Annali di Matematica 180, 451–482 (2002)

DOI (Digital Object Identifier) 10.1007/s102310100023

Michael Dreher · Ingo Witt

Edge Sobolev spaces and weakly hyperbolic equations

Received: October 3, 2000
Published online: December 19, 2001 –

Summary. Edge Sobolev spaces are proposed as a main new tool for the investigation of weakly hyperbolic equations. The well-posedness of the linear and semilinear Cauchy problem in the class of these edge Sobolev spaces is proved. An application to the propagation of singularities for solutions to the semilinear problem is considered.

Mathematics Subject Classification (2000). Primary: 35L80; Secondary: 46E35, 35L15

1. Introduction

This paper is devoted to the study of weakly hyperbolic Cauchy problems

$$Lu = f(u), \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{1.1}$$
$$Lv = 0, \qquad v(0,x) = u_0(x), \quad v_t(0,x) = u_1(x), \tag{1.2}$$

where

$$L = \partial_t^2 + 2\sum_{j=1}^n \lambda(t)c_j(t)\partial_t\partial_{x_j} - \sum_{i,j=1}^n \lambda(t)^2 a_{ij}(t)\partial_{x_i}\partial_{x_j} + \sum_{j=1}^n \lambda'(t)b_j(t)\partial_{x_j} + c_0(t)\partial_t \tag{1.3}$$

and $\lambda(t) = t^{l_*}$ for some $l_* \in \mathbb{N}_+ = \{1, 2, 3, \dots\}$.

The special choice of the exponents of t in (1.3) reflects so-called Levi conditions which are necessary and sufficient for the C^∞ well-posedness of the linear Cauchy problem, see [10], [14]. As the basic new ingredient, solutions to (1.1), (1.2) are sought in edge Sobolev spaces, a concept which has been initially invented in the analysis of elliptic pseudodifferential equations near edges, see [8], [18].

M. Dreher: Institute of Mathematics, University of Tsukuba, Tsukuba–shi, Ibaraki 305, Japan, e-mail: dreher@math.tsukuba.ac.jp

I. Witt: Institute of Mathematics, University of Potsdam, PF 60 15 53, D-14415 Potsdam, Germany, e-mail: ingo@math.uni-potsdam.de

The assumptions are as follows:

$$c_j, a_{ij}, b_j, c_0 \in C^\infty([0, T_0], \mathbb{R}), \tag{1.4}$$

$$\left(\sum_{j=1}^n c_j(t)\xi_j \right)^2 + \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \ge \alpha_0|\xi|^2, \quad \alpha_0 > 0, \quad \forall (t, \xi), \tag{1.5}$$

$$f = f(u) \text{ is an entire function, } f(0) = 0. \tag{1.6}$$

We employ standard notations like $D = -i\nabla$, $D_t = -i\partial_t$.

The operator L can be written in the form

$$L = \left(\frac{\Lambda(t)}{\lambda(t)} \right)^{-\mu} P\left(t, \frac{\Lambda(t)}{\lambda(t)}\partial_t, \Lambda(t)\partial_x\right), \tag{1.7}$$

where $\Lambda(t) = \int_0^t \lambda(t')\,dt'$ and $P(t, \tau, \xi)$ is a certain polynomial in τ, ξ of degree $\mu = 2$ with coefficients depending on t smoothly up to $t = 0$. Operators with such a structure arise in the investigation of edge pseudodifferential problems on manifolds with cuspidal edges, where cusps are described by means of the function $\lambda(t)$. The singularity of the manifold requires the use of adapted classes of Sobolev spaces, so-called *edge Sobolev spaces*. The principles of forming these edge Sobolev spaces are expounded in Sect. 2.

A lot of results concerning the well-posedness of the Cauchy problem for weakly hyperbolic operators have been provided over the last decades. Under suitable assumptions on the data, the right-hand side, and the coefficients, the solutions have been proved to belong to the spaces $C^k([0, T], H^s(\mathbb{R}^n))$ ([5], [12], [14], [17]), $C^k([0, T], C^\infty(\mathbb{R}^n))$ ([3], [4]), and $C^k([0, T], \gamma^{(s)}(\mathbb{R}^n))$ ([3], [11]), respectively, where $\gamma^{(s)}(\mathbb{R}^n)$ denotes the Gevrey space of order s.

All these function spaces, however, have the disadvantage that their elements have different smoothness with respect to t and x. We do not know any result concerning the weakly hyperbolic Cauchy problem stating that solutions belong to a function space that embeds into the Sobolev spaces $H^s_{\text{loc}}((0, T) \times \mathbb{R}^n)$, for some $s \in \mathbb{R}$, under the assumption that the initial data and the right-hand side themselves belong to appropriate function spaces of the same kind.

We are going to introduce Sobolev spaces $H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$, where $s \ge 0$ denotes the Sobolev smoothness and $\delta \in \mathbb{R}$ figures as an additional parameter, in which unique solutions u, v to (1.1), (1.2) exist provided that the initial data belong to suitable Sobolev spaces on $\mathbb{R}^n$. These spaces possess the property that

$$\begin{aligned} H^s_{\text{comp}}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n} &\subset H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n) \\ &\subset H^s_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n}, \end{aligned}$$

with continuous embeddings. Furthermore, the space of all smooth functions on $[0, T] \times \mathbb{R}^n$ with bounded support is dense in $H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$.

