$H^{1/2}$ MAPS WITH VALUES INTO THE CIRCLE: MINIMAL CONNECTIONS, LIFTING, AND THE GINZBURG–LANDAU EQUATION

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1. Introduction

Let $G \subset \mathbf{R}^3$ be a smooth bounded domain with $\Omega = \partial G$ simply connected. We are concerned with the properties of the space

$$H^{1/2}(\Omega; S^1) = \{g \in H^{1/2}(\Omega; \mathbf{R}^2);\ |g| = 1 \text{ a.e. on } \Omega\}.$$

Recall (*see* [12]) that there are functions in $H^{1/2}(\Omega; S^1)$ which cannot be written in the form $g = e^{\imath\varphi}$ with $\varphi \in H^{1/2}(\Omega; \mathbf{R})$. For example, we may assume that locally, near a point on Ω, say 0, Ω is a disc B_1; then take

$$g(x, y) = (x, y)/(x^2 + y^2)^{1/2} \quad \text{on} \quad B_1. \tag{1.1}$$

Recall also (*see* [25]) that there are functions in $H^{1/2}(\Omega; S^1)$ which cannot be approximated in the $H^{1/2}$-norm by functions in $C^\infty(\Omega; S^1)$. Consider, for example, again a function g which is the same as in (1.1) near 0.

It is therefore natural to introduce the classes

$$X = \{g \in H^{1/2}(\Omega; S^1);\ g = e^{\imath\varphi} \text{ for some } \varphi \in H^{1/2}(\Omega; \mathbf{R})\}$$

and

$$Y = \overline{C^\infty(\Omega; S^1)}^{H^{1/2}}.$$

Clearly, we have

$$X \subset Y \subset H^{1/2}(\Omega; S^1).$$

Moreover, these inclusions are strict. Indeed, any function $g \in H^{1/2}(\Omega; S^1)$ which satisfies (1.1) does not belong to Y. On the other hand, the function

$$g(x, y) = \begin{cases} e^{2\imath\pi/r^\alpha}, & \text{on } B_1 \\ 1, & \text{on } \Omega \setminus B_1 \end{cases}$$

with $r = (x^2 + y^2)^{1/2}$ and $1/2 \le \alpha < 1$, belongs to Y, but not to X (*see* [12]).

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To every map $g \in \mathrm{H}^{1/2}(\Omega; \mathbf{R}^2)$ we associate a distribution $\mathrm{T} = \mathrm{T}(g) \in \mathscr{D}'(\Omega; \mathbf{R})$. When $g \in \mathrm{H}^{1/2}(\Omega; \mathrm{S}^1)$, the distribution $\mathrm{T}(g)$ describes the location and the topological degree of its singularities. This is the analogue of a tool introduced by Brezis, Coron and Lieb [19] in the framework of $\mathrm{H}^1(\mathrm{G}; \mathrm{S}^2)$ (*see* the discussion following Lemma 2 below). In the context of $\mathrm{H}^{1/2}(\Omega; \mathrm{S}^1)$, the distribution $\mathrm{T}(g)$ and the corresponding number $\mathrm{L}(g)$ (defined after Lemma 1) were originally introduced by the authors in 1996 and these concepts were presented in various lectures.

Given $g \in \mathrm{H}^{1/2}(\Omega; \mathbf{R}^2)$ and $\varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R})$, consider any $\mathrm{U} \in \mathrm{H}^1(\mathrm{G}; \mathbf{R}^2)$ and any $\Phi \in \mathrm{Lip}\ (\mathrm{G}; \mathbf{R})$ such that

$$\mathrm{U}_{|\Omega} = g \text{ and } \Phi_{|\Omega} = \varphi. \tag{1.2}$$

Set

$$\mathrm{H} = 2(\mathrm{U}_y \wedge \mathrm{U}_z, \mathrm{U}_z \wedge \mathrm{U}_x, \mathrm{U}_x \wedge \mathrm{U}_y);$$

this H is independent of the choice of direct orthonormal bases in $\mathbf{R}^3$ (to compute derivatives) and in $\mathbf{R}^2$ (to compute $\wedge$-products). Next, consider

$$\int_{\mathrm{G}} \mathrm{H} \cdot \nabla \Phi. \tag{1.3}$$

It is not difficult to show (*see* Section 2) that (1.3) is independent of the choice of U and Φ; it depends only on g and φ. We may thus define the distribution $\mathrm{T}(g) \in \mathscr{D}'(\Omega; \mathbf{R})$ by

$$\langle \mathrm{T}(g), \varphi \rangle = \int_{\mathrm{G}} \mathrm{H} \cdot \nabla \Phi.$$

If there is no ambiguity, we will simply write T instead of $\mathrm{T}(g)$.

When g has a little more regularity, we may also express T in a simpler form:

Lemma **1.** — *If* $g \in \mathrm{H}^{1/2}(\Omega; \mathbf{R}^2) \cap \mathrm{W}^{1,1}(\Omega; \mathbf{R}^2) \cap \mathrm{L}^\infty(\Omega; \mathbf{R}^2)$, *then*

$$\langle \mathrm{T}(g), \varphi \rangle = \int_{\Omega} \big((g \wedge g_x)\varphi_y - (g \wedge g_y)\varphi_x \big), \quad \forall \varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R}).$$

The integrand is computed pointwise in any orthonormal frame (x, y) such that (x, y, n) is direct, where n is the outward normal to G *– and the corresponding quantity is frame-invariant.*

By analogy with the results of [19] and [6] we introduce, for every $g \in H^{1/2}(\Omega; \mathbf{R}^2)$, the number

$$\begin{aligned} L(g) &= \frac{1}{2\pi} \text{ Sup } \{ \langle T(g), \varphi \rangle \,;\, \varphi \in \text{Lip } (\Omega; \mathbf{R}), |\varphi|_{\text{Lip}} \leq 1 \} \\ &= \frac{1}{2\pi} \text{ Max } \{...\}, \end{aligned}$$

where $|\varphi|_{\text{Lip}} = \operatorname{Sup}_{x \neq y} |\varphi(x) - \varphi(y)|/d(x, y)$ refers to a given metric d on Ω. There are three (equivalent) metrics on Ω which are of interest:

$$\begin{aligned} &d_{\mathbf{R}^3}(x, y) = |x - y|, \\ (\mathbf{1.4}) \qquad &d_{G}(x, y) = \text{the geodesic distance in } \bar{G}, \\ &d_{\Omega}(x, y) = \text{the geodesic distance in } \Omega. \end{aligned}$$

When dealing with a specified metric, we will write $L_{\mathbf{R}^3}, L_G$ or L_Ω. Otherwise, we will simply write L (note that all these $L's$ are equivalent). It is easy to see that

$$(\mathbf{1.5}) \qquad 0 \leq L(g) \leq C\|g\|^2_{H^{1/2}}, \quad \forall g \in H^{1/2}(\Omega; \mathbf{R}^2)$$

and

$$(\mathbf{1.6}) \qquad |L(g) - L(h)| \leq C\|g - h\|_{H^{1/2}}(\|g\|_{H^{1/2}} + \|h\|_{H^{1/2}}), \quad \forall g, h \in H^{1/2}(\Omega; \mathbf{R}^2).$$

When g takes its values into S^1 and has only a finite number of singularities, there are very simple expressions for $T(g)$ and $L(g)$:

Lemma **2.** — *If* $g \in H^{1/2}(\Omega; S^1) \cap H^1_{\text{loc}}\left(\Omega \setminus \cup_{j=1}^k \{a_j\}; S^1\right)$, *then*

$$T(g) = 2\pi \sum_{j=1}^{k} d_j \delta_{a_j},$$

where $d_j = \deg(g, a_j)$. *Moreover* $L(g)$ *is the length of the minimal connection associated to the configuration* (a_j, d_j) *and to the specific metric on* Ω *(in the sense of [19];* see *also [27]).*

Remark **1.1.** — Here, $\deg(g, a_j)$ denotes the topological degree of g restricted to any small circle around a_j, positively oriented with respect to the outward normal. It is well defined using the degree theory for maps in $H^{1/2}(S^1; S^1)$ (*see* [17] and [22]).

By the definition of $T(g)$, we see that $\langle T(g), 1\rangle = 0$. Therefore, if g is as in Lemma 2, then $\sum d_j = 0$. Thus we may write the collection of points (a_j), repeated with their multiplicity d_j, as $(P_1, ..., P_k, N_1, ..., N_k)$, where $k = 1/2 \sum |d_j|$ (we exclude from this collection the points of degree 0). A point a_j is counted among the P's if it has positive degree and among the N's otherwise. Then $L(g) = \operatorname{Inf}_{\sigma} \sum d(P_j, N_{\sigma(j)})$. Here, the Inf is taken over all the permutations σ of $\{1, ..., k\}$ and d is one of the metrics in (1.4).

The conclusion of Lemma 2 is reminiscent of a concept originally introduced by Brezis, Coron and Lieb [19]. There, u is a map from $G \subset \mathbf{R}^3$ into S^2 with a finite number of singularities $a_j \in G$. To such a map u, one associates a distribution $T(u)$ describing the location and the topological charge of the singular set of u. More precisely, if $u \in H^1(G; S^2)$, set

$$\mathscr{D} = (u \cdot u_y \wedge u_z, \quad u \cdot u_z \wedge u_x, \quad u \cdot u_x \wedge u_z)$$

and $T(u) = \operatorname{div}\mathscr{D}$.

If u is smooth except at the a_j's, it is proved in [19] that

$$T(u) = 4\pi \sum d_j \delta_{a_j}.$$

Here, d_j is the topological degree of u around a_j.

Using a density result of T. Rivière (*see* [38] and Lemma 11 in Section 2; *see* also the proof of Lemma 23, Remark 5.1 and Appendix B), we will extend Lemma 2 to general functions in $H^{1/2}(\Omega; S^1)$:

Theorem **1.** — *Given any* $g \in H^{1/2}(\Omega; S^1)$, *there are two sequences of points* (P_i) *and* (N_i) *in* Ω *such that*

$$\sum_i |P_i - N_i| < \infty \tag{1.7}$$

and

$$\langle T(g), \varphi\rangle = 2\pi \sum_i \big(\varphi(P_i) - \varphi(N_i)\big), \quad \forall \varphi \in \operatorname{Lip}(\Omega; \mathbf{R}). \tag{1.8}$$

In addition, for any metric d *in (1.4)*

$$L(g) = \operatorname{Inf} \sum_i d(P_i, N_i),$$

where the infimum is taken over all possible sequences $(P_i), (N_i)$ *satisfying (1.7), (1.8).*

If the distribution T *is a measure (of finite total mass), then*

$$T(g) = 2\pi \sum_{\text{finite}} d_j \delta_{a_j}$$

with $d_j \in \mathbf{Z}$ *and* $a_j \in \Omega$.

Remark **1.2.** — There are always infinitely many representations of $T(g)$ as a sum satisfying (1.7)–(1.8) and such representations need not be equivalent modulo a permutation of points. For example, a dipole $\delta_P - \delta_Q$ may be represented as $\delta_P - \delta_{Q_1} + \sum_{j\geq 1}(\delta_{Q_j} - \delta_{Q_{j+1}})$ for any sequence (Q_j) rapidly converging to Q. The last assertion in Theorem 1 is the $H^{1/2}$-analogue of a result of Jerrard and Soner [28, 29] (*see* also Hang and Lin [28]) concerning maps in $W^{1,1}(\Omega; S^1)$.

Maps in Y can be characterized in terms of the distribution T:

Theorem **2** *(Rivière [38]). — Let* $g \in H^{1/2}(\Omega; S^1)$. *Then* $T(g) = 0$ *if and only if* $g \in Y$.

This result is the $H^{1/2}$-counterpart of a well-known result of Bethuel [3] characterizing the closure of smooth maps in $H^1(B^3; S^2)$ (*see* also Demengel [24]).

The implication $g \in Y \implies T(g) = 0$ is trivial, using e.g. (1.6). The converse is more delicate; it uses the "dipole removing" technique of Bethuel [3] and we refer the reader to [38]; for convenience we present in Section 4 a slightly different proof.

As was mentioned earlier, functions in Y need not belong to X, i.e., they need not have a lifting in $H^{1/2}(\Omega; \mathbf{R})$. However, we have

Theorem **3.** — *For every* $g \in Y$ *there exists* $\varphi \in H^{1/2}(\Omega; \mathbf{R}) + W^{1,1}(\Omega; \mathbf{R})$, *which is unique (modulo* 2π*), such that* $g = e^{\imath\varphi}$. *Conversely, if* $g \in H^{1/2}(\Omega; S^1)$ *can be written as* $g = e^{\imath\varphi}$ *with* $\varphi \in H^{1/2} + W^{1,1}$, *then* $g \in Y$.

The existence will be proved in Section 3 with the help of paraproducts (in the sense of J.-M. Bony and Y. Meyer). The heart of the matter is the estimate

$$\|\varphi\|_{H^{1/2}+W^{1,1}} \leq C_\Omega \|e^{\imath\varphi}\|_{H^{1/2}}(1 + \|e^{\imath\varphi}\|_{H^{1/2}}), \qquad (\mathbf{1.9})$$

which holds for any smooth real-valued function φ; here C_Ω depends only on Ω.

Using Theorem 3 and the basic estimate (1.9), we will prove that, for every $g \in H^{1/2}(\Omega; S^1)$, there exists $\varphi \in H^{1/2}(\Omega; \mathbf{R}) + BV(\Omega; \mathbf{R})$ such that $g = e^{\imath\varphi}$ (*see* Section 4). Of course, this φ is not unique. There is an interesting link between all possible liftings of g and the minimal connection of g:

Theorem **4.** — *For every* $g \in H^{1/2}(\Omega; S^1)$ *we have*

$$\mathrm{Inf}\ \left\{|\varphi_2|_{BV}; g = e^{\imath(\varphi_1+\varphi_2)}; \varphi_1 \in H^{1/2} \ \text{and}\ \varphi_2 \in BV\right\} = 4\pi L_\Omega(g),$$

where $|\varphi_2|_{BV} = \int_\Omega |D\varphi_2|$.

Another useful fact about the structure of $H^{1/2}(\Omega; S^1)$ is the following factorization result:

Theorem **5.** — *We have*

$$H^{1/2}(\Omega; S^1) = (X) \cdot (H^{1/2} \cap W^{1,1}),$$

i.e., every $g \in H^{1/2}(\Omega; S^1)$ *may be written as* $g = e^{\imath\varphi} h$, *with* $\varphi \in H^{1/2}(\Omega; \mathbf{R})$ *and* $h \in H^{1/2}(\Omega; S^1) \cap W^{1,1}(\Omega; S^1)$. *Moreover we have the control*

$$\|\varphi\|^2_{H^{1/2}} + \|h\|_{W^{1,1}} \leq C_\Omega \|g\|^2_{H^{1/2}}.$$

The interplay between the Ginzburg–Landau energy and minimal connections has been first pointed out in the important work of T. Rivière [37] (*see* also [34] and [38]) in the case of boundary data with a finite number of singularities. We are concerned here with a general boundary condition g in $H^{1/2}$.

Given $g \in H^{1/2}(\Omega; S^1)$, set

$$e_{\varepsilon,g} = e_\varepsilon = \operatorname*{Min}_{H^1_g(G;\mathbf{R}^2)} E_\varepsilon(u), \tag{1.10}$$

where

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_G (|u|^2 - 1)^2$$

and

$$H^1_g(G; \mathbf{R}^2) = \{u \in H^1(G; \mathbf{R}^2);\ u = g \text{ on } \Omega\}.$$

Theorem **6.** — *For every* $g \in H^{1/2}(\Omega; S^1)$ *we have, as* $\varepsilon \to 0$,

$$e_\varepsilon = \pi L_G(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)). \tag{1.11}$$

This result and some variants are proved in Section 5. For special $g's$ (namely $g's$ with finite number of singularities), formula (1.11) was first proved by T. Rivière in [37]. For a general $g \in H^{1/2}(\Omega; S^1)$, it was established in [12] that

$$e_\varepsilon \leq C(g) \log(1/\varepsilon)$$

where $C(g) = C(G)\|g\|^2_{H^{1/2}(\Omega)}$; another proof of the same inequality is given in [38].

Using Theorem 6, we may characterize the classes X and Y in terms of the behavior of the Ginzburg–Landau energy as $\varepsilon \to 0$. Indeed, Theorem 6 implies that

$$Y = \{ g \in H^{1/2}(\Omega; S^1)\ ;\ e_\varepsilon = o(\log(1/\varepsilon)) \}.$$

On the other hand, it is easy to see that

$$X = \{ g \in H^{1/2}(\Omega; S^1) \, ; \; e_\varepsilon = O(1) \}.$$

Next, we present various estimates for minimizers u_ε in (1.10). In Section 6, we discuss the following theorem (originally announced in [13] and subsequently established with a simpler proof in [5]):

Theorem **7.** — *For every* $g \in H^{1/2}(\Omega; S^1)$ *we have*

$$\|u_\varepsilon\|_{W^{1,p}(G)} \le C_p, \quad \forall\, 1 \le p < 3/2. \tag{1.12}$$

In fact, we will prove the following slight generalization of Theorem 7:

Theorem **7′.** — *For every* $g \in H^{1/2}(\Omega; S^1)$*, the family* (u_ε) *is relatively compact in* $W^{1,p}$ *for every* $p < 3/2$.

Remark **1.3.** — It is very plausible that Theorem 7 still holds when $p = 3/2$. However, the conclusion fails for $p > 3/2$; *see* the discussion in Section 9.

In Section 7, we will establish stronger *interior* estimates:

Theorem **8.** — *For every* $g \in H^{1/2}(\Omega; S^1)$*, we have*

$$\|u_\varepsilon\|_{W^{1,p}(K)} \le C_{p,K}, \quad \forall\, 1 \le p < 2, \quad \forall\, K \text{ compact in } G. \tag{1.13}$$

Consequently, (u_ε) *is relatively compact in* $W^{1,p}_{loc}$ *for every* $p < 2$.

Remark **1.4.** — The conclusion of Theorem 8 fails for $p = 2$. Here is an example, with $G = B_1$, the unit ball in $\mathbf{R}^3$, and $g(x_1, x_2, x_3) = (x_1, x_2)\big/\sqrt{x_1^2 + x_2^2}$. T. Rivière [37] (*see* also F.H. Lin and T. Rivière [34]) has proved that in this case $u_\varepsilon \to u = (x_1, x_2)\big/\sqrt{x_1^2 + x_2^2}$, and clearly this u does not belong to $H^1_{loc}(G)$.

Finally, we have a very precise result concerning the limit of u_ε when $g \in Y$:

Theorem **9.** — *For every* $g \in Y$*, write (as in Theorem 3)* $g = e^{\imath\varphi}$*, with* $\varphi \in H^{1/2} + W^{1,1}$*. Then we have*

$$u_\varepsilon \to u_* = e^{\imath\tilde\varphi} \text{ in } W^{1,p}(G) \cap C^\infty(G), \quad \forall p < 3/2,$$

where $\widetilde\varphi$ *is the harmonic extension of* φ.

Theorem 9 and some of its variants are presented in Section 8. In Section 9 we prove some partial results about estimates in $W^{1,p}$ when $p = 3/2$. In Section 10 we list some open problems.

Most of the results in this paper were announced in [13].

The paper is organized as follows:

2. Elementary properties of the minimal connection. Proof of Theorem 1

To every $g \in H^{1/2}(\Omega; \mathbf{R}^2)$ we associate a distribution $T(g) \in \mathscr{D}'(\Omega; \mathbf{R})$ in the following way: consider any $U \in H^1(G; \mathbf{R}^2)$ such that

$$U_{|\Omega} = g.$$

Given $\varphi \in \text{Lip}\,(\Omega; \mathbf{R})$, let $\Phi \in \text{Lip}\,(G; \mathbf{R})$ be such that

$$\Phi_{|\Omega} = \varphi.$$

Set

$$H = 2(U_y \wedge U_z, U_z \wedge U_x, U_x \wedge U_y).$$

Lemma **3.** — *The quantity* $\int_G H \cdot \nabla\Phi$ *depends only on* g *and* φ.

Proof. — We first claim that $\int_G H \cdot \nabla\Phi$ does not depend on the choice of Φ. Observe that, if $U \in C^\infty(\bar{G}; \mathbf{R}^2)$, then

$$\text{div } H = 0.$$

By density, we find that

$$\operatorname{div} H = 0 \text{ in } \mathscr{D}'(G)$$

for any $U \in H^1(G; \mathbf{R}^2)$. It follows easily that

$$\int_G H \cdot \nabla\Psi = 0, \qquad \forall \Psi \in \text{ Lip } (G; \mathbf{R}) \text{ with } \Psi = 0 \text{ on } \Omega.$$

This implies the above claim.

Next, we verify that $\int_G H \cdot \nabla\Phi$ does not depend on the choice of U. Let V be another choice in $H^1(G; \mathbf{R}^2)$ such that $V_{|\Omega} = g$. Set $W = V - U \in H^1_0$. Then, with obvious notation,

$$\int_G H_V \cdot \nabla\Phi = \int_G H_U \cdot \nabla\Phi + \int_G R_1 \cdot \nabla\Phi + \int_G R_2 \cdot \nabla\Phi,$$

with $R_1 = (W_y \wedge U_z + U_y \wedge W_z, \ldots), \quad R_2 = (W_y \wedge W_z, \ldots)$.

We complete the proof of Lemma 3 with the help of

Lemma **4.** — *For each* $U \in H^1(G; \mathbf{R}^2)$ *and* $W \in H^1_0(G; \mathbf{R}^2)$ *we have*

$$\int_G R_1 \cdot \nabla\Phi = 0, \qquad \forall \Phi \in \text{ Lip } (G; \mathbf{R}).$$

Proof of Lemma 4. — By density, it suffices to prove the above equality for $U \in C^\infty(\bar{G}; \mathbf{R}^2)$, $W \in C^\infty_0(\bar{G}; \mathbf{R}^2)$ and $\Phi \in C^\infty(\bar{G}; \mathbf{R})$. For such U and W, note that

$$W_y \wedge U_z + U_y \wedge W_z = (W \wedge U_z)_y + (U_y \wedge W)_z.$$

Therefore,

$$\int_G R_1 \cdot \nabla\Phi = -\int_G [(W \wedge U_z)\Phi_{xy} + (U_y \wedge W)\Phi_{xz} + \cdots] = 0.$$

As a consequence of Lemma 3, the map

$$\varphi \longmapsto \int_G H \cdot \nabla\Phi$$

is a continuous linear functional on Lip $(\Omega; \mathbf{R})$. In particular, it is a distribution. Again by Lemma 3, this distribution depends only on $g \in H^{1/2}(\Omega; \mathbf{R}^2)$. We will denote it $T(g)$.

Remark **2.1.** — It is important to note that T has a "local" character. More precisely, if $g_1, g_2 \in H^{1/2}(\Omega; \mathbf{R}^2)$ are such that $g_1 = g_2$ in ω (where ω is an open subset of Ω), then

$$\langle T(g_1), \varphi\rangle = \langle T(g_2), \varphi\rangle, \qquad \forall\varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R}),\ \text{with}\ \ \mathrm{supp}\ \varphi \subset \omega.$$

This is an easy consequence of Lemma 3 and of the fact that, if supp $g \cap$ supp $\varphi = \emptyset$, then one may extend g to $U \in H^1$ and φ to $\Phi \in$ Lip such that supp $U \cap$ supp $\Phi = \emptyset$. Thus, one may define a local version of T as follows: if $g \in H^{1/2}_{\mathrm{loc}}(\omega; \mathbf{R}^2)$, set

$$\langle T(g), \varphi\rangle = \langle T(h), \varphi\rangle, \qquad \forall\varphi \in C^1_0(\omega; \mathbf{R}),$$

where h is any map in $H^{1/2}(\Omega; \mathbf{R}^2)$ such that $h = g$ in a neighborhood of supp φ.

Remark **2.2.** — Another important property is the invariance under diffeomorphisms. More precisely, let Ω, G, g, φ be as above and let $\xi : \widetilde{\Omega} \to \Omega$ be an orientation-preserving diffeomorphism. Then

$$\langle T(g), \varphi\rangle = \langle T(\tilde{g}), \widetilde{\varphi}\rangle,$$

where $\tilde{g} = g \circ \xi$ and $\widetilde{\varphi} = \varphi \circ \xi$. Clearly, ξ extends as an orientation-preserving diffeomorphism (still denoted ξ) from a small tubular neighborhood of $\widetilde{\Omega}$ in $\widetilde{G}$ to a tubular neighborhood of Ω in G (as in the proof of Lemma 5 below).

We have

$$\langle T(g), \varphi\rangle = \int_G H \cdot \nabla\Phi = 2\int_G \mathrm{Jac}\ (\Phi, U),$$

since

$$H = 2(U_y \wedge U_z, U_z \wedge U_x, U_x \wedge U_y).$$

We may choose U and Φ supported in a small tubular neighborhood of Ω and set $\widetilde{U} = U \circ \xi$ and $\widetilde{\Phi} = \Phi \circ \xi$. Then, with obvious notation,

$$\begin{aligned}\langle T(\tilde{g}), \widetilde{\varphi}\rangle &= \int_{\widetilde{G}} \widetilde{H} \cdot \nabla\widetilde{\Phi} = 2\int_{\widetilde{G}} \mathrm{Jac}\ (\widetilde{\Phi}, \widetilde{U}) \\ &= 2\int_G \mathrm{Jac}\ (\Phi, U) = \langle T(g), \varphi\rangle.\end{aligned}$$

Similarly, if ω is an open subset of Ω and $\xi : \tilde{\omega} \to \omega$ is an orientation-preserving diffeomorphism, then (using Remark 2.1) we have

$$\langle T(g), \varphi \rangle = \langle T(\tilde{g}), \tilde{\varphi} \rangle$$

for every $g \in H^{1/2}_{loc}(\omega; \mathbf{R}^2)$ and $\varphi \in C^1_0(\omega; \mathbf{R})$. This is extremely useful because we can always choose a local diffeomorphism with $\widetilde{\Omega}$ flat near a point. More precisely, let (ω_i) be a finite covering of Ω with each ω_i diffeomorphic to a disc D via $\xi_i : D \to \omega_i$. Let (α_i) be a corresponding partition of unity. Then, $\forall \varphi \in$ Lip $(\Omega; \mathbf{R})$,

$$\langle T(g), \varphi \rangle = \sum \langle T(g), \alpha_i \varphi \rangle$$

and we may compute each term $\langle T(g), \alpha_i \varphi \rangle$ in D using the fact that

$$\langle T(g), \alpha_i \varphi \rangle = \langle T(g \circ \xi_i), (\alpha_i \varphi) \circ \xi_i \rangle.$$

Here is a noticeable fact about $T(g)$:

Lemma **5.** — *Let* $g \in H^{1/2}(\Omega; \mathbf{R}^2)$. *Then there exists an* L^1*-section* F *of the tangent bundle* $T(\Omega)$ *such that*

$$\langle T(g), \varphi \rangle = \int_\Omega F \cdot \nabla \varphi, \quad \forall \varphi \in \text{Lip } (\Omega; \mathbf{R}).$$

Proof of Lemma 5. — For $\beta > 0$, let

$$G_\beta = \{X \in G; \quad \delta(X) < \beta\}, \quad \Omega_\beta = \{X \in G; \quad \delta(X) = \beta\},$$

where $\delta(X) = \text{dist}\,(X, \Omega)$. Assuming that β is sufficiently small, say $\beta < \beta_0$, for every $X \in G_\beta$ there exists a unique point $\sigma(X) \in \Omega$ such that $\delta(X) = |X - \sigma(X)|$. Let $\Pi : G_\beta \to (0, \beta) \times \Omega$ be the mapping defined by $\Pi(X) = (\delta(X), \sigma(X))$. This mapping is a C^2-diffeomorphism and its inverse is given by

$$\Pi^{-1}(t, \sigma) = \sigma - tn(\sigma), \qquad \forall (t, \sigma) \in (0, \beta) \times \Omega,$$

where $n(\sigma)$ is the outward unit normal to Ω at σ. For $0 < t < \beta_0$, let K_t denote the mapping $\Pi^{-1}(t, \cdot)$ of Ω onto Ω_t.

Since $n(\sigma)$ is orthogonal to $\Omega_t = \Pi^{-1}(t, \Omega)$ at $\sigma - tn(\sigma)$, it follows that, for every integrable non-negative function f in G_β,

$$\int_{G_\beta} f = \int_0^\beta dt \int_{\Omega_t} f d\sigma_t = \int_0^\beta dt \int_\Omega f(K_t(\sigma))(\text{Jac } K_t) d\sigma,$$

where $d\sigma$, $d\sigma_t$ denote surface elements on Ω, Ω_t respectively.

We now make a special choice of U and Φ. Let

$$\Phi(X) = \varphi(\sigma(X))\zeta(\delta(X)),$$

where $\varphi \in C^1(\Omega; \mathbf{R})$ is the given test function and

$$\zeta(t) = \begin{cases} 1, & \text{for } 0 \le t \le \beta_0/2 \\ 0, & \text{for } t \ge \beta_0. \end{cases}.$$

We take U to be any H^1 extension of g such that $U(X) = 0$ if $\delta(X) \ge \beta_0/2$. Hence

$$\begin{aligned}\langle T(g), \varphi\rangle &= \int_G H \cdot \nabla\Phi = \int_{G_{\beta_0/2}} H \cdot \nabla\Phi \\ &= \int_0^{\beta_0/2} dt \int_\Omega H \cdot \nabla\Phi(K_t(\sigma))(\operatorname{Jac} K_t)d\sigma.\end{aligned} \tag{2.1}$$

For every $\sigma \in \Omega$, fix a frame $\mathscr{F}_\sigma = (x, y)$ as in Lemma 1. We already observed that $H \cdot \nabla\Phi$ can be computed (pointwise) in any direct orthonormal frame of $\mathbf{R}^3$. We choose, at any points $X \in G_{\beta_0/2}$, the special frame $(\mathscr{F}_{\sigma(X)}, n(\sigma(X))$. Then, we have, $\forall t \in (0, \beta_0/2), \forall \sigma \in \Omega$,

$$(H \cdot \nabla\Phi)(K_t(\sigma)) = 2(U_y \wedge U_z)(K_t(\sigma))\varphi_x(\sigma) + 2(U_z \wedge U_x)(K_t(\sigma))\varphi_y(\sigma). \tag{2.2}$$

We now insert (2.2) into (2.1) and obtain the conclusion of Lemma 5 with $F(\sigma) = F_1(\sigma)\frac{\partial}{\partial x} + F_2(\sigma)\frac{\partial}{\partial y}$, where

$$F_1(\sigma) = 2\int_0^{\beta_0/2} (U_y \wedge U_z)(K_t(\sigma))(\operatorname{Jac} K_t)dt$$

and

$$F_2(\sigma) = 2\int_0^{\beta_0/2} (U_z \wedge U_x)(K_t(\sigma))(\operatorname{Jac} K_t)dt.$$

We now turn to the

Proof of Lemma 1. — It suffices to prove that

$$\int_G H \cdot \nabla\Phi = \int_\Omega [(g \wedge g_x)\varphi_y - (g \wedge g_y)\varphi_x]$$

when $U \in C^\infty(\bar{G}; \mathbf{R}^2)$ and $\Phi \in C^\infty(\bar{G}; \mathbf{R})$. We write

$$H = \big((U \wedge U_z)_y + (U_y \wedge U)_z, (U \wedge U_x)_z + (U_z \wedge U)_x, \\ (U \wedge U_y)_x + (U_x \wedge U)_y\big).$$

Integration by parts yields

$$\int_G H \cdot \nabla \Phi = \int_\Omega U \wedge \det(\nabla U, \nabla \Phi, \vec{n}).$$

By Lemma 3, we may assume further that $\dfrac{\partial U}{\partial n} = 0$ and $\dfrac{\partial \Phi}{\partial n} = 0$.

For each $\sigma \in \Omega$, we compute $\det(\nabla U, \nabla \Phi, \vec{n})$ in the frame given by Lemma 1. We have

$$\det(\nabla U, \nabla \Phi, \vec{n}) = \frac{\partial U}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial \Phi}{\partial x} = g_x \varphi_y - g_y \varphi_x,$$

and the conclusion follows.

Here are some straightforward variants and consequences of Lemma 1 and Remarks 2.1–2.2:

Lemma **6.** — *Let ω be an open subset of Ω. Let*

$$g \in H^{1/2}(\omega; \mathbf{R}^2) \cap W^{1,1}(\omega) \cap L^\infty(\omega).$$

Then

$$\langle T(g), \varphi \rangle = \int_\omega [(g \wedge g_x)\varphi_y - (g \wedge g_y)\varphi_x], \qquad \forall \varphi \in C_0^1(\omega; \mathbf{R}). \tag{2.3}$$

Lemma **7.** — *Let ω be an open subset of Ω. Let $g \in H^{1/2}(\omega; S^1) \cap VMO(\omega; S^1)$. Then*

$$\langle T(g), \varphi \rangle = 0, \quad \forall \varphi \in C_0^1(\omega; \mathbf{R}).$$

Proof of Lemma 7. — In view of Remark 2.2, we may assume that ω is a disc. There is a sequence $(g_n) \in C^\infty(\omega; S^1)$ such that $g_n \to g$ in $H^{1/2}_{loc}(\omega)$ (*see* [22]). Hence $\langle T(g_n), \varphi \rangle \to \langle T(g), \varphi \rangle$, $\forall \varphi \in C_0^1(\omega; \mathbf{R})$, by (2.5) below. On the other hand, by Lemma 6,

$$\langle T(g_n), \varphi \rangle = \int_\omega [(g_n \wedge g_{nx})\varphi_y - (g_n \wedge g_{ny})\varphi_x] \\ = 2\int_\omega (g_{nx} \wedge g_{ny})\varphi = 0$$

since $|g_n| = 1$ on ω.

There is yet another representation formula for T:

Lemma **8.** — *Let* $g = (g_1, g_2) \in \mathrm{H}^{1/2}(\Omega; \mathbf{R}^2)$. *Then if* $\omega \subset \Omega$ *is diffeomorphic to a disc* $\tilde{\omega}$ *as in Remark 2.2, we have,* $\forall \varphi \in \mathrm{C}_0^\infty(\omega; \mathbf{R})$,

$$
\begin{aligned}
\langle \mathrm{T}(g), \varphi \rangle = {} & \langle \tilde{g}_1, (\tilde{g}_2 \tilde{\varphi}_y)_x - (\tilde{g}_2 \tilde{\varphi}_x)_y \rangle_{\mathrm{H}^{1/2}, \mathrm{H}^{-1/2}} \\
& - \langle \tilde{g}_2, (\tilde{g}_1 \tilde{\varphi}_y)_x - (\tilde{g}_1 \tilde{\varphi}_x)_y \rangle_{\mathrm{H}^{1/2}, \mathrm{H}^{-1/2}}.
\end{aligned} \tag{2.4}
$$

Observe that, e.g. $\tilde{g}_2 \tilde{\varphi}_y \in \mathrm{H}^{1/2}(\tilde{\omega})$, so that $(\tilde{g}_2 \tilde{\varphi}_y)_x \in \mathrm{H}^{-1/2}(\tilde{\omega})$.

Proof of Lemma 8. — When g is smooth, (2.4) coincides with (2.3). The general case is obtained by approximation.

We now describe some elementary but useful facts about T and L:

Lemma **9.** — *We have, for* $g, h \in \mathrm{H}^{1/2}(\Omega; \mathbf{R}^2)$, $\varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R})$,

$$
|\langle \mathrm{T}(g) - \mathrm{T}(h), \varphi \rangle| \le \mathrm{C} |g - h|_{\mathrm{H}^{1/2}} (|g|_{\mathrm{H}^{1/2}} + |h|_{\mathrm{H}^{1/2}}) |\varphi|_{\mathrm{Lip}}, \tag{2.5}
$$

$$
|\mathrm{L}(g) - \mathrm{L}(h)| \le \mathrm{C} |g - h|_{\mathrm{H}^{1/2}} (|g|_{\mathrm{H}^{1/2}} + |h|_{\mathrm{H}^{1/2}}) \tag{2.6}
$$

and, in particular,

$$
\mathrm{L}(g) \le \mathrm{C} |g|_{\mathrm{H}^{1/2}}^2.
$$

If, in addition, g *and* h *are* S^1*-valued, then*

$$
\mathrm{T}(gh) = \mathrm{T}(g) + \mathrm{T}(h), \tag{2.7}
$$

$$
\mathrm{L}(g\bar{h}) \le \mathrm{C} |g - h|_{\mathrm{H}^{1/2}} (|g|_{\mathrm{H}^{1/2}} + |h|_{\mathrm{H}^{1/2}}) \tag{2.8}
$$

and

$$
\mathrm{L}(gh) \le \mathrm{L}(g) + \mathrm{L}(h). \tag{2.9}
$$

Here, we have identified $\mathbf{R}^2$ with $\mathbf{C}$ and gh denotes complex multiplication, while $|\ |_{\mathrm{H}^{1/2}}$ denotes the canonical seminorm on $\mathrm{H}^{1/2}$:

$$
|g|_{\mathrm{H}^{1/2}}^2 = \int_\Omega \int_\Omega \frac{|g(x) - g(y)|^2}{d(x,y)^3} dx dy.
$$

The constant C in this lemma depends only on Ω.

Proof. — Let $U, V \in H^1(G; \mathbf{R}^2)$ be the harmonic extensions of g, respectively h. Then clearly, $\forall \Phi \in \text{Lip}\ (G; \mathbf{R})$,

$$\int_G H_U \cdot \nabla\Phi \\ \leq \int_G H_V \cdot \nabla\Phi + C\|\nabla U - \nabla V\|_{L^2}(\|\nabla U\|_{L^2} + \|\nabla V\|_{L^2})\|\nabla\Phi\|_{L^\infty},$$

so that (2.5) follows. Moreover, we find that

$$L(g) \leq L(h) + C|g - h|_{H^{1/2}}(|g|_{H^{1/2}} + |h|_{H^{1/2}}).$$

Reversing the roles of g and h, yields (2.6).

