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Eigenvalue asymptotics for a class of md-elliptic ψdo's on manifolds with cylindrical exits

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Abstract. We obtain a Weyl formula for a class of positive md-elliptic operators on manifolds with finitely many cylindrical exits. Weyl formula is deduced by a classical Tauberian theorem through the asymptotic expansion at $t = 0$ of the trace of the heat parametrix. The constant of the leading term is expressed invariantly by means of the usual principal symbol and exit symbols.

1. Introduction

This paper is devoted to the asymptotic behaviour for large λ of the counting function

$$\mathcal{N}(\lambda, A) := \sum_{\lambda_j \leq \lambda} 1 \tag{1.1}$$

associated with certain L^2-unbounded operators A on a class of non-compact Riemannian manifolds M, nevertheless endowed with a spectrum made of real eigenvalues $\{\lambda_j\}$ of finite multiplicity, clustering at infinity.

Spectral asymptotics has an old tradition. A wide literature regarding the case of compact manifolds is available, starting from the pioneering Weyl's, Carleman's and Gärding's works ([23], [3], [7]) about the leading term of the expansion of (1.1). Since then, most efforts have been directed towards both the improvement of the remainder estimates and a deeper understanding of the interplay between geometrical and analytical aspects; see for instance [10], [1], [4], [6].

On the other hand, a great number of results about eigenvalue asymptotics for Schrödinger-type operators on $\mathbb{R}^n$ has been accurately obtained by using different techniques (variational principles, approximate spectral projector method, ...), see [2], [11], [8], [13], [22] and the references therein.

A remarkable point in the study of operators on unbounded domains is the necessity of specifying assumptions on the growth of the corresponding symbols $a(x, \xi)$ with respect to the x variable as well. This allows us to gain compact embeddings of weighted Sobolev spaces into $L^2(M)$ and, in the end, the discreteness of the spectrum.

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The prominent local model on $\mathbb{R}^n$ that highlights the peculiar features of the operators we are interested in is given by

$$A(x, D_x) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha, \qquad |x| > \delta > 0, \tag{1.2}$$

with smooth coefficients satisfying the following growth condition at infinity

$$a_\alpha(x) = |x|^\mu c_\alpha\left(\frac{x}{|x|}\right) + r_\alpha(x) \qquad \forall|\alpha| \geq 0,$$

for some $\mu \in]0, +\infty[$, $c_\alpha \in C^\infty(\mathbb{S}^{n-1})$ and $\partial_x^\beta r_\alpha(x) = \mathcal{O}(\langle x\rangle^{\mu-|\beta|-1})$ $\forall\beta \in \mathbb{N}^n$. We also assume that the following global (or md-) ellipticity conditions hold:

1) $a_m(x, \xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0 \qquad \forall|x| > \delta, \forall\xi \in \mathbb{R}^n \setminus \{0\}$;
2) $a_\infty(\omega, \xi) := \sum_{|\alpha|\leq m} c_\alpha(\omega)\xi^\alpha \neq 0 \qquad \forall\omega \in \mathbb{S}^{n-1}, \forall\xi \in \mathbb{R}^n$;
3) $a_\sharp(\omega, \xi) := \sum_{|\alpha|=m} c_\alpha(\omega)\xi^\alpha \neq 0 \qquad \forall\omega \in \mathbb{S}^{n-1}, \forall\xi \in \mathbb{R}^n \setminus \{0\}$.

The prototype we keep in mind is $A(x, D_x) = |x|^\mu(-\Delta + 1)$ ($-\Delta$ is the Laplacian) and the kind of manifolds M we extend this model to are non-compact in the following sense: they can be obtained by removing finitely many open disks, D_k, with disjoint closures, from a compact n-manifold M_0 and gluing along each boundary component an infinitely extended cylindrical handle (the *exit*). The differential structure on M is induced by M_0 out of the disks and defined by means of the gluing diffeomorphisms f on the infinite exits in such a way that M becomes a SG-manifold [17]. We will be more precise at the beginning of Section 3. It is to be pointed out that, in our framework, only homogeneous transition maps of degree one are allowed; moreover, the Riemannian metric on M is the pullback via f of the standard Euclidean one on the cylindrical exit.

We are therefore allowed to bring up the general SG-calculus for operators on M that admit local representation, on each exit chart, having weighted symbols (see Definition 2.3), according to the theory developed in [5] and [17]. In Sections 2 and 3 of this paper we deal with the class $ECL^{m,\mu}(M)$ of md-elliptic pseudodifferential operators A on M of order $(m, \mu) \in \mathbb{R}^2_+$ for which we claim and prove the existence of both an *"internal" principal symbol* a_m and *"exit" principal symbols* a_∞ and $a_\sharp$, globally defined as

$$a_m \in C^\infty(T^*M \setminus 0), \qquad a_\infty \in C^\infty(T^*_X M), \qquad a_\sharp \in C^\infty(T^*_X M \setminus 0),$$

where $X = \partial(M_0 \setminus \cup D_k)$ and $T^*_X M$ is the restriction to X of the cotangent bundle T^*M, endowed with a natural contact structure. This operator class has been conisedred by Melrose for example in [14].

Incidentally, md-ellipticity in this class is yielded by algebraic conditions involving the triplet, $\{a_m, a_\infty, a_\sharp\}$, analogously to the previous Conditions 1), 2), 3). A similar class of symbols has been considered by Schulze in [19].

In Section 4 the SG-machinery is used to carry out a detailed construction of the heat parametrix, $U(t)$, associated with A. As long as $t > 0$, $U(t)$ is smoothing and trace-class. Thereafter, by Karamata's Tauberian Theorem, the expansion of its trace for small times leads, in Section 5, to the Weyl formula (i.e. to the expression of the asymptotic behaviour of the counting function for large λ).

The main focus of this work is to detect the constant C of the leading term in Weyl formula for positive self-adjoint $A \in ECL^{m,\mu}(M)$ (Corollary 5.5). We throw light on the different asymptotic behaviour of $\mathcal{N}(\lambda, A)$ according to the ratio between m and μ. A precise expression of C is given, invariantly, in terms of a_m, a_∞ and $a_\sharp$, respectively when $\mu > m$, $\mu < m$ and $\mu = m$.

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2. Weighted symbols

In this section we develop the local model for the calculus on a manifold with finitely many cylindrical exits. Since each exit is not reducible to the whole $\mathbb{R}^n$, we are naturally led to consider unbounded open sets of the form $U_\delta = \{x \in \mathbb{R}^n : |x| > \delta\}$, $\delta > 0$ and define suitable function spaces which play the same role that do Schwartz spaces of rapidly decreasing smooth functions in the standard global theory on $\mathbb{R}^n$. We recall the bare essentials from the calculus of weighted symbols, leaving out most of the proofs, for which we refer to [5], [15], [17]. Above all, the class $ECL^{m,\mu}(U_\delta)$ is introduced in detail and its invariance with respect to suitable diffeomorphisms is shown.

Definition 2.1. *By $\mathcal{S}_0(U_\delta)$ we shall denote the space of all functions $f \in \mathcal{S}(\mathbb{R}^n)$ having support strictly contained in U_δ, i.e.*

$$\mathcal{S}_0(U_\delta) := \bigcup_{\delta' \searrow \delta} \{f \in \mathcal{S}(\mathbb{R}^n) : supp\ f \subseteq \overline{U}_{\delta'}\}.$$

The topology of $\mathcal{S}_0(U_\delta)$ is the inductive limit one. It can be made clearer through the notion of convergence, that is, $f_n \to f$ in $\mathcal{S}_0(U_\delta)$ if and only if $f_n \to f$ in $\mathcal{S}(\mathbb{R}^n)$ and $(1-\phi) f_n \to (1-\phi) f$ in $C_0^\infty(U_\delta)$ for every $\phi \in C^\infty(\mathbb{R}^n)$, supp $\phi \subset U_{\delta'}$ and $\phi = 1$ on $U_{\delta''}$ for some $\delta'' > \delta' > \delta$.

Definition 2.2. *By $\mathcal{S}(U_\delta)$ we shall denote the space of all functions $f \in C^\infty(U_\delta)$ such that for every $\alpha, \beta \in \mathbb{N}^n$ and for every $\delta' > \delta$ the following estimate holds:*

$$\sup_{x \in U_{\delta'}} |x^\alpha D^\beta f(x)| < \infty.$$

$\mathcal{S}(U_\delta)$ is a Fréchet space with the seminorms implicit in the definition. Observe that if $f \in \mathcal{S}(U_\delta)$, then $\phi f \in \mathcal{S}_0(U_\delta)$ for every $\phi \in C^\infty(\mathbb{R}^n)$ with supp $\phi \subset U_\delta$, ϕ and all its derivatives "polinomially growing at infinity" (with the obvious meaning), and for short, "p.g.i.".

The understanding of dual spaces $\mathcal{S}'_0(U_\delta)$ and $\mathcal{S}'(U_\delta)$ is carried out in the next theorems. Notice that $C_0^\infty(U_\delta) \subset \mathcal{S}_0(U_\delta) \subset \mathcal{S}(U_\delta)$, thereby yielding $S'(U_\delta) \subset S'_0(U_\delta) \subset \mathcal{D}'(U_\delta)$.

Theorem 2.1. *The dual space $\mathcal{S}_0'(U_\delta)$ can be identified with the space*

$$E_\delta := \left\{u \in \mathcal{D}'(U_\delta) : \forall\phi \in C^\infty(\mathbb{R}^n),\ \phi \text{ p.g.i., } \operatorname{supp}\phi \subset U_\delta \Rightarrow \phi u \in \mathcal{S}'(\mathbb{R}^n)\right\}.$$

Proof. The inclusion $\mathcal{S}_0'(U_\delta) \subseteq E_\delta$ is easily verified; in fact, if $u \in \mathcal{S}_0'(U_\delta)$ and ϕ is as above, we can set the duality $\langle \phi u, f\rangle_{\mathcal{S}',\mathcal{S}} := \langle u, \phi f\rangle_{\mathcal{S}_0',\mathcal{S}_0}$ for every $f \in \mathcal{S}(\mathbb{R}^n)$.

Conversely if $u \in E_\delta$, we can choose ϕ such that $\phi = 1$ for $|x|$ large enough and define

$$\langle u_\phi, f\rangle_{\mathcal{S}_0',\mathcal{S}_0} := \langle \phi u, f\rangle_{\mathcal{S}',\mathcal{S}} + \langle u, (1-\phi)f\rangle_{\mathcal{D}',\mathcal{D}}, \qquad \text{for every } f \in \mathcal{S}_0(U_\delta).$$

The continuity of the functional u_ϕ on $\mathcal{S}_0(U_\delta)$ is straightforward. Let us now observe that if ψ is another C^∞ function on $\mathbb{R}^n$ which has the same properties of ϕ, we have (notice that $1-\phi, 1-\psi \in C_0^\infty(\mathbb{R}^n)$)

$$\langle \phi u, f\rangle_{\mathcal{S}',\mathcal{S}} = \langle u_\psi, \phi f\rangle_{\mathcal{S}_0',\mathcal{S}_0},$$
$$\langle u, (1-\phi)f\rangle_{\mathcal{D}',\mathcal{D}} = \langle u_\psi, (1-\phi)f\rangle_{\mathcal{S}_0',\mathcal{S}_0}.$$

This means that the distribution u_ϕ does not depend on ϕ. □

Theorem 2.2. $\mathcal{S}'(U_\delta) = \{u \in \mathcal{S}'(\mathbb{R}^n) : $ *supp* $u \subset U_\delta\}$.

Proof. Let u be in $\mathcal{S}'(U_\delta)$. Then for every $\beta \in C_0^\infty(\mathbb{R}^n)$ with $\beta = 1$ in $B_{\delta'}(0)$ (we denote by $B_r(0)$ the Euclidean ball of radius r and centre 0), $\beta = 0$ in $U_{\delta''}$, $\delta < \delta' < \delta''$, we can write $u = \beta u + (1-\beta)u$, where $\beta u \in \mathcal{E}'(B_{\delta''}(0) \setminus \overline{B}_\delta(0))$, while $(1-\beta)u \in \mathcal{S}'(\mathbb{R}^n)$ with supp $(1-\beta)u \subset U_\delta$, so $\mathcal{S}'(U_\delta) \subseteq \{u \in \mathcal{S}'(\mathbb{R}^n) :$ supp $u \subset U_\delta\}$.