The spaces $H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$ additionally reflect the loss of Sobolev regularity observed when passing from the Cauchy data to the solution. Namely, there are traces

$$H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n) \to H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n), \quad u(t, x) \mapsto (\partial_t^j u)(0, x),$$

for all $j \in \mathbb{N}$, $j < s - 1/2$, $\beta = 1/(l_* + 1)$, leading to a higher regularity at $t = 0$ if $\delta > 0$. The phenomenon of the loss of regularity was first recognized by Qi [15] for the equation

$$Lv = v_{tt} - t^2 v_{xx} - (4m + 1)v_x = 0, \quad m \in \mathbb{N}, \tag{1.8}$$

with initial data $v(0, x) = u_0(x)$, $v_t(0, x) = 0$. He found an explicit representation of the solution v,

$$v(t, x) = \sum_{j=0}^{m} C_{jm} t^{2j} (\partial_x^j u_0)(x + t^2/2), \quad C_{mm} \neq 0. \tag{1.9}$$

Then the assumption $u_0 \in H^{s+m}(\mathbb{R})$ implies $v(t, .) \in H^s(\mathbb{R})$, and this is the best possible. The phenomenon of the loss of regularity makes, for example, the investigation of the semilinear problem $Lu = f(u)$ delicate, since the usual iteration approach in standard function spaces, for example, in $C([0, T], H^s(\mathbb{R}))$, does not work.

The surprising fact, however, is that the iteration approach is applicable if we employ the edge Sobolev spaces $H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$. We will show that the solutions u and v belong to the same edge Sobolev space. A discussion of this result in the case of Qi's operator is given in Example 5.4. In other words, the nonlinearity does not induce an additional loss of regularity. Similar results have been proved in [6], [7], where function spaces

$$B_\vartheta = \left\{u\colon \vartheta(t, x, D_x)u(t, x) \in C([0, T], L^2(\mathbb{R}^n))\right\}$$

have been utilized. Here $\vartheta = \vartheta(t, x, \xi)$ is a suitably chosen elliptic pseudodifferential symbol of variable order. These spaces generalize the spaces $C([0, T], H^s(\mathbb{R}^n))$. In the present article, the spaces B_ϑ are replaced by the edge Sobolev spaces $H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$ which admit a more uniform treatment of space and time variables.

A future challenge consists in developing a calculus of pseudodifferential operators of the form (1.7) acting in the spaces $H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$. The primary aim is both to admit operators L that depend on the spatial variable x and to come closer to the interesting branching phenomena arising in the propagation of singularities, as observed for the linear Cauchy problem, for example, in [1], [2], [21]. It is known that these branching phenomena crucially depend on the lower-order terms of the operator L, see, for example, Qi's example. The pseudodifferential calculus to be developed has to be organized in part as a calculus of pseudodifferential operators on cuspidal wedges for which, in determining the ellipticity, besides the invertibility of usual principal pseudodifferential symbol $\sigma_\psi^\mu(L)(t, x, \tau, \xi)$, the invertibility of the so-called principal edge symbol $\sigma_\wedge^\mu(L)(x, \xi)$ living on $T^*\mathbb{R}^n \setminus 0$ and taking values in a certain class of pseudodifferential operators on $\mathbb{R}_+$ enters. In [6], the Cauchy problem for operators L with x-dependent coefficients has been treated in the spaces B_ϑ. Elements of a calculus of pseudodifferential operators on cuspidal wedges have been developed in [16], [19].

The paper is organized as follows. In Sect. 2, we introduce the Sobolev spaces $H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$ for $T > 0$ and derive their basic properties. Well-posedness of the linear Cauchy problem in these classes of Sobolev spaces is shown in Sect. 3. In Sect. 4, we then prove that the spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ are algebras under pointwise multiplication if s, δ are sufficiently large. This enables us to consider the semilinear Cauchy problem (1.1) in Sect. 5, where we prove uniqueness and local in time existence of the solution u in the same Sobolev space as v. We conclude with an application to the theory of the propagation of mild singularities.

2. Edge Sobolev spaces

Here we are concerned with the spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$. Details on the abstract approach to edge Sobolev spaces can be found, for example, in [8], [18].

2.1. Weighted Sobolev spaces on $\mathbb{R}_+$

For $s\in\mathbb{N}$, $\delta\in\mathbb{R}$, the weighted Sobolev space $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$ consists of all $v\in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that

$$t^{j-\delta}\partial_t^j v\in L^2(\mathbb{R}_+),\quad \forall j\in\mathbb{N},\ j\le s.$$

For general s, $\delta\in\mathbb{R}$, the space $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$ is defined by interpolation and duality. A norm on the space $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$ is given by

$$\|v\|_{\mathcal{H}^{s,\delta}(\mathbb{R}_+)}=\left\{\frac{1}{2\pi i}\int_{\operatorname{Re}z=1/2-\delta}\langle z\rangle^{2s}|Mv(z)|^2\,dz\right\}^{1/2},\tag{2.1}$$

where $Mv(z)=\int_0^\infty t^{z-1}v(t)\,dt$ is the Mellin transform. Recall that $M\colon L^2(\mathbb{R}_+)\to L^2\big(\{z\in\mathbb{C}\colon\ \operatorname{Re}z=1/2\};(2\pi i)^{-1}dz\big)$ is an isometry and

$$M\{(-t\partial_t)v\}(z)=zMv(z),$$
$$M\{t^{-\delta}v\}(z)=Mv(z-\delta).$$

Furthermore, the space $C^\infty_{\mathrm{comp}}(\mathbb{R}_+)$ is dense in $\mathcal{H}^{s,\delta}(\mathbb{R}_+)$.