The proof of (2.7)–(2.9) relies on the following

Lemma **10.** — *For* $g, h \in H^{1/2}(\Omega; \mathbf{R}^2) \cap L^\infty$, *we have,* $\forall \varphi \in C_0^\infty(\omega; \mathbf{R})$, *with the same notation as in Lemma 8,*

$$\begin{aligned}\langle T(gh), \varphi\rangle = \quad & \langle |\tilde h|^2 \tilde g_1, (\tilde g_2 \tilde\varphi_y)_x - (\tilde g_2 \tilde\varphi_x)_y\rangle_{H^{1/2}, H^{-1/2}} \\ & - \langle |\tilde h|^2 \tilde g_2, (\tilde g_1 \tilde\varphi_y)_x - (\tilde g_1 \tilde\varphi_x)_y\rangle_{H^{1/2}, H^{-1/2}} \\ & + \langle |\tilde g|^2 \tilde h_1, (\tilde h_2 \varphi_y)_x - (\tilde h_2 \tilde\varphi_x)_y\rangle_{H^{1/2}, H^{-1/2}} \\ & - \langle |\tilde g|^2 \tilde h_2, (\tilde h_1 \tilde\varphi_y)_x - (\tilde h_1 \tilde\varphi_x)_y\rangle_{H^{1/2}, H^{-1/2}}.\end{aligned}$$

Note that the above equality makes sense since $H^{1/2} \cap L^\infty$ is an algebra.

Proof of Lemma 10. — When g and h are smooth, the above equality is clear by Lemma 8. The general case follows by approximation, using the fact that, if $g_n \to g$ in $H^{1/2}$, $h_n \to h$ in $H^{1/2}$, $\|g_n\|_{L^\infty} \leq C$, $\|h_n\|_{L^\infty} \leq C$, then $g_n h_n \to gh$ in $H^{1/2}$ (this is proved using dominated convergence).

Proof of Lemma 9 completed. — When $|g| = |h| = 1$, we find that $T(gh) = T(g) + T(h)$, by combining Lemma 8 and Lemma 10. Also in this case, we have

$$T(g\bar h) = T(g) + T(\bar h) = T(g) - T(h).$$

Using (2.5), we find that

$$L(g\bar h) = \operatorname*{Sup}_{|\varphi|_{\text{Lip}} \leq 1} \langle T(g) - T(h), \varphi\rangle \leq C|g - h|_{H^{1/2}}\ (|g|_{H^{1/2}} + |h|_{H^{1/2}}).$$

Finally, inequality (2.9) is a trivial consequence of (2.7).

Remark **2.3.** — There is an alternative proof of (2.7)–(2.9), which consists of combining Lemma 2 (proved below) with the density result of T. Rivière [38]; *see* Lemma 11.

We now consider the special case where $g \in H^{1/2}(\Omega; S^1)$ is "smooth" except at a finite number of singularities:

Proof of Lemma 2. — The proof consists of 3 steps:

Step **1.** — Supp $T(g) \subset \cup_{j=1}^{k}\{a_j\}$

This is a trivial consequence of Lemma 7.

Step **2.** — $T(g) = \sum_{j=1} c_j \delta_{a_j}$.

In view of Remark 2.2 we may assume that Ω is flat near each a_j. We first note that, by a celebrated result of L. Schwartz, $T(g)$ is a finite sum of the form $T(g) = \sum_{j,\alpha} c_{j,\alpha} D^\alpha \delta_{a_j}$.

We want to prove that $c_{j,\alpha} = 0$ if $\alpha \neq 0$. For this purpose, it suffices to check that $\langle T(g), \varphi\rangle = 0$ if $\varphi(a_j) = 0, \forall j$. Let φ be any such function. Then, clearly, there is a sequence $(\varphi_n) \subset C_0^1(\Omega \setminus \cup_{j=1}^{k} \{a_j\})$ such that $\nabla\varphi_n \to \nabla\varphi$ a.e. and $\|\nabla\varphi_n\|_{L^\infty} \leq C$. Using Lemma 5, we obtain, by dominated convergence, that $\langle T(g), \varphi_n\rangle \to \langle T(g), \varphi\rangle$. On the other hand, $\langle T(g), \varphi_n\rangle = 0$ by Step 1.

Step **3.** — We have $c_j = 2\pi d_j$ where $d_j = \deg(g, a_j)$.

Let φ be a smooth function on Ω such that

$$\varphi(x) = \begin{cases} 1, & \text{for } |x - a_j| < R/2 \\ 0, & \text{for } |x - a_j| \geq R \end{cases},$$

where $R > 0$ is sufficiently small.

Note that $\nabla\varphi = 0$ outside the annulus $\mathscr{A} = \{x \in \Omega; |x - a_j| \in [R/2, R]\}$ and, moreover, that $g \in H^1$ on the same annulus. By Lemma 8 we find that

$$\langle T(g), \varphi\rangle = \int_{\mathscr{A}} g_1[(g_2\varphi_y)_x - (g_2\varphi_x)_y] - \int_{\mathscr{A}} g_2[(g_1\varphi_y)_x - (g_1\varphi_x)_y].$$

Integration by parts yields

$$\langle T(g), \varphi\rangle = \int_{\mathscr{A}} [(g_y \wedge g)\varphi_x + (g \wedge g_x)\varphi_y].$$

If g is smooth on $\mathscr{A}$, and if we integrate by parts once more, we find that

$$\langle \mathrm{T}(g), \varphi \rangle = -\int\limits_{\Sigma} (g_y \wedge g)\nu_x - \int\limits_{\Sigma} (g \wedge g_x)\nu_y,$$

where $\sum = \{x \in \Omega; |x - a_j| = \mathrm{R}/2\}$ and ν is the inward normal to $\mathscr{A}$ on $\sum$. With τ the direct tangent vector on $\sum$, we have

$$-(g_y \wedge g)\nu_x - (g \wedge g_x)\nu_y = g \wedge g_\tau .$$

Since g is S^1-valued, we find that

$$\langle \mathrm{T}(g), \varphi \rangle = 2\pi \deg(g, a_j).$$

For a general $g \in \mathrm{H}^1(\mathscr{A}; \mathrm{S}^1)$, we use the fact that $\mathrm{C}^\infty(\bar{\mathscr{A}}; \mathrm{S}^1)$ is dense in $\mathrm{H}^1(\mathscr{A}; \mathrm{S}^1)$ (*see* [41], [10] and [22]) and the stability of the degree under $\mathrm{H}^{1/2}$-convergence (*see* [17] and [22]), to conclude that $\langle \mathrm{T}(g), \varphi \rangle = 2\pi \deg(g, a_j)$.

We now recall a useful density result due to T. Rivière, which is the $\mathrm{H}^{1/2}$ analogue of a result of Bethuel and Zheng [10] concerning H^1 maps from B^3 to S^2 (*see* also a related result of Bethuel [4] concerning fractional Sobolev spaces).

Lemma **11** *(Rivière [38]). — Let* $\mathscr{R}$ *denote the class of maps belonging to* $\mathrm{W}^{1,p}(\Omega; \mathrm{S}^1)$, $\forall p < 2$, *which are* C^∞ *on* Ω *except at a finite number of points. Then* $\mathscr{R}$ *is dense in* $\mathrm{H}^{1/2}(\Omega; \mathrm{S}^1)$.

Remark **2.4.** — The above assertion does not appear in Rivière [38] but it is implicit in his proof; for the convenience of the reader we present a simple proof in Remark 5.1 – *see* also Appendix B for a more precise statement.

Remark **2.5.** — Similar density results hold in greater generality. Let $\Omega \subset \mathbf{R}^2$ be a smooth bounded domain. Let $0 < s < \infty$, $1 < p < \infty$ and

$$\mathscr{R}^{s,p} = \{u \in \mathrm{W}^{s,p}(\Omega; \mathrm{S}^1); u \text{ is } \mathrm{C}^\infty \text{ except at a finite number of points}\}.$$

Then $\mathscr{R}^{s,p}$ is dense in $\mathrm{W}^{s,p}(\Omega; \mathrm{S}^1)$ for all values of s and p (*see* [16]); this extends earlier results in [10], [25] and [4].

The density result combined with Lemma 2 yields "concrete" representations of the distribution $\mathrm{T}(g)$ and of the length of a minimal connection $\mathrm{L}(g)$ for a general $g \in \mathrm{H}^{1/2}(\Omega; \mathrm{S}^1)$; this is the content of Theorem 1.

Proof of Theorem 1. — We start by recalling a result of Brezis, Coron and Lieb [19] (*see* also [18]).

Lemma **12** *(Brezis, Coron and Lieb [19]). — Let* (X, d) *be a metric space. Let* $P_1, \ldots, P_k$, *and* $N_1, \ldots, N_k$ *be two collections of k points in* X. *Then*

$$L = \operatorname*{Min}_{\sigma \in S_k} \sum d(P_j, N_{\sigma(j)}) = \operatorname{Max}\left\{ \sum_j (\varphi(P_j) - \varphi(N_j));\ |\varphi|_{\mathrm{Lip}} \leq 1 \right\},$$

where S_k *denotes the group of permutation of* $\{1, 2, \ldots, k\}$.

The analogue of Lemma 12 for infinite sequences, which we need, is

Lemma **12′.** — *Let* (X, d) *be a metric space. Let* (P_i), (N_i) *be two infinite sequences such that* $\sum d(P_i, N_i) < \infty$.
Let

$$L = \operatorname*{Sup}_{\varphi} \left\{ \sum_i (\varphi(P_i) - \varphi(N_i));\ |\varphi|_{\mathrm{Lip}} \leq 1 \right\}. \tag{2.10}$$

Then

$$L = \operatorname*{Inf}_{(\widetilde{N}_i)} \left\{ \sum_i d(P_i, \widetilde{N}_i);\ \sum_i (\delta_{P_i} - \delta_{\widetilde{N}_i}) = \sum_i (\delta_{P_i} - \delta_{N_i}) \right\}.$$

Here, and throughout the rest of the paper, the equality

$$\sum_i (\delta_{\widetilde{P}_i} - \delta_{\widetilde{N}_i}) = \sum_i (\delta_{P_i} - \delta_{N_i})$$

for sequences $(\widetilde{P}_i)$, $(\widetilde{N}_i)$, (P_i), (N_i) such that

$$\sum_i d(\widetilde{P}_i, \widetilde{N}_i) < \infty \text{ and } \sum_i d(P_i, N_i) < \infty$$

means that

$$\sum_i (\varphi(\widetilde{P}_i) - \varphi(\widetilde{N}_i)) = \sum_i (\varphi(P_i) - \varphi(N_i)), \quad \forall \varphi \in \mathrm{Lip}.$$

Remark **2.6.** — A slightly different way of stating Lemma 12′ is the following. Given sequences (P_i), (N_i) in a metric space X with $\sum_i d(P_i, N_i) < \infty$, then

$$\begin{aligned} L &= \operatorname*{Inf}_{(\widetilde{P}_i),(\widetilde{N}_i)} \left\{ \sum_i d(\widetilde{P}_i, \widetilde{N}_i);\ \sum_i (\delta_{\widetilde{P}_i} - \delta_{\widetilde{N}_i}) = \sum_i (\delta_{P_i} - \delta_{N_i}) \right\} \\ &= \operatorname*{Sup}_{\varphi} \left\{ \sum_i (\varphi(P_i) - \varphi(N_i));\ \varphi \in \mathrm{Lip}\ (X; \mathbf{R}) \text{ and } |\varphi|_{\mathrm{Lip}} \leq 1 \right\}. \end{aligned} \tag{2.10′}$$

It is easy to see that the supremum in (2.10′) is always achieved. (Let (φ_n) be a maximizing sequence. By a diagonal process, we may assume that $\varphi_n(P_i)$ and $\varphi_n(N_i)$ converge for every i to limits which define a function ψ_0 on the set $\{P_i, N_i, i = 1, 2, ...\}$ with $|\psi_0|_{\text{Lip}} \le 1$. Next, ψ_0 is defined on all of X by a standard extension technique preserving the condition $|\psi|_{\text{Lip}} \le 1$). A natural question is whether the infimum in (2.10′) is achieved. The answer is negative. An interesting example, with $X = [0, 1]$, has been constructed by A. Ponce [36].

Proof of Lemma 12′. — Let $(\widetilde{N}_i)$ be such that

$$\sum(\delta_{P_i} - \delta_{\widetilde{N}_i}) = \sum(\delta_{P_i} - \delta_{N_i}).$$

Then

$$\sum_i(\varphi(P_i) - \varphi(N_i)) \le \sum_i d(P_i, \widetilde{N}_i)$$

and thus

$$L \le \sum_i d(P_i, \widetilde{N}_i).$$

Conversely, given $\varepsilon > 0$, we will construct a sequence $(\widetilde{N}_i)$ such that $\sum_i d(P_i, \widetilde{N}_i) \le L + \varepsilon$ and $\sum_i(\delta_{P_i} - \delta_{\widetilde{N}_i}) = \sum_i(\delta_{P_i} - \delta_{N_i})$.

Let n_0 be such that $\sum_{j>n_0} d(P_j, N_j) < \varepsilon/2$. Let σ_0 be a permutation of the integers $\{1, 2, ..., n_0\}$ which achieves

$$\operatorname*{Min}_{\sigma} \sum_{j=1}^{n_0} d(P_j, N_{\sigma(j)}).$$

Set

$$\widetilde{N}_j = \begin{cases} N_{\sigma_0(j)}, & \text{for} \quad 1 \le j \le n_0 \\ N_j, & \text{for} \quad j > n_0 \end{cases}.$$

Clearly,

$$\sum_{j\ge1}\left(\delta_{P_j} - \delta_{\widetilde{N}_j}\right) = \sum_{j\ge1}\left(\delta_{P_j} - \delta_{N_j}\right).$$

By definition of L, we have

$$\begin{aligned} L &= \operatorname*{Sup}_{|\varphi|_{\text{Lip}}\le1} \sum_{j\ge1}(\varphi(P_j) - \varphi(N_j)) \\ &\ge \operatorname*{Max}_{|\varphi|_{\text{Lip}}\le1} \sum_{j=1}^{n_0}(\varphi(P_j) - \varphi(N_j)) - \varepsilon/2 \\ &= \sum_{j=1}^{n_0} d(P_j, \widetilde{N}_j) - \varepsilon/2, \end{aligned}$$

by Lemma 12. Thus

$$\sum_{j\geq 1} d(\mathrm{P}_j, \widetilde{\mathrm{N}}_j) \leq \mathrm{L} + \varepsilon/2 + \varepsilon/2.$$

Proof of Theorem 1 continued. — For $g \in \mathscr{R}$ we have

$$\mathrm{L}(g) = \sum_{j=1}^{k} d(\mathrm{P}_j, \mathrm{N}_j)$$

and

$$\langle \mathrm{T}(g), \varphi \rangle = 2\pi \sum_{j=1}^{k} \big(\varphi(\mathrm{P}_j) - \varphi(\mathrm{N}_j)\big)$$

for some suitable integer k depending on g and suitable points $\mathrm{P}_1, \ldots, \mathrm{P}_k, \mathrm{N}_1, \ldots, \mathrm{N}_k$ in Ω. Let now $g \in \mathrm{H}^{1/2}(\Omega; \mathrm{S}^1)$ and consider a sequence $(g_n) \subset \mathscr{R}$ such that $|g_n - g|_{\mathrm{H}^{1/2}} \leq 1/2^n$.

By Lemma 2, $\mathrm{T}(g_{n+1}) - \mathrm{T}(g_n)$ is a finite sum of the form $2\pi \sum(\delta_{\mathrm{Q}_j} - \delta_{\mathrm{S}_j})$. By Lemma 12, after relabeling the points (Q_j) and (S_j), we may assume that

$$\mathrm{T}(g_1) = 2\pi \sum_{j=1}^{k_1} (\delta_{\mathrm{P}_j} - \delta_{\mathrm{N}_j})$$

and

$$\mathrm{T}(g_{n+1}) - \mathrm{T}(g_n) = 2\pi \sum_{j=k_n+1}^{k_{n+1}} (\delta_{\mathrm{P}_j} - \delta_{\mathrm{N}_j}), \forall n \geq 1$$

with

$$\begin{aligned} 2\pi \sum_{k_n+1}^{k_{n+1}} d(\mathrm{P}_j, \mathrm{N}_j) &= \mathrm{Sup}\, \{\langle \mathrm{T}(g_{n+1}) - \mathrm{T}(g_n), \varphi \rangle; \\ &\qquad\qquad \varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R}), |\varphi|_{\mathrm{Lip}} \leq 1\} \\ &\leq \mathrm{C}|g_{n+1} - g_n|_{\mathrm{H}^{1/2}}(|g_{n+1}|_{\mathrm{H}^{1/2}} + |g_n|_{\mathrm{H}^{1/2}}) \leq \mathrm{C}/2^n \text{ (by (2.5))}. \end{aligned}$$

We find that $\mathrm{T}(g_n) = 2\pi \sum_{j=1}^{k_n} (\delta_{\mathrm{P}_j} - \delta_{\mathrm{N}_j})$ and that $\sum_{j\geq 1} d(\mathrm{P}_j, \mathrm{N}_j) < \infty$.

Then for every $\varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R})$, the sequence $\big(\langle \mathrm{T}(g_n), \varphi \rangle\big)$ converges to $2\pi \sum_{j\geq 1} (\varphi(\mathrm{P}_j) - \varphi(\mathrm{N}_j))$. By Lemma 9, we find that $\mathrm{T}(g) = 2\pi \sum_{j\geq 1} (\delta_{\mathrm{P}_j} - \delta_{\mathrm{N}_j})$.

The second assertion in Theorem 1 is an immediate consequence of Lemma 12′ and Remark 2.6.

The last property in Theorem 1, namely the fact that, if $T(g)$ is a measure, then $T(g)$ may be represented as a *finite* sum of the form $2\pi \sum_j (\delta_{P_j} - \delta_{N_j})$, was originally announced in [13] and established using a technique of Jerrard and Soner [31], [32], which was based on the (Jacobian) structure of $T(g)$. We do not reproduce this argument since Smets [43] has proved the following general result:

Theorem **10** *(Smets [43]). — Let* X *be a compact metric space and let* $(P_j), (N_j) \subset X$ *be infinite sequences such that* $\sum d(P_j, N_j) < \infty$. *Assume that*

$$\left| \sum_j \big(\varphi(P_j) - \varphi(N_j)\big) \right| \le C \operatorname*{Sup}_{x \in X} |\varphi(x)|, \qquad \forall \varphi \in \text{Lip } (X).$$

Then one may find two finite *collections of points* $(Q_1, \ldots, Q_k)$ *and* $(M_1, \ldots, M_k)$, *such that*

$$\sum_{j=1}^{\infty} \big(\varphi(P_j) - \varphi(N_j)\big) = \sum_{i=1}^{k} \big(\varphi(Q_i) - \varphi(M_i)\big), \qquad \forall \varphi \in \text{Lip } (X).$$

We refer to [43] and to [36] for more general results.

Remark **2.7.** — A final word about the possibility of defining a minimal connection $L(g)$ when $g \in W^{s,p}(\Omega; S^1)$, for $0 < s < \infty$ and $1 \le p < \infty$. Recall (*see* [16] and Remark 2.5) that $\mathscr{R}^{s,p}$ is always dense in $W^{s,p}(\Omega; S^1)$ and note that we may always define $L(g)$ for $g \in \mathscr{R}^{s,p}$. A natural question is whether there is a continuous extension of L to $W^{s,p}$:

a) When $sp < 1$, the answer is negative. Indeed, let $g \in \mathscr{R}^{s,p}$ be a map with singularities of nonzero degree, so that $L(g) > 0$. There is a sequence (g_n) in $C^\infty(\Omega; S^1)$ such that $g_n \to g$ in $W^{s,p}$ (*see* Escobedo [25]). Clearly, $L(g_n) = 0$, $\forall n$, and $L(g_n)$ does not converge to $L(g)$.

b) When $sp \ge 2$, the answer is positive since $L(g) = 0$, $\forall g \in \mathscr{R}^{s,p}$ (any singularity in $W^{s,p}$ must have zero degree since $W^{s,p} \subset \text{VMO}$).

c) When $1 \le sp < 2$, the answer is positive. For $s > 1/2$ the proof is easy (indeed if $s \in (1/2, 1)$, then $W^{s,p}(\Omega; S^1) \subset H^{1/2}$, while if $s \ge 1$, then $W^{s,p} \subset W^{1,1}$ and we may apply the result of Demengel [24] which asserts the existence of a minimal connection in $W^{1,1}$). The case where $s \le 1/2$ is delicate and studied in [16].

3. Lifting for $g \in Y$. Characterization of Y. Proof of Theorem 3

The main ingredient in this Section is the following estimate, whose proof has already been presented in Bourgain-Brezis [11]. We reproduce it here for the convenience of the reader.

Theorem **3′.** — *Let* ψ *be a smooth real-valued function on the* d*-dimensional torus* $\mathbf{T}^d$ *and set* $g = e^{\imath\psi}$. *Then*

$$|\psi|_{H^{1/2}+W^{1,1}} \le C(d)(1+|g|_{H^{1/2}})|g|_{H^{1/2}}. \qquad (\mathbf{3.1})$$

Here, $|\ |$ denotes the canonical seminorm on $H^{1/2}$ (respectively $H^{1/2}+W^{1,1}$).

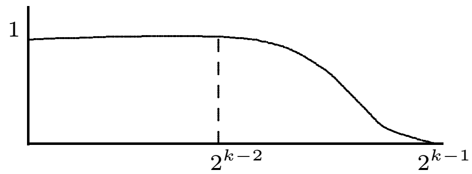
Proof of Theorem 3′. — Write $g - \fint g$ as a Fourier series,

$$g - \fint g = \sum_{\xi\in\mathbf{Z}^d\setminus\{0\}} \hat{g}(\xi)e^{\imath x\cdot\xi}.$$

The $H^{1/2}$-component in the decomposition of ψ will be obtained as a paraproduct of $g - \fint g$ and $\bar{g} - \fint \bar{g}$. Let

$$P = \sum_k \Big[\sum_{\xi_2} \lambda_k(|\xi_2|)\overline{\hat{g}(\xi_2)}e^{-\imath x\cdot\xi_2}\Big]\Big[\sum_{2^k\le|\xi_1|<2^{k+1}} \hat{g}(\xi_1)e^{\imath x\cdot\xi_1}\Big], \qquad (\mathbf{3.2})$$

where, for each k, we let $0 \le \lambda_k \le 1$ be a smooth function on $\mathbf{R}_+$ as below:



We claim that

$$|P|_{H^{1/2}} \le C\|g\|_\infty |g|_{H^{1/2}} \qquad (\mathbf{3.3})$$

and

$$|\psi - \frac{1}{\imath}P|_{W^{1,1}} \le C|g|^2_{H^{1/2}}. \qquad (\mathbf{3.4})$$

Proof of (3.3). — This is totally obvious from the construction since, with $\|\ \|_p$ standing for the L^p-norm, we have

$$(\mathbf{3.5}) \qquad \begin{aligned} |P|^2_{H^{1/2}} &\sim \sum_k 2^k \Big\| \Big[\sum_{\xi_2} \lambda_k(|\xi_2|)\overline{\hat{g}(\xi_2)}e^{-\imath x\cdot\xi_2}\Big]\Big[\sum_{2^k\le|\xi_1|<2^{k+1}} \hat{g}(\xi_1)e^{\imath x\cdot\xi_1}\Big]\Big\|_2^2 \\ &\le \sum_k 2^k \Big\| \sum \lambda_k(|\xi|)\overline{\hat{g}(\xi)}e^{-\imath x\cdot\xi}\Big\|^2_\infty \Big[\sum_{|\xi|\sim 2^k} |\hat{g}(\xi)|^2\Big] \\ &\le C\|g\|^2_\infty |g|^2_{H^{1/2}}. \end{aligned}$$

Proof of (3.4). — We estimate, for instance,

$$\left\| \partial_1 \psi - \frac{1}{\imath}\partial_1 P \right\|_{L^1}. \tag{3.6}$$

Thus, letting $\xi = (\xi^1, ..., \xi^d) \in \mathbf{Z}^d$, we have

$$\partial_1 \psi = \frac{1}{\imath}\bar{g}\partial_1 g = \sum_{\xi_1,\xi_2\in\mathbf{Z}^d} \xi_1^1 \hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}\, e^{\imath x\cdot(\xi_1-\xi_2)} \tag{3.7}$$

and, by (3.2), we find

$$\frac{1}{\imath}\partial_1 P = \sum_k \sum_{\substack{2^k\le|\xi_1|<2^{k+1}\\ \xi_2\in\mathbf{Z}^d}} \left(\xi_1^1 - \xi_2^1\right)\lambda_k(|\xi_2|)\hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}e^{\imath x\cdot(\xi_1-\xi_2)} \tag{3.8}$$

and

$$\partial_1 \psi - \frac{1}{\imath}\partial_1 P = \sum_k \sum_{\substack{2^k\le|\xi_1|<2^{k+1}\\ \xi_2\in\mathbf{Z}^d}} m_k(\xi_1,\xi_2)\hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}e^{\imath x\cdot(\xi_1-\xi_2)}. \tag{3.9}$$

Here, by definition of λ_k,

$$m_k(\xi_1,\xi_2) = \xi_1^1 - \lambda_k(|\xi_2|)\left(\xi_1^1 - \xi_2^1\right) = \begin{cases} \xi_2^1, & \text{if } |\xi_2| \le 2^{k-2} \\ \xi_1^1, & \text{if } |\xi_2| \ge 2^{k-1} \end{cases}. \tag{3.10}$$

Estimate

$$\left\| \partial_1 \psi - \frac{1}{\imath}\partial_1 P \right\|_1 \le \sum_{k_1,k_2} \left\| \sum_{|\xi_1|\sim 2^{k_1},|\xi_2|\sim 2^{k_2}} m_{k_1}(\xi_1,\xi_2)\hat{g}(\xi_1)\overline{\hat{g}(\xi_2)}e^{\imath x\cdot(\xi_1-\xi_2)} \right\|_1. \tag{3.11}$$

We split the right-hand side of (3.11) as

$$\sum_{k_1\sim k_2} + \sum_{k_1<k_2-4} + \sum_{k_1>k_2+4} = (3.12) + (3.13) + (3.14).$$

Clearly, $2^{-k}m_k(\xi_1,\xi_2)$ restricted to $[|\xi_1| \sim 2^k] \times [|\xi_2| \sim 2^k]$ is a smooth multiplier satisfying the usual derivative bounds. Therefore,

$$(3.12) \le C \sum_k 2^k \left\| \sum_{|\xi_1|\sim 2^k} \hat{g}(\xi_1)e^{\imath x\cdot\xi_1} \right\|_2 \left\| \sum_{|\xi_2|\sim 2^k} \hat{g}(\xi_2)e^{\imath x\cdot\xi_2} \right\|_2 \sim |g|^2_{H^{1/2}}. \tag{3.15}$$

If $k_1 < k_2 - 4$, then $|\xi_2| > 2^{k_1}$ and $m_{k_1}(\xi_1, \xi_2) = \xi_1^1$, by (3.10). Therefore

$$
\begin{aligned}
(3.13) &= \sum_{k_1 < k_2 - 4} \Big\| \sum_{|\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}} \xi_1^1 \hat{g}(\xi_1) \overline{\hat{g}(\xi_2)} e^{\imath x \cdot (\xi_1 - \xi_2)} \Big\|_1 \\
&\leq \sum_{k_1 < k_2 - 4} 2^{k_1} \Big\| \sum_{|\xi_1| \sim 2^{k_1}} \hat{g}(\xi_1) e^{\imath x \cdot \xi_1} \Big\|_2 \cdot \Big\| \sum_{|\xi_2| \sim 2^{k_2}} \hat{g}(\xi_2) e^{\imath x \cdot \xi_2} \Big\|_2 \qquad (\mathbf{3.16}) \\
&\leq \sum_{k_1 < k_2} 2^{k_1} \Big(\sum_{|\xi_1| < 2^{k_1}} |\hat{g}(\xi_1)|^2 \Big)^{1/2} \Big(\sum_{|\xi_2| \sim 2^{k_2}} |\hat{g}(\xi_2)|^2 \Big)^{1/2} \leq C |g|^2_{H^{1/2}}.
\end{aligned}
$$

If $k_1 > k_2 + 4$, then $|\xi_2| < 2^{k_1 - 2}$ and $m_{k_1}(\xi_1, \xi_2) = \xi_2^1$ and the bound on (3.14) is similar.

We now derive a consequence of Theorem 3′:

Corollary **1.** — *Let* G *be a smooth bounded domain in* $\mathbf{R}^{d+1}$ *such that* $\Omega = \partial G$ *is connected. Let* ψ *be a Lipschitz real-valued function on* Ω *and set* $g = e^{\imath\psi}$. *Then*

$$|\psi|_{H^{1/2}+W^{1,1}} \leq C_\Omega (1 + |g|_{H^{1/2}}) |g|_{H^{1/2}}.$$

Proof of Corollary 1. — It is convenient to divide the argument into 4 steps.

Step **1.** — The conclusion of Theorem 3′ still holds if ψ is Lipschitz. This is clear by density.

Step **2.** — The conclusion of Theorem 3′ holds if $\mathbf{T}^d$ is replaced by a d-dimensional cube Q and $\psi \in \operatorname{Lip}(Q)$. This is done by standard reflections and extensions by periodicity.

As a consequence, we have

Step **3.** — The conclusion of Step 2 holds when Q is replaced by a domain in Ω diffeomorphic to a cube.

Step **4.** — Proof of Corollary 1. Consider a finite covering (U_α) of Ω by domains diffeomorphic to cubes. Note that, if $U_\alpha \cap U_\beta \neq 0$, then

$$|\psi|_{H^{1/2}+W^{1,1}(U_\alpha \cup U_\beta)} \sim |\psi|_{H^{1/2}+W^{1,1}(U_\alpha)} + |\psi|_{H^{1/2}+W^{1,1}(U_\beta)}.$$

Using the connectedness of Ω, we find that

$$|\psi|_{H^{1/2}+W^{1,1}(\Omega)} \sim \sum_\alpha |\psi|_{H^{1/2}+W^{1,1}(U_\alpha)}.$$

The conclusion now follows from Step 3.

Proof of Theorem 3. — First, let $g \in Y$ and consider a sequence $(g_n) \subset C^\infty(\Omega; S^1)$ such that $g_n \to g$ in $H^{1/2}$. Since Ω is simply connected, we may write $g_n = e^{\imath\psi_n}$, with $\psi_n \in C^\infty(\Omega; \mathbf{R})$.

Applying Corollary 1 to $g_n \bar{g}_m$, we find

$$|\psi_n - \psi_m|_{H^{1/2}+W^{1,1}} \leq C(1 + |g_n \bar{g}_m|_{H^{1/2}})|g_n \bar{g}_m|_{H^{1/2}}.$$

Since $g_n \to g$ in $H^{1/2}$ and $|g_n| \equiv 1$, we have $|g_n \bar{g}_m|_{H^{1/2}} \to 0$ as $m, n \to \infty$ (*see* the proof of Lemma 10). Therefore, $(\psi_n - \fint_\Omega \psi_n)$ converges in $H^{1/2} + W^{1,1}$ to a map ζ. Then, with C an appropriate constant, $\psi = \zeta + C \in H^{1/2} + W^{1,1}, g = e^{\imath\psi}$ and ψ satisfies the estimate

$$|\psi|_{H^{1/2}+W^{1,1}} \leq C(1 + |g|_{H^{1/2}})|g|_{H^{1/2}}.$$

The uniqueness of ψ is an immediate consequence of the following

Lemma **13.** — *Let Ω be a connected open set in $\mathbf{R}^d$. Let $f : \Omega \to \mathbf{Z}$ be such that $f = f_0 + \sum_j f_j$, with $f_0 \in W^{1,1}_{loc}(\Omega; \mathbf{R})$ and $f_j \in W^{s_j,p_j}_{loc}(\Omega; \mathbf{R})$, where $0 < s_j < 1,\ 1 < p_j < \infty$, $s_j p_j \geq 1$. Then f is a constant.*

The proof of Lemma 13 is given in [12], Appendix B, Step 2. The argument is by dimensional reduction, observing that the restriction of f to almost every line is $\mathbf{Z}$-valued and VMO; thus it is constant (*see* [22]). This implies (*see* e.g. Lemma 2 in [20]) that f is locally constant in Ω.

We now prove the last assertion in Theorem 3. Let $g \in H^{1/2}(\Omega; S^1)$ be such that $g = e^{\imath\psi}$ for some $\psi \in H^{1/2} + W^{1,1}(\Omega; \mathbf{R})$. Let $\psi = \psi_1 + \psi_2$, with $\psi_1 \in H^{1/2}$ and $\psi_2 \in W^{1,1}$. Set $g_j = e^{\imath\psi_j}, j = 1, 2$. Clearly, $g_1 \in X$, so that $g_1 \in Y$ and thus $T(g_1) = 0$. On the other hand, $g_2 \in H^{1/2} \cap W^{1,1}$, since $g_2 = g\bar{g}_1 \in H^{1/2}$. Therefore, we may use the representation of $T(g_2)$ given by Lemma 1 and find, after localization, as in Remark 2.2,

$$\langle T(g_2), \varphi\rangle = \int_\omega (\psi_{2x}\varphi_y - \psi_{2y}\varphi_x) = 0, \quad \forall\varphi \in C^1_0(\omega; \mathbf{R}).$$

Hence $T(g_2) = 0$. By (2.7) in Lemma 9, we obtain that $T(g) = 0$. Using Theorem 2, we derive that $g \in Y$.

Remark **3.1.** — Theorem 3 is not fully satisfactory since, whenever $\psi \in W^{1,1}$, the function $e^{\imath\psi}$ need not belong to $H^{1/2}$ (but "almost", since $e^{\imath\psi} \in W^{1,1} \cap L^\infty$, which is almost contained in $H^{1/2}$, but not quite). Here is an example: take some $\psi \in W^{1,1} \cap L^\infty$ with $\psi \notin H^{1/2}$. We may assume $|\psi| \leq 1$. Then

$$|e^{\imath\psi(x)} - e^{\imath\psi(y)}| \sim |\psi(x) - \psi(y)|,$$

so that

$$|e^{\imath\psi}|_{H^{1/2}} \sim |\psi|_{H^{1/2}} = +\infty.$$

4. Lifting for a general $g \in H^{1/2}$. Optimizing the BV part of the phase. Proof of Theorems 4 and 5

Assume g is a general element in $H^{1/2}(\Omega; S^1)$. This g need not be in Y and thus need not have a lifting in $H^{1/2} + W^{1,1}$. However, g has a lifting in the larger space $H^{1/2} + BV$. This is an immediate consequence of Theorem 3 (and estimate (1.9)) and of the following result of T. Rivière [38] (which is the analogue of a similar result of Bethuel [3] for H^1 maps from B^3 to S^2).

Lemma **14** *(Rivière [38]). — Let* $g \in H^{1/2}(\Omega; S^1)$. *Then there is a sequence* $(g_n) \subset C^\infty(\Omega; S^1)$ *such that* $g_n \rightharpoonup g$ *weakly in* $H^{1/2}$.

Remark **4.1.** — Lemma 14 implies that $g \mapsto T(g)$ and $g \mapsto L(g)$ are not continuous under weak $H^{1/2}$ convergence.

Here is a refined version of Lemma 14 which will be proved at the end of Section 4.2:

Lemma **14′.** — *Let* $g \in H^{1/2}(\Omega; S^1)$. *Then there is a sequence* $(g_n) \subset C^\infty(\Omega; S^1)$ *such that* $g_n \rightharpoonup g$ *weakly in* $H^{1/2}$ *and*

$$\limsup_{n\to\infty} |g_n|^2_{H^{1/2}} \leq |g|^2_{H^{1/2}} + C_\Omega L(g),$$

for some constant C_Ω *depending only on* Ω. *Moreover, for* **every** *sequence* (g_n) *in* Y *such that* $g_n \to g$ *a.e., we have*

$$\liminf_{n\to\infty} |g_n|^2_{H^{1/2}} \geq |g|^2_{H^{1/2}} + C'_\Omega L(g),$$

for some positive constant C'_Ω *depending only on* Ω.

Existence of a lifting in $H^{1/2} + BV$

Let $g \in H^{1/2}(\Omega; S^1)$. For g_n as in the above Lemma 14, write, using Corollary 1, $g_n = e^{\imath\varphi_n}$, with $\varphi_n \in C^\infty(\Omega; S^1)$ and

$$|\varphi_n|_{H^{1/2}+W^{1,1}} \leq C_\Omega\big(|g_n|_{H^{1/2}} + |g_n|^2_{H^{1/2}}\big).$$

Then, up to a subsequence, there is some $\zeta \in H^{1/2} + BV$ such that $\varphi_n - \fint \varphi_n \to \zeta$ a.e. We find that $g = e^{\imath\varphi}$, with $\varphi = \zeta + C$ and C some appropriate constant. Moreover, we may write $\varphi = \varphi_1 + \varphi_2$, with

$$(\mathbf{4.1}) \qquad |\varphi_1|_{H^{1/2}} + |\varphi_2|_{BV} \leq C_\Omega\big(|g|_{H^{1/2}} + |g|^2_{H^{1/2}}\big).$$

An additional information about the decomposition is contained in Theorem 4. On the other hand note that estimate (4.1) implies that every $g \in H^{1/2}$ may be written as $g = g_1 g_2$, with

$$g_1 = e^{\imath\varphi_1} \in X \text{ and } g_2 = e^{\imath\varphi_2} \in H^{1/2} \cap BV,$$
$$\text{i.e., } H^{1/2} = (X) \cdot (H^{1/2} \cap BV).$$

A finer assertion is $H^{1/2} = (X) \cdot (H^{1/2} \cap W^{1,1})$, which is the content of Theorem 5.

The proofs of Theorems 4 and 5 require a number of ingredients:

a) the dipole construction (*see* Section 4.1). This is inspired by the dipole construction in the $H^1(B^3; S^2)$ context (*see* [19] and [3]);

b) the construction of a map $g \in H^{1/2}(\Omega; S^1) \cap W^{1,1}$ having *prescribed* singularities (with control of the norms). This is done in Section 4.2;

c) lower bound estimates for the BV part of the phase, which are presented in Section 4.3, in the spirit of [19], [2], [27]. This is a typical phenomenon in the context of relaxed energies and/or Cartesian Currents. More precisely, if one considers the Sobolev space $X = W^{s,p}(U; S^k)$, $U \subset \mathbf{R}^N$, and if smooth maps are *not* dense in X for the strong topology, then the relaxed energy is defined by

$$E(g) = \operatorname{Inf}\Big\{ \liminf_{n\to\infty} \|g_n\|^p_{W^{s,p}};\ (g_n) \subset C^\infty(\bar{U}; S^k),\ g_n \to g \text{ a.e.}\Big\}.$$

The gap $E(g) - \|g\|^p_{W^{s,p}} \geq 0$ has often a geometrical interpretation in terms of the singular set of g. For example, in the $H^1(B^3; S^2)$ context, the gap is $8\pi L(g)$, where $L(g)$ is the length of a minimal connection associated with the singularities of g (*see* [19]). We will consider, in Section 4.3, similar lower bounds for S^1-valued maps on Ω.

4.1. *The dipole construction*

Throughout this section, the metric d denotes the geodesic distance d_Ω in Ω and $L(g) = L_\Omega(g)$.