On the other hand, let us take $u \in \mathcal{S}'(\mathbb{R}^n)$, with supp $u \subset U_\delta$ and ϕ p.g.i., supp $\phi \subset U_\delta$, $\phi = 1$ in a neighbourhood of supp u. Thus we can define $\langle \tilde{u}, f\rangle := \langle u, \phi f\rangle_{\mathcal{S}',\mathcal{S}}$ for every $f \in \mathcal{S}(U_\delta)$. It is readily seen that $\tilde{u}$ is a continous linear map from $\mathcal{S}(U_\delta)$ to $\mathbb{C}$ and it does not depend on ϕ having the properties stated above. □

Now we run over the basic facts from the so-called SG-calculus, going through weighted symbols, operators and kernels, as far as a Sobolev continuity result. In what follows $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$.

Definition 2.3. *(i) For real m, μ_1, μ_2, let $S^{m,\mu_1,\mu_2}(U_\delta \times U_\delta)$ denote the class of all complex-valued amplitudes $p \in C^\infty(U_\delta \times U_\delta \times \mathbb{R}^n)$ such that for all $\alpha, \beta, \gamma \in \mathbb{N}^n$ and for every $\delta' > \delta$ there exists a positive constant $C_{\alpha,\beta,\gamma}(\delta')$ such that*

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma p(x,y,\xi)\right| \le C_{\alpha,\beta,\gamma} \langle\xi\rangle^{m-|\alpha|}\langle x\rangle^{\mu_1-\beta}\langle y\rangle^{\mu_2-|\gamma|}, \qquad (x,y,\xi) \in U_{\delta'} \times U_{\delta'} \times \mathbb{R}^n.$$

(ii) For real m and μ, let $S^{m,\mu}(U_\delta)$ be the linear subspace of $S^{m,\mu,0}(U_\delta \times U_\delta)$ consisting of all amplitudes which are independent of y. They will be referred to as symbols.

$S^{m,\mu}(U_\delta)$ is a Fréchet space for every real m, μ and the increasing filtration given by the embeddings $S^{m',\mu'}(U_\delta) \hookrightarrow S^{m,\mu}(U_\delta)$ $\forall m' \leq m$, $\mu' \leq \mu$ is continuous.

For any $p \in S^{m,\mu_1,\mu_2}(U_\delta \times U_\delta)$ we define a linear operator $\mathrm{Op}(p)$ as follows: $\forall u \in \mathcal{S}_0(U_\delta)$

$$\mathrm{Op}(p)u(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p(x, y, \xi)u(y)dy\,đ\xi,$$

where the double integral is understood as an oscillatory integral and $đ\xi = (2\pi)^{-n}d\xi$.

It can be shown that $\mathrm{Op}(p)$, as a continous linear operator from $\mathcal{S}_0(U_\delta)$ to $\mathcal{S}_0'(U_\delta)$, has a Schwartz distributional kernel $K \in \mathcal{S}_0'(U_\delta \times U_\delta)$.

Theorem 2.3 (Continuity). *Let $p \in S^{m,\mu_1,\mu_2}(U_\delta \times U_\delta)$. Then we have the continuous map*

$$\mathrm{Op}(p) : \mathcal{S}_0(U_\delta) \to \mathcal{S}(U_\delta).$$

Proof. We have to prove that for every $\delta' > \delta$ and $\alpha, \beta \in \mathbb{N}^n$ there exist $C_{\alpha,\beta}(\delta')$ and N such that

$$\sup_{x\in\bar{U}_{\delta'}} \left|x^\alpha \partial_x^\beta(\mathrm{Op}(p)(f)(x))\right| \leq C_{\alpha,\beta}(\delta') \sum_{|\alpha'|,|\beta'|\leq N} \sup_x \left|x^{\alpha'}\partial_x^{\beta'} f(x)\right|.$$

We proceed in the standard way. Let L be the differential operator defined as

$$L = \frac{1}{1+|\xi|^2}(1 + \langle \xi, D_y\rangle).$$

Then, if k is large enough and x lies in U_δ, we can write, after an integration by parts,

$$x^\alpha\partial_x^\beta \mathrm{Op}(p)(f)(x) = \int x^\alpha \partial_x^\beta \left(e^{i(x-y)\cdot\xi}\,({}^tL)^k\,[p(x, y, \xi) f(y)]\right) dy\,đ\xi.$$

If we split the integration domain, we are led to

$$\left|x^\alpha\partial_x^\beta \mathrm{Op}(p)(f)(x)\right| \leq |I_1| + |I_2|,$$

where

$$I_1 = \int_{|x-y|\leq|x|/2} x^\alpha \partial_x^\beta \left(e^{i(x-y)\cdot\xi}\,({}^tL)^k[p(x, y, \xi) f(y)]\right) dy\,đ\xi$$

and

$$I_2 = \int_{|x-y|\geq|x|/2} x^\alpha \partial_x^\beta \left(e^{i(x-y)\cdot\xi}\,({}^tL)^k[p(x, y, \xi) f(y)]\right) dy\,đ\xi.$$

Now, observe that $|I_1|$ is bounded by a linear combination of terms of the form

$$\int_{|x-y|\leq|x|/2} |x^\alpha|\langle\xi\rangle^{-k+m+|\beta|}\langle x\rangle^{\mu_1}\langle y\rangle^{\mu_2} \sum_{|\gamma|\leq k} \left|\partial_y^\gamma f(y)\right| dy\,đ\xi.$$

Choosing k large enough and taking into account that when $|x-y| \leq |x|/2$ we have $|x|^\alpha = O(|y|^\alpha)$, we get

$$|I_1| \leq C \sum_{|\alpha'|,|\beta'|<N} \langle y\rangle^{|\alpha'|} \left|\partial_y^{\beta'} f(y)\right|.$$

The same kind of estimate can be found for $|I_2|$ by approximating it with a sequence I_2^ϵ, whose terms can be estimated (uniformly in ϵ) as we did with I_1. $\square$

Corollary 2.4. Op*(p) can be extended to a continuous linear map*

$$\mathrm{Op}(p) : \mathcal{S}'(U_\delta) \longrightarrow \mathcal{S}_0'(U_\delta).$$

We put

$$L^{-\infty,-\infty}(U_\delta) = \mathrm{Op}(S^{-\infty,-\infty}(U_\delta)) , \qquad \text{where} \qquad S^{-\infty,-\infty}(U_\delta) = \bigcap_{m,\mu} S^{m,\mu}(U_\delta) .$$

Next, we denote by $L^{m,\mu}(U_\delta)$ the class of all continuous linear maps $A : S_0(U_\delta) \to S_0'(U_\delta)$ for which there exists $a \in S^{m,\mu}(U_\delta)$ such that $A - \mathrm{Op}(a) \in L^{-\infty,-\infty}(U_\delta)$. Such an A will be referred to as pseudodifferential operator of double order (m,μ) on U_δ.

It can be shown that $L^{-\infty,-\infty}(U_\delta)$ coincides with the class of integral operators having smooth kernel in $\mathcal{S}(U_\delta \times U_\delta)$ and that every $A \in L^{m,\mu}(U_\delta)$ has a distribution kernel K in $C^\infty(U_\delta \times U_\delta \setminus \Delta) \cap \mathcal{S}(U_\delta \times U_\delta \setminus I_\Delta)$, where $\Delta = \{(x,x) : x \in U_\delta\}$ and I_Δ is any conic neighbourhood of Δ.

Definition 2.4. *Let $a_j \in S^{m_j,\mu_j}(U_\delta)$, $\{m_j\}_{j\geq 0}$, $\{\mu_j\}_{j\geq 0}$ decreasing to $-\infty$. The formal series $\sum_{j\geq 0} a_j$ is said to be asymptotically summable if there exists $a \in S^{m_0,\mu_0}(U_\delta)$ such that:*

$$a - \sum_{j\leq N-1} a_j \in S^{m_N,\mu_N}(U_\delta), \qquad \text{for every } N > 0.$$

We shall write this relation as $a \sim \sum_{j\geq 0} a_j$.

It is a remarkable fact that every formal series (which in general is not absolutely convergent) is asymptotically summable and the asymptotic sum is uniquely determined up to an additive term in $S^{-\infty,-\infty}(U_\delta)$.

Theorem 2.5. *For $p \in S^{m,\mu_1,\mu_2}(U_\delta \times U_\delta)$ there exists a symbol $a \in S^{m,\mu_1+\mu_2}(U_\delta)$ such that:* $\mathrm{Op}(p) - \mathrm{Op}(a) \in L^{-\infty,-\infty}(U_\delta)$. *Moreover,*

$$a(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} D_\xi^\alpha \partial_y^\alpha p(x,y,\xi)|_{y=x}.$$

Theorem 2.6. *Let $a \in S^{m,\mu}(U_\delta)$ and $b \in S^{m',\mu'}(U_\delta)$. Then* $\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(c)$, *where $c \in S^{m+m',\mu+\mu'}(U_\delta)$. Moreover,*

$$c(x,\xi) \sim \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} D_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi).$$

Proofs of these results can be found in [5] and [17].

Let us define $\Lambda^{s,\nu} = \mathrm{Op}(\langle\xi\rangle^s\langle x\rangle^\nu)$ and recall that $u \in H^{s,\nu}(\mathbb{R}^n) \Leftrightarrow \Lambda^{s,\nu}u \in L^2(\mathbb{R}^n)$, where $H^{s,\nu}(\mathbb{R}^n)$ is the weighted Sobolev space defined in [15]. We now "localize" these spaces and consider

$$H^{s,\nu}_{(0)}(U_\delta) := \{u \in H^{s,\nu}(\mathbb{R}^n) : \operatorname{supp} u \subset U_\delta\} = \{u \in \mathcal{S}'(U_\delta) : \Lambda^{s,\nu}u \in L^2(U_\delta)\} \tag{2.1}$$

endowed with the inductive limit topology and

$$H^{s,\nu}_{loc}(U_\delta) := \{u \in \mathcal{S}'_0(U_\delta) : \forall\phi \in C^\infty(U_\delta),\ |\partial_x^\alpha\phi(x)| \le C_\alpha\langle x\rangle^{-|\alpha|}, \operatorname{supp}\phi \subset U_\delta \Rightarrow \phi u \in H^{s,\nu}(\mathbb{R}^n)\} \tag{2.2}$$

with the natural Fréchet topology.

Theorem 2.7. *Let $a \in S^{m,\mu}(U_\delta)$. Then $\forall s, \nu \in \mathbb{R}$ we have the continuous map*

$$\mathrm{Op}(a) : H^{s,\nu}_{(0)}(U_\delta) \longrightarrow H^{s-m,\nu-\mu}_{loc}(U_\delta).$$

Proof. For every ϕ, $|\partial_x^\alpha\phi(x)| \le C_\alpha\langle x\rangle^{-|\alpha|}$, supported in U_δ, and every $u \in \mathcal{S}_0(U_\delta)$ we have $\phi(x)\mathrm{Op}(a)u(x) = \mathrm{Op}(\phi a)u(x)$, where $\phi(x)a(x,\xi) \in S^{m,\mu}(\mathbb{R}^n)$ and $\mathrm{Op}(\phi a)$ is continuous from $H^{s,\nu}(\mathbb{R}^n)$ to $H^{s-m,\nu-\mu}(\mathbb{R}^n)$ (see [15]). □

Corollary 2.8. *If $A \in L^{-\infty,-\infty}(U_\delta)$ then it continuously maps $\mathcal{S}'(U_\delta)$ into $\mathcal{S}(U_\delta)$ (and it will therefore be said to be a "regularizing" or "infinitely smoothing" operator).*

Proof. It follows from the topological equality $\bigcap_{s,\nu} H^{s,\nu}_{loc}(U_\delta) = \mathcal{S}(U_\delta)$. □

We state now the invariance of the class $S^{m,\mu}(U_\delta)$ with respect to suitable diffeomorphisms. To do this, we refer to [17].