Let $H^s(\mathbb{R}_+\times\mathbb{R}^n)=\big\{v\big|_{\mathbb{R}_+\times\mathbb{R}^n}\colon\ v\in H^s(\mathbb{R}^{1+n})\big\}$ and $H^s_0(\overline{\mathbb{R}}_+\times\mathbb{R}^n)=\big\{v\in H^s(\mathbb{R}^{1+n})\colon\ \operatorname{supp}v\subseteq\overline{\mathbb{R}}_+\times\mathbb{R}^n\big\}$.

Example 2.1. For $s\ge 0$, $H^s_0(\overline{\mathbb{R}}_+)=\mathcal{H}^{0,0}(\mathbb{R}_+)\cap\mathcal{H}^{s,s}(\mathbb{R}_+)$.

2.2. Abstract edge Sobolev spaces

A Hilbert space $(E,\{\kappa_\nu\}_{\nu>0})$ with a strongly continuous group action is a Hilbert space E together with a strongly continuous group $\{\kappa_\nu\}_{\nu>0}$ of isomorphisms acting on E. In particular, $\kappa_\nu\kappa_{\nu'}=\kappa_{\nu\nu'}$ for $\nu,\ \nu'>0$ and $\kappa_1=\mathrm{id}_E$.

For $s \in \mathbb{R}$, the abstract edge Sobolev space $\mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu\}_{\nu>0}\big)\big)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n; E)$ such that $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^n; E)$ and the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^n;(E,\{\kappa_\nu\}_{\nu>0}))} = \left\{ \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \left\| \kappa(\xi)^{-1} \hat{u}(\xi) \right\|_E^2 d\xi \right\}^{1/2} \tag{2.2}$$

is finite. Here $\hat{u}(\xi) = Fu(\xi) = \int e^{-ix\xi} u(x)\, dx$ is the Fourier transform of u and $\kappa(\xi) = \kappa_{\langle\xi\rangle}$. $\mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu\}_{\nu>0}\big)\big)$ equipped with the norm (2.2) is a Hilbert space.

Example 2.2. For $s \geq 0$,

$$\begin{aligned} H^s(\mathbb{R}_+ \times \mathbb{R}^n) &= \mathcal{W}^s\big(\mathbb{R}^n; \big(H^s(\mathbb{R}_+), \{\bar{\kappa}_\nu\}_{\nu>0}\big)\big), \\ H^s_0(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) &= \mathcal{W}^s\big(\mathbb{R}^n; \big(H^s_0(\overline{\mathbb{R}}_+), \{\bar{\kappa}_\nu\}_{\nu>0}\big)\big), \end{aligned} \tag{2.3}$$

where $\bar{\kappa}_\nu v(t) = \nu^{1/2} v(\nu t)$, $\nu > 0$. See, for example, [18].

In applications, the spaces E often consist of functions $v = v(t)$ on $\mathbb{R}_+$, where characteristic features of such functions are expressed by prescribing a different behaviour as $t \to +0$ and $t \to \infty$, respectively.

In the following, let $\omega = \omega(t)$ be a cut-off function close to $t = 0$, i.e., $\omega \in C^\infty(\overline{\mathbb{R}}_+)$, $\operatorname{supp}\omega$ is bounded, and $\omega(t) = 1$ for t close to 0.

Lemma 2.3. *Let $s \in \mathbb{R}$, E_0, E_1 be Hilbert spaces of functions on $\mathbb{R}_+$ such that*

$$H^s_{\mathrm{comp}}(\mathbb{R}_+) \subset E_i \subset H^s_{\mathrm{loc}}(\mathbb{R}_+), \quad i = 0, 1,$$

with continuous embeddings. Furthermore, the multiplication operators $E_0 \to E_0$, $u_0 \mapsto \omega u_0$ and $E_1 \to E_1$, $u_1 \mapsto (1-\omega)u_1$ should be continuous, where ω is a cut-off function as above. Then the space

$$E = \big\{\omega u_0 + (1-\omega)u_1 \colon\ u_0 \in E_0,\ u_1 \in E_1\big\},$$

equipped with the norm

$$\|u\|_E = \left\{ \|\omega u\|^2_{E_0} + \|(1-\omega)u\|^2_{E_1} \right\}^{1/2},$$

is a Hilbert space.

Proof. By virtue of the open mapping theorem, the spaces $\{u_i \in E_i \colon\ \operatorname{supp} u_i \subseteq [a_0, a_1]\}$ for $i = 0, 1$ and $\{u \in H^s(\mathbb{R}_+) \colon\ \operatorname{supp} u \subseteq [a_0, a_1]\}$ coincide algebraically and topologically, for any $0 < a_0 < a_1 < \infty$. In particular, the space E is independent of the choice of the cut-off function ω, up to the equivalence of norms.

