appendix A) yields the full topological conjugacy.

1.4. *Comparison with earlier approaches.* — In the case of stationary combinatorics, two approaches were previously used to construct the fixed point f_* and to prove convergence to f_* in the hybrid class $\mathcal{H}(f_*)$. The first one, due to Sullivan, is based on ideas of Teichmüller theory (see [S, MvS]); the other one, due to McMullen, is based on a geometric theory of *towers* and their quasiconformal rigidity [McM2].

The Teichmüller approach, albeit beautiful and natural, faces a number of subtle technical issues. Also, it does not seem to lead to the exponential contraction.[2] The geometric tower approach can be carried out all the way to prove exponential contraction [McM2]. On the other hand, in [L3], exponential contraction was obtained by combining towers rigidity (as a source of contraction, but without the rate) with the Schwarz Lemma in Banach spaces.

Both approaches generalize without problem to the bounded combinatorics case. The tower approach can be carried further to "essentially bounded" combinatorics [Hi]. The remaining "high" combinatorics case (as well as the oscillating situation) was handled in [L4] using the geometric property of growth of moduli in the Principal Nest of the Yoccoz puzzle [L2]. This is a powerful geometric property which is valid only in the quadratic case. So, this method is not sufficient in the higher degree case (at least, it would require further non-trivial geometric analysis).

The approaches developed in this paper use much softer geometric input (only beau bounds) and treat all the combinatorial cases in a unified way.

Remark **1.3.** — There are also computer-assisted methods going back to the classical paper by Lanford [La], as well as approaches that do not rely on holomorphic dynamics [E, Ma2]. These methods can be important for dealing with the case of fractional degree d. The almost periodicity idea can possibly contribute to it, too.

[2] It has been suggested that this relates to the fact the Teichmüller approach naturally deals with conformal (rather than affine) equivalence between the polynomial-like germs. However, our Schwarz Lemma argument seems to work equally well for conformal classes.

1.5. *Basic Notation.* — $\mathbf{D} = \{z : |z| < 1\}$ is the unit disk,
$\mathbf{D}_r = \{z : |z| < r\}$ is the disk of radius r,
and $\mathbf{T} = \{z : |z| = 1\}$ is the unit circle;
$p_c : z \mapsto z^d + c$ is the unicritical polynomial family.

We assume the reader's familiarity with the basic theory of quasiconformal ("qc") maps. We let $\operatorname{Dil} h$ be the dilatation of a qc map.

2. Hybrid classes, external maps, and renormalization

The theory of polynomial-like maps was laid down in [DH] and further developed, particularly in the quadratic-like setting, in [McM1, L3]. In this section we will refine the basic theory in the case of unicritical polynomial-like maps of arbitrary degree.

2.1. *Holomorphic motions.* — Given a domain $D \subset \mathbf{C}$ with a base point λ_0 and a set $X_0 \subset \mathbf{C}$, a *holomorphic motion* of X_0 over D is a family of injections $h_\lambda : X_0 \to \mathbf{C}$, $\lambda \in D$, such that $h_{\lambda_0} = \mathrm{id}$ and $h_\lambda(z)$ is holomorphic in λ for any $z \in X_0$. Let $X_\lambda = h_\lambda(X_0)$.

We will summarize fundamental properties of holomorphic motions which are usually referred to as the λ-*lemma*. It consists of two parts: extension of the motion and transverse quasiconformality, which will be stated separately.

Extension λ-Lemma (See [Sl]). — A holomorphic motion $h_\lambda : X_0 \to X_\lambda$ of a set $X_0 \subset \mathbf{C}$ over the disk $\mathbf{D}$ admits an extension to a holomorphic motion $\hat{h}_\lambda : \mathbf{C} \to \mathbf{C}$ of the whole complex plane over $\mathbf{D}$.

Remark **2.1.** — We will usually keep the same notation, h_λ, for the extended motion.

Quasiconformality λ-Lemma (See [MSS]). — Let $h_\lambda : U_0 \to U_\lambda$ be a holomorphic motion of a domain $U_0 \subset \mathbf{C}$ over the disk $\mathbf{D}$, based on 0. Then over any smaller disk $\mathbf{D}_r$, $r < 1$, all the maps h_λ are $K(r)$-qc, where $K(r) = \frac{1+r}{1-r}$.

2.2. *Polynomial-like maps and germs.* — A *polynomial-like map* ("p-l map") of degree $d \geq 2$ is a holomorphic branched covering $f : U \to V$ of degree d between quasidisks $U \Subset V$. Its filled Julia set is $K(f) = \bigcap_{n\geq 0} f^{-n}(U)$, and the Julia set is $J(f) = \partial K(f)$. In what follows, the letter d will be reserved for the degree of f.

A p-l map is called *unicritical* if it has a unique critical point (of local degree d). We will normalize unicritical polynomial-like maps so that $0 \in U$ is the critical point and $f(z) = z^d + c + O(z^{d+1})$ near 0. In what follows, polynomial-like maps under consideration will be assumed unicritical. The annulus $V \setminus U$ is called the *fundamental annulus* of a p-l map $f : U \to V$ (the corresponding open and closed annuli, $V \setminus \bar{U}$ and $\bar{V} \setminus U$ will also be called "fundamental").

Basic examples of p-l maps are provided by appropriate restrictions of unicritical polynomials $p_c : z \mapsto z^d + c$, e.g., $p_c : \mathbf{D}_r \to p_c(\mathbf{D}_r)$ for $r > 1 + |c|$.

The *Basic Dichotomy* asserts that the (filled) Julia set of f is either connected or a Cantor set, and the former happens iff $0 \in \mathrm{K}(f)$.

Given a polynomial-like map f with connected Julia set, the corresponding *polynomial-like germ* is an equivalence class of polynomial-like maps $\tilde{f}$ such that $\mathrm{K}(f) = \mathrm{K}(\tilde{f})$ and $f = \tilde{f}$ in a neighborhood of 0 (hence, by analytic continuation, also in a neighborhood of $\mathrm{K}(f)$). We will not make notational distinction between polynomial-like maps and the corresponding germs. Let

$$\operatorname{mod} f = \sup \operatorname{mod}(\mathrm{V} \setminus \mathrm{U}),$$

where the supremum is taken over all p-l representatives $\mathrm{U} \to \mathrm{V}$ of f.

We let $\mathcal{C} = \mathcal{C}_d$ be the set of all polynomial-like germs with connected Julia set.

A unicritical polynomial $p_c : z \mapsto z^d + c$ defines an element of $\mathcal{C}$ if and only if c belongs to the Multibrot set $\mathcal{M} = \{c \in \mathbf{C} : \sup |p_c^n(0)| < \infty\}$. Those are the only (normalized) germs with infinite modulus.

We will use superscript $\mathbf{R}$ for the *real slice* of a certain space. For instance $\mathcal{C}^{\mathbf{R}}$ stands for germs of *real* polynomial-like maps $f : \mathrm{U} \to \mathrm{V}$ (with connected Julia set), i.e., such that f preserves the real line and domains U, V are $\mathbf{R}$-symmetric.

2.3. *Topology.* — For a quasidisk $\mathrm{U} \subset \mathbf{C}$, let $\mathcal{B}_\mathrm{U}$ stand for the Banach space of functions holomorphic in U and continuous in $\bar{\mathrm{U}}$. The norm in this space will be denoted by $\|\cdot\|$. Let $\mathcal{B}_\mathrm{U}(f, \epsilon)$ stand for the Banach ball in $\mathcal{B}_\mathrm{U}$ centered at f of radius ϵ.

We introduce a topology in $\mathcal{C}$ as follows. We say that $f_n \to f$ if there exists a quasidisk neighborhood W of $\mathrm{K}(f)$ such that (some representatives of) the germs f_n are defined on $\bar{\mathrm{W}}$ for sufficiently large n, and f_n converges to (an appropriate restriction of a representative of) f in the Banach space $\mathcal{B}_\mathrm{W}$. The topology in $\mathcal{C}$ is defined by declaring its closed sets to be the ones which are sequentially closed. It is easy to see that $\mathrm{K}(f)$ depends upper semi-continuously (in the Hausdorff topology) on $f \in \mathcal{C}$, while its boundary $\mathrm{J}(f)$ depends lower semi-continuously.

We let $\mathcal{C}(\epsilon)$ be the set of all $f \in \mathcal{C}$ with $\operatorname{mod}(f) \geq \epsilon$. Then $\mathcal{C}(\epsilon)$ is compact, and any compact subset $\mathcal{K}$ of $\mathcal{C}$ is contained in some $\mathcal{C}(\epsilon)$ (see [McM1]).

2.4. *Hybrid classes.* — Notice that the Multibrot set $\mathcal{M}$ has rotational symmetry of order $d-1$ coming from the fact that polynomials p_c and $p_{\varepsilon c}$ are affinely equivalent for $\varepsilon = e^{2\pi i/(d-1)}$. In fact, the *moduli space* of unicritical polynomials of degree d (that is, the space of these polynomial moduli affine conjugacy) is the orbifold $\mathbf{C}/\langle\varepsilon\rangle$ with order $d-1$ cone point at the origin.

We say that two polynomial-like germs $f, g \in \mathcal{C}$ are *hybrid equivalent* if there exists a quasiconformal map $h : \mathbf{C} \to \mathbf{C}$, such that $h(\mathrm{K}(f)) = \mathrm{K}(g)$, $h \circ f = g \circ h$ in a neighborhood of $\mathrm{K}(f)$ (for any representatives of f and g), and $\bar{\partial} h = 0$ on $\mathrm{K}(f)$. We call h a *hybrid conjugacy* between f and g.

By the Douady-Hubbard *Straightening Theorem*, every $f \in \mathcal{C}$ is hybrid conjugate to some p_c with $c \in \mathcal{M}$. However, in the higher degree case ($d > 2$) c may not be uniquely defined. Indeed, polynomials p_c and $p_{\varepsilon c}$ are affinely equivalent, so they belong to the same hybrid class. Vice versa, one can show that hybrid equivalent polynomials p_b and p_c are affinely equivalent, so $b = \varepsilon^k c$ for some $k \in \mathbf{Z}/(d-1)\mathbf{Z}$. We will see later how to define a single polynomial straightening associated to each germ (the resolution of the apparent ambiguity involves global considerations).

We let $\tilde{\mathcal{H}}_c$ be the hybrid class containing p_c.

2.4.1. *Beltrami paths.* — A path $f_\lambda \in \mathcal{C}$, $\lambda \in \mathbf{D}_r$, is called a *Beltrami path* if there exists a holomorphic motion $h_\lambda : \mathbf{C} \to \mathbf{C}$ over $\mathbf{D}_r$, based on 0, that provides a hybrid conjugacy between f_0 and f_λ (the continuity of $\lambda \mapsto f_\lambda$ is in fact automatic). In this case, the pair (f_λ, h_λ) is called a *guided Beltrami path*. The guided Beltrami paths with a fixed initial point f_0, are in one-to-one correspondence with holomorphic families of Beltrami differentials $\mu_\lambda = \bar{\partial} h_\lambda / \partial h_\lambda$ on $\mathbf{C}$ such that $\mu_0 \equiv 0$, and the differentials μ_λ vanish a.e. on $\mathrm{K}(f_0)$ and are f_0-invariant near $\mathrm{K}(f_0)$. So, in what follows our treatment of Beltrami paths will freely switch from one point of view to the other.

Obviously, any Beltrami path lies entirely in a path connected component of a hybrid class.

2.4.2. *Hybrid leaves.* — Given maps $f_0, f \in \tilde{\mathcal{H}}_c$, let us consider a hybrid conjugacy $h : \mathbf{C} \to \mathbf{C}$ between them. Let μ be the Beltrami differential of h with L^∞-norm $\kappa = \|\mu\|_\infty < 1$. The family of Beltrami differentials $\lambda\mu$, $|\lambda| < 1/\kappa$, generates a guided Beltrami path (f_λ, h_λ) in $\tilde{\mathcal{H}}_c$, with f_1 affinely conjugate to f. In particular each map in $\mathcal{C}$ can be connected to one of its straightenings by a Beltrami path.

Let $\mathcal{H}_c$ be the path connected component of $\tilde{\mathcal{H}}_c$ containing p_c. The $\mathcal{H}_c$ will be called *hybrid leaves*. By the previous discussion, $\tilde{\mathcal{H}}_c$ is the union of the hybrid leaves $\mathcal{H}_{\varepsilon^k c}$, $k \in \mathbf{Z}/(d-1)\mathbf{Z}$. We will later see that for $c \neq 0$, the hybrid leaves $\mathcal{H}_{\varepsilon^k c}$, $k \in \mathbf{Z}/(d-1)\mathbf{Z}$ (which by definition either coincide or are disjoint), are in fact all distinct.

2.5. *Expanding circle maps.* — A real analytic circle map $g : \mathbf{T} \to \mathbf{T}$ is called *expanding* if there exists $n \geq 1$ such that $|\mathrm{D}f^n(z)| > 1$ for every $z \in \mathbf{T}$.

Let $\mathcal{E} = \mathcal{E}_d$ be the space of real analytic expanding circle maps $g : \mathbf{T} \to \mathbf{T}$ of degree d normalized so that $g(1) = 1$. Such a map admits a holomorphic extension to a covering $\mathrm{U} \to \mathrm{V}$ of degree d, where $\mathrm{U} \Subset \mathrm{V}$ are annuli neighborhoods of $\mathbf{T}$. Such extensions will be called *annuli representatives of g* and will be denoted by the same letter. We define

$$\operatorname{mod}(g) = \sup \operatorname{mod}(\mathrm{V} \setminus (\mathrm{U} \cup \mathbf{D}))$$

where the supremum is taken over all annuli representatives $g : \mathrm{U} \to \mathrm{V}$.

Lifting a map $g \in \mathcal{E}$ to the universal covering of $\mathbf{T}$, we obtain a real analytic function $\tilde{g} : \mathbf{R} \to \mathbf{R}$ such that $\tilde{g}(x) = d\,x + \phi(x)$ where $\phi(x)$ is a 1-periodic real analytic function with $\phi(0) = 0$. Let $\mathcal{A}$ be the space of all such functions, and let $\mathcal{A}_n$ be the subspace of the ϕ that admit a holomorphic extension to the strip $|\operatorname{Im} z| < 1/n$ continuous up to the boundary. As the latter spaces are Banach, $\mathcal{A}$ is realized as an inductive limit of Banach spaces, and we can endow it with the inductive limit topology. It induces a topology on the space $\mathcal{E}$. In this topology, a sequence $g_n \in \mathcal{E}$ converges to $g \in \mathcal{E}$ if there is a neighborhood W of $\mathbf{T}$ such that all the g_n admit a holomorphic extension to W, and $g_n \to g$ uniformly on W.

Let $\mathcal{E}^{\mathbf{R}}$ stand for the subspace of $\mathbf{R}$-symmetric expanding circle maps $g : \mathbf{T} \to \mathbf{T}$ (i.e., commuting with the complex conjugacy $z \mapsto \bar{z}$).

Lemma **2.1.** — *The spaces $\mathcal{E}$ and $\mathcal{E}^{\mathbf{R}}$ are contractible.*

Proof. — Let us work with the lifts $g : \mathbf{R} \to \mathbf{R}$ of the maps $g \in \mathcal{E}$ without making a notational difference between them. Let $\mathcal{E}_1$ stand for the set of $g \in \mathcal{E}$ such that $|\mathrm{D}g| > 1$ through $\mathbf{R}$. This is a convex functional space, so it can be contracted to a point through the affine homotopy.

The space $\mathcal{E}_1$ contains the set $\mathcal{E}_*$ of maps $g \in \mathcal{E}$ preserving the Lebesgue measure, so $\mathcal{E}_*$ can be contracted through $\mathcal{E}_1$.

To deal with the whole $\mathcal{E}$, let us make use of the fact (see, e.g., [KS]) that any $g \in \mathcal{E}$ has an absolutely continuous invariant measure $d\mu = \rho\, d\theta$ with real analytic density $\rho(\theta) > 0$. Let us consider a real analytic circle diffeomorphism fixing 0

$$h(t) = \int_0^t \rho(\theta)\, d\theta$$

such that $h_*(d\mu) = d\theta$. Then the map $\mathrm{G} = h \circ g \circ h^{-1}$ preserves the Lebesgue measure, so $\mathrm{G} \in \mathcal{E}_*$. So, we obtain a projection $\pi : \mathcal{E} \to \mathcal{E}_*$, $g \mapsto \mathrm{G}$.

But the space $\mathcal{F}$ of diffeomorphisms h fixing 0 is identified with the space of densities ρ, which is also convex, and hence contractible. It follows that $\mathcal{E}_*$ is a deformation retract for $\mathcal{E}$, and the conclusion follows.

In case of $\mathcal{E}^{\mathbf{R}}$, just notice that all the above homotopies can be made equivariant (with respect to the complex conjugacy). □

2.6. *External map, mating and product structure.* — Given $f \in \mathcal{C}$, let $\psi : \mathbf{C} \setminus \overline{\mathbf{D}} \to \mathbf{C} \setminus \mathrm{K}(f)$ be the Riemann mapping. The map $g = \psi^{-1} \circ f \circ \psi$ induces (by the Schwarz reflection) an expanding circle endomorphism of degree d called the *external map* of f. It is unique up to conjugacy by a circle rotation, so it can be normalized so that $g \in \mathcal{E}$. For $d = 2$, this normalization is unique, but in the higher degree case, there are generally $d - 1$ ways of normalizing g. Irrespective of this issue, it is clear that the quality of the analytic extensions of the germ and of its external map are related by $\operatorname{mod}(f) = \operatorname{mod}(g)$.

Remark **2.2.** — Maps with symmetries have fewer normalizations, e.g., $z \mapsto z^d$ has the maximal possible symmetry group $\mathbf{Z}/(d-1)\mathbf{Z}$ and hence has a unique normalization. Note that this is the external map of any polynomial p_c, $c \in \mathcal{M}$.

Left inverses to the external map construction are provided by the *matings* between polynomials p_c, $c \in \mathcal{M}$, and expanding maps $g \in \mathcal{E}$. It goes as follows. Choose a quasiconformal homeomorphism $h : \mathbf{C} \setminus \bar{\mathbf{D}} \to \mathbf{C} \setminus \mathrm{K}(p_c)$ such that $h \circ g = p_c \circ h$ near the circle. Consider the Beltrami differential μ equal to $\bar{\partial} h^{-1}/\partial h^{-1}$ on $\mathbf{C} \setminus \mathrm{K}(p_c)$ and vanishing on $\mathrm{K}(p_c)$. It is invariant under p_c on some Jordan disk D containing $\mathrm{K}(f)$. Let $\phi : \mathbf{C} \to \mathbf{C}$ be the solution of the Beltrami equation $\bar{\partial}\phi/\partial\phi = \mu$. Then the map $f = \phi \circ p_c \circ \phi^{-1}$ is polynomial-like on some neighborhood of $\phi(\mathrm{K}(p_c))$, with filled Julia set $\phi(\mathrm{K}(p_c))$, so up to normalization, it defines a germ in $\mathcal{C}$.

It is possible to show that, except for the normalization, the germ f does not depend on the various choices made in the construction, and it depends continuously on $g \in \mathcal{E}$ and $c \in \mathcal{M}$ (see Lemma 2.4 below). In Section 2.7 we will carry out formally the details of the construction, to obtain the following result:

Theorem **2.2.** — *There is a canonical choice of the straightening $\chi(f) \in \mathcal{M}$ and an external map $\pi(f) \in \mathcal{E}$ associated to each germ $f \in \mathcal{C}$ and depending continuously on f. It has the following properties:*

(1) *For each $c \in \mathcal{M}$, the hybrid leaf $\mathcal{H}_c$ is the fiber $\chi^{-1}(c)$, and the external map π restricts to a homeomorphism $\mathcal{H}_c \to \mathcal{E}$, whose inverse is denoted by i_c and called the (canonical) mating,*
(2) *$(\pi, \chi) : \mathcal{C} \to \mathcal{E} \times \mathcal{M}$ is a homeomorphism,*
(3) *(Compatibility between matings and Beltrami paths) For $c, c' \in \mathcal{M}$, if f_λ is a Beltrami path in $\mathcal{H}_c$ then $i_{c'} \circ i_c^{-1}(f_\lambda)$ is a Beltrami path in $\mathcal{H}_{c'}$.*
(4) *External map, straightening and mating are equivariant with respect to complex conjugation.*

Except for the need to introduce some novelties to handle $\mathbf{Z}/(d-1)\mathbf{Z}$-ambiguities, the argument follows the quadratic case [L3].

One way to resolve these ambiguities (that show up in both the external map and the mating constructions) is to introduce *markings*. Each germ $f \in \mathcal{C}$ has $d-1$ distinct β-*fixed points*, which do not disconnect $\mathrm{K}(f)$, and a marking of f is just a choice of a preferred β-fixed point. The external map of a marked germ inherits a marking as well, that is, one of its fixed points is distinguished. Reciprocally, mating a marked expanding map with a marked polynomial leads to a well defined marked polynomial-like germ. Marking also allows us to resolve the ambiguities inherent to the straightening, since the straightening of a marked germ is a marked polynomial.

Both expanding maps and polynomials have "natural markings": for expanding maps we choose 1 as the preferred fixed point, and for polynomials we choose the landing point of the external ray of angle 0. As it turns out, the natural marking of polynomials can be extended continuously, in a unique way, through the entire $\mathcal{C}$ (this global property is related to the simple topology of $\mathcal{E}$, see Lemma 2.1). Thus, keeping in mind the natural marking, we end up with natural external map, mating, and straightening constructions. By design, the mating construction provides an inverse to the external map and straightening constructions, so that the connectedness locus $\mathcal{C}$ inherits a product structure from $\mathcal{E} \times \mathcal{M}$.

We will discuss markings in more details later in Section 2.8 (as they are not formally needed for the proof of Theorem 2.2).

The reader who is mostly interested in the quadratic case can skip the next two sections.

2.7. *Proof of Theorem 2.2.* — For $c \in \mathcal{M}$, let $\xi_c : \mathbf{C} \setminus \overline{\mathbf{D}} \to \mathbf{C} \setminus \mathrm{K}(p_c)$ be the univalent map tangent to the identity at ∞: it satisfies $\xi_c \circ p_0 = p_c \circ \xi_c$ on the complement of $\mathrm{K}(p_0) = \overline{\mathbf{D}}$.

Define the canonical mating $i_c(g) \in \mathcal{H}_c$ between any polynomial p_c, $c \in \mathcal{M}$, and any expanding map $g \in \mathcal{E}$ as follows. Choose a continuous path g_t, $t \in [0, 1]$ connecting $g_0 : z \mapsto z^d$ to $g_1 = g$, and a continuous family of quasiconformal maps $h_t : \mathbf{C} \setminus \mathbf{D} \to \mathbf{C} \setminus \mathbf{D}$, with continuously depending Beltrami differentials ν_t, satisfying $h_0 = \mathrm{id}$ and $h_t \circ g_0 = g_t \circ h_t$ near the circle $\mathbf{T}$.[3] Let μ_t be the extension of the Beltrami differential of $h_t \circ \xi_c^{-1}$ to the whole complex plane, obtained by letting it be 0 on $\mathrm{K}(p_c)$. It is invariant under p_c in a neighborhood of $\mathrm{K}(p_c)$. Let $\phi_t : \mathbf{C} \to \mathbf{C}$ be the solution of the Beltrami equation $\bar{\partial}\phi_t/\partial\phi_t = \mu_t$. By invariance of μ_t, $f_t = \phi_t \circ p_c \circ \phi_t^{-1}$ is holomorphic in a neighborhood of $\mathrm{K}(f_t) = \phi_t(\mathrm{K}(p_c))$, and if ϕ_t is appropriately normalized it defines a germ in $\tilde{\mathcal{H}}_c$. We choose the normalization so that ϕ_t depends continuously on t and $\phi_0 = \mathrm{id}$.