Lemma **15.** — *Let* $P, N \in \Omega$, $P \neq N$. *Given any* $\varepsilon > 0$ *there exists some* $g (= g_\varepsilon)$ *such that*

(**4.2**) $$g \in W^{1,\infty}_{loc}(\Omega \backslash \{P, N\}; S^1) \cap W^{1,p}(\Omega; S^1),\ \forall p \in [1, 2),$$

(**4.3**) $$T(g) = 2\pi(\delta_P - \delta_N),$$

(**4.4**) $$|g|_{W^{1,1}} \leq 2\pi d(P, N) + \varepsilon,$$

(**4.5**) $|g|^2_{H^{1/2}} \le C_\Omega d(P, N)$ *where* C_Ω *depends only on* Ω,

(**4.6**) $\begin{cases} \textit{there is a function } \psi(=\psi_\varepsilon) \in BV(\Omega; \mathbf{R}) \textit{ such that } g = e^{\imath\psi}, \\ \textit{with } \operatorname{supp} \psi \subset \Lambda = \{x \in \Omega;\, d(x,\gamma) < \varepsilon\} \textit{ and } |\psi|_{BV} \le 4\pi d(P,N) + \varepsilon, \end{cases}$

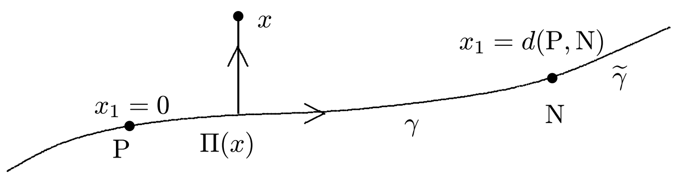
where γ *is a geodesic curve joining* P *and* N,

(**4.7**) $g = 1$ *outside* Λ.

Proof. — Extend γ smoothly beyond P and N; denote this extension by $\tilde{\gamma}$. For $\varepsilon_0 > 0$ sufficiently small (depending on $\tilde{\gamma}$), the projection Π of

$$\Gamma = \{x \in \Omega;\, d(x, \gamma) < \varepsilon_0\}$$

onto $\widetilde{\gamma}$ is well-defined and smooth. Let x_1 be the arclength coordinate on $\widetilde{\gamma}$, such that $x_1(P) = 0$, $x_1(N) = d(P, N) = L$.



For $x \in \Gamma$, let $x_1 = x_1(\Pi(x))$ be the arclength coordinate of $\Pi(x)$ on $\tilde{\gamma}$ and let $x_2 = \pm d(x, \tilde{\gamma})$, where we choose "+" if the basis formed by the (oriented) tangent vector at $\Pi(x)$ to $\tilde{\gamma}$, the (oriented) tangent vector at $\Pi(x)$ to the geodesic segment $[\Pi(x), x]$ and the exterior normal n at $\Pi(x)$ to G is direct in $\mathbf{R}^3$; we choose "–" otherwise. Define the mapping

$$x \in \Gamma \mapsto \Phi(x) = (x_1, x_2) \in \mathbf{R}^2.$$

Let $0 < \delta < \varepsilon_0$ and consider the domain in $\mathbf{R}^2$

$$\widetilde{\Gamma}_\delta = \left\{ (t_1, t_2) \in \mathbf{R}^2;\, 0 < t_1 < L \text{ and } |t_2| < \frac{2\delta}{L} \min(t_1, L - t_1) \right\}.$$

and the corresponding domain Γ_δ in Ω,

$$\Gamma_\delta = \{x \in \Gamma;\, \Phi(x) \in \widetilde{\Gamma}_\delta\}.$$

Set, on $\mathbf{R}^2$,

$$\tilde{g}(t) = \tilde{g}(t_1, t_2) = \begin{cases} \exp(\imath\varphi(Lt_2/2\delta \min(t_1, L - t_1)), & \text{on } \widetilde{\Gamma}_\delta, \\ 1, & \text{outside } \widetilde{\Gamma}_\delta, \end{cases}$$

where φ is defined by $\varphi(s) = \begin{cases} \pi(s+1)^+, & \text{if } s \le 1 \\ 2\pi, & \text{if } s > 1 \end{cases}$.

An easy computation shows that

$$\tilde{g} \in W^{1,\infty}_{loc}(\mathbf{R}^2 \setminus \{\widetilde{P}, \widetilde{N}\}; S^1) \cap W^{1,p}_{loc}(\mathbf{R}^2; S^1), \quad \forall\, 1 \le p < 2,$$

where $\widetilde{P} = \Phi(P) = (0,0)$ and $\widetilde{N} = \Phi(N) = (L, 0)$. More precisely, we have

$$|\tilde{g}|^p_{W^{1,p}(\tilde{\Gamma}_\delta)} = 4 \int_0^{L/2} \left(\frac{L}{2\delta t_1}\right)^{p-1} dt_1 \int_0^{+1} \pi^p \left(\left(\frac{2\delta s}{L}\right)^2 + 1\right)^{p/2} ds.$$

In particular, we find

(**4.8**) $$|\tilde{g}|_{W^{1,1}(\tilde{\Gamma}_\delta)} \le 2\pi\,(L + \delta)$$

and, for every $1 \le p < 2$,

(**4.9**) $$|\tilde{g}|_{W^{1,p}(\tilde{\Gamma}_\delta)} \le C_p (L\delta)^{1/p} \left(\frac{1}{\delta} + \frac{1}{L}\right).$$

For later purpose, it is also convenient to observe that, for any $1 \le q \le \infty$,

(**4.10**) $$\|\tilde{g} - 1\|_{L^q(\tilde{\Gamma}_\delta)} \le 2(L\delta)^{1/q}.$$

We now transport the function $\tilde{g}$ on Ω and define

$$g(x) = \begin{cases} \tilde{g}(\Phi(x)), & \text{if } x \in \Gamma_\delta \\ 1, & \text{outside } \Gamma_\delta \end{cases}.$$

It is not difficult to see that Φ is a C^2-diffeomorphism on Γ and

(**4.11**) $$|\,\mathrm{Jac}\ \Phi(x) - 1| \le C_\gamma \delta \quad \text{on } \Gamma_\delta,$$

where C_γ is a constant depending on γ. Combining (4.8)–(4.11) yields

(**4.12**) $$|g|_{W^{1,1}(\Omega)} \le 2\pi(L+\delta)(1 + C_\gamma\delta),$$

(**4.13**) $$|g|_{W^{1,p}(\Omega)} \le C_p(L\delta)^{1/p}\left(\frac{1}{\delta} + \frac{1}{L}\right)(1 + C_\gamma \delta), \quad 1 \le p < 2,$$

and

(**4.14**) $$\|g - 1\|_{L^q(\Omega)} \le 2(L\delta)^{1/q}(1 + C_\gamma\delta).$$

From a variant of the Gagliardo–Nirenberg inequality (*see* e.g. [21] and the references therein) we know that, if $1 < p < \infty$ and

(**4.15**) $$\frac{1}{p} + \frac{1}{q} = 1,$$

then

(**4.16**) $$|g|^2_{H^{1/2}(\Omega)} \le C(p, \Omega)|g|_{W^{1,p}(\Omega)}\|g\|_{L^q(\Omega)}.$$

We now check properties (4.2)–(4.7): (4.2), (4.3) and (4.7) are clear. Estimate (4.4) (resp. (4.5)) follows from (4.12) (resp. (4.16) applied e.g. with $p = 3/2$) provided δ is sufficiently small (depending on ε and γ).

Construction of ψ and estimate (4.6)

In the region where $\tilde{g} \equiv 1$, we take $\tilde{\psi} \equiv 0$. In the region $\widetilde{\Gamma}_\delta$ where $\tilde{g}$ lives, we take

$$\tilde{\psi}(t_1, t_2) = \begin{cases} \varphi(\mathrm{L}t_2/2\delta \min(t_1, \mathrm{L} - t_1)), & \text{if } t_2 \leq 0 \\ \varphi(\mathrm{L}t_2/2\delta \min(t_1, \mathrm{L} - t_1)) - 2\pi, & \text{if } t_2 > 0 \end{cases}.$$

Set

$$\psi(x) = \begin{cases} \tilde{\psi}(\Phi(x)), & \text{if } x \in \Gamma_\delta \\ 0, & \text{outside } \Gamma_\delta \end{cases}.$$

Then $|\mathrm{D}\psi| = |\mathrm{D}g| + 2\pi\delta_\gamma$, where δ_γ is the $1 - d$ Hausdorff measure uniformly distributed on γ. Thus

$$|\psi|_{\mathrm{BV}} = \int_\Omega |\mathrm{D}\psi| = \int_\Omega |\mathrm{D}g| + 2\pi\mathrm{L} \leq 4\pi\mathrm{L} + \varepsilon.$$

4.2. *Construction of a map with prescribed singularities*

Let (P_i), (N_i) be two sequences of points in $\Omega = \partial\mathrm{G}$ such that $\sum d_\Omega(\mathrm{P}_i, \mathrm{N}_i) < \infty$. Define

$$\mathrm{T} = 2\pi \sum_i (\delta_{\mathrm{P}_i} - \delta_{\mathrm{N}_i})$$

and

$$\mathrm{L} = \mathrm{L}_\Omega = \frac{1}{2\pi} \sup\{\langle \mathrm{T}, \varphi\rangle; \varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R}), |\varphi|_{\mathrm{Lip}} \leq 1\}.$$

Lemma **16.** — *a) For every* $g \in \mathrm{W}^{1,1}(\Omega; \mathrm{S}^1) \cap \mathrm{H}^{1/2}(\Omega; \mathrm{S}^1)$ *such that* $\mathrm{T}(g) = \mathrm{T}$, *we have*

$$\int_\Omega |\mathrm{D}g| \geq 2\pi\mathrm{L} \quad \textit{and} \quad |g|^2_{\mathrm{H}^{1/2}} \geq \mathrm{C}_\Omega \mathrm{L},$$

where C_Ω *is a positive constant depending only on* Ω.

b) For every $\varepsilon > 0$, *there is some* $g (= g_\varepsilon) \in \mathrm{W}^{1,1}(\Omega; \mathrm{S}^1) \cap \mathrm{H}^{1/2}(\Omega; \mathrm{S}^1)$ *such that*

$$\mathrm{T}(g) = \mathrm{T}, \tag{4.17}$$

$$|g|_{\mathrm{W}^{1,1}} \leq 2\pi(\mathrm{L} + \varepsilon), \tag{4.18}$$

$$|g|^2_{\mathrm{H}^{1/2}} \leq \mathrm{C}_\Omega \mathrm{L}, \tag{4.19}$$

$$\text{(4.20)} \qquad \begin{cases} \textit{there is a function } \psi(=\psi_\varepsilon) \in \mathrm{BV}(\Omega; \mathbf{R}) \textit{ such that} \\ g = e^{\imath\psi}, \textit{ and } |\psi|_{\mathrm{BV}} \le 4\pi(\mathrm{L}+\varepsilon) \end{cases},$$

$$\text{(4.21)} \qquad \text{meas (Supp } \psi) = \text{meas (Supp } (g-1)) \le \varepsilon.$$

In the proof of Lemma 16 we will use:

Lemma **17.** — *Let* (u_n) *be a bounded sequence in* $\mathrm{H}^{1/2}(\Omega; \mathbf{C}) \cap \mathrm{L}^\infty$ *such that* $u_n \to 1$ *a.e. Then for every* $v \in \mathrm{H}^{1/2}(\Omega; \mathbf{C}) \cap \mathrm{L}^\infty$ *we have*

$$|u_n v|^2_{\mathrm{H}^{1/2}} = \int_\Omega\int_\Omega |v(x)|^2 \frac{|u_n(x)-u_n(y)|^2}{d(x,y)^3} + |v|^2_{\mathrm{H}^{1/2}} + o(1) \qquad \textit{as } n \to \infty.$$

Proof of Lemma 17. — We have

$$\begin{aligned} |u_n v|^2_{\mathrm{H}^{1/2}} &= \int_\Omega\int_\Omega |v(x)|^2 \frac{|u_n(x)-u_n(y)|^2}{d(x,y)^3} + \int_\Omega\int_\Omega |u_n(y)|^2 \frac{|v(x)-v(y)|^2}{d(x,y)^3} + 2\mathrm{I}_n \\ &= \int_\Omega\int_\Omega |v(x)|^2 \frac{|u_n(x)-u_n(y)|^2}{d(x,y)^3} + |v|^2_{\mathrm{H}^{1/2}} + 2\mathrm{I}_n + o(1), \end{aligned}$$

where

$$\mathrm{I}_n = \int_\Omega\int_\Omega \frac{(v(x)(u_n(x)-u_n(y))) \cdot (u_n(y)(v(x)-v(y)))}{d(x,y)^3},$$

so that it suffices to prove that

$$\mathrm{J}_n = \int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)||v(x)-v(y)|}{d(x,y)^3} \to 0.$$

Fix some $\varepsilon > 0$. Then

$$\begin{aligned} \mathrm{J}_n &= \iint_{d(x,y)\ge\varepsilon} \frac{|u_n(x)-u_n(y)||v(x)-v(y)|}{d(x,y)^3} + \iint_{d(x,y)<\varepsilon} \frac{|u_n(x)-u_n(y)||v(x)-v(y)|}{d(x,y)^3} \\ &= o(1) + \iint_{d(x,y)<\varepsilon} \frac{|u_n(x)-u_n(y)||v(x)-v(y)|}{d(x,y)^3} \\ &\le o(1) + |u_n|_{\mathrm{H}^{1/2}} \left(\iint_{d(x,y)<\varepsilon} \frac{|v(x)-v(y)|^2}{d(x,y)^3} \right)^{1/2}, \end{aligned}$$

so that $\mathrm{J}_n \to 0$.

Proof of Lemma 16. — a) By Lemma 1, we have

$$\langle \mathrm{T}(g), \varphi \rangle = \int_{\Omega} g \wedge (g_x \varphi_y - g_y \varphi_x), \quad \forall \varphi \in \mathrm{Lip}\ (\Omega; \mathbf{R}),$$

so that

$$|\langle \mathrm{T}(g), \varphi \rangle| \leq \int_{\Omega} |g|\ |\mathrm{D}g|\ |\mathrm{D}\varphi| \leq \int_{\Omega} |\mathrm{D}g|$$

if $|\varphi|_{\mathrm{Lip}} \leq 1$. Taking the Sup over all such φ's yields the first inequality.

The second inequality in a), namely $\mathrm{L} \leq \mathrm{C}_{\Omega}|g|^2_{\mathrm{H}^{1/2}}$, was already established in Lemma 9.

b) Let $\varepsilon < \mathrm{L}$. By Lemma 12′, we may find a sequence $(\widetilde{\mathrm{N}}_j)$ such that

$$(\mathbf{4.22}) \qquad \mathrm{T} = 2\pi \sum_i (\delta_{\mathrm{P}_i} - \delta_{\mathrm{N}_i}) = 2\pi \sum_j (\delta_{\mathrm{P}_j} - \delta_{\widetilde{\mathrm{N}}_j})$$

and

$$(\mathbf{4.23}) \qquad \sum_j d(\mathrm{P}_j, \widetilde{\mathrm{N}}_j) < \mathrm{L} + \varepsilon/4\pi.$$

By the dipole construction (Lemma 15), for each j and for each $\varepsilon_j > 0$, there is some $g_j = g_{j,\varepsilon_j}$ such that

$$(\mathbf{4.24}) \qquad \mathrm{T}(g_j) = 2\pi(\delta_{\mathrm{P}_j} - \delta_{\widetilde{\mathrm{N}}_j}),$$

$$(\mathbf{4.25}) \qquad \int_{\Omega} |\mathrm{D}g_j| \leq 2\pi d(\mathrm{P}_j, \widetilde{\mathrm{N}}_j) + \varepsilon_j,$$

$$(\mathbf{4.26}) \qquad |g_j|^2_{\mathrm{H}^{1/2}} \leq \mathrm{C}_{\Omega} d(\mathrm{P}_j, \widetilde{\mathrm{N}}_j),$$

$(\mathbf{4.27})$ there is a function $\psi_j \in \mathrm{BV}$ such that $g_j = e^{\imath\psi_j}$,

with

$$(\mathbf{4.28}) \qquad |\psi_j|_{\mathrm{BV}} \leq 4\pi d(\mathrm{P}_j, \widetilde{\mathrm{N}}_j) + \varepsilon_j$$

and

$$(\mathbf{4.29}) \qquad \mathrm{meas}\ (\mathrm{Supp}\ \psi_j) = \mathrm{meas}\ (\mathrm{Supp}\ (g_j - 1)) \leq \varepsilon_j.$$

We claim that $g = \prod_{j=1}^{\infty} g_j$ and $\psi = \sum_{j=1}^{\infty} \psi_j$ have all the required properties if we choose the ε_j's appropriately.

Fix $\varepsilon_1 < \varepsilon/2$ and let $g_1 = g_{1,\varepsilon_1}$. By Lemma 17, we have

$$\limsup_{\varepsilon\to 0} |g_1 g_{2,\varepsilon}|^2_{H^{1/2}} \leq |g_1|^2_{H^{1/2}} + \limsup_{\varepsilon\to 0} |g_{2,\varepsilon}|^2_{H^{1/2}}.$$

Thus, we may choose $\varepsilon_2 < \varepsilon/4$ and $g_2 = g_{2,\varepsilon_2}$ such that (using (4.5))

$$|g_1 g_2|^2_{H^{1/2}} \leq C_\Omega(d(P_1, \widetilde{N}_1) + d(P_2, \widetilde{N}_2)) + \varepsilon/2.$$

Using repeatedly Lemma 17, we choose $\varepsilon_3, \varepsilon_4, \ldots,$ such that

$$\varepsilon_j \leq \varepsilon 2^{-j} \quad \forall j \geq 1, \tag{4.30}$$

and, for every $k \geq 2$,

$$\begin{aligned} \left|\prod_{j=1}^{k} g_j\right|^2_{H^{1/2}} &\leq C_\Omega \sum_{j=1}^{k} d(P_j, \widetilde{N}_j) + \varepsilon \sum_{j=1}^{k-1} 2^{-j} \\ &\leq C_\Omega(L + \varepsilon) + \varepsilon \leq C'_\Omega L, \end{aligned} \tag{4.31}$$

since $\varepsilon < L$.

We claim that $\left(\prod_{j=1}^{k} g_j\right)$ converges in $W^{1,1}$. Indeed, set $H = \sum_{j\geq 1} |Dg_j|$. Then clearly $H \in L^1$ and

$$\left|D\left(\prod_{j=1}^{k} g_j\right)\right| \leq H.$$

On the other hand, for $k_2 \geq k_1 \geq 1$, we have, by (4.25),

$$\int_\Omega \left|D\left(\prod_{j=k_1}^{k_2} g_j\right)\right| \leq \sum_{j\geq k_1} \int |Dg_j| \leq 2\pi \sum_{j\geq k_1} d(P_j, \widetilde{N}_j) + \varepsilon 2^{-k_1+1}.$$

Thus

$$\begin{aligned} \left|\prod_{j=1}^{k} g_j - \prod_{j=1}^{k+\ell} g_j\right|_{W^{1,1}} &\leq \int_\Omega H\left|1 - \prod_{j=k+1}^{k+\ell} g_j\right| + 2\pi \sum_{j\geq k+1} d(P_j, \widetilde{N}_j) + \varepsilon 2^{-k} \\ &\leq 2 \int_{\cup_{j>k}\{x; g_j(x)\neq 1\}} H + 2\pi \sum_{j\geq k+1} d(P_j, \widetilde{N}_j) + \varepsilon 2^{-k}. \end{aligned}$$

Since meas $\left(\bigcup_{j>k} \text{Supp } (g_j - 1)\right) \leq \varepsilon 2^{-k}$ and $\sum d(\mathrm{P}_j, \widetilde{\mathrm{N}}_j) < \infty$, we conclude that $\left(\prod_{j=1}^{k} g_j\right)$ is a Cauchy sequence in $\mathrm{W}^{1,1}$ (note that it is clearly a Cauchy sequence in L^1, by (4.29)).

Set $g = \prod_{j=1}^{\infty} g_j$. By construction

$$
\begin{aligned}
|g|_{\mathrm{W}^{1,1}} &\leq \int_\Omega \mathrm{H} \leq 2\pi \sum_{j=1}^{\infty} d(\mathrm{P}_j, \widetilde{\mathrm{N}}_j) + \varepsilon \\
&\leq 2\pi \left(\mathrm{L} + \frac{\varepsilon}{4\pi}\right) + \varepsilon \quad \text{(by (4.23))} \ \leq 2\pi(\mathrm{L} + \varepsilon).
\end{aligned}
$$

This proves (4.18).

On the other hand, by (4.31), the sequence $\left(\prod_{j=1}^{k} g_j\right)$ is bounded in $\mathrm{H}^{1/2}$, so that $g \in \mathrm{H}^{1/2}$ and $|g|^2_{\mathrm{H}^{1/2}} \leq \mathrm{C}'_\Omega \mathrm{L}$; this proves (4.19).

We now turn to (4.17). By (2.7) and (4.24), we have

$$
\mathrm{T}\left(\prod_{j=1}^{k} g_j\right) = 2\pi \sum_{j=1}^{k} (\delta_{\mathrm{P}_j} - \delta_{\widetilde{\mathrm{N}}_j}).
$$

By Lemma 1 and the convergence of $(\prod_{j=1}^{k} g_j)$ to g in $\mathrm{W}^{1,1}$ as $k \to \infty$, we have

$$
\langle \mathrm{T}\left(\prod_{j=1}^{k} g_j\right), \varphi \rangle \to \langle \mathrm{T}(g), \varphi \rangle, \quad \forall \varphi \in \text{ Lip } (\Omega; \mathbf{R}).
$$

Thus,

$$
\langle \mathrm{T}(g), \varphi \rangle = 2\pi \sum_{j=1}^{\infty} (\varphi(\mathrm{P}_j) - \varphi(\widetilde{\mathrm{N}}_j)), \quad \forall \varphi \in \text{ Lip } (\Omega; \mathbf{R}).
$$

From (4.22) we conclude that

$$
\mathrm{T}(g) = 2\pi \sum_i (\delta_{\mathrm{P}_i} - \delta_{\mathrm{N}_i}).
$$

Properties (4.20) and (4.21) are immediate consequences of (4.23), (4.28) and (4.29).

We now derive some consequences of the above results. We start with a simple

Proof of Theorem 2. — Let $g \in H^{1/2}(\Omega; S^1)$ be such that $L(g) = 0$. We must show that $g \in Y = \overline{C^\infty(\Omega; S^1)}^{H^{1/2}}$. By Lemma 11 there exists a sequence (g_n) in $\mathscr{R}$ such that $g_n \to g$ in $H^{1/2}$, and thus $L(g_n) \to 0$. Since each g_n has only finitely many singularities, it follows from the dipole construction there exists a sequence (h_n) such that

$$h_n \in W^{1,\infty}_{\mathrm{loc}}\left(\Omega\backslash\Sigma_n; S^1\right) \cap W^{1,p}(\Omega; S^1), \forall p \in [1, 2), T(h_n) = T(g_n),$$

where Σ_n is the singular set of $g_n(\Sigma_n$ is a finite set), and moreover

$$|h_n|^2_{H^{1/2}} \leq C_\Omega L(h_n) \to 0,$$

$$h_n \to 1 \text{ a.e. on } \Omega.$$

Clearly $k_n = g_n\overline{h_n} \in W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\Sigma_n; S^1) \cap W^{1,p}(\Omega; S^1), \forall p \in [1, 2)$ and $T(k_n) = T(g_n) - T(h_n) = 0$. By Lemma 2, we have $\deg(k_n, a) = 0 \quad \forall a \in \Sigma_n$. Therefore k_n admits a well-defined lifting on $\Omega, k_n = e^{i\varphi_n}$, with $\varphi_n \in W^{1,\infty}_{\mathrm{loc}}(\Omega\backslash\Sigma_n; \mathbf{R}) \cap W^{1,p}(\Omega; \mathbf{R}), \forall p \in [1, 2)$. In particular, $k_n \in X \subset Y$. In order to prove that $g \in Y$ it suffices to check that $k_n \to g$ in $H^{1/2}$. Write

$$\begin{aligned} |k_n - g|_{H^{1/2}} &= |g_n\overline{h_n} - g|_{H^{1/2}} = |(g_n - g)\overline{h_n} + g(\overline{h_n} - 1)|_{H^{1/2}} \\ &\leq |(g_n - g)\overline{h_n}|_{H^{1/2}} + |g(\overline{h_n} - 1)|_{H^{1/2}}. \end{aligned}$$

But

$$|(g_n - g)\overline{h_n}|_{H^{1/2}} \leq |g_n - g|_{H^{1/2}} + 2|h_n|_{H^{1/2}} \to 0$$

and

$$|g(\overline{h_n} - 1)|^2_{H^{1/2}} \leq C \int_\Omega \int_\Omega \frac{|g(x) - g(y)|^2}{d(x, y)^3} |h_n(x) - 1|^2 dxdy + C|h_n|^2_{H^{1/2}} \to 0.$$

Corollary **2.** — *Given any* $g \in H^{1/2}(\Omega; S^1)$, *there exist* $h \in Y, k \in H^{1/2}(\Omega; S^1) \cap W^{1,1}(\Omega; S^1)$ *and* $\psi \in BV(\Omega; \mathbf{R})$ *such that*

$$g = hk \text{ and } k = e^{\imath\psi}.$$

Moreover, for every $\varepsilon > 0$, *one may choose* h, k, ψ *such that*

$$\int_\Omega |Dk| \leq 2\pi L(g) + \varepsilon, \quad |k|^2_{H^{1/2}} \leq C_\Omega L(g),$$

$$|h|^2_{H^{1/2}} \leq |g|^2_{H^{1/2}} + C_\Omega L(g)$$

and

$$|\psi|_{BV} \leq 4\pi L(g) + \varepsilon.$$

Proof. — By Lemma 16 there exists a sequence (k_n) in $H^{1/2}(\Omega; S^1) \cap W^{1,1}$ such that

$$T(k_n) = T(g), \quad \forall n,$$

$$\limsup_{n\to\infty} |k_n|_{W^{1,1}} \le 2\pi L(g),$$

$$|k_n|^2_{H^{1/2}} \le C_\Omega L(g), \quad \forall n,$$

and

$$k_n \to 1 \quad \text{a.e. on } \Omega.$$

Set $h_n = g\bar{k}_n$, so that $T(h_n) = 0$, $\forall n$, and thus $h_n \in Y$. By Lemma 17 we have

$$\limsup_{n\to\infty} |h_n|^2_{H^{1/2}} \le |g|^2_{H^{1/2}} + C_\Omega L(g).$$

The conclusion of Corollary 2 is now clear with $k = k_n$, $h = h_n$ and n sufficiently large.

Proof of Theorem 5. — As in the proof of Corollary 2 write $g = h_n k_n$. Since $h_n \in Y$, we may apply Theorem 3 and write $h_n = e^{\imath(\varphi_n+\psi_n)}$, with $\varphi_n \in H^{1/2}$ and $\psi_n \in W^{1,1}$. An inspection of the proof of Theorem 3 shows that

$$|\varphi_n|_{H^{1/2}} \le C_\Omega |h_n|_{H^{1/2}} \le C'_\Omega |g|_{H^{1/2}}$$

and

$$|\psi_n|_{W^{1,1}} \le C_\Omega |h_n|^2_{H^{1/2}} \le C'_\Omega |g|^2_{H^{1/2}}.$$

Thus

$$g = e^{\imath\varphi_n}(e^{\imath\psi_n} k_n),$$

which is the desired decomposition since $e^{\imath\psi_n}k_n \in W^{1,1}$ and

$$|e^{\imath\psi_n}k_n|_{W^{1,1}} \le |\psi_n|_{W^{1,1}} + |k_n|_{W^{1,1}} \le C''_\Omega |g|^2_{H^{1/2}}.$$

Proof of the upper bound in Theorem 4. — We have to show that, for every $g \in H^{1/2}(\Omega; S^1)$,

$$\operatorname{Inf}\Big\{|\psi|_{BV};\, g = e^{\imath(\varphi+\psi)},\, \varphi \in H^{1/2},\, \psi \in BV\Big\} \le 4\pi L(g),$$

i.e., for every $\varepsilon > 0$, we must find $\varphi_\varepsilon \in H^{1/2}$ and $\psi_\varepsilon \in BV$ such that $g = e^{\imath(\varphi_\varepsilon+\psi_\varepsilon)}$ and

$$|\psi_\varepsilon|_{BV} \le 4\pi L(g) + \varepsilon.$$

Going back to the proof of Corollary 2 and Theorem 5, we may write, by (4.20), $k_n = e^{\imath\eta_n}$, with $\eta_n \in \mathrm{BV}$ and

$$\limsup_{n\to\infty} |\eta_n|_{\mathrm{BV}} \leq 4\pi \mathrm{L}(g).$$

On the other hand, since $C^\infty(\Omega; \mathbf{R})$ is dense in $W^{1,1}(\Omega; \mathbf{R})$, we may choose $\tilde{\psi}_n \in C^\infty(\Omega; \mathbf{R})$ such that

$$\|\psi_n - \tilde{\psi}_n\|_{W^{1,1}} < 1/n.$$

Finally, we may write

$$g = h_n k_n = e^{\imath(\varphi_n + \psi_n + \eta_n)} = e^{\imath(\varphi_n + \tilde{\psi}_n) + \imath(\psi_n - \tilde{\psi}_n + \eta_n)},$$

with $\varphi_n + \tilde{\psi}_n \in H^{1/2}$, $\psi_n - \tilde{\psi}_n + \eta_n \in \mathrm{BV}$ and

$$\limsup |\psi_n - \tilde{\psi}_n + \eta_n|_{\mathrm{BV}} \leq 4\pi \mathrm{L}(g),$$

which is the desired conclusion.

We now turn to the

Proof of Lemma 14′. — For the first assertion, we proceed as in the proof of Corollary 2. Since $h_n \in Y$, $\forall n$, we may find a sequence $(\tilde{h}_n)$ in $C^\infty(\Omega; S^1)$ such that

$$\|\tilde{h}_n - h_n\|^2_{H^{1/2}} \to 0 \text{ as } n \to \infty.$$

Recall that

$$h_n = g\bar{k}_n \longrightarrow g \text{ a.e.}$$

Thus, by Lemma 17, we find

$$\limsup |\tilde{h}_n|^2_{H^{1/2}} \leq |g|^2_{H^{1/2}} + C_\Omega \mathrm{L}(g)$$

and (passing to a subsequence)

$$\tilde{h}_n \longrightarrow g \text{ a.e.,} \quad \tilde{h}_n \rightharpoonup g \text{ weakly in } H^{1/2}.$$

To prove the second assertion, let (g_n) be any sequence in Y such that $g_n \longrightarrow g$ a.e. Writing $g_n = (g_n\bar{g})g$ and observing that $g_n\bar{g} \to 1$ a.e., we deduce from Lemma 17 that

$$|g_n|^2_{H^{1/2}} = |g|^2_{H^{1/2}} + |g_n\bar{g}|^2_{H^{1/2}} + o(1) \text{ as } n \to \infty.$$

On the other hand (*see* Lemma 9),

$$L(g_n \bar{g}) \le C_\Omega |g_n \bar{g}|^2_{H^{1/2}}.$$

But $L(g_n \bar{g}) = L(\bar{g})$, since $L(g_n) = 0$, and thus

$$|g_n|^2_{H^{1/2}} \ge |g|^2_{H^{1/2}} + C'_\Omega L(g) + o(1).$$

Remark **4.2.** — We have now at our disposal two different techniques for lifting a general $g \in H^{1/2}(\Omega; S^1)$ in the form

$$g = e^{\imath(\varphi+\psi)} \text{ with } \varphi \in H^{1/2} \text{ and } \psi \in BV.$$

The first method, described at the beginning of Section 4, yields some $\varphi \in H^{1/2}$ and $\psi \in BV$ such that

$$g = e^{\imath(\varphi+\psi)},$$

with the estimate

$$|\varphi|_{H^{1/2}} \le C_\Omega |g|_{H^{1/2}} \tag{4.32}$$

and

$$|\psi|_{BV} \le C_\Omega |g|^2_{H^{1/2}}. \tag{4.33}$$

The second method, described in the proof of Theorem 4 (upper bound), yields, for every $\varepsilon > 0$, some $\varphi_\varepsilon \in H^{1/2}$ and $\psi_\varepsilon \in BV$ such that

$$g = e^{\imath(\varphi_\varepsilon+\psi_\varepsilon)},$$

with

$$|\psi_\varepsilon|_{BV} \le 4\pi L(g) + \varepsilon \tag{4.34}$$

and **no estimate** for φ_ε in $H^{1/2}$.

A natural question is whether one can achieve a decomposition of the phase in the form

$$g = e^{\imath\left(\varphi^\#_\varepsilon+\psi^\#_\varepsilon\right)}$$

with the double control

$$\left|\varphi^\#_\varepsilon\right|_{H^{1/2}} \le C(\varepsilon, |g|_{H^{1/2}})$$

and

$$\left|\psi_\varepsilon^\#\right|_{BV} \le 4\pi L(g) + \varepsilon \ ?$$

The answer is negative even with $g \in Y$. To see this, we may use an example studied in [15]. Assume that, locally, near a point of Ω, say 0, the square $Q = I^2$, with $I = (-1, +1)$, is contained in Ω. Consider the function $\gamma_\delta(x)$ defined on I by

$$\gamma_\delta(x) = \begin{cases} 0, & \text{if} \quad -1 < x < 0 \\ 2\pi x/\delta, & \text{if} \quad 0 < x < \delta \\ 2\pi, & \text{if} \quad \delta < x < 1 \end{cases},$$

where δ is small.

On Q, set

$$g_\delta(x, y) = e^{\imath\gamma_\delta(x)} \text{ for } (x, y) \in Q.$$

Clearly, we have $g_\delta \in Y$, so that $L(g_\delta) = 0$. We claim that

$$\|g_\delta\|_{H^{1/2}(Q)} \le C, \quad \forall\, \delta, \tag{4.35}$$

and that there exist absolute positive constants c_* and C_* such that, if

$$g_\delta = e^{\imath(\varphi_\delta + \psi_\delta)}, \ \ \varphi_\delta \in H^{1/2}(Q), \ \psi_\delta \in BV(Q), \tag{4.36}$$

with

$$|\psi_\delta|_{BV(Q)} \le C_*, \tag{4.37}$$

then

$$|\varphi_\delta|^2_{H^{1/2}(Q)} \ge c_* \log(1/\delta) \text{ as } \delta \to 0. \tag{4.38}$$

The verification of (4.35) is easy. Indeed, by scaling we have

$$|g_\delta(\cdot, y)|_{H^{1/2}(I)} \le C, \quad \forall\, \delta, \ \forall\, y,$$

and recall (*see* e.g. [1], Lemma 7.44) that

$$\int_I |f(\cdot, y)|^2_{H^{1/2}(I)} dy + \int_I |f(x, \cdot)|^2_{H^{1/2}(I)} dx \sim |f|^2_{H^{1/2}(Q)}, \tag{4.39}$$

so that (4.35) follows.

We now turn to the proof of (4.38) under the assumptions (4.36) and (4.37). By Theorem 2 in [15] we know that, for a.e. $y \in I$,

$$|\varphi_\delta(\cdot, y) + \psi_\delta(\cdot, y)|_{H^s(I)} \geq c(\log(1/\delta))^{1/2} \tag{4.40}$$

for some absolute constant $c > 0$, where

$$2s = 1 - (\log 1/\delta)^{-1}. \tag{4.41}$$

On the other hand, it is easy to see that

$$|f|^2_{H^\sigma(I)} \leq \frac{C}{1-2\sigma}|f|^2_{BV(I)}, \quad \forall f \in BV(I),\ \forall \sigma < 1/2 \tag{4.42}$$

and

$$|f|_{H^\sigma(I)} \leq C|f|_{H^{1/2}(I)}, \quad \forall f \in H^{1/2},\ \forall \sigma \leq 1/2, \tag{4.43}$$

with constants C independent of σ. Combining (4.40), (4.41), (4,42) and (4.43) yields, for a.e. $y \in I$,

$$|\varphi_\delta(\cdot, y)|_{H^{1/2}(I)} + (\log(1/\delta))^{1/2}|\psi_\delta(\cdot, y)|_{BV(I)} \geq c(\log(1/\delta))^{1/2}. \tag{4.44}$$

Integrating (4.44) in y and using the inequalities

$$\begin{aligned}\int_I |f(\cdot, y)|_{H^{1/2}(I)} dy &\leq \left(2\int_I |f(\cdot, y)|^2_{H^{1/2}(I)} dy\right)^{1/2} \\ &\leq C|f|_{H^{1/2}(Q)}, \quad \forall f \in H^{1/2}(Q),\end{aligned}$$

and

$$\int_I |f(\cdot, y)|_{BV(I)} dy \leq C|f|_{BV(Q)}, \quad \forall f \in BV(Q),$$

together with (4.37), we obtain

$$|\varphi_\delta|_{H^{1/2}(Q)} + C_*(\log 1/\delta)^{1/2} \geq c(\log 1/\delta)^{1/2},$$

and (4.38) follows, provided C_* is sufficiently small.

4.3. *Lower bound estimates for the BV part of the phase*

We start with a simple lemma about maps from S^1 into S^1.

Lemma **18.** — *Let* $(g_n) \subset BV(S^1; S^1) \cap C^0(S^1; S^1)$ *be such that* $g_n \to g$ *a.e. for some* $g \in BV(S^1; S^1) \cap C^0(S^1; S^1)$ *and* $\|g_n\|_{BV} \le C$. *Then*

$$\liminf_{n\to\infty} \left(\int_{S^1} |\dot{g}_n| - 2\pi |\deg g_n - \deg g| \right) \ge \int_{S^1} |\dot{g}|.$$

Here, $\dot{g}$ *denotes the measure* $\dfrac{\partial g}{\partial \theta}$.

Proof. — (We thank Augusto Ponce for simplifying our original proof). For $g \in BV(S^1; S^1) \cap C^0(S^1; S^1)$, let $f \in C^0([0, 2\pi]; \mathbf{R})$ be such that $g(\exp(\imath\theta)) = \exp(\imath f(\theta))$. Then $\deg g = \dfrac{1}{2\pi}\big(f(2\pi) - f(0)\big)$. Moreover, we have $f \in BV$ and

$$\int_0^{2\pi} |f'| = \int_{S^1} |\dot{g}|, \tag{4.45}$$

where f' is the measure $\dfrac{df}{dx}$. Indeed, since g is continuous, we have

$$\begin{aligned} \int_{S^1} |\dot{g}| &= \operatorname{Sup} \left\{ \sum_{j=1}^{n} |g(\exp(\imath t_{j+1})) - g(\exp(\imath t_j))|;\ 0 \le t_1 < \cdots < t_n \le 2\pi \right\} \\ &= \operatorname{Sup} \left\{ \sum_{j=1}^{n-1} |g(\exp(\imath t_{j+1})) - g(\exp(\imath t_j))|;\ 0 \le t_1 < \cdots < t_n \le 2\pi \right\} \end{aligned} \tag{4.46}$$

(with the convention $t_{n+1} = t_1$).