It remains to show completeness of the norm $\|.\|_E$. So let $\{u^j\}_{j\in\mathbb{N}} \subset E$ be a sequence such that $\omega u^j \to u_0$ in E_0 and $(1-\omega)u^j \to u_1$ in E_1. Set $u = u_0 + u_1 \in E$. Then $\omega(1-\omega)u^j \to \omega u_1$ in E; thus in $H^s(\mathbb{R}_+)$ and in E_0. We obtain $\omega u^j = \omega^2 u^j + \omega(1-\omega)u^j \to \omega u_0 + \omega u_1 = \omega u$ in E_0. Therefore, $\omega u = u_0$ and $(1-\omega)u = u_1$. □

We will also need the following result:

Lemma 2.4. *Let* $(E, \{\kappa_\nu\}_{\nu>0})$, $(\tilde{E}, \{\tilde{\kappa}_\nu\}_{\nu>0})$ *be Hilbert spaces with strongly continuous group actions. Further let* $a\colon \mathbb{R}^n \to \mathcal{L}(E, \tilde{E})$ *be measurable with*

$$\left\|\tilde{\kappa}(\xi)^{-1} a(\xi)\kappa(\xi)\right\|_{\mathcal{L}(E,\tilde{E})} \leq C\,\langle\xi\rangle^{\mu}, \quad \xi \in \mathbb{R}^n \text{ a.e.},$$

for some $\mu \in \mathbb{R}$ *and some constant* $C > 0$. *Then, for each* $s \in \mathbb{R}$,

$$\mathrm{Op}(a)\colon \mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu\}_{\nu>0}\big)\big) \to \mathcal{W}^{s-\mu}\big(\mathbb{R}^n; \big(\tilde{E}, \{\tilde{\kappa}_\nu\}_{\nu>0}\big)\big)$$

continuously, where $\mathrm{Op}(a)u = F^{-1}_{\xi\to x}\big\{a(\xi)\hat{u}(\xi)\big\}$.

2.3. The spaces $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$

Below we will use the group $\{\kappa_\nu^{(\delta)}\}_{\nu>0}$,

$$\kappa_\nu^{(\delta)} v(t) = \nu^{\beta/2-\beta\delta l_*} v(\nu^\beta t), \quad \nu > 0,$$

$\beta = 1/(l_* + 1)$. Here $\delta \in \mathbb{R}$ is a parameter to be specified later on.

Lemma 2.5. *Let* $s \geq 0$ *and the space* E *be as in Lemma* 2.3 *with* $E_1 = \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+) \cap \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)$. (*Here the space* E_0 *satisfying the conditions of Lemma* 2.3 *is arbitrary otherwise.*) *Then, for each* $a > 0$,

$$\begin{aligned}\mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big)\big|_{(a,\infty)\times\mathbb{R}^n} \\ = \big\{\lambda(t)^{1/2+\delta} v(\Lambda(t), x)\colon\ v \in H^s(\mathbb{R}_+ \times \mathbb{R}^n)\big\}\big|_{(a,\infty)\times\mathbb{R}^n}\end{aligned}$$

holds algebraically and topologically, where $\big|_{(a,\infty)\times\mathbb{R}^n}$ *means restriction of functions* $u = u(t, x)$ *from the corresponding function space to* $(a, \infty) \times \mathbb{R}^n$. *In particular,*

$$H^s_{\mathrm{comp}}(\mathbb{R}_+ \times \mathbb{R}^n) \subset \mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big) \subset H^s_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^n),$$

with continuous embeddings.

Proof. If $\operatorname{supp} v \subseteq [a, \infty)$ for $v = v(t)$ and some $a > 0$, then $\operatorname{supp}(\kappa_\nu^{-1} v) \subseteq [a, \infty)$ for each $\nu \geq 1$. We obtain

$$\begin{aligned}c_0\, \|\kappa_\nu^{-1} v\|_{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\cap\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)} \leq \|\kappa_\nu^{-1} v\|_E \\ \leq c_1\, \|\kappa_\nu^{-1} v\|_{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\cap\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)}, \quad \nu \geq 1,\end{aligned}$$

with certain constants $0 < c_0 < c_1$ depending on $a > 0$ provided that $\operatorname{supp} v \subseteq [a, \infty)$.

Therefore, the norm of the space $\big\{u \in \mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big)\colon \operatorname{supp} u \subseteq [a,\infty)\times\mathbb{R}^n\big\}$ is equivalent to

$$\begin{aligned}
&\left\{\int \langle\xi\rangle^{2s} \|\kappa^{(\delta)}(\xi)^{-1}\hat{u}(\cdot,\xi)\|^2_{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\cap\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)}\, d\xi\right\}^{1/2}\\
&\quad= \left\{\int \langle\xi\rangle^{2s} \|\kappa^{(\delta)}(\xi)^{-1}\hat{u}(\cdot,\xi)\|^2_{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)}\, d\xi\right.\\
&\qquad\left. + \int \langle\xi\rangle^{2s} \|\kappa^{(\delta)}(\xi)^{-1}\hat{u}(\cdot,\xi)\|^2_{\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)}\, d\xi\right\}^{1/2}\\
&\quad= \left\{\int \langle\xi\rangle^{2s} \|\hat{u}(\cdot,\xi)\|^2_{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)}\, d\xi\right.\\
&\qquad\left. + \int \|\hat{u}(\cdot,\xi)\|^2_{\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)}\, d\xi\right\}^{1/2},
\end{aligned}$$

since $\|\kappa_\nu v\|_{\mathcal{H}^{s,\gamma(l_*+1)+\delta l_*}(\mathbb{R}_+)} = \nu^\gamma\, \|v\|_{\mathcal{H}^{s,\gamma(l_*+1)+\delta l_*}(\mathbb{R}_+)}$ for $s\in\mathbb{R}$, $\nu>0$. (Use the norm (2.1).) Thus the latter space coincides with the space

$$\begin{aligned}
&\big\{u \in H^s(\mathbb{R}^n; \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)) \cap H^0(\mathbb{R}^n; \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+))\colon\\
&\qquad \operatorname{supp} u \subseteq [a,\infty)\times\mathbb{R}^n\big\}.
\end{aligned}$$