Let us check that g_t is an external map of f_t for every $t \in [0, 1]$. Let $\psi_t : \mathbf{C} \setminus \overline{\mathbf{D}} \to \mathbf{C} \setminus \mathrm{K}(f_t)$ be the continuous family of univalent maps normalized so that $\psi_0 = \xi_c$, and $\tilde{g}_t := \psi_t^{-1} \circ f_t \circ \psi_t$ (which extends analytically across the circle by the Schwarz reflection) fixes 1, hence $\tilde{g}_t$ is an external map of f_t. Then $\sigma_t := \psi_t^{-1} \circ \phi_t \circ \xi_c : \mathbf{C} \setminus \mathbf{D} \to \mathbf{C} \setminus \mathbf{D}$ is a quasiconformal map conjugating g_0 to $\tilde{g}_t$ whose Beltrami differential coincides with that of h_t. Hence $\lambda_t := \sigma_t \circ h_t^{-1}$ is a rotation conjugating g_t to $\tilde{g}_t$. Since 1 is a fixed point of g_t, $\lambda_t(1)$ is one of the fixed points of $\tilde{g}_t$ for any $t \in [0, 1]$. But $\lambda_0 = \mathrm{id}$, so by continuity, $\lambda_t(1) = 1$ (which is one of the fixed points of $\tilde{g}_t$) for all $t \in [0, 1]$. We conclude that $\lambda_t = \mathrm{id}$ and hence $g_t = \tilde{g}_t$ for every t.

Next, we will show that the germ $f = f_1$ depends only on c and g, but not on the various choices we have made, which would allow us to define the mating by $i_c(g) = f$. Let us first show that once the connecting path g_t is chosen, the path f_t does not depend on the choice of the conjugacies h_t. Indeed, let h'_t be another choice, with the Beltrami

[3] Such a family h_t can be constructed as follows. For large n, g_t is close to $g_{k/n}$, for every $k \in 0, \dots, n-1$ and $t \in [k/n, (k+1)/n]$. Considering persistent fundamental annuli for the $g_{k/n}$, define a conjugacy $h_{k,t}$, $t \in [k/n, (k+1)/n]$, between $g_{k/n}$ and g_t (first on the fundamental domain, then extended by pulling back) satisfying $h_{k,k/n} = \mathrm{id}$ and the continuity requirements. Then h_t can be defined in each interval $[k/n, (k+1)/n]$ by $h_t = h_{k,t} \circ h_{k-1,k/n} \circ \cdots \circ h_{0,1/n}$.

differential ν'_t, so that the map $\rho_t := h_t^{-1} \circ h'_t$ commutes with g_0 near $\mathbf{T}$. Then the map $\zeta_t := \xi_c \circ \rho_t \circ \xi_c^{-1}$ commutes with p_c near $\mathrm{K}(p_c)$ (outside it). Let us extend ζ_t to the entire plane by setting $\zeta_t|\mathrm{K}(p_c) = \mathrm{id}$.

Lemma **2.3.** — *The map ζ_t is a quasiconformal homeomorphism.*

Proof. — This is a version of the pullback argument, see e.g., [MvS], Chapter 6, Section 4. Choose a quasidisk $\mathrm{V} \supset \mathrm{K}(p_c)$ such that $\mathrm{U} = p_c^{-1}(\mathrm{V}) \Subset \mathrm{V}$ and $\zeta_t \circ p_c = p_c \circ \zeta_t$ on U. Consider a continuous family of quasiconformal maps $\zeta_t^{(0)} : \mathbf{C} \to \mathbf{C}$, such that $\zeta_0^{(0)} = \zeta_0 = \mathrm{id}$, $\zeta_t^{(0)} = \zeta_t$ outside U, and $\zeta_t^{(0)} = \mathrm{id}$ near $\mathrm{K}(p_c)$. We can then set by induction $\zeta_t^{(k+1)}$ as the unique lift (under p_c) of $\zeta_t^{(k)}$ such that $\zeta_t^{(k+1)} = \mathrm{id}$ near $\mathrm{K}(p_c)$. Clearly $\zeta_0^{(k)} = \mathrm{id}$, and by continuity in t, we see that $\zeta_t^{(k)} = \zeta_t$ outside $p_c^{-k}(\mathrm{U})$ for every k. Hence $\zeta_t^{(k)} \to \zeta_t$ pointwise. Since the dilatations of the $\zeta_t^{(k)}$ do not depend on k, they form a precompact family of quasiconformal maps. It follows that the limit map ζ_t is quasiconformal. □

Remark **2.3.** — By the Bers Lemma (see [DH], Lemma 2, p. 303), in order to show that ζ_t is quasiconformal, it is enough to check its continuity, i.e., that the points $z \in \mathbf{C} \setminus \mathrm{K}(p_c)$ near $\mathrm{K}(p_c)$ are not moved much by ζ_t. This can be verified directly by a hyperbolic contraction argument (using that ζ_t commutes with p_c): in fact, the hyperbolic distance (in the complement of $\mathrm{K}(p_c)$) between z and $\zeta_t(z)$ remains bounded as z approaches $\mathrm{K}(p_c)$, see (see [DH], Lemma 1, p. 302).

We let $\mu'_t = (\xi_c)_* \nu'_t$ outside $\mathrm{K}(p_c)$ and $\mu'_t \equiv 0$ on $\mathrm{K}(p_c)$. Since $\nu_t = (\rho_t)_* \nu'_t$, we have: $\mu_t = (\zeta_t)_* \mu'_t$ (outside $\mathrm{K}(p_c)$ and, obviously, on it). Hence μ'_t is the Beltrami differential for $\phi'_t := \phi_t \circ \zeta_t$. It follows that the map $f'_t := \phi'_t \circ p_c \circ (\phi'_t)^{-1}$ is the mating of p_c and g_t corresponding to the conjugacy h'_t. But since ζ_t commuted with p_c near $\mathrm{K}(p_c)$, we conclude that $f'_t = f_t$ near $\mathrm{K}(f_t)$, as was asserted.

Let us now show that the endpoint $f = f_1$ does not depend on the choice of the path g_t connecting g_0 and g. Since $\mathcal{E}$ is simply connected, given another connecting path g'_t, we can fix a homotopy (fixing endpoints) g_t^s with $g_t^0 = g_t$ and $g_t^1 = g'_t$. The mating construction then provides germs $f_t^s \in \mathcal{C}$, and it also allows us to choose hybrid conjugacies ϕ_t^s between p_c and f_t^s depending continuously on s and t. By the previous discussion, g is an external map representative of f_1^s for every s, and in fact there is a continuous family $\psi_1^s : \mathbf{C} \setminus \overline{\mathbf{D}} \to \mathbf{C} \setminus \mathrm{K}(f_1^s)$ of univalent maps conjugating f_1^s to g. Define $\zeta^s : \mathbf{C} \to \mathbf{C}$ by $\zeta^s = ((\psi_1^s)^{-1} \circ \phi_1^s)^{-1} \circ ((\psi_1^0)^{-1} \circ \phi_1^0)$ outside $\mathrm{K}(p_c)$ and $\zeta^s = \mathrm{id}$ on $\mathrm{K}(p_c)$. Then ζ^s commutes with p_c in an outer neighborhood of $\mathrm{K}(p_c)$, and by the same argument as in Lemma 2.3, we see that ζ^s is a global quasiconformal homeomorphism. Hence

$$\tau^s := \phi_1^s \circ \zeta^s \circ (\phi_1^0)^{-1} = \psi_1^s \circ (\psi_1^0)^{-1}$$

is a hybrid conjugacy between f_1^0 and f_1^s which is also conformal outside $\mathrm{K}(f_1^0)$, so it is affine. Moreover, note that 1) τ^s is the identity at $s = 0$ and 2) the germs f_1^s are normalized, so for each $s \in [0,1]$, there is $k \in \mathbf{Z}/(d-1)\mathbf{Z}$ such that τ^s is tangent to $z \mapsto e^{2\pi ik/(d-1)}z$ at 0. We conclude that $\tau^s = \mathrm{id}$ for all s. Hence $f_1^s = f_1^0$ for all s, and in particular $f_1^1 = f_1^0$, so the mating $i_c(g) = f$ is indeed well defined.

Lemma **2.4.** — *The mating* $(g, c) \mapsto i_c(g)$ *is a homeomorphism* $\mathcal{E} \times \mathcal{M} \to \mathcal{C}$.

Proof. — Let us begin with continuity of the mating. It is easy to see that it is continuous in g, uniformly with respect to c. Also, it satisfies $\mathrm{mod}(i_c(g)) = \mathrm{mod}(g)$.

So, it is enough to show that for a given $g \in \mathcal{E}$, it is continuous with respect to c. Consider a sequence $c_n \to c \in \mathcal{M}$. Choose a path g_t connecting $g_0 : z \to z^d$ to $g_1 = g$ in $\mathcal{E}$. Passing to a subsequence, we may assume that the paths $f_{t,n} = i_{c_n}(g_t)$ converge uniformly to a path f_t.

Then f_t is a path in $\mathcal{H}_c$. Indeed, since the $\mathrm{mod}(f_{t,n})$ are bounded away from 0, the $f_{t,n}$ are K-qc conjugate to $f_{0,n}$ (with some K independent of t). Compactness of the space of K-qc maps implies that the f_t are K-qc conjugate to f_0. Let us show that f_t is actually hybrid conjugate to $f_0 = p_c$. If this is not the case then p_c must be qc conjugate to a unicritical polynomial (any straightening of f_t) which is not itself hybrid conjugate to p_c, i.e., c is not qc rigid. But this implies that $\mathrm{K}(p_{c'})$ moves holomorphically for c' in a neighborhood of c, $\mathrm{K}(p_{c'}) = h_{c'}(\mathrm{K}(p_c))$. It follows that the characteristic function of $\mathrm{K}(p_{c_n})$ converges in measure to that of $\mathrm{K}(p_c)$,[4] which readily implies that the limit H of hybrid conjugacies H_n between p_{c_n} and $f_{t,n}$ must be a hybrid conjugacy between p_c and f_t (since $\bar\partial \mathrm{H}_n \to \bar\partial \mathrm{H}$ weakly in L^2).

Let $\psi_{t,n} : \mathbf{C} \setminus \overline{\mathbf{D}} \to \mathbf{C} \setminus \mathrm{K}(f_{t,n})$ be as in the above construction of the external map, i.e., it is the continuous family of conformal maps such that $\psi_{t,n}^{-1} \circ f_{t,n} \circ \psi_{t,n} = g_t$ and $\psi_{0,n} = \xi_{c_n}$. Then $t \mapsto \psi_{t,n}$ is clearly uniformly continuous in t (with respect to the topology of uniform convergence on compact subsets). So we can take a limiting continuous family $\psi_t : \mathbf{C} \setminus \overline{\mathbf{D}} \to \mathbf{C} \setminus \mathrm{K}(f_t)$ (though $\mathrm{K}(f_{t,n})$ need not converge to $\mathrm{K}(f_t)$, any limit is contained in $\mathrm{K}(f_t)$ and its boundary contains $\mathrm{J}(f_t) = \partial \mathrm{K}(f_t)$, which is enough here). Then $\psi_t^{-1} \circ f_t \circ \psi_t = g_t$, and $\psi_0 = \xi_c$, i.e., g_t is a path that determines the external map of f_t. So $f_t = i_c(g_t)$ and hence $\lim i_{c_n}(g) = \lim f_{1,n} = f_1 = i_c(g)$. This proves continuity.

Let us now show that the mating is bijective. Notice first that each polynomial p_c has a single preimage (p_0, c), since it can only be obtained by mating with its single external map representative.

Consider a path f_t connecting p_c and an arbitrary map f in $\mathcal{H}_c$. Then $f_t = i_c(g_t)$ where g_t is the determination of the external map constructed above. Since $i_c^{-1}(p_c) = \{p_0\}$, this path lifting property implies that each $i_c : \mathcal{E} \to \mathcal{H}_c$ is a bijection. In particular, $\mathcal{H}_c = i_c(\mathcal{E})$ contains a single polynomial, $i_c(p_0) = p_c$, so all hybrid leaves are distinct. Since $\mathcal{C}$ is the union of hybrid leaves, this implies that the mating is bijective.

---

[4] To check it, use the following: since h_c as an element of the Sobolev space $\mathrm{W}^{1,2}$ depends holomorphically on c, $\mathrm{Jac}\, h_c = |\partial h_c|^2 - |\bar\partial h_c|^2$ depends continuously on c weakly in L^1.

Since the mating is continuous and bijective, it restricts to a homeomorphism $\mathcal{E}(\epsilon) \times \mathcal{M} \to \mathcal{C}(\epsilon)$ for each $\epsilon > 0$, by compactness, and this implies that the mating is a homeomorphism $\mathcal{E} \times \mathcal{M} \to \mathcal{C}$. □

We obtain the canonical external map π and the canonical straightening χ by setting (π, χ) as the inverse of the mating. All constructions are clearly equivariant with respect to complex conjugation. One also checks directly that $i_{c'} \circ i_c^{-1}$ takes Beltrami paths in $\mathcal{H}_c$ to Beltrami paths in $\mathcal{H}_{c'}$. □

2.8. *Marking.* — Let us take a polynomial-like map $f : U \to V$ with connected Julia set. Let $A = \bar{V} \setminus U$ and let $\Gamma = \partial U$. Select an arc $\gamma_0 \subset A$ connecting a point $a_{-1} \in f(\Gamma) = \partial V$ to one of its preimages, $a_0 \in \Gamma$. It can be lifted to an arc $\gamma_1 \subset f^{-1}A$ connecting a point $a_1 \in f^{-1}(\Gamma)$ to a_0. In turn, this curve can be lifted to an arc $\gamma_{-2} \subset f^{-2}A$ connecting some point $a_2 \in f^{-2}(\Gamma)$ to a_1. Continuing this way we obtain a sequence of arcs γ_n concatenating a curve $\gamma \subset \bar{V} \setminus K(f)$ such that $f(\gamma \cap \bar{U}) = \gamma$ (we will refer to such a curve as "invariant"). A standard hyperbolic contraction argument shows that this curve can accumulate only at fixed points, and hence lands at some "preferred" fixed point of g. Fixed points that arise in this way are called *β-fixed points*. A *marking* of g is a choice of such an invariant curve up to equivariant homotopy. This notion descends to germs, by identifying markings of polynomial-like representatives of g which coincide up to truncation.

If $c \in \mathcal{M}$, we can mark the corresponding germ p_c with an *invariant external ray* of p_c. No two invariant external rays can land at the same point,[5] so those markings are indeed distinct. On the other hand, it is easy to see that any invariant curve is equivariantly homotopic to some invariant external ray. Since such a ray has external argument $k/(d-1)$ with $k \in \mathbf{Z}/(d-1)\mathbf{Z}$, this procedure shows that there are exactly $d-1$ different markings, and that markings are in bijection with the β-fixed points.

Since a hybrid equivalence between polynomial-like maps gives a correspondence between the markings, the bijection between markings and β-fixed points holds through $\mathcal{C}$ as well. The family of β-fixed points depends continuously through $\mathcal{C}$,[6] so each β-fixed point (or equivalently, each choice of marking) of a germ admits an unique local continuation to every sufficient small connected neighborhood a germ.

Similarly to polynomial-like maps, a circle map $g : U \to V$ of class $\mathcal{E}$ can be *marked* by choosing an invariant curve $\gamma \subset V \setminus \mathbf{D}$ up to equivariant homotopy. Such a curve lands at a fixed point of g which depends only on the marking. Vice versa, a fixed point determines the marking, so there are exactly $d-1$ distinct markings of any circle map $g \in \mathcal{E}$.

[5] Otherwise the sector bounded by those two rays and which does not contain the critical point would be invariant by the maximum principle (notice that the image of the sector does not contain the critical value).

[6] By means of straightening, we can restrict attention to polynomials, for which it is readily checked that repelling β-fixed points are persistent, and repelling non-β-fixed points are persistent as well. Thus a discontinuity might only arise at a parabolic bifurcation, where both candidates to be a β-fixed point are close.

The marking of $g \in \mathcal{E}$ corresponding to the fixed point 1 is called *natural*. It provides us with a continuous global marking of the space $\mathcal{E}$.

Due to the product structure $\mathcal{C} \approx \mathcal{E} \times \mathcal{M}$, the natural marking of $\mathcal{E}$ can be pulled back to a natural marking of $\mathcal{C}$.

2.9. *Control of quasiconformal dilatation.* — We say that two polynomial-like germs $f, \tilde{f} \in \mathcal{C}$ are (C, ϵ)-*close* if there exist polynomial-like representatives $f : U \to V$ and $\tilde{f} : \tilde{U} \to \tilde{V}$ with $\mathrm{mod}(V \setminus U) > \epsilon$, $\mathrm{mod}(\tilde{V} \setminus \tilde{U}) > \epsilon$, and a quasiconformal homeomorphism $h : \mathbf{C} \setminus U \to \mathbf{C} \setminus \tilde{U}$ respecting the natural marking of f and $\tilde{f}$[7] with $\mathrm{Dil}(h) < C$ such that $h \circ f = \tilde{f} \circ h$ on ∂U. Notice that (C, ϵ)-closeness only depends on the (canonical) external maps $\pi(f)$ and $\pi(\tilde{f})$.

Standard arguments (c.f. the proof of Lemma 2.3) yield:

Lemma **2.5.** — *If f and $\tilde{f}$ are (C, ϵ)-close by means of h and are hybrid equivalent, then h extends, in a unique way, to a hybrid conjugacy between f and $\tilde{f}$ with dilatation bounded by* C.

By compactness one has:

Lemma **2.6.** — *For every $\epsilon_0 > \epsilon > 0$ there exists $C > 1$ such that if $f, \tilde{f} \in \mathcal{C}(\epsilon_0)$ then f and $\tilde{f}$ are (C, ϵ)-close.*

For nearby germs, the constant C can be taken close to 1:

Lemma **2.7.** — *Let $f_n, \tilde{f}_n \in \mathcal{C}$ be converging sequences with the same limit. Then there exists $\epsilon > 0$ and $C_n \searrow 1$ such that f_n and $\tilde{f}_n$ are (C_n, ϵ)-close for every n sufficiently large.*

Proof. — Let $f = \lim f_n = \lim \tilde{f}_n$. By definition of convergence, there exists a polynomial-like representative $f : U \to V$ such that f_n extends holomorphically to U for every n sufficiently large, $f_n | U$ converges uniformly to f, and $\tilde{f}_n \to f$ uniformly on U.

Let W be the quasidisk bounded by the equator (i.e., the simple closed hyperbolic geodesic) of $V \setminus \overline{U}$, and let $\Omega := f^{-1}(W)$, $\Omega_n := f_n^{-1}(W)$ and $\tilde{\Omega}_n = \tilde{f}_n^{-1}(W)$. Then the Jordan curves $\partial\Omega_n$ and $\partial\tilde{\Omega}_n$ converge in C^∞ topology to the curve $\partial\Omega$. It follows that for n sufficiently large, the maps $f_n : \Omega_n \to W$ and $\tilde{f}_n : \tilde{\Omega}_n \to W$ are polynomial-like, and the $\mathrm{mod}(W \setminus \Omega_n)$, $\mathrm{mod}(W \setminus \tilde{\Omega}_n)$ approach $2\epsilon := \mathrm{mod}(W \setminus \Omega)$. Hence there exist C^∞ diffeomorphisms $h_n : \mathbf{C} \to \mathbf{C}$, such that $h_n \,|\, \mathbf{C} \setminus W = \mathrm{id}$ and $f_n \circ h_n = \tilde{f}_n \equiv h_n \circ \tilde{f}_n$ on $\partial\Omega_n$, approaching the identity in the C^∞ topology. Thus $\mathrm{Dil}\, h_n \to 1$. Moreover h_n, being close to the identity, preserves the natural marking and we are done. □

[7] This condition makes sense since the marking of f can be given by a curve $\gamma \subset \bar{V} \setminus U$ connecting a point $z \in \partial U$ to its image $f(z) \in \partial V$ up to homotopy rel the endpoints.

2.10. *Renormalization and a priori bounds.* — A unicritical polynomial-like map $f : U \to V$ (of degree d) is called *renormalizable* with period $p > 1$ if there exists a topological disk $W \ni 0$ with the following properties:

R1 The map $g = f^p \mid W$ is a unicritical polynomial-like map of degree d (onto its image W'); it is called the *pre-renormalization* of f.
R2 The *little Julia set* $K(g)$ is connected;
R3 $K(g)$ does not touch its images $f^m(K(g))$, $m = 1, \dots, p-1$, except perhaps at one of its β-fixed points.

Note that these images are also Julia sets $K(g_m)$ for appropriate degree d polynomial-like restrictions $g_m : W_m \to W'_m$ of f^p. They are also referred to as "little Julia sets of f".

By [McM1, Theorem 5.11], the polynomial-like germ of the pre-renormalization is well defined: it does not depend on the choice of the domain W above.

In fact, there is a standard combinatorial choice of the domain W. Namely, let us consider the little Julia set $K(g_1)$ around the critical value $f(0)$. Among its β-fixed points, there is a *dividing* point β_1, i.e., the landing point of more than one external rays[8] for f (see [Mi, Theorems 1.2 and 1.4]). Two of these rays bound a sector containing $f(0)$, the *characteristic sector* $\mathcal{S}_1$. The renormalization range W'_1 is obtained by truncating $\mathcal{S}_1$ by an equipotential and slightly "thickening" it (see [D] or [Mi, Section 8]). The domain $W_1 \ni f(0)$ is the pullback of W'_1 by f^p. The domains $W' \supset W \ni 0$ are the pullbacks of the $W'_1 \supset W_1$ under f.

Note that the dividing fixed point β_1 is uniquely defined. Indeed, as the characteristic sector $\mathcal{S}_1$ has size less than $1/d$,[9] it does not contain the critical point 0 and hence $\partial\mathcal{S}_1$ separates 0 from $f(0)$. Since the little Julia set $K(g_1) \ni f(0)$ is connected and the rays landing at the β-fixed points of g_1 do not cut through $K_1(g)$, there cannot be more than one separating point.

We will mark β_1 on the little Julia set $K(g_1)$ and the corresponding fixed point $\beta = f^{p-1}(\beta_1) \in f^{-1}(\beta_1)$ on the little Julia set $K(g)$. Notice that if $f \in \mathcal{C}^{\mathbf{R}}$, these points lie in the real line (by symmetry).

Now, the *renormalization* of f is obtained by normalizing the pre-renormalization with minimal possible period,

$$Rf(z) = \lambda^{-1} g(\lambda z) : z \mapsto c + z^d + h.o.t.$$

There is no ambiguity in the choice of normalization since the pre-renormalization g is marked with the β-fixed point. In case $f \in \mathcal{C}^{\mathbf{R}}$, we have $Rf \in \mathcal{C}^{\mathbf{R}}$ as well, since β is real.

The renormalization is called *primitive* if the little Julia sets $K(g_m)$ do not touch, and is called *satellite* otherwise.

[8] External rays for polynomial-like maps are defined by means of the straightening.
[9] In fact, $\mathcal{S}_1$ is the *minimal* sector into which the rays landing on orb β_1 divide the plane.

The set of angles of the external rays (defined with help of the canonical straightening of f) landing at the distinguished β-fixed points of g determine the "renormalization combinatorics".[10] A classical theorem by Douady and Hubbard [DH] asserts that the renormalizable unicritical polynomials p_c with the same combinatorics form a "little copy $\mathcal{M}'$ of the Multibrot set" (or "$\mathcal{M}$-copy"), except that the roots of $\mathcal{M}'$ may or may not be renormalizable. Thus, the renormalization combinatorics can be labeled by the little copies themselves.