For a given $\delta > 0$, we have

$$(1-\delta)|f(t_{j+1}) - f(t_j)| \le |g(\exp(\imath t_{j+1})) - g(\exp(\imath t_j))| \le |f(t_{j+1}) - f(t_j)|, \tag{4.47}$$

provided the partition (t_j) is sufficiently fine. We obtain (4.45) by combining (4.46) and (4.47).

Let $f_n \in BV([0, 2\pi]; \mathbf{R}) \cap C^0([0, 2\pi]; \mathbf{R})$ be such that $g_n(\exp(\imath\theta)) = \exp(\imath f_n(\theta))$ and $\|f_n\|_{BV} \le C$. Up to a subsequence, we may assume that $f_n \to h$ a.e. and in L^1 for some $h \in BV$.

Since $g = e^{\imath h} = e^{\imath f}$, we find that $h = f + k$, where $k \in \mathrm{BV}([0, 2\pi]; 2\pi\mathbf{Z})$. Thus k must be of the form

$$k = 2\pi \sum_{j=1}^{p} \alpha_j \chi_{\mathrm{I}_j} \text{ a.e.},$$

where $\alpha_j \in \mathbf{Z}$, $\mathrm{I}_j = (a_j, a_{j+1})$, $0 = a_1 < \cdots < a_{p+1} = 2\pi$. Therefore

$$h' = f' + \sum_{j=2}^{p} \alpha_j \delta_{a_j}. \tag{4.48}$$

We have to prove that

$$\liminf_{n\to\infty} \left(\int_0^{2\pi} |f_n'| - \left| \int_0^{2\pi} (f_n' - f') \right| \right) \geq \int_0^{2\pi} |f'|. \tag{4.49}$$

It suffices to show that

$$\liminf_{n\to\infty} \left(\int_0^{2\pi} |f_n'| + \int_0^{2\pi} (f_n' - f') \right) \geq \int_0^{2\pi} |f'|. \tag{4.50}$$

Indeed, (4.50) applied to $\bar{g}_n$ gives

$$\liminf_{n\to\infty} \left(\int_0^{2\pi} |f_n'| - \int_0^{2\pi} (f_n' - f') \right) \geq \int_0^{2\pi} |f'| \tag{4.51}$$

and the combination of (4.50) and (4.51) is equivalent to (4.49). We may rewrite (4.50) as

$$\liminf_{n\to\infty} \int_0^{2\pi} (f_n')^+ \geq \int_0^{2\pi} (f')^+. \tag{4.52}$$

Let $\varphi \in C_0^\infty(0, 2\pi)$, $0 \leq \varphi \leq 1$. Then

$$-\int_0^{2\pi} f_n \varphi' = \int_0^{2\pi} f_n' \varphi \leq \int_0^{2\pi} (f_n')^+$$

and thus

$$-\int_0^{2\pi} h\varphi' \leq \liminf_{n\to\infty} \int_0^{2\pi} (f_n')^+.$$

Taking the supremum over such φ's yields

$$\liminf_{n\to\infty} \int_0^{2\pi} \big(f_n'\big)^+ \geq \int_0^{2\pi} (h')^+ = \int_0^{2\pi} \big(f' + \sum \alpha_j \delta_{a_j}\big)^+ \quad \text{by (4.48)}.$$

We conclude with the help of the following elementary

Lemma **19.** — *Let* $f \in \mathrm{BV}([0, 2\pi]) \cap \mathrm{C}^0([0, 2\pi])$. *Then*

$$\int_0^{2\pi} \big(f' + \sum_{\text{finite}} \alpha_j \delta_{a_j}\big)^+ = \int_0^{2\pi} (f')^+ + \sum (\alpha_j)^+$$

for any choice of distinct points $a_j \in (0, 2\pi)$ *and of* α_j *in* $\mathbf{R}$.

Proof of Lemma 19. — It suffices to consider the case of a single point $a \in (0, 2\pi)$. Let $\zeta_n = \zeta\big(n(x - a)\big)$, where ζ is a fixed cutoff function with $\zeta(0) = 1$, $0 \leq \zeta \leq 1$. For any fixed $\psi \in \mathrm{C}^1\big([0, 2\pi]\big)$, we claim that

$$\int_0^{2\pi} f(\zeta_n \psi)' \to 0.$$

Indeed,

$$\int_0^{2\pi} f(\zeta_n \psi)' = \int_0^{2\pi} \big(f - f(a)\big)(\zeta_n \psi)',$$

so that

$$\left| \int_0^{2\pi} f(\zeta_n \psi)' \right| \leq \int_0^{2\pi} |f - f(a)| \; |(\zeta_n \psi)'| \xrightarrow{n} 0,$$

since f is continuous at a.

Let $\varepsilon > 0$. Fix some $\psi \in C_0^1\big((0, 2\pi)\big)$, $0 \le \psi \le 1$, such that

$$-\int_0^{2\pi} f\psi' \ge \int_0^{2\pi} (f')^+ - \varepsilon.$$

Then, with $0 \le t \le 1$,

$$\int_0^{2\pi} (f' + \alpha\delta_a)[(1-\zeta_n)\psi + t\zeta_n] =$$
$$-\int_0^{2\pi} f[(1-\zeta_n)\psi + t\zeta_n]' + t\alpha \overset{n}{\to} -\int_0^{2\pi} f\psi' + t\alpha.$$

Since $0 \le (1-\zeta_n)\psi + t\zeta_n \le 1$, we find that

$$\int_0^{2\pi} (f' + \alpha\delta_a)^+ \ge \int_0^{2\pi} (f')^+ + t\alpha - \varepsilon, \quad \forall\, \varepsilon > 0,\ \forall\, t \in [0, 1],$$

and thus

$$\int_0^{2\pi} (f' + \alpha\delta_a)^+ \ge \int_0^{2\pi} (f')^+ + \alpha^+.$$

The opposite inequality

$$\int_0^{2\pi} (f' + \alpha\delta_a)^+ \le \int_0^{2\pi} (f')^+ + \alpha^+$$

being clear, the proof of Lemma 19 is complete.

Remark **4.3.** — The assumption $\|g_n\|_{BV} \le C$ in Lemma 18 is essential (A. Ponce, personal communication).

Corollary **3.** — *Let* $\Gamma \subset \mathbf{R}^N$ *be an oriented curve. Let* $(g_n) \subset BV(\Gamma; S^1) \cap C^0(\Gamma; S^1)$ *be such that* $g_n \to g$ *a.e. and* $\|g_n\|_{BV} \le C$, *where* $g \in BV(\Gamma; S^1) \cap C^0(\Gamma; S^1)$. *Then*

$$\liminf_{n\to\infty} \left(\int_\Gamma |\dot{g}_n| - 2\pi\, |\deg g_n - \deg g| \right) \ge \int_\Gamma |\dot{g}|.$$

In particular, if $\deg g_n = 0, \forall n$, *then*

$$\liminf_{n\to\infty} \int_\Gamma |\dot{g}_n| \geq 4\pi |\deg g|$$

(the assumption $\|g_n\|_{BV} \leq C$ *is not required here).*

Here, Γ need not be connected. If $\Gamma = \bigcup_j \gamma_j$, with each γ_j simple, we set

$$\deg g = \sum_j \deg(g; \gamma_j),$$

where γ_j has the orientation inherited from that of Γ.

Remark **4.4.** — It can be easily seen that the constants 2π in Lemma 18 and 4π in Corollary 3 cannot be improved.

We now prove a coarea type formula (in the spirit of [2]) used in the proof of the lower bound in Theorem 4.

Lemma **20.** — *Let* $g \in H^{1/2}(\Omega; S^1)$ *and* $\zeta \in C^\infty(\Omega; \mathbf{R})$. *If* $\lambda \in \mathbf{R}$ *is a regular value of* ζ, *let*

$$\Gamma_\lambda = \{x \in \Omega; \zeta(x) = \lambda\}.$$

We orient Γ_λ *such that, for each* $x \in \Gamma_\lambda$, *the basis* $\big(\tau(x), D\zeta(x), n(x)\big)$ *is direct, where* $n(x)$ *is the outward normal to* Ω *at* x. *Then*

$$\langle T(g), \zeta\rangle = 2\pi \int_{\mathbf{R}} \deg(g; \Gamma_\lambda)\, d\lambda.$$

Remark **4.5.** — For a.e. λ we have $g_{|\Gamma_\lambda} \in H^{1/2} \subset VMO$. Therefore, $\deg(g; \Gamma_\lambda)$ makes sense for a.e. λ (*see* [22]). In general, Γ_λ is a union of simple curves, $\Gamma_\lambda = \bigcup \gamma_j$. In this case, we set

$$\deg(g; \Gamma_\lambda) = \sum \deg(g; \gamma_j),$$

where on each γ_j we consider the orientation inherited from Γ_λ.

Proof of Lemma 20. — We write $g = g_1 h$, with $g_1 \in X$ and $h \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$. For a.e. λ, we have $h_{|\Gamma_\lambda} \in W^{1,1}$ and $g_{1\,|\Gamma_\lambda} \in H^{1/2}$.

Since $g_1 = e^{\imath\varphi_1}$ for some $\varphi_1 \in H^{1/2}(\Omega; \mathbf{R})$, for a.e. λ we have $\deg(g_1; \Gamma_\lambda) = 0$, so that $\deg(g; \Gamma_\lambda) = \deg(h; \Gamma_\lambda)$ for a.e. λ. Moreover, we have $T(g) = T(h)$. It suffices therefore to prove the statement of the lemma for $h \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$. In this case, we have

$$\langle T(h), \zeta\rangle = \int_\Omega |D\zeta| h \wedge \left(Dh \wedge \frac{D\zeta}{|D\zeta|}\right)$$

(*see* Lemma 1 in the introduction).

We recall the coarea formula (*see*, e.g., Federer [26], Simon [42])

$$(\mathbf{4.53}) \qquad \int_\Omega f|D\varphi| = \int_{\mathbf{R}} \left(\int_{\varphi=\lambda} f ds\right) d\lambda, \quad \varphi \in C^\infty(\Omega; \mathbf{R}),\ f \in L^1(\Omega; \mathbf{R}).$$

Applying (4.53) with $\varphi = \zeta$, $f = h \wedge \left(Dh \wedge \dfrac{D\zeta}{|D\zeta|}\right) = h \wedge \dfrac{\partial h}{\partial \tau}$ (where τ is the oriented tangent unit vector to Γ_λ) we find

$$\langle T(h), \zeta\rangle = \int_{\mathbf{R}} \left(\int_{\Gamma_\lambda} h \wedge \frac{\partial h}{\partial \tau} ds\right) d\lambda = 2\pi \int_{\mathbf{R}} \deg(h; \Gamma_\lambda) d\lambda.$$

The final ingredient in the proof of Theorem 4 is the lower bound given by

Lemma **21.** — *Let* $g \in H^{1/2}(\Omega; S^1)$. *If* $g = e^{\imath(\varphi+\psi)}$ *with* $\varphi \in H^{1/2}(\Omega; \mathbf{R})$ *and* $\psi \in BV(\Omega; \mathbf{R})$, *then*

$$\int_\Omega |D\psi| \geq 4\pi L(g).$$

Proof. — Let $h = e^{-\imath\varphi} g \in H^{1/2}(\Omega; S^1)$. Let (ψ_n) be a sequence of smooth real-valued functions such that $\psi_n \to \psi$ a.e. and

$$\int_\Omega |D\psi_n| \to \int_\Omega |D\psi|.$$

Fix some $\zeta \in C^\infty(\Omega; \mathbf{R})$ and let, for λ a regular value of ζ, $\Gamma_\lambda = \{x \in \Omega; \zeta(x) = \lambda\}$. Let $h_n = e^{i\psi_n}$. For a.e. λ we have $h_{n|\Gamma_\lambda} \to h_{|\Gamma_\lambda}$ a.e. and $h_{|\Gamma_\lambda} \in H^{1/2} \cap BV$. For any such λ we have $h_{|\Gamma_\lambda} \in BV \cap C^0$. Indeed, since $k = h_{|\Gamma_\lambda} \in BV$, k has finite limits from the left and from the right at each point. These limits must coincide, since $H^{1/2} \subset VMO$ in dimension 1 (*see* e.g. [17] and [22]) and non-trivial characteristic functions are not in VMO.

By the second assertion in Corollary 3, we find that, for a.e. λ,

$$\liminf_{n\to\infty} \int_{\Gamma_\lambda} |\dot{h}_n| \geq 4\pi |\deg(h; \Gamma_\lambda)|.$$

Thus, if $|D\zeta| \leq 1$, we have by the coarea formula,

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega |Dh_n| &\geq \liminf_{n\to\infty} \int_\Omega |Dh_n|\ |D\zeta| = \liminf_{n\to\infty} \int_{\mathbf{R}} \left(\int_{\Gamma_\lambda} |Dh_n| ds \right) d\lambda \\ &\geq \liminf_{n\to\infty} \int_{\mathbf{R}} \left(\int_{\Gamma_\lambda} |\dot{h}_n| ds \right) d\lambda \geq 4\pi \int_{\mathbf{R}} |\deg(h; \Gamma_\lambda)| d\lambda \\ &\geq 4\pi \left| \int_{\mathbf{R}} \deg(h; \Gamma_\lambda) d\lambda \right|.\end{aligned}$$

On the other hand, by Lemma 20, we have

$$4\pi \left| \int_{\mathbf{R}} \deg(h; \Gamma_\lambda) d\lambda \right| = 2|\langle T(h), \zeta \rangle|.$$

Thus, if $\zeta \in C^\infty(\Omega; \mathbf{R})$ is such that $|D\zeta| \leq 1$, we have

$$\begin{aligned}\int_\Omega |D\psi| &= \liminf_{n\to\infty} \int_\Omega |D\psi_n| \\ &= \liminf_{n\to\infty} \int_\Omega |Dh_n| \geq 2|\langle T(h), \zeta\rangle| = 2|\langle T(g), \zeta \rangle|.\end{aligned} \tag{4.54}$$

We conclude by taking in (4.54) the supremum over all such ζ's.

5. Minimal connection and Ginzburg–Landau energy for $g \in H^{1/2}$. Proof of Theorem 6

Throughout this section, the metric d denotes d_G, the geodesic distance (on Ω) relative to G, and $L = L_G$.

Proof of Theorem 6. — We start by deriving some elementary inequalities. For $g \in H^{1/2}(\Omega; \mathbf{R}^2)$, let

$$e_{\varepsilon,g} = \operatorname{Min}\left\{ E_\varepsilon(u);\ u \in H^1_g(G; \mathbf{R}^2) \right\}.$$

Let $g_1, g_2 \in H^{1/2}(\Omega; S^1)$ and let $u_j \in H^1_{g_j}(G; B^2)$ be such that $e_{\varepsilon,g_j} = E_\varepsilon(u_j)$, $j = 1, 2$. Then $u_1 u_2 \in H^1_{g_1 g_2}(G; \mathbf{R}^2)$. We find that, for each $\delta > 0$, we have

$$
\begin{aligned}
e_{\varepsilon,g_1 g_2} &\leq E_\varepsilon(u_1 u_2) \leq \frac{1}{2}\int_G (|\nabla u_1| + |\nabla u_2|)^2 + \frac{1}{4\varepsilon^2}\int_G (1 - |u_1 u_2|^2)^2 \\
&\leq \frac{1+\delta}{2}\int_G |\nabla u_1|^2 + \frac{C(\delta)}{2}\int_G |\nabla u_2|^2 \\
&\quad + \frac{1}{4\varepsilon^2}\int_G ((1-|u_1|^2) + (1-|u_2|^2))^2 \\
&\leq (1+\delta) e_{\varepsilon,g_1} + C(\delta) e_{\varepsilon,g_2}.
\end{aligned} \tag{5.1}
$$

Similarly, we have

$$
e_{\varepsilon,g_1 g_2} \geq (1-\delta) e_{\varepsilon,g_1} - C(\delta) e_{\varepsilon,g_2}. \tag{5.2}
$$

The upper bound $e_{\varepsilon,g} \leq \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon))$

We will use Lemma A.1 in Appendix A, which asserts that, if $g \in \mathscr{R}_1$, then

$$
e_{\varepsilon,g} \leq \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)) \text{ as } \varepsilon \to 0. \tag{5.3}
$$

The class $\mathscr{R}_1$, which is dense in $H^{1/2}(\Omega; S^1)$, is defined in Appendix A. Inequality (5.3) was essentially established by Sandier [40].

Another ingredient needed in the proof is the following upper bound, valid for $g \in H^{1/2}(\Omega; S^1)$, and already mentioned in the Introduction (*see* [12], Theorem 5 and Remark 8; *see* also [38], Proposition II.1 for a different proof):

$$
e_{\varepsilon,g} \leq C|g|^2_{H^{1/2}}(1 + \log(1/\varepsilon)), \tag{5.4}
$$

for some $C = C(G)$.

We now turn to the proof of the upper bound. Let $g \in H^{1/2}(\Omega; S^1)$. By Lemma B.1 in Appendix B, there is a sequence (g_k) in $\mathscr{R}_1$ such that $g_k \to g$ in $H^{1/2}$. On the one hand, since $H^{1/2} \cap L^\infty$ is an algebra, we find that $|g/g_k|_{H^{1/2}} \to 0$. On the other hand, recall that $L(g_k) \to L(g)$. Fix some $\tilde{\delta} > 0$. By (5.4) applied to g/g_k, we find that

$$
e_{\varepsilon,g/g_k} \leq \tilde{\delta} \log(1/\varepsilon) \quad \text{for } \varepsilon \text{ sufficiently small}, \tag{5.5}
$$

if k is sufficiently large. Using (5.3) for g_k, where k is sufficiently large, we obtain

$$
e_{\varepsilon,g_k} \leq \pi(L(g) + \delta) \log(1/\varepsilon). \tag{5.6}
$$

The upper bound follows by combining (5.1), (5.5) and (5.6).

The lower bound $e_{\varepsilon,g} \geq \pi L(g)\log(1/\varepsilon) + o(\log(1/\varepsilon))$

We rely on the corresponding lower bound in [40] (Theorem 3.1, part 1): if $g \in \mathscr{R}_0$ (where the class $\mathscr{R}_0$, dense in $H^{1/2}(\Omega; S^1)$, is defined in Appendix A), then

$$e_{\varepsilon,g} \geq \pi L(g)\log(1/\varepsilon) + o(\log(1/\varepsilon)) \quad \text{for } \varepsilon \text{ sufficiently small} \tag{5.7}$$

(no geometrical assumption is made on Ω or g). We fix some $\delta > 0$. Applying (5. 7) to g_k for k sufficiently large, we find that

$$e_{\varepsilon,g_k} \geq \pi(L(g) - \delta)\log(1/\varepsilon) \quad \text{for } \varepsilon \text{ sufficiently small.} \tag{5.8}$$

The lower bound is a consequence of (5.2), (5.5) and (5.8).

There is a variant of Theorem 6 when the boundary condition depends on ε. Let $g \in H^{1/2}(\Omega; S^1)$ and let $g_\varepsilon \in H^{1/2}(\Omega; \mathbf{R}^2)$ be such that

$$g_\varepsilon \to g \text{ in } H^{1/2}, \tag{5.9}$$

$$|g_\varepsilon| \leq 1, \tag{5.10}$$

$$\| |g_\varepsilon| - 1 \|_{L^2} \leq C\sqrt{\varepsilon}. \tag{5.11}$$

Set

$$e_{\varepsilon,g_\varepsilon} = \operatorname{Min}\left\{E_\varepsilon(u);\, u \in H^1_{g_\varepsilon}(G; \mathbf{R}^2)\right\}.$$

Theorem **6′.** — *Assume (5.9), (5.10) and (5.11). Then we have*

$$e_{\varepsilon,g_\varepsilon} = \pi L(g)\log(1/\varepsilon) + o(\log(1/\varepsilon)) \ \text{as } \varepsilon \to 0. \tag{5.12}$$

The main ingredients in the proof of (5.12) are the following Lemmas 22 and 23.

Lemma **22.** — *Let* $\varphi \in H^{1/2}(\Omega; \mathbf{R}^2)$ *and let* $u (= u_\varepsilon)$ *be the solution of the linear problem*

$$-\Delta u + \frac{1}{\varepsilon^2}u = 0 \quad \text{in } G, \tag{5.13}$$

$$u = \varphi \quad \text{on } \Omega = \partial G. \tag{5.14}$$

Then, for sufficiently small $\varepsilon > 0$,

$$\int_G |\nabla u|^2 + \frac{1}{\varepsilon^2}\int_G |u|^2 \leq C_G\left(|\varphi|^2_{H^{1/2}(\Omega)} + \frac{1}{\varepsilon}\int_\Omega |\varphi|^2\right). \tag{5.15}$$

Proof of Lemma 22. — Let Φ be the harmonic extension of φ and fix some $\zeta \in C_0^\infty(\mathbf{R})$ with $\zeta(0) = 1$. Set

$$v(x) = \Phi(x)\zeta(\text{dist }(x, \Omega)/\varepsilon).$$

Using, for $0 < \delta < \delta_0(G)$, the standard estimate

$$\int_{\{x;\,\text{dist }(x,\Omega)=\delta\}} \Phi^2 \leq C \int_\Omega \varphi^2,$$

it is easy to see that, for $0 < \varepsilon < \varepsilon_0(G)$, we have

$$\int_G |\nabla v|^2 + \frac{1}{\varepsilon^2}\int_G |v|^2 \leq C_G\left(|\varphi|^2_{H^{1/2}} + \frac{1}{\varepsilon}\int_G |\varphi|^2\right),$$

and the conclusion follows, since u is a minimizer so that,

$$\int_G |\nabla u|^2 + \frac{1}{\varepsilon^2}\int_G |u|^2 \leq \int_G |\nabla v|^2 + \frac{1}{\varepsilon^2}\int_G |v|^2.$$

For later use, we mention a related estimate, whose proof is similar and left to the reader:

Lemma **22′.** — *For* $0 < \varepsilon < \varepsilon_0(G)$, *set*

$$G_\varepsilon = \{\, x \in \mathbf{R}^3 \setminus G\,;\ \textit{dist}\ (x, \Omega) < \varepsilon \,\}.$$

Let $\varphi \in H^{1/2}(\Omega; \mathbf{R}^2)$ *and let* $u (= u_\varepsilon)$ *be the solution of the linear problem*

(**5.16**) $$-\Delta u + \frac{1}{\varepsilon^2} u = 0 \quad \textit{in } G_\varepsilon,$$

(**5.17**) $$u = \varphi \quad \textit{on } \Omega = \partial G,$$

(**5.18**) $$u = 0 \quad \textit{on } \partial G_\varepsilon \setminus \partial G.$$

Then

(**5.19**) $$\int_{G_\varepsilon} |\nabla u|^2 + \frac{1}{\varepsilon^2}\int_{G_\varepsilon} |u|^2 \leq C_G\left(|\varphi|^2_{H^{1/2}} + \frac{1}{\varepsilon}\int_\Omega |\varphi|^2\right).$$

Lemma **23.** — *Let* (g_ε) *in* $H^{1/2}(\Omega; \mathbf{R}^2)$ *satisfy (5.10), (5.11) and*

$$\|g_\varepsilon\|_{H^{1/2}} \le C. \tag{5.20}$$

Then there is (h_ε) *in* $H^{1/2}(\Omega; S^1)$ *such that*

$$\|h_\varepsilon\|_{H^{1/2}} \le C \tag{5.21}$$

and

$$\|g_\varepsilon - h_\varepsilon\|_{L^2} \le C\sqrt{\varepsilon}. \tag{5.22}$$

Moreover if, in addition,

$$g_\varepsilon \to g \text{ in } H^{1/2}, \tag{5.23}$$

then

$$h_\varepsilon \to g \text{ in } H^{1/2}. \tag{5.24}$$

Proof. — We divide the proof in 4 steps

Step **1.** — Let $g_\varepsilon^1 = g_\varepsilon * P_\varepsilon$ be an ε-smoothing of g_ε.
Clearly

$$\left\|g_\varepsilon - g_\varepsilon^1\right\|_{L^2} \le \sqrt{\varepsilon}\|g_\varepsilon\|_{H^{1/2}} \le C\sqrt{\varepsilon} \tag{5.25}$$

and from (5.11), (5.25) we have

$$\left\|1 - \left|g_\varepsilon^1\right|\right\|_{L^2} \le C\sqrt{\varepsilon}. \tag{5.26}$$

Also

$$\left\|g_\varepsilon^1\right\|_{H^{1/2}} \le C, \tag{5.27}$$

and

$$\left\|g_\varepsilon^1\right\|_{H^1} \le C\varepsilon^{-1/2}\|g_\varepsilon\|_{H^{1/2}} \le C\varepsilon^{-1/2}. \tag{5.28}$$

Step **2.** — Given a point $a \in \mathbf{R}^2$ with $|a| < 1/10$, let $\pi_a : \mathbf{R}^2 \setminus \{a\} \to S^1$ be the radial projection onto S^1 with vertex at a, i.e.,

$$\pi_a(\xi) = a + \lambda(\xi - a), \;\; \xi \in \mathbf{R}^2 \setminus \{a\}$$

where $\lambda \in \mathbf{R}$ is the unique positive solution of

$$|a + \lambda(\xi - a)| = 1.$$

It is also convenient to note that

$$\pi_a(\xi) = j_a^{-1}\left(\frac{\xi - a}{|\xi - a|}\right) \text{ for } \xi \neq a$$

where $j_a : \mathrm{S}^1 \to \mathrm{S}^1, j_a(z) = \dfrac{z-a}{|z-a|}$, is a smooth diffeomorphism.

In particular,

$$(\mathbf{5.29}) \qquad |\mathrm{D}\pi_a(\xi)| \leq \frac{\mathrm{C}}{|\xi - a|} \quad \forall \xi \in \mathbf{R}^2 \setminus \{a\},$$

and π_a is lipschitzian on $\{|\xi| \geq 1/2\}$ with a uniform Lipschitz constant (independent of a).

We claim that

$$(\mathbf{5.30}) \qquad h_{a,\varepsilon} = \pi_a \circ g_\varepsilon^1 : \Omega \to \mathrm{S}^1$$

satisfies all the required properties for an appropriate choice of $a = a_\varepsilon$, $|a_\varepsilon| < 1/10$.

For this purpose, it is useful to introduce a smooth function $\psi : [0, \infty) \to [0, 1]$ such that

$$\psi(t) = \begin{cases} 0 & \text{if } t \leq 1/4, \\ 1 & \text{if } t \geq 1/2, \end{cases}$$

and to write

$$(\mathbf{5.31}) \qquad h_{a,\varepsilon} = \pi_a\big(g_\varepsilon^1\big)\psi\big(\big|g_\varepsilon^1\big|\big) + \pi_a\big(g_\varepsilon^1\big)\big(1 - \psi\big(\big|g_\varepsilon^1\big|\big)\big) = u_{a,\varepsilon} + v_{a,\varepsilon}.$$

Note that, in general, $h_{a,\varepsilon}$ is not well-defined since g_ε^1 may take the value a on a large set. However, if a is chosen to be a *regular value* of g_ε^1, then

$$\Sigma_\varepsilon = \big\{x \in \Omega;\, g_\varepsilon^1(x) = a\big\}$$

consists of a finite number of points and $h_{a,\varepsilon}$ is smooth on $\Omega \setminus \Sigma_\varepsilon$, and we have, using (5.29),

$$(\mathbf{5.32}) \qquad \big|\nabla\big(\pi_a\big(g_\varepsilon^1\big)\big)\big| \leq \mathrm{C}\frac{\big|\nabla g_\varepsilon^1\big|}{\big|g_\varepsilon^1 - a\big|} \text{ on } \Omega \setminus \Sigma_\varepsilon.$$

Moreover, near every point $\sigma \in \Sigma_\varepsilon$, we have $|g_\varepsilon^1(x) - a| \geq c|x-\sigma|$, $c > 0$, and thus

$$\left|\nabla\big(\pi_a\big(g_\varepsilon^1\big)\big)\right| \leq \frac{C_\varepsilon}{|x-\sigma|}.$$

In particular $h_{a,\varepsilon} \in W^{1,p}(\Omega; S^1)$, $\forall p < 2$.

Clearly, the function $\pi_a(z)\psi(|z|)$ is well-defined and lipschitzian on $\mathbf{R}^2$ for any a, $|a| < 1/10$, with a uniform Lipschitz constant independent of a. Therefore, (5.27) yields

$$(\mathbf{5.33}) \qquad \|u_{a,\varepsilon}\|_{H^{1/2}} \leq C\left\|g_\varepsilon^1\right\|_{H^{1/2}} \leq C,$$

where C is independent of a and ε.

Next, we turn to $v_{a,\varepsilon}$, which is well-defined only if a is a regular value of g_ε^1. On $\Omega \setminus \Sigma_\varepsilon$, we have

$$\begin{aligned}|\nabla v_{a,\varepsilon}| &\leq C\frac{\left|\nabla g_\varepsilon^1\right|}{\left|g_\varepsilon^1 - a\right|}(1-\psi)\big(\left|g_\varepsilon^1\right|\big) + \left|\psi'\big(\left|g_\varepsilon^1\right|\big)\right|\left|\nabla g_\varepsilon^1\right| \\ &\leq C\frac{\left|\nabla g_\varepsilon^1\right|}{\left|g_\varepsilon^1 - a\right|}\chi_{[|g_\varepsilon^1|<1/2]},\end{aligned}$$

with C independent of a and ε.

We now make use of an averaging device due to H. Federer and W. H. Fleming [FF] and adapted by R. Hardt, D. Kinderlehrer and F. H. Lin [29] in the context of Sobolev maps with values into spheres. Recall that, by Sard's theorem, the regular values of g_ε^1 have full measure and thus

$$(\mathbf{5.34}) \qquad \int_{B_{1/10}}\int_\Omega |\nabla v_{a,\varepsilon}|^p dx da \leq C_p \int_{[|g_\varepsilon^1|<1/2]} |\nabla g_\varepsilon^1|^p dx, \text{ for any } p < 2.$$

By Hölder, (5.34), (5.26) and (5.28) we find

$$(\mathbf{5.35}) \qquad \int_{B_{1/10}}\int_\Omega |\nabla v_{a,\varepsilon}|^p dx da \leq \left\|g_\varepsilon^1\right\|_{H^1}^p \left|\left[\left|g_\varepsilon^1\right| < 1/2\right]\right|^{1-\frac{p}{2}} \leq C\varepsilon^{-\frac{p}{2}}\varepsilon^{1-\frac{p}{2}} \leq C\varepsilon^{1-p}.$$

Next, fix any $1 < p < 2$ and estimate (*see* e.g. [21])

$$(\mathbf{5.36}) \qquad \|v_{a,\varepsilon}\|_{H^{1/2}} \leq C\|v_{a,\varepsilon}\|_{L^{p'}}^{1/2}\|v_{a,\varepsilon}\|_{W^{1,p}}^{1/2}.$$

From the definition of ψ we have

$$|v_{a,\varepsilon}| \leq \chi_{[|g_\varepsilon^1|<1/2]}$$

and, using (5.26), we obtain

$$(\mathbf{5.37}) \qquad \|v_{a,\varepsilon}\|_{L^{p'}} \leq C\varepsilon^{1/p'}.$$

Substitution of (5.37) and (5.35) in (5.36) yields

$$(\mathbf{5.38}) \qquad \int_{B_{1/10}} \|v_{a,\varepsilon}\|^{2p}_{H^{1/2}} da \leq C\varepsilon^{p-1}\varepsilon^{1-p} \leq C.$$

In view of (5.38) we may now choose $a = a_\varepsilon \in B_{1/10}$, a regular value of g^1_ε, such that

$$(\mathbf{5.39}) \qquad \|v_{a_\varepsilon,\varepsilon}\|_{H^{1/2}} \leq C.$$

Returning to (5.31), and using (5.33) and (5.39), we obtain (5.21) with $h_\varepsilon = h_{a_\varepsilon,\varepsilon}$.

Step **3.** — Write $Z_\varepsilon = [|g^1_\varepsilon| > 1/2]$. For any regular value a of g^1_ε we have

$$\begin{aligned} \left\|h_{a,\varepsilon} - g^1_\varepsilon\right\|^2_{L^2(\Omega)} &= \left\|h_{a,\varepsilon} - g^1_\varepsilon\right\|^2_{L^2(|g^1_\varepsilon|\leq 1/2)} + \left\|h_{a,\varepsilon} - g^1_\varepsilon\right\|^2_{L^2(Z_\varepsilon)} \\ &\leq C\varepsilon + \left\|h_{a,\varepsilon} - g^1_\varepsilon\right\|^2_{L^2(Z_\varepsilon)} \text{ by (5.26).} \end{aligned}$$

Next we estimate

$$\begin{aligned} \left\|h_{a,\varepsilon} - g^1_\varepsilon\right\|_{L^2(Z_\varepsilon)} &\leq \left\|h_{a,\varepsilon} - \frac{g^1_\varepsilon}{|g^1_\varepsilon|}\right\|_{L^2(Z_\varepsilon)} + \left\|\frac{g^1_\varepsilon}{|g^1_\varepsilon|} - g^1_\varepsilon\right\|_{L^2(Z_\varepsilon)} \\ &= \left\|\pi_a\big(g^1_\varepsilon\big) - \pi_a\left(\frac{g^1_\varepsilon}{\big|g^1_\varepsilon\big|}\right)\right\|_{L^2(Z_\varepsilon)} + \left\|\frac{g^1_\varepsilon}{\big|g^1_\varepsilon\big|} - g^1_\varepsilon\right\|_{L^2(Z_\varepsilon)}. \end{aligned}$$

Since $\pi_a(\xi)$ is lipschitzian on $[|\xi| \geq 1/2]$ we obtain

$$\begin{aligned} \left\|h_{a,\varepsilon} - g^1_\varepsilon\right\|_{L^2(Z_\varepsilon)} \leq C\left\|g^1_\varepsilon - \frac{g^1_\varepsilon}{\big|g^1_\varepsilon\big|}\right\|_{L^2(Z_\varepsilon)} &\leq C\left\|1 - \big|g^1_\varepsilon\big|\right\|_{L^2(Z_\varepsilon)} \\ &\leq C\sqrt{\varepsilon}, \text{ by (5.26).} \end{aligned}$$

Therefore

$$(\mathbf{5.40}) \qquad \left\|h_{a,\varepsilon} - g^1_\varepsilon\right\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}$$

with C independent of a and ε.

Combining (5.25) and (5.40) yields

$$\|h_{a,\varepsilon} - g_\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon},$$

which is (5.22) when choosing $a = a_\varepsilon$.

Step **4.** — Suppose now, in addition, that $g_\varepsilon \to g$ in $H^{1/2}$. We claim that $h_\varepsilon \to g$ in $H^{1/2}$.

Indeed, we have

$$\begin{aligned}\left\|g_\varepsilon^1\right\|_{H^1} &\le \|(g_\varepsilon - g) * P_\varepsilon\|_{H^1} + \|g * P_\varepsilon\|_{H^1}\\ &\le C\varepsilon^{-1/2}\|g_\varepsilon - g\|_{H^{1/2}} + \|g * P_\varepsilon\|_{H^1}\\ &= o(\varepsilon^{-1/2}).\end{aligned}$$

Returning to (5.35) and (5.38) we now find

$$\int_{B_{1/10}}\int_\Omega |\nabla v_{a,\varepsilon}|^p\, dx da \to 0 \text{ as } \varepsilon \to 0$$

and we may choose a_ε so that

$$\|v_{a_\varepsilon,\varepsilon}\|_{H^{1/2}} \to 0 \text{ as } \varepsilon \to 0.$$

It remains to show that

$$u_{a_\varepsilon,\varepsilon} \to g \text{ in } H^{1/2} \text{ as } \varepsilon \to 0. \tag{5.41}$$

Recall that

$$u_{a_\varepsilon,\varepsilon} = \pi_{a_\varepsilon}\left(g_\varepsilon^1\right)\psi\left(\left|g_\varepsilon^1\right|\right) = L_\varepsilon\left(g_\varepsilon^1\right),$$

where $L_\varepsilon : \mathbf{R}^2 \to \mathbf{R}^2$ are lipschitzian maps with a uniform Lipschitz constant.

We have

$$\begin{aligned}\left\|g_\varepsilon^1 - g\right\|_{H^{1/2}} &= \|(g_\varepsilon - g) * P_\varepsilon + (g * P_\varepsilon) - g\|_{H^{1/2}}\\ &\le C\|g_\varepsilon - g\|_{H^{1/2}} + \|(g * P_\varepsilon) - g\|_{H^{1/2}},\end{aligned}$$

so that

$$\|g_\varepsilon^1 - g\|_{H^{1/2}} \to 0. \tag{5.42}$$

Finally we use the following claim:

$$\begin{cases}\text{If } (k_n) \text{ is a sequence in } H^{1/2}(\Omega;\mathbf{R}^2) \text{ such that } k_n \to k \text{ in } H^{1/2} \text{ and}\\ L_n : \mathbf{R}^2 \to \mathbf{R}^2 \text{ satisfy a uniform Lipschitz condition, then}\\ L_n(k_n) - L_n(k) \to 0 \text{ in } H^{1/2}.\end{cases} \tag{5.43}$$

Proof of (5.43). — It suffices to argue on subsequences. Since

$$|k_n - k|^2_{H^{1/2}} = \int_\Omega \int_\Omega \frac{|k_n(x) - k(x) - k_n(y) + k(y)|^2}{d(x,y)^3} dxdy \to 0,$$

there is, (modulo a subsequence), some fixed $h(x, y) \in L^1(\Omega \times \Omega)$ such that

$$\frac{|k_n(x) - k_n(y)|^2}{d(x,y)^3} \le h(x,y), \quad \forall n.$$

We have

$$\begin{aligned} &|L_n(k_n) - L_n(k)|^2_{H^{1/2}} \\ &\quad = \int_\Omega \int_\Omega \frac{|L_n(k_n(x)) - L_n(k(x)) - L_n(k_n(y)) + L_n(k(y))|^2}{d(x,y)^3} dxdy, \end{aligned}$$

and the integrand $I_n(x, y)$ satisfies

$$\begin{aligned} I_n(x,y) &\le C\frac{(|k_n(x) - k_n(y)|^2 + |k(x) - k(y)|^2)}{d(x,y)^3} \\ &\le Ch(x,y), \end{aligned}$$

and also,

$$I_n(x,y) \le C\frac{(|k_n(x) - k(x)|^2 + |k_n(y) - k(y)|^2)}{d(x,y)^3}.$$

Therefore, by dominated convergence,

$$|L_n(k_n) - L_n(k)|_{H^{1/2}} \to 0.$$

This proves (5.43).