The space $H^s(\mathbb{R}^n; \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)) \cap H^0(\mathbb{R}^n; \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+))$, however, is readily seen to be equal to the space $\{\lambda(t)^{1/2+\delta} v(\Lambda(t),x)\colon v\in H^s_0(\overline{\mathbb{R}}_+\times\mathbb{R}^n)\}$, since $\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\cap\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+) = \{\lambda(t)^{1/2+\delta} v(\Lambda(t))\colon v\in H^s_0(\overline{\mathbb{R}}_+)\}$ in view of Example 2.1 and by employing (2.3) of Example 2.2. □

Lemma 2.6. *Let the assumptions of the previous lemma be fulfilled.*

(a) $\mathcal{S}(\overline{\mathbb{R}}_+\times\mathbb{R}^n)$ *is contained and dense in* $\mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big)$ *provided that* $\mathcal{S}(\overline{\mathbb{R}}_+)$ *is contained and dense in* E.

(b) $C^\infty_{\text{comp}}(\mathbb{R}_+\times\mathbb{R}^n)$ *is dense in* $\mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big)$ *if and only if* $C^\infty_{\text{comp}}(\mathbb{R}_+)$ *is dense in* E.

Proof. We prove (a); (b) is similar.

Since $\mathcal{S}(\overline{\mathbb{R}}_+)$ is dense in E and $\mathcal{S}(\overline{\mathbb{R}}_+\times\mathbb{R}^n) = \mathcal{S}(\overline{\mathbb{R}}_+)\hat{\otimes}_\pi\mathcal{S}(\mathbb{R}^n)$, we obtain that the space

$$\Big\{F^{-1}_{\xi\to x}\big\{\langle\xi\rangle^{\beta/2-\beta\delta l_*}\hat{u}(\langle\xi\rangle^\beta t,\xi)\big\}\colon u\in\mathcal{S}(\overline{\mathbb{R}}_+\times\mathbb{R}^n)\Big\}, \tag{2.4}$$

where $\hat{u}(t,\xi) = F_{x\to\xi}\{u(t,x)\}$, is dense in $\mathcal{W}^s\big(\mathbb{R}^n; \big(E, \{\kappa_\nu\}_{\nu>0}\big)\big)$. But (2.4) is the Schwartz space $\mathcal{S}(\overline{\mathbb{R}}_+\times\mathbb{R}^n)$, as shown by elementary estimates. □

Definition 2.7. (a) For $s\geq 0$, $\delta\in\mathbb{R}$, the space $H^{s,\delta;\lambda}(\mathbb{R}_+)$ is defined to be the space E from Lemma 2.3 with $E_0 = H^s(\mathbb{R}_+)$ and $E_1 = \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\cap\mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)$. Analogously, $H^{s,\delta;\lambda}_0(\overline{\mathbb{R}}_+)$ is defined to be the space E from Lemma 2.3 with $E_0 = H^s_0(\overline{\mathbb{R}}_+)$ and E_1 as before.

(b) For $s \geq 0$, $\delta \in \mathbb{R}$, we set

$$H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) = \mathcal{W}^s\big(\mathbb{R}^n; \big(H^{s,\delta;\lambda}(\mathbb{R}_+), \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big),$$
$$H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) = \mathcal{W}^s\big(\mathbb{R}^n; \big(H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+), \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big).$$

We summarize results obtained so far.

Proposition 2.8. *Let $s \geq 0$, $\delta \in \mathbb{R}$.*

(a) $H^s_{\mathrm{comp}}(\mathbb{R}_+ \times \mathbb{R}^n) \subset H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) \subseteq H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \subset H^s_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)$ *with continuous embeddings. The spaces in the middle coincide if and only if $s < 1/2$.*

(b) $\mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ *is dense in* $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$.

(c) *The space $H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ is closed in $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ if and only if $s \notin 1/2 + \mathbb{N}$. In this case, $H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ is the closure of $C^\infty_{\mathrm{comp}}(\mathbb{R}_+ \times \mathbb{R}^n)$ in $H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$.*

Lemma 2.9. $\big\{H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) :\ s \geq 0\big\}$ *for $\delta \in \mathbb{R}$ forms an interpolation scale with respect to the complex interpolation method.*

Proof. Obviously, $\big\{H^{s,\delta;\lambda}(\mathbb{R}_+) :\ s \geq 0\big\}$ forms an interpolation scale with respect to the complex interpolation method. It remains to apply the functor $\mathcal{W}^s\big(\mathbb{R}^n; \big(\cdot, \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big)\big)$. □

Proposition 2.10. *Let $s \geq 0$, $\delta \in \mathbb{R}$. Then, for each $j \in \mathbb{N}$, $j < s - 1/2$, the map $\mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $u \mapsto (\partial_t^j u)(0, x)$, extends by continuity to a map*

$$\tau_j \colon H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n). \tag{2.5}$$

Furthermore, the map

$$H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to \prod_{j<s-1/2} H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n), \quad u \mapsto \big\{\tau_j u\big\}_{j<s-1/2}$$

is surjective.