In case of a renormalizable real map, all the above notions can be described in purely real terms. The real traces of the little Julia sets are intervals that are permuted under the dynamics. The order of these intervals on the line describes the renormalization combinatorics. The set of renormalizable maps with a given combinatorics is a parameter interval $\mathcal{M}' \cap \mathbf{R}$ called the *renormalization window*. Note that the boundary points of a renormalization window renormalize to a map with either *parabolic* (more precisely, it has a parabolic fixed point with multiplier 1) or *Ulam-Neumann* (in this case, $f^2(0)$ is the β-fixed point, such maps are also called *Chebyshev*) combinatorics. In particular, *the boundary maps are not twice renormalizable*. (In case of doubling renormalization, the parabolic boundary map is not renormalizable in the complex sense, but can be viewed as renormalizable on the real line.)

A polynomial-like germ $f \in \mathcal{C}$ is called renormalizable, if it has a renormalizable representative. The renormalization descends naturally to the level of germs. Whether a germ is renormalizable or not, and even its renormalization combinatorics, only depends on its hybrid leaf. The renormalization operator acts nicely at the level of hybrid leaves:[11]

Lemma **2.8.** — *The renormalization operator maps hybrid leaves into hybrid leaves, and takes Beltrami paths to Beltrami paths.*

Proof. — It is enough to prove the last statement. Let (f_λ, h_λ) be a guided Beltrami path in a renormalizable hybrid leaf. Let $f_0 : U_0 \to V_0$ be a p-l representative of f_0. We may assume that V_0 is small enough so that $\mu_\lambda = \overline{\partial} h_\lambda / \partial h_\lambda$ is f_0-invariant for every $\lambda \in \mathbf{D}$. Let $g_0 = f_0^p : U_0' \to V_0'$ be a pre-renormalization of f_0. Then μ_λ is g_0-invariant for every $\lambda \in \mathbf{D}$. It follows that $h_\lambda \circ g_0 \circ h_\lambda^{-1}$ is a pre-renormalization of $f_\lambda : h_\lambda(U_0) \to h_\lambda(V_0)$. If A_0 is the affine map conjugating g_0 and Rf_0, there is a unique holomorphic continuation A_λ which normalizes g_λ, which is readily seen to conjugate g_λ and Rf_λ. Thus $(Rf_\lambda, A_\lambda \circ h_\lambda \circ A_0^{-1})$ is a guided Beltrami path. □

One can now naturally define n times renormalizable maps, including $n = \infty$. The combinatorics of an infinitely renormalizable map can be labeled by a sequence of little

[10] An alternative point of view is the following. The relative positions of the little Julia sets $K(g_m)$ inside the big one, $K(f)$, can be described in terms of a graph called the *Hubbard tree*. This graph determines the renormalization combinatorics up to symmetry.

[11] As for the hybrid classes, we notice that affinely conjugate renormalizable germs have the same renormalization.

Mandelbrot copies $\mathcal{M}'_n$, $n \in \mathbf{N}$ (describing the combinatorics of the consecutive renormalizations). It incorporates the sequence $\{p_n\}$ of the (relative) renormalization periods. This can be an arbitrary sequence of natural numbers > 1. We say that f has a *bounded combinatorics* if the sequence of periods p_n is bounded.

We say that an infinitely renormalizable germ f has *a priori bounds* if its renormalizations $R^n f$ have definite moduli: $\operatorname{mod}(R^n f) \geq \epsilon > 0$.

Let us note, for further use, a simple consequence (Lemma 2.10) of the a priori bounds. We will need the following topological preparation:

Lemma **2.9.** — *Let $f' : U' \to V'$ be a p-l representative of a pre-renormalization (not necessarily the first) of the p-l map $f : U \to V$, of total period q. If $V' \subset V$ then $f^k(U') \subset U$ for $0 \leq k < q$ and $f' = f^q|U'$.*

Proof. — The connected component of $f^{-q}(V')$ containing 0 is a simply connected domain taken by f^q onto V' as a proper map which coincides with f' near $K(f')$. By analytic continuation, such connected component must coincide with U' and we have $f' = f^q|U'$. □

Lemma **2.10.** — *Let $f \in \mathcal{C}$ be infinitely renormalizable with a priori bounds, and let f_n be the sequence of pre-renormalizations (of total period q_n). Then there exist $C > 0$, $\lambda < 1$ (only depending on the a priori bounds) such that*

$$\max_{m \in \mathbf{Z}/q_n\mathbf{Z}} \operatorname{diam} K_m(f_n) \leq C\lambda^n.$$

Proof. — We are going to show that there exists $\delta > 0$, only depending on the a priori bounds, such that for every m, m', n, n' such that $n' > n$ and $K_m(f_n) \supset K_{m'}(f_{n'})$, the Hausdorff distance between $K_m(f_n)$ and $K_{m'}(f_{n'})$ is at least $\delta \operatorname{diam}(K_m(f_n))$. This clearly implies that there exists $k > 0$ such that

$$\operatorname{diam}(K_{m'}(f_{n'})) < \operatorname{diam}(K_m(f_n))/2$$

provided $n' \geq n + k$, and the exponential decay follows.[12]

Let $f : U \to V$ and $f_{n'} : U' \to V'$ be polynomial-like representatives, with

$$\operatorname{mod}(V \setminus U) \geq \operatorname{mod}(V' \setminus U') = \epsilon.$$

Up to replacing ϵ by ϵ/d^t and (U', V') by $(f_{n'}^{-t}(U'), f_{n'}^{-t}(V'))$ (with t only depending on ϵ) we may assume that $V' \subset V$. By Lemma 2.9, $f_{n'} = f^{q_{n'}}|U'$.

[12] Indeed, we can choose $m_j \in \mathbf{Z}/q_{n+j}\mathbf{Z}$, $0 \leq j \leq k$ with $m_0 = m$ and $m_k = m'$, such that the $K_{m_j}(f_{n+j})$ are nested. After suitable translation and rescaling by $\operatorname{diam}(K_m(f_n))^{-1}$, one gets $k+1$ compact subsets of the closed unit disk which are pairwise $\delta/2$-separated in the Hausdorff metric. Since the set of compact subsets of the closed disk is compact, k is bounded in terms of δ.

Then $f^{q_{n'}}$ has a single critical point in $\tilde{U} = f^{m'}(f_{n'}^{-1}(U'))$, where we represent m' in the range $1 \leq m' \leq q_{n'}$. For each $1 \leq l \leq q_{n'}$, there exists a unique $z_l \in K_l(f_{n'})$ such that $f^{q_{n'}-l}(z_l) = 0$. It follows that $\tilde{U}$ contains at most one of the z_l. Since $K_m(f_n)$ contains at least two distinct z_l, we conclude that $K_m(f_n) \not\subset \tilde{U}$. But $\operatorname{mod}(\tilde{U} \setminus K_{m'}(f_{n'})) \geq \epsilon/d$, so $\tilde{U}$ is a $\delta \operatorname{diam}(K_{m'}(f_{n'}))$-neighborhood of $K_{m'}(f_{n'})$, with δ only depending on ϵ. □

3. Path holomorphic spaces, the Carathéodory metric and the Schwarz Lemma

A *path holomorphic structure* on a space X is a family Hol(X) of maps $\gamma : \mathbf{D} \to X$, called holomorphic paths, which contains the constants and is invariant under holomorphic reparametrizations: for every $\gamma \in \mathrm{Hol}(X)$ and every holomorphic (in the usual sense) map $\phi : \mathbf{D} \to \mathbf{D}$, $\gamma \circ \phi \in \mathrm{Hol}(X)$. Natural examples of path holomorphic spaces are complex Banach manifolds, where holomorphic paths are taken as the paths which are holomorphic in the usual sense.

If X, Y are path holomorphic spaces, a map $\Phi : X \to Y$ is called path holomorphic if for every holomorphic path $\gamma : \mathbf{D} \to X$, $\Phi \circ \gamma : \mathbf{D} \to Y$ is a holomorphic path. Let Hol(X, Y) be the space of path holomorphic maps from X to Y. Notice that $\mathrm{Hol}(\mathbf{D}, X) = \mathrm{Hol}(X)$. In case of complex Banach manifolds, this coincides with the usual notion of being holomorphic, as long as the maps are continuous. Obviously, composition of path holomorphic maps is path holomorphic.

Given a path holomorphic space X, any subset $Y \subset X$ can be naturally considered as a path holomorphic space: if $i : Y \to X$ is the inclusion, then Hol(Y) consists of all $\phi : \mathbf{D} \to Y$ with $i \circ \phi \in \mathrm{Hol}(X)$.

Let $h(x, y)$ be the hyperbolic metric on $\mathbf{D}$ (normalized to be twice the Euclidean metric at 0). Introduce a metric $d(x, y)$ on $\mathbf{D}$ by taking $d = \frac{e^h - 1}{e^h + 1}$ (this is a metric by convexity). This is the unique metric invariant under the group of conformal automorphisms of $\mathbf{D}$ and such that $d(0, z) = |z|$.

By the usual Schwarz Lemma, any holomorphic map $\phi : \mathbf{D} \to \mathbf{D}$ weakly contracts d, i.e., $d(\phi(x), \phi(y)) \leq d(x, y)$.

Let X be a path holomorphic space. Then we can define the following *Carathéodory pseudo-metric*:

$$(\mathbf{3.1}) \qquad d_X(x, y) = \sup_{\phi \in \mathrm{Hol}(X, \mathbf{D})} d(\phi(x), \phi(y)).$$

(Obviously, in this definition we can consider only ϕ normalized so that $\phi(x) = 0$.) By the usual Schwarz Lemma we have

$$(\mathbf{3.2}) \qquad d_{\mathbf{D}}(x, y) = d(x, y).$$

If $Y \subset X$, we let $\operatorname{diam}_X Y$ denote the diameter of Y in the pseudo-metric d_X. We say that Y is *small* in X if $\operatorname{diam}_X Y < 1$.

The Carathéodory pseudo-metric d_X is a metric if and only if the set of bounded path holomorphic functions $X \to \mathbf{C}$ separates points. In this case, X is called *Carathéodory hyperbolic.*

The definitions immediately imply:

Schwarz Lemma (Weak form). — Any path holomorphic map $\Phi : X \to Y$ *is weakly contracting:*

$$d_Y(\Phi(x), \Phi(y)) \leq d_X(x, y). \tag{3.3}$$

It follows that any subset of a Carathéodory hyperbolic space is Carathéodory hyperbolic.

The universal class of Carathéodory hyperbolic spaces are given by Banach balls:

Lemma **3.1.** — *The unit ball* $\mathcal{B}(1)$ *in a complex Banach space* $\mathcal{B}$ *is Carathéodory hyperbolic and* $d_{\mathcal{B}_1}(x, 0) = \|x\|$. *A path holomorphic space* X *is Carathéodory hyperbolic if and only if* X *holomorphically injects into a Banach ball.*

Proof. — Normalized linear functionals $\phi \in \mathcal{B}_1^*$ are holomorphic maps $\mathcal{B}_1 \to \mathbf{D}$. By definition of the Carathéodory metric and the Hahn-Banach Theorem,

$$d_{\mathcal{B}_1}(x, 0) \geq \sup_{\phi \in \mathcal{B}^*(1)} |\phi(x)| = \|x\|.$$

The opposite inequality is obtained by applying the Schwarz Lemma to the embedding $\mathbf{D} \to \mathcal{B}_1$, $\lambda \mapsto \lambda x/\|x\|$ at $\lambda = \|x\|$.

The Schwarz Lemma shows that a space which is not Carathéodory hyperbolic can not inject into one that is. Vice versa, assume X is Carathéodory hyperbolic. Let $S \subset \mathrm{Hol}(X, \mathbf{D})$ be any subset which separates points (e.g., $S = \mathrm{Hol}(X, \mathbf{D})$). Then X holomorphically injects into the unit ball of $\ell^\infty(S)$, the Banach space of bounded functions $S \to \mathbf{C}$, via the map $x \mapsto (\phi(x))_{\phi \in S}$. □

Small subsets of a hyperbolic space X have "definitely stronger" Carathéodory metrics:

Lemma **3.2.** — *Let* X *be a path holomorphic space and let* $Y \subset X$. *Then for any* $x, y \in Y$,

$$d_X(x, y) \leq \mathrm{diam}_X(Y) d_Y(x, y). \tag{3.4}$$

Proof. — Let $r > \mathrm{diam}_X Y$. By the weak Schwarz Lemma for any path holomorphic function $\phi : (X, x) \to (\mathbf{D}, 0)$ we have $\phi(Y) \subset \mathbf{D}_r$. Hence the function $\tilde{\phi} := r^{-1}\phi \,|\, Y$ belongs to $\mathrm{Hol}(Y, \mathbf{D})$, and we obtain:

$$d_Y(x, y) \geq \sup_{\phi} |\tilde{\phi}(y)| = \frac{1}{r} d_X(x, y).$$

The conclusion follows. □

Putting this together with the weak Schwarz Lemma, we obtain:

Schwarz Lemma (Strong form). — Any path holomorphic map $\Phi : Y \to X$ *with small image is strongly contracting:*

$$d_X(\Phi(x), \Phi(y)) \le \operatorname{diam}_X(\Phi(Y)) \cdot d_Y(x, y).$$

Proof. — Decompose Φ as $i \circ \Phi_0$ where $i : \Phi(Y) \to X$ is the inclusion, $\Phi_0 = \Phi : Y \to \Phi(Y)$. Then apply the weak Schwarz Lemma to Φ_0 and Lemma 3.2 to i. □

Remark **3.1.** — One can consider the (stronger) Kobayashi metric on path holomorphic spaces as well. Though the Kobayashi hyperbolicity is a more general notion, the spaces of interest in this paper turn out to be already Carathéodory hyperbolic. More importantly, the Carathéodory metric is much better adapted to our purposes, since strong contraction can be derived from a very simple smallness criterion.

4. Hybrid leaves as Carathéodory hyperbolic spaces

4.1. *Path holomorphic structure on hybrid leaves.* — For any $c \in \mathcal{M}$, we introduce a path holomorphic structure on the hybrid leaf $\mathcal{H}_c$ as follows. A continuous family

$$(f_\lambda : U_\lambda \to V_\lambda) \in \mathcal{H}_c, \quad \lambda \in \mathbf{D},$$

is said to be a *holomorphic path* if there exists a holomorphic motion $h_\lambda : \mathbf{C} \to \mathbf{C}$ based at the origin such that $h_\lambda(K(f_0)) = K(f_\lambda))$, $\overline{\partial} h_\lambda = 0$ a.e. on $K(f_0)$ (which makes sense since the h_λ are qc by the Quasiconformality λ-Lemma) and $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ on $K(f_0)$ (*equivariance* property).

Clearly every Beltrami path is a holomorphic path. Though the notion of a Beltrami path is in principle stronger, they coincide at least locally. Indeed, let f_λ be a holomorphic path and let h_λ be the corresponding motion of $K(f_\lambda)$. For each $\lambda_0 \in \mathbf{D}$, we can make a choice of a fundamental annulus $V_\lambda \setminus U_\lambda$ which moves holomorphically with λ in a small disk D around λ_0. This holomorphic motion can be then extended (using the Extension λ-Lemma) to $\mathbf{C} \setminus U_\lambda$ and then (uniquely) to a holomorphic motion on $\mathbf{C} \setminus K(f_\lambda)$ that is equivariant on $U_\lambda \setminus K(f)$. Matching it with the original motion of $K(f_\lambda)$ we obtain a holomorphic motion of $\mathbf{C}$ over D, which provides a hybrid conjugacy.

Remark **4.1.** — Yet another way to look at holomorphic paths is the following: a continuous family $f_\lambda \in \mathcal{H}_c$, $\lambda \in \mathbf{D}$, is a holomorphic path if and only if the map $(\lambda, z) \mapsto f_\lambda(z)$ extends to a holomorphic map in a neighborhood of

$$\bigcup_{\lambda \in \mathbf{D}} \{\lambda\} \times K(f_\lambda).$$

However, this point of view will play no role in our analysis of hybrid classes.

Through the local characterization of holomorphic paths as Beltrami paths, we can translate Theorem 2.2 (item 3) and Lemma 2.8 to path holomorphicity statements:

Lemma **4.1.** —

(1) *All hybrid leaves are path holomorphically equivalent: For every* $c, c' \in \mathcal{M}$, $i_{c'} \circ i_c^{-1} : \mathcal{H}_c \to \mathcal{H}_{c'}$ *is path holomorphic.*
(2) *The renormalization operator is leafwise path holomorphic: if* $\mathcal{H}_c$ *and* $\mathcal{H}_{c'}$ *are such that* $\mathrm{R}(\mathcal{H}_c) \subset \mathcal{H}_{c'}$ *then* $\mathrm{R} : \mathcal{H}_c \to \mathcal{H}_{c'}$ *is path holomorphic.*

4.2. *Carathéodory hyperbolicity.*

Theorem **4.2.** — *For every* $c \in \mathcal{M}$, $\mathcal{H}_c$ *is Carathéodory hyperbolic.*

Proof. — In order to prove Carathéodory hyperbolicity of the hybrid leaves, it is enough, by Lemma 4.1, to prove it for any one of them. The most convenient one will be $\mathcal{H}_0$, since in this case the Julia set traps a "definite domain of holomorphicity":

$$\mathbf{D}_{1/4} \subset \mathrm{K}(f) \quad \text{if } f \in \mathcal{H}_0. \tag{4.1}$$

Indeed, there exists a univalent map ψ_f from $\operatorname{int}\mathrm{K}(f)$ onto $\mathbf{D}$ (the Böttcher coordinate), such that $\psi_f(f(z)) = \psi_f(z)^d$ and $\mathrm{D}\psi_f(0) = 1$,[13] and this implies (4.1) by the Koebe-1/4 Theorem.

We will now show that $\mathcal{H}_0$ holomorphically injects in a Banach ball, which is equivalent to Carathéodory hyperbolicity by Lemma 3.1. As the target space, we take $\mathcal{B}_{\mathbf{D}_\rho}$ (the space of bounded holomorphic functions on $\mathbf{D}_\rho$ which are continuous up to the boundary) for an arbitrary $0 < \rho < 1/4$.

Clearly the restriction operator $\mathrm{I}_\rho : \mathcal{H}_0 \to \mathcal{B}_{\mathbf{D}_\rho}$ is injective, by analytic continuation. It is also bounded: the branch of the d-th root of $f | \operatorname{int}\mathrm{K}(f)$ tangent to the identity at 0 restricts to a univalent map on $\mathbf{D}_{1/4}$, so that $f | \mathbf{D}_\rho$ can be bounded in terms of ρ. Let us show that

$$f_\lambda(z) \text{ is holomorphic in } (\lambda, z) \in \mathbf{D} \times \mathbf{D}_{1/4} \text{ if } f_\lambda \text{ is a holomorphic path in } \mathcal{H}_0, \tag{4.2}$$

as this clearly implies that I_ρ is path holomorphic.

Indeed, if f_λ is a holomorphic path in $\mathcal{H}_0$ then there exists a holomorphic motion $h_\lambda : \mathbf{C} \to \mathbf{C}$ centered on 0 such that $h_\lambda(\mathrm{K}(f_0)) = \mathrm{K}(f_\lambda)$, h_λ is holomorphic on $\operatorname{int}\mathrm{K}(f_0)$, and h_λ conjugates f_0 and f_λ on their filled Julia sets. By separate holomorphicity, we see that $(\lambda, z) \mapsto (\lambda, h_\lambda(z))$ is holomorphic in $\mathbf{D} \times \operatorname{int}\mathrm{K}(f_0) \to \bigcup_{\lambda \in \mathbf{D}} \{\lambda\} \times \operatorname{int}\mathrm{K}(f_\lambda)$. This implies (4.2), since we can write $f_\lambda(z) = h_\lambda \circ f_0 \circ h_\lambda^{-1}(z)$. □

[13] One way to obtain the Böttcher coordinate is by restricting a hybrid conjugacy between f and p_0. The fact that the derivative at 0 can be taken as 1 is immediate from the normalization.

4.3. *Carathéodory vs Montel.* — The following discussion is not actually needed for our proofs of exponential contraction of renormalization with respect to the Carathéodory metric, but allows us to reinterpret this result in more familiar terms (the Montel metrics of [L3]).

A compact subset $\mathcal{K} \subset \mathcal{C}$ is called *sliceable* if there exist an open quasidisk W and $C > 0$ such that $\overline{\bigcup_{f \in \mathcal{K}} K(f)} \subset W$ and every $f \in \mathcal{K}$ has a holomorphic extension to W bounded by C. Notice that if $L \Subset W$ is a neighborhood of 0, then the uniform metric on $C^0(L)$ induces a distance on $\mathcal{K}$, and different choices of L lead to Hölder equivalent distances, by Hadamard's Three Circles Theorem. In particular, all those distances define the same topology on $\mathcal{K}$, which is easily seen to coincide with the natural topology of $\mathcal{K}$ (as a subset of $\mathcal{C}$).

A metric d defined on a compact subset $\mathcal{K} \subset \mathcal{C}$ will be called *Montel* if on each sliceable subset of $\mathcal{K}$, it is Hölder equivalent to the uniform metric on all sufficiently small compact neighborhoods of 0. Notice that it is enough to check this last condition on any family of sliceable subsets whose $\mathcal{K}$-interiors cover $\mathcal{K}$. Thus Montel metrics can be constructed by gluing appropriately metrics on finitely many sliceable subsets. They are all Hölder equivalent and compatible with the topology.

Remark **4.2.** — We need to go through sliceable subsets since two different germs $f, \tilde{f} \in \mathcal{C}$ may coincide in a neighborhood of 0 (which cannot happen when f and $\tilde{f}$ are in the same sliceable set).

Example **4.1.** — Let us sketch a construction of a pair of quadratic-like germs $f_\pm \in \mathcal{C}$ that coincide in a neighborhood of 0: indeed f_+ and f_- are restrictions of the same analytic map defined in $K(f_+) \cup K(f_-)$.

Let us start with the map

$$F : \mathbf{T} \to \mathbf{T}, \quad z \mapsto \exp(\pi(z - z^{-1})/2), \quad \text{or} \quad \theta \mapsto \pi \sin\theta,$$
$$\text{where } z = e^{i\theta} \in \mathbf{T}.$$

The upper and lower half-circles $\mathbf{T}_\pm$ are invariant under F, and the unimodal maps $F\,|\,\mathbf{T}_\pm$ admit quadratic-like extensions that are hybrid equivalent to the Chebyshev map $z \mapsto z^2 - 2$.

Let us now consider the analytic real-symmetric immersion $\phi : \mathbf{T} \to \mathbf{C}$ satisfying

$$\frac{1}{2\pi^2}\phi(z)^2 = 1 + F(z), \qquad \phi(-1) = 2\pi.$$

Its image S is a real-symmetric and 0-symmetric "figure eight" with double point at 0. The map F lifts to an analytic real-symmetric map $f : S \to S$, $f \circ \phi = \phi \circ F$. The segments $S_\pm := \phi(\mathbf{T}_\pm)$ are invariant under f and the maps $f\,|\,S_\pm$ admit quadric-like extensions that are hybrid equivalent to the Chebyshev map. So, they define two different quadratic-like germs in $\mathcal{H}_{-2}$.

However, they define the same germ at 0.

Theorem **4.3.** — *For every* $\epsilon > 0$, $d_{\mathcal{H}_c}$ *defines a Montel metric on* $\mathcal{H}_c(\epsilon)$, *for each* $c \in \mathcal{M}$. *Moreover those metrics are uniformly Hölder equivalent to the restriction of any fixed Montel metric on* $\mathcal{C}(\epsilon)$.