Let $\mathrm{Diff}(U_\delta)$ denote the class of all diffeomorphisms on U_δ. Take $F \in \mathrm{Diff}(U_\delta)$ such that for every $\alpha \in \mathbb{N}^n$ $|\partial_x^\alpha F(x)| = \mathcal{O}(|x|^{1-|\alpha|})$, $|\partial_y^\alpha F^{-1}(y)| = O(|y|^{1-|\alpha|})$. Define an isomorphism $\phi \to \tilde{\phi}$ from the function space $\mathcal{S}_0(U_\delta)$ into itself by putting $\tilde{\phi} = \phi \circ F$. Thus, there exists a unique extention $u \to \tilde{u}$ to the distribution space $\mathcal{S}_0^{'}(U_\delta)$ such that

$$\langle\tilde{u}, \phi\rangle = \langle u, (\phi \circ F^{-1}) \cdot |\det dF^{-1}|\,\rangle.$$

For every linear operator $A : \mathcal{S}_0(U_\delta) \to \mathcal{S}'_0(U_\delta)$ we denote by $\tilde{A}$ the pull-back operator of A through F implicitly defined by $\tilde{A}\tilde{u}(x) := (Au)(F(x))$.

Theorem 2.9. *Let $A = \mathrm{Op}(a)$, with $a \in S^{m,\mu}(U_\delta)$ and $\tilde{A}$ its pullback through F as above. Then there exists $\tilde{a} \in S^{m,\mu}(U_\delta)$ such that $\tilde{A} = \mathrm{Op}(\tilde{a}) + L^{-\infty,-\infty}(U_\delta)$. Moreover, the following statement is true:*

$$\tilde{a}(x,\xi) - a(F(x), {}^t dF_x^{-1}(\xi)) \in S^{m-1,\mu-1}(U_\delta)\,.$$

Remark 2.1. A straightforward computation when F is of the form $F(\rho\omega) = \rho\,\Theta(\omega)$, $\rho = |x|$ and $\omega = x/|x|$, for some $\Theta \in \mathrm{Diff}(\mathbb{S}^{n-1})$, once denoted ${}^t dF_{\rho\omega}(\zeta) = \xi$, yields $\forall v \in T_{\rho\omega}U_\delta$, $v = v_{rad} \oplus v_{tg}$, with $v_{rad} = \langle v, \omega\rangle\,\omega$ and $v_{tg} \in T_\omega\mathbb{S}^{n-1}$:

$$\langle \xi, v\rangle = \langle \zeta, \Theta(\omega)\rangle\,|v_{rad}| + \langle {}^t d\Theta_\omega(\zeta'),\ v_{tg}\rangle,$$

where ζ' denotes the restriction of ζ onto $T_{\Theta(\omega)}\mathbb{S}^{n-1}$. This shows that in some sense ${}^t dF_{\rho\omega}$ preserves the radial component of covectors and acts on their tangential component through ${}^t d\Theta_\omega$.

We close this section by focusing on the class $ECL^{m,\mu}(U_\delta)$ which contains, in particular, differential operators of the form (1.2).

Definition 2.5. *Let $a(x,\xi) \in S^{m,\mu}(U_\delta)$. We shall write $a \in CS^{m,\mu}(U_\delta)$, $a \to \{a_m, a_\infty, a_\sharp\}$, and say that a is classical with exit symbols if:*

i) $\exists a_m \in C^\infty(U_\delta \times \mathbb{R}^n \setminus \{0\})$, *homogeneous of degree m in ξ, such that*

$$a(x,\xi) - a_m(x,\xi) = \mathcal{O}(\langle\xi\rangle^{m-1}\langle x\rangle^{\mu}) \qquad (x,\xi) \in U_\delta \times \mathbb{R}^n \setminus \{0\};$$

ii) $\exists a_\infty \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{R}^n)$ *such that*

$$a(x,\xi) - |x|^\mu a_\infty\left(\frac{x}{|x|}, \xi\right) = \mathcal{O}(\langle\xi\rangle^{m}\langle x\rangle^{\mu-1}) \qquad (x,\xi) \in U_\delta \times \mathbb{R}^n;$$

iii) $\exists a_\sharp \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{R}^n \setminus \{0\})$, *homogeneous of degree m in ξ, such that*

$$|x|^\mu\left[a_\infty\left(\frac{x}{|x|}, \xi\right) - |\xi|^m a_\sharp\left(\frac{x}{|x|}, \frac{\xi}{|\xi|}\right)\right] = \mathcal{O}(\langle\xi\rangle^{m-1}\langle x\rangle^{\mu})$$
$$(x,\xi) \in U_\delta \times \mathbb{R}^n \setminus \{0\},$$
$$|\xi|^m\left[a_m\left(x, \frac{\xi}{|\xi|}\right) - |x|^\mu a_\sharp\left(\frac{x}{|x|}, \frac{\xi}{|\xi|}\right)\right] = \mathcal{O}(\langle\xi\rangle^{m}\langle x\rangle^{\mu-1})$$
$$(x,\xi) \in U_\delta \times \mathbb{R}^n \setminus \{0\}.$$

Note that "compatibility" conditions iii) do not follow from i) and ii). Besides, $a_\sharp$ does not allow us to determine a_∞ and a_m uniquely. Vice versa, the triplet $\{a_\infty, a_m, a_\sharp\}$ determines a symbol in $S^{m,\mu}(U_\delta)$ up to a lower order term which is in $S^{m-1,\mu}(U_\delta) \cap S^{m,\mu-1}(U_\delta)$.

We point out that a_m is the (internal) principal symbol of $\mathrm{Op}(a)$ as long as we look at a as a standard symbol of single order m (see [9]). Global information at infinity is gathered by both a_∞ and $a_\sharp$, thus we think of them as the exit principal symbols of $\mathrm{Op}(a)$.

These quantities turn out to be stable under the composition of operators; indeed, if $a \in CS^{m,\mu}(U_\delta)$ and $b \in CS^{m',\mu'}(U_\delta)$, with $a \mapsto \{a_m, a_\infty, a_\sharp\}$, $b \mapsto \{b_{m'}, b_\infty, b_\sharp\}$, then $\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(c)$, with $c \in CS^{m+m',\mu+\mu'}(U_\delta)$ and $c \mapsto \{a_m b_{m'}, a_\infty b_\infty, a_\sharp b_\sharp\}$.

We mention now a subclass of the diffeomorphisms allowed in Theorem 2.9 (see Remark 2.1.1 above), "preserving" $CS^{m,\mu}(U_\delta)$.

Theorem 2.10. *Let $F \in \mathrm{Diff}(U_\delta)$ be of the form $F(x) = |x|\Theta(x/|x|)$, $\Theta \in \mathrm{Diff}(\mathbb{S}^{n-1})$ and $A = \mathrm{Op}(a)$, with $a \in CS^{m,\mu}(U_\delta)$, $a \mapsto \{a_m, a_\infty, a_\sharp\}$. Let $\tilde{A}$ be the pullback of A through F. Then $\tilde{A} - \mathrm{Op}(\tilde{a}) \in L^{-\infty,-\infty}(U_\delta)$, where $\tilde{a} \in CS^{m,\mu}(U_\delta)$, $\tilde{a} \mapsto \{\tilde{a}_m, \tilde{a}_\infty, \tilde{a}_\sharp\}$. As usual, we have $\tilde{a}_m(x,\xi) = a_m(F(x), {}^t dF_x^{-1}(\xi))$. Moreover,*

$$\tilde{a}_\infty(\omega,\xi) = a_\infty\big(\Theta(\omega), {}^t dF_\omega^{-1}(\xi)\big), \ \ \text{and} \ \ \tilde{a}_\sharp(\omega,\xi) = a_\sharp\big(\Theta(\omega), {}^t dF_\omega^{-1}(\xi)\big). \quad (2.3)$$

Proof. We are looking for a candidate for $\tilde{a}_\infty(\frac{x}{|x|},\xi)$ such that $\tilde{a}(x,\xi) - |x|^\mu \tilde{a}_\infty(\frac{x}{|x|},\xi) \in S^{m,\mu-1}(U_\delta)$. By hypothesis, $a(y,\eta) - |y|^\mu a_\infty(\frac{y}{|y|},\eta) \in S^{m,\mu-1}(U_\delta)$. Replace $y = F(x)$ and $\eta = {}^t dF_x^{-1}(\xi)$, observing that $|F(x)| = |x|$ and $\frac{F(x)}{|F(x)|} = \Theta\left(\frac{x}{|x|}\right)$, we get

$$a\big(F(x), {}^t dF_x^{-1}(\xi)\big) - |x|^\mu a_\infty\left(\Theta\left(\frac{x}{|x|}\right), {}^t dF_{\frac{x}{|x|}}^{-1}(\xi)\right) \in S^{m,\mu-1}(U_\delta),$$

where the class of symbols does not change because F and dF_x are homogeneous of degree 1 and 0, respectively. From Theorem 2.9 we deduce

$$\tilde{a}(x,\xi) - |x|^\mu a_\infty\left(\Theta\left(\frac{x}{|x|}\right), {}^t dF_{\frac{x}{|x|}}^{-1}(\xi)\right) \in S^{m,\mu-1}(U_\delta).$$

This proves the first part. Analogously, we know by hypothesis that

$$a_\infty(\omega',\eta) - a_\sharp(\omega',\eta) = \mathcal{O}(\langle\eta\rangle^{m-1})$$
$$a_m(y,\eta) - |y|^\mu a_\sharp\left(\frac{y}{|y|},\eta\right) = \mathcal{O}(\langle y\rangle^{\mu-1}\langle\eta\rangle^m).$$

Again replacing $\omega' = \Theta(\omega)$, $y = F(x)$ and $\eta = {}^t dF_x^{-1}(\xi)$, we obtain

$$a_\infty\big(\Theta(\omega), {}^t dF_x^{-1}(\xi)\big) - a_\sharp\big(\Theta(\omega), {}^t dF_\omega^{-1}(\xi)\big) = \mathcal{O}(\langle\xi\rangle^{m-1})$$
$$a_m\big(F(x), {}^t dF_x^{-1}(\xi)\big) - |x|^\mu a_\sharp\left(\Theta\left(\frac{x}{|x|}\right), {}^t dF_{\frac{x}{|x|}}^{-1}(\xi)\right) = \mathcal{O}(\langle x\rangle^{\mu-1}\langle\xi\rangle^m).$$

Once more, Theorem 2.9 yields

$$\tilde{a}_\infty(\omega,\xi) - a_\sharp\big(\Theta(\omega), {}^t dF_{\frac{x}{|x|}}^{-1}(\xi)\big) = \mathcal{O}(\langle\xi\rangle^{m-1})$$
$$\tilde{a}_m(x,\xi) - |x|^\mu a_\sharp\left(\Theta\left(\frac{x}{|x|}\right), {}^t dF_{\frac{x}{|x|}}^{-1}(\xi)\right) = \mathcal{O}(\langle x\rangle^{\mu-1}\langle\xi\rangle^m).$$

This proves the theorem. □

Definition 2.6. *Let $a \in S^{m,\mu}(U_\delta)$. We shall say that a is globally elliptic (or md-elliptic) of order (m,μ) if there exist positive constants c, C such that*

$$|a(x,\xi)| \geq C\langle x\rangle^\mu \langle\xi\rangle^m, \qquad \textit{when } |x| + |\xi| \geq c\,.$$

It is proved in [5] that global ellipticity of the symbol $a(x,\xi)$ of order (m,μ) is equivalent to the existence of a (two-sided) parametrix of $\mathrm{Op}(a)$ in $L^{-m,-\mu}(U_\delta)$.

The point about global ellipticity in the class CS is that it can be expressed by means of algebraic conditions both on the usual principal symbol and the exit principal symbols.

Proposition 2.11. *Let $a \in CS^{m,\mu}(U_\delta)$, $a \mapsto \{a_m, a_\infty, a_\sharp\}$. Then a is globally elliptic if and only if the following three conditions hold:*

1) $a_m(x,\xi) \neq 0 \quad \forall(x,\xi) \in U_\delta \times \mathbb{R}^n \setminus \{0\}$;
2) $a_\infty(\omega,\xi) \neq 0 \quad \forall(\omega,\xi) \in \mathbb{S}^{n-1} \times \mathbb{R}^n$;
3) $a_\sharp(\omega,\xi) \neq 0 \quad \forall(\omega,\xi) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \setminus \{0\}$.