Proof. By Lemma 2.9, we may assume that $s \notin 1/2 + \mathbb{N}$. Then

$$H^{s,\delta;\lambda}(\mathbb{R}_+) = \Bigg\{\sum_{j<s-1/2} \omega(t)\,\frac{t^j}{j!}\,d_j :\ d_j \in \mathbb{C}\ \forall j\Bigg\} \oplus H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+),$$

where ω is a cut-off function as above. We get

$$\begin{aligned} &H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \\ &= \Bigg\{\sum_{j<s-1/2} F^{-1}_{\xi\to x}\Bigg\{\langle\xi\rangle^{\beta/2-\beta\delta l_*}\,\omega(\langle\xi\rangle^\beta t)\,\frac{(\langle\xi\rangle^\beta t)^j}{j!}\,\hat{d}_j(\xi)\Bigg\} : \\ &\qquad d_j \in H^s(\mathbb{R}^n)\ \forall j\Bigg\} \oplus H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) \\ &= \Bigg\{\sum_{j<s-1/2} F^{-1}_{\xi\to x}\big\{\omega(\langle\xi\rangle^\beta t)\hat{c}_j(\xi)\big\}\,\bar{\omega}(t)\,\frac{t^j}{j!} : \\ &\qquad c_j \in H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n)\ \forall j\Bigg\} \oplus H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n), \end{aligned}$$

where $\bar{\omega} = \bar{\omega}(t)$ is another cut-off function such that $\omega(\langle\xi\rangle^{\beta} t)\bar{\omega}(t) = \omega(\langle\xi\rangle^{\beta} t)$ for all $\xi \in \mathbb{R}^n$. Now, for $u \in \mathcal{S}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n) \subset H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ written in the form

$$u(t,x) = \sum_{j<s-1/2} F^{-1}_{\xi\to x}\left\{\omega(\langle\xi\rangle^{\beta} t)\hat{c}_j(\xi)\right\}\bar{\omega}(t)\,\frac{t^j}{j!} + u_0(t,x)$$

with uniquely determined coefficients $c_j \in \mathcal{S}(\mathbb{R}^n) \subset H^{s-\beta j+\beta\delta l_*-\beta/2}(\mathbb{R}^n)$ and $u_0 \in H_0^{s,\delta;\lambda}(\overline{\mathbb{R}}_+ \times \mathbb{R}^n)$, we obviously have

$$c_j(x) = \left(\partial_t^j u\right)(0,x)$$

for all $j \in \mathbb{N}$, $j < s - 1/2$. This yields the desired result. □

Proposition 2.11. *For $s \geq 0$, $\delta \in \mathbb{R}$, we have continuity of the following maps:*

(a) $\partial_t\colon H^{s+1,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta+1;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$;
(b) $t^l\colon H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta+l/l_*;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ *for* $l = 0, 1, \ldots, l_*$;
(c) $\partial_{x_j}\colon H^{s+1,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ *for* $1 \leq j \leq n$;
(d) $\varphi\colon H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ *for each* $\varphi = \varphi(t) \in \mathcal{S}(\overline{\mathbb{R}}_+)$.

Here t^l means the operator of multiplication with t^l; similarly for φ. In particular, the differential operator L from (1.3) *is continuous from $H^{s+2,\delta;\lambda}((0,T_0) \times \mathbb{R}^n)$ to $H^{s,\delta+2;\lambda}((0,T_0) \times \mathbb{R}^n)$, where the space $H^{s,\delta;\lambda}((0,T_0) \times \mathbb{R}^n)$ for $s \geq 0$, $\delta \in \mathbb{R}$ is defined in* (2.6) *below.*

Proof. These are consequences of Lemma 2.4 and

(a) $\partial_t\colon H^{s+1,\delta;\lambda}(\mathbb{R}_+) \to H^{s,\delta+1;\lambda}(\mathbb{R}_+)$, $\kappa^{(\delta+1)}(\xi)^{-1}\partial_t\kappa^{(\delta)}(\xi) = \langle\xi\rangle\,\partial_t$;
(b) $t^l\colon H^{s,\delta;\lambda}(\mathbb{R}_+) \to H^{s,\delta+l/l_*;\lambda}(\mathbb{R}_+)$, $\kappa^{(\delta+l/l_*)}(\xi)^{-1}t^l\kappa^{(\delta)}(\xi) = t^l$;
(c) $i\xi_j\colon H^{s+1,\delta;\lambda}(\mathbb{R}_+) \to H^{s,\delta;\lambda}(\mathbb{R}_+)$, $\kappa^{(\delta)}(\xi)^{-1}i\xi_j\kappa^{(\delta)}(\xi) = i\xi_j$;
(d) $\varphi(\langle\xi\rangle^{-\beta}t)\colon H^{s,\delta;\lambda}(\mathbb{R}_+) \to H^{s,\delta;\lambda}(\mathbb{R}_+)$ is uniformly bounded in $\xi \in \mathbb{R}^n$, $\kappa^{(\delta)}(\xi)^{-1}\varphi(t)\kappa^{(\delta)}(\xi) = \varphi(\langle\xi\rangle^{-\beta}t)$.