We will need two preliminary results:

Lemma **4.4.** — *For every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $d_{\mathcal{H}_c(\delta)}$ *defines a Montel metric on* $\mathcal{H}_c(\epsilon)$, *for each* $c \in \mathcal{M}$. *Moreover those metrics are uniformly Hölder equivalent to the restriction of any fixed Montel metric on* $\mathcal{C}(\epsilon)$.

Proof. — By Theorem 4.2 and the Schwarz Lemma, $\mathcal{H}_c(\delta)$ is Carathéodory hyperbolic, so that $d_{\mathcal{H}_c(\delta)}$ is a metric.

Given $\delta > 0$, there exists $\rho > 0$ such that all $f \in \mathcal{C}(\delta)$ extend holomorphically to a holomorphic function on $\mathbf{D}_\rho$ bounded by ρ^{-1}. It then follows, by analytic continuation, that for every $c \in \mathcal{M}$ and $z \in \mathbf{D}_\rho$ the function $f \mapsto f(z)$ is holomorphic on $\mathcal{H}_c(\delta)$. This shows that $2\rho^{-1} d_{\mathcal{H}_c(\delta)}$ dominates a Montel metric on each sliceable subset $\mathcal{K} \subset \mathcal{H}_c(\delta)$ (take the Montel metric given by the uniform distance on each sufficiently small neighborhood L of 0).

Let us now show that if $\delta > 0$ is sufficiently small, then for each $f \in \mathcal{H}_c(\epsilon)$, $d_{\mathcal{H}_c(\delta)}$ is Hölder dominated by a Montel metric in an $\mathcal{H}_c(\epsilon)$-neighborhood of f. In a neighborhood of f there exist open quasidisks $V \Subset V'$ such that if $f_0, f_1 \in \mathcal{H}_c(\epsilon)$ are γ close with respect to the Montel metric, then they are $C\gamma^\theta$ close over V' and there are polynomial-like extensions $f_0 : U_0 \to V$ and $f_1 : U_1 \to V$ with modulus uniformly bounded from below by some $\kappa > 0$. For small γ, this implies the existence of a quasiconformal homeomorphism $h : \mathbf{C} \to \mathbf{C}$ which is the identity outside V and conjugates $f_0 : \partial U_0 \to \partial V$ and $f_1 : \partial U_1 \to \partial V$. Moreover the Beltrami differential of h has L^∞ norm bounded by $C'\gamma^\theta$.[14] If f_0 and f_1 are hybrid equivalent, we conclude (via the pullback argument) that h can be turned into a hybrid conjugacy preserving the natural marking. This yields a Beltrami path parametrized by $\mathbf{D}_{C'^{-1}\gamma^{-\theta}}$ connecting f_0 to f_1. This Beltrami path, restricted to $\mathbf{D}_{C'^{-1}\gamma^{-\theta}/2}$, lies in $\mathcal{H}_c(\kappa/3)$, by the Quasiconformality λ-Lemma. By the Schwarz Lemma, if $\delta \le \kappa/3$ we get $d_{\mathcal{H}_c(\delta)}(f_0, f_1) \le 2C'\gamma^\theta$, as desired.

The uniformity on c is clear from the argument. □

Lemma **4.5.** — *For every* $\epsilon > 0$, $d_{\mathcal{H}_0}$ *is a Montel metric on* $\mathcal{H}_0(\epsilon)$.

Proof. — By the Schwarz Lemma $d_{\mathcal{H}_0}$ is dominated by the Montel metric $d_{\mathcal{H}_0(\delta)}$ for $\delta > 0$. For fixed $0 < \rho < 1/4$, the restriction $I_\rho : \mathcal{H}_0 \to \mathcal{B}_{\mathbf{D}_\rho}$ is a bounded path holo-

[14] To see this, consider the holomorphic family $f_\lambda = f_0 + \lambda f_1$ over a disk of radius $(C'\gamma^\theta)^{-1}$ around 0 and apply the Schwarz Lemma to the map $\lambda \mapsto \mu_\lambda$ where μ_λ is the Beltrami differential of the holomorphic motion of the fundamental annulus.

morphic map for (c.f. the proof of Theorem 4.2), so another application of the Schwarz Lemma shows that $d_{\mathcal{H}_0}$ dominates a multiple of the uniform metric on $\overline{\mathbf{D}_\rho}$, which is Montel. □

Proof of Theorem 4.3. — Since both $d_{\mathcal{H}_0}$ and $d_{\mathcal{H}_0(\delta)}$ are Montel over $\mathcal{H}_0(\epsilon)$, they are Hölder equivalent. Since $i_c \circ i_0^{-1}$ is a biholomorphic map $(\mathcal{H}_0(\epsilon), \mathcal{H}_0(\delta), \mathcal{H}_0) \to (\mathcal{H}_c(\epsilon), \mathcal{H}_c(\delta), \mathcal{H}_c)$, $d_{\mathcal{H}_c}$ and $d_{\mathcal{H}_c(\delta)}$ are also Hölder equivalent (with the same constants). Since the latter is (uniformly) Montel, the former is as well. □

5. From beau bounds to exponential contraction

A priori bounds are called *beau* (over a family $\mathcal{F}$ of infinitely renormalizable maps under consideration) if there exists $\epsilon_0 > 0$ such that for any $\delta > 0$ there exists a moment n_δ such that for any $f \in \mathcal{F}$ with $\mathrm{mod}(f) \geq \delta$ we have: $\mathrm{mod}(\mathrm{R}^n f) \geq \epsilon_0$ for $n \geq n_\delta$.

The works [K, KL1, KL2] supply a big class of infinitely renormalizable maps with beau bounds. In this class the little $\mathcal{M}$-copies $\mathcal{M}'_n$ describing the combinatorics should stay away from the "main molecule" of $\mathcal{M}$ (which comprises the main cardioid of $\mathcal{M}$ and all hyperbolic components obtained from it via a cascade of bifurcations). For instance, this class contains all infinitely renormalizable maps of bounded primitive type and all real infinitely renormalizable maps with all renormalization periods $p_n \neq 2$. Let us emphasize that the approach to the Main Theorem we will develop in Sections 6–8 does not rely at all on [K, KL1, KL2] (and it will cover all real combinatorics, including period doubling, in a unified way).

We will show that beau bounds through *complex* hybrid classes imply exponential contraction of the renormalization:

Theorem **5.1.** — *Let $\mathcal{F} \subset \mathcal{C}$ be a family of infinitely renormalizable maps with beau bounds which is forward invariant under renormalization. If $\mathcal{F}$ is a union of entire hybrid leaves then there exists $\lambda < 1$ such that whenever $f, \tilde{f} \in \mathcal{F}$ are in the same hybrid leaf, we have*

$$d_{\mathcal{H}_{c_n}}(\mathrm{R}^n(f), \mathrm{R}^n(\tilde{f})) \leq \mathrm{C}\lambda^n, \quad n \in \mathbf{N},$$

where $c_n = \chi(\mathrm{R}^n(f)) = \chi(\mathrm{R}^n(\tilde{f}))$ and $\mathrm{C} > 0$ only depends on $\mathrm{mod}(f)$ *and* $\mathrm{mod}(\tilde{f})$.

Remark **5.1.** — We will actually show that $\mathrm{C}(f, \tilde{f})$ is small when f is close to $\tilde{f}$, and indeed if $\mathrm{mod}(f), \mathrm{mod}(\tilde{f}) \geq \delta$ we can take

$$\mathrm{C}(f, \tilde{f}) = \mathrm{C}(\delta) d_{\mathcal{H}_c(\delta)}(f, \tilde{f}). \tag{5.1}$$

The proof is based on the Schwarz Lemma and the following easy "smallness" estimate.

Lemma **5.2.** — *For every* $\epsilon > 0$ *there exist* $\delta \in (0, \epsilon)$ *and* $\gamma < 1$ *such that for all* $c \in \mathcal{M}$, *we have:*

$$\text{diam}_{\mathcal{H}_c(\delta)} \mathcal{H}_c(\epsilon) < \gamma. \tag{5.2}$$

Proof. — There exists $r = r(\epsilon) < 1$ with the following property (see Lemmas 2.5 and 2.6). For any p-l germs $f, \tilde{f} \in \mathcal{H}_c(\epsilon)$, there exist p-l representatives $f : U \to V$, $\tilde{f} : \tilde{U} \to \tilde{V}$ and a hybrid conjugacy (respecting the natural marking) $h : \mathbf{C} \to \mathbf{C}$ between f and $\tilde{f}$ such that $\text{mod}(V \setminus U) > \frac{\epsilon}{2}$ and the Beltrami differential $\mu = \bar{\partial} h / \partial h$ has L^∞-norm bounded by $r(\epsilon)$.

Let us consider a Beltrami path $\mathbf{D}_\rho \to \mathcal{H}_c$,

$$\lambda \mapsto f_\lambda = h_{\lambda\mu} \circ f \circ h_{\lambda\mu}^{-1}, \quad \text{where } \rho = \rho(\epsilon) = \frac{1+r}{2r} \in \left(1, \frac{1}{r}\right), \tag{5.3}$$

where $h_{\lambda\mu}$ is a suitably normalized solution of the Beltrami equation $\bar{\partial} h / \partial h = \lambda\mu$.

As $\|\lambda\mu\|_\infty \leq (1+r)/2$, we have

$$\text{Dil}\, h_{\lambda\mu} \leq K = K(\epsilon) = \frac{r+3}{1-r}, \quad \lambda \in \mathbf{D}_\rho.$$

Hence the fundamental annulus of f_λ has modulus at least $\delta = \delta(\epsilon) := \epsilon/2K$, so $f_\lambda \in \mathcal{H}_c(\delta)$. By the (weak) Schwarz Lemma, $d_{\mathcal{H}_c(\delta)}(f, \tilde{f}) \leq d_{\mathbf{D}_\rho}(0, 1) = \rho^{-1}$. □

Proof of Theorem 5.1. — Let $\epsilon_0 > 0$ be the "beau bound" for $\mathcal{F}$, so that for every $\delta > 0$ there exists n_δ such that $\text{mod}(R^n(f_0)) \geq \epsilon_0$ whenever $f_0 \in \mathcal{F}$, $\text{mod}(f_0) \geq \delta$ and $n \geq n_\delta$.

Using Lemma 5.2, choose $0 < \delta_0 < \epsilon_0$ and $\lambda < 1$ such that

$$\text{diam}_{\mathcal{H}_c(\delta_0)} \mathcal{H}_c(\epsilon_0) < \lambda^{n_{\delta_0}}$$

for every $c \in \mathcal{M}$.

The Schwarz Lemma gives for $f, \tilde{f} \in \mathcal{H}_c(\delta)$

$$d_{\mathcal{H}_{c_n}}(R^n(f), R^n(\tilde{f})) \leq \min\{d_{\mathcal{H}_c(\delta)}(f, \tilde{f}), d_{\mathcal{H}_{c_n}(\delta_0)}(R^n(f), R^n(\tilde{f}))\},$$

$$d_{\mathcal{H}_{c_n}(\delta_0)}(R^n(f), R^n(\tilde{f})) \leq d_{\mathcal{H}_c(\delta)}(f, \tilde{f}), \quad n \geq n_\delta,$$

$$d_{\mathcal{H}_{c_n}(\delta_0)}(R^n(f), R^n(\tilde{f})) \leq \lambda^{n_{\delta_0}} d_{\mathcal{H}_{c_{n-n_{\delta_0}}}(\delta_0)}(R^{n-n_{\delta_0}}(f), R^{n-n_{\delta_0}}(\tilde{f})), \quad n \geq n_{\delta_0} + n_\delta,$$

which combined yields the result with $C_{f,\tilde{f}}$ as in (5.1). (Check it first for $n = kn_{\delta_0} + n_\delta$, $k = 1, 2, \ldots$, and then for the intermediate moments.) □

6. From beau bounds for real maps to uniform contraction

It is a difficult problem to prove beau bounds for complex maps. However, it is more tractable for real maps: in that case, the beau bounds were established a while ago (see [S, MvS] for bounded combinatorics and [LvS, LY] for the general case).

Theorem **6.1** *(Beau bounds for real maps). — There exists* $\epsilon_0 > 0$ *with the following property. For every* $\delta > 0$ *there exist* $\epsilon = \epsilon(\delta) > 0$ *and* $N = N(\delta)$ *such that for any* $f \in \mathcal{C}^{\mathbf{R}}(\delta)$, *we have*

$$R^n(f) \in \mathcal{C}(\epsilon), \quad n = 0, 1, \dots \quad \textit{and}$$
$$R^n(f) \in \mathcal{C}(\epsilon_0), \quad n = N, N+1, \dots.$$

From the point of view of the previous discussion, the main shortcoming of this result is that it does not provide enough compactness for complex maps, which is crucial for the Schwarz Lemma application. In this section we will overcome this problem using some ideas from functional analysis and differential topology (Theorem 6.1 remaining the only ingredient from "hard analysis"), by proving:

Theorem **6.2** *(Beau bounds and macroscopic contraction for complex maps in the real hybrid classes). — There exists* $\epsilon_0 > 0$ *with the following property. For any* $\gamma > 0$ *and* $\delta > 0$ *there exists* $N = N(\gamma, \delta)$ *such that for any two maps* $f, \tilde{f} \in \mathcal{C}(\delta)$ *in the same real-symmetric hybrid leaf we have*

$$R^n f, R^n \tilde{f} \in \mathcal{C}(\epsilon_0), \quad \textit{and} \quad d_{\mathcal{H}_{c_n}}(R^n f, R^n \tilde{f}) < \gamma, \quad n \geq N,$$

with $c_n = \chi(R^n f) = \chi(R^n \tilde{f})$.

The proof of this result will take this and the next two sections.

Through the sequel, let $\mathcal{I}$ be the subspace of infinitely renormalizable p-l maps in $\mathcal{C}$, and let $\mathcal{I}(\delta) = \mathcal{I} \cap \mathcal{C}(\delta)$. Let $\mathcal{C}^{(\mathbf{R})}$ be the space of polynomial like maps with connected Julia set which are hybrid equivalent to real p-l maps. We will use superscript **(R)** for the slices of various spaces by $\mathcal{C}^{(\mathbf{R})}$, e.g., $\mathcal{I}^{(\mathbf{R})}(\delta) := \mathcal{I} \cap \mathcal{C}^{(\mathbf{R})}(\delta)$.

6.1. *Cocycle setting.* — We will now abstract properties of the renormalization operator that will be sufficient for Theorem 6.2.

Let $\mathcal{S}$ be a semigroup, and let $Q = \{(m, n) \in \mathbf{N} \times \mathbf{N} : n > m\}$. An $\mathcal{S}$-*cocycle* is a map $G : Q \to \mathcal{S}$, $(m, n) \mapsto G^{m,n}$, such that $G^{m,n} G^{l,m} = G^{l,n}$.

Letting $F_n := G^{n,n+1} \in \mathcal{S}$, we obtain

$$G^{m,n} = F_{n-1} \circ \dots \circ F_m, \tag{6.1}$$

and vice versa, any sequence $F_n \in \mathcal{S}$ determines a cocycle by means of (6.1).

Let $\mathrm{Hol}(\mathcal{H}_0, \mathcal{H}_0)$ be the semigroup of continuous path holomorphic maps $F : \mathcal{H}_0 \to \mathcal{H}_0$. Let $\mathrm{Hol}^{\mathbf{R}}(\mathcal{H}_0, \mathcal{H}_0)$ be the sub-semigroup of those F such that $F(\mathcal{H}_0^{\mathbf{R}}) \subset \mathcal{H}_0^{\mathbf{R}}$.

Theorem **6.3.** — *Let* $\mathcal{G}$ *be a family of cocycles with values in* $\mathrm{Hol}^{\mathbf{R}}(\mathcal{H}_0, \mathcal{H}_0)$ *satisfying:*

H1. *A priori bounds for complex maps: For every* $\delta > 0$, *there exists* $\epsilon = \epsilon(\delta) > 0$ *such that*

$$\text{If } f \in \mathcal{H}_0(\delta) \quad \text{then } \mathrm{G}^{m,n}(f) \in \mathcal{H}_0(\epsilon)$$
$$\text{for every } \mathrm{G} \in \mathcal{G} \text{ and } (m, n) \in \mathrm{Q}.$$

H2. *Beau bounds for nearly real maps: There exists* $\epsilon_0 > 0$ *such that for every* $\delta > 0$, *there exists* $\mathrm{N} = \mathrm{N}(\delta)$ *and* $\eta = \eta(\delta) > 0$ *such that:*

$$\text{If } f \in \mathcal{H}_0^{\mathbf{R}}(\delta), \tilde{f} \in \mathcal{H}_0(\delta) \text{ and } d_{\mathcal{H}_0}(f, \tilde{f}) \leq \eta$$
$$\text{then } \mathrm{G}^{m,n}(\tilde{f}) \in \mathcal{H}_0(\epsilon_0)$$
$$\text{for every } \mathrm{G} \in \mathcal{G} \text{ and } (m, n) \in \mathrm{Q} \text{ with } n - m \geq \mathrm{N}.$$

Then we have:

C1. *Macroscopic contraction: For every* $\delta > 0$ *and* $\gamma > 0$ *there exists* $\mathrm{N} = \mathrm{N}(\delta, \gamma)$ *such that if* $f, \tilde{f} \in \mathcal{H}_0(\delta)$ *then*

$$d_{\mathcal{H}_0}(\mathrm{G}^{m,n}(f), \mathrm{G}^{m,n}(\tilde{f})) < \gamma$$
$$\text{for every } \mathrm{G} \in \mathcal{G} \text{ and } (m, n) \in \mathrm{Q} \text{ with } n - m \geq \mathrm{N}.$$

C2. *Beau bounds for complex maps: For every* $\delta > 0$ *there exists* $\mathrm{N} = \mathrm{N}(\delta)$ *such that*

$$\mathrm{G}^{m,n}(\mathcal{H}_0(\delta)) \subset \mathcal{H}_0(\epsilon_0)$$
$$\text{for every } \mathrm{G} \in \mathcal{G} \text{ and } (m, n) \in \mathrm{Q} \text{ with } n - m \geq \mathrm{N}.$$

6.2. *Reduction to the cocycle setting.* — Let $\Pi = i_0 \circ \pi : \mathcal{C} \to \mathcal{H}_0$, so that for each $c \in \mathcal{M}$, Π restricts to a homeomorphism $\mathcal{H}_c \to \mathcal{H}_0$ which is path holomorphic and preserves the modulus. For each infinitely renormalizable hybrid leaf $\mathcal{H}_c$, we can associate a cocycle $\mathrm{G} = \mathrm{G}_c$ with values in $\mathrm{Hol}(\mathcal{H}_0, \mathcal{H}_0)$, by the formula

$$\mathrm{G}^{m,n}(\Pi(f)) = \Pi(\mathrm{R}^{n-m}(f)), \quad f \in \mathcal{H}_{c_m}, \tag{6.2}$$

where $c_m = \chi(\mathrm{R}^m(p_c))$.

Let $\mathcal{G}$ be the set of all such cocycles which correspond to real-symmetric hybrid leaves. Once we show that hypothesis (H1–H2) of Theorem 6.3 are satisfied for $\mathcal{G}$, the conclusions (C1–C2) translate precisely into beau bounds and macroscopic contraction in real hybrid classes (Theorem 6.2).

Let us start with (H1):

Lemma **6.4.** — *For every* $\delta > 0$ *there exists* $\epsilon = \epsilon(\delta) > 0$ *such that:*

$$\text{If } f \in \mathcal{I}^{(\mathbf{R})}(\delta) \quad \text{then } \mathrm{R}^n f \in \mathcal{C}(\epsilon), \quad n = 0, 1, \ldots.$$

Proof. — Let $g = p_c : z \mapsto z^d + c$, $c \in \mathbf{R}$, be the straightening of f, and let g_n denote its nth pre-renormalizations. Let q_n be the corresponding periods, that is, g_n is a p-l restriction of g^{q_n}.

By Lemmas 2.5 and 2.6, there exist p-l representatives $g : \mathrm{U}' \to \mathrm{V}'$, $f : \mathrm{U} \to \mathrm{V}$ with $\mathrm{mod}(\mathrm{V}' \setminus \mathrm{U}')$, $\mathrm{mod}(\mathrm{V} \setminus \mathrm{U}) > \delta/2$ and a C-qc map $h : (\mathbf{C}, \mathrm{U}) \to (\mathbf{C}, \mathrm{U}')$ with $\mathrm{C} = \mathrm{C}(\delta)$ conjugating f to g, $h \circ f = g \circ h$ in U. Since $\mathrm{mod}(\mathrm{V}' \setminus \mathrm{U}') > \delta/2$,

$$\inf_{y \in \partial \mathrm{U}'} \inf_{x \in \mathrm{K}(g)} |y - x| \geq \mathrm{A}\,\mathrm{diam}\,\mathrm{K}(g),$$

for some $\mathrm{A} = \mathrm{A}(\delta) > 0$.

By the *a priori bounds* for real maps, there exists $\eta > 0$, depending only on the degree d, such that

$$\mathrm{mod}\, g_n \geq \eta, \qquad n = 0, 1, \ldots .$$

It follows from compactness of $\mathcal{C}(\eta)$ that we can select $\eta' = \eta'(\mathrm{A}, \eta) > 0$ so small that each germ g_n has a (non-normalized) p-l representative $\mathrm{U}_n \to \mathrm{V}_n$ with $\mathrm{mod}(\mathrm{V}_n \setminus \mathrm{U}_n) > \eta'$ and

$$\begin{aligned}\sup_{y \in \partial g^k(\mathrm{U}_n)} \inf_{x \in g^k(\mathrm{K}(g_n))} |y - x| &\leq \mathrm{A}\,\mathrm{diam}\, g^k(\mathrm{K}(g_n)) \\ &\leq \mathrm{A}\,\mathrm{diam}\,\mathrm{K}(g), \qquad k = 0, 1, 2, \ldots, q_n - 1.\end{aligned}$$

We conclude that $g^k(\mathrm{U}_n) \subset \mathrm{U}'$ for $0 \leq k \leq q_n - 1$. Consequently, the p-l map

$$h^{-1} \circ g_n \circ h : \; h^{-1}(\mathrm{U}_n) \to h^{-1}(\mathrm{V}_n)$$

is a (non-normalized) representative of the n-th pre-renormalization of f. Since

$$\mathrm{mod}(h^{-1}(\mathrm{V}_n) \setminus h^{-1}(\mathrm{U}_n)) > \eta'/\mathrm{C},$$

we obtain *a priori* bounds for f with $\epsilon = \eta'/\mathrm{C}$. □

In order to show that hypothesis (H2) is satisfied, we will use the following.

Lemma **6.5.** — *Let* $f_n, \tilde{f}_n \in \mathcal{I}$ *and* $\chi(f_n) = \chi(\tilde{f}_n)$. *Assume that the sequences* f_n *and* $\tilde{f}_n$ *converge to the same limit* f. *If* $k_n \to \infty$ *then* $\liminf_{n\to\infty} \mathrm{mod}(\mathrm{R}^{k_n}(f_n)) = \liminf_{n\to\infty} \mathrm{mod}(\mathrm{R}^{k_n}(\tilde{f}_n))$.