Proof. Definition 2.5 gives, for all $x \in U_\delta$:

$$\frac{\left|a_\infty\left(\frac{x}{|x|},\xi\right)\right|}{\langle\xi\rangle^m} = \frac{|a(x,\xi)|}{|x|^\mu\langle\xi\rangle^m} + O(|x|^{-1}) , \qquad \frac{|a_m(x,\xi)|}{|x|^\mu\langle\xi\rangle^m} = \frac{|a(x,\xi)|}{|x|^\mu\langle\xi\rangle^m} + O(\langle\xi\rangle^{-1}).$$

Global ellipticity applies to the former right member for large $|x|$ and to the latter for large $|\xi|$; thereby 1) and 2) hold. Taking into account just one of the relationships iii) of Definition 2.5 between $a_\sharp$ and each of the symbols a_∞ and a_m, Condition 3) follows.

On the other hand, from Hypotheses 2), 3) and the first part of iii) of Definition 2.5 we get the important estimate:

$$\left|a_\infty\left(\frac{x}{|x|},\xi\right)\right| \geq C'\ \langle\xi\rangle^m \qquad \forall(x,\xi) \in U_\delta \times \mathbb{R}^n\ .$$

Global ellipticity of a is then an easy consequence of Hypothesis 1) and the same relations used above which, once more, can be rewritten in this way:

$$\frac{|a(x,\xi|}{|a_m(x,\xi)|} = 1 + O(\langle\xi\rangle^{-1}), \qquad \frac{|a(x,\xi)|}{|x|^\mu\left|\alpha_\infty\left(\frac{x}{|x|},\xi\right)\right|} = 1 + O(\langle x\rangle^{-1}),$$

for large $|\xi|$ and large $|x|$, respectively. This completes the proof. □

Remark 2.2. Let B be the parametrix of $\mathrm{Op}(a)$, with $a \in CS^{m,\mu}(U_\delta)$ globally elliptic. Then $B = \mathrm{Op}(b)$ with $b \in CS^{-m,-\mu}(U_\delta)$ and $b \to \{\frac{1}{a_m}, \frac{1}{a_\infty}, \frac{1}{a_\sharp}\}$.

Definition 2.7. *Let $a \in CS^{m,\mu}(U_\delta)$, $a \mapsto \{a_\infty, a_m, a_\sharp\}$. We shall write $a \in ECS^{m,\mu}(U_\delta)$, or equivalently $\mathrm{Op}(a) \in ECL^{m,\mu}(U_\delta)$, if Conditions 1), 2), 3) of the previous proposition hold with $>$ instead of $\neq$.*

3. Manifolds with cylindrical exits

The aim of this section is to make precise the notion of manifold with exits, M, and establish invariantly the formal calculus for the class $ECL^{m,\mu}(M)$.

Definition 3.1. *We call an n-manifold with an exit the triplet $(M, X, [f])$ such that*

i) $M = (M_0 \backslash D) \bigcup \mathcal{C}$, *where*
- M_0 *is a compact n-manifold (without boundary) and* D *an open disk of* M_0;
- $\mathcal{C}$ *is an n-manifold with boundary* $\partial\mathcal{C} = X$;
- $\bigcup$ *means gluing by identification along the boundaries;*

ii) $f : [\delta_f, \infty)\times\mathbb{S}^{n-1} \to \mathcal{C}$ *is a diffeomorphism,* $\delta_f > 0$ *and* $f(\{\delta_f\}\times\mathbb{S}^{n-1}) = X$*;*
iii) $[f]$ *denotes the equivalence class with respect to the following relation:* $f \sim g$ *if and only if there exists* $\Theta \in \mathrm{Diff}(\mathbb{S}^{n-1})$ *such that,*

$$(g^{-1} f)(\rho, \omega) = (\rho, \Theta(\omega)), \tag{3.1}$$

for every $\rho \geq max(\delta_f, \delta_g)$ *and for every* $\omega \in \mathbb{S}^{n-1}$.

We shall refer to $\mathcal{C}$ as the cylindrical exit of M and to X as its base. We put $\dot{\mathcal{C}} = \mathcal{C}\setminus X$ and $\dot{f} : (\delta_f, \infty) \times \mathbb{S}^{n-1} \to \dot{\mathcal{C}}$. Let $\pi(x) = (|x|, \frac{x}{|x|})$, then $\dot{f}_\pi = \dot{f} \circ \pi : U_{\delta_f} \to \dot{\mathcal{C}}$ and $f_\pi = f \circ \pi : \overline{U}_{\delta_f} \to \mathcal{C}$ is a natural parametrization of the exit. We shall refer to $\dot{f}_\pi^{-1}$ as the exit chart. Then (3.1) is equivalent to

$$g_\pi^{-1} \circ f_\pi = F, \text{ with } F(x) = |x|\ \Theta\left(\frac{x}{|x|}\right). \tag{3.2}$$

We fix a system of local charts $\mathcal{A} = \{\Omega_j, \phi_j\}_{j=1,\cdots N}$, given by any finite atlas for $(M_0\setminus D)\cup f([\delta_f, \delta_f+\epsilon)\times\mathbb{S}^{n-1})$ joined with the exit chart $(\Omega_N, \phi_N) = (\dot{C}, \dot{f}_\pi^{-1})$. It is easily recognisable as an SG-differential structure on M (uniquely determined by any representative of the class $[f]$) according to Schrohe's framework described in [17]. We assume that the Riemannian structure on M is also induced by the pullback via f of the standard Euclidean metric on $\mathbb{R}^n$.

Remark 3.1. Definition 3.1 can be immediately generalized to manifolds with finitely many cylindrical exits $(M; X_1, \ldots, X_N; [f_1], \ldots, [f_N])$. For the sake of simplicity, we continue considering the case of only one exit, which is no loss of generality.

Remark 3.2 (Compatible partition of unity). Let $\mathcal{A} = \{(\Omega_j, \phi_j)\}_{j\in\mathbb{J}}$ be the SG-atlas on M. According to [5] and [17], there exist a smooth partition of unity $\{\psi_j\}$ subordinated to $\mathcal{A}$ and a smooth family $\{\theta_j\}$, supported in the local charts, with $\theta_j = 1$ on supp ψ_j for every $j \in \{1, \ldots, N\}$, such that, $\forall\alpha \in \mathbb{N}^n$, the following growth conditions hold on the exit:

$$\left|\partial_x^\alpha \psi_N(x)\right| \leq C_\alpha \langle x\rangle^{-|\alpha|}, \qquad \left|\partial_x^\alpha \theta_N(x)\right| \leq C'_\alpha \langle x\rangle^{-|\alpha|}.$$

For short we will denote by M a manifold with one cylindrical exit. Let $C^\infty(M)$ be the space of all smooth complex-valued functions on M. A natural space of rapidly decreasing functions well suited to our manifold is the following:

$$\mathcal{S}(M) = \{u \in C^\infty(M) : \forall \text{ exit chart } \dot{f}_\pi^{-1} : \dot{\mathcal{C}} \to U_\delta \qquad u \circ \dot{f}_\pi \in \mathcal{S}(U_\delta)\}.$$

Let $\mathcal{S}'(M)$ denote its topological dual space of tempered distribution on M. We are now ready to deal with operators on M.

Definition 3.2. *Let* $A : \mathcal{S}(M) \to \mathcal{S}'(M)$ *be a continuous linear map. We shall say that* A *is a pseudodifferential operator on* M *of order* (m, μ) *– and write* $A \in L^{m,\mu}(M)$ *– if:*

1) *for every chart* $\psi : W \to V$, *with* $\overline{V} \subset \mathbb{R}^n$ *compact, there exists a standard symbol* $a^{(\psi)} \in S^m(V)$ *such that for the pullback of A through* ψ^{-1} *(that is, the transfer of the restriction of A to W) we have the representation*

$$\tilde{A}u(x) = \int e^{i(x-y)\cdot\xi} a^{(\psi)}(x,\xi)u(y)dy\,d\!\!\bar{}\,\xi, \qquad \forall u \in C_0^\infty(V);$$

2) *for every exit chart* $\dot{f}_\pi^{-1} : \dot{\mathcal{C}} \to U_{\delta_f}$ *there exists a weighted symbol* $a^{(f)} \in S^{m,\mu}(U_{\delta_f})$ *such that for the pullback of A through* $\dot{f}_\pi^{-1}$ *we have*

$$\tilde{A}u(x) = \mathrm{Op}(a^{(f)})u(x), \qquad \forall u \in \mathcal{S}_0(U_{\delta_f});$$

3) *the distribution kernel* K_A *has the following property:*

$$K_A \in C^\infty(M \times M \setminus \Delta) \bigcap \mathcal{S}(\dot{\mathcal{C}} \times \dot{\mathcal{C}} \setminus W),$$

where Δ *is the diagonal of* $M \times M$ *and* $W = (\dot{f}_\pi \times \dot{f}_\pi)(V)$, *with f any exit chart and V any conic neighbourhood of the diagonal of* $U_{\delta_f} \times U_{\delta_f}$.

Observe that Condition 3) on the kernel of A guarantees on one hand that A is pseudolocal (i.e. $\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u$, $\forall u \in \mathcal{S}'(M)$); on the other that A is regularizing at infinity off the diagonal. We now give "good" definitions of exit symbols for an operator on the manifold in order to show their global meaning.

Definition 3.3. *We shall denote by* $CL^{m,\mu}(M)$ *the set of all operators* $A \in L^{m,\mu}(M)$ *with principal symbol* $a_m \in C^\infty(T^*M\setminus 0)$ *such that for every diffeomorphism f of the exit:*

i) $\exists a_\infty^{(f)} \in C^\infty(\mathbb{S}^{n-1} \times \mathbb{R}^n)$ *such that*

$$a^{(f)}(x,\xi) - |x|^\mu a_\infty^{(f)}\left(\frac{x}{|x|}, \xi\right) \in S^{m,\mu-1}(U_{\delta_f});$$

ii) $\exists a_\sharp^{(f)} \in C^\infty(\mathbb{S}^{n-1} \times \dot{\mathbb{R}}^n)$ *positively homogeneous in* ξ *of degree m such that*

$$\chi(\xi)|x|^\mu \left[a_\infty^{(f)}\left(\frac{x}{|x|}, \xi\right) - a_\sharp^{(f)}\left(\frac{x}{|x|}, \xi\right)\right] \in S^{m-1,\mu}(U_{\delta_f}),$$

and

$$\chi(\xi)\left[a_m^{(f)}(x,\xi) - |x|^\mu\, a_\sharp^{(f)}\left(\frac{x}{|x|}, \xi\right)\right] \in S^{m,\mu-1}(U_{\delta_f}),$$

for every $\chi \in C^\infty(\mathbb{R}^n)$ *which is zero near the origin and identically* 1 *for large* ξ.

We now introduce the geometrical object which allows us to interpret a_∞ and $a_\sharp$ in a global sense, namely the restriction to X of the cotangent bundle of M:

$$T_X^*M = \{(x,\xi) : x \in X,\ \xi \in T_x^*M\},$$

X being the base of the exit of M. It is worthwhile to note that, here, covectors are n-dimensional, while base points are $(n-1)$-dimensional; therefore $\dim T_X^*M = 2n-1$. Of course, there is a natural projection between the vector bundles (both over X) $T_X^*M \longrightarrow T^*X$: the restriction of covectors $(x,\xi) \longmapsto \big(x, \xi|_{T_xX}\big)$.