Another possibility in proving this lemma consists in utilizing the norm (2.7). □

Remark 2.12. For $s \in \mathbb{N}$, the norm of the space $H^{s,-s;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ can be shown to be equivalent to the norm

$$\|u\|_{H^{s,-s;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)} \sim \left\{\int_0^\infty \left(\sum_{j+l\leq s} \|t^{k_{jl}}\,\partial_t^j u\|^2_{H^l(\mathbb{R}^n)} + \|u\|^2_{H^{\beta s}(\mathbb{R}^n)}\right) dt\right\}^{1/2},$$

where $k_{jl} = \max\{0, -s + j + (l_* + 1)l\}$. Without the additional term $\|u\|_{H^{\beta s}(\mathbb{R}^n)}$, this norm has been used for treating degenerate elliptic operators of type 4 in [13] .

2.4. The spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$

For $T>0$, we define

$$H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n) = H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)\big|_{(0,T)\times\mathbb{R}^n} \tag{2.6}$$

and equip this space, for the time being, with its infimum norm.

Lemma 2.13. *For $s\in\mathbb{N}$, $\delta\in\mathbb{R}$, and $T>0$, the infimum norm of the space $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ is equivalent to the norm $\|.\|_{H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)}$, where*

$$\begin{aligned}
&\|u\|^2_{H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)} \\
&\quad = \sum_{l=0}^{s}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2s-2l}\int_0^{\min\{\langle\xi\rangle^{-\beta},T\}}\lambda(\langle\xi\rangle^{-\beta})^{-2l-2\delta}|\partial_t^l\hat{u}(t,\xi)|^2\,dt\,d\xi \\
&\qquad + \sum_{l=0}^{s}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2s-2l}\int_{\min\{\langle\xi\rangle^{-\beta},T\}}^{T}\lambda(t)^{-2l-2\delta}|\partial_t^l\hat{u}(t,\xi)|^2\,dt\,d\xi.
\end{aligned} \tag{2.7}$$

Proof. First of all, note that the norm $\|.\|_{H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)}$ is given by

$$\begin{aligned}
&\|u\|^2_{H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)} \\
&\quad = \sum_{l=0}^{s}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2s}\int_0^{\infty}\left|\partial_t^l\omega(t)\langle\xi\rangle^{-\beta/2+\beta\delta l_*}\hat{u}(\langle\xi\rangle^{-\beta}t,\xi)\right|^2dt\,d\xi \\
&\qquad + \sum_{l=0}^{s}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2s} \\
&\qquad\quad \times\int_0^{\infty}\left|\lambda(t)^{-\delta-l}\partial_t^l(1-\omega(t))\langle\xi\rangle^{-\beta/2+\beta\delta l_*}\hat{u}(\langle\xi\rangle^{-\beta}t,\xi)\right|^2dt\,d\xi \\
&\quad \sim \sum_{l=0}^{s}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2s-2l}\int_0^{\min\{\langle\xi\rangle^{-\beta},T\}}\lambda(\langle\xi\rangle^{-\beta})^{-2l-2\delta}|\partial_t^l\hat{u}(t,\xi)|^2\,dt\,d\xi \\
&\qquad + \sum_{l=0}^{s}\int_{\mathbb{R}^n_\xi}\langle\xi\rangle^{2s-2l}\int_{\min\{\langle\xi\rangle^{-\beta},T\}}^{T}\lambda(t)^{-2l-2\delta}|\partial_t^l\hat{u}(t,\xi)|^2\,dt\,d\xi,
\end{aligned}$$

where ω is a cut-off function as above. Thus, for each $u\in H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)$, the norm (2.7) evaluated for $u\big|_{(0,T)\times\mathbb{R}^n}$ is finite.

Conversely, suppose that, for some u, the norm (2.7) is finite. We choose a measurable family $\left\{\Pi_\xi\colon H^s(0,T)\to H^s(\mathbb{R}_+)\colon \xi\in\mathbb{R}^n\right\}$ of extension operators, i.e., we have $\Pi_\xi v\big|_{(0,T)}=v$ for all v, such that

$$\begin{aligned}
&\sum_{l=0}^{s}\int_0^{\langle\xi\rangle^{-\beta}}\lambda(\langle\xi\rangle^{-\beta})^{-2\delta-2l}|\partial_t^l\Pi_\xi v(t)|^2\,dt \\
&\qquad + \sum_{l=0}^{s}\int_{\langle\xi\rangle^{-\beta}}^{\infty}\lambda(t)^{-2\delta-2l}|\partial_t^l\Pi_\xi v(t)|^2\,dt
\end{aligned}$$

$$\leq C^2 \sum_{l=0}^{s} \int_0^{\min\{\langle\xi\rangle^{-\beta},T\}} \lambda(\langle\xi\rangle^{-\beta})^{-2\delta-2l} |\partial_t^l v(t)|^2 \, dt$$
$$+ C^2 \sum_{l=0}^{s} \int_{\min\{\langle\xi\rangle^{-\beta},T\}}^{T} \lambda(t)^{-2\delta-2l} |\partial_t^l v(t)|^2 \, dt$$

with some constant $C > 0$ independent of ξ, i.e., in the indicated norms of the spaces $H^s(0,T)$ and $H^s(\mathbb{R}_+)$, respectively, the operator norm of Π_ξ does not exceed C. To get rid of the possibly unrestricted growth of the factors $\lambda(t)^{\mp(2\delta+2l)}$ as $t \to \infty$, the extension operators Π_ξ are constructed in such a way that $\operatorname{supp} v \subseteq (0, T_1]$ for all v and some $T_1 > T$.