Proof. — It is enough to show that for every $\epsilon > 0$,

$$\text{if } \liminf_{n\to\infty} \mathrm{mod}(\mathrm{R}^{k_n}(f_n)) > \epsilon \quad \text{then } \liminf_{n\to\infty} \mathrm{mod}(\mathrm{R}^{k_n}(\tilde{f}_n)) > \epsilon.$$

Let $f_n : \mathrm{U}_n \to \mathrm{V}_n$ and $\tilde{f}_n : \tilde{\mathrm{U}}_n \to \tilde{\mathrm{V}}_n$ be p–l representatives of the germs f_n and $\tilde{f}_n$ that Carathéodory converge to a p-l map $f : \mathrm{U} \to \mathrm{V}$. By Lemma 2.7, there exist $\mathrm{C}_n \to 1$

and C_n-qc maps $h_n : V_n \to \tilde{V}_n$ conjugating f_n to $\tilde{f}_n$ (maybe after a slight adjustment of the domains).

Let $f'_n : U'_n \to V'_n$ be p-l representatives of the k_n-th pre-renormalizations of the f_n with $\liminf \operatorname{mod}(V'_n \setminus U'_n) > \epsilon$ and filled Julia sets K_n. By Lemma 2.10, $\operatorname{diam}(K_n) \to 0$, so we may choose V'_n contained in V_n. By Lemma 2.9, $f'_n = f_n^{q_{k_n}} | U'_n$.

Let $\tilde{U}'_n = h_n(U'_n)$ and $\tilde{V}'_n = h_n(V'_n)$. Then the map $\tilde{f}_n^{q_{k_n}} : \tilde{U}'_n \to \tilde{V}'_n$ is a well defined p-l representative of the k_n-th pre-renormalization of $\tilde{f}_n$. Moreover, $\operatorname{mod}(\tilde{V}'_n \setminus \tilde{U}'_n) > \operatorname{mod}(V'_n \setminus U'_n)/C_n$, and the conclusion follows. □

Take ϵ_0 from Theorem 6.1. If (H2) does not hold for $\epsilon_0/2$, we can find a $\delta > 0$ and sequences $f_n \in \mathcal{I}^{\mathbf{R}} \cap \mathcal{C}(\delta)$, $\tilde{f}_n \in \mathcal{H}_{\chi(f_n)}(\delta)$, $k_n \to \infty$, with $d_{\mathcal{H}_{\chi(f_n)}}(f_n, \tilde{f}_n) \to 0$ such that $R^{k_n}\tilde{f}_n \notin \mathcal{C}(\epsilon_0/2)$. Passing to a subsequence, we can assume that the f_n converge, and thus the $\tilde{f}_n$ must converge to the same limit. By the previous lemma, it follows that $\liminf \operatorname{mod}(R^{k_n} f_n) \le \epsilon_0/2$, contradicting Theorem 6.1.

We have reduced Theorem 6.2 to Theorem 6.3.

6.3. *Retractions and the proof of Theorem 6.3.* — Recall that a continuous map $P : X \to X$ in a topological space X is called a *retraction* if $P^2 = P$. In other words, there exists a closed subset $Y \subset X$ (a "retract") such that $P(X) \subset Y$ and $P | Y = \mathrm{id}$. Linear retractions in topological vector spaces are called projections. A retraction is naturally called *constant* if its image is a single point.

The proof of Theorem 6.3 has two main parts. The first shows, using (H1), that lack of uniform contraction in the leafwise dynamics allows one to construct a retraction towards a non-trivial "attractor":

Theorem **6.6.** — *Let $\mathcal{G}$ be a family of cocycles with values in* $\mathrm{Hol}(\mathcal{H}_0, \mathcal{H}_0)$. *Assume that property* (H1) *holds but* (C1) *fails. Then there exist sequences* $G_k \in \mathcal{G}$, $(m_k, n_k) \in Q$ *with* $n_k - m_k \to \infty$, *and a non-constant retraction* $P \in \mathrm{Hol}(\mathcal{H}_0, \mathcal{H}_0)$ *such that* $G_k^{m_k, n_k}(f) \to P(f)$ *for every* $f \in \mathcal{H}_0$.

The second shows, in general, that non-trivial real-symmetric retractions cannot be "too compact" in the real direction.

Theorem **6.7.** — *Let* $P \in \mathrm{Hol}^{\mathbf{R}}(\mathcal{H}_0, \mathcal{H}_0)$ *be a retraction, and assume the following compactness property for nearly real maps*:

(P) *There exists a compact set* $\mathcal{K} \subset \mathcal{H}_0$ *such that if* $f_n \in \mathcal{H}_0$ *is a sequence converging to* $f \in \mathcal{H}_0^{\mathbf{R}}$, *then* $P(f_n) \in \mathcal{K}$ *for n large.*

Then P *is constant.*

Those two results put together imply Theorem 6.3:

Proof of Theorem 6.3. — Let $\mathcal{G}$ be a family of cocycles with values in $\mathrm{Hol}^{\mathbf{R}}(\mathcal{H}_0, \mathcal{H}_0)$ such that (H1) and (H2) hold, but (C1) does not. By Theorem 6.6, there exist a sequence $G_k \in \mathcal{G}$ and $(m_k, n_k) \in Q$ with $n_k - m_k \to \infty$, and a non-constant retraction $P \in \mathrm{Hol}(\mathcal{H}_0, \mathcal{H}_0)$, such that $G_k^{m_k,n_k}(f) \to P(f)$ for every $f \in \mathcal{H}_0$. It satisfies the hypothesis of Theorem 6.7: since each $G_k^{m_k,n_k}$ preserves $\mathcal{H}_0^{\mathbf{R}}$, P also does, while (H2) immediately gives

(P′) *For any* $f \in \mathcal{H}_0^{\mathbf{R}}(\delta)$ *and* $\tilde{f} \in \mathcal{H}_0(\delta)$ *with* $d_{\mathcal{H}_0}(\tilde{f}, f) < \eta$ *we have* $P(\tilde{f}) \in \mathcal{H}_0(\epsilon_0)$

which clearly implies the crucial property (P). By Theorem 6.7, P is constant, yielding the desired contradiction.

This concludes the proof of (C1). Together with (H2), it implies (C2). □

The essentially independent proofs of Theorems 6.7 (of differential topology nature) and 6.6 (dynamical) will be given in the next two sections.

7. Triviality of retractions

7.1. *Plan of the proof of Theorem 6.7.* — Let us describe the plan of the proof of Theorem 6.7. Let $P^{\mathbf{R}} = P \,|\, \mathcal{H}_0^{\mathbf{R}}$. Let $\mathcal{Z}^{\mathbf{R}} = \mathrm{Im}\, P^{\mathbf{R}} = \mathrm{Fix}\, P^{\mathbf{R}}$. By Property (P), $\mathcal{Z}^{\mathbf{R}} \subset \mathcal{K}$, and hence $\mathcal{Z}^{\mathbf{R}} = P(\mathcal{K} \cap \mathcal{H}_0^{\mathbf{R}})$, which is compact. We will complete the argument in three consecutive steps:

Step 1. $\mathcal{Z}^{\mathbf{R}}$ is a finite-dimensional topological manifold (by the Implicit Function Theorem),
Step 2. $\mathcal{Z}^{\mathbf{R}}$ is a single point (by a Brower-like topological argument),
Step 3. $\mathcal{Z} := \mathrm{Im}\, P = \mathrm{Fix}\, P$ is a single point, too (by analytic continuation).

The first and third steps would be immediate to carry out if we were dealing with Banach spaces. For instance, corresponding to the first step we have:

Lemma **7.1** *(See [Ca]). — Let* $\mathcal{B}$ *be a complex (respectively, real) Banach space and let* P *be a holomorphic (respectively, real analytic) map from an open set of* $\mathcal{B}$ *to* $\mathcal{B}$ *such that* $P(0) = 0$. *Assume that* $DP(0)$ *is compact and* $P^2 = P$ *near* 0. *Then for any sufficiently small open ball* B *around* 0 *in* $\mathcal{B}$, $P(B)$ *is a complex (respectively, real analytic) finite-dimensional manifold.*

Proof. — This is a particular case of [Ca] but we will give the argument for the reader's convenience. Let $h = \mathrm{id} - DP(0) - P$. Since $P^2 = P$, we have $DP(0) = DP(0)^2$ and hence $Dh(0)^2 = \mathrm{id}$, so h is a local diffeomorphism near 0. Obviously $h \circ P = DP(0) \circ h$, so $P(B) = h^{-1}(DP(0)(h(B)))$ if B is a sufficiently small ball around 0. Since $DP(0)$ is compact and $DP(0) = DP(0)^2$, it has finite rank so $DP(0) \cdot h(B)$ is an open subset of a finite dimensional subspace. □

In order to translate this more familiar analysis to our context, we will use *Banach slices* (first introduced in [L3]).

7.2. *Banach slices.* — Let $f \in \mathcal{H}_0$. We call an open quasidisk W f-admissible if $W \supset K(f)$ and f extends holomorphically to W and continuously to the boundary. Let $\mathcal{B}^*_W \subset \mathcal{B}_W$ be the Banach space of all holomorphic maps $w : W \to \mathbf{C}$ such that $w(z) = O(z^{d+1})$ near 0 and w extends continuously to the boundary.

The set of f-admissible quasidisks can be partially ordered by inclusion. This partial order is directed in the sense that any finite set has a lower bound, thus it makes sense to speak of "sufficiently small" f-admissible W.

Let $B_{W,r}$ be the open ball around 0 in $\mathcal{B}^*_W$ of radius r. The following lemma is a straightforward consequence of the definition of the topology in $\mathcal{H}_0$.

Lemma **7.2.** — *For every $\epsilon > 0$, and $f \in \mathcal{H}_0(\epsilon)$, for every sufficiently small f-admissible* W*, for every $r > 0$, there exists a neighborhood $\mathcal{V}$ of f in $\mathcal{H}_0(\epsilon)$ such that for every $f' \in \mathcal{V}$,* W *is f'-admissible and $f' - f \in B_{W,r}$.*

It is easy to see that there exists $\epsilon_0 = \epsilon_0(f, W)$ and $r_0 = r_0(f, W) > 0$ such that if $w \in B_{W,r_0}$ then $f' = f + w$ admits a polynomial-like restriction $f' : U \to V$ with $K(f') \subset U \subset W$ and $\mathrm{mod}(V \setminus U) > \epsilon_0$. Since $f'(0) = 0$, f' defines a germ in $\mathcal{H}_0(\epsilon_0)$ denoted by $j_{f,W,r_0}(w)$. The map $j_{f,W,r_0} : B_{W,r_0} \to \mathcal{H}_0$ is readily seen to be continuous and injective.

Theorem **7.3.** — *Let f,* W *and r_0 be as above, and let $\lambda \mapsto w_\lambda$ be a continuous map $\mathbf{D} \to B_{W,r_0}$. Then $\lambda \mapsto w_\lambda$ is holomorphic if and only if $f_\lambda = j_{f,W,r_0}(w_\lambda)$ is a holomorphic path in $\mathcal{H}_0$.*

For the proof, we will need a preliminary result. As in the proof of Theorem 4.2, for $0 < R < 1/4$ we let $I_R : \mathcal{H}_0 \to \mathcal{B}_{\mathbf{D}_R}$ be the restriction operator, which is well defined by (4.1).

Lemma **7.4.** — *Let $0 < R < 1/4$ and let $\lambda \mapsto f_\lambda$ be a continuous map $\mathbf{D} \to \mathcal{H}_0$. Then f_λ is a holomorphic path in $\mathcal{H}_0$ if and only if $\lambda \mapsto I_R(f_\lambda)$ is a holomorphic path in $\mathcal{B}_{\mathbf{D}_R}$.*

Proof. — The only if part (equivalent to the path holomorphicity of I_R) was established in the proof of Theorem 4.2.

Assume that $I_R(f_\lambda)$ is a holomorphic path in $\mathcal{B}_{\mathbf{D}_R}$. Since f_λ is assumed to be continuous, in order to show that it is a holomorphic path, it is enough to construct a holomorphic motion $h_\lambda : \mathrm{int}\, K(f_0) \to \mathrm{int}\, K(f_\lambda)$ such that for each λ, h_λ is holomorphic and conjugates f_0 and f_λ: by the Extension λ-Lemma Theorem, it extends to a holomorphic motion $\mathbf{C} \to \mathbf{C}$, which, by continuity, conjugates $f_0|K(f_0)$ and $f_\lambda|K(f_\lambda)$. For the construction, we will make use of the Böttcher coordinate (c.f. proof of Theorem 4.2) $\psi_f : \mathrm{int}\, K(f) \to \mathbf{D}$ associated to any map $f \in \mathcal{H}_0$: the desired holomorphic motion is then given by $h_\lambda = \psi_{f_\lambda}^{-1} \circ \psi_{f_0}$. It is obviously injective and holomorphic in z, and conjugates f_0 and f_λ, for each $\lambda \in \mathbf{D}$, so we just need to show that $\psi_{f_\lambda}^{-1}(z)$ is a holomorphic function $\mathbf{D} \times \mathbf{D} \to \mathbf{C}$.

Holomorphicity of $I_R(f_\lambda)$ implies that $f_\lambda(z)$ is a holomorphic function $\mathbf{D} \times \mathbf{D}_R \to \mathbf{C}$. By the Koebe-1/4 Theorem, if $z \in \mathbf{D}_{R^3}$ then $\psi_{f_\lambda}(z) \in \mathbf{D}_{R^2}$, so $\psi_{f_\lambda}(z)^d \in \mathbf{D}_{R^4}$, and (again by the Koebe-1/4 Theorem), $f_\lambda(z) = \psi_{f_\lambda}^{-1}(\psi_{f_\lambda}(z)^d) \in \mathbf{D}_{R^3}$. It follows that $f_\lambda^n(z) \in \mathbf{D}_{R^3} \subset \mathbf{D}_R$ for every $z \in \mathbf{D}_{R^3}$, $n \geq 1$. By holomorphicity of the iteration we conclude that for every $n \geq 1$, $f_\lambda^n(z)$ is holomorphic in $(\lambda, z) \in \mathbf{D} \times \mathbf{D}_{R^3}$.

Let $\psi_{f_\lambda,n} : \operatorname{int} K(f_\lambda) \to \mathbf{C}$ be such that $\psi_{f_\lambda,n}(z)^{d^n} = f_\lambda^n(z)$ and $D\psi_{f_\lambda,n}(0) = 1$. It is easy to see that $\psi_{f_\lambda,n}$ converges to ψ_{f_λ} uniformly on compacts of $\operatorname{int} K(f_\lambda)$ (actually one usually constructs the Böttcher coordinate ψ_{f_λ} directly as the limit of the $\psi_{f_\lambda,n}$). Over $(\lambda, z) \in \mathbf{D} \times \mathbf{D}_{R^3}$, the holomorphicity of $f_\lambda^n(z)$ implies, successively, that $\psi_{f_\lambda,n}(z)$ and $\psi_{f_\lambda}(z)$ are also holomorphic.

By the Koebe-1/4 Theorem, $\psi_{f_\lambda}^{-1}(\mathbf{D}_{R^4}) \subset \mathbf{D}_{R^3}$, and it follows that $\psi_{f_\lambda}^{-1}(z)$ is a holomorphic function of $(\lambda, z) \in \mathbf{D} \times \mathbf{D}_{R^4}$. Since for each fixed $\lambda \in \mathbf{D}$, $\psi_{f_\lambda}^{-1}$ is a holomorphic function of $\mathbf{D}$, Hartog's Theorem implies that $\psi_{f_\lambda}^{-1}(z)$ is in fact a holomorphic function of (λ, z) through $\mathbf{D} \times \mathbf{D}$. □

Proof of Theorem 7.3. — Assume that f_λ is a holomorphic path in $\mathcal{H}_0$, and let us show that for every bounded linear functional $L : \mathcal{B}_W^* \to \mathbf{C}$, $\lambda \mapsto L(w_\lambda)$ is holomorphic: since w_λ takes values in a ball, this implies that $\lambda \mapsto w_\lambda$ is holomorphic. By (4.1), $f_\lambda(z)$, and hence $w_\lambda(z) = f_\lambda(z) - f(z)$, is holomorphic in $\mathbf{D} \times \mathbf{D}_{1/4}$. By Hartog's Theorem, it is then holomorphic in $\mathbf{D} \times W$, and since it is continuous in z up to ∂W, and bounded in both variables, we see that $\lambda \mapsto f_\lambda(z)$ is a holomorphic function for every $z \in \overline{W}$. By Riesz's Theorem, there exists a complex measure of finite mass μ, supported on $\overline{W}$, such that $L(w) = \int w(z)\, d\mu(z)$, so $\lambda \mapsto L(w_\lambda)$ is holomorphic.

Assume now that $\lambda \mapsto w_\lambda$ is holomorphic. Since j_{f,W,r_0} is continuous, $\lambda \mapsto f_\lambda$ is continuous as well. Fix $0 < R < 1/4$. By (4.1), $\mathbf{D}_{1/4} \subset K(f) \subset W$, hence the restriction operator $I_{R,W} : \mathcal{B}_W^* \to \mathcal{B}_{\mathbf{D}_R}$ is holomorphic. Since $w_\lambda \in \mathcal{B}_W^*$ depends holomorphically on λ, it follows that $I_R(f_\lambda) = I_R(f) + I_{R,W}(w_\lambda) \in \mathcal{B}_R$ also depends holomorphically on λ. By Lemma 7.4, f_λ is a holomorphic path. □

7.3. *Proof of Theorem 6.7.* — We will carry out the three steps of the plan of proof described in Section 7.1. We will use the notation introduced therein.

Let $\epsilon > 0$ be such that $\mathcal{K} \subset \mathcal{H}_0(\epsilon)$. Let $f \in \mathcal{Z}^{\mathbf{R}}$. Let us consider a neighborhood $\mathcal{U}$ of f in $\mathcal{H}_0(\epsilon/2)$. If it is small enough then all the maps $g \in \bar{\mathcal{U}}$ are well defined on some admissible neighborhood of $W \supset K(f)$ (see Section 7.2), so $\bar{\mathcal{U}}$ naturally embeds into some Banach ball $B_r := B_{W,r}$. Let $J : \bar{\mathcal{U}} \to B_r(f)$, $J(g) = g - f \mid W$ denote this embedding. Since $\mathcal{H}_0(\epsilon/2)$ is compact, $J(\bar{\mathcal{U}})$ is compact as well.

On the other hand, by Property (P) and continuity of P, there is a $\rho > 0$ such that $P(j_{f,W,\rho}(B_\rho)) \subset \mathcal{U}$. Hence $P_f := J \circ P \circ j_{f,W,\rho} : B_\rho \to B_r$ is a compact Banach holomorphic retraction.

Let us consider its real-symmetric part $P_f^{\mathbf{R}} : B_\rho^{\mathbf{R}} \to B_r^{\mathbf{R}}$. It is a compact real analytic Banach retraction, so by Lemma 7.1, the set $\mathrm{Fix}\, P_f^{\mathbf{R}}$ near f is a real analytic finite-dimensional submanifold of B_ρ. But since $\mathcal{Z}^{\mathbf{R}}$ is compact, the topology induced on it from the Banach ball coincides with its own topology (induced from the whole space $\mathcal{H}_0$). Hence $\mathcal{Z}^{\mathbf{R}}$ is a finite-dimensional topological manifold near f. Since $f \in \mathcal{Z}^{\mathbf{R}}$ is arbitrary, the first step of the proof is completed.

By Lemma 2.1, the space $\mathcal{E}^{\mathbf{R}}$ is contractible. Since $\mathcal{H}_0^{\mathbf{R}}$ is homeomorphic to it, it is also contractible. Since $\mathcal{Z}^{\mathbf{R}}$ is a retract of $\mathcal{H}_0^{\mathbf{R}}$, it is contractible as well. (If h_t is a homotopy that contracts $\mathcal{H}_0^{\mathbf{R}}$ to a point, then $P \circ h_t : \mathcal{Z}^{\mathbf{R}} \to \mathcal{Z}^{\mathbf{R}}$ does the same to $\mathcal{Z}^{\mathbf{R}}$.) But the only contractible compact finite-dimensional manifold (without boundary) is a point (since otherwise the top homology group $H^n(M)$ is non-trivial.) This concludes the 2nd step.

Thus, $P_f(B_\rho^{\mathbf{R}}) = \{0\}$. Since $P_f : B_\rho \to B_r$ is holomorphic (as a Banach map), $P_f(B_\rho) = \{0\}$.

Let us show that a small neighborhood $\mathcal{V}$ of f in $\mathcal{Z}$ is contained in the neighborhood $\mathcal{U} \subset \mathcal{H}_0(\epsilon/2)$ considered above. Otherwise $f \in \mathrm{cl}(\mathcal{Z} \smallsetminus \mathcal{U})$. Since the notion of closedness in $\mathcal{H}_0$ is given in terms of sequences, there would exist a sequence $f_n \in \mathcal{Z} \smallsetminus \mathcal{U}$ converging to f. By Property (P), the maps $f_n = Pf_n$ would eventually belong to $\mathcal{H}_0(\epsilon/2)$, and hence to the neighborhood $\mathcal{U}$—contradiction.

Thus, we have $\mathcal{V} \subset \mathcal{U} \subset B_r$. Shrinking $\mathcal{V}$ if needed, we make $J(\mathcal{V}) \subset B_\rho$ and hence

$$J(\mathcal{V}) = J(P(\mathcal{V})) \subset P_f(B_\rho) = \{0\}.$$

Since J is injective, $\mathcal{V} = \{f\}$. Thus, f is an isolated point in $\mathcal{Z}$. But since $\mathcal{Z} = P(\mathcal{H}_0)$ is connected, we conclude that $\mathcal{Z} = \{f\}$, which completes the last step of the proof. □

8. Almost periodicity and retractions

We now turn to the dynamical construction of retractions. The presence of enough compactness, together with the non-expansion of the Carathéodory metric, allows us to implement the notion of *Almost Periodicity* (adapted appropriately to the cocycle setting).

8.1. *Almost periodic cocycles.* — We will now discuss cocycles with values in a Hausdorff topological semi-group $\mathcal{S}$. Since we aim to eventually take $\mathcal{S} = \mathrm{Hol}^{\mathbf{R}}(\mathcal{H}_0, \mathcal{H}_0)$, we allow for the possibility that $\mathcal{S}$ is not metrizable, neither satisfy the First Countability Axiom, however we will always assume that $\mathcal{S}$ is *sequential* in the sense that the notions of continuity, closedness and compactness can be defined in terms of sequences.

A *subcocycle* is the restriction G^{k_m,k_n} of a cocycle G to a subsequence of $\mathbf{N}$. More formally, it is a pullback of G under a strictly monotone embedding $k : \mathbf{N} \to \mathbf{N}$.

The ω-limit set of a cocycle, $\omega(G)$, is the set of all existing $\lim G^{m,n}$ as $m \to \infty$ and $n - m \to \infty$.

A cocycle is called *almost periodic* if the family $\{G^{m,n}\}_{(m,n)\in Q}$ is precompact in $\mathcal{S}$. The ω-limit set of an almost periodic cocycle is compact.

We say that a cocycle *converges* if there is the limit $G^{m,\infty} := \lim_{n\to\infty} G^{m,n}$ for every $m \in \mathbf{N}$. The cocyclic rule extends to the limits of converging cocycles:

$$G^{m,\infty} = G^{n,\infty} G^{m,n}, \quad (m, n) \in Q.$$

We say that a cocycle *double converges* if there exists the limit $G^{\infty,\infty} := \lim_{m\to\infty} G^{m,\infty}$. The cocyclic rule extends to the limits of double converging cocycles:

$$G^{m,\infty} = G^{\infty,\infty} G^{m,\infty}, \quad G^{\infty,\infty} = (G^{\infty,\infty})^2.$$

In particular, $G^{\infty,\infty}$ is an *idempotent*.