We now return to the proof of (5.41). Applying (5.43) to $L_n(\xi) = \pi_{a_{\varepsilon_n}}(\xi)\psi(|\xi|)$ and to $k_n = g^1_{\varepsilon_n} \to g$ in $H^{1/2}$ by (5.42), we find that

$$L_n\left(g^1_{\varepsilon_n}\right) - L_n(g) \to 0 \text{ in } H^{1/2}.$$

But $L_n(g) = g \quad \forall n$ since $|g| = 1$. Thus we are led to $L_n(g^1_{\varepsilon_n}) \to g$ in $H^{1/2}$, which is (5.41).

This completes the proof of Lemma 23.

Remark **5.1.** — It is interesting to observe that the construction used in the proof of Lemma 23 gives a simple proof of Rivière's Lemma 11. In fact, we have a more precise statement. Fix any element $g \in H^{1/2}(\Omega; S^1)$ and apply the construction described above with $g_\varepsilon \equiv g$. The sequence

$$h_\varepsilon = \pi_{a_\varepsilon}(g * P_\varepsilon)$$

satisfies the following properties:

$$h_\varepsilon \in W^{1,p}(\Omega; S^1), \quad \forall p < 2, \forall \varepsilon, \tag{5.44}$$

$$h_\varepsilon \to g \text{ in } H^{1/2} \text{ as } \varepsilon \to 0, \tag{5.45}$$

$$\begin{cases} h_\varepsilon \text{ is smooth except on a finite set } \Sigma_\varepsilon \subset \Omega \text{ and} \\ |\nabla h_\varepsilon(x)| \leq \dfrac{C_\varepsilon}{\operatorname{dist}(x, \Sigma_\varepsilon)}, \quad \forall x \in \Omega \setminus \Sigma_\varepsilon, \end{cases} \tag{5.46}$$

$$\begin{cases} \text{for each } \sigma \in \Sigma_\varepsilon, \text{ there is a smooth diffeomorphism } \gamma = \gamma_{\varepsilon,\sigma}, \\ \text{from the unit circle in } T_\sigma(\Omega) \text{ onto } S^1, \text{ such that, assuming} \\ \Omega \text{ flat near } \sigma \text{ (for simplicity), we have} \\ \left| h_\varepsilon(x) - \gamma\left(\dfrac{x-\sigma}{|x-\sigma|}\right)\right| \leq C_\varepsilon |x-\sigma| \text{ for } x \in \Omega \text{ near } \sigma. \end{cases} \tag{5.47}$$

Here, $T_\sigma(\Omega)$ denotes the tangent space to Ω at σ. Note that (5.47) implies that $\deg(g, \sigma) = \pm 1$ for each singularity σ.

All the above properties are clear from the proof of Lemma 23, except possibly (5.47). Taylors's expansion near $\sigma \in \Sigma_\varepsilon$ gives

$$g_\varepsilon^1(x) = g_\varepsilon^1(\sigma) + M(x-\sigma) + O(|x-\sigma|^2)$$

where $g_\varepsilon^1(\sigma) = a_\varepsilon$ and $M = M_{\varepsilon,\sigma} = Dg_\varepsilon^1(\sigma)$ is a bounded invertible linear operator from $T_\sigma(\Omega)$ onto $\mathbf{R}^2$ (since a_ε is a regular value of g_ε^1). Thus

$$\frac{g_\varepsilon^1(x) - a_\varepsilon}{\left|g_\varepsilon^1(x) - a_\varepsilon\right|} = \frac{M(x-\sigma)}{|M(x-\sigma)|} + O(|x-\sigma|)$$

and therefore

$$h_\varepsilon(x) = j_{a_\varepsilon}^{-1}\left(\frac{g_\varepsilon^1(x) - a_\varepsilon}{\left|g_\varepsilon^1(x) - a_\varepsilon\right|}\right) = j_{a_\varepsilon}^{-1}\left(\frac{M(x-\sigma)}{|M(x-\sigma)|}\right) + O(|x-\sigma|),$$

where $j_{a_\varepsilon}(\xi) = \dfrac{\xi - a_\varepsilon}{|\xi - a_\varepsilon|} : S^1 \to S^1$. This proves (5.47) with

$$\gamma(z) = j_{a_\varepsilon}^{-1}\left(\frac{Mz}{|Mz|}\right), z \in T_\sigma(\Omega).$$

Clearly, γ is a smooth diffeomorphism from the unit circle in $T_\sigma(\Omega)$ onto S^1. We will present in Appendix B a more precise statement.

Remark **5.2.** — The averaging process over a in the proof of Lemma 23 can be done on any ball B_ρ, $0 < \rho \leq 1/10$, with ρ possibly depending on ε. In particular, when $g_\varepsilon \to g$ in $H^{1/2}$, one may choose some special $\rho_\varepsilon \to 0$ and obtain a corresponding a_ε with $a_\varepsilon \to 0$. Then

$$\tilde{h}_{a_\varepsilon,\varepsilon} = \frac{g^1_\varepsilon - a_\varepsilon}{|g^1_\varepsilon - a_\varepsilon|}$$

has all the desired properties without having to consider

$$h_{a_\varepsilon,\varepsilon} = j^{-1}_{a_\varepsilon}\tilde{h}_{a_\varepsilon,\varepsilon}.$$

The argument is similar, with a minor modification in Step 3.

Proof of Theorem 6′. — Let $k_\varepsilon \in H^{1/2}(\Omega; \mathbf{R}^2)$ with $|k_\varepsilon| \leq 1$. We claim that

$$(\mathbf{5.48}) \qquad e_{\varepsilon,k_\varepsilon} \leq C_\Omega \left(|k_\varepsilon|^2_{H^{1/2}} + \frac{1}{\varepsilon}\|k_\varepsilon - 1\|^2_{L^2} \right).$$

Indeed, let $u = u_\varepsilon$ be the solution of (5.13), (5.14) corresponding to $\varphi = k_\varepsilon - 1$. Using the function $(u_\varepsilon + 1)$ as a test function in the definition of $e_{\varepsilon,k_\varepsilon}$, we find

$$(\mathbf{5.49}) \qquad e_{\varepsilon,k_\varepsilon} \leq \frac{1}{2}\int_G |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}\int_G \left(|u_\varepsilon + 1|^2 - 1\right)^2.$$

From (5.15), we have

$$(\mathbf{5.50}) \qquad \int_G |\nabla u_\varepsilon|^2 \leq C \left(|k_\varepsilon|^2_{H^{1/2}} + \frac{1}{\varepsilon}\|k_\varepsilon - 1\|^2_{L^2} \right).$$

On the other hand, by the maximum principle, we have

$$\|u_\varepsilon\|_{L^\infty(G)} \leq \|k_\varepsilon - 1\|_{L^\infty(\Omega)} \leq 2,$$

and thus, by (5.15),

$$(\mathbf{5.51}) \qquad \begin{aligned} \int_G (|u_\varepsilon + 1|^2 - 1)^2 &= \int_G (|u_\varepsilon + 1| - 1)^2(|u_\varepsilon + 1| + 1)^2 \leq 16\int_G |u_\varepsilon|^2 \\ &\leq C\varepsilon^2 \left(|k_\varepsilon|^2_{H^{1/2}} + \frac{1}{\varepsilon}\|k_\varepsilon - 1\|^2_{L^2} \right). \end{aligned}$$

Combining (5.49), (5.50) and (5.51) yields (5.48).

Next, we write, using h_ε from Lemma 23,

$$g_\varepsilon = (g_\varepsilon \bar{h}_\varepsilon)(h_\varepsilon \bar{g})g$$

and apply (5.1) to find

$$e_{\varepsilon,g_\varepsilon} \leq (1+\delta)e_{\varepsilon,g} + C(\delta)(e_{\varepsilon,h_\varepsilon\bar{g}} + e_{\varepsilon,g_\varepsilon\bar{h}_\varepsilon}). \tag{5.52}$$

We deduce from (5.48) (applied to $k_\varepsilon = g_\varepsilon \bar{h}_\varepsilon$) that

$$\begin{aligned} e_{\varepsilon,g_\varepsilon\bar{h}_\varepsilon} &\leq C\left(|g_\varepsilon\bar{h}_\varepsilon|^2_{H^{1/2}} + \frac{1}{\varepsilon}\|g_\varepsilon\bar{h}_\varepsilon - 1\|^2_{L^2}\right) \\ &\leq C\left(|g_\varepsilon|^2_{H^{1/2}} + |h_\varepsilon|^2_{H^{1/2}} + \frac{1}{\varepsilon}\|g_\varepsilon - h_\varepsilon\|^2_{L^2}\right) \leq C. \end{aligned} \tag{5.53}$$

Applying (5.4) (with g replaced by $h_\varepsilon\bar{g}$) yields

$$e_{\varepsilon,h_\varepsilon\bar{g}} \leq C|h_\varepsilon\bar{g}|^2_{H^{1/2}}(1+\log(1/\varepsilon)). \tag{5.54}$$

Recall that $|h_\varepsilon\bar{g}|_{H^{1/2}} \to 0$ as $\varepsilon \to 0$ (by (5.24)). By Theorem 6, we know that

$$e_{\varepsilon,g} = \pi L(g)\log(1/\varepsilon) + o(\log(1/\varepsilon)). \tag{5.55}$$

Combining (5.52)–(5.55) we finally obtain

$$\limsup_{\varepsilon\to 0} \frac{e_{\varepsilon,g_\varepsilon}}{\log(1/\varepsilon)} \leq \pi L(g)(1+\delta), \quad \forall\, \delta > 0.$$

The lower bound

$$\liminf_{\varepsilon\to 0} \frac{e_{\varepsilon,g_\varepsilon}}{\log(1/\varepsilon)} \geq \pi L(g)(1-\delta), \quad \forall\, \delta > 0,$$

is deduced in the same way via (5.2). This completes the proof of Theorem 6′.

6. $W^{1,p}(G)$ compactness for $p < 3/2$ and $g \in H^{1/2}$. Proof of Theorem 7′

Proof of Theorem 7′. — The estimate

$$\|u_\varepsilon\|_{W^{1,p}(G)} \leq C_p, \quad \forall\, 1 \leq p < 3/2,$$

was established in [5]. We will now show that a simple adaptation of the argument there yields compactness. We rely on the following

Lemma **24.** — *The family* $(u_\varepsilon \wedge du_\varepsilon)$ *is compact in* $L^p(G)$, $1 \le p < 3/2$.

Proof of Lemma 24. — Let $X_\varepsilon = u_\varepsilon \wedge du_\varepsilon$. Since $\operatorname{div}(X_\varepsilon) = 0$, we may write $X_\varepsilon =$ curl H_ε. As explained in Section 3 of [5], we may choose H_ε of the form $H_\varepsilon = H^1_\varepsilon + H^2$. Here $H^2 \in W^{1,p}(G)$, $1 \le p < 3/2$, depends only on g, while H^1_ε is a linear operator acting on X_ε satisfying the estimate

$$\|H^1_\varepsilon\|_{W^{1,p}(G)} \le C_p \|dX_\varepsilon\|_{[W^{1,q}(G)]^*}, \ \ 1 \le p < 3/2, \ \ \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore, it suffices to prove that (dX_ε) is relatively compact in $[W^{1,q}(G)]^*$.

For $1 \le p < 3/2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, let $0 < \beta < \alpha = 1 - \dfrac{3}{q}$. Then the imbedding $W^{1,q}(G) \subset C^{0,\beta}(\overline{G})$ is compact. Hence the imbedding $(C^{0,\beta}(\overline{G}))^* \subset (W^{1,q}(G))^*$ is compact. The conclusion of Lemma 24 follows now easily from the bound $\|dX_\varepsilon\|_{[C^{0,\beta}(\overline{G})]^*} \le C$ derived in [5]; *see* Theorem 2bis in [5].

Proof of Theorem 7′ completed. — Let $A = A_\varepsilon = \{x \in G\,;\, |u_\varepsilon(x)| \le 1/2\}$. Since $E_\varepsilon(u_\varepsilon) \le C\log(1/\varepsilon)$, we have $|A_\varepsilon| \le C\varepsilon^2\log(1/\varepsilon)$. In $G\backslash A_\varepsilon$, we have

$$du_\varepsilon = \frac{\imath u_\varepsilon}{|u_\varepsilon|^2} u_\varepsilon \wedge du_\varepsilon + \frac{u_\varepsilon}{|u_\varepsilon|} d|u_\varepsilon|. \tag{6.1}$$

We may thus write in G

$$du_\varepsilon = \chi_{A_\varepsilon} du_\varepsilon + \chi_{G\backslash A_\varepsilon}\Big(\frac{\imath u_\varepsilon}{|u_\varepsilon|^2} u_\varepsilon \wedge du_\varepsilon + \frac{u_\varepsilon}{|u_\varepsilon|} d|u_\varepsilon|\Big).$$

Note that

$$\int_{A_\varepsilon} |du_\varepsilon|^p \le \Big(\int_{A_\varepsilon} |du_\varepsilon|^2\Big)^{p/2} |A_\varepsilon|^{1-p/2} \overset{\varepsilon}{\to} 0, \quad 1 \le p < 2.$$

Recall the following estimate (*see* [9], Proposition VI. 4):

$$\int_G |d|u_\varepsilon||^p \overset{\varepsilon}{\to} 0, \quad 1 \le p < 2.$$

Applying (6.1) and Lemma 24 we see that (u_ε) is bounded in $W^{1,p}$, $p < 3/2$. In particular, up to a subsequence, we have $u_\varepsilon \overset{\varepsilon}{\to} u_0$ a.e. for some u_0. Moreover, we see that $|u_\varepsilon| \overset{\varepsilon}{\to} 1$ a.e., since

$$\frac{1}{\varepsilon^2}\int_G (1 - |u_\varepsilon|^2)^2 \le C\log(1/\varepsilon),$$

so that $|u_0| = 1$. Thus, up to a subsequence, we find

$$du_\varepsilon - \imath u_0(u_\varepsilon \wedge du_\varepsilon) \xrightarrow{\varepsilon} 0 \text{ in } L^p, \quad 1 \le p < 2.$$

Finally, Lemma 24 implies that, up to a further sequence, (du_ε) converges in $L^p(G)$, $1 \le p < 3/2$.

The proof of Theorem 7′ is complete.

As in the case of Theorem 6, Theorem 7′ generalizes to the situation where the boundary data is not fixed anymore:

Theorem **7″.** — *Assume that the maps* $g_\varepsilon \in H^{1/2}(\Omega; \mathbf{R}^2)$ *are such that:*

$$|g_\varepsilon|_{H^{1/2}} \le C, \tag{6.2}$$

$$|g_\varepsilon| \le 1 \quad \textit{on } \Omega, \tag{6.3}$$

and

$$\| |g_\varepsilon| - 1 \|_{L^2} \le C\sqrt{\varepsilon}. \tag{6.4}$$

Let u_ε *be a minimizer of* E_ε *in* $H^1_{g_\varepsilon}(G; \mathbf{R}^2)$. *Then* $E_\varepsilon(u_\varepsilon) \le C \log(1/\varepsilon)$ *and* (u_ε) *is relatively compact in* $W^{1,p}(G)$, $1 \le p < 3/2$.

An easy variant of the proof of Theorem 6′ yields the bound $E_\varepsilon(u_\varepsilon) \le C \log(1/\varepsilon)$. To establish compactness in $W^{1,p}$ we rely on the following variant of Lemma 24:

Lemma **24′.** — *The family* $(u_\varepsilon \wedge du_\varepsilon)$ *is compact in* $L^p(G)$, $1 \le p < 3/2$.

Proof of Lemma 24′. — With $X_\varepsilon = u_\varepsilon \wedge du_\varepsilon$, we may write $X_\varepsilon = \operatorname{curl} H_\varepsilon$, where H_ε is a linear operator acting on $(X_\varepsilon, g_\varepsilon \wedge d_T g_\varepsilon)$ and satisfying the estimate

$$\|H_\varepsilon\|_{W^{1,p}} \le C(\|dX_\varepsilon\|_{[W^{1,q}(G)]^*} + \|g_\varepsilon \wedge d_T g_\varepsilon\|_{[W^{1-1/q,q}(\Omega)]^*}),$$
$$1 \le p < 3/2, \ \frac{1}{p} + \frac{1}{q} = 1$$

(*see* [5]). Here, d_T stands for the tangential differential operator on Ω.

The proof of Lemma 2 in [5] implies that $(g_\varepsilon \wedge d_T g_\varepsilon)$ is bounded in $[W^{\sigma,q}(\Omega)]^*$ provided $\sigma > 1/2$ and $\sigma q > 2$. If we choose $\sigma > 1/2$ such that $\dfrac{2}{q} < \sigma < 1 - \dfrac{1}{q}$, we find that $(g_\varepsilon \wedge d_T g_\varepsilon)$ is compact in $[W^{1-1/q,q}(\Omega)]^*$.

It remains to prove that (dX_ε) is compact in $[W^{1,q}(G)]^*$. As in the proof of Lemma 24, it suffices to prove that (dX_ε) is bounded in $[C^{0,\alpha}(\overline{G})]^*$ for $0 < \alpha < 1$.

For this purpose, we construct an appropriate extension of u_ε to a larger domain. Let, for $0 < \varepsilon < \varepsilon_0(\mathrm{G})$, Π_ε be the projection onto Ω of the set

$$\Omega_\varepsilon = \{ x \in \mathbf{R}^3 \setminus \Omega\, ;\ \mathrm{dist}\ (x, \Omega) = \varepsilon \}.$$

Set $\widetilde{h}_\varepsilon = h_\varepsilon \circ \Pi_\varepsilon \in \mathrm{H}^{1/2}(\Omega_\varepsilon)$ (where h_ε is defined in Lemma 23) and let K_ε be the harmonic extension of $\widetilde{h}_\varepsilon$ to

$$\mathrm{G} \cup \{ x \in \mathbf{R}^3\, ;\ \mathrm{dist}\ (x, \Omega) < \varepsilon \}.$$

By standard estimates, we have

$$\|h_\varepsilon - \mathrm{K}_{\varepsilon|\Omega}\|_{\mathrm{L}^2} \leq \mathrm{C}_\mathrm{G} |h_\varepsilon|_{\mathrm{H}^{1/2}} \varepsilon^{1/2},$$

so that

$$\|g_\varepsilon - \mathrm{K}_{\varepsilon|\Omega}\|_{\mathrm{L}^2} \leq \mathrm{C}\varepsilon^{1/2}.$$

By Lemma 22′ applied to $\varphi = g_\varepsilon - \mathrm{K}_{\varepsilon|\Omega}$, we may find a map $v_\varepsilon : \mathrm{G}_\varepsilon \to \mathbf{C}$ such that

$$\int_{\mathrm{G}_\varepsilon} |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_{\mathrm{G}_\varepsilon} |v_\varepsilon|^2 \leq \mathrm{C},$$

$$v_\varepsilon = g_\varepsilon - \mathrm{K}_{\varepsilon|\Omega} \quad \text{on } \Omega, \quad v_\varepsilon = 0 \quad \text{on } \Omega_\varepsilon$$

and

$$|v_\varepsilon| \leq 2 \quad \text{in } \mathrm{G}_\varepsilon.$$

Set

$$\mathrm{U}_\varepsilon = \begin{cases} u_\varepsilon, & \text{in G} \\ v_\varepsilon + \mathrm{K}_\varepsilon, & \text{in } \mathrm{G}_\varepsilon \end{cases},$$

which satisfies $\mathrm{U}_\varepsilon = \widetilde{h}_\varepsilon$ on Ω_ε. Since, for $0 < \delta < \varepsilon$, we have

$$\begin{aligned} \int_{\Omega_\delta} (1 - |\mathrm{U}_\varepsilon|^2)^2 &\leq \int_{\Omega_\delta} (|1 - |\mathrm{K}_\varepsilon|| + |v_\varepsilon|)^2 (1 + |\mathrm{K}_\varepsilon| + |v_\varepsilon|)^2 \\ &\leq 32 \int_{\Omega_\delta} (|h_\varepsilon \circ \Pi_\delta - \mathrm{K}_\varepsilon|^2 + |v_\varepsilon|^2), \end{aligned}$$

we find by standard estimates that

$$\int_{\Omega_\delta} (1 - |\mathrm{U}_\varepsilon|^2)^2 \leq \mathrm{C}\Big(\varepsilon |h_\varepsilon|^2_{\mathrm{H}^{1/2}} + \int_{\Omega_\delta} |v_\varepsilon|^2 \Big). \tag{6.5}$$

Integration of (6.5) over δ combined with the obvious bound

$$\|K_\varepsilon\|_{H^1(G\cup G_\varepsilon)} \le C$$

yields

$$E_\varepsilon(U_\varepsilon; G_\varepsilon) \le C. \tag{6.6}$$

As we already mentioned, an easy variant of the proof of Theorem 6′ gives

$$E_\varepsilon(u_\varepsilon; G) \le C\log(1/\varepsilon)$$

and thus

$$E_\varepsilon(U_\varepsilon; G\cup G_\varepsilon) \le C\log(1/\varepsilon). \tag{6.7}$$

Let now $R > 0$ be such that

$$\overline{G\cup G_{\varepsilon_0(G)}} \subset B_R.$$

A straightforward adaptation of Proposition 4 in [5] implies that, for $0 < \varepsilon < \varepsilon_0(G)$, there is a map $w_\varepsilon \in H^1(B_R \setminus (G\cup G_\varepsilon))$ such that

$$w_\varepsilon = \widetilde{h}_\varepsilon \quad \text{on } \Omega_\varepsilon, \quad w_\varepsilon = 1 \quad \text{on } \partial B_R, \tag{6.8}$$

$$E_\varepsilon(w_\varepsilon) \le C\log(1/\varepsilon), \tag{6.9}$$

and

$$\int_{B_R\setminus(G\cup G_\varepsilon)} |\mathrm{Jac}\ w_\varepsilon| \le C. \tag{6.10}$$

Set

$$V_\varepsilon = \begin{cases} U_\varepsilon, & \text{in } G\cup G_\varepsilon \\ w_\varepsilon, & \text{in } B_R \setminus (G\cup G_\varepsilon) \end{cases}.$$

By (6.7) and (6.9), we have

$$E_\varepsilon(V_\varepsilon; B_R) \le C\log(1/\varepsilon),$$

so that $\mathrm{Jac}\, V_\varepsilon$ is bounded in $[C^{0,\alpha}_{loc}(B_R)]^*$ for $0 < \alpha < 1$ (*see* [33]). As in the proof of Theorem 2bis in [5], we may now establish the boundedness of dX_ε in $[C^{0,\alpha}(\overline{G})]^*$ for

$0 < \alpha < 1$. Indeed, let $\delta > 0$ be sufficiently small. For $\zeta \in C^{0,\alpha}(\overline{G}; \wedge^1(\mathbf{R}))$, let ψ be an extension of ζ to $\mathbf{R}^3$ such that $\|\psi\|_{C^{0,\alpha}(\mathbf{R}^3)} \le C\|\zeta\|_{C^{0,\alpha}(\overline{G})}$ and Supp $\psi \subset \overline{B}_{R-\delta}$. Then

$$\left|\int_G dX_\varepsilon \wedge \zeta\right| \le \left|\int_{B_R} d(V_\varepsilon \wedge dV_\varepsilon) \wedge \psi\right| + \int_{B_R\setminus G} \left|d(V_\varepsilon \wedge dV_\varepsilon) \wedge \psi\right|$$
$$\le C_\alpha \|\psi\|_{C^{0,\alpha}(\overline{G})} + \|\psi\|_{L^\infty} \int_{B_R\setminus G} |\text{Jac } V_\varepsilon| \le C\|\zeta\|_{C^{0,\alpha}(\overline{G})},$$

by (6.6) and (6.10).

The proof of Lemma 24′ is complete.

Proof of Theorem 7″. — An inspection of the proof of Theorem 7′ shows that it suffices to establish the estimate

$$\textbf{(6.11)} \qquad \int_G |\nabla |u_\varepsilon||^p \to 0 \text{ as } \varepsilon \to 0, \quad \forall\ 1 \le p < 2.$$

We adapt the proof of Proposition VI.4 in [9]. Set $\eta = \eta_\varepsilon = 1 - |u_\varepsilon|^2$, which satisfies

$$\textbf{(6.12)} \qquad -\Delta\eta + \frac{2}{\varepsilon^2}|u_\varepsilon|^2\eta = 2|\nabla u_\varepsilon|^2 \quad \text{in } G,$$

$$\textbf{(6.13)} \qquad \eta \ge 0 \qquad \text{on } \Omega.$$

Let $\widetilde{\eta}$ be the solution of

$$\textbf{(6.14)} \qquad -\Delta\widetilde{\eta} + \frac{2}{\varepsilon^2}|u_\varepsilon|^2\widetilde{\eta} = 2|\nabla u_\varepsilon|^2 \quad \text{in } G,$$

$$\textbf{(6.15)} \qquad \widetilde{\eta} = 0 \qquad \text{on } \Omega,$$

so that

$$\textbf{(6.16)} \qquad 1 - |u_\varepsilon|^2 = \eta \ge \widetilde{\eta} \ge 0,$$

by the maximum principle. Set $\overline{\eta} = \text{Min }(\widetilde{\eta}, \varepsilon^{1/2})$. Multiplying (6.14) by $\overline{\eta}$, we find

$$\textbf{(6.17)} \qquad \int_{\{\widetilde{\eta}<\varepsilon^{1/2}\}} |\nabla\widetilde{\eta}|^2 \le 2\varepsilon^{1/2} \int_G |\nabla u_\varepsilon|^2 \to 0 \text{ as } \varepsilon \to 0.$$

On the other hand, we have

$$\textbf{(6.18)} \qquad \{x;\ \widetilde{\eta}(x) \ge \varepsilon^{1/2}\} \subset \{x;\ |u_\varepsilon(x)|^2 \le 1 - \varepsilon^{1/2}\}.$$

Set $\zeta = \eta - \widetilde{\eta}$, which satisfies

$$(\mathbf{6.19}) \qquad -\Delta\zeta + \frac{2}{\varepsilon^2}|u_\varepsilon|^2\zeta = 0 \quad \text{in } G,$$

$$(\mathbf{6.20}) \qquad \zeta = \varphi_\varepsilon \quad \text{on } \Omega,$$

where $\varphi_\varepsilon = 1 - |g_\varepsilon|^2$. Clearly, we have $|\varphi_\varepsilon|_{H^{1/2}} \leq C$ and by (6.4)

$$(\mathbf{6.21}) \qquad \|\varphi_\varepsilon\|_{L^2} \leq C\varepsilon^{1/2}.$$

By the proof of Lemma 22, we find that

$$(\mathbf{6.22}) \qquad \int_G |\nabla\zeta|^2 \leq C.$$

We claim that

$$(\mathbf{6.23}) \qquad \int_G |\nabla\zeta|^p \to 0 \text{ as } \varepsilon \to 0, \quad \forall p < 2.$$

Indeed, by the maximum principle, $0 \leq \zeta \leq \hat{\zeta}$ where $\hat{\zeta}$ is the solution of

$$\begin{aligned} -\Delta\hat{\zeta} &= 0 \quad \text{in } G, \\ \hat{\zeta} &= \varphi_\varepsilon \quad \text{on } \Omega. \end{aligned}$$

In particular, from (6.21) we see that

$$(\mathbf{6.24}) \qquad \int_G |\hat{\zeta}|^2 \to 0 \text{ as } \varepsilon \to 0.$$

Let $\chi \in C_0^\infty(G)$ with $0 \leq \chi \leq 1$ on G. Multiplying (6.19) by $\zeta\chi$ and integrating we obtain

$$\int_G |\nabla\zeta|^2\chi \leq \frac{1}{2}\int_G \zeta^2|\Delta\chi| \leq \frac{1}{2}\int_G \hat{\zeta}^2|\Delta\chi|.$$

Combining this with (6.24) yields

$$(\mathbf{6.25}) \qquad \int_G |\nabla\zeta|^2\chi \to 0 \quad \forall\chi \in C_0^\infty(G), 0 \leq \chi \leq 1.$$

From (6.22) and (6.25) we deduce (6.23).

We now claim that

$$\int_G |\nabla \eta|^p \to 0 \text{ as } \varepsilon \to 0, \quad \forall p < 2. \tag{6.26}$$

Since $\eta = \zeta + \tilde{\eta}$, in view of (6.17) and (6.23) it suffices to prove that

$$\int_{Z_\varepsilon} |\nabla \tilde{\eta}|^p \to 0.$$

where $Z_\varepsilon = \{x\,;\ |u_\varepsilon(x)|^2 \le 1 - \varepsilon^{1/2}\}$. But

$$\int_G (1 - |u_\varepsilon|^2)^2 \le C\varepsilon^2 \log(1/\varepsilon),$$

and thus

$$|Z_\varepsilon| \le C\varepsilon \log(1/\varepsilon), \tag{6.27}$$

so that, by Hölder and (6.14)–(6.15),

$$\begin{aligned} \int_{Z_\varepsilon} |\nabla \tilde{\eta}|^p &\le \|\nabla \tilde{\eta}\|_{L^2}^p |Z_\varepsilon|^{(2-p)/2} \\ &\le C\|\nabla u_\varepsilon\|_{L^2}^p |Z_\varepsilon|^{(2-p)/2} \le C\varepsilon^{(2-p)/2}(\log(1/\varepsilon)) \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{6.28}$$

Hence we have established (6.26). Similarly,

$$\int_{Z_\varepsilon} |\nabla u_\varepsilon|^p \le \|\nabla u_\varepsilon\|_{L^2}^p |Z_\varepsilon|^{(2-p)/2} \le C\varepsilon^{(2-p)/2} \log(1/\varepsilon)) \to 0 \text{ as } \varepsilon \to 0. \tag{6.29}$$

Finally, we note that, for ε sufficiently small, we have

$$|\nabla |u_\varepsilon|| \le |\nabla u_\varepsilon| \chi_{Z_\varepsilon} + |\nabla \eta|, \tag{6.30}$$

so that (6.11) follows by combining (6.26), (6.29) and (6.30).

The proof of Theorem 7″ is complete.

7. Improved interior estimates. $W^{1,p}_{loc}(G)$ compactness for $p < 2$ and $g \in H^{1/2}$. Proof of Theorem 8

Remark **7.1.** — As in the proof of Theorems 7′ and 7″, it suffices to establish the estimate

$$\|u_\varepsilon \wedge du_\varepsilon\|_{L^p(K)} \le C, \quad 3/2 \le p < 2, \quad K \text{ compact in } G. \tag{7.1}$$

Estimate (7.1) will be proved under the following assumptions:

$$E_\varepsilon(u_\varepsilon) \le C \log(1/\varepsilon)$$

and

$$u_\varepsilon \text{ is bounded in } W^{1,r}(G), \qquad \text{for some } 4/3 < r < 3/2.$$

In view of Theorems 6, 7 and of their variants, we find that Theorem 8 extends to minimizers u_ε of E_ε when the variable boundary conditions satisfy (6.1)–(6.3).

Proof of Theorem 8. — In what follows, we establish (7.1) when K is any compact subset of the unit ball B.

Fix some $3/2 \le p < 2$ and $0 < \gamma < 1$. Fix

$$4/3 < r < 3/2. \tag{7.2}$$

Denote $u = u_\varepsilon$. Since, by Theorems 6 and 7, we have

$$\|u\|_{W^{1,r}(B)} \le C \quad \text{and} \quad \|u\|_{H^1(B)} \le C(\log(1/\varepsilon))^{1/2},$$

we may choose

$$1 - \gamma < \rho < 1 - \gamma/2$$

such that

$$\|u\|_{W^{1,r}(\partial B_\rho)} \le C_\gamma \tag{7.3}$$

and

$$\|u\|_{H^1(B_\rho)} \le C_\gamma (\log(1/\varepsilon))^{1/2}. \tag{7.4}$$

Set now $p = 2 - s$, so that $s > 0$ and the conjugate exponent of p is

$$2 < q = \frac{2-s}{1-s} \le 3. \tag{7.5}$$

Perform on B_ρ a Hodge decomposition

$$\frac{u \wedge du}{|u \wedge du|^s} = d^* k + d\mathrm{L},$$

where

$$\mathrm{L} = \text{ 0-form}, \quad \mathrm{L} = 0 \text{ on } \partial B_\rho \tag{7.6}$$

and

$$\begin{aligned} k = \text{ 2-form}, \quad \|k\|_{W^{1,q}} \leq C \left\| \frac{u \wedge du}{|u \wedge du|^s} \right\|_q &= C\|u \wedge du\|_p^{1-s} \\ &= C\|u \wedge du\|_p^{p-1}; \end{aligned} \tag{7.7}$$

here, we use the notation $\| \quad \|_p = \| \quad \|_{L^p(B_\rho)}$.

Recalling the fact that $\operatorname{div}(u \wedge du) = 0$, we find that

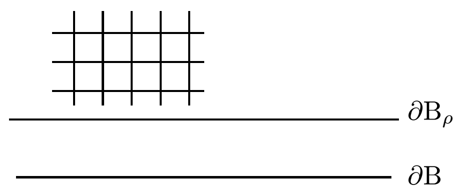
$$\|u \wedge du\|_p^p = \int_{B_\rho} (d^* k) \cdot (u \wedge du) + \int_{B_\rho} d\mathrm{L} \cdot (u \wedge du) = \int_{B_\rho} (d * k) \wedge (u \wedge du), \tag{7.8}$$

since, by (7.6), we have $\mathrm{L} = 0$ on ∂B_ρ.

Let

$$\delta = \varepsilon^{10^{-3}}. \tag{7.9}$$

Assuming, for simplicity, ∂B to be flat near some point, consider a partition of B_ρ in δ-cubes Q



(we will average over translates of this grid in later estimates).

Define

$$\mathscr{F} = \left\{ Q | Q \cap \left[|u| < \frac{1}{2} \right] \neq \emptyset \right\}.$$

We are going to estimate the number of cubes in $\mathscr{F}$ with the help of the η-ellipticity property of T. Rivière [37], that we state in a more precise form, proved in [8]:

Lemma **25.** — *Let u_ε be a minimizer of* E_ε *in* B_R *with respect to its own boundary condition. Then there is a universal constant* C *such that, for every* $\eta > 0$, $0 < \varepsilon < 1$ *and* $R > 0$ *we have*

$$E_\varepsilon(u_\varepsilon; B_R) \leq \eta R \log(R/\varepsilon) \Rightarrow |u_\varepsilon(0)| \geq 1 - C\eta^{1/60}.$$

Let, for $Q \in \mathscr{F}$, $\tilde{Q}$ be the cube having the same center as Q and the size twice the one of Q. From the η-ellipticity property, we have

$$\int_{\tilde{Q}} e_\varepsilon(u) \geq C\delta \log(\delta/\varepsilon) \sim \delta \log(1/\varepsilon), \quad \forall Q \in \mathscr{F}, \tag{7.10}$$

so that

$$\#\mathscr{F} \leq C\delta^{-1} \quad \text{and} \quad \Big|\bigcup_{Q\in\mathscr{F}} Q\Big| \leq C\delta^2. \tag{7.11}$$

Define

$$\Omega = B_\rho \backslash \bigcup_{Q\in\mathscr{F}} Q, \tag{7.12}$$

on which $|u| > 1/2$.

We have, by (7.8),

$$\begin{aligned} \|u \wedge du\|_p^p &= \int_\Omega (d * k) \wedge (u \wedge du) + \int_{B_\rho \backslash \Omega} (d * k) \wedge (u \wedge du) \\ &\leq \int_\Omega (d * k) \wedge (u \wedge du) + 2\|k\|_{W^{1,q}} \|\nabla u\|_2 (B_\rho \backslash \Omega)^{1/2 - 1/q}. \end{aligned} \tag{7.13}$$

By (7.7) and (7.11), the second term of (7.13) is bounded by

$$C(\log(1/\varepsilon))^{1/2} \cdot \delta^{1-2/q} \|u \wedge du\|_p^{1-s} \leq \|u \wedge du\|_p^{1-s}, \tag{7.14}$$

provided ε is sufficiently small.

For the first term of (7.13), we use the identity

$$u \wedge du = \frac{u}{|u|} \wedge \left(d\left(\frac{u}{|u|}\right)\right) + \left(1 - \frac{1}{|u|^2}\right)(u \wedge du) \quad \text{in } \Omega$$

and the fact that

$$d\left(\frac{u}{|u|} \wedge \left(d\left(\frac{u}{|u|}\right)\right)\right) = 0,$$

to get

$$\textbf{(7.15)}\qquad \int_\Omega (d*k)\wedge(u\wedge du) = \int_{\partial\Omega} (*k)\wedge\left(\frac{u}{|u|}\wedge d\left(\frac{u}{|u|}\right)\right) + O(\|k\|_{W^{1,q}}\|\nabla u\|_2\|1-|u|^2\|_{2q/(q-2)}).$$

Since $|u|\le 1$ and

$$\|1-|u|^2\|_2 \le 2\varepsilon(E_\varepsilon(u_\varepsilon))^{1/2} \le C\varepsilon(\log(1/\varepsilon))^{1/2},$$

the second term of (7.15) bounded by

$$\textbf{(7.16)}\qquad C\|u\wedge du\|_p^{1-s}(\log(1/\varepsilon))^{1-1/q}\varepsilon^{1-2/q} \le \|u\wedge du\|_p^{1-s},$$

provided ε is sufficiently small.