Theorem 3.1. *Let $A \in CL^{m,\mu}(M)$. Then, besides $a_m \in C^\infty(T^*M\backslash 0)$, there exist in addition two globally defined functions*

$$a_\infty \in C^\infty(T^*_X M) \qquad \text{and} \qquad a_\sharp \in C^\infty(T^*_X M\backslash 0),$$

such that, for any diffeomorphism f on the exit,

$$a_\infty(x,\xi) = a_\infty^{(f)}\big(f_\pi^{-1}(x),\, {}^t df_\pi(x)(\xi)\big), \qquad x \in X,\ \xi \in T^*_x M \tag{3.3}$$

and

$$a_\sharp(x,\xi) = a_\sharp^{(f)}\left(f_\pi^{-1}(x),\, {}^t df_\pi(x)(\xi)\right), \qquad x \in X,\ \xi \in T^*_x M\backslash\{0\}. \tag{3.4}$$

Proof. The crucial point to be proved is the independence of (3.3) and (3.4) of the diffeomorphism. To verify this, suppose that $g \sim f$ and that (3.2) holds. Assume also that $x = f_\pi(\omega) = g_\pi(\omega')$, that is $\omega' = F(\omega) = \Theta(\omega)$. We have to prove that $a_\infty^{(f)}(\omega, {}^t d\tilde{f}(x)\xi)$ and $a_\infty^{(g)}(\omega', {}^t d\tilde{g}(x)\xi)$ are equal.

Notice that $\mathrm{Op}(a^f)$ is, up to regularizing operators, nothing but the pullback of $\mathrm{Op}(a^g)$ through F. So, from Theorem 2.10, it follows that

$$a_\infty^{(f)}(\omega,\eta) = a_\infty^{(g)}(\Theta(\omega), {}^t dF^{-1}(\omega)\eta).$$

Now take $\eta = {}^t df_\pi(x)\xi$ and observe that, since $F = g_\pi^{-1} \circ f_\pi$, we have

$${}^t dF^{-1}(\omega) \circ {}^t df_\pi(x)|_{x=f_\pi(\omega)} = {}^t dg_\pi(x)|_{x=g_\pi(\omega')}.$$

Thereby

$$\begin{aligned} a_\infty^{(f)}(\omega, {}^t df_\pi(f_\pi(\omega))\xi) &= a_\infty^{(g)}(\Theta(\omega), {}^t dF^{-1}(\omega) \circ {}^t df_\pi(f_\pi(\omega))\xi) \\ &= a_\infty^{(g)}(\omega', {}^t dg_\pi(g_\pi(\omega'))\xi). \end{aligned}$$

Do this analogously for $a_\sharp$. This completes the proof. □

Ellipticity (see Definition 2.6 and Proposition 2.11) has an invariant meaning with respect to diffeomorphisms that are transition maps of our manifold, so it is consistent to give the following:

Definition 3.4. *We denote by $ECL^{m,\mu}(M)$ the class of all operators $A \in CL^{m,\mu}(M)$ whose principal symbol a_m is elliptic of order m in the standard way and, furthermore, for every diffeomorphism f of the exit, we have $a^{(f)} \in ECS^{m,\mu}(U_{\delta_f})$.*

Of course, we can construct the operator $\Lambda^{m,\mu} \in ECL^{m,\mu}(M)$ for any $m, \mu \in \mathbb{R}$ by gluing the corresponding operators on local charts having symbol $\langle\xi\rangle^m\langle x\rangle^\mu$ and a parametrix of it is given by $\Lambda^{-m,-\mu}$. We can now turn to *weighted Sobolev spaces* on M by putting

$$H^{s,\nu}(M) = \{u \in \mathcal{S}'(M) : \Lambda^{s,\nu}u \in L^2(M)\}.$$

If we consider $\bigcup_{s,\nu} H^{s,\nu}(M)$ with the inductive limit topology and $\bigcap_{s,\nu} H^{s,\nu}(M)$ with the projective limit topology, we have the following algebraic and topological equalities:

$$\bigcup_{s,\nu} H^{s,\nu}(M) \cong \mathcal{S}'(M), \qquad \bigcap_{s,\nu} H^{s,\nu}(M) \cong \mathcal{S}(M). \tag{3.5}$$

By using local continuity results it is not difficult to prove the following theorem:

Theorem 3.2. *If $A \in L^{m,\mu}(M)$, then $\forall s, \nu \in \mathbb{R}$ we have the following continuous map:*

$$A : H^{s,\nu}(M) \to H^{s-m,\nu-\mu}(M).$$

Furthemore, $H^{s,\nu}(M) \hookrightarrow H^{s',\nu'}(M)$ is a compact embedding whenever $s' < s$, $\nu' < \nu$.

Let us finally sort out spectral properties of the class $ECL^{m,\mu}(M)$.

By $\varrho(A)$ we denote the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ maps the domain of A bijectively onto $L^2(M)$. The spectrum of A is given by $\sigma(A) = \mathbb{C} \setminus \varrho(A)$.

Theorem 3.3 (Spectral theorem). *Let $A \in ECL^{m,\mu}(M)$, $m, \mu > 0$ and $A^* = A$, be regarded as a closed unbounded operator on $L^2(M)$ with dense domain $H^{m,\mu}(M)$. Then:*

i) *$(\lambda I - A)^{-1}$ is a compact operator on $L^2(M)$ for every $\lambda \in \varrho(A)$. More precisely, $(\lambda I - A)^{-1}$ is an extension by continuity from $\mathcal{S}(M)$ or a restriction from $\mathcal{S}'(M)$ of an operator in $ECL^{-m,-\mu}(M)$.*

ii) *$\sigma(A)$ consists of a sequence of real isolated eigenvalues $\{\lambda_j\}$ with finite multiplicity clustering at infinity; the orthonormal system of eigenfunctions $\{e_j\}_{j\geq 1}$ is complete in $L^2(M)$, moreover, $e_j \in \mathcal{S}(M)$.*

iii) *$-A$ is the infinitesimal generator of an analytic semigroup of bounded operators on $L^2(M)$, $H(t) = e^{-tA}$, $t \geq 0$, called the heat semigroup, with kernel*

$$H(t, x, y) = \sum_j e^{-t\lambda_j} e_j(x)\overline{e_j(y)}.$$

iv) *$H(t)$ is trace class when $t > 0$ and*

$$\text{Trace } H(t) = \int_M H(t, x, x)dx = \sum_j e^{-t\lambda_j}\,. \tag{3.6}$$

Proof. i) Pick $\lambda \in \varrho(A)$. Global ellipticity of A yields the existence of a parametrix $B \in ECL^{-m,-\mu}(M)$ of $\lambda I - A$ such that

$$(\lambda I - A)B = I - R_1 \qquad \text{and} \qquad B(\lambda I - A) = I - R_2,$$

with R_1, R_2 smoothing. It follows that $(\lambda I - A)^{-1} = B + (\lambda I - A)^{-1}R_1$, so it remains for us to verify that $(\lambda I - A)^{-1}R_1$ is smoothing. From Theorem 3.2, B maps $H^{s,\nu}(M)$ into $H^{s+m,\nu+\mu}(M)$ continuously. If we assume $(\lambda I - A)u \in H^{s,\nu}(M)$ and write

$$u = B(\lambda I - A)u + R_2(u),$$

we have $u \in H^{s+m,\nu+\mu}(M)$. This shows that $(\lambda I - A)^{-1}$ maps $H^{s,\nu}(M)$ into $H^{s+m,\nu+\mu}(M)$ for any s, ν as well. In particular $(\lambda I - A)^{-1}$ maps $\mathcal{S}(M)$ into itself, thereby $(\lambda I - A)^{-1}R_1$ is smoothing. On the other hand, the map $(\lambda I - A)^{-1} : L^2(M) \to H^{m,\mu}(M)$ is continuous by the closed graph theorem.

Now, from Theorem 3.2 it follows the compactness of the composition

$$L^2(M) \overset{(\lambda I - A)^{-1}}{\longrightarrow} H^{m,\mu}(M) \overset{\iota}{\hookrightarrow} L^2(M).$$

Compactness of $(\lambda' I - A)^{-1}$ for every other $\lambda' \in \varrho(A)$ follows by the resolvent identity.

ii) From self-adjointness $\sigma(A)$ is a non-empty subset of $\mathbb{R}$. The remaining part is a consequence of the Riesz–Schauder theory. Notice that if $(\lambda I - A)^{-1} e_j = r_j e_j$, then $r_j \to 0$ because of part i) and we can rewrite $A e_j = (r_j^{-1} + \lambda) e_j$. This shows that $e_j \in \mathcal{S}(M)$ and the same e_j's are eigenfunctions of A with eigenvalues $\lambda_j = r_j^{-1} + \lambda$. Semiboundedness of A yields a lower bound for the $\sigma(A)$, hence necessarily $\lambda_j \to +\infty$.

iii) Every semibounded self-adjoint operator is sectorial, therefore it generates an analytic semigroup (see [16]).

iv) This comes immediately from ii) and iii). □

Remark 3.3. If we drop the self-adjointness assumption from the hypotheses of Theorem 3.3, it is still possible to prove that $\varrho(A) \neq \emptyset$ and $\sigma(A)$ is contained in a sector of the complex plane of the form $\{\lambda \in \mathbb{C} : |\lambda| > R,\ |Arg\,\lambda| \leq \theta\}$ for some $R > 0$ and $\theta \in]0, \frac{\pi}{2}[$. Thus Theorem 3.3 remains true.

4. Heat parametrix

We shall assume that the following hypotheses are fulfilled:

(H1) $(M, X, [f])$ is a manifold with one cylindrical exit, $\dim M = n$;
(H2) $A \in ECL^{m,\mu}(M)$, with $m, \mu > 0$;
(H3) A is positive self-adjoint, i.e. $A^* = A > 0$.

We shall deal with *t-regular* one-parameter families of bounded linear operators, $\{T(t)\}_{t \geq 0}$, on $L^2(M)$ in the sense that

$$T \in C^\infty(]0, T[; \mathcal{L}(L^2(M))), \qquad \forall u \in L^2(M) \quad t \to T(t)u \text{ is } C([0, T[; L^2(M)).$$

We shall also need t-regular families of (x-)smoothing operators in the sense that

$$T \in C^\infty(]0, T[; \mathcal{L}(\mathcal{S}'(M); \mathcal{S}(M))).$$

This section is mostly concerned with the solution of the following Cauchy problem for the generalized heat equation associated with our operator A:

$$\begin{cases} \dfrac{d}{dt} U(t) + A\, U(t) \equiv 0 & \text{for } 0 < t < T \\ U(0) \equiv I, & \end{cases} \tag{4.1}$$

where by solution $U(t)$ we mean an equivalence class of t-regular families of bounded linear operators from $L^2(M)$ into itself such that $U(t)u \in D(A)$, $\forall t > 0$, $\forall u \in L^2(M)$. Moreover, $U(t) \sim U'(t)$ if and only if $U(t) - \tilde{U}(t)$ is a t-regular family of x-smoothing operators. $U(t)$ will always be referred to as the heat parametrix of A.

We first establish a result of uniqueness via the semigroup approach, drawing a comparison with the heat semigroup e^{-tA}. Then we develop a detailed construction of $U(t)$ through a convenient pseudodifferential representation, highlighting its singularity in $t = 0$ and its smoothing effect for positive t.