Lemma **8.1.** — *An almost periodic cocycle has a double converging subcocycle.*

Proof. — A converging subcocycle is extracted by means of the diagonal process. Selecting then a converging subsequence of the $G^{m,\infty}$, we obtain a double converging subcocycle. □

Corollary **8.2.** — *The ω-limit set of an almost periodic cocycle contains an idempotent.*

We endow the space of cocycles with the pointwise convergence topology:

$$G_k \to G \quad \text{if } G_k^{m,n} \to G^{m,n} \quad \text{for all } (m, n) \in Q.$$

The *shift* T in the space of cocycles is induced by the embedding $\mathbf{N} \to \mathbf{N}$, $n \mapsto n + 1$. In other words, $(TG)^{m,n} = G^{m+1,n+1}$, $(m, n) \in Q$.

If G is almost periodic then all its translates $\{T^n G\}_{n=0}^{\infty}$ form a precompact family of cocycles.

Given a function $\rho : \mathcal{S} \to \mathbf{R}_{\geq 0}$, we say that a cocycle is *uniformly ρ-contracting* if for any $\gamma > 0$ there exists an N such that $\rho(G^{m,n}) < \gamma$ for any $(m, n) \in Q$ with $n - m \geq N$.

A continuous function $\rho : \mathcal{S} \to \mathbf{R}_{\geq 0}$ is called *Lyapunov*

$$\rho(F_l F F_r) \leq \rho(F) \quad \text{for any } F, F_l, F_r \in \mathcal{S}.$$

The next assertion will not be directly used but can serve as a model for what follows:

Proposition **8.3.** — *Let* G *be an almost periodic cocycle and let ρ be a Lyapunov function. If $\rho(e) = 0$ for any limit idempotent $e \in \omega(G)$ then the cocycle is uniformly ρ-contracting.*

We leave it as an exercise.

We will need a more general form of the above proposition. Assume that we have two continuous functions, $\rho' \geq \rho \geq 0$ on $\mathcal{S}$, which are not assumed to be individually Lyapunov, but rather possess a joint Lyapunov property (adapted to the cocycle):

$$(\mathbf{8.1}) \qquad \rho(G^{l,n}) \leq \rho(G^{l,m}) \quad \text{and} \quad \rho(G^{l,n}) \leq \rho'(G^{m,n}) \quad \text{for any } l < m < n.$$

We call it a *Lyapunov pair* for the cocycle.

We will also need a uniform version of the above lemma, over a family of cocycles. Let $\mathcal{G}$ be a family of cocycles G_s labeled by an element s of some set Σ. We say that $\mathcal{G}$ is *uniformly almost periodic* if the whole family $G_s^{m,n}$, $s \in \Sigma$, $(m, n) \in Q$, is precompact in $\mathcal{S}$. Then $\omega(\mathcal{G}) \subset \mathcal{S}$ stands for the set of the limits of all converging sequences $G_{s_k}^{m_k,n_k}$ as $m_k \to \infty$ and $n_k - m_k \to \infty$.

We say that the family is *uniformly ρ-contracting* if for any $\gamma > 0$ there exists an N such that $\rho(G_s^{m,n}) < \gamma$ for any $s \in \Sigma$, $m \in \mathbf{N}$ and $n \geq m + N$.

Lemma **8.4.** — *Let $\mathcal{G}$ be a uniformly almost periodic family of cocycles. Let (ρ, ρ') be a Lyapunov pair for all cocycles in $\mathcal{G}$. If $\rho'(e) = 0$ for any limit idempotent $e \in \omega(\mathcal{G})$ then $\mathcal{G}$ is uniformly ρ-contracting.*

Proof. — Otherwise there exists a $\gamma > 0$, a sequence $s_k \in \Sigma$ and two non-decreasing sequences $q_k \in \mathbf{N}$ and $n_k \to \infty$ such that $\rho(G_{s_k}^{q_k,q_k+n_k}) \geq \gamma$. Since $\mathcal{G}$ is uniformly almost periodic, the sequence of cocycles $T^{q_k} G_{s_k}$ admits a converging subsequence. Let G be a limit cocycle. Then $\rho(G^{0,n}) \geq \gamma > 0$ for any $n > 0$ (by continuity of ρ and the first part of (8.1)). By the second part of (8.1), we have $\rho'(G^{m,n}) \geq \gamma$ for all $(m, n) \in Q$. Hence $\rho'(\phi) \geq \gamma$ for all $\phi \in \omega(G)$. In particular, $\rho'(e) > 0$ for any idempotent $e \in \omega(G)$ from Lemma 8.2. Since $\omega(G) \subset \omega(\mathcal{G})$, we arrive at a contradiction. □

8.2. *Tame spaces.* — Let X be a (sequential) topological space endowed with a continuous metric $d : X \times X \to \mathbf{R}_{\geq 0}$ that is compatible with the topology on compact subsets of X (but not necessarily on X). We say that X is *tame* if the following properties hold:

(1) There exists a filtration of compact subsets, $X_1 \subset X_2 \subset \cdots \subset X$ such that $\bigcup X_i = X$;
(2) Each compact set in X is contained in some X_i;
(3) A set is open in X if and only if its intersection with any compact subset of X is relatively open.

A family of continuous maps $F_s : X \to X'$, $s \in \Sigma$, between tame spaces (with metrics d and d' respectively) is called *equicompact* if for every compact set $K \subset X$ there exists a compact set $K' \subset X'$ such that $F_s(K) \subset K'$ for all $s \in \Sigma$.

- An equicompact family $\{F_s\}$ is called *equicontinuous on compact sets* if for every compact set $K \subset X$, for every $x \in K$ and $\epsilon > 0$ there exists $\delta = \delta(K, x, \epsilon) > 0$ (in fact, by compactness of K, one may take $\delta = \delta(\mathrm{K}, \epsilon)$ here) such that for every $s \in \Sigma$, we have

$$\text{if } d(x, y) < \delta \text{ and } y \in \mathrm{K} \quad \text{then } d'(\mathrm{F}_s(x), \mathrm{F}_s(y)) < \epsilon.$$

- A sequence $\{\mathrm{F}_n\}$ is called *uniformly converging on compact sets* if $\{\mathrm{F}_n\}$ is equicompact and there exists a continuous map $\mathrm{F} : \mathrm{X} \to \mathrm{X}'$ such that for every compact set $\mathrm{K} \subset \mathrm{X}$, we have

$$\lim_{n\to\infty} \sup_{x\in\mathrm{K}} d'(\mathrm{F}_n(x), \mathrm{F}(x)) = 0.$$

  Notice that in this case F_n is necessarily equicontinuous on compact sets.
- A family of cocycles G_s, $s \in \Sigma$, is called *uniformly contracting on compact sets* if for any compact subset $\mathrm{K} \subset \mathrm{X}$ and any $\gamma > 0$, there exists a compact set K' and an N such that

$$\mathrm{G}_s^{m,n}(\mathrm{K}) \subset \mathrm{K}' \quad \text{and} \quad \operatorname{diam}(\mathrm{G}_s^{m,n}(\mathrm{K})) < \gamma$$
$$\text{for all } s \in \Sigma,\ m \in \mathbf{N} \text{ and } n \geq m + \mathrm{N}.$$

The space of continuous maps $\mathrm{F} : \mathrm{X} \to \mathrm{X}'$ between two tame spaces is endowed with the topology of uniform convergence on compact subsets.

Lemma **8.5.** — *A sequence of maps* $\mathrm{F}_n : \mathrm{X} \to \mathrm{X}'$ *between tame spaces is precompact if and only if it is equicontinuous on compact sets.*

Proof. — In the non-trivial direction, it follows from the Ascoli-Arzela's Theorem on each X_i and the diagonal argument. □

Given a tame space X, let $\mathcal{S} \equiv \mathcal{S}_{\mathrm{X}}$ be the topological semigroup of all continuous weak contractions of X (endowed with the topology of uniform convergence on compact subsets). Idempotents in this semigroup are retractions.

Lemma **8.6.** — *Let* X *be a tame space, and let* $\mathcal{G}$ *be a uniformly almost periodic family of cocycles* $\mathrm{G}_s^{m,n}$, $s \in \Sigma$, *with values in the semigroup* $\mathcal{S}_{\mathrm{X}}$. *If all limit retractions* $\mathrm{P} \in \omega(\mathcal{G})$ *are constants, then* $\mathcal{G}$ *is uniformly contracting on compact sets.*

Proof. — Since $\mathcal{G}$ is uniformly almost periodic, the family of maps $\mathrm{G}_s^{m,n}$ is equicompact: for any $i \in \mathbf{N}$ there exists $j = j(i) \geq i$ such that $\mathrm{G}_s^{m,n}(\mathrm{X}_i) \subset \mathrm{X}_j$ for all $s \in \Sigma$, $(m, n) \in \mathrm{Q}$.

Let $\rho_i : \mathcal{S} \to \mathbf{R}_{\geq 0}$ be defined as $\rho_i(\mathrm{G}) = \operatorname{diam} \mathrm{G}(\mathrm{X}_i)$. Obviously, these functions are continuous and form a monotonically increasing sequence. Moreover, for any $i \in \mathbf{N}$, the functions $\rho := \rho_i$ and $\rho' := \rho_{j(i)}$ form a Lyapunov pair for any cocycle $G \in \mathcal{G}$. Indeed, for any $l < m < n$ we have:

$$\rho(G^{l,n}) = \operatorname{diam} G^{l,n}(X_i) \le \operatorname{diam} G^{l,m}(X_i) = \rho(G^{l,m})$$

(where the estimate holds since the semigroup $\mathcal{S}$ consists of weakly contracting maps)

$$\rho(G^{l,n}) = \operatorname{diam} G^{l,n}(X_i) \le \operatorname{diam} G^{m,n}(X_j) = \rho'(G^{m,n}).$$

(where the estimate holds since $G^{l,m}(X_i) \subset X_j$).

Since all retractions $P \in \omega(\mathcal{G})$ are constants, we have $\rho'(P) = 0$ for any of them. By Lemma 8.4, the family $\mathcal{G}$ of cocycles is uniformly ρ-contracting, so for any $\gamma > 0$, there exists an N such that $\operatorname{diam} G_s^{m,n}(X_i) < \gamma$ as long as $n \ge m + N$.

Since i is arbitrary, we are done. □

8.3. *Proof of Theorem 6.6.* — Notice that $\mathcal{H}_0$, with the Carathéodory metric, is tame in the sense of Section 8.2 (take, e.g., $X_i := \mathcal{H}_0(2^{-i})$ as a filtration). By the Schwarz Lemma, $\operatorname{Hol}(\mathcal{H}_0, \mathcal{H}_0)$ is a sub-semigroup of $\mathcal{S}_{\mathcal{H}_0}$, which turns out to be closed:

Lemma **8.7.** — *If* $F_n \in \operatorname{Hol}(\mathcal{H}_0, \mathcal{H}_0)$ *converges uniformly on compact sets to a map* F *then* $F \in \operatorname{Hol}(\mathcal{H}_0, \mathcal{H}_0)$.

Proof. — We have to show that if $\gamma : \mathbf{D} \to \mathcal{H}_0$ is a holomorphic path then $F \circ \gamma$ is a holomorphic path as well. Let $0 < \rho < 1/4$, and let $I_\rho : \mathcal{H}_0 \to \mathcal{B}_{\mathbf{D}_\rho}$ be the restriction operator (c.f. (4.1)). The sequence of maps $\{I_\rho \circ F_n \circ \gamma\}_n$ converges uniformly on compact sets to $I_\rho \circ F \circ \gamma$. By Lemma 7.4, each $I_\rho \circ F_n \circ \gamma$ is holomorphic in the usual Banach sense, so the limit $I_\rho \circ F \circ \gamma$ is holomorphic as well. By Lemma 7.4, $F \circ \gamma$ is path holomorphic. □

Property (H1) implies that the family $\mathcal{G}$ is uniformly almost periodic. If (C1) does not hold then by Lemma 8.6, there exists a sequence $G_k \in \mathcal{G}$ and $(m_k, n_k) \in Q$ with $n_k - m_k \to \infty$, such that $G_k^{m_k,n_k}$ converges uniformly on compact sets to a non-constant retraction $P \in \mathcal{S}_{\mathcal{H}_0}$. By Lemma 8.7, P is path holomorphic, concluding the proof. □

9. Horseshoe

9.1. *Beau bounds and rigidity yield the horseshoe.*

9.1.1. *Complex case.* — Let $\mathcal{F}$ be a family of disjoint little Multibrot sets $\mathcal{M}_k$ (encoding certain renormalization combinatorics). We say that *beau bounds are valid for* $\mathcal{F}$, if they are valid for the family of infinitely renormalizable maps f whose renormalizations $R^n f$ have combinatorics $\bar{\mathcal{M}} = (\mathcal{M}_n)_{n=0}^\infty$ with $\mathcal{M}_n \equiv \mathcal{M}_{k_n} \in \mathcal{F}$. (We will loosely say that "$\bar{\mathcal{M}}$ is in $\mathcal{F}$".)

We say that $\mathcal{F}$ *is rigid* if for any combinatorics $\bar{\mathcal{M}}$ in $\mathcal{F}$, there exists a unique polynomial $p_c : z \mapsto z^d + c$ which is infinitely renormalizable with this combinatorics.[15]

Remark **9.1.** — It is conjectured that any family $\mathcal{F}$ is in fact rigid (which would imply that the Multibrot set is locally connected at all infinitely renormalizable parameter values, and hence would prove MLC, for all unicritical families).

In [Ch], it is shown that beau bounds implies rigidity for a large class of combinatorics, and in fact for all combinatorics for which beau bounds have been proved [K, KL1, KL2].

Remark **9.2.** — In the quadratic case, this rigidity result had been established in [L2]. The work [Ch] makes use of recent advances: a new version of the *Pullback argument* developed in [AKLS].

A *semi-conjugacy* between two dynamical systems $F : X \to X$ and $G : Y \to Y$ is a continuous *surjection* $h : X \to Y$ such that $h \circ F = G \circ h$.

Let $\Sigma \equiv \Sigma_{\mathcal{F}} = \mathcal{F}^{\mathbf{Z}}$ be the symbolic space with symbols from $\mathcal{F}$, and let $\sigma : \Sigma \to \Sigma$ be the corresponding two-sided shift. A map $h : \Sigma \to \mathcal{C}$ is called *combinatorially faithful* if for any combinatorics $\bar{\mathcal{M}} = (\mathcal{M}_n)_{n=-\infty}^{\infty} \in \Sigma$, the image $f = h(\bar{\mathcal{M}})$ is renormalizable with combinatorics $\mathcal{M}_0$, and h semi-conjugates σ and $R|\mathcal{A}$.

Theorem **9.1.** — *Assume a family $\mathcal{F}$ has beau bounds and is rigid. Then there exists a precompact R-invariant set $\mathcal{A} \subset \mathcal{C}$ and a combinatorially faithful semi-conjugacy $h : \Sigma \to \mathcal{A}$. Moreover, R is exponentially contracting along the leaves of the hybrid lamination of $\mathcal{A}$, with respect to the Carathéodory metric.*[16]

Proof. — Since $\mathcal{F}$ has beau bounds, there exists $\epsilon > 0$ such that

$$\operatorname{mod} R^n p \geq \epsilon, \quad n = 0, 1, \ldots$$

for any polynomial $p : z \mapsto z^d + c$ which is infinitely renormalizable with combinatorics $\mathcal{F}$.

Let $\overline{\mathcal{M}} = (\mathcal{M}_n)_{n\in\mathbf{Z}} \in \Sigma$. For any $-n \in \mathbf{Z}_-$, there exists a polynomial p_{c_n} which is infinitely renormalizable with combinatorics $(\mathcal{M}_{-n}, \mathcal{M}_{-n+1}, \ldots)$. Then for any $l \in \mathbf{Z}$, $l \geq -n$, the germ $f_{n,l} := R^{n+l} p_{c_{-n}} \in \mathcal{C}(\epsilon)$ is infinitely renormalizable with combinatorics $(\mathcal{M}_l, \mathcal{M}_{l+1}, \ldots)$. As this family of germs is precompact, for any l we can select a subsequence $f_{n(i),l}$ converging to some $f_l \in \mathcal{C}(\epsilon)$ as $n(i) \to -\infty$. This map is infinitely renormalizable with combinatorics $(\mathcal{M}_l, \mathcal{M}_{l+1}, \ldots)$.[17] Using the diagonal procedure (going

[15] Note that according to our definition (see Sect. 2.10) affinely conjugated renormalizable maps are distinguished by the combinatorics of the first renormalization.

[16] It follows from rigidity that the hybrid lamination of $\mathcal{A}$ consists of all infinitely renormalizable maps whose renormalizations have combinatorics in $\mathcal{F}$.

[17] This actually needs a little check-up: compare the argument four paragraphs down.

backwards in l) we ensure that $Rf_{l-1} = f_l$. Thus, we obtain a bi-infinite sequence of maps $f_l \in \mathcal{C}(\epsilon)$ such that f_l is renormalizable with combinatorics $\mathcal{M}_l$ and $Rf_{l-1} = f_l$.

Assume there exist two such sequences, $(f_l)_{l\in\mathbf{Z}}$ and $(\tilde{f}_l)_{l\in\mathbf{Z}}$. Since $\mathcal{F}$ is rigid, the hybrid class of any f_l is uniquely determined by the renormalization combinatorics $(\mathcal{M}_l, \mathcal{M}_{l+1}, \dots)$, so for any $l \in \mathbf{Z}$, the germs f_l and $\tilde{f}_l$ are hybrid equivalent. But by Theorem 5.1, the renormalization is exponentially contracting with respect to the Carathéodory metric in the hybrid lamination. Hence there exist $C > 0$ and $\lambda \in (0, 1)$ such that

$$d_{\chi(f_0)}(f_0, \tilde{f}_0) \le C(\epsilon)\lambda^n d_{\chi(f_n)}(f_{-n}, \tilde{f}_{-n}). \tag{9.1}$$

Letting $n \to \infty$ we see that $f_0 = \tilde{f}_0$. For the same reason, $f_l = \tilde{f}_l$ for any $l \in \mathbf{Z}$.

Thus, we obtain a well defined equivariant map $h : \overline{\mathcal{M}} \mapsto f$, where $f \equiv f_0$ is a polynomial-like germ for which there exists a bi-infinite sequence $f_l \in \mathcal{C}(\epsilon)$, $l \in \mathbf{Z}$, such that f_l is renormalizable with combinatorics $\mathcal{M}_l$. Let $\mathcal{A} \subset \mathcal{C}$ consist of all such germs, which makes h surjective by definition.

To see that h is continuous, consider a sequence $\overline{\mathcal{M}}^{(k)} \to \overline{\mathcal{M}}$ in Σ, and let $f_l^{(k)} = h(\sigma^l(\overline{\mathcal{M}}^{(k)}))$. We need to show that $h(\overline{\mathcal{M}}^{(k)}) \to h(\overline{\mathcal{M}})$. By passing through an arbitrary subsequence, we may assume that for each $l \in \mathbf{Z}$, $f_l^{(k)}$ converges in $\mathcal{C}(\epsilon)$ to some f_l. By definition of convergence in Σ, for each $l \in \mathbf{Z}$ and for each k sufficiently large, $f_l^{(k)}$ is renormalizable with combinatorics in $\mathcal{M}_l$, i.e., $\chi(f_l^{(k)}) \in \mathcal{M}_l$.

Let us show that f_l is renormalizable with combinatorics $\mathcal{M}_l$. If $\mathcal{M}_l$ is a primitive copy, then it is closed, which readily implies that $\chi(f_l) \in \mathcal{M}_l$. If $\mathcal{M}_l$ is a satellite copy, its closure is obtained by adding the root, so we also need to guarantee that $\chi(f_l^{(k)})$ does not converge to the root. But for k large, $\chi(f_l^{(k)})$ belongs to a subcopy of $\mathcal{M}_l$ (consisting of those polynomials in $\mathcal{M}_l$ whose renormalization has combinatorics $\mathcal{M}_{l+1}$), which is at definite distance from the root of $\mathcal{M}_l$, so we can again conclude that $\chi(f_l) \in \mathcal{M}_l$.

By continuity of any renormalization operator with fixed combinatorics, we also conclude $Rf_l = f_{l+1}$. By the definition of h, $h(\overline{\mathcal{M}}) = f_0 = \lim f_0^{(k)} = \lim h(\overline{\mathcal{M}}^{(k)})$, as desired. □

9.1.2. *Real horseshoe.* — Let $\mathcal{F}^{\mathbf{R}}$ stand for the family of all real renormalization combinatorics with minimal periods. Let $\sigma : \Sigma \to \Sigma$ be the corresponding shift. While neither beau bounds, nor (complex) rigidity have been established for $\mathcal{F}^{\mathbf{R}}$,[18] the proven beau bounds and rigidity for *real-symmetric germs* is enough to construct the renormalization horseshoe. Moreover, we will show that in this case, the combinatorially faithful semi-conjugacy h is actually a homeomorphism:

[18] They have been established, however, for the family of primitive real combinatorics [KL1], which covers all real combinatorics except period doubling.

Theorem **9.2.** — *For the family $\mathcal{F}^{\mathbf{R}}$, there is an* **R**-*invariant set $\mathcal{A} \subset \mathcal{C}^{\mathbf{R}}$ and a combinatorially faithful homeomorphism $h : \Sigma \to \mathcal{A}$. Moreover,* R *is exponentially contracting along the leaves of the hybrid lamination of $\mathcal{A}$ (endowed with the Carathéodory metric), which contains all infinitely renormalizable real-symmetric germs.*

Proof. — The construction of the horseshoe $\mathcal{A}$, along with the combinatorially faithful semi-conjugacy h, is basically the same as in Theorem 9.1, with the following adjustments:

(1) The polynomials p_{c_n} should be selected to be real, $c_n \in \mathbf{R}$;
(2) Theorem 6.1 (*beau bounds for real maps*) provides the needed compactness for the construction of the maps f_l;
(3) Rigidity of $\mathcal{F}$ is replaced with *rigidity for real polynomials*, obtained in [L2, GS] (quadratic case) and in [KSS] (arbitrary degree)[19]: any real renormalization combinatorics determines a single *real-symmetric* hybrid leaf;
(4) *Exponential contraction* is obtained by combining Theorems 6.2 and 5.1.

The injectivity of h follows from the injectivity of the renormalization operator acting on real p-l maps [MvS, p. 440]. What is left, is to verify continuity of h^{-1}. Since convergence in Σ means coordinatewise convergence, it is equivalent to the following statement:

Lemma **9.3.** — *Let $(\bar{\mathcal{M}}^{(j)})_{j=1}^{\infty}$ be a sequence of symbolic strings $\bar{\mathcal{M}}^{(j)} = (\mathcal{M}_n^{(j)})_{n\in\mathbf{Z}}$ in Σ such that the corresponding germs $f_j \equiv h(\bar{\mathcal{M}}^{(j)})$ converge to some $f_\infty \in \mathcal{A}$. Then for any $n \in \mathbf{Z}$, the combinatorics $\mathcal{M}_n^{(j)}$ of $\mathrm{R}^n f_j$ eventually coincides with that of $\mathrm{R}^n(f_\infty)$.*

Notice that Lemma 9.3 is clear for $n = 0$: a real perturbation of a twice renormalizable real map is renormalizable (at least once) with the same combinatorics (since on the boundary of the renormalization windows the maps are not twice renormalizable, see Section 2.10). Since the renormalization operator acts continuously on $\mathcal{A}$, we have for any $n \geq 0$ that $\mathrm{R}^n f_j \to \mathrm{R}^n f_\infty$ as $j \to \infty$. It follows that Lemma 9.3 holds for all $n \geq 0$ as well.