Let $\varphi : D = [|z|\le 1] \to D$ be a smooth map such that $\varphi(\bar z) = \overline{\varphi(z)}$ and $\varphi(z) = z/|z|$ if $|z| > 1/10$. Thus

$$\int_{\partial\Omega} *k\wedge\left(\frac{u}{|u|}\wedge d\left(\frac{u}{|u|}\right)\right) = \int_{\partial B_\rho} *k\wedge(\varphi(u)\wedge d\varphi(u)) - \sum_{Q\in\mathscr{F}}\int_{\partial Q} *k\wedge(\varphi(u)\wedge d\varphi(u)) = (7.17) - (7.18).$$

Using (7.3) and the fact that, by (7.5), we have $q > 2$, we find that

$$\textbf{(7.19)}\qquad \begin{aligned}(7.17) &\le C\|u\|_{W^{1,r}(\partial B_\rho)}\|k\|_{L^{r'}(\partial B_\rho)} \le C\|k\|_{L^{r'}(\partial B_\rho)} \le C\|k\|_{H^{1-2/r'}(\partial B_\rho)}\\ &\le C\|k\|_{H^{3/2-2/r'}(B_\rho)} \le C\|k\|_{W^{1,q}(B_\rho)} \le C\|u\wedge du\|_p^{1-s}.\end{aligned}$$

In order to estimate the term (7.18) we replace, on each cube Q, k by its mean $\bar k_Q$. The error is of the order of

$$\sum_{Q\in\mathscr{F}}\int_{\partial Q}|k-\bar k_Q||\nabla u| \le \int_{\partial B_\rho}|k|\cdot|\nabla u| + \sum_{\substack{Q\in\mathscr{F}\\ Q\cap\partial B_\rho\neq\emptyset}}|\bar k_Q|\int_{\partial Q\cap\partial B_\rho}|\nabla u| + \sum_{Q\in\mathscr{F}}\int_{\partial Q\setminus\partial B_\rho}|k-\bar k_Q||\nabla u| = (7.20)+(7.21)+(7.22).$$

As for (7.17), we find that

(**7.23**) $$(7.20) \leq C\|u \wedge du\|_p^{1-s}.$$

Since

$$|\bar{k}_Q| \leq \delta^{-3} \int_Q |k| \leq \delta^{-3/r'} \left(\int_Q |k|^{r'} \right)^{1/r'}$$

and

$$\int_{\partial Q \cap \partial B_\rho} |\nabla u| \leq \delta^{2/r'} \left(\int_{\partial Q \cap \partial B_\rho} |\nabla u|^r \right)^{1/r},$$

we have

$$\begin{aligned}
(7.21) &\leq C\delta^{-1/r'} \sum_{\substack{Q \in \mathscr{F} \\ Q \cap \partial B_\rho \neq \emptyset}} \left(\int_Q |k|^{r'} \right)^{1/r'} \left(\int_{\partial Q \cap \partial B_\rho} |\nabla u|^r \right)^{1/r} \\
&\leq C\delta^{-1/r'} \|u\|_{W^{1,r}(\partial B_\rho)} \cdot \left(\int_{\substack{\cup Q \\ Q \in \mathscr{F} \\ Q \cap \partial B_\rho \neq \emptyset}} |k|^{r'} \right)^{1/r'} \\
&\leq C\delta^{-1/r'} \Bigg| \bigcup_{Q \in \mathscr{F}, Q \cap \partial B_\rho \neq \emptyset} Q \Bigg|^{1/r' - 1/6} \cdot \|k\|_6.
\end{aligned}$$

In view of (7.11) one may clearly choose $1 - \gamma < \rho < 1 - \gamma/2$ such that

(**7.24**) $$\#\{Q \in \mathscr{F} | Q \cap \partial B_\rho \neq \emptyset\} \lesssim 1/\gamma,$$

and therefore

$$\Bigg| \bigcup_{Q \in \mathscr{F}, Q \cap \partial B_\rho \neq \emptyset} Q \Bigg| \leq C\delta^3.$$

This gives

(**7.25**) $$(7.21) \leq C\delta^{-1/r'} \delta^{3/r' - 1/2} \|k\|_{W^{1,q}} \leq C\delta^{2/r' - 1/2} \|k\|_{W^{1,q}} < \|u \wedge du\|_p^{1-s},$$

provided ε is sufficiently small.

To bound (7.22), we use averaging over the grids. For $\lambda \in \mathbf{R}^3$ with $|\lambda| < \delta$, consider the grid of δ–cubes having λ as one of the vertices and let $\mathscr{F}_\lambda$ be the corresponding collection of bad cubes. Then

$$\begin{aligned}
\delta^{-3} \int\limits_{|\lambda|<\delta} (7.22) &\leq \delta^{-3} \int\limits_{|\lambda|<\delta} \delta^{-3} \sum_{Q\in\mathscr{F}_\lambda} \int_{\partial Q\setminus\partial B_\rho} dx \int\limits_{Q} dy |k(x)-k(y)||\nabla u(x)| \\
&\leq C\delta^{-4} \sum_{Q\in\mathscr{F}_0} \iint\limits_{\tilde{Q}\times\tilde{Q}} dxdy |k(x)-k(y)||\nabla u(x)| \\
&\leq C\delta^{1/2-6/q} \sum_{Q\in\mathscr{F}_0} \|\nabla u\|_{L^2(\tilde{Q})} \|k(x)-k(y)\|_{L^q(\tilde{Q}\times\tilde{Q})} \\
&\leq C\delta^{-5/q} \|\nabla u\|_{L^2(B_\rho)} \Big[\sum_{Q\in\mathscr{F}_0} \int\limits_{\tilde{Q}\times\tilde{Q}}\int |k(x)-k(y)|^q dxdy \Big]^{1/q} \\
&\leq C\delta^{1-2/q} (\log(1/\varepsilon))^{1/2} \Big[\sum_{Q\in\mathscr{F}_0} \int\limits_{\tilde{Q}} |\nabla k|^q \Big]^{1/q} \\
&\leq \|u\wedge du\|_p^{1-s},
\end{aligned}$$

provided ε is sufficiently small. Therefore, by choosing the proper grid, we may assume that

$$(7.22) \leq C\|u\wedge du\|_p^{1-s}. \tag{7.26}$$

Combining (7.23), (7.25) and (7.26), it follows that

$$(7.20) + (7.21) + (7.22) \leq C\|u\wedge du\|_p^{1-s}. \tag{7.27}$$

By (7.13), (7.14), (7.16) and (7.27), we have

$$\|u\wedge du\|_p^p = (7.29) + O\big(\big\|u\wedge du\big\|_p^{1-s}\big), \tag{7.28}$$

where

$$(7.29) = -\sum_{Q\in\mathscr{F}} \int\limits_{\partial Q} *\tilde{k}_Q \wedge (\varphi(u)\wedge d\varphi(u)).$$

For $i = 1, 2, 3$, let π_i be the projection onto the axis $0x_i$. For $x_i \in \pi_i(\partial Q)$, let

$$\Gamma_{x_i} = (\pi_i)^{-1}(x_i) \cap \partial Q.$$

Then

$$\textbf{(7.30)} \qquad |(7.29)| \leq \sum_{i=1}^{3} \sum_{Q \in \mathscr{F}} |\hbar_Q| \int_{\pi_i(Q)} \left| \int_{\Gamma_{x_i}} \varphi(u) \wedge \partial\varphi(u)/\partial\tau \right| dx_i.$$

Denote $\widetilde{\Gamma}$ the δ-square with $\partial\,\widetilde{\Gamma} = \Gamma$ and let

$$\textbf{(7.31)} \qquad \delta_1 = \delta^3, \delta_2 = \delta^4.$$

Consider "good" sections Γ, i.e., such that

$$\textbf{(7.32)} \qquad \text{dist}\,(\Gamma, [|u| < 1/2]) > \delta_1$$

and, with

$$e_\varepsilon(u) = e_\varepsilon(u)(x) = |\nabla u(x)|^2 + \frac{1}{\varepsilon^2}(1 - |u|^2)^2(x),$$

$$\textbf{(7.33)} \qquad \int_{\widetilde{\Gamma}} e_\varepsilon(u) < \delta_2 \varepsilon^{-1}.$$

Condition (7.33) implies that

$$\textbf{(7.34)} \qquad \frac{1}{\varepsilon^2} \int_{\widetilde{\Gamma}} (1 - |u|^2)^2 < \delta_2 \varepsilon^{-1}.$$

Since $|\nabla u| \leq C/\varepsilon$, it follows that the set $\widetilde{\Gamma} \cap [|u| < 1/2]$ may be covered by a family $\mathscr{G}$ of ε–squares such that

$$\#\mathscr{G} \leq C_0 \delta_2/\varepsilon$$

and

$$\textbf{(7.35)} \qquad \sum_{S \in \mathscr{G}} \text{length}(S) \leq C_0 \varepsilon \delta_2/\varepsilon = C_0 \delta_2.$$

We next invoke the following estimate (*see* the proposition in Section 1 in [39]):

Lemma **26** *(Sandier [39]). — Under the assumptions (7.32) and (7.35) we have, with* C_0 *the constant in (7.35),*

$$\int_{\widetilde{\Gamma} \cap [|u| \geq 1/2]} \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 dx \geq K|d| \log(\delta_1/(2C_0\delta_2)),$$

where d *is the degree of* $u_{|\Gamma}$ *and* K *is some universal constant.*

By Lemma 26 and our choice of δ_1, δ_2, we find that

$$\left|\int_\Gamma \varphi(u)\wedge d\varphi(u)\right| = \left|\deg\left(\frac{u}{|u|},\Gamma\right)\right| \le C\int_{\widetilde{\Gamma}} |\nabla u|^2/\log(1/\varepsilon). \tag{7.36}$$

On the other hand, recall the monotonicity formula of T. Rivière (*see* Lemma 2.5 in [37]):

Lemma **27** *(Rivière [37]). — Let* $x\in G$*. Then, for* $0<r<dist(x,\Omega)$*, the map*

$$r\mapsto \frac{1}{r}\int_{B_r(x)}\left(|\nabla u_\varepsilon(x)|^2+\frac{3}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)$$

is non-increasing.

By combining (7.36) and Lemma 27, we see that the collected contribution of the good sections in the r.h.s. of (7.30) is bounded by

$$C\sum_{Q\in\mathscr{F}}|\hbar_Q|\int_Q |\nabla u|^2/\log(1/\varepsilon)\le C\delta\sum_{Q\in\mathscr{F}}|\hbar_Q|\lesssim \delta^{-2}\int_{B_\rho}|k|\Big(\sum_{Q\in\mathscr{F}}\chi_Q\Big). \tag{7.37}$$

We consider an extension, denoted by h, of $|k|$ to $\mathbf{R}^3$, such that

$$\|h\|_{W^{1,q}(\mathbf{R}^3)}\le C\||k|\|_{W^{1,q}(B_\rho)}.$$

We estimate the integral in (7.37) using the $(B^1_{q,q}, B^{-1}_{p,p})$–duality (for the definition of the Besov spaces $B^\sigma_{p,q}$, *see* e.g. H. Triebel [45]), where

$$\|f\|_{B^\sigma_{r,r}}=\Big[2^{\sigma r}\|f*P_1\|_r^r+\sum_{j\ge 2}(2^{\sigma j}\|f*P_{2^{-j}}-f*P_{2^{-j+1}}\|_r)^r\Big]^{1/r}. \tag{7.38}$$

We let here $P_1\ge 0$ be a suitable L^1–normalized smooth bump function supported in the unit cube of $\mathbf{R}^3$, and denote $P_h(x)=h^{-3}P_1(h^{-1}x)$.

On the one hand, since $q>2$ we have

$$\|h\|_{B^1_{q,q}}\le C\|h\|_{W^{1,q}}\le C\|k\|_{W^{1,q}}\le C\|u\wedge du\|_p^{1-s}. \tag{7.39}$$

Letting $f=\sum_{Q\in\mathscr{F}}\chi_Q$, we estimate next $\|f\|_{B^{-1}_{p,p}}$. Without any loss of generality, we may assume that $B_6\subset G$.

Assume first that j is such that $1\ge 2^{-j}\ge\delta$. If $Q_1\subset B_3$ is a 2^{-j}–cube, then

$$\int_{Q_1} e_\varepsilon(u)\le C2^{-j}\log(1/\varepsilon), \tag{7.40}$$

by Lemma 27. On the other hand, if $Q \in \mathscr{F}$, then (7.10) holds. Therefore

$$\#\{Q \in \mathscr{F};\, Q \subset Q_1\} \le C 2^{-j}\delta^{-1}. \tag{7.41}$$

Also, if $Q_1 \cap \mathscr{F} \neq \emptyset$, the η-ellipticity lemma implies

$$\int_{\tilde{Q}_1} e_\varepsilon(u) \ge C 2^{-j} \log(1/\varepsilon), \tag{7.42}$$

and hence the set $[|u| \le 1/2]$ intersects at most $C2^j$ cubes Q_1 of size 2^{-j}. Thus

$$\begin{aligned} \|(f * P_{2^{-j}}) - (f * P_{2^{-j+1}})\|_p &\lesssim \|f * P_{2^{-j}}\|_p \\ &\lesssim \Big\| \sum_{Q_1, Q_1 \cap \mathscr{F} \neq \emptyset} \frac{1}{|Q_1|} \chi_{\tilde{Q}_1} \int_{\tilde{Q}_1} f \Big\|_p \\ &\lesssim \Big[\sum_{Q_1, Q_1 \cap \mathscr{F} \neq \emptyset} 2^{-3j} (2^{3j} |\tilde{Q}_1 \cap \mathscr{F}|)^p \Big]^{1/p} \\ &\lesssim \Big[\sum_{Q_1 \cap \mathscr{F} \neq \emptyset} 2^{-3j} (2^{3j} \cdot \delta^3 \cdot 2^{-j} \delta^{-1})^p \Big]^{1/p} \text{ by (7.41)} \\ &\lesssim 2^{-2j/p} 2^{2j} \delta^2 = \delta^2 4^{j/q}. \end{aligned} \tag{7.43}$$

Assume now that $2^{-j} < \delta$. Estimate then

$$|f * (P_{2^{-j}} - P_{2^{-j+1}})| \le \sum_{Q \in \mathscr{F}} |\chi_Q * (P_{2^{-j}} - P_{2^{-j+1}})|.$$

In this case, it is easy to see that

$$|\chi_Q * (P_{2^{-j}} - P_{2^{-j+1}})| \le C\chi_A,$$

where

$$A = \{ x;\ \operatorname{dist}(x, \partial Q) \le 2^{-j} \}.$$

In particular, each point in $\mathbf{R}^3$ belongs to at most 8 A′s. Thus

$$\Big\| \sum_{Q \in \mathscr{F}} \chi_Q * (P_{2^{-j}} - P_{2^{-j+1}}) \Big\|_p^p \le C \sum_{Q \in \mathscr{F}} \|\chi_Q * (P_{2^{-j}} - P_{2^{-j+1}})\|_p^p \le C\delta 2^{-j}. \tag{7.44}$$

From (7.43), (7.44)

$$\begin{aligned} \|f\|_{B^{-1}_{p,p}} &\le C \Big[\sum_{2^{-j} \ge \delta} (2^{-j} \delta^2 4^{j/q})^p + \sum_{2^{-j} < \delta} (2^{-j} \delta^{1/p} 2^{-j/p})^p \Big]^{1/q'} \\ &\lesssim (\delta^{2p} + \delta^{2+p})^{1/p} < \delta^2. \end{aligned} \tag{7.45}$$

Here, we have used the fact that $p < 2 < q$.

From (7.37), (7.39) and (7.45), we find that

(**7.46**) $$(7.37) \leq C\|u \wedge du\|_p^{1-s}.$$

Next, we analyze the contribution of the "bad" sections Γ_{x_i} in (7.30). A bad section $\Gamma_{x_i} = \Gamma$ fails either (7.32) or (7.33).

Fix $i = 1, 2, 3$ and $Q \in \mathscr{F}$. Define

(**7.47**) $$J'_Q = \{x_i \in \pi_i(Q);\ \Gamma_{x_i} \text{ fails } (7.32)\},$$

(**7.48**) $$J''_Q = \{x_i \in \pi_i(Q);\ \Gamma_{x_i} \text{ fails } (7.33)\},$$

and the surfaces

(**7.49**) $$\mathfrak{S}' = \mathfrak{S}'_i = \bigcup_Q \bigcup_{x_i \in J'_Q} \Gamma_{x_i}$$

(**7.50**) $$\mathfrak{S}'' = \mathfrak{S}''_i = \bigcup_Q \bigcup_{x_i \in J''_Q} \Gamma_{x_i}.$$

Estimate the contribution of the bad sections in (7.30) by

(**7.51**) $$\Big(\max_{Q \in \mathscr{F}} |\bar{k}_Q|\Big) \sum_{i=1}^{3} \int_{\mathfrak{S}'_i \cup \mathfrak{S}''_i} |\nabla u|.$$

Estimate

(**7.52**) $$\begin{aligned} |\bar{k}_Q| \leq \delta^{-3} \int_Q |k| \leq \delta^{-3}|Q|^{5/6}\|k\|_{L^6(B_\rho)} &\lesssim \delta^{-1/2}\|k\|_{W^{1,q}(B_\rho)} \\ &\lesssim \delta^{-1/2}\|u \wedge du\|_p^{1-s}. \end{aligned}$$

Consider, for $\lambda \in \mathbf{R}^3$, the grid of δ–cubes having λ as one of the edges and let $\mathscr{G}_\lambda$ be the grid defined by the boundaries of these cubes. For each λ, we have

(**7.53**) $$\begin{aligned} \int_{\mathfrak{S}'_i \cup \mathfrak{S}''_i} |\nabla u| &\leq \Big(\int_{\mathscr{G}_\lambda} |\nabla u|^2\Big)^{1/2} (|\mathfrak{S}'_i| + |\mathfrak{S}''_i|)^{1/2} \\ &\leq C\Big(\int_{\mathscr{G}_\lambda} |\nabla u|^2\Big)^{1/2} \Big(\delta \sum_{Q \in \mathscr{F}_\lambda} (|J'_Q| + |J''_Q|)\Big)^{1/2}. \end{aligned}$$

Since (7.33) fails for $x_i \in J''_Q$, we have

$$\int_Q e_\varepsilon(u) \geq \int_{\substack{\cup \tilde{\Gamma}_{x_i} \\ x_i \in J''_Q}} e_\varepsilon(u) \geq |J''_Q| \delta_2 \varepsilon^{-1}.$$

Thus

$$\sum_{Q \in \mathscr{F}_\lambda} |J''_Q| \lesssim \varepsilon \delta_2^{-1} \log(1/\varepsilon). \tag{7.54}$$

To estimate (7.53), we use again an average over the grids $\mathscr{G}_\lambda$. Denote this averaging by Av_τ (τ refers to the translation).

Thus, taking (7.54) into account, we obtain

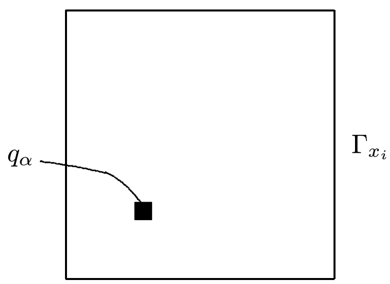
$$(7.53) \lesssim \Big[Av_\tau \int_{\mathscr{G}_\lambda} |\nabla u|^2 \Big]^{1/2} \Big[\delta \delta_2^{-1} \varepsilon \log(1/\varepsilon) + \delta Av_\tau \Big(\sum_{Q \in \mathscr{F}_\lambda} |J'_Q| \Big) \Big]^{1/2}. \tag{7.55}$$

Notice that the J'_Q-intervals of points x_i such that dist $\big(\Gamma_{x_i}, \big[|u| < \frac{1}{2}\big]\big) < \delta_1$ do depend on the grid translation – a fact that will be exploited next.

First, recalling (7.4), we have

$$Av_\tau \int_{\mathscr{G}_\tau} |\nabla u|^2 \leq \int_{\partial B_\rho} |\nabla u|^2 + \frac{1}{\delta} \int_{B_\rho} |\nabla u|^2 \lesssim \frac{\log 1/\varepsilon}{\delta}. \tag{7.56}$$

By the η–ellipticity lemma, we may cover $[|u| < 1/2] \cap B$ with at most $C\delta_1^{-1}$ δ_1–cubes q_α, $\alpha \leq C\delta_1^{-1}$. We fix such a covering (independent of λ). Fix i, Q. If dist $(\Gamma_{x_i}, [|u| < 1/2]) < \delta_1$, then clearly $x_i \in \pi_i(\tilde{q}_\alpha)$ for some $q_\alpha \subset \tilde{Q}$ with dist $(q_\alpha, \mathscr{G}_\lambda) < \delta_1$.



Hence

$$|J'_Q| \leq 2\delta_1 \cdot \#\{\alpha;\, q_\alpha \subset \tilde{Q},\ \text{dist}\,(q_\alpha, \mathscr{G}_\lambda) < \delta_1\} \tag{7.57}$$

and

$$\sum_{Q} |J'_Q| \leq C\delta_1 \cdot \#\{\alpha;\, \mathrm{dist}\,(q_\alpha, \mathscr{G}_\lambda) < \delta_1\}. \tag{7.58}$$

We now average over the grid translation. On the one hand, for fixed α, the inequality

$$\mathrm{dist}\,(q_\alpha, \mathscr{G}_\lambda \setminus \partial B_\rho) < \delta_1$$

holds with τ–probability $\sim \delta_1/\delta$. On the other hand, for fixed α and $1-\gamma < \rho < 1-\gamma/2$, the inequality

$$\mathrm{dist}\,(q_\alpha, \partial B_\rho) < \delta_1$$

holds with ρ–probability $\sim \delta_1/\gamma$.

Hence, by choosing ρ properly, we may assume that

$$\#\{\alpha;\, \mathrm{dist}\,(q_\alpha, \partial B_\rho) < \delta_1\} \leq C.$$

For any such ρ, we have

$$Av_\tau(7.58) \lesssim \delta_1 \cdot \frac{1}{\delta_1} \cdot \frac{\delta_1}{\delta} + C \lesssim \frac{\delta_1}{\delta}. \tag{7.59}$$

Hence

$$Av_\tau\Big(\sum |J'_Q|\Big) \leq C\frac{\delta_1}{\delta}. \tag{7.60}$$

Substitution of (7.56), (7.60) into (7.55) yields, for small ε,

$$(7.55) \lesssim \Big(\frac{\log(1/\varepsilon)}{\delta}\Big)^{1/2}\Big(\delta\delta_2^{-1}\varepsilon\log(1/\varepsilon) + \delta_1\Big)^{1/2} < \delta^{3/4}, \tag{7.61}$$

by (7.9) and (7.31).

From (7.52) and (7.61),

$$(7.51) \leq \delta^{3/4}\delta^{-1/2}\|u \wedge du\|_p^{1-s} \leq C\|u \wedge du\|_p^{1-s}. \tag{7.62}$$

This completes the analysis. Indeed, by collecting the estimates (7.28), (7.30), (7.37), (7.46), (7.51) and (7.62), it follows that

$$\|u \wedge du\|^p_{L^p(B_\rho)} \leq C_\gamma \|u \wedge du\|^{1-s}_{L^p(B_\rho)}, \tag{7.63}$$

and thus

$$\|u \wedge du\|_{L^p(B_{1-\gamma})} \leq C_\gamma.$$

Since $0 < \gamma < 1$ and $3/2 \leq p < 2$ are arbitrary, the proof of Theorem 8 is complete.

8. Convergence for $g \in Y$. Proof of Theorem 9

Proof of Theorem 9. — We already know that a subsequence of (u_ε) converges in $W^{1,p}(G)$, $1 \le p < 3/2$. The main novelties in Theorem 9 are:

a) the identification of the limit

$$u_* = e^{\imath\tilde{\varphi}},$$

where $g = e^{\imath\varphi}$, $\varphi \in H^{1/2} + W^{1,1}$ and $\tilde{\varphi}$ is the harmonic extension of φ;

b) $u_\varepsilon \to u_*$ in $C^\infty(G)$.

We first discuss b), which is easier. In view of a), it suffices to prove that (u_ε) is bounded in $C^k(K)$ for every integer k and every compact subset K of G. Since $E_\varepsilon(u_\varepsilon) = o(\log 1/\varepsilon)$, by Theorem 6, we find, with the help of the η–ellipticity Lemma 24 that, for every compact K in G, we have

$$|u_\varepsilon| \ge \frac{1}{2}$$

in K for small ε.

We next recall Theorem IV.1 in [9].

Lemma **28.** — *Let u_ε be a solution of*

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \text{ in } B_1$$

such that

$$E_\varepsilon(u_\varepsilon; B_1) \le C. \tag{8.1}$$

Then (u_ε) is bounded in $C^k(B_{1/2})$, for every $k \in \mathbf{N}$.

We now complete the proof of b) by establishing (8.1) on every ball B compactly contained in G.

We write $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ in B. Let ζ be a cutoff function with $\zeta \equiv 1$ in B. We start by multiplying the equation for φ_ε

$$\operatorname{div}(\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = 0$$

by $\zeta^2(\varphi_\varepsilon - \fint_B \varphi_\varepsilon)$.

We find that

$$\int \rho_\varepsilon^2 |\nabla\varphi_\varepsilon|^2 \zeta^2 \le 2 \int \rho_\varepsilon^2 |\nabla\varphi_\varepsilon|\ |\zeta|\ |\nabla\zeta|\ |\varphi_\varepsilon - \fint_B \varphi_\varepsilon|$$
$$\le C\Big(\int \rho_\varepsilon^2 |\nabla\varphi_\varepsilon|^2\zeta^2\Big)^{1/2} \Big(\int |\nabla\varphi_\varepsilon|^{6/5}\Big)^{5/6},$$

by the Sobolev imbedding $W^{1,6/5} \subset L^2$, We obtain that φ_ε is bounded in H^1_{loc}, since $|\nabla\varphi_\varepsilon| \le 2|\nabla u_\varepsilon|$ in B and u_ε is bounded in $W^{1,6/5}$ by Theorem 7.

Next consider the equation for ρ_ε,

$$-\Delta\rho_\varepsilon + \rho_\varepsilon|\nabla\varphi_\varepsilon|^2 = \frac{1}{\varepsilon^2}\rho_\varepsilon\big(1-\rho_\varepsilon^2\big).$$

Multiplying by $(1-\rho_\varepsilon)\zeta$, we find that

$$\int|\nabla\rho_\varepsilon|^2\zeta + \frac{1}{\varepsilon^2}\int\big(1-\rho_\varepsilon^2\big)^2\zeta \le C\Big(\int|\nabla\rho_\varepsilon| + \int|\nabla\varphi_\varepsilon|^2\Big).$$

We conclude by noting that

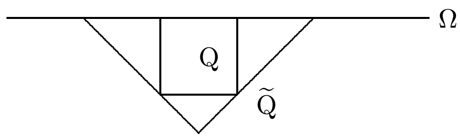
$$E_\varepsilon(u_\varepsilon; B) \le \int_B|\nabla\rho_\varepsilon|^2 + \int_B|\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_B\big(1-\rho_\varepsilon^2\big)^2 \le C_B.$$

We now turn to the proof of a).

We start by constructing an appropriate domain $G_\varepsilon \subset G$ on which $|u_\varepsilon| \sim 1$. For simplicity, we assume Ω flat near some point. Fix some $0 < \delta_0 < 1$ to be determined later. Let $0 < \delta < \delta_0$ and $u = u_\varepsilon$. Set

(**8.2**) $$A_\delta = \{x \in G;\ \mathrm{dist}\,(x,\Omega) \ge \sqrt{\varepsilon},\ |u(x)| \le 1-\delta\}.$$

For $x \in A_\delta$, let Q be the cube centered at x such that one of its faces is contained in Ω and let $\widetilde{Q}$ be the conical domain



Let also $Q^\#$ be the cube centered at x having the size a third the one of Q. By Vitali's lemma, we may choose a finite family $(Q_\alpha^\#)$ of disjoint cubes such that $A_\delta \subset \cup Q_\alpha$. By the η-ellipticity property, there is some $\eta(\delta) > 0$ such that we have, with δ_α the size of Q_α,

(**8.3**) $$E_\varepsilon(u, Q_\alpha^\#) \ge \eta(\delta)\delta_\alpha\log(\delta_\alpha/\varepsilon) \ge 1/2\eta(\delta)\delta_\alpha\log(1/\varepsilon),$$

since $\delta_\alpha \ge \sqrt{\varepsilon}$. Thus

(**8.4**) $$\sum\delta_\alpha < \frac{2}{\eta(\delta)}\frac{E_\varepsilon(u,G)}{\log(1/\varepsilon)}.$$

Since, by Theorem 6, we have $E_\varepsilon(u, G) = o(\log(1/\varepsilon))$, we find that

$$\sum \delta_\alpha < \delta, \tag{8.5}$$

provided ε is sufficiently small.

We now set

$$G_\varepsilon = \{x \in G; \operatorname{dist}(x, \Omega) \geq \sqrt{\varepsilon}\} \setminus \cup \widetilde{Q}_\alpha,$$

so that $|u_\varepsilon| \geq 1 - \delta$ in G_ε.

By (8.5) and the construction of G_ε, there is a Lipschitz homeomorphism $\Phi_\varepsilon : G_\varepsilon \to G$ such that

$$\begin{aligned} &\|D\Phi_\varepsilon\|_{L^\infty} \leq C, \left\|D\left(\Phi_\varepsilon^{-1}\right)\right\|_{L^\infty} \leq C, \\ &\Phi_{\varepsilon|\partial G_\varepsilon} = \Pi_{|\partial G_\varepsilon}, \Phi_{\varepsilon|\{x\in G; \operatorname{dist}(x,\Omega)\geq 2\delta\}} = \mathrm{id}, \end{aligned} \tag{8.6}$$

provided δ_0 is sufficiently small, with constants C independent of ε.

Here, Π is the projection on Ω. In particular, G_ε is simply connected. We may thus write in G_ε

$$u = \rho e^{\imath\psi}, \rho = |u|, \psi \in C^\infty. \tag{8.7}$$

Assuming further that $\delta_0 < 1/2$, we have $\rho \geq 1/2$ in G_ε and thus

$$|\psi|^2_{H^1(G_\varepsilon)} \leq 4|u|^2_{H^1(G_\varepsilon)} \leq 4|u|^2_{H^1(G)} \leq \delta \log(1/\varepsilon), \tag{8.8}$$

provided ε is sufficiently small. Moreover, by Theorem 7, we have

$$|\psi|_{W^{1,p}(G_\varepsilon)} \leq 2|u|_{W^{1,p}(G_\varepsilon)} \leq 2|u|_{W^{1,p}(G)} \leq C_p, 1 \leq p < 3/2. \tag{8.9}$$

We are now going to prove that $\psi|_{\partial G_\varepsilon}$ is almost equal to $\varphi \circ \Pi_{|\partial G_\varepsilon}$, where $\varphi \in H^{1/2} + W^{1,1}(\Omega; \mathbf{R})$ is such that $g = e^{\imath\varphi}$.

Let $\eta > 0$ be to be determined later. Since $g \in Y$, we may find some $h \in C^\infty(\Omega; S^1)$ such that $\|g - h\|_{H^{1/2}} < \eta$. Let $\zeta \in C^\infty(\Omega; \mathbf{R})$ be such that $h = e^{i\zeta}$. Let $T_\varepsilon = \Phi_\varepsilon\big|_{\partial G_\varepsilon}$ and $U_\varepsilon = T_\varepsilon^{-1} : \Omega \to \partial G_\varepsilon$. Fix a smooth map $\pi : \mathbf{C} \to \mathbf{C}$ such that $\pi(z) = z/|z|$ if $|z| \geq 1/2$ and let

$$\xi(x) = g(x) - e^{\imath\psi(U_\varepsilon(x))}, \; x \in \Omega,$$

so that

$$\xi(x) = \pi(g(x)) - \pi(e^{i\psi(U_\varepsilon(x))}), x \in \Omega \setminus \cup\widetilde{Q}_\alpha. \tag{8.10}$$

Therefore, we have

$$(\mathbf{8.11}) \qquad \int_{\Omega\setminus\cup\widetilde{Q}_\alpha} |\xi(x)|dx \leq C(G) \int_{\{x;\,\mathrm{dist}\,(x,\partial\Omega)\leq\sqrt{\varepsilon}\}} |Du| \leq C\|Du\|_{L^2}\varepsilon^{1/4}$$
$$\leq C\varepsilon^{1/4}(\log 1/\varepsilon)^{1/2} \leq 1/2\varepsilon^{1/5},$$

provided ε is sufficiently small. It follows that

$$(\mathbf{8.12}) \qquad \int_{\Omega\setminus\cup\widetilde{Q}_\alpha} |h(x) - e^{i\psi(U_\varepsilon(x))}|dx < \varepsilon^{1/5},$$

provided η is sufficiently small. Thus, with $\lambda = \zeta - \psi \circ U_\varepsilon$, we have

$$(\mathbf{8.13}) \qquad \|e^{i\lambda} - 1\|_{L^1(\Omega\setminus\cup\widetilde{Q}_\alpha)} < \varepsilon^{1/5}.$$

By combining (8.6) and (8.8) (resp. (8.6) and (8.9)), we find that

$$(\mathbf{8.14}) \qquad |\lambda|_{H^{1/2}(\Omega)} \leq \|\zeta\|_{H^{1/2}(\Omega)} + C\|\psi\|_{H^1(G_\varepsilon)} < \delta^{1/2}(\log(1/\varepsilon))^{1/2}$$

and

$$(\mathbf{8.15}) \qquad \|\lambda\|_{W^{1/4,4/3}(\Omega)} \leq \|\zeta\|_{W^{1/4,4/3}(\Omega)} + C\|\psi\|_{W^{1,4/3}(G_\varepsilon)} \leq C,$$

provided ε is sufficiently small. In particular, we have

$$(\mathbf{8.16}) \qquad \|\lambda\|_{L^{4/3}(\Omega)} \leq C.$$

By Lemma C.1 in Appendix C, if δ_0 is sufficiently small and λ satisfies (8.13), (8.14) and (8.15), while the squares $\widetilde{Q}_\alpha \cap \Omega$ satisfy (8.5), then there is some integer a such that

$$(\mathbf{8.17}) \qquad \|\lambda - 2\pi a\|_{L^1(\Omega)} < \delta^{1/18}.$$

Without restricting the generality, we may assume that $a = 0$, so that

$$(\mathbf{8.18}) \qquad \|\xi - \psi \circ U_\varepsilon\|_{L^1(\Omega)} < \delta^{1/18}.$$

We actually claim that

$$(\mathbf{8.19}) \qquad \|\varphi - \psi \circ U_\varepsilon\|_{L^1(\Omega)} < \delta^{1/20},$$

if we choose the lifting φ of g properly. Indeed, by estimate (1.9) in Theorem 3, the map $g\bar{h} \in Y$ has a lifting $\chi \in H^{1/2} + W^{1,1}$ such that

$$(\mathbf{8.20}) \qquad |\chi|_{H^{1/2}+W^{1,1}} \leq C(G)|g\bar{h}|_{H^{1/2}}(1 + |g\bar{h}|_{H^{1/2}}).$$

Since

$$|g\bar{h}|_{H^{1/2}} = |\bar{h}(g-h)|_{H^{1/2}} \to 0 \text{ as } h \to g,$$

we may choose η sufficiently small in order to have

$$\text{(8.21)} \qquad \|\chi - ⨍ \chi\|_{L^1(\Omega)} < \delta^{1/18}.$$

Using the fact that

$$\|g\bar{h} - e^{i ⨍ \chi}\|_{L^1} = \|e^{i\chi} - e^{i ⨍ \chi}\|_{L^1} \leq \|\chi - ⨍ \chi\|_{L^1} < \delta^{1/18}$$

and

$$\|g\bar{h} - 1\|_{L^1} < \delta^{1/18},$$

provided η is sufficiently small, we find that, modulo $2\pi\mathbf{Z}$, we may assume that

$$\text{(8.22)} \qquad \|⨍ \chi\|_{L^1(\Omega)} < 2\delta^{1/18}.$$

Since $g = e^{i(\chi+\xi)}$, inequality (8.19) follows by combining (8.20)–(8.22), provided δ_0 is sufficiently small.

We now prove that ψ and $\tilde{\varphi}$ are close on compact sets of G. Set $\tilde{\psi} = \psi \circ \Phi_\varepsilon^{-1}$, $\tilde{\rho} = \rho \circ \Phi_\varepsilon^{-1}$, so that $\tilde{\psi}, \tilde{\rho}$ are defined on G and, in the set

$$M = \{x \in G; \operatorname{dist}(x, \Omega) \geq 2\delta\},$$

we have $\tilde{\psi} = \psi$ and $\tilde{\rho} = \rho$.

Recall that ψ satisfies the equation $\operatorname{div}(\rho^2 \nabla \psi) = 0$ in G_ε. Transporting this equation on G and using (8.6), we see that ψ satisfies

$$\text{(8.23)} \qquad \begin{cases} \operatorname{div}(A(x)\tilde{\rho}^2 \nabla \tilde{\psi}) = 0 & \text{in } G \\ \tilde{\psi} = \psi \circ U_\varepsilon & \text{on } \Omega \end{cases},$$

with

$$\text{(8.24)} \qquad C^{-1}|\xi|^2 \leq < A(x)\xi, \xi > \leq C|\xi|^2, \ \tilde{\rho}(x) = \rho(x) \text{ and } A(x) = I \text{ if } x \in M.$$

Therefore, the function

$$f = \tilde{\varphi} - \tilde{\psi}$$

satisfies

$$\text{(8.25)}\qquad \begin{cases} \Delta f = \operatorname{div}\,((\mathrm{I}-\mathrm{A}(x)\tilde{\rho}^2)\nabla\tilde{\psi}) & \text{in G} \\ f = \varphi - \psi\circ \mathrm{U}_\varepsilon & \text{on } \partial\mathrm{G} \end{cases}.$$

Thus, for $1 \leq p < 3/2$ and K compact in G, we have

$$\text{(8.26)}\qquad \|f\|_{\mathrm{W}^{1,p}(\mathrm{K})} \leq \mathrm{C}_\mathrm{K}(\|(\mathrm{I}-\mathrm{A}(x)\tilde{\rho}^2)\nabla\psi\|_{\mathrm{L}^p(\mathrm{G})} + \|\varphi - \psi\circ\mathrm{U}_\varepsilon\|_{\mathrm{L}^1(\Omega)}).$$

As we already observed in the proof of part b) of the theorem, we have $\rho \to 1$ uniformly on the compacts of G. Thus

$$\text{(8.27)}\qquad \|(\mathrm{I}-\mathrm{A}(x)\tilde{\rho}^2)\nabla\tilde{\psi}\|_{\mathrm{L}^p(\mathrm{M})} \to 0.$$

as $\varepsilon \to 0$. On the other hand, we have

$$\text{(8.28)}\qquad \|(\mathrm{I}-\mathrm{A}(x)\tilde{\rho}^2)\nabla\tilde{\psi}\|_{\mathrm{L}^p(\mathrm{G}\setminus\mathrm{M})} \leq \mathrm{C}\|\nabla\tilde{\psi}\|_{\mathrm{L}^p(\mathrm{G}\setminus\mathrm{M})} \leq \mathrm{C}\|\nabla u\|_{\mathrm{L}^p(\mathrm{G}\setminus\mathrm{M})}.$$

If we choose some r with $p < r < 3/2$, we find that

$$\text{(8.29)}\qquad \|(\mathrm{I}-\mathrm{A}(x)\tilde{\rho}^2)\nabla\tilde{\psi}\|_{\mathrm{L}^p(\mathrm{G}\setminus\mathrm{M})} \leq \mathrm{C}\|\nabla u\|_{\mathrm{L}^r(\mathrm{G}\setminus\mathrm{M})}|\mathrm{G}\setminus\mathrm{M}|^{\frac{r-p}{r}} \leq \mathrm{C}\delta^{\frac{r-p}{r}},$$

by Theorem 7. By combining (8.19), (8.26), (8.27) and (8.29) we find that, for some $0 < \alpha < 1$ fixed, we have

$$\text{(8.30)}\qquad \|f\|_{\mathrm{W}^{1,p}(\mathrm{K})} \leq \delta^\alpha,$$

provided ε is sufficiently small.