Theorem 4.1 (Uniqueness). *If $U(t)$ is a solution of* (4.1)*, then $U(t)$ is unique (in the sense of the equivalence class precised above).*

Proof. Let e^{-tA} be the heat semigroup generated by $-A$ (Theorem 3.3). Now pick any $u_0 \in L^2(M)$ and suppose $U(t)$ is a solution of (4.1). Then $u(t) := U(t)u_0$ must satisfy the following problem:

$$\begin{cases} u'(t) + Au(t) = R(t)u_0 \\ u(0) = u_0 + \tilde{R}(0)u_0, \end{cases}$$

for some t-regular maps R, $\tilde{R}$ of smoothing operators. From Duhamel's Theorem there exists a unique solution given by

$$u(t) = e^{-tA}(u_0 + \tilde{R}(0)u_0) + \int_0^t e^{-(t-\tau)A}\ R(\tau)u_0\, d\tau.$$

Taking into account that the semigroup e^{-tA} is analytic, it follows that

$$e^{-tA}v \in \bigcap_{k\geq 0} D(A^k) = \bigcap_{k\geq 0} H^{km,k\mu}(M) = \mathcal{S}(M) \qquad \forall v \in L^2(M).$$

Thus we can write

$$U(t)u_0 = e^{-tA}u_0 + Q(t)u_0,$$

for some t-regular family of x-smoothing $Q(t)$. This amounts to saying that $U(t)$ is in the same equivalence class of e^{-tA}. The theorem is proved. □

Theorem 4.2 (Existence). *If Conditions* (H1)*,* (H2) *and* (H3) *are fulfilled, then there exists a t-regular map $U(t)$ of bounded linear operators on $L^2(M)$ solution of* (4.1) *such that in each local chart $(\Omega; x_1, \ldots, x_n)$ of M (a representative of the class) $U(t)$ is "pseudodifferential" of the form*

$$U(t)\phi(x) = \int e^{i(x-y)\cdot\xi} u(t, x, \xi)\ \phi(y)\ dy đ\xi,$$

whose symbol has the following properties:

i) $u \in C^\infty(]0, T[\times\Omega \times \mathbb{R}^n)$*;*

ii) for any given integers $N, l \geq 0$ and any $\alpha, \beta \in \mathbb{N}^n$ there exists $C > 0$ such that

$$\left|\partial_t^l \partial_\xi^\alpha \partial_x^\beta u(t, x, \xi)\right| \leq C\, t^{-N} \langle\xi\rangle^{(l-N)m-|\alpha|} \langle x\rangle^{(l-N)\mu-|\beta|},$$

for all $(t, x, \xi) \in]0, T] \times \Omega' \times \mathbb{R}^n$, where $\Omega' = \Omega$ whenever Ω is an exit chart; otherwise Ω' is any compact subset of Ω and C may depend on Ω'. In all cases $C = C_{Nl\alpha\beta}(\Omega')$ does not depend upon t, once some $T > 0$ has been fixed.

In particular, $U(t)$ is smoothing for positive t.

Proof. The parametrix is constructed locally in each chart map (Ω, ϕ) by identifying Ω with an open set $U \subset \mathbb{R}^n$ through a coordinate system $(x_1, \ldots, x_n)$ on Ω. We refer to [21, pp. 134–138] for the case of relatively compact charts.

It suffices then to carry out the construction in an exit chart and we no longer label the operator $U(t)$, nor the symbol $u(t, x, \xi)$, by the name of the chart. It is natural to look for the symbol of $U(t)$ in the form of an asymptotic sum:

$$\sum_{j\geq 0} u_j(t, x, \xi) \sim u(t, x, \xi),$$

with $u_j \in S^{-j,-j}(U)$. Without loss of generality we can assume that for the symbol $a(x, \xi)$ the estimates of Definition 2.3 do hold uniformly in $U \times \mathbb{R}^n$ (by shrinking the chart, if necessary). The composition formula allows us to convert Problem (4.1) in terms of the symbols of A and $U(t)$ as follows:

$$\begin{cases} \partial_t u(t, x, \xi) \;+ \displaystyle\sum_{|\alpha|\geq 0} \frac{1}{\alpha!} D_\xi^\alpha a(x, \xi) \partial_x^\alpha u(t, x, \xi) = 0 \\ u(0, x, \xi) = 1, \end{cases}$$

and replace the asymptotic expansion of u to obtain also

$$\begin{cases} \displaystyle\sum_{j\geq 0} \partial_t u_j(t, x, \xi) \;+ \sum_{l,|\alpha|\geq 0} \frac{1}{\alpha!} D_\xi^\alpha a(x, \xi)\, \partial_x^\alpha u_l(t, x, \xi) \;=\; 0 \\ u_0(0, x, \xi) = 1,\; u_j(0, x, \xi) = 0, \quad \forall j \geq 1. \end{cases}$$

Taking into account the uniqueness of the parametrix and regrouping the terms of the same order, we can deduce the following transport equations, coupled with the respective initial conditions:

$$\begin{cases} \partial_t u_0(t, x, \xi) + a(x, \xi) u_0(t, x, \xi) = 0 \\ u_0(0, x, \xi) = 1, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \partial_t u_j(t, x, \xi) + a u_j + \displaystyle\sum_{\substack{l+|\alpha|=j \\ 0\leq l\leq j-1}} \frac{1}{\alpha!} D_\xi^\alpha a(x, \xi) \partial_x^\alpha u_l(t, x, \xi) = 0 \\ u_j(0, x, \xi) = 0, \end{cases} \quad \text{for all j} \geq 1. \tag{4.3}$$

Freezing x and ξ, (4.2) and (4.3) turn out to be Cauchy problems for first-order ordinary linear equations in the variable t. From (4.2) we immediately get

$$u_0(t, x, \xi) = e^{-ta(x,\xi)}. \tag{4.4}$$

For $j = 1$, Problem (4.3) is simply

$$\begin{cases} \partial_t u_1 = -a u_1 + e^{-ta} \sum_{|\alpha|=1} D_\xi^\alpha a \partial_x^\alpha u_0 \\ u_1(0) = 0. \end{cases}$$

We now exploit the following useful extension of the Leibniz formula:

$$\partial_x^\alpha(e^{-ta}) = e^{-ta}\sum_{s=1}^{|\alpha|} t^s S^{sm,s\mu-|\alpha|} \qquad \forall|\alpha|\geq 1, \tag{4.5}$$

where $t^l S^{h,k}$ stands for a function $t^l g(x,\xi)$ for some $g\in S^{h,k}(U)$. In particular, for $|\alpha|=1$, we have $D_\xi^\alpha a\cdot\partial_x^\alpha u_0\in S^{m-1,\mu}(U)\cdot e^{-ta}tS^{m,\mu-1}(U)=e^{-ta}tS^{2m-1,2\mu-1}(U)$. Hence, an immediate integration gives

$$u_1(t,x,\xi)=e^{-ta(x,\xi)}t^2S^{2m-1,2\mu-1}.$$

Proceeding by induction it is easily proved that

$$u_j(t,x,\xi)=e^{-ta(x,\xi)}\sum_{k=2}^{2j} t^k S^{km-j,k\mu-j}, \qquad \text{for j}\geq 1. \tag{4.6}$$

In order to make this formal construction meaningful and prove the regularity properties with respect to the variable t, we observe that from (4.4) and (4.6) we have $\forall j\geq 0$, $\forall\alpha,\beta\in\mathbb{N}^n$, $\forall l\in\mathbb{N}$

$$\partial_\xi^\alpha\partial_x^\beta\partial_t^l u_j(t,x,\xi)=e^{-t\,a(x,\xi)}\sum_{s=s^*}^{2j+|\alpha+\beta|}(t\,a(x,\xi))^s\; S^{lm-j-|\alpha|,l\mu-j-|\beta|},$$

where $s^*=s^*(\alpha,\beta,l,j)\geq 0$. It readily follows from the ellipticity that, $\forall N\in\mathbb{N}$,

$$\left|t^N\partial_t^l\partial_x^\beta\partial_\xi^\alpha u_j(t,x,\xi)\right|\leq C\,\langle\xi\rangle^{(l-N)m-j-|\alpha|}\,\langle x\rangle^{(l-N)\mu-j-|\beta|},$$

from which

$$\left|t^N\partial_t^l\partial_x^\beta\partial_\xi^\alpha\left(u-\sum_{j\leq J-1}u_j\right)\right|\leq C\,\langle\xi\rangle^{(l-N)m-J-|\alpha|}\langle x\rangle^{(l-N)\mu-J-|\beta|}.$$

This suffices to conclude the proof of the theorem. □

Remark 4.1. To convince ourselves that the construction of a local parametrix actually yields a global one, let us suppose that (Ω,ϕ) and (Ω_1,ϕ_1) are local charts of M such that $\Omega\cap\Omega_1\neq\emptyset$ and $U^\phi(t)$ has been constructed as above satisfying

$$\begin{cases}\dfrac{d}{dt}U^\phi(t)v+A^\phi\cdot U^\phi v=R(t)v\\[2mm] U^\phi(0)v=v+R'(0)v,\end{cases}\qquad \forall v\in L^2(\Omega\cap\Omega_1). \tag{4.7}$$

Let $U^{\phi_1}(t)$ be constructed analogously. For the pullback, $\tilde U^\phi(t)$, of $U^\phi(t)$ via the transition map between Ω and Ω_1 we can say that it satisfies the following:

$$\begin{cases}\dfrac{d}{dt}\tilde U^\phi(t)v+\tilde A^\phi\cdot\tilde U^\phi v=\tilde R(t)v\\[2mm] \tilde U^\phi(0)v=v+\tilde R'(0)v,\end{cases} \tag{4.8}$$

in as much as $(\frac{d}{dt}U^{\phi(t)})^{\sim} = \frac{d}{dt}\tilde{U}^{\phi}(t)$ and $(A^{\phi}\cdot U^{\phi})^{\sim} = \tilde{A}^{\phi}\cdot\tilde{U}^{\phi}$. Now, from Theorem 2.9 there exists $Q \in L^{-\infty,-\infty}(\Omega\cap\Omega_1)$ such that $\tilde{A}^{\phi} = A^{\phi_1} + Q$, whereby,

$$\begin{cases} \dfrac{d}{dt}\tilde{U}^{\phi}(t)v + A^{\phi_1}\cdot\tilde{U}^{\phi}v = (\tilde{R}(t) - Q\cdot\tilde{U}^{\phi}(t))v \\ \tilde{U}^{\phi}(0)v = v + \tilde{R}'(0)v. \end{cases} \tag{4.9}$$

Consequently, from Theorem 4.1 we can deduce that $\tilde{U}^{\phi}(t) \equiv U^{\phi_1}(t)$.

It is a standard procedure now to glue together the local parametrices, once we have a compatible partition of unity (Remark 3.2).

5. Trace asymptotics

We assume that Hypotheses (H1), (H2), (H3) of the previous section are fulfilled. The heat parametrix of A, $U(t)$, constructed there is regularizing, hence trace class [18], when $t > 0$. In particular, it has the same singularity in $t = 0$ as the heat semigroup, for the difference is a t-regular family of x-smoothing operators. From Theorem 3.3 we then have

$$\text{Trace}\, U(t) = \int_0^{+\infty} e^{-t\lambda} d\mathcal{N}(\lambda, A) + \mathcal{O}(1), \qquad t \to 0^+. \tag{5.1}$$

The study of the asymptotic behaviour of Trace $U(t)$, which we are going to carry out, will therefore provide, by means of Karamata's Tauberian Theorem [20], the Weyl's estimate for $\mathcal{N}(\lambda, A)$ we are seeking. Let us state the most important result of the section:

Theorem 5.1. *Assume $A \in ECL^{m,\mu}(M)$, $A^* = A > 0$ and $m, \mu > 0$. Then the trace of the corresponding heat parametrix $U(t)$ can be estimated for small t by:*

$$\text{Trace}\, U(t) = \Gamma\left(1 + \frac{n}{m}\right) C_m\, t^{-\frac{n}{m}} + o(t^{-\frac{n}{m}}) \qquad \text{if } \mu > m;$$

$$\text{Trace}\, U(t) = \Gamma\left(1 + \frac{n}{\mu}\right) C_\infty\, t^{-\frac{n}{\mu}} + o(t^{-\frac{n}{\mu}}) \qquad \text{if } \mu < m;$$

and, finally,

$$\text{Trace}\, U(t) = \Gamma\left(1 + \frac{n}{m}\right) C_\sharp\, t^{-\frac{n}{m}} \log(t^{-1}) + o(t^{-\frac{n}{m}} \log(t^{-1})), \qquad \text{if } \mu = m;$$

where $\Gamma(\cdot)$ is the gamma function and

$$C_m := \int_{\{a_m \le 1\}} dx đ\xi \tag{5.2}$$

$$C_\infty := \frac{1}{n}\int_{T_X^* M} a_\infty^{-n/\mu}\, d\omega đ\xi \tag{5.3}$$

$$C_\sharp := \frac{1}{m}\int_{\{a_\sharp \le 1\}} d\omega đ\xi, \tag{5.4}$$

with $dxđ\xi$ the volume element on T^*M and $d\omega đ\xi$ the one on T_X^*M given by restriction on X of the volume on T^*M.