In order to prove it inductively for $n < 0$, it is enough to show that the hypothesis of Lemma 9.3 imply that $\mathrm{R}^{-1} f_j \to \mathrm{R}^{-1} f_\infty$. To this end, it is sufficient to prove that the *renormalization combinatorics of the germs* $\mathrm{R}^{-1} f_j$ *are bounded*. Indeed, in this case any limit g of these germs is renormalizable and $\mathrm{R}g = f_\infty$. By injectivity of the renormalization operator, $g = \mathrm{R}^{-1} f_\infty$, and the conclusion follows.

Boundedness of the renormalization combinatorics follows from an analysis of the domain of analyticity of limits of renormalized germs:

[19] This result also follows from the combination of [AKLS] (dealing with at most finitely renormalizable maps) and [Ch] (dealing with the infinitely renormalizable situation).

Lemma **9.4.** — *Let $\tilde{f}_j \in \mathcal{C}^{\mathbf{R}}$, $j \geq 1$, be a sequence of renormalizable germs. If the renormalization periods of the $\tilde{f}_j$ go to infinity, then any limit of the renormalizations $R\tilde{f}_j$ is either a unicritical polynomial or its real trace has a bounded domain of analyticity.*

See Appendix A for a proof (unlike the previous parts of this paper, it relies on the fine combinatorial and geometric structure of *one* renormalization).

Let us apply Lemma 9.4 to $\tilde{f}_j = R^{-1} f_j$. Notice that $f_\infty = \lim R\tilde{f}_j$ cannot be a unicritical polynomial since those are never anti-renormalizable. Moreover, since f_∞ is *infinitely* anti-renormalizable with *a priori* bounds, it is well known (see [McM2], Sect. 7.3) that its real trace extends analytically to $\mathbf{R}$: If f_∞ is the n-th pre-renormalization of $f_{(n)}$, then f_∞ extends explicitly to $K(f_{(n)}) \cap \mathbf{R}$ (as an appropriate iterate of $f_{(n)}$), and $\bigcup_n K(f_{(n)})$ contains $\mathbf{R}$, as diam $K(f_{(n)})$ is growing exponentially with n (by Lemma 2.10), while the distance from the critical point to the boundary of $K(f_{(n)}) \cap \mathbf{R}$ remains comparable with $\operatorname{diam}(K(f_{(n)}))$ (since, up to normalization, $f_{(n)}$ belongs to a compact subset of $\mathcal{C}^{\mathbf{R}}$). So, both options offered by Lemma 9.4 are impossible in our situation, and hence the renormalization periods of the germs $\tilde{f}_j$ must be bounded. This concludes the proof of Lemma 9.3, and thus of Theorem 9.2. □

Acknowledgements

The first author would like to thank the hospitality of the University of Toronto, the Fields Institute and the Stony Brook University. This research was partially conducted during the period the first author served as a Clay Research Fellow. The second author was partially supported by the NSF and NSERC.

Appendix A: Analytic continuation of the first renormalization

A.1 *Principal nest and scaling factors.* — Through this section, we consider a renormalizable unimodal map $f : I \to I$, of period p, with a polynomial-like extension in $\mathcal{C}^{\mathbf{R}}(\epsilon_0)$ for some fixed $\epsilon_0 > 0$. For simplicity of notation, we will also assume that f is even. We will also assume that we can write $f(x) = \psi(x^d)$ for some diffeomorphism ψ with non-positive Schwarzian derivative. All arguments below can be carried out without the extra assumptions with only technical changes, but in the situation arising in our application (f is infinitely anti-renormalizable with *a priori* bounds) they are indeed automatically satisfied.

Below $C > 1$ stands for a constant which may only depend on ϵ_0.

Recall that a closed interval $T \subset I$ which is symmetric (i.e., $f(\partial T)$ is a single point) is called *nice* if $f^k(\partial T) \cap \operatorname{int} T = \emptyset$, $k \geq 1$. If the critical point returns to the interior of a nice interval T, then we let T' be the central component of the first return map to T. We let $\lambda(T) = |T'|/|T|$ be the *scaling factor*.

We define the *principal nest* I_n, $n \geq 0$ as follows. Since f is renormalizable, it has a unique orientation reversing fixed point ρ. Its preimage $\{\rho, -\rho\}$ bounds a nice interval, which we denote I_0. Then we define I_{n+1} inductively as I'_n, i.e., I_{n+1} is the central component of the first return map to I_n.

Let us assume from now on that $p > 2$. Under this condition, $I_{n+1} \Subset \operatorname{int} I_n$ for every $n \geq 0$. Let $\lambda_n = |I_{n+1}|/|I_n|$ be the corresponding scaling factors.

Let g_n be the first return map to I_n. We say that g_n is *central* if $g_n(0) \in I_{n+1}$. We define a sequence $(j_k)_{k\geq 0}$ inductively so that $j_0 = 0$ and j_{k+1} is the minimum $n > j_k$ such that g_{n-1} is non-central. Since f is renormalizable, g_n is central for all sufficiently large n, so the sequence (j_k) terminates at some $N = j_\kappa$. We call κ the *height* of f.

We have the following basic estimates on the scaling factors (see [Ma1]):

A priori bounds. — We have: $\lambda_{j_k} \leq 1 - C^{-1}$ *for every* k. *Moreover. the maps* $g_{j_k} : I_{j_{k+1}} \to I_{j_k}$ *are compositions of power maps* $x \mapsto x^d$ *and diffeomorphisms with bounded distortion.*

Corollary **A.1.** —
$\lambda_{n+1} \leq C\lambda_n^{1/d}$,
$\lambda_{j_{k+1}} \leq C \prod_{n=j_k}^{j_{k+1}-1} \lambda_n^{1/d}$.

Let v_n be the *principal return times*, i.e. $f^{v_n}|I_{n+1} = g_n$. In particular, $v_N = p$ is the renormalization period. We let $g = f^p$. Then $g : J \to J$ is the *unimodal pre-renormalization* of f, where $J = \bigcap_{n\geq 0} I_n$.

We say that $n \geq 0$ is *admissible* if $f^n(0) \in I_0$. For admissible n, let T_n be the closure of the connected component of $f^{-n}(\operatorname{int} I_0)$ containing 0. In particular $T_0 = I_0$. More generally, letting $w_n = \sum_{k=0}^{n-1} v_k$, we obtain $T_{w_k} = I_k$.

Lemma **A.2.** — *We have* $T_p \subset I_{\max\{0,N-1\}}$.

Proof. — If $\kappa = 0$ then clearly $T_p = I_1 \subset I_0$.

Assume that $\kappa \geq 1$ and hence $N \geq 1$. The interval T_p is the smallest interval containing 0 whose boundary is taken by f^p to the boundary of I_0. Hence it is enough to show that $f^p(\partial I_{N-1}) \subset \partial I_0$.

It is easy to see that for $k \geq 1$ we have

$$f^{sv_{j_{k-1}}}(0) \in I_{j_k-s} \setminus I_{j_k+1-s} \quad \text{for } 1 \leq s \leq j_k - j_{k-1}.$$

We conclude that

$$v_{j_k} \geq v_{j_{k-1}-1} + (j_k - j_{k-1})v_{j_{k-1}} = \sum_{n=j_{k-1}-1}^{j_k-1} v_n = v_{j_{k-1}} + \sum_{n=j_{k-1}-1}^{j_k-2} v_n, \quad \text{for } j_k \geq 2,$$

which implies inductively that

$$v_{j_k} \geq \sum_{n=0}^{j_k-2} v_n \quad \text{for } j_k \geq 2. \tag{A.1}$$

Letting $k = \kappa$ (so that $j_\kappa = N$ and $v_N = p$) we obtain:

$$t := p - \sum_{n=0}^{N-2} v_n \geq 0.$$

Since $f^{v_n}(\partial I_{n+1}) \subset \partial I_n$, it follows that $f^p(\partial I_{N-1}) \subset f^t(\partial I_0) \subset \partial I_0$. □

A.2 *Transition maps.* — The geometric considerations made below are all contained in [L1], though we do not need the finest part of that argument, dealing with growth of geometry for Fibonacci-like cascades (with or without saddle-node subcascades), which is not valid in higher degree anyway.

If n is admissible we let $A_n : T_n \to \mathbf{I}$ be the orientation preserving affine homeomorphism, where $\mathbf{I} = [-1, 1]$.

We say that T_n is a *pullback* of T_m if $n > m$ and $f^{n-m}(0) \in T_m$. In this case, f^{n-m} restricts to a map $(T_n, \partial T_n) \to (T_m, \partial T_m)$, and we let $G_{n,m} = A_m \circ f^{n-m} \circ A_n^{-1}$, which we call a *transition map*.

We say that T_n is a *kid* of T_m if T_n is a pullback of T_m but is not a pullback of any T_k with $m < k < n$. Notice that in this case, $f^{n-m-1}|f(T_n)$ extends to an analytic diffeomorphism onto T_m. If T_n is a kid of T_m, the transition map $G_{n,m}$ is called *short*, otherwise it is called *long*.

A short transition map $G_{n,m}$ is called δ*-good* if $f^{n-m-1}|f(T_n)$ extends to an analytic diffeomorphism onto a $\delta|T_m|$-neighborhood of T_m. The usual Koebe space argument (see [MvS]) yields:

Lemma **A.3.** — *For every $\delta > 0$, any δ-good transition map belongs to a compact set $\mathcal{K} = \mathcal{K}(\delta, \epsilon_0) \subset C^\omega(\mathbf{I}, \mathbf{I})$, only depending on ϵ_0 and δ.*

Here C^ω stands for the space of analytic maps, with the usual inductive limit topology.

There is a unique *canonical decomposition* of a long transition map into short transition maps: letting $m = n_1 < \cdots < n_l = n$ be the sequence of moments such that $f^{n-n_j}(0) \in T_{n_j}$, then $T_{n_{j+1}}$ is a kid of T_{n_j} for $0 \leq j \leq l-1$ and $G_{n,m} = G_{n_2,n_1} \circ \cdots \circ G_{n_l,n_{l-1}}$.

A *central cascade* is a sequence $T_{n_1}, \ldots, T_{n_l}$ such that $T_{n_{j+1}}$ is the first kid of T_{n_j} and $f^{n_{j+1}-n_j}(0) \in T_{n_{j+1}}$ for $1 \leq j \leq l-1$. Notice that in this case $n_{j+1} - n_j$ is independent of $j \in [1, l-1]$. If $n_2 - n_1 < p$ then we distinguish the *saddle-node* and *Ulam-Neumann* types of central cascades according to whether $0 \notin f^{n_2-n_1}(T_{n_2})$ or $0 \in f^{n_2-n_1}(T_{n_2})$.

A long transition map is called saddle-node/Ulam-Neumann if its canonical decomposition $G_{n_2,n_1} \circ \cdots \circ G_{n_l,n_{l-1}}$ is such that $T_{n_1}, \ldots, T_{n_l}$ is a saddle-node/Ulam-Neumann cascade.

We say that a long transition map G_{n_l,n_1} is δ-good if all the components G_{n_{j+1},n_j} of its canonical decomposition are δ-good. Notice that if G_{n_l,n_1} is central then this is equivalent to δ-goodness of the top level G_{n_2,n_1}.

Besides δ-goodness, an important role is also played by two parameters associated to a long transition map of saddle-node type: the scaling factors of the top and bottom levels, $\lambda_{\mathrm{top}} = |T_{n_2}|/|T_{n_1}|$ and $\lambda_{\mathrm{bot}} = |T_{n_l}|/|T_{n_{l-1}}|$.

Lemma **A.4.** — *For every* $0 < \underline{\lambda} < \overline{\lambda} < 1$, $\delta > 0$, *any* δ*-good long transition maps of saddle-node type with parameters* $\lambda_{\mathrm{bot}}, \lambda_{\mathrm{top}} \in [\underline{\lambda}, \overline{\lambda}]$ *belongs to a compact set* $\mathcal{K}(\underline{\lambda}, \overline{\lambda}, \delta, \epsilon_0) \subset C^\omega(\mathbf{I}, \mathbf{I})$.

Proof. — For such a transition map $G_{n,m}$, let us consider a maximal saddle-node cascade $T_{n_1}, \ldots, T_{n_L}$ such that $m = n_1$ and $n = n_l$ for some $l \leq L$.

By Lemma A.3, we only risk losing compactness when l, and hence L, is large, which is related to the presence of a *nearly parabolic fixed point* in T_{n_2} for $F = f^{n_2 - n_1}$ (since there is in fact no fixed point, the terminology means that a parabolic fixed point appears after a small perturbation of F). In this case we have the basic geometric estimate, due to Yoccoz:

$$\text{(A.2)} \qquad \frac{|T_{n_i}|}{|T_{n_{i+1}}|} - 1 \sim \max\{i, L - i\}^{-2}, \quad 1 \leq i \leq L - 1$$

(the implied constants depending on the bounds $\underline{\lambda}, \overline{\lambda}$ on scaling factors). See [FM], Section 4.1, for a discussion of almost parabolic dynamics and the statement of Yoccoz's Lemma.

In particular, either l is bounded (and we are fine) or $L - l$ is bounded. Assuming that $l \geq 4$, $F^3(T_{n_l})$ is contained in a connected component J of $T_{n_{l-3}} \setminus T_{n_{L-1}}$. Since Lemma A.3 provides bounds on $F^3|T_{n_l}$, we just have to show that $F^{l-4}|J$ is under control. But the map F^{l-4} maps J onto a connected component of $T_{n_1} \setminus T_{n_{L-l+3}}$, and extends analytically to a diffeomorphism onto a connected component J' of $T^\delta_{n_1} \setminus T_{n_{L-l+4}}$, where $T^\delta_{n_1}$ is a $\delta|T_{n_1}|$-neighborhood of T_{n_1}. By (A.2), J' is a $\delta'|F^{l-4}(J)|$-neighborhood of $F^{l-4}(J)$ for some $\delta' > 0$, so $F^{l-4}|J$ is under Koebe control. □

A.3 *Small scaling factors.* — In [L1], several combinatorial properties are shown to yield small scaling factors. We will need somewhat simpler estimates, which we will obtain from the following:

Lemma **A.5.** — *For every* $\epsilon > 0$, *there exists* $\delta = \delta(\epsilon, \epsilon_0) > 0$ *with the following property. Assume that the postcritical set intersects a connected component* D *of the first landing map to* $I_{j_{k+1}}$ *such that* $D \subset I_{j_k} \setminus I_{j_k+1}$ *and* $|D|/|I_{j_k}| < \delta$. *Then* $|I_{j_\kappa+1}|/|I_{j_k}| < \epsilon$.

Proof. — We may assume that λ_{j_k} is not small and that $\kappa - k$ is bounded. Let $r > 0$ be minimal such that $f^r(0) \in D$. For definiteness, we will discuss in detail only the case $k > 0$, and will then indicate how to treat the simpler case $k = 0$.

We will use the well known combinatorial fact that each branch of the first landing map to I_{j_k} extends analytically to a diffeomorphism onto $I_{j_{k-1}}$. Notice that if $f^s : U \to I_{j_k}$ is a branch of the first return map to I_{j_k} and U' is the connected component of $f^{-s}(I_{j_{k-1}})$ containing U, then $U' \subset I_{j_k}$. In this setting we get, in particular:

- If $V \subset I_{j_k}$ is an interval much smaller than I_{j_k}, then the connected component of $f^{-s}(V)$ contained in U is much smaller than U', and hence much smaller than I_{j_k}.

In order to deal with central cascades, we will use a variation of this estimate:

- If $1 \le l \le j_{k+1} - j_k - 2$, $V \subset I_{j_k+1} \setminus I_{j_k+2}$ is an interval much smaller than I_{j_k} and V' is a connected component of $f^{-lv_{j_k}}(V)$ contained in I_{j_k+l}, then $|V'|$ is much smaller than $|I_{j_k}|$.

Indeed, $f^{lv_{j_k}}|V'$ extends analytically to a diffeomorphism from a connected component L' of $I_{j_k+l} \setminus I_{j_k+l+2}$ onto a connected component L of $I_{j_k} \setminus I_{j_k+2}$. Then V is small compared to both connected components of $L \setminus V$, so that $|V'|$ is small compared to $|L'| \le |I_{j_k}|$.

We now proceed by considering two distinct cases. Assume first $r < v_{j_{k+1}}$, i.e., orb 0 lands in D before landing in $I_{j_{k+1}}$.

Let $0 < r_1 < \cdots < r_u = r$ be the successive landing times of orb 0 in $I_{j_k} \setminus I_{j_{k+1}}$ up to the first entry in D. If u is bounded, we can use repeatedly the above estimates (a bounded number of times) to conclude that the connected component of $f^{-r}(D)$ containing 0, is much smaller than I_{j_k}. This concludes, since under the hypothesis $r < v_{j_{k+1}}$, this connected component is just $I_{j_{k+1}+1}$. If u is large, let U_i, $1 \le i \le u$, be the component of the domain of the first return map to I_{j_k} containing $f^{r_i}(0)$.

Denoting by $\tilde{U}_i$ the connected component of $I_{j_k} \setminus I_{j_k+1}$ containing U_i, we claim that U_i has definite Koebe space inside $\tilde{U}_i$, i.e., each of the connected components of $\tilde{U}_i \setminus U_i$ are not much smaller than $|U_i|$. Indeed, let s_i be the return time of U_i to I_{j_k}, and let Z_i be the connected component of $f^{-s_i}(I_{j_{k-1}})$ containing U_i. Let Z be the connected component of $f^{-v_{j_k}}(I_{j_{k-1}})$ containing I_{j_k+1}. Then Z_i and Z are contained in I_{j_k}, U_i has definite Koebe space in Z_i and I_{j_k+1} has definite Koebe space in Z. In particular, the distance from U_i to ∂I_{j_k} is not too small compared with $|U_i|$. If Z_i does not intersect I_{j_k+1} then we can also conclude that the distance from U_i to ∂I_{j_k+1} is not too small compared with $|U_i|$. But if Z_i intersects Z then either $Z_i \subset Z \setminus \operatorname{int} I_{j_k+1}$ or $Z \subset Z_i \setminus \operatorname{int} U_i$ (according to whether $s_i > v_{j_k}$ or $s_i < v_{j_k}$).[20] In particular, if Z_i intersects I_{j_k+1} then Z does not intersect U_i. Since I_{j_k+1} has

[20] Recall that there are intervals $Z'_i \supset f(Z_i)$ and $Z' \supset f(Z)$ such that $f^{s_i-1} : Z'_i \to I_{j_{k-1}}$ and $f^{v_{j_k}-1} : Z' \to I_{j_{k-1}}$ are diffeomorphisms. Assume, say, that $s_i < v_{j_k}$, the other case being analogous (by the assumption that Z_i intersects Z, $v_{j_k} = s_i$ would imply $Z_i = Z$ and hence $U_i = I_{j_k+1}$ which is impossible). Since $I_{j_{k-1}}$ is nice, $Z' \subset Z'_i$, and we need to show that

definite Koebe space in Z, we conclude that the distance from U_i to ∂I_{j_k+1} is not too small compared with $|I_{j_k+1}|$, which is comparable to $|I_{j_k}| \geq |U_i|$. This concludes the claim.

Let $D_i \subset U_i$ be the component of the domain of the first landing map to $I_{j_{k+1}}$ containing $f^{r_i}(0)$. Then $f^{r_{i+1}-r_i} : D_i \to D_{i+1}$ extends to a diffeomorphism onto $\tilde{U}_{i+1}$ for $1 \leq i \leq u-1$. Let $\tilde{D}_j$ be the connected component of $f^{r_1-r_j}(\tilde{U}_j)$ containing D_1. Then $D_1 \subset \tilde{D}_u$, $\tilde{D}_j \supset \tilde{D}_{j+1}$ for $1 \leq j \leq u-1$, and $\tilde{D}_1 = \tilde{U}_1$. Moreover, for $1 \leq j \leq u-1$, $f^{r_j-r_1} : \tilde{D}_j \to \tilde{U}_j$ is a diffeomorphism taking $\tilde{U}_{j+1}$ inside U_j, so that $\tilde{D}_{j+1}$ has definite Koebe space inside $\tilde{D}_j$. It follows that $|D_1|/|I_{j_k}|$ is exponentially small in u. Since u is large, $|D_1|$ is indeed small and we can then apply the previous case, replacing D by D_1 (which replaces u by 1), to conclude.

Assume now that $r > v_{j_{k+1}}$, i.e., orb 0 lands in $I_{j_{k+1}}$ before landing in D. Let $s \in (0, r)$ be its last landing moment in $I_{j_{k+1}}$ before landing in D. Then

$$f^s(0) \in I_{j_{k+1}} \setminus I_{j_{k+1}+1}, \quad \text{for otherwise} \quad f^{r-s}(0) \in f^{r-s}(I_{j_{k+1}+1}) \subset D,$$

contradicting the minimality of r.

Let Δ be the pullback of D under f^{r-s} containing $f^s(0)$ (which is the component of the domain of the first return map to $I_{j_{k+1}}$ containing $f^s(0)$). Applying the previous argument, one sees that $|\Delta|$ is much smaller than $|I_{j_k}|$, and hence much smaller than $|I_{j_{k+1}}|$. All the more, the component Δ' of the domain of the first landing map to $I_{j_{k+2}}$ containing $f^s(0)$ is small compared with $I_{j_{k+1}}$. We can now start the procedure over with k replaced by $k+1$ and D replaced by Δ'. Since $\kappa - k$ is assumed to be bounded, this process must eventually produce a small scaling factor.

Let us now sketch how the argument can be modified to handle the case $k = 0$. Notice first that we can assume that v_0 is bounded (otherwise I_1 is readily seen to be small). Any branch of the first landing map to I_0 extends to a diffeomorphism onto the connected component I' of $I \setminus \{f(0)\}$ containing 0, and I_0 has definite Koebe space in I', which gives some distortion control.

Assume first $r < v_{j_1}$. If D has lots of Koebe space inside the connected component of $I_0 \setminus I_1$ containing it, we just pullback the Koebe space. Otherwise, D must be close to either ∂I_0 or ∂I_1. Let $0 < r_1 < \cdots < r_u = r$ and D_j, U_j, s_j, $1 \leq j \leq u$, be defined as before. If $r = r_1$, we can use the distortion control to show that the connected component of $f^{-r}(D)$ containing 0, i.e., I_{j_1+1}, is small, which allows us to conclude. If $u > 1$, the distortion control allows one to show that D_{u-1} and $f^{s_{u-1}}(D_{u-1})$ are small. If $f^{s_{u-1}}(D_{u-1})$ is contained in I_1, then this implies that D_{u-1} has lots of Koebe space in U_{u-1}, which allows us to conclude. Otherwise, we can still get that D_{u-1} has lots of Koebe space in U_{u-1} unless D is close to ∂I_0. If D_{u-1} is not close to ∂I_0, we conclude then by repeating the argument with D_{u-1} instead of D. If D_{u-1} is also close to ∂I_0, then U_{u-1} is a connected component

---

$f^{s_i-1}(Z')$ does not intersect int I_{j_k}. If it did intersect, then $0 \in I_{j_k} \subset f^{s_i-1}(Z')$ (since both $f^{s_i-1}(Z')$ and I_{j_k} are connected components of preimages of the nice interval $I_{j_{k-1}}$), so that $f^{v_{j_k}-1}|Z'$ is not a diffeomorphism, contradiction.

of the domain of the first return map to I_0 which is adjacent to one of the boundary points of I_0 (i.e., $s_{u-1} = 2$). This shows that if D_{u-1} is close to ∂I_0, then it must be even closer to ∂I_0 then D was. Moreover D_{u-1} is not bigger than the (small) distance of D to ∂I_0. Then either the sequence $D_u, \ldots, D_1$ gets always closer to ∂I_0, and we can conclude by replacing D with D_1, or at some point we get some small D_j away from ∂I_0, and we can conclude by replacing D with D_j.