Since, for $\delta_0 = \delta_0(\mathrm{K})$ sufficiently small, we have $f = \varphi - \psi$ in K, we find that, as $\varepsilon \to 0$, $\tilde{\varphi} - \psi \to 0$ in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathrm{G})$, $1 \leq p < 3/2$. Using once more the fact that $\rho \to 1$ in $\mathrm{C}^k_{\mathrm{loc}}(\mathrm{G})$, we find that $u_\varepsilon \to u_*$ in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathrm{G})$. This proves Theorem 9.

Remark **8.1.** — Under the assumptions of Theorem 9 it is not true in general that $|u_\varepsilon| \to 1$ uniformly on $\bar{\mathrm{G}}$. Indeed, if this were true, then $u_\varepsilon/|u_\varepsilon|$ would belong to $\mathrm{H}^1(\mathrm{G};\mathrm{S}^1)$ for ε sufficiently small. Thus $u_\varepsilon/|u_\varepsilon|$ admits a lifting $\varphi_\varepsilon \in \mathrm{H}^1(\mathrm{G};\mathbf{R})$ and $g = e^{i\varphi_{\varepsilon|\Omega}}$. Hence g must necessarily belong to X. But, even when $g \in \mathrm{X}$ it is unlikely that $|u_\varepsilon| \to 1$ uniformly on $\bar{\mathrm{G}}$.

Remark **8.2.** — Let $g \in \mathrm{H}^{1/2}(\Omega;\mathrm{S}^1)$ with $\mathrm{L}(g) = 0$ and write $g = e^{i\varphi}$ with $\varphi \in \mathrm{H}^{1/2} + \mathrm{W}^{1,1}$. Let $\tilde{\varphi}$ be the harmonic extension of φ. One may wonder whether

$$\text{(8.31)}\qquad \|u_\varepsilon e^{-i\tilde{\varphi}}\|_{\mathrm{W}^{1,p}} \leq \mathrm{C} \quad \forall p < 2 \text{ as } \varepsilon \to 0?$$

The answer is negative. The argument relies on the following

Lemma **29.** — *Fix* ε *and let* u_ε *be a minimizer for* E_ε, *with* $u_\varepsilon = g$ *on* Ω. *Then*

$$u_\varepsilon = \tilde{g} + \psi \tag{8.32}$$

where $\tilde{g}$ *is the harmonic extension of* g *and*

$$|\psi(x)| \leq C\varepsilon^{-1}\operatorname{dist}(x, \Omega). \tag{8.33}$$

Proof. — Clearly $\psi = 0$ on Ω, $|\psi| \leq 2$, and $|\Delta\psi| \leq C\varepsilon^{-2}$ on G. By interpolation one deduces that $|\nabla\psi| \leq C\varepsilon^{-1}$ (*see* e.g. [7]) and the conclusion follows.

1. Using (8.32), write

$$\begin{aligned} |\nabla(u_\varepsilon e^{-i\tilde{\varphi}})| &\geq |u_\varepsilon|\ |\nabla\tilde{\varphi}| - |\nabla u_\varepsilon| \\ &\geq |\tilde{g}|\ |\nabla\tilde{\varphi}| - |\psi|\ |\nabla\tilde{\varphi}| - |\nabla u_\varepsilon|. \end{aligned} \tag{8.34}$$

We have

$$\|\nabla u_\varepsilon\|_{L^2(G)} \lesssim \left(\log\frac{1}{\varepsilon}\right)^{1/2} < \infty$$

and, by (8.33)

$$\begin{aligned} \int_G (|\psi|\ |\nabla\tilde{\varphi}|)^2 &\leq C\varepsilon^{-2}\sum_{s\geq 0} 4^{-s}\int_{\operatorname{dist}(x,\Omega)\sim 2^{-s}} |(\nabla\tilde{\varphi})(x)|^2 \\ &\leq C\varepsilon^{-2}\sum_{s\geq 0} 4^{-s}.\, 4^{s}.\, 2^{-s}\|\varphi\|^2_{L^2(\Omega)} \leq C\varepsilon^{-2} < \infty. \end{aligned}$$

Consequently, assuming (8.31) were true for some $p < 2$, we necessarily must have, by (8.34), that

$$|\tilde{g}|\ |\nabla\tilde{\varphi}| \in L^p(G) \tag{8.35}$$

whenever $g = e^{i\varphi} \in H^{1/2}(\Omega, S^1)$.

This statement relates only to g and we show next that (8.35) *cannot* hold for $p > 3/2$.

2. Let $0 < \delta < 1$ be small and take $0 \leq \varphi \leq (\frac{1}{\delta})^{1-}$ such that

$$\operatorname{supp}\varphi \subset B(0, 2\delta) \subset \Omega \text{ (identified with the } x_1, x_2\text{-plane)}, \tag{8.36}$$

$$\varphi = \left(\frac{1}{\delta}\right)^{1-} \text{ on } B(0, \delta), \tag{8.37}$$

$$|\nabla\varphi| \leq \left(\frac{1}{\delta}\right)^{2-}. \tag{8.38}$$

Hence

$$\|e^{i\varphi}\|_{H^{1/2}} < C.$$

Also, from (8.1)

$$\|1 - e^{i\varphi}\|_{L^1} \leq C\delta^2.$$

Hence for $x_3 > C\delta$

$$|1 - \tilde{g}(x_1, x_2, x_3)| \leq \int |1 - e^{i\varphi}|(x_1', x_2')P_x(x_1', x_2')dx_1 dx_2 \leq C\delta^2\|P_x\|_\infty < \frac{1}{10}. \tag{8.39}$$

Thus from (8.39)

$$\begin{aligned} \|\tilde{g}. |\nabla\tilde{\varphi}|\ \|_{L^p} &\gtrsim \|\nabla\tilde{\varphi}\|_{L^p(x_1,x_2;x_3>C\delta)} \\ &\sim \left\| \int_{\mathbf{R}^2} |\xi|\hat{\varphi}(\xi)e^{i(x_1\xi_1+x_2\xi_2)}e^{-x_3|\xi|}d\xi \right\|_{L^p(x_1,x_2;x_3>C\delta)} \\ &\geq \left\| \|\, |\xi|\hat{\varphi}(\xi)e^{-x_3|\xi|}\|_{L^{p'}_\xi} \right\|_{L^p(x_3>C\delta)} \quad (8.40) \\ &\geq c\Big[\|\, |\xi|\hat{\varphi}(\xi)\|_{L^{p'}_{|\xi|\sim\frac{1}{10\delta}}}\Big].\, \delta^{\frac{1}{p}} \\ &\sim \delta^{-1}\hat{\varphi}(0)\cdot\left(\frac{1}{\delta}\right)^{\frac{2}{p'}}\delta^{1/p} \\ &\sim \delta^{\frac{1}{p}-\frac{2}{p'}+}. \quad (8.41) \end{aligned}$$

In (8.40), we use Hausdorff–Young inequality and (8.41) follows from (8.36), (8.37).

Since $\frac{1}{p}-\frac{2}{p'} < 0$ for $p > 3/2$, a gluing construction with the preceding as building block and $\delta \to 0$ will clearly violate (8.35).

As in the previous sections and with some more work, we may prove the following variant of Theorem 9:

Theorem **9′.** — *Assume* $g \in Y$, *and let* g_ε *be as in Theorem 6′ of Section 5. Let* u_ε *be a minimizer of* E_ε *in* $H^1_{g_\varepsilon}$. *Then*

$$u_\varepsilon \to u_* \text{ in } W^{1,p}(G) \cap C^\infty(G), \quad \forall p < 3/2,$$

where u_* *is the same as in Theorem 9.*

9. Further thoughts about $p = 3/2$

Let $g \in H^{1/2}(\Omega; S^1)$ and let (u_ε) be a minimizer for E_ε in H^1_g. In Section 6 we have established that (u_ε) is relatively compact in $W^{1,p}(G)$ for every $p < 3/2$. It is plausible that (u_ε) is bounded and possibly even relatively compact in $W^{1,3/2}$; *see* Open Problem 2 in Section 10.

There are two directions of evidence suggesting that, indeed, (u_ε) is bounded in $W^{1,3/2}$.

The first one relies on a conjectured strengthening of the Jerrard–Soner inequality mentioned below.

The second one is a complete proof of the fact that any limit (in $W^{1,p}$, $p < 3/2$) of (u_ε) belongs to $W^{1,3/2}$; *see* Theorem 12.

9.1. *Jerrard–Soner revisited*

First recall the following immediate consequence of a result in [33]:

Proposition **1** *(Jerrard and Soner [33]). — Let* (v_ε) *be a sequence in* $H^1(Q; \mathbf{R}^2)$, $Q \subset \mathbf{R}^3$ *a cube, satisfying*

$$E_\varepsilon(v_\varepsilon; Q) = \int_Q \left[\frac{1}{2}[\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2}||v_\varepsilon|^2 - 1|^2\right] \le C \log 1/\varepsilon \tag{9.1}$$

for all $\varepsilon < \varepsilon_0$. *Then for* $\zeta \in C_0^\infty(\omega), \bar{\omega} \subset Q$, *we have the inequality*

$$\left| \int J(v_\varepsilon)\zeta \right| \le K\|\zeta\|_{W^{1,q}(Q)} \tag{9.2}$$

where $J(v_\varepsilon)$ *is any* 2×2 *Jacobian determinant of* v_ε, $q > 3$, *and* $K = K(C, q, \omega)$.

Remark **9.1.** — In fact in [33] one obtains a stronger estimate with the norm $\|\zeta\|_{W^{1,q}}$ replaced by any $\|\zeta\|_{C^{0,\alpha}}$-norm, $\alpha > 0$.

In this subsection, we will show that:

a) The conclusion of Proposition 1 fails for any $q < 3$.
b) The validity of Proposition 1 for $q = 3$ (which we conjecture) would imply the boundedness in $W^{1,3/2}$ of the minimizers (u_ε) of the Ginzburg–Landau problem in G with boundary data g controlled in $H^{1/2}(\Omega; S^1)$, $\Omega = \partial G$.

A basic tool is the following construction of an extension of g outside G.

Lemma **30.** — *Assume* $\overline{G} \subset Q$ *and* $g \in H^{1/2}(\Omega; S^1)$. *Then there is* $w_\varepsilon \in H^1(Q\backslash G; \mathbf{R}^2)$ *satisfying*

(**9.3**) $\qquad w_\varepsilon = g$ *on* ∂G *and* $w_\varepsilon \equiv 1$ *in some fixed neighborhood of* ∂Q,

(**9.4**) $\qquad E_\varepsilon(w_\varepsilon; Q\backslash G) \le C\|g\|_{H^{1/2}} \log 1/\varepsilon$,

(**9.5**) $\qquad \|w_\varepsilon\|_{W^{1,p}(Q\backslash G)} \le C_p\|g\|_{H^{1/2}}$ *for every* $p < 2$,

(**9.6**) $\qquad w_{\varepsilon_n} \to w$ *in* $W^{1,p}(Q \setminus G)$ *for every* $p < 2$ *with* $w \in W^{1,p}(Q \setminus G)$, $\forall p < 2$

(**9.7**) $\qquad |w_\varepsilon| \le 1$ *in* $Q \setminus G$.

Proof. — We follow the same construction as in [5] which we briefly recall here. First, let H be any smooth function in $Q\backslash G$ with $H \in H^1(Q \setminus G; \mathbf{R}^2)$ satisfying the boundary conditions $H = g$ on $\Omega = \partial G$, $H \equiv 1$ near ∂Q, and $\|H\|_{H^1} \le C\|g\|_{H^{1/2}}$.

Using the same notation as in the proof of Lemma 23, define

$$w_{\varepsilon,a}(x) = \psi\left(\frac{|H(x) - a|}{\varepsilon}\right)\pi_a(H(x)).$$

It may be shown as in [5] (or as in the proof of Lemma 23) that for some $a = a_\varepsilon \in \mathbf{C}$, $|a_\varepsilon| < 1/10$, the functions $(w_{\varepsilon,a_\varepsilon})$ satisfy all the required properties.

Next, we establish the following

Proposition **2.** — *Assume that the conclusion of Proposition 1 is valid for some* $2 < q \le 3$. *Let* (u_ε) *be a sequence of minimizers of* E_ε *in* G *as above. Then* (u_ε) *is bounded in* $W^{1,q'}(G)$ *with* $q' = q/(q-1)$.

Proof. — As in Section 6, it suffices to establish the boundedness of $u_\varepsilon \wedge du_\varepsilon$ in the space $L^{q'}(G)$. Proceeding by duality, consider $\zeta \in L^q(G; \mathbf{R}^3)$, $\|\zeta\|_q \le 1$ and take its Hodge decomposition as

$$\begin{cases} \zeta = \text{curl } k + \nabla L \text{ in } G \\ L = 0 \text{ on } \Omega, \\ \text{with } \|k\|_{W^{1,q}(G)} + \|L\|_{W^{1,q}(Q)} \le C \end{cases} \qquad (\mathbf{9.8})$$

(*see* e.g. [30] or [27]). Recall that, with the notations of differential forms we used earlier, curl $= d^*$ and $\nabla = d$. Let Q be a cube with $\overline{G} \subset Q$ and let ω be an open set such that

$$\overline{G} \subset \omega \text{ and } \overline{\omega} \subset Q.$$

Next, extend k to $\tilde{k}$ on Q, $\tilde{k} = 0$ on $Q \setminus \omega$, with control of $\|\tilde{k}\|_{W^{1,q}(Q)}$. We extend u_ε to Q defining

$$v_\varepsilon = \begin{cases} u_\varepsilon \text{ in } G \\ w_\varepsilon \text{ in } Q \backslash G \end{cases}$$

where w_ε is provided by Lemma 30.

Recall that $\operatorname{div}(u_\varepsilon \wedge du_\varepsilon) = 0$, and thus

$$\int_G (u_\varepsilon \wedge du_\varepsilon) \cdot \zeta = \int_G (u_\varepsilon \wedge du_\varepsilon) \cdot \operatorname{curl} k.$$

Hence

$$\left| \int_G (u_\varepsilon \wedge du_\varepsilon) \cdot \zeta \right| \le \left| \int_Q (v_\varepsilon \wedge dv_\varepsilon) \cdot \operatorname{curl} \tilde{k} \right| + \int_{Q \backslash G} |\nabla w_\varepsilon| \, |\nabla \tilde{k}|. \tag{9.9}$$

From (9.5), the last term in (9.9) is bounded by $C\|w_\varepsilon\|_{W^{1,q'}(Q \backslash G)}$, hence by $C'\|g\|_{H^{1/2}}$, since $q' < 2$.

For the first term, perform an integration by part ($\tilde{k} = 0$ on ∂Q) to get

$$\left| \int_Q (v_\varepsilon \wedge dv_\varepsilon) \cdot \operatorname{curl} \tilde{k} \right| = 2 \left| \int_Q J(v_\varepsilon) \cdot \tilde{k} \right| \tag{9.10}$$

and this quantity is bounded, by assumption, by $C\|\tilde{k}\|_{W^{1,q}(Q)}$ (since $\operatorname{supp} \tilde{k} \subset \overline{\omega}$).

This proves Proposition 2.

Remark **9.2.** — The proof of Proposition 2 also provides an alternative quick proof of Theorem 7.

Corollary **4.** — *The conclusion of Proposition 1 fails for every* $q < 3$.

Proof. — By Proposition 2, one would otherwise obtain the boundedness of the Ginzburg–Landau minimizers in $W^{1,p}(G)$ for some $p > 3/2$. This is not true in general, even for certain $g \in Y$. Arguing by contradiction, one would otherwise obtain that the limit u_* obtained in Theorem 9 belongs to $W^{1,p}$ with $p > 3/2$. However, this is false. Indeed

Remark **9.3.** — In general $u_* \notin W^{1,t}$ for $t > 3/2$. Here is an example (*see* [5]): Suppose Ω is flat near 0 and choose $g(r) = e^{i/r^\alpha}$ with $\alpha < 1, \alpha$ close to 1 and g smooth away from 0. This g belongs to Y. It is easy to see that the harmonic extension of $1/r^\alpha$ does not belong to $W^{1,t}$, for $t > 3/(\alpha+1)$. Thus $u_* \notin W^{1,t}$.

Remark **9.4.** — The preceding also shows that the improved interior estimates from Section 7 can not be established via a strengthening of Jerrard–Soner but requires additional structure (in particular the monotonicity formula).

9.2. $W^{1,3/2}$ – *estimate of the limit*

We start with the simple case when $g \in Y$.

Theorem **11.** — *Assume* $g \in Y$ *and let* u_* *be as in Theorem 9. Then* $u_* \in W^{1,3/2}$.

Proof of Theorem 11. — Recall that $u_* = e^{\imath\tilde{\varphi}}$ where $\tilde{\varphi}$ is the harmonic extension of $\varphi \in H^{1/2} + W^{1,1}$. Therefore, it suffices to apply the following imbedding result, which is an immediate consequence of Theorem 1.5 in Cohen, Dahmen, Daubechies and DeVore [23]:

Lemma **31.** — *In 2-dimensions we have* $W^{1,1}(\Omega) \subset W^{\frac{1}{3},\frac{3}{2}}(\Omega)$.

For completeness we will prove a slightly more general form of this result in Appendix D.

We now turn to the case of a general $g \in H^{1/2}(\Omega; S^1)$.

Theorem **12.** — *Let* $g \in H^{1/2}(\Omega; S^1)$ *and let* (u_ε) *be a minimizer of* E_ε *in* $H^1_g(G; \mathbf{R}^2)$. *In view of Theorem 7′ we may assume that (modulo a subsequence)*

$$u_{\varepsilon_n} \to U \text{ in } W^{1,p}(G), \quad \forall p < 3/2.$$

Then

$$U \in W^{1,3/2}(G).$$

Proof of Theorem 12. — In the proof we will not fully use the fact that u_ε is a minimizer. We will only make use of the properties

$$\text{div}(u_\varepsilon \wedge du_\varepsilon) = 0 \text{ in } G, \tag{9.0.1}$$

$$e_\varepsilon = E_\varepsilon(u_\varepsilon) \le C \log 1/\varepsilon, \tag{9.0.2}$$

$$u_{\varepsilon_n} \to U \text{ in } W^{1,p}(G), \quad \forall p < 3/2, \tag{9.0.3}$$

$$u_{\varepsilon|\Omega} = g \in H^{1/2}(\Omega; S^1). \tag{9.0.4}$$

Claim.

$$U \wedge dU \text{ belongs to } L^{3/2}(G). \tag{9.0.5}$$

This implies that $U \in W^{1,3/2}$. Indeed we have

$$|b|^2 = |a \wedge b|^2 + |a \cdot b|^2$$

for any vectors a, b in $\mathbf{R}^2$ with $|a| = 1$; applying this with $a = U$ and $b = \frac{\partial U}{\partial x_i}$ yields $|dU| = |U \wedge dU|$ since $U \cdot \frac{\partial U}{\partial x_i} = 0$.

In order to prove the Claim (9.0.5) we will check that, for every $\vec{\zeta} \in L^3(G; \mathbf{R}^3)$, we have

$$\left| \int_G \vec{\zeta} \cdot (U \wedge dU) \right| \le C \|\vec{\zeta}\|_{L^3}. \tag{9.0.6}$$

Clearly, it suffices to verify (9.0.6) when $\vec{\zeta} \in C_0^\infty$. Consider the Hodge decomposition of $\vec{\zeta}$ as above, i.e.,

$$\vec{\zeta} = \operatorname{curl} \vec{k} + \nabla L \qquad \text{in } G, \tag{9.0.7}$$

$$L = 0 \qquad \text{on } \partial G, \tag{9.0.8}$$

$$\|\vec{k}\|_{W^{1,3}(G)} \le C \|\vec{\zeta}\|_{L^3}. \tag{9.0.9}$$

Then, by (9.0.1) and (9.0.8),

$$\int_G \nabla L \cdot (U \wedge dU) = 0$$

and thus

$$\int_G \vec{\zeta} \cdot (U \wedge dU) = \int_G (\operatorname{curl} \vec{k}) \cdot (U \wedge dU). \tag{9.0.10}$$

We will establish the bound

$$\left| \int_G (\operatorname{curl} \vec{k}) \cdot (U \wedge dU) \right| \le C \|\vec{k}\|_{W^{1,3}} \tag{9.0.11}$$

in 5 steps. The desired estimate (9.0.6) will be consequence of (9.0.10) and (9.0.11).

Step **1.** — Extensions.

Let Q be a cube such that $\overline{G} \subset Q$. Let $\tilde{k} \in W^{1,3}(Q; \mathbf{R}^3)$ be such that supp $\tilde{k}$ is contained in a fixed compact subset of Q,

$$\tilde{k} = \vec{k} \text{ in } G,$$

and

$$\|\tilde{k}\|_{W^{1,3}(Q)} \le C \|\vec{k}\|_{W^{1,3}(G)}.$$

Next, we extend g to $Q \setminus G$ using Lemma 30. Thus, we obtain a family $w_\varepsilon \in H^1(Q \setminus G; \mathbf{R}^2)$ satisfying

(**9.1.1**) $$w_{\varepsilon|\partial G} = g,$$

(**9.1.2**) $$w_\varepsilon \equiv 1 \text{ in some fixed neighborhood of } \partial Q,$$

(**9.1.3**) $$E_\varepsilon(w_\varepsilon; Q \setminus G) \leq C \log 1/\varepsilon$$

(**9.1.4**) $$\|w_\varepsilon\|_{W^{1,p}(Q\setminus G)} \leq C_p, \quad \forall p < 2$$

(**9.1.5**) $$w_{\varepsilon_n} \longrightarrow w \text{ in } W^{1,p}(Q \setminus G), \quad \forall p < 2,$$

for some $w \in W^{1,p}(Q \setminus G; S^1)$, $\forall p < 2$.

Set

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } G \\ w_\varepsilon & \text{in } Q \setminus G, \end{cases}$$

so that $\tilde{u}_\varepsilon \in H^1(Q; \mathbf{R}^2)$ and

(**9.1.6**) $$\tilde{u}_{\varepsilon_n} \longrightarrow \widetilde{U} \text{ in } W^{1,p}(Q), \quad \forall p < 3/2,$$

where

$$\widetilde{U} = \begin{cases} u & \text{in } G \\ w & \text{in } Q \setminus G \end{cases}$$

and $\widetilde{U} \in W^{1,p}(Q; S^1)$, $\forall p < 3/2$.

Clearly,

(**9.1.7**) $$E_\varepsilon(\tilde{u}_\varepsilon; Q) \leq C \log 1/\varepsilon.$$

It is convenient to introduce the following distribution denoted $\widetilde{U}_{x_i} \wedge \widetilde{U}_{x_j}$, $i \neq j$

$$\widetilde{U}_{x_i} \wedge \widetilde{U}_{x_j} = \frac{1}{2}(\widetilde{U}_{x_i} \wedge \widetilde{U})_{x_j} + \frac{1}{2}(\widetilde{U} \wedge \widetilde{U}_{x_j})_{x_i}$$

acting on functions $C_0^\infty(Q; \mathbf{R})$.

An immediate computation shows that

(**9.1.8**) $$-\frac{1}{2}\int_Q (\text{curl } \tilde{k}) \cdot \widetilde{U} \wedge d\widetilde{U} = \langle \widetilde{U}_{x_2} \wedge \widetilde{U}_{x_3}, \tilde{k}_1 \rangle + \langle \widetilde{U}_{x_3} \wedge \widetilde{U}_{x_1}, \tilde{k}_2 \rangle + \langle \widetilde{U}_{x_1} \wedge \widetilde{U}_{x_2}, \tilde{k}_3 \rangle.$$

We will prove e.g. that

$$(\mathbf{9.1.9}) \qquad \left| < \widetilde{U}_{x_1} \wedge \widetilde{U}_{x_2}, k > \right| \le C\|k\|_{W^{1,3}}.$$

for every $k \in C_0^\infty(Q; \mathbf{R})$ and similarly for the other terms.

Assuming (9.1.9) we then have

$$(\mathbf{9.1.10}) \qquad \left| \int_Q (\operatorname{curl} \tilde{k}) \cdot (\widetilde{U} \wedge d\widetilde{U}) \right| \le C\|\tilde{k}\|_{W^{1,3}(Q)}$$

and thus

$$(\mathbf{9.1.11}) \qquad \begin{aligned} \left| \int_G (\operatorname{curl} \vec{k}) \cdot (U \wedge dU) \right| &\le \left| \int_{Q\setminus G} (\operatorname{curl} \tilde{k}) \cdot w \wedge dw \right| + C\|\tilde{k}\|_{W^{1,3}(Q)} \\ &\le \|\tilde{k}\|_{W^{1,3}(Q\setminus G)} \|w\|_{L^{3/2}(Q\setminus G)} + C\|\tilde{k}\|_{W^{1,3}(Q)}. \end{aligned}$$

Finally we obtain, by (9.1.4),

$$(\mathbf{9.1.12}) \qquad \left| \int_G (\operatorname{curl} \vec{k}) \cdot (U \wedge dU) \right| \le C\|\vec{k}\|_{W^{1,3}(G)}$$

which is the desired estimate (9.0.11).

The rest of the argument is devoted to the proof of (9.1.9).

Step **2.** — Use of a result of Jerrard–Soner.

For any $\bar{x}_3 \in \mathbf{R}$ set

$$\Sigma_{\bar{x}_3} = Q \cap (\mathbf{R}^2 \times \{\bar{x}_3\}).$$

Consider $\bar{x}_3$ such that

$$(\mathbf{9.2.1}) \qquad \liminf_{\varepsilon \to 0} \frac{E_\varepsilon(\tilde{u}_\varepsilon \big| \Sigma_{\bar{x}_3})}{\log 1/\varepsilon} < \infty$$

and

$$(\mathbf{9.2.2}) \qquad \widetilde{U}_{\varepsilon_n | \Sigma_{\bar{x}_3}} \longrightarrow \widetilde{U}_{|\Sigma_{\bar{x}_3}} \text{ in } W^{1,\frac{3}{2}-}(\Sigma_{\bar{x}_3}).$$

From (9.1.6), (9.1.7), this is the case for almost all $\bar{x}_3$.

It follows then from Theorem 3.1 in [33] that $(\tilde{u}_{\varepsilon_n})_{x_1} \wedge (\tilde{u}_{\varepsilon_n})_{x_2}$ converges in $\mathscr{D}'(\Sigma_{\bar{x}_3})$ to $\widetilde{U}_{x_1} \wedge \widetilde{U}_{x_2}$ and that

$$(\mathbf{9.2.3}) \qquad \widetilde{U}_{x_1} \wedge \widetilde{U}_{x_2} = \pi \sum_i d_i \delta_{a_i}$$

where $d_i = d_i(\bar{x}_3) \in \mathbf{Z}$, $a_i = a_i(\bar{x}_3) \in \sum_{\bar{x}_3}$ satisfy

$$(\mathbf{9.2.4}) \qquad \pi \sum_i |d_i(\bar{x}_3)| \leq \liminf_{\varepsilon \to 0} \frac{\mathrm{E}_\varepsilon(\tilde{u}_\varepsilon \big| \Sigma_{\bar{x}_3})}{\log 1/\varepsilon}.$$

Thus, from (9.1.7)

$$(\mathbf{9.2.5}) \qquad \sum_i \int |d_i(x_3)| dx_3 \leq \mathrm{C}$$

and we may write

$$(\mathbf{9.2.6}) \qquad < \widetilde{\mathrm{U}}_{x_1} \wedge \widetilde{\mathrm{U}}_{x_2}, k >= \pi \int dx_3 \Big\{ \sum_i d_i(x_3) k\big(a_i(x_3)\big) \Big\}.$$

To bound (9.2.6), we will need, besides (9.2.5), also certain cancellations that have to do with the sign of d_i's.

Step **3.** — Use of minimal connections.

Take $\bar{x}_3$ as in Step 2 and consider the domain

$$\Omega_{\bar{x}_3} = \mathrm{Q} \cap [x_3 \leq \bar{x}_3] \qquad (\text{or } x_3 \geq \bar{x}_3).$$

Since $\tilde{u}_{\varepsilon_n} \to \widetilde{\mathrm{U}}$ in $\mathrm{W}^{1,\frac{3}{2}-}(\partial\Omega_{\bar{x}_3})$, $\tilde{u}_{\varepsilon_n} \to \widetilde{\mathrm{U}}$ in $\mathrm{H}^{1/2}(\partial\Omega_{\bar{x}_3})$. Remark also that, since $\widetilde{\mathrm{U}} = 1$ on $\partial\mathrm{Q}$, the singularities of $\widetilde{\mathrm{U}}$ on $\partial\Omega_{\bar{x}_3}$ are necessarily in $\Sigma_{\bar{x}_3}$.

Invoke next Theorem 6′ to claim that

$$(\mathbf{9.3.1}) \qquad \pi \mathrm{L}(\widetilde{\mathrm{U}}_{|\Sigma_{\bar{x}_3}}) = \pi \mathrm{L}(\widetilde{\mathrm{U}}_{|\partial\Omega_{\bar{x}_3}}) \leq \liminf_{\varepsilon \to 0} \frac{\mathrm{E}_\varepsilon(\tilde{u}_{\varepsilon|\Omega_{\bar{x}_3}})}{\log 1/\varepsilon} \leq \sup \frac{\mathrm{E}_\varepsilon(\tilde{u}_\varepsilon)}{\log 1/\varepsilon} \leq \mathrm{C}.$$

Note that assumption (5.11) is satisfied since

$$\frac{1}{\varepsilon^2} \int_{\mathrm{Q}} (|\tilde{u}_\varepsilon|^2 - 1)^2 \leq \mathrm{C} \log 1/\varepsilon$$

implies

$$\frac{1}{\varepsilon} \int_{\mathrm{Q}} (|\tilde{u}_\varepsilon|^2 - 1)^2 = \frac{1}{\varepsilon} \int dx_3 \int_{\Sigma_{x_3}} (|\tilde{u}_\varepsilon|^2 - 1)^2 \longrightarrow 0$$

and then

$$\frac{1}{\varepsilon_n}\int_{\Sigma_{x_3}}(|\tilde{u}_{\varepsilon_n}|-1)^2 \le h(x_3)$$

for some fixed function $h \in L^1$.

Thus, by (9.3.1), there is a reordering

$$\{a_i(d_i)\} = \{p_1, \dots, p_\ell\} \cup \{n_1, \dots, n_\ell\}$$

with possible repetition, such that

$$\sum_j |p_j(\bar{x}_3) - n_j(\bar{x}_3)| \le C \qquad (\mathbf{9.3.2})$$

and (9.2.5), (9.2.6) may be rewritten as

$$\int \ell(x_3)dx_3 \le C \qquad (\mathbf{9.3.3})$$

(where $2\ell(x_3) = \sum |d_i(x_3)|$)
and

$$< \widetilde{U}_{x_1} \wedge \widetilde{U}_{x_2}, k >= \pi \int dx_3 \Big\{ \sum_j [k(p_j(x_3)) - k(n_j(x_3))] \Big\}. \qquad (\mathbf{9.3.4})$$

We will now establish the desired bound (9.1.9) with the help of the following

Proposition **3.** — *Assume (9.3.3) and (9.3.4), then, for every* $k \in C_0^\infty(Q; \mathbf{R})$,

$$\left| \int dx_3 \Big\{ \sum_j \big[k(p_j(x_3)) - k(n_j(x_3))\big] \Big\} \right| \le C\|k\|_{W^{1,3}(Q)}. \qquad (\mathbf{9.3.5})$$

Step **4.** — Decomposition of $W^{1,3}(\mathbf{R}^3)$-function.

Let $k \in W^{1,3}(\mathbf{R}^3)$, $\|k\|_{W^{1,3}} \le 1$ and let

$$k = \sum_{s \ge 0} \Delta_s k$$

be a usual Littlewood–Paley decomposition (we assume $\operatorname{supp} k \subset Q$).

Thus

$$\sum 8^s \|\Delta_s k\|_3^3 < C. \qquad (\mathbf{9.4.1})$$

Denote

(**9.4.2**) $$\lambda_s = 8^s \|\Delta_s k\|_3^3;$$

hence

(**9.4.3**) $$\sum \lambda_s < C.$$

First we estimate for fixed $\rho > 0$

(**9.4.4**) $$\text{meas } [x_3; \sup_{x_1,x_2} |\Delta_s k(x_1,x_2,x_3)| > \rho].$$

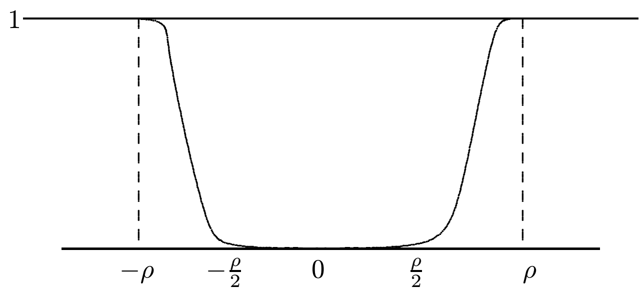
Clearly, for fixed x_3,

$$\|\Delta_s k(x_3)\|_{L^\infty_{x_1,x_2}} \le C4^{s/3}\|\Delta_s k(x_3)\|_{L^3_{x_1,x_2}}$$

so that

(**9.4.5**) $$(9.4.4) \le \rho^{-3}\int (\|\Delta_s k(x_3)\|_{L^\infty_{x_1,x_2}})^3 dx_3 \le C\rho^{-3}4^s\|\Delta_s k\|_3^3 \le C\rho^{-3}2^{-s}\lambda_s.$$

Denote ζ_ρ the function on **R**



Fix s_0 and decompose for $s \ge s_0 + 1$

$$\Delta_s k = k^1_{s,s_0} + k^2_{s,s_0} \quad \text{with} \quad k^1_{s,s_0} = \Delta_s k(1 - \zeta_{1/(s-s_0)^2})(\Delta_s k).$$

Hence

$$\left|k^1_{s,s_0}\right| \le |\Delta_s k| \ \chi_{\left[|\Delta_s k| < (s-s_0)^{-2}\right]}$$

$$\left|k^2_{s,s_0}\right| \le |\Delta_s k| \ \chi_{\left[|\Delta_s k| > \frac{1}{2}(s-s_0)^{-2}\right]}.$$

Therefore

(**9.4.6**) $$\sum_{s \ge s_0+1} \left|k^1_{s,s_0}\right| < C$$

and by (9.4.5)

(**9.4.7**) $$\text{meas }_{x_3}\left(\text{Proj}_{x_3}\left(\text{supp}\, k^2_{s,s_0}\right)\right) \le C(s-s_0)^6 \ 2^{-s}\lambda_s.$$

Step **5.** — Estimation of (9.3.5).

Using the decomposition of Step 4, estimate

$$(9.3.5) \le \int dx_3 \Big\{ \sum_{s_0} \sum_{j||p_j - n_j| \sim 2^{-s_0}} \big| k(p_j(x_3)) - k\big(n_j(x_3)\big) \big| \Big\} \tag{9.5.0}$$

and

$$|k(p_j) - k(n_j)| \le \sum_{s \le s_0} |\Delta_s k(p_j) - \Delta_s k(n_j)| \tag{9.5.1}$$

$$+ \sum_{s > s_0} \left(\left| k^1_{s,s_0}(p_j) \right| + \left| k^1_{s,s_0}(n_j) \right| \right) \tag{9.5.2}$$

$$+ \sum_{s > s_0} \left(\left| k^2_{s,s_0}(p_j) \right| + \left| k^2_{s,s_0}(n_j) \right| \right). \tag{9.5.3}$$

Contribution of (9.5.1)

Estimate

$$|\Delta_s k(p_j) - \Delta_s k(n_j)| \le \|\Delta_s k\|_{\mathrm{Lip}} \, |p_j - n_j| \le C 2^{s - s_0}.$$

Thus the contribution in (9.5.0) is bounded by

$$\int dx_3 \Big[\sum_{s_0, s \le s_0} 2^{s - s_0} (\#\{ j | \ |p_j(x_3) - n_j(x_3)| \sim 2^{-s_0} \}) \Big]$$
$$\le \int \ell(x_3) dx_3 < C$$

by (9.3.3).

Contribution of (9.5.2)

Same, since (9.5.2) < C from (9.4.6).

Contribution of (9.5.3)

This is the crux of the argument.

Estimate, using (9.3.2) and the fact that $|k^2_{s,s_0}| \le C$,

$$\sum_{j| \ |p_j - n_j| \sim 2^{-s_0}} |k^2_{s,s_0}\big(p_j(x_3)\big)| \le \|k^2_{s,s_0}\|_\infty \cdot \chi_{\mathrm{Proj}_{x_3}(\mathrm{supp}\, k^2_{s,s_0})}(x_3)$$
$$\cdot [\#\{ j | \ |p_j(x_3) - n_j(x_3)| \sim 2^{-s_0} \}]$$
$$< C 2^{s_0} \chi_{\mathrm{Proj}_{x_3}(\mathrm{supp}\, k^2_{s,s_0})}(x_3).$$

Integration in x_3 gives therefore, using (9.4.7),

$$(\mathbf{9.5.4}) \qquad C(s-s_0)^6\ 2^{-(s-s_0)}\lambda_s$$

which, by (9.4.3), is summable in $\sum_{s_0, s>s_0}$.

This completes the proof of (9.3.5), and thus of Theorem 12.

9.3. *A geometric estimate related to Proposition 3*

With the same technique as in the proof of Proposition 3 we may derive the following estimate which has an interesting geometric flavour. It may be used to provide an alternative proof of Theorem 12 as in [BOS1].

Proposition **4.** — *Let* Γ *be a closed, oriented, rectifiable curve in* $\mathbf{R}^3$*, and denote by* $\vec{t}$ *the unit tangent vector along* Γ*; let* $\vec{k} \in W^{1,3}(\mathbf{R}^3; \mathbf{R}^3)$*. Then*

$$\left| \int_\Gamma \vec{k}\cdot\vec{t}\, \right| \le C\|k\|_{W^{1,3}}|\Gamma|.$$

Proof. — Part of the argument is a repetition of the proof of Proposition 3, but we have kept it for the convenience of the reader who wishes to concentrate on Proposition 4 independently of the rest of the paper. Assume $|\Gamma| = 1$ and let $\gamma : [0,1] \longrightarrow \Gamma$ be the arclength parametrization ($|\dot\gamma| = 1$).