The proof of this result will follow from a sequence of propositions. To reach the goal it is enough to consider the exit chart $\dot{f}_\pi^{-1} : \dot{\mathcal{C}} \to U_{\delta_f}$ where we study the asymptotic behaviour of

$$\int_{\mathbb{R}^n} \int_{U_{\delta_f}} u^{(f)}(t, x, \xi) dx đ\xi,$$

giving the local expression of (5.1), seen in Part iv) of Theorem 3.3. The contributions from relatively compact local charts are taken for granted and can be found in [18]. In order to simplify the notation, we "drop the f" from the names of the symbols and write U_δ instead of U_{δ_f}.

In what follows we set, when $b > 0$ and $x \in \mathbb{R}$, $\Gamma(b, x) := \int_x^\infty e^{-\tau} \tau^{b-1} d\tau$ for the incomplete gamma function. Each asymptotic formula, unless othewise stated, is meant for $t \to 0^+$. We start by proving a general lemma.

Lemma 5.2. *Let $Y, Z \subset \mathbb{R}^n$ be open sets, with $Z = \{z : |z| > R\}$ for some $R > 0$. Assume also that $\phi \in C^\infty(Y \times Z)$ and $\exists l > 0$ such that $\phi(y, tz) = t^l \phi(y, z)$ for every $(y, z) \in Y \times Z$ and $t > 1$. For any fixed $k \geq 0$ define $k^* := (k + n)/l$.*

I) If $\phi > 0$, then we have the following identity:

$$\int_Z e^{-t\phi(y,z)} |z|^k dz = \frac{t^{-k^*}}{l} \int_{\mathbb{S}^{n-1}} \Gamma(k^*, tR^l \phi(y, \omega)) \phi(y, \omega)^{-k^*} d\omega.$$

II) If $\phi(y, \omega) \geq c\langle y\rangle^p$ on Y for some $p > \frac{n}{k^*}$, then we can estimate

$$\int_Y \int_Z e^{-t\phi(y,z)} |z|^k dz dy = \frac{\Gamma(k^*)}{l} t^{-k^*} \int_Y \int_{\mathbb{S}^{n-1}} \phi(y, \omega)^{-k^*} d\omega dy + o(t^{-k^*}).$$

Proof. Switching to polar coordinates, $z = \sigma\omega$ and putting $\sigma = (\frac{\rho}{t\phi(y,\omega)})^{1/l}$, we obtain

$$\begin{aligned}\int_Z e^{-t\phi(y,z)} |z|^k dz &= \int_{\mathbb{S}^{n-1}} \int_R^\infty e^{-t\sigma^l \phi(y,\omega)} \sigma^{k+n-1} d\sigma d\omega \\ &= \frac{1}{l} \int_{\mathbb{S}^{n-1}} \left(\int_{tR^l\phi(y,\omega)}^\infty e^{-\rho} \rho^{k^*-1} (t\phi(y, \omega))^{-k^*} d\rho \right) d\omega,\end{aligned}$$

so Part I) is proved. Part II) is a consequence of I) by means of the convergent expansion of $\Gamma(b, x)$ in ascending powers of x: $\Gamma(b, x) = \Gamma(b) - \sum_{k=0}^{+\infty} c_k(b) x^{b+k}$, with $c_k(b) = (-1)^k/[(b + k)k!])$ for all x, and an application of Lebesgue's dominated convergence theorem. □

Corollary 5.3. *For each $a \in ECS^{m,\mu}(U_\delta)$, $a \mapsto \{a_\infty, a_m, a_\sharp\}$, we have, for small t,*

$$\int_{|\xi|\geq 1} \int_{U_\delta} e^{-ta_m(x,\xi)} dx đ\xi = \Gamma\left(1 + \frac{n}{m}\right) \tilde{C}_m t^{-\frac{n}{m}} + o(t^{-\frac{n}{m}}) \quad \text{if } \mu > m$$

$$\int_{\mathbb{R}^n} \int_{U_\delta} e^{-t|x|^\mu a_\infty(\frac{x}{|x|},\xi)} dx đ\xi = \Gamma\left(1 + \frac{n}{\mu}\right) \tilde{C}_\infty t^{-\frac{n}{\mu}} + o(t^{-\frac{n}{\mu}}) \quad \text{if } \mu < m,$$

and finally, when $m = \mu$,

$$\int_{|\xi|\geq 1}\int_{U_\delta} e^{-t|x|^\mu a_\sharp(\frac{x}{|x|},\xi)} dx đ\xi = \Gamma\left(1+\frac{n}{m}\right) \tilde{C}_\sharp\, t^{-\frac{n}{m}} \log\frac{1}{t} + o\left(t^{-\frac{n}{m}} \log\frac{1}{t}\right),$$

where

$$\tilde{C}_m = \frac{(2\pi)^{-n}}{n}\int_{\mathbb{S}^{n-1}}\int_{U_\delta} a_m^{-\frac{n}{m}}(x,\theta)dx d\theta \tag{5.5}$$

$$\tilde{C}_\infty = \frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} a_\infty^{-\frac{n}{\mu}}(\omega,\xi) d\omega đ\xi \tag{5.6}$$

$$\tilde{C}_\sharp = \frac{(2\pi)^{-n}}{nm}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} a_\sharp^{-\frac{n}{m}}(\omega,\theta) d\omega d\theta. \tag{5.7}$$

Proof. The first two estimates are obtained by applying Part II) of Lemma 5.2, twice, respectively, with $l = m$, $k = 0$, $Y = U_\delta$, $Z = \{|\xi| > 1\}$ and with $l = \mu$, $k = 0$, $Y = \mathbb{R}^n$, $Z = U_\delta$.

To prove the last estimate, we apply Part I) of Lemma 5.2 with $l = m = \mu$, $k = 0$, $Y = \{|\xi| > 1\}$, $Z = U_\delta$) and move to polar coordinates $x = \rho'\omega$ and $\xi = \sigma'\eta/\rho'$, obtaining

$$\begin{aligned}&\int_{|\xi|\geq 1}\int_{U_\delta} e^{-t|x|^m a_\sharp(\frac{x}{|x|},\xi)} dx đ\xi \\ &\quad = \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} d\omega đ\eta \int_\delta^{+\infty} d\rho' \int_{\rho'}^{+\infty} \frac{d\sigma'}{\rho'} (\sigma')^{n-1} e^{-t(\sigma')^m a_\sharp(\omega,\eta)}.\end{aligned} \tag{5.8}$$

We put $t(\sigma')^m a_\sharp(\omega,\eta) = \sigma$ and $t(\rho')^m a_\sharp(\omega,\eta) = \rho$, from which it follows that

$$(\sigma')^{n-1} d\sigma = \frac{1}{m}\,(t a_\sharp)^{-n/m} \sigma^{n/m-1} d\sigma, \qquad d\rho'/\rho' = \frac{1}{m}\, d\rho/\rho;$$

hence (5.8) becomes

$$\frac{t^{-\frac{n}{m}}}{m^2}\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} a_\sharp(\omega,\eta)^{-n/m}\, d\omega đ\eta \int_{t\delta^m a_\sharp(\omega,\eta)}^{+\infty} \frac{1}{\rho}\,\Gamma\left(\frac{n}{m},\rho\right)\, d\rho.$$

Let us note that

$$\begin{aligned}&\int_{t\delta^m a_\sharp(\omega,\theta)}^{+\infty} \frac{1}{\rho}\Gamma\left(\frac{n}{m};\rho\right) d\rho = \int_{t\delta^m a_\sharp(\omega,\theta)}^{1} \frac{1}{\rho}\Gamma\left(\frac{n}{m};\rho\right) d\rho + O(1) \\ &= -\log(t\delta^m a_\sharp(\omega,\theta))\Gamma\left(\frac{n}{m}; t a_\sharp(\omega,\theta)\right) + \int_{t\delta^m a_\sharp(\omega,\theta)}^{1} \log(\rho)\rho^{\frac{n}{m}-1} e^{-\rho} d\rho + O(1).\end{aligned}$$

It is easily seen that $\int_{t\delta^m a_\sharp(\omega,\theta)}^{1} \log(\rho)\rho^{\frac{n}{m}-1} e^{-\rho} d\rho = O(1)$.

This makes the proof complete, taking into account the expansion of the incomplete gamma function. □

Proposition 5.4. *For each $a \in ECS^{m,\mu}(U_\delta)$, $a \mapsto \{a_\infty, a_m, a_\sharp\}$, we have, for small t:*

(i) if $\mu > m$, then

$$\int_{\mathbb{R}^n}\int_{U_\delta} e^{-ta(x,\xi)}dx đ\xi = \int_{|\xi|\geq 1}\int_{U_\delta} e^{-ta_m(x,\xi)}dx đ\xi + o(t^{-\frac{n}{m}});$$

(ii) if $\mu < m$, then

$$\int_{\mathbb{R}^n}\int_{U_\delta} e^{-ta(x,\xi)}dx đ\xi = \int_{\mathbb{R}^n}\int_{U_\delta} e^{-t|x|^\mu a_\infty(\frac{x}{|x|},\xi)}dx đ\xi + o(t^{-\frac{n}{\mu}});$$

(iii) if $\mu = m$, then

$$\int_{\mathbb{R}^n}\int_{U_\delta} e^{-ta(x,\xi)}dx đ\xi = \int_{|\xi|\geq 1}\int_{U_\delta} e^{-t|x|^\mu a_\sharp(\frac{x}{|x|},\xi)}dx đ\xi + O(t^{-\frac{n}{m}}).$$

Proof. (ii) We prefer to begin with the case $\mu < m$. For any $R > \delta$ we can write

$$\int_{\mathbb{R}^n}\int_{U_\delta} e^{-ta(x,\xi)}dx đ\xi - \int_{\mathbb{R}^n}\int_{U_\delta} e^{-t|x|^\mu a_\infty(\frac{x}{|x|},\xi)}dx đ\xi$$

$$= o(t^{-\frac{n}{\mu}}) + \int_{\mathbb{R}^n}\int_{|x|\geq R}\left(e^{-ta(x,\xi)} - e^{-t|x|^\mu a_\infty(\frac{x}{|x|},\xi)}\right)dx đ\xi.$$

In fact the contribution of the integration over any bounded subset of U_δ turns out to be $\mathcal{O}(t^{-\frac{n}{m}}) = o(t^{-\frac{n}{\mu}})$ in view of the ellipticity property and Lemma 5.2 (with $k = 0$, $l = m$, Y bounded; $k^* = n/m$). Thus, what is to be proved is that for a sufficiently large R we have

$$t^{\frac{n}{\mu}}\int_{\mathbb{R}^n}\int_{|x|\geq R}\left(e^{-ta(x,\xi)} - e^{-t|x|^\mu a_\infty(\frac{x}{|x|},\xi)}\right)dx đ\xi \overset{t\to 0^+}{\longrightarrow} 0. \tag{5.9}$$

To justify the passage to the limit as $t \to 0^+$ under the sign, we put $x = \rho\omega$ and then $t\rho^\mu a_\infty(\omega,\xi) = \sigma$, so that $\rho^{n-1}d\rho = \frac{1}{\mu}(ta_\infty(\omega,\xi))^{-\frac{n}{\mu}}\sigma^{\frac{n}{\mu}-1}d\sigma$ and the left member of (5.9) turns into

$$\frac{1}{\mu}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} a_\infty(\omega,\xi)^{-\frac{n}{\mu}}\int_{tR^\mu a_\infty(\omega,\xi)}^{\infty}\sigma^{\frac{n}{\mu}-1}\left[e^{-t\tilde{a}(t,\sigma,\omega,\xi)} - e^{-\sigma}\right]d\sigma d\omega đ\xi, \tag{5.10}$$

where

$$\tilde{a}(t,\sigma,\omega,\xi) := a((\sigma/t)^{\frac{1}{\mu}}a_\infty(\omega,\xi)^{-\frac{1}{\mu}}\omega,\xi).$$

We now define

$$\chi(t,\sigma,\omega,\xi) := \frac{\tilde{a}(t,\sigma,\omega,\xi) - (\sigma/t)}{\sigma/t},$$

so as to write $e^{-t\tilde{a}(t,\sigma,\omega,\xi)} = e^{-\sigma(1+\chi(t,\sigma,\omega,\xi))}$. Notice that from i) of Definition 2.5 and the ellipticity we obtain

$$|\chi(t,\sigma,\omega,\xi)| \leq \frac{C(\sigma/t)^{\frac{\mu-1}{\mu}} a_\infty(\omega,\xi)^{-\frac{\mu-1}{\mu}} \langle\xi\rangle^m}{\sigma/t} \leq C''(R^\mu\langle\xi\rangle^m)^{-\frac{1}{\mu}}\langle\xi\rangle^{\frac{m}{\mu}} = CR^{-1}.$$

Here we have also used the fact that, on the integration domain, $\frac{\sigma}{t} \geq R^\mu a_\infty(\omega,\xi) \geq cR^\mu\langle\xi\rangle^m$. So, for a fixed $\epsilon \in]0,1[$ we can find an R so large to have $\chi(t,\sigma,\omega,\xi) \geq -\epsilon$. Then the integrand function of (5.10) can be estimated from above, independently of t, by

$$a_\infty(\omega,\xi)^{-\frac{n}{\mu}}\sigma^{\frac{n}{\mu}-1}\left[e^{-\sigma(1+\epsilon)} - e^{-\sigma}\right],$$

which is clearly summable with respect to σ, ω and ξ. An application of Lebesgue's dominated convergence theorem yields the desired result.