If $r > v_{j_1}$, we conclude (as in the original argument) that there is a connected component $\Delta' \subset I_{j_1} \setminus I_{j_1+1}$ of the domain of the first landing map to I_{j_2} which intersects the postcritical set and is much smaller than I_{j_1}. Replacing D by Δ', we get in the case $k > 0$ and the result follows. □

One important situation in our analysis corresponds to the critical orbit hitting deep inside a long central cascade.

More precisely, we say that T_m is *k-deep* ($k \geq 2$) inside a central cascade if there is a central cascade $T_{n_1}, \ldots, T_{n_l=m}$,$T_{n_{l+1}} = T'_m, \ldots, T_{n_L}$ such that $k \leq l \leq L - k$. We say that the critical orbit *hits* T_m if there is $r \geq 0$ such that $f^r(0) \notin T'_m$, but $f^{r+n_2-n_1}(0) \in T_m \setminus T'_m$.

Lemma **A.6.** — *For every* $\epsilon > 0$ *there exists* $k = k(\epsilon, \epsilon_0) > 0$ *with the following property. Assume that the critical orbit hits some* T_m *which is k-deep inside a central cascade. Then* $|I_{j_\kappa+1}|/|T_m| < \epsilon$.

Proof. — It is no loss of generality to assume that $T_{n_1} = I_{j_i}$ for some i (since between any interval T_n and its kid T'_n, there must be an interval of the principal nest).

We may assume that λ_{j_i} is not small. Let D′ be the component of the domain of the first landing map to $I_{j_{i+1}}$ containing $f^{r+n_2-n_1}(0)$. Then $|D'|/|I_{j_i}|$ is small, see (A.2). Let $s < r + n_2 - n_1$ be maximal with $f^s(0) \in I_{j_i} \setminus I_{j_i+1}$. Let D be the component of the domain of the first landing map to $I_{j_{i+1}}$ containing $f^s(0)$. Pulling back the Koebe space, we get $|D|/|I_{j_i}|$ small. The result follows from Lemma A.5. □

A simpler situation involves long Ulam-Neumann cascades:

Lemma **A.7.** — *For every* $\epsilon > 0$ *there exists* $k = k(\epsilon, \epsilon_0) > 0$ *with the following property. Let* $T_{n_1}, \ldots, T_{n_k}$ *be a central cascade of Ulam-Neumann type. Then there exists* $I_t \subset T_{n_k}$ *such that* $\lambda_t < \epsilon$.

Proof. — As in Lemma A.6, we may assume that $T_{n_1} = I_{j_i}$. Due to the long Ulam-Neumann cascade, $\lambda_{j_{i+1}-1}$ is close to 1, see [L2, Lemma 8.3]. In particular, the domain D of the first landing map to $I_{j_{i+1}}$ containing $f^{v_{j_i}}(0)$ has lots of Koebe space in I_{j_i}. Pulling back by $f^{v_{j_i}}$, we conclude that $|I_{j_{i+1}+1}|/|I_{j_i+1}|$ is small, which implies that $\lambda_{j_{i+1}}$ is small. □

We will also need the following easy criterion. Let us say that T_n is δ-safe if the postcritical set does not intersect a $\delta|T_n|$-neighborhood of ∂T_n. Notice that if T_n is δ-safe and T_m is a kid of T_n then $G_{m,n}$ is δ-good.

Lemma **A.8.** — *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 \leq k \leq \kappa$ is such that I_{j_k} is not δ-safe, then $|I_{j_\kappa+1}|/|I_{j_k}| < \epsilon$.*

Proof. — Consider first the case of "postcritical set inside", i.e., for some $r > 0$, $f^r(0) \in I_{j_k}$ is near ∂I_{j_k}. Let D be the component of the first landing map to $I_{j_{k+1}}$ containing $f^r(0)$. Since it has Koebe space inside I_{j_k} (the landing map $D \to I_{j_{k+1}}$ extends to a diffeomorphism $I_{j_k} \supset D' \to I_{j_k}$), $|D|/|I_{j_k}|$ is small, so the result follows from Lemma A.5.

Consider now the case of "postcritical set outside", i.e., for some $r > 0, f^r(0) \notin I_{j_k}$ is near ∂I_{j_k}. We may assume that $k > 0$ (otherwise applying f once produces "postcritical set inside" reducing to the previous case). We may assume further that λ_{j_k} is not small (otherwise the result is obvious), and that if $j_k - j_{k-1}$ is large then this is due to a saddle-node cascade (otherwise we can apply Lemma A.7). Letting $r' = r + (j_k - j_{k-1} - 1)v_{j_{k-1}}$, it follows that $f^{r'}(0)$ is just outside $\partial I_{j_{k-1}+1}$. Let D be the connected component of the first landing map to I_{j_k} containing $f^{r'}(0)$. Since it has Koebe space inside $I_{j_{k-1}} \setminus I_{j_{k-1}+1}$, this implies that $|D|/|I_{j_{k-1}}|$ is small, and the result follows from Lemma A.5. □

A.4 *Main precompactness.*

Lemma **A.9.** — *Let us consider a composition of transition maps $G_{m_2,m_1} \circ \cdots \circ G_{m_r,m_{r-1}}$, where each G_{m_{j+1},m_j} is either short or saddle-node. Assume that T_{m_1} is δ-safe, $\lambda(T_{m_1}) < 1 - \delta$ and $|T_{m_r}|/|T_{m_1}| > \delta$. Assume also that whenever G_{m_{j+1},m_j} is saddle-node of length at least l then $T_{m_{j+1}}$ is not l-deep inside a central cascade. Then there exists $\delta' = \delta(\epsilon_0, \delta, l, r) > 0$ and a compact subset $\mathcal{K} = \mathcal{K}(\epsilon_0, \delta, l, r) \subset C^\omega(\mathbf{I}, \mathbf{I})$ such that for $2 \leq j \leq r$ we have*

(1) *T_{m_j} is δ'-safe,*
(2) *$\lambda(T_{m_j}) < 1 - \delta'$,*
(3) *$G_{m_j,m_{j-1}} \in \mathcal{K}$.*

Proof. — Induction reduces considerations to the case $r = 2$.

Notice that if we show that G_{m_2,m_1} is in a compact class, it will follow that T_{m_2} is δ'-safe, as any postcritical set near ∂T_{m_2} would be taken by $f^{m_2-m_1}$ to postcritical set near ∂T_{m_1}. Moreover, it will also follow that $\lambda(T_{m_2}) < 1 - \delta'$: the map $f^{m_2-m_1}$ takes T'_{m_2} into a connected component of the first landing map to T'_{m_1} and any such component must have Koebe space inside T_{m_1} since $\lambda(T_{m_1}) < 1 - \delta$.

Thus we just have to show that G_{m_2,m_1} is in a compact class. Notice that G_{m_2,m_1} is δ-good. If G_{m_2,m_1} is short, the conclusion follows from Lemma A.3. If G_{m_2,m_1} is saddle-node, this will follow from Lemma A.4 once we show that λ_{top} and λ_{bot} are bounded away from 0 and 1. Clearly both are at least δ and moreover $\lambda_{top} = \lambda(T_{m_1}) < 1 - \delta$.

Let $T_{n_1=m_1}, \ldots, T_{n_s=m_2}, \ldots, T_{n_L}$ be the maximal central cascade starting at T_{n_1}. As in Lemma A.4, see (A.2), we see that if λ_{bot} is close to 1 then s and $L - s$ are large. But by hypothesis $\min\{s, L - s\} \leq l$, giving the result. □

The following two similar estimates will be proved simultaneously:

Lemma **A.10.** — *For* $\epsilon > 0$ *there exists a compact subset* $\mathcal{K} = \mathcal{K}(\epsilon, \epsilon_0) \subset C^\omega(\mathbf{I}, \mathbf{I})$ *with the following property. Assume that* $|I_{j_\kappa+1}|/|I_{j_k}| > \epsilon$. *Then* $G_{w_{j_k}+p, w_{j_k}} \in \mathcal{K}$.

Lemma **A.11.** — *For* $\epsilon > 0$ *and* $b_0 \in \mathbf{N}$, *there exists a compact subset* $\mathcal{K} = \mathcal{K}(\epsilon, \epsilon_0, b_0) \subset C^\omega(\mathbf{I}, \mathbf{I})$ *with the following property. Assume that* $|I_{j_\kappa+1}|/|I_0| > \epsilon$. *If* $0 \le b < \min\{p, b_0\}$ *is such that* $p - b$ *is admissible then* $G_{p-b,0} \in \mathcal{K}$.

A.4.1 *Proof of Lemmas A.10 and A.11.* — The proofs of both lemmas follow a basically parallel path. In both cases we need to estimate a map of type $G_{w_{j_k}+p-b, w_{j_k}}$, where in the setting of Lemma A.10 we set $b = 0$, while in the setting of Lemma A.11 we set $k = 0$. Write it as a composition of a minimal number of either short transition maps or long transition maps of Ulam-Neumann or saddle-node type. Clearly the number of elements of this decomposition is bounded in terms of $\kappa - k$, which in turn is bounded in terms of ϵ.

Let us now consider a finer decomposition $G_{m_2,m_1} \circ \cdots \circ G_{m_r,m_{r-1}}$, where we split the Ulam-Neumann pieces into short transition maps (so that each G_{m_{j+1},m_j} is either short or saddle-node). The Ulam-Neumann cascades have bounded length (by Lemma A.7), so r is also bounded.

In order to conclude, it is enough to show that the conditions of Lemma A.9 are satisfied. Since $T_{m_1} = I_{j_k}$, $\lambda(T_{m_1})$ is indeed bounded away from 1, and it is δ-safe by Lemma A.8. So we just have to check that for $1 \le j \le r-1$, if G_{m_{j+1},m_j} is saddle-node with big length then $T_{m_{j+1}}$ is not too deep inside a central cascade.

The following combinatorial estimate will be key to the analysis.

Lemma **A.12.** — *Suppose that* G_{m_{j+1},m_j} *is saddle-node, and let* $T_{n_1=m_j}, \ldots, T_{n_l=m_{j+1}}$ *be the associated cascade. If* $l \ge 4 + b$ *then* $f^{m_r - m_j}(0) \notin T_{n_{4+b}}$.

Proof. — Let $j_k + 1 \le s \le j_\kappa$ be minimal such that $T_{m_j} \supset I_s$. It follows that

$$(\mathbf{A.3}) \qquad m_j - m_1 \le \sum_{n=j_k}^{s-1} v_n.$$

Since $I_s \subset T_{n_1} \subset \operatorname{int} I_{s-1}$, it follows that $I_{s+1} \subset T_{n_2} \subset \operatorname{int} I_s$ and $T_{n_3} \subset \operatorname{int} I_{s+1}$. Then $n_3 - n_2 \ge v_s \ge n_2 - n_1$, and since $n_{t+1} - n_t = n_2 - n_1$ for $1 \le t \le l-1$, we see that $n_2 - n_1 = v_s$.

Assume first that $s = j_t$ for some $k < t \le \kappa$. Then (A.1) and (A.3) imply $m_j - m_1 \le 2v_s$, so that $m_j - m_1 + bv_s \le (2+b)v_s$. Thus,

$$\begin{aligned} p &= m_r - m_1 + bv_s = (m_r - m_j) + (m_j - m_1) + bv_s \\ &= (m_r - m_j) + q, \quad \text{where } q \le (2+b)v_s. \end{aligned}$$

If $x := f^{m_r - m_j}(0) \in T_{n_{3+b}}$ then $f^{nv_s}(x) \in f^{v_s} T_{n_2}$ for $n \leq 2 + b$. Hence $f^p(0) = f^q(x)$ is either in $f^{v_s}(T_{n_2})$ or it is outside T_{n_1}. In any case, it can not belong to the renormalization interval $\bigcap_{n \geq 1} I_n$—contradiction.

Assume now that $j_t + 1 = s < j_{t+1}$ for some $k \leq t < \kappa$. Arguing as before, we see that $m_j - m_1 \leq 3v_s$, and if $f^{m_r - m_j}(0) \in T_{n_{4+b}}$ we arrive at a similar contradiction.

Assume now that $j_t + 2 \leq s < j_{t+1}$ for some $k \leq t < \kappa$. Then the map $f^{n_2 - n_1}$ has a unimodal extension to the interval $I_{s-1} \supset T_{n_1}$. Hence T_{n_1} is a kid of the interval $T_{n_0} \subset I_{s-2}$ of depth $n_0 = 2n_1 - n_2$. But then G_{m_{j+1}, m_j} is not a maximal saddle-node transition map in the decomposition of G_{m_r, m_1}, contradicting the definition of m_j. □

Let $T_{n_1 = m_j}, \ldots, T_{n_l = m_{j+1}}, \ldots, T_{n_L}$ be the maximal continuation of the saddle-node cascade associated to G_{m_{j+1}, m_j}, and assume that l and $L - l$ are large. By Lemma A.12, $f^{m_r - m_j}(0) \notin T_{n_{4+b}}$, which implies that $f^{m_r - m_{j+1}}(0) \in T_{n_s} \setminus T_{n_{s+1}}$ for some $l \leq s \leq l + 2 + b$. Indeed, $f^{m_r - m_{j+1}}(0) \in T_{m_{j+1}} = T_{n_l}$, but $f^{m_r - m_{j+1}}(0) \notin T_{n_{l+3+b}}$, for otherwise

$$f^{m_r - m_j}(0) = f^{m_{j+1} - m_j}(f^{m_r - m_{j+1}}(0)) \in f^{n_l - n_1}(T_{n_{l+3+b}}) \subset T_{n_{b+4}}.$$

On the other hand, since G_{m_{j+1}, m_j} is a maximal saddle-node cascade in the decomposition of G_{m_r, m_1}, we must have $f^{m_r - m_{j+1} + n_1 - n_2}(0) \notin T_{n_{s+1}}$. We can then apply Lemma A.6 to conclude that $|I_{j_\kappa + 1}| / |I_{j_k}|$ is small, contradiction. This establishes that either l or $L - l$ must be small, as desired. □

A.5 *Proof of Lemma 9.4.* — We may assume that the sequence $f_n = \tilde{f}_n$ converges to some f_∞. Let p_n be the period of f_n. Let Λ_n be the affine map such that $\Lambda_n \circ (f_n)^{p_n} \circ \Lambda_n^{-1}$ is normalized.

Assume first that the pre-renormalization intervals of f_n do not have length bounded from below: following the terminology of Corollary A.1 we will say that the combinatorics of the f_n is not essentially bounded. Then either $\inf \lambda_{N(f_n)}(f_n) = 0$ or $\sup \kappa(f_n) = \infty$ by [L1].

If $\inf \lambda_{N(f_n)}(f_n) = 0$ then f_∞ is a unicritical polynomial [L1].

Consider now the case $\inf \lambda_{N(f_n)}(f_n) > 0$ and $\sup \kappa(f_n) = \infty$. We may assume that $\lim \kappa(f_n) = \infty$. Passing through a subsequence we may assume that for each $k \geq 0$, $\Lambda_n(I_{j_{\kappa(f_n)-k}}(f_n))$ converges to a closed interval D_k. Clearly each D_k is a bounded interval (scaling factors minorated) and $\bigcup D_k = \mathbf{R}$ (scaling factors bounded away from 1). We may also assume that $\Lambda_n(T_{w_{j_{\kappa(f_n)-k}} + p_n}(f_n))$ converges to a closed interval D'_k. Then $D'_k \subset D_1$ by Lemma A.2. By Lemma A.10, for every $k \geq 0$, f_∞ has an analytic extension $D'_k \to D_k$ which is proper. It follows that f_∞ has a maximal analytical extension to $\bigcup_{k \geq 0} D'_k \subset D_1$.

Assume now that f_n has essentially bounded, but unbounded, combinatorics. We may assume that $p_n \to \infty$. Let $0 = \beta_n^0 < \beta_n^1 < \cdots$ be the sequence of admissible moments, i.e., such that $f_n^{\beta_n^i}(0) \in I_0(f_n)$. Clearly $\beta_n^{i+1} - \beta_n^i \leq \beta_n^1$ (since the critical point returns to I_0 no earlier than any other point $x \in I_0$). Moreover, β_n^1 is bounded (otherwise the combinatorics is close to the Chebyshev one and we would already have $\inf |I_1(f_n)| = 0$).

Notice that if $0 \le \beta_n^i < p_n$, then $f_n^{p_n}(\partial T_{\beta_n^i})$ is the orientation reversing fixed point of f_n. Let $l_n \ge p_n/\beta_n^1$ be such that $p_n = \beta_n^{l_n}$, and for $0 \le i \le l_n$, let $b_n^i = p_n - \beta_n^{l_n - i}$. Then $b_n^i \le i\beta_n^1$ and $f_n^{p_n}$ has at least $2i-1$ critical points in $T_{p_n - b_n^i}(f_n)$ (counted with multiplicity) for $1 \le i \le l_n$.

We may assume that the intervals $\Lambda_n(T_{p_n - b_n^i})$ converge to intervals D_i for each i. Clearly $\bigcup D_i$ is a bounded interval. By Lemma A.11, f_∞ has an analytic extension to $\bigcup D_i$, and restricted to each D_i it has at least $2i-1$ critical points. So f_∞ cannot extend beyond $\bigcup D_i$. □

REFERENCES

[AKLS] A. Avila, J. Kahn, M. Lyubich, and W. Shen, Combinatorial rigidity for unicritical polynomials, *Ann. Math.*, **170** (2009), 783–797.

[ALM] A. Avila, M. Lyubich, and W. de Melo, Regular or stochastic dynamics in real analytic families of unimodal maps, *Invent. Math.*, **154** (2003), 451–550.

[ALS] A. Avila, M. Lyubich, and W. Shen, Parapuzzle of the Multibrot set and typical dynamics of unimodal maps, *J. Eur. Math. Soc.*, **13** (2011), 27–56.

[Ca] H. Cartan, Sur les rétractions d'une variété, *C. R. Acad. Sci. Paris Sér. I, Math.*, **303** (1986), 715.

[Ch] D. Cheraghi, Combinatorial rigidity for some infinitely renormalizable unicritical polynomials, *Conform. Geom. Dyn.*, **14** (2010), 219–255.

[Cv] P. Cvitanović, *Universality in Chaos*, Adam Hilger, Bristol, 1984.

[D] A. Douady, Chirurgie sur les applications holomorphes, in *Proceedings of ICM-86*, pp. 724–738, AMS, Providence, 1987.

[DH] A. Douady and J. H. Hubbard, On the dynamics of polynomial-like maps, *Ann. Sci. Ecole Norm. Super.*, **18** (1985), 287–343.

[E] H. Epstein, Fixed points of composition operators II, *Nonlinearity*, **2** (1989), 305–310.

[F] M. J. Feigenbaum, Quantitative universality for a class of non-linear transformations, *J. Stat. Phys.*, **19** (1978), 25–52.

[FM] E. de Faria and W. de Melo, Rigidity of critical circle mappings I, *J. Eur. Math. Soc.*, **1** (1999), 339–392.

[GS] J. Graczyk and G. Swiatek, Generic hyperbolicity in the logistic family, *Ann. Math. (2)*, **146** (1997), 1–52.

[Hi] B. Hinkle, Parabolic limits of renormalization, *Ergod. Theory Dyn. Syst.*, **20** (2000), 173–229.

[K] J. Kahn, A priori bounds for some infinitely renormalizable quadratics: I. Bounded primitive combinatorics. *Preprint IMS at Stony Brook*, # 5 (2006).

[KL1] J. Kahn and M. Lyubich, A priori bounds for some infinitely renormalizable quadratics: II. Decorations, *Ann. Sci. Ecole Norm. Super.*, **41** (2008), 57–84.

[KL2] J. Kahn and M. Lyubich, A priori bounds for some infinitely renormalizable quadratics, III. Molecules, in D. Schleicher (ed.), *Complex Dynamics: Families and Friends. Proceeding of the conference dedicated to Hubbard's 60th birthday*, AK Peters, Wellesley, 2009.

[KSS] O. Kozlovski, W. Shen, and S. van Strien, Rigidity for real polynomials, *Ann. Math.*, **165** (2007), 749–841.

[KS] K. Krzyzewski and W. Szlenk, On invariant measures for expanding differential mappings, *Studia Math.*, **33** (1969), 83–92.

[La] O. E. Lanford III, A computer assisted proof of the Feigenbaum conjectures, *Bull. Am. Math. Soc.*, **6** (1982), 427–434.

[LvS] G. Levin and S. van Strien, Local connectivity of Julia sets of real polynomials, *Ann. Math.*, **147** (1998), 471–541.

[L1] M. Lyubich, Combinatorics, geometry and attractors of quasi-quadratic maps, *Ann. Math.*, **140** (1994), 347–404.

[L2] M. Lyubich, Dynamics of quadratic polynomials. I, II, *Acta Math.*, **178** (1997), 185–297.

[L3] M. Lyubich, Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture, *Ann. Math. (2)*, **149** (1999), 319–420.

[L4] M. Lyubich, Almost every real quadratic map is either regular or stochastic, *Ann. Math. (2)*, **156** (2002), 1–78.

[LY] M. Lyubich and M. Yampolsky, Dynamics of quadratic polynomials: complex bounds for real maps, *Ann. Inst. Fourier*, **47** (1997), 1219–1255.

[Lju1] Y. I. Ljubich, *Introduction to the Theory of Banach Representations of Groups*, Birkhäuser, Basel, 1988.

[Lju2] Y. I. Ljubich, Dissipative actions and almost periodic representations of abelian semigroups, *Ukr. Math. J.*, **40** (1988), 58–62.

[Ma1] M. Martens, Distortion results and invariant Cantor sets for unimodal maps, *Ergod. Theory Dyn. Syst.*, **14** (1994), 331–349.

[Ma2] M. Martens, The periodic points of renormalization, *Ann. Math.*, **147** (1998), 543–584.

[McM1] C. McMullen, *Complex Dynamics and Renormalization*, Annals of Math. Studies, vol. 135, Princeton University Press, Princeton, 1994.

[McM2] C. McMullen, *Renormalization and 3-Manifolds which Fiber over the Circle*, Annals of Math. Studies, vol. 135, Princeton University Press, Princeton, 1996.

[MvS] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer, Berlin, 1993.

[Mi] J. Milnor, Periodic orbits, external rays, and the Mandelbrot set: expository lectures, in *Géometrie complexe et systémes dynamiques, Volume in Honor of Douady's 60th Birthday*. Astérisque, vol. 261, pp. 277–333, 2000.

[MSS] R. Mañé, P. Sad, and D. Sullivan, On the dynamics of rational maps, *Ann. Sci. Ecole Norm. Super.*, **16** (1983), 193–217.

[Sl] Z. Slodkowski, Holomorphic motions and polynomial hulls, *Proc. Am. Math. Soc.*, **111** (1991), 347–355.

[S] D. Sullivan, *Bounds, Quadratic Differentials, and Renormalization Conjectures*, AMS Centennial Publications, vol. 2, 1992. Mathematics into Twenty-first Century.

[TC] C. Tresser and P. Coullet, Itération d'endomorphismes et groupe de renormalisation, *C. R. Acad. Sci. Paris A*, **287** (1978), 577–580.

A. A.
CNRS UMR 7586,
Institut de Mathématiques de Jussieu,
175 rue du Chevaleret,
75013 Paris, France

and

IMPA,
Estrada Dona Castorina, 110,
22460-320 Rio de Janeiro, RJ, Brazil
artur@math.sunysb.edu
url: www.impa.br/~avila

M. L.
Department of Mathematics,
Stony Brook University,
Stony Brook, NY 11794-3651, USA
mlyubich@math.sunysb.edu

Manuscrit reçu le 15 juin 2010
Manuscrit accepté le 23 mai 2011
publié en ligne le 9 août 2011.