We need to bound

$$(\mathbf{9.6.1}) \qquad \int_\Gamma k_3(\gamma(s))\dot\gamma_3(s)ds = \int dx_3 \Bigg[\sum_{x\in\Gamma_{x_3}} \sigma(x)k_3(x)\Bigg],$$

where $\Gamma_{x_3} = \Gamma \cap [x = x_3]$ is assumed finite (by choice of coordinate system) and $\sigma(\gamma(s)) = \operatorname{sign}\dot\gamma_3(s)$.

Thus $\Gamma_{x_3} = \{P_1, \ldots, P_r\} \cup \{N_1, \ldots, N_r\}$, where $\sigma(P_i) = 1$ and $\sigma(Q_i) = -1$. Also,

$$r = r(x_3) = \frac{1}{2}\operatorname{card}(\Gamma_{x_3})$$

and

$$\int r(x_3)dx_3 = \frac{1}{2}\int |\dot\gamma_3(s)|ds < 1,$$

$$(\mathbf{9.6.3}) \qquad \sum_i |P_i - N_i| \le |\Gamma| = 1.$$

Write k for k_3 and assume $\|k\|_{W^{1,3}} \leq 1$. Write, for fixed x_3,

$$\left| \sum_{x\in\Gamma_{x_3}} \sigma(x)k(x) \right| \leq \sum_{i=1}^{r(x_3)} |k(P_i) - k(N_i)| = \sum_{s_0} \sum_{|P_i - N_i|\sim 2^{-s_0}} |k(P_i) - k(N_i)|. \tag{9.6.4}$$

To estimate (9.6.4), we perform again the same decomposition of $k \in W^{1,3}$. Thus, for fixed s_0,

$$k = k_{s_0} + \sum_{s>s_0} k^1_{s_0,s} + \sum_{s>s_0} k^2_{s_0,s}$$

satisfying

$$|\nabla k_{s_0}| \lesssim 2^{s_0} \tag{9.6.5}$$

$$\left|k^1_{s_0,s}\right| \lesssim (s-s_0)^{-2} \tag{9.6.6}$$

$$\begin{cases} \left|k^2_{s_0,s}\right| \lesssim 1 \text{ and} \\ \operatorname{supp} k^2_{s_0,s} \text{ contained in the union of } \lesssim \sigma_s (s-s_0)^6 \text{ cubes of size } 2^{-s} \end{cases} \tag{9.6.7}$$

with

$$\sum \sigma_s < C \tag{9.6.8}$$

(in fact $\sigma_s^{1/3} = \|\Delta_s k\|_{W^{1,3}}$, $k = \sum \Delta_s k$, Littlewood-Paley decomposition).

Returning to (9.6.4), we get for fixed s_0,

$$\sum_{|P_i - N_i|\sim 2^{-s_0}} |k_{s_0}(P_i) - k_{s_0}(N_i)| \tag{9.6.9}$$

$$+ \sum_{s>s_0} \sum_{|P_i - N_i|\sim 2^{-s_0}} \left|k^1_{s_0,s}(P_i)\right| + \left|k^1_{s_0,s}(N_i)\right| \tag{9.6.10}$$

$$+ \sum_{s>s_0} \sum_{|P_i - N_i|\sim 2^{-s_0}} \left|k^2_{s_0,s}(P_i)\right| + \left|k^2_{s_0,s}(N_i)\right|. \tag{9.6.11}$$

Contribution of (9.6.9)

$$(9.6.5) \Rightarrow (9.6.9) \lesssim \#\left\{i \,\middle|\, |P_i - N_i| \sim 2^{-s_0}\right\}.$$

Sum in $s_0 \Rightarrow r(x_3)$ satisfying (9.6.2).

Contribution of (9.6.10)

$$(9.6.6) \Rightarrow \sum_{s>s_0} \left|k^1_{s_0,s}\right| < C.$$

Hence

$$(9.6.10) \lesssim \#\left\{i \middle| |P_i - N_i| \sim 2^{-s_0}\right\}.$$

Contribution of (9.6.11)

For fixed $s > s_0$, we need to restrict x_3 to $\mathrm{Proj}_{x_3}(\mathrm{supp}\, k^2_{s_0,s}) \subset \mathbf{R}$ of measure $\lesssim \sigma_s(s-s_0)^6 2^{-s}$ by (9.6.7).

By (9.6.3), $\#\left\{i \middle| |P_i - N_i| \sim 2^{-s_0}\right\} \le 2^{s_0}, \quad \forall x_3.$

Thus,

$$\int dx_3 \Bigg[\sum_{|P_i - N_i| \sim 2^{-s_0}} \left|k^2_{s_0,s}(P_i)\right| + \dots \Bigg] \le \sigma_s (s-s_0)^6 2^{-(s-s_0)},$$

summable in $s, s_0, s > s_0$, taking also (9.6.8) into account.

10. Open problems

OP **1.** — Let u_ε be a minimizer of E_ε in H^1_g with $g \in H^{1/2}(\Omega; S^1)$. Is it true that

$$\int_G |u_{\varepsilon x_i} \wedge u_{\varepsilon x_j}| \le C \quad \forall i,j \text{ as } \varepsilon \to 0\,?$$

OP **2.** — Let u_ε be a minimizer of E_ε in H^1_g with $g \in H^{1/2}(\Omega; S^1)$. Is it true that

$$\|u_\varepsilon\|_{W^{1,3/2}(G)} \le C \text{ as } \varepsilon \to 0\,?$$

Is (u_ε) relatively compact in $W^{1,3/2}$?

OP **3.** — Assume $u_\varepsilon : B \to \mathbf{R}^2$ (B unit ball in $\mathbf{R}^3$) is smooth and satisfies

$$\int_B |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_B (|u_\varepsilon|^2 - 1)^2 \le C \log(1/\varepsilon).$$

Is it true that for every compact subset $K \subset B$,

$$\left| \int_B (u_{\varepsilon x} \wedge u_{\varepsilon y})\varphi \right| \leq C_K \|\varphi\|_{W^{1,3}} \quad \forall \varphi \in C_0^\infty(K)?$$

(As explained in Section 9.1 a positive solution of OP3 yields a positive answer to the first question in OP2)

OP **4.** — Let u_ε be a minimizer of E_ε in H^1_g with $g \in H^{1/2}(\Omega; S^1)$. Is it true that

$|u_\varepsilon|$ is bounded in $H^1(G)$?

11. Appendices

Appendix A. The upper bound for the energy

With G and $\Omega = \partial G$ as in Section 1, consider the following distinguished classes in $H^{1/2}(\Omega; S^1)$:

$$\mathscr{R} = \left\{ \begin{array}{l} g \in g \in W^{1,p}(\Omega; S^1), \forall p < 2; g \text{ is smooth away from} \\ \text{a finite set } \Sigma \text{ of singularities} \end{array} \right\},$$

$$\mathscr{R}_0 = \left\{ \begin{array}{l} g \in \mathscr{R}; |\nabla g(x)| \leq C/|x - \sigma| \text{ near each } \sigma \in \Sigma \\ \text{and } \deg(g, \sigma) = \pm 1, \ \forall \sigma \in \Sigma \end{array} \right\},$$

$$\mathscr{R}_1 = \left\{ g \in \mathscr{R}_0 \left| \begin{array}{l} \text{for each } \sigma \in \Sigma, \text{ there is some } R \in \mathscr{O}(3) \text{ such that} \\ \left| g(x) - R\left(\frac{x-\sigma}{|x-\sigma|}\right) \right| \leq C|x - \sigma| \text{for } x \text{ near } \sigma \end{array} \right. \right\},$$

where $\mathscr{O}(3)$ denotes the group of linear isometries of $\mathbf{R}^3$. Here, we identify $S^1 \subset \mathbf{R}^2$ with $S^1 \times \{0\}$ viewed as a subset of $\mathbf{R}^3$. From the definition of $\mathscr{R}_1$ we see that R must map the tangent plane $T_\sigma(\Omega)$ into $\mathbf{R}^2 \times \{0\}$ and thus $R(n(\sigma)) = (0, 0, \pm 1)$, where $n(\sigma)$ is the outward unit normal to Ω. Clearly, $\deg(g, \sigma) = +1$ if R is orientation-preserving and -1 otherwise.

This appendix is devoted to the proof of the following

Lemma **A.1.** — *Let $g \in \mathscr{R}_1$ and let L_G be the length of a minimal connection corresponding to the geodesic distance in* G. *Then*

$$\text{(A.1)} \quad \begin{aligned} &\operatorname{Min} \left\{ E_\varepsilon(u); u \in H^1_g(G; \mathbf{R}^2) \right\} \\ &\quad \leq \pi L_G(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)) \text{ as } \varepsilon \to 0. \end{aligned}$$

The proof we present below uses some arguments from [40], Section 1.

Proof. — Given $\delta > 0$ small, we first construct a domain G_δ and a diffeomorphism ξ_δ: $G \to G_\delta$ (with $\xi_\delta : \partial G \to \partial G_\delta$) such that

$$\|D\xi_\delta - I\| \leq C\delta \text{ on } G \tag{A.2}$$

and ∂G_δ is flat in a δ-neighborhood of each singularity $\xi_\delta(a_j)$ of $g_\delta = g \circ \xi_\delta^{-1}$.

The construction of ξ_δ is standard. Assume, for simplicity, that 0 is a singular point of g on Ω and that, near 0, the graph of Ω is given by $x_3 = \psi(x_1, x_2)$ with ψ smooth and $\nabla\psi(0) = 0$. Set

$$\eta(x_1, x_2, x_3) = (x_1, x_2, x_3 - \psi(x_1, x_2))$$

so that $\|D\eta(x) - I\| \leq C|x|$ near 0. Let $\zeta \in C_0^\infty(B_1)$ with $\zeta = 1$ on $B_{1/2}$. Then

$$\xi_\delta(x) = x + \zeta(x/\delta)(\eta(x) - x), x \in G$$

has all the required properties relative to one singularity. We proceed similarly for the other singularities.

We now write G and g instead of G_δ and g_δ, so that we may assume that Ω is flat in a δ-neighborhood of each singularity.

After relabeling the singularities of g, we may assume that $L_G(g) = \sum_{j=1}^k$ length (γ_j), where γ_j connects (in G) P_j and N_j. We now introduce a second parameter λ, $0 < \lambda < \delta$, and we choose some disjoint smooth curves Γ_j having the following properties:

a) $\sum_{j=1}^k$ length $(\Gamma_j) \leq L_G(g) + \lambda$;
b) Γ_j is a simple curve;
c) Γ_j is contained in G except for its endpoints P_j and N_j;
d) the curve Γ_j is orthogonal to Ω in a λ-neighborhood of its endpoints.

Moreover, we may assume that Γ_j is parametrized in such a way that the tangent vector at P_j is outward and the one at N_j is inward. We take the arclength as parameter. We may thus write $\Gamma_j = \{X_j(t); t \in [0, T_j]\}$, with $X_j(0) = N_j$, $X_j(T_j) = P_j$, where X_j is smooth, into and an immersion, and $T_j = \text{length}(\Gamma_j)$.

We consider the unit tangent vector to Γ_j, $e(X_j(t)) = X_j'(t)$. We may find two smooth vector fields f, g on Γ_j such that $\{f(X_j(t)), g(X_j(t)), e(X_j(t))\}$ is a direct orthonormal basis for each t.

We now define the map $\Phi_j : [0, T_j] \times \overline{B}_\lambda \to \mathbf{R}^3$ by

$$\Phi_j(t, u, v) = X_j(t) + uf(X_j(t)) + vg(X_j(t)),$$

where $B_\lambda = \{(u, v) \in \mathbf{R}^2; u^2 + v^2 \leq \lambda^2\}$.

Clearly,

$$\text{(A.3)} \qquad \|D\Phi_j(t,u,v) - M(t)\| \le C\lambda \text{ on } [0,T_j]\times B_\lambda,$$

where $M(t) \in \mathcal{O}(3)$. Thus, for λ sufficiently small, Φ_j is a diffeomorphism from $[0,T_j] \times \overline{B}_\lambda$ onto a λ-tubular neighborhood U_j of Γ_j. Moreover $U_j \subset \overline{G}$ for λ small.

It is easy to see that the restriction of g to $\Omega \setminus \cup_j U_j$ has a smooth S^1-valued extension, $\tilde{g}$, to $\overline{G} \setminus \cup_j U_j$. Indeed, let $\zeta_j : G \to \mathbf{R}^3$ be a diffeomorphism onto $\zeta_j(G)$ with $\zeta_j(G) \subset B_R \times [0,T_j]$ and $\zeta_j(U_j) = \overline{B}_\lambda \times [0,T_j]$. Consider the function $k : \mathbf{R}^3 \to S^1$ defined by

$$k(x,y,z) = (x,y)\big/(x^2+y^2)^{1/2}.$$

Then

$$k_j = k \circ \zeta_j : G \setminus U_j \to S^1$$

is smooth and

$$q = \Pi_{j=1}^k k_j : G \setminus \bigcup_j U_j \to S^1$$

is also smooth. Moreover

$$\deg\left(q, C_j^\pm\right) = \pm 1 \quad \forall j$$

where $C_j^+ = \{x \in \Omega;\ |x-P_j| = \lambda\}$ and $C_j^- = \{x \in \Omega;\ |x-N_j| = \lambda\}$. Therefore

$$\deg\left(g/q, C_j^\pm\right) = 0 \quad \forall j.$$

Hence the function g/q restricted to $\Omega \setminus \bigcup_j U_j$ admits a smooth extension $f : \Omega \to S^1$. Then f extends to a smooth map $\tilde{f} : \overline{G} \to S^1$. Finally, the map $\tilde{g} = \tilde{f} q$ has the desired properties.

Clearly we have

$$\text{(A.4)} \qquad E_\varepsilon(\tilde{g}; G \setminus \bigcup_j U_j) \le C_\lambda.$$

Consider the map $h_j : \partial([0,T_j]\times\overline{B}_\lambda) \to S^1$ defined by

$$h_j = \begin{cases} \tilde{g} \circ \Phi_j, & \text{on } [0,T_j]\times\partial\overline{B}_\lambda \\ g \circ \Phi_j, & \text{on } \{0\}\times\overline{B}_\lambda \text{ and on } \{T_j\}\times\overline{B}_\lambda \end{cases}.$$

Then h_j is smooth on $\partial([0, T_j] \times B_\lambda)$ except at the points $(0, 0, 0)$ and $(T_j, 0, 0)$. From the construction in [40] we know that

$$\text{(A.5)} \qquad \begin{aligned} &\text{Min } \left\{ E_\varepsilon(u; (0, T_j) \times B_\lambda)); u \in H^1_{h_j}\big((0, T_j) \times B_\lambda; \mathbf{R}^2\big) \right\} \\ &\quad \leq \pi T_j \log(1/\varepsilon) + C_\lambda. \end{aligned}$$

Using (A.5) and (A.3) we return to U_j via Φ_j and obtain a map

$$v = v_{j,\varepsilon,\lambda} : U_j \to \mathbf{R}^2$$

such that $v = g$ on $(\partial U_j) \cap \Omega$ and

$$\text{(A.6)} \qquad E_\varepsilon(v; U_j) \leq (\pi T_j \log(1/\varepsilon) + C_\lambda)(1 + C\lambda).$$

Gluing the maps $v_{j,\varepsilon,\lambda}$ defined above with the map $\tilde{g}_{|\overline{G} \setminus \cup_j U_j}$, we obtain a map $w_{\varepsilon,\lambda} : G \to \mathbf{R}^2$ satisfying

$$w_{\varepsilon,\lambda} = g \text{ on } \Omega$$

and (by (A.4) and (A.6)),

$$\text{(A.7)} \qquad E_\varepsilon(w_{\varepsilon,\lambda}; G) \leq \Big(\pi\big(\sum T_j\big) \log(1/\varepsilon) + C_\lambda\Big)(1 + C\lambda) + C_\lambda.$$

Returning to the original notation G_δ and $\Omega_\delta = \partial G_\delta$, we have just constructed a map $w_{\varepsilon,\lambda} : G_\delta \to \mathbf{R}^2$ satisfying

$$w_{\varepsilon,\lambda} = g_\delta = g \circ \xi_\delta^{-1} \text{ on } \Omega_\delta$$

and

$$\text{(A.8)} \qquad E_\varepsilon(w_{\varepsilon,\lambda}; G_\delta) \leq \pi(L_{G_\delta}(g_\delta) + \lambda) \log(1/\varepsilon)(1 + C\lambda) + C'_\lambda.$$

Finally, coming back to the original domain G via ξ_δ, we obtain some $\tilde{w}_{\varepsilon,\lambda,\delta} \in H^1_g(G; \mathbf{R}^2)$ such that

$$\text{(A.9)} \qquad E_\varepsilon(\tilde{w}_{\varepsilon,\lambda,\delta}; G) \leq \big[\pi(L_{G_\delta}(g_\delta) + \lambda) \log(1/\varepsilon)(1 + C\lambda) + C'_\lambda\big](1 + C\delta).$$

It is easy to see that

$$\big|L_{G_\delta}(g_\delta) - L_G(g)\big| \leq C\delta$$

and thus we arrive at

$$\text{(A.10)} \qquad E_\varepsilon(\tilde{w}_{\varepsilon,\lambda,\delta}; G) \leq \pi L_G(g) \log(1/\varepsilon)(1 + C\lambda + C\delta) + C'_{\lambda,\delta},$$

which yields the desired conclusion (A.1) since $\lambda < \delta$ are arbitrarily small.

Appendix B. A variant of the density result of T. Rivière

We use the same notation as in Appendix A for $\mathscr{R}, \mathscr{R}_0$ and $\mathscr{R}_1$. Recall that $\mathscr{R}_0$ is dense in $H^{1/2}(\Omega; S^1)$; *see* Rivière [38], quoted as Lemma 11, and *see* Remark 5.1 for a proof. This appendix is devoted to the following improvement:

Lemma **B.1.** — *The class* $\mathscr{R}_1$ *is dense in* $H^{1/2}(\Omega; S^1)$.

Proof. — Given $g \in H^{1/2}(\Omega; S^1)$ and $\varepsilon > 0$ we first use the density of $\mathscr{R}_0$ to construct a map $h \in \mathscr{R}_0$ such that $\|h - g\|_{H^{1/2}} < \varepsilon$.

Next, write, as usual, the singular set Σ of h as

$$\Sigma = \{P_1, P_2, ..., P_k, N_1, N_2, ..., N_k\}.$$

For every $\sigma \in \Omega$, let $T_\sigma(\Omega)$ denote the tangent plane to Ω at σ; we orient it using the outward normal $n(\sigma)$ to G. Let P_Ω denote the projection onto Ω defined in a tubular neighborhood of Ω in $\mathbf{R}^3$.

For each $i = 1, 2, ..., k$, *fix* two smooth maps:

$$\gamma_i^+ : \{\xi \in T_{P_i}(\Omega); |\xi| = 1\} \to S^1,$$
$$\gamma_i^- : \{\xi \in T_{N_i}(\Omega); |\xi| = 1\} \to S^1,$$

such that

$$\deg(\gamma_i^+) = +1 \text{ and } \deg(\gamma_i^-) = -1. \tag{B.1}$$

The conclusion of Lemma B.1 is an immediate consequence of the following more general:

Claim. — With h as above, there is a sequence (h_n) in $H^{1/2}(\Omega; S^1)$ such that:

$$h_n \to h \text{ in } H^{1/2} \tag{B.2}$$

$$h_n \in C^\infty(\Omega \setminus \Sigma; S^1), \quad \forall n, \tag{B.3}$$

$$h_n \in W^{1,p}(\Omega \setminus \Sigma; S^1), \quad \forall n, \quad \forall p < 2, \tag{B.4}$$

$$|\nabla h_n(x)| \leq C_n / \operatorname{dist}(x, \Sigma), \quad \forall n, \quad \forall x \in \Omega \setminus \Sigma, \tag{B.5}$$

for all $0 < t < t_0$ (sufficiently small, depending only on Ω) and all $i = 1, 2, ...k$, we have:

$$|h_n(P_\Omega(P_i + t\xi)) - \gamma_i^+(\xi)| \leq C_n t, \quad \forall n, \forall \xi \in T_{P_i}(\Omega), |\xi| = 1, \tag{B.6}$$

$$|h_n(P_\Omega(N_i + t\xi)) - \gamma_i^-(\xi)| \leq C_n t, \quad \forall n, \forall \xi \in T_{N_i}(\Omega), |\xi| = 1. \tag{B.7}$$

Proof of the Claim. — Fix an *arbitrary* function $k \in C^\infty(\Omega \setminus \Sigma; S^1) \cap W^{1,p}(\Omega, S^1)$, $\forall p < 2$ satisfying

$$|\nabla k(x)| \leq C \operatorname{dist}(x, \Sigma), \quad \forall x \in \Omega \setminus \Sigma, \tag{B.8}$$

$$|k(P_\Omega(P_i + t\xi)) - \gamma_i^+(\xi)| \leq Ct, \tag{B.9}$$

$$|k(P_\Omega(N_i + t\xi)) - \gamma_i^-(\xi)| \leq Ct, \tag{B.10}$$

for all t, i, ξ as in (B.6)–(B.7).

The existence of k is proved as in Appendix A. First we define it on $\partial B_1 \times [0, T]$ using the parameter t to homotopy γ_i^+ to the complex conjugate of γ_i^-. We then extend it to $B_1 \times [0, T]$ by homogeneity of degree 0 and transfer it to a "tube-like" region U_i in G connecting P_i to N_i. Finally, we extend these functions smoothly to $G \setminus U_i$, take their complex product, and restrict it to Ω.

To complete the proof of the claim, note that $T(h) = T(k) = 2\pi \sum_{i=1}^k (\delta_{P_i} - \delta_{N_i})$. Thus $T(h\bar{k}) = 0$ and, by Theorem 2, there exists a sequence $r_n \in C^\infty(\Omega; S^1)$ such that $r_n \to h\bar{k}$ in $H^{1/2}$. Using the fact that points have zero H^1-capacity in $2 - d$ (and thus zero $H^{1/2}$ - capacity), we may also assume that $r_n(P_i) = r_n(N_i) = 1, \forall n, \forall i$. Clearly, the sequence $h_n = kr_n$ has all the desired properties (B.2)–(B.7).

Lemma B.1 is obtained by choosing, in the claim, as γ_i^+ and γ_i^- any isometries from $T_{P_i}(\Omega)$ and $T_{N_i}(\Omega)$ onto $\mathbf{R}^2$.

Appendix C: Almost Z-valued functions

The purpose of this section is to prove the following fact used earlier in Section 8.

Lemma **C.1.** — *Assume* $\varphi \in H^{1/2}((0, 1) \times (0, 1))$ *and* $\{Q_\alpha\}$ *a collection of squares in* $(0, 1)^2$ *such that*

$$\|\varphi\|_{L^{4/3}} \leq C \tag{C.1}$$

$$\|e^{\imath\varphi} - 1\|_{L^1([0,1]^2 \setminus \cup Q_\alpha)} \leq \varepsilon \tag{C.2}$$

$$|\varphi|_{H^{1/2}} \leq \delta(\log(1/\varepsilon))^{1/2} \tag{C.3}$$

$$\sum_\alpha \sigma_\alpha \leq \delta, \tag{C.4}$$

where $\varepsilon < \delta \ll 1$ *and* σ_α *denotes the size of* Q_α.

Then there is some $a \in \mathbf{Z}$ *such that*

$$\|\varphi - 2\pi a\|_{L^1} \leq C\delta^{1/8}. \tag{C.5}$$

The proof will rely on the following inequality (*see* also [15] and [35] for related results).

Lemma **C.2.** — *Let* $Q = (0,1)^2, f \in L^1(Q)$. *Then for all* $0 < \rho < \rho_0$, ρ_0 *sufficiently small,*

$$\left\| f - \int f \right\|_{L^1} \leq C|\log \rho|^{-1} \iint_{Q\times Q} \frac{|f(x) - f(y)|}{|x-y|(|x-y|+\rho)^2} dxdy \tag{C.6}$$

with C *some constant.*

Proof of Lemma C.1. — It follows from (C2) that we may write Q as a *disjoint* union

$$Q = \bigcup Q_\alpha \cup Z_0 \cup \bigcup_{j\in\mathbf{Z}} A_j.$$

where

$$A_j \subset [|\varphi - 2\pi j| < \varepsilon^{1/8}] \tag{C.7}$$

$$|Z_0| < \varepsilon^{3/4}. \tag{C.8}$$

Apply Lemma C.2 to $f = \chi_{A_j}$ with $\rho = \varepsilon^{1/20}$. Hence, denoting $Z = Z_0 \cup \bigcup_\alpha Q_\alpha$,

$$\begin{aligned}
|A_j|(1-|A_j|) &\leq C|\log\varepsilon|^{-1} \iint_{A_j\times(Q\setminus A_j)} |x-y|^{-1}(|x-y|+\rho)^{-2} \\
&\leq C|\log\varepsilon|^{-1} \sum_{k\neq j} \iint_{A_j\times A_k} |x-y|^{-3} + C|\log\varepsilon|^{-1} \\
&\times \iint_{A_j\times Z} |x-y|^{-1}(|x-y|+\rho)^{-2} \\
&\leq C|\log\varepsilon|^{-1} \iint_{\substack{A_j\times\cup A_k\\ k\neq j}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^3} + C|\log\varepsilon|^{-1} \\
&\times \iint_{A_j\times Z} |x-y|^{-1}(|x-y|+\rho)^{-2}.
\end{aligned}$$

Summation over j gives

$$\begin{aligned}\sum_j |A_j|(1-|A_j|) &\leq C|\log\varepsilon|^{-1}\,\|\varphi\|^2_{H^{1/2}}\\ &\quad + C|\log\varepsilon|^{-1}\iint_{Z\times(Q\setminus Z)} |x-y|^{-1}(|x-y|+\rho)^{-2}\\ &\overset{\text{by (C.3)}}{\leq} C\delta^2 + C|\log\varepsilon|^{-1}\\ &\quad \times\Big[\sum_\alpha \iint_{Q_\alpha\times(Q\setminus Q_\alpha)} |x-y|^{-1}(|x-y|+\rho)^{-2}\Big]\\ &\quad + C|Z_0|.\,\varepsilon^{-\frac{1}{10}}.\end{aligned} \tag{C.9}$$

For fixed α, estimate

$$\iint_{Q_\alpha\times(Q\setminus Q_\alpha)} |x-y|^{-1}(|x-y|+\rho)^{-2}. \tag{C.10}$$

Since for fixed $x\in Q_\alpha$, $|x-y| > \operatorname{dist}(x,\partial Q_\alpha)$, we get easily

$$\text{(C.10)} \leq C\int_{Q_\alpha}[\operatorname{dist}(x,\partial Q_\alpha)+\rho]^{-1}dx < C|\log\varepsilon|\sigma_\alpha$$

with σ_α the size of Q_α.

Substitute in (C.9) and use (C.4), (C.8) to bound

$$\sum_j |A_j|(1-|A_j|) \leq C\delta^2 + C\sum\sigma_\alpha + \varepsilon^{\frac{3}{4}-\frac{1}{10}} \leq C\delta + \varepsilon^{3/5}. \tag{C.11}$$

Take j_0 with $|A_{j_0}| = \max|A_j|$. Thus $|A_j| \leq \frac{1}{2}$ for $j\neq j_0$ and by (C.11)

$$\sum_{j\neq j_0}|A_j| \leq C(\delta+\varepsilon^{3/5}). \tag{C.12}$$

Taking $a = j_0$, finally estimate using (C.1), (C.7)

$$\begin{aligned}\|\varphi-2\pi a\|_1 &\leq \|\varphi-2\pi j_0\|_{L^1(A_{j_0})} + \|\varphi\|_{L^1(Q\setminus A_{j_0})} + 2\pi|a|\;|Q\setminus A_{j_0}|\\ &\leq \varepsilon^{\frac{1}{8}} + C|Q\setminus A_{j_0}|^{\frac{1}{4}} + 2\pi|a|\;|Q\setminus A_{j_0}|\end{aligned}$$

where, by (C.4), (C.8), (C.12)

$$\begin{aligned}|Q\setminus A_{j_0}| &\leq \sum|Q_\alpha| + |Z_0| + \sum_{j\neq j_0}|A_j| \leq \sum\sigma_\alpha^2 + \varepsilon^{3/4} + C(\delta+\varepsilon^{3/5})\\ &\leq C(\delta+\varepsilon^{3/5}).\end{aligned}$$

Hence

$$\|\varphi - 2\pi a\|_1 \le C(\varepsilon^{1/8} + \delta^{1/4}) + C|a|(\delta + \varepsilon^{3/5})$$

implying

$$2\pi|a| \le \|\varphi\|_1 + 1 + |a|$$

so that

$$|a| \le C \text{ and } \|\varphi - 2\pi a\|_1 \le C(\delta^{1/4} + \varepsilon^{1/8}) \le C\delta^{1/8}$$

which is (C.5).

Proof of Lemma C.2. — We will derive the inequality by contradiction, using Theorem 4 in [14]. Let thus (f_n) be a sequence in $L^1(Q)$ and $(\varepsilon_n) \downarrow 0$ such that

(**C.13**) $$|\log \varepsilon_n|^{-1} \iint\limits_{Q\times Q} \frac{|f_n(x) - f_n(y)|}{|x-y|(|x-y| + \varepsilon_n)^2} dxdy \le 1$$

and

(**C.14**) $$\|f_n - \int f_n\|_{L^1} \to \infty.$$

Denote by ρ_n the radial mollifier on $\mathbf{R}^2$

(**C.15**) $$\rho_n(x) = c_n|\log \varepsilon_n|^{-1}(|x| + \varepsilon_n)^{-2}$$

with c_n such that $\int \rho_n = 1$ (hence $c_n \sim 1$). Applying Theorem 4 from [14], with $p = 1$, it follows that (f_n) is relatively compact in $L^1(Q)$, contradicting (C.14). This proves (C.6).

Appendix D. Sobolev imbeddings for BV

It is well-known that, if $p > 1$ and $0 < s < 1$, then

$$W^{1,p}(\Omega) \subset W^{s,q}(\Omega), \;\; \Omega \subset \mathbf{R}^d$$

with

$$\frac{1}{q} = \frac{1}{p} - \frac{(1-s)}{d}.$$

This imbedding fails for $p = 1$ and $d = 1$, i.e., $W^{1,1}$ is *not* contained in $W^{1/q,q}$ for $q > 1$. Surprisingly, the imbedding holds when $p = 1$ and $d \ge 2$.

Lemma **D.1.** — *Assume* $d \geq 2$ *and* $0 < s < 1$. *Then*

$$\mathrm{BV}(\mathbf{R}^d) \subset \mathrm{W}^{s,p}(\mathbf{R}^d)$$

with

$$\frac{1}{p} = 1 - \frac{1-s}{d}. \tag{D.1}$$

When $d = 2$, this result is an immediate consequence of an interpolation result of Cohen, Dahmen, Daubechies and DeVore [23]. It also seems to be contained in an earlier work of V. A. Solonnikov [44] although the condition $d \geq 2$ does not appear in his paper. We thank V. Maz'ya and T. Shaposhnikova for calling our attention to the paper of Solonnikov and for confirming that the assumption $d \geq 2$ is indeed used there implicitly; they have also devised another proof of Solonnikov's inequality (personal communication).

Our proof relies on the following one-dimensional elementary inequality:

Lemma **D.2.** — *Let* $1 < p < \infty$ *and* $0 < s < 1/p$. *Then, for every* $f \in \mathrm{C}_0^\infty(\mathbf{R})$,

$$|f|^p_{\mathrm{W}^{s,p}(\mathbf{R})} \leq \mathrm{C}\|f\|^{p(1-sp)}_{\mathrm{L}^p(\mathbf{R})}\|f'\|^{sp^2}_{\mathrm{L}^1(\mathbf{R})}, \tag{D.2}$$

where C *depends only on* p *and* s.

Here, $|\ |_{\mathrm{W}^{s,p}(\mathbf{R})}$ denotes the canonical semi-norm on $\mathrm{W}^{s,p}(\mathbf{R})$, i.e.,

$$|f|^p_{\mathrm{W}^{s,p}(\mathbf{R})} = \int_{\mathbf{R}} dx \int_0^\infty \frac{|f(x+h) - f(x)|^p}{h^{1+sp}} dh.$$

Proof. — Write, for $\lambda > 0$,

$$\begin{aligned}
|f|^p_{\mathrm{W}^{s,p}} &= \int_{\mathbf{R}} dx \int_0^\lambda \cdots dh + \int_{\mathbf{R}} dx \int_\lambda^\infty \cdots dh \\
&\leq 2^{p-1}\|f\|^{p-1}_{\mathrm{L}^\infty}\|f'\|_{\mathrm{L}^1}\frac{\lambda^{1-sp}}{1-sp} + 2^{p-1}\|f\|^p_{\mathrm{L}^p}\frac{\lambda^{-sp}}{sp} \\
&\leq 2^{p-1}\left(\|f'\|^p_{\mathrm{L}^1}\frac{\lambda^{1-sp}}{1-sp} + \|f\|^p_{\mathrm{L}^p}\frac{\lambda^{-sp}}{sp}\right),
\end{aligned}$$

since $sp < 1$. Minimizing in λ yields (D.2) with $\mathrm{C} = 2^{p-1}/sp(1-sp)$.

Proof of Lemma D.1. — Let $u \in C_0^\infty(\mathbf{R}^d)$. We will use the following equivalent norm on $W^{s,p}$ (*see* e.g. Adams [1], Lemma 7.44)

$$\|u\|_{W^{s,p}}^p \sim \|u\|_{L^p}^p + \sum_{j=1}^d \int_{\mathbf{R}^d} dx \int_0^\infty \frac{|u(x+he_j)-u(x)|^p}{h^{1+sp}} dh. \tag{D.3}$$

Note that $BV \subset L^1 \cap L^{d/(d-1)}$ and thus we may estimate (via Hölder)

$$\|u\|_{L^p} \le C\|u\|_{BV},$$

since

$$\frac{1}{p} = 1 - \frac{(1-s)}{d} = \frac{s}{1} + \frac{1-s}{d/(d-1)}. \tag{D.4}$$

We now turn to the second term in (D.3); without loss of generality we may take $j = 1$. We apply Lemma D.1 to the function

$$f(\cdot) = u(\cdot, x_2, x_3, \dots, x_d)$$

(note that, by (D.4), $sp < 1$) and we obtain

$$\begin{aligned} &\int_{\mathbf{R}} dx_1 \int_0^\infty \frac{|u(x_1+h, x_2, \dots, x_d) - u(x_1, x_2, \dots, x_d)|^p}{h^{1+sp}} dh \\ &\le C\|f\|_{L^p(\mathbf{R})}^{p(1-sp)} \|f'\|_{L^1(\mathbf{R})}^{sp^2} \le C\|f\|_{L^1}^{sp(1-sp)} \|f\|_{L^{d/(d-1)}}^{(1-s)p(1-sp)} \|f'\|_{L^1}^{sp^2}. \end{aligned} \tag{D.5}$$

On the one hand, we have

$$\int_{\mathbf{R}^{d-1}} \|f'\|_{L^1(\mathbf{R})} dx_2 dx_3 \dots dx_d \le \int_{\mathbf{R}^d} |\nabla u| dx. \tag{D.6}$$

On the other hand, the imbedding $BV \subset L^{d/(d-1)}$ gives, with $q = d/(d-1)$,

$$\int_{\mathbf{R}^{d-1}} \|f\|_{L^q(\mathbf{R})}^q dx_2 dx_3 \dots dx_d = \|u\|_{L^q(\mathbf{R}^d)}^q \le C\left(\int_{\mathbf{R}^d} |\nabla u| dx\right)^q. \tag{D.7}$$

Finally we claim that

$$\int_{\mathbf{R}^{d-1}} \|f\|_{L^1(\mathbf{R})}^{(d-1)/(d-2)} dx_2 dx_3 \dots dx_d \le C\left(\int_{\mathbf{R}^d} |\nabla u| dx\right)^{(d-1)/(d-2)}; \tag{D.8}$$

when $d = 2$, inequality (D.8) reads

$$\|f\|_{L^\infty_{x_2}(L^1_{x_1})} \leq \int_{\mathbf{R}^2} |\nabla u|.$$

To prove (D.8) we use once more the imbedding $BV \subset L^r$, but this time in $\mathbf{R}^{d-1}$, with $r = (d-1)/(d-2)$, and we obtain

$$\text{(D.9)} \qquad \|f(x_1, \cdot)\|_{L^r(\mathbf{R}^{d-1})} \leq C \int_{\mathbf{R}^{d-1}} |\nabla u(x_1, \cdot)| dx_2 dx_3 \dots dx_d.$$

Next, we have

$$\begin{aligned} \|f\|_{L^r(\mathbf{R}^{d-1}; L^1(\mathbf{R}))} &= \left\| \int_{\mathbf{R}} |f(x_1, \cdot)| dx_1 \right\|_{L^r(\mathbf{R}^{d-1})} \\ &\leq \int_{\mathbf{R}} \|f(x_1, \cdot)\|_{L^r(\mathbf{R}^{d-1})} dx_1 \qquad \text{by the triangle inequality} \\ &\leq C \int_{\mathbf{R}^d} |\nabla u(x)| dx \qquad \text{by (D.9).} \end{aligned}$$

Finally, we return to (D.5), integrate in $dx_2 dx_3 \dots dx_d$, and apply Hölder with exponents P, Q, R such that

$$\begin{aligned} Psp(1-sp) &= (d-1)/(d-2), \\ Q(1-s)p(1-sp) &= d/(d-1), \\ Rsp^2 \qquad &= 1. \end{aligned}$$

[A straightforward computation shows that $\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} = 1$]. From (D.8), (D.7) and (D.6) we deduce that

$$|u|^p_{W^{s,p}(\mathbf{R}^d)} \leq C \left(\int_{\mathbf{R}^d} |\nabla u| dx \right)^p.$$

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Added in proof:

1) After our work was completed some of our results were generalized to higher dimensions in [ABO].

2) F. Bethuel, G. Orlandi and D. Smets have solved our Open Problem 3 (and thereby also the first part of Open Problem 2) in Section 10; see [BOS1] and [BOS2].

3) J. Van Schaftingen [VS] has given an elementary proof of our Proposition 4, which extends easily to higher dimensions. His proof follows the same strategy as ours, except that he uses the Morrey-Sobolev imbedding in place of a Littlewood Paley decomposition.

4) An alternative approach to Proposition 4 is to use a new estimate for the div-curl system (see [BB]), namely

$$\|u\|_{L^{3/2}} \leq C\|\text{curl } u\|_{L^1}, \forall u \text{ with div } u = 0.$$

5) An interesting extension of Lemma C.2 may be found in [P].

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