(i) The case $\mu > m$ is similar to the previous one. For any $r > 0$ we have

$$\int_{\mathbb{R}^n}\int_{U_\delta} e^{-ta(x,\xi)}dx đ\xi - \int_{|\xi|>1}\int_{U_\delta} e^{-ta_m(x,\xi)}dx đ\xi$$

$$= o(t^{-\frac{n}{m}}) + \int_{|\xi|>r}\int_{U_\delta}\left(e^{-ta(x,\xi)} - e^{-ta_m(x,\xi)}\right) dx đ\xi.$$

In fact integration over ξ-bounded subsets can now be estimated by $O(t^{-\frac{n}{\mu}}) = o(t^{-\frac{n}{m}})$ (again using ellipticity and Lemma 5.2 with $l = \mu$).

The proof that, for a suitable positive r, we have

$$t^{\frac{n}{m}}\int_{|\xi|>r}\int_{U_\delta}\left(e^{-ta(x,\xi)} - e^{-ta_m(x,\xi)}\right) dx đ\xi \xrightarrow{t\to 0^+} 0 \tag{5.11}$$

uses the same arguments of (i).

(iii) Let $\mu = m$. Arguing as we did in the previous cases, we have only to prove that, for r and R large enough,

$$t^{-\frac{n}{m}}\int_{|\xi|\geq r}\int_{|x|\geq R}\left(e^{-ta(x,\xi)} - e^{-t|x|^m a_\sharp(\frac{x}{|x|},\xi)}\right) dx đ\xi = O(1).$$

The term to be estimated can be split into the sum of

$$t^{\frac{n}{m}}\int_{|\xi|\geq r}\int_{|x|\geq R}\left(e^{-ta(x,\xi)} - e^{-t|x|^m a_\infty(\frac{x}{|x|},\xi)}\right) dx đ\xi \tag{5.12}$$

and

$$t^{\frac{n}{m}}\int_{|\xi|\geq r}\int_{|x|\geq R}\left(e^{-t|x|^m a_\infty(\frac{x}{|x|},\xi)} - e^{-t|x|^m a_\sharp(\frac{x}{|x|},\xi)}\right) dx đ\xi. \tag{5.13}$$

The conclusion about (5.12) follows from Part (i), while (5.13) can be estimated by using the same techniques we used in (i) and (ii). This completes the proof. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. From what we have seen in the previous section, the symbol of the heat parametrix $U(t)$ on the local chart has the asymptotic expansion $u \sim \sum_{j\geq 0} u_j$. The trace formula can therefore be formulated, locally, as follows:

$$\int_{U_\delta}\int_{U_\delta} u(t,x,\xi)dx đ\xi = \int_{U_\delta}\int_{U_\delta} e^{-ta(x,\xi)}dx đ\xi + \int_{U_\delta}\int_{U_\delta}\sum_{j=1}^{N-1} u_j(t,x,\xi)dx đ\xi + \int_{U_\delta}\int_{U_\delta} p_N(t,x,\xi)dx đ\xi,$$

where $p_N(t,x,\xi) \sim \sum_{j\geq N} u_j(t,x,\xi)$. Corollary 5.3 and Proposition 5.4 give

$$\int_{U_\delta}\int_{U_\delta} e^{-ta(x,\xi)}dx đ\xi \sim \begin{cases} \Gamma\big(1+\frac{n}{m}\big)\,\tilde{C}_m\, t^{-\frac{n}{m}} + o(t^{\frac{n}{m}}) & \text{if } \mu > m \\ \Gamma\big(1+\frac{n}{\mu}\big)\,\tilde{C}_\infty\, t^{-\frac{n}{\mu}} + o(t^{-\frac{n}{\mu}}) & \text{if } \mu < m \\ \Gamma\big(1+\frac{n}{m}\big)\,\tilde{C}_\sharp\, t^{-\frac{n}{m}}\, \log\frac{1}{t} + o\big(t^{-\frac{n}{m}}\, \log\frac{1}{t}\big) & \text{if } \mu = m, \end{cases}$$

where the constants $\tilde{C_m}$, $\tilde{C}_\infty$ and $\tilde{C}_\sharp$ have been defined in (5.6), (5.5) and (5.7), respectively.

Recalling that, for any $N > 0$, $p_N(t,x,\xi) \in C^\infty([0,T); S^{-N,-N}(\mathbb{R}^n))$, we can conclude, choosing $N = n+1$ once for all, that

$$|\int_{U_\delta}\int_{U_\delta}\sum_{j\geq N} u_j(t,x,\xi)dxd\xi| \leq C\int_{U_\delta}\int_{U_\delta}\langle x\rangle^{-N}\langle\xi\rangle^{-N}dxd\xi,$$

with C independent of t. In order to have the proof completed we have to analyse the contribution of the finite sum

$$R(t) = \sum_{1\leq j\leq n}\int_{U_\delta}\int_{U_\delta} u_j(t,x,\xi)dx đ\xi.$$

Taking into account the expression of u_j, $R(t)$ is the sum of terms such as

$$R_{j,h}(t) = t^h\int_{U_\delta}\int_{U_\delta} e^{-ta(x,\xi)}\, S^{hm-j,h\mu-j}dx đ\xi.$$

The summation is taken over $1 \leq j \leq N-1$ e $1 \leq h \leq 2j$. Lemma 5.2 ensures that $|R_{j,h}(t)| = I^{j,h}(t) + O(t^{-\frac{n}{m^*}+\frac{j}{m^*}})$, where $m^* = \min\{m,\mu\}$ and

$$I^{j,h}(t) := t^h\int_{|\xi|\geq r}\int_{|x|\geq R} e^{-ta(x,\xi)}|\xi|^{hm-j}|x|^{h\mu-j}dxd\xi.$$

Here r and R are arbitrary positive constants. After a change of coordinates, we can estimate $I^{j,h}(t)$ as follows:

$$I^{j,h}(t) \leq Ct^{-n/m+j/m}\int_{tR^\mu r^m}^{+\infty}\rho^{(n-j)/(\frac{1}{m}-\frac{1}{\mu})-1}\,\Gamma\left(\frac{n}{\mu}-\frac{j}{\mu}+h,\rho\right)d\rho,$$

from which

$$I^{j,h}(t) = \begin{cases} O(t^{-\frac{n}{m^*}+\frac{j}{m^*}}) & \text{if } m \neq \mu \\ O(t^{-\frac{n}{m}+\frac{j}{m}} \log t^{-1}) & \text{if } m = \mu. \end{cases}$$

All this shows that on any local chart U_δ

$$\int_{U_\delta}\int_{U_\delta} u(t,x,\xi)dx đ\xi \sim \begin{cases} \Gamma\big(1+\frac{n}{m}\big)\,\tilde{C}_m\, t^{-\frac{n}{m}} + o(t^{\frac{n}{m}}) & \text{if } \mu > m \\ \Gamma\big(1+\frac{n}{\mu}\big)\,\tilde{C}_\infty\, t^{-\frac{n}{\mu}} + o(t^{-\frac{n}{\mu}}) & \text{if } \mu < m \\ \Gamma\big(1+\frac{n}{m}\big)\,\tilde{C}_\sharp\, t^{-\frac{n}{m}} \log\frac{1}{t} + o\big(t^{-\frac{n}{m}} \log\frac{1}{t}\big) & \text{if } \mu = m. \end{cases}$$

We now prove the invariant meaning of the constants $\tilde{C}_\infty$, $\tilde{C}_m$ and $\tilde{C}_\sharp$ showing that these are, respectively, the local expression of the quantities defined by (5.3), (5.2) and (5.4). Set

$$\tilde{C}_\infty^{(f)} = \frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} a_\infty^{(f)}(\omega,\xi)^{-n/\mu} d\omega đ\xi,$$

where $a_\infty^{(f)}(\omega,\xi)$ is the local expression of the exit symbol a_∞ of A with respect to the exit chart f. It suffices to prove that, if $g \sim f$, then $\tilde{C}_\infty^{(g)} = \tilde{C}_\infty^{(f)}$.

In fact, let $f^{-1}g(\rho,\omega) = (\rho, \Theta(\omega))$ and $F(x) = |x|\Theta(\frac{x}{|x|})$; then from Theorem 3.1, it follows that

$$\begin{aligned}
\tilde{C}_\infty^{(g)} &= \frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} a_\infty^{(g)}(\sigma,\eta)^{-n/\mu} d\sigma đ\eta \\
&= \frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} a_\infty^{(f)}\big(\Theta(\omega),{}^t\, dF_\omega^{-1}(\xi)\big)^{-n/\mu} \det|d\Theta(\omega)| \cdot \big|\det dF_\omega^{-1}\big| d\omega đ\xi \\
&= \frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} a_\infty^{(f)}(\omega,\xi)^{-n/\mu} |\det d(\Theta\circ F^{-1})(\omega)| d\omega đ\xi.
\end{aligned}$$

Notice that $\Theta \circ F^{-1}(\omega) = \Theta(\pi^{-1}\circ(Id,\Theta)^{-1}\circ\pi)(\omega) = \omega$ and therefore

$$\tilde{C}_\infty^{(g)} = \frac{1}{n}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} a_\infty^{(f)}(\omega,\xi)^{-n/\mu}\cdot 1\, d\omega đ\xi = \tilde{C}_\infty^{(f)}.$$

As far as $\tilde{C}_m$ and $\tilde{C}_\sharp$ are concerned, it is readily seen ([18, proof of Lemma 13.1, p. 110]) that

$$\tilde{C}_m = \int_{\{a_m(x,\xi)\le 1\}} dx đ\xi, \qquad \tilde{C}_\sharp = \frac{1}{m}\int_{\{a_\sharp(\omega,\xi)\le 1\}} d\omega đ\xi.$$

Theorem 3.1 allows us to interpret these quantities as local expressions of the global constants (5.2) and (5.4). This completes the proof of the theorem. □

Corollary 5.5 (Weyl formula). *Let $A \in ECL^{m,\mu}(M)$, $A^* = A > 0$ and $m, \mu > 0$. Then the corresponding counting function $\mathcal{N}(\lambda, A)$ can be estimated for large λ as follows:*

$$\mathcal{N}(\lambda,A) = C_m\,\lambda^{\frac{n}{m}} + o(\lambda^{\frac{n}{m}}) \qquad \text{if } \mu > m,$$

$$\mathcal{N}(\lambda,A) = C_\infty\,\lambda^{\frac{n}{\mu}} + o(\lambda^{\frac{n}{\mu}}) \qquad \text{if } \mu < m;$$

and, finally,

$$\mathcal{N}(\lambda, A) = C_\sharp \lambda^{\frac{n}{m}} \log \lambda + o(\lambda^{\frac{n}{m}} \log \lambda) \qquad \text{if } \mu = m.$$

Proof. Apply Karamata's Tauberian Theorem ([20, p. 89]). □

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