Now, letting U be defined by $\hat{U}(t,\xi) = \Pi_\xi \hat{u}(t,\xi) = \Pi_\xi(\hat{u}(\cdot,\xi))(t)$, we get that $U \in H^{s,\delta;\lambda}(\mathbb{R}_+ \times \mathbb{R}^n)$ is an extension of u, for

$$\begin{aligned}
&\|U\|^2_{H^{s,\delta;\lambda}(\mathbb{R}_+\times\mathbb{R}^n)} \\
&\sim \sum_{l=0}^{s} \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s-2l} \int_0^{\langle\xi\rangle^{-\beta}} \lambda(\langle\xi\rangle^{-\beta})^{-2l-2\delta} |\partial_t^l \Pi_\xi \hat{u}(t,\xi)|^2 \, dt \, d\xi \\
&\quad + \sum_{l=0}^{s} \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s-2l} \int_{\langle\xi\rangle^{-\beta}}^{\infty} \lambda(t)^{-2l-2\delta} |\partial_t^l \Pi_\xi \hat{u}(t,\xi)|^2 \, dt \, d\xi \\
&\leq C^2 \sum_{l=0}^{s} \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s-2l} \int_0^{\min\{\langle\xi\rangle^{-\beta},T\}} \lambda(\langle\xi\rangle^{-\beta})^{-2l-2\delta} |\partial_t^l \hat{u}(t,\xi)|^2 \, dt \, d\xi \\
&\quad + C^2 \sum_{l=0}^{s} \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s-2l} \int_{\min\{\langle\xi\rangle^{-\beta},T\}}^{\infty} \lambda(t)^{-2l-2\delta} |\partial_t^l \hat{u}(t,\xi)|^2 \, dt \, d\xi \\
&= C^2 \, \|u\|^2_{H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof. □

Lemma 2.14. *For $s,\ s' \geq 0$, $\delta,\ \delta' \in \mathbb{R}$, and $T > 0$,*

$$H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n) \subseteq H^{s',\delta';\lambda}((0,T)\times\mathbb{R}^n) \tag{2.8}$$

if and only if

$$s \geq s', \qquad s + \beta\delta l_* \geq s' + \beta\delta' l_*. \tag{2.9}$$

In particular, for $s \geq \beta l_*$, $\delta \in \mathbb{R}$,

$$H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n) \subseteq H^{s-1+\beta,\delta+1;\lambda}((0,T)\times\mathbb{R}^n).$$

Proof. Necessity. By the embeddings in Proposition 2.8 (a) and the trace theorems in Proposition 2.10, (2.8) forces (2.9) to hold.

Sufficiency. Here we treat the case $s,\ s' \in \mathbb{N}$. The general case is treated in the appendices. If (2.9) is fulfilled, then

$$\begin{aligned}
\langle\xi\rangle^{s'-l}\lambda(\langle\xi\rangle^{-\beta})^{-\delta'-l} &\leq C\,\langle\xi\rangle^{s-l}\lambda(\langle\xi\rangle^{-\beta})^{-\delta-l}, \quad 0 \leq t \leq \langle\xi\rangle^{-\beta}, \\
\langle\xi\rangle^{s'-l}\lambda(t)^{-\delta'-l} &\leq C\,\langle\xi\rangle^{s-l}\lambda(t)^{-\delta-l}, \qquad \langle\xi\rangle^{-\beta} \leq t \leq T,
\end{aligned}$$

for all $0 < t < T$, $\xi \in \mathbb{R}^n$, $0 \le l \le s$, and some constant $C = C(\delta, \delta', T)$,

$$C(\delta, \delta', T) = \begin{cases} 1 & \text{if } \delta \le \delta', \\ T^{(\delta-\delta')l_*} & \text{otherwise.} \end{cases}$$

This implies (2.8) with the embedding constant $C(\delta, \delta', T)$ for the norms (2.7). □

The norm eventually used for the space $H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$ is $\|.\|_{s,\delta;T}$, where

$$\begin{aligned} \|u\|^2_{s,\delta;T} = \sum_{l=0}^{s} T^{2l-1} \times & \\ & \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s-2l} \int_0^{\min\{\langle\xi\rangle^{-\beta},T\}} \lambda(\langle\xi\rangle^{-\beta})^{-2l-2\delta} |\partial_t^l \hat{u}(t,\xi)|^2 \, dt \, d\xi \\ + \sum_{l=0}^{s} T^{2l-1} & \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s-2l} \int_{\min\{\langle\xi\rangle^{-\beta},T\}}^{T} \lambda(t)^{-2l-2\delta} |\partial_t^l \hat{u}(t,\xi)|^2 \, dt \, d\xi. \end{aligned}$$

For later reference note that the embedding in (2.8) with constant 1 in the case that $s \ge s'$, $s + \beta\delta l_* \ge s' + \beta\delta' l_*$, and $\delta \le \delta'$ remains valid for the norms $\|.\|_{s,\delta;T}$.

3. The linear Cauchy problem

We start our considerations with the linear Cauchy problem

$$Lw(t,x) = g(t,x), \qquad w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x). \tag{3.1}$$

The partial Fourier transform $\hat{w}(t,\xi) = F_{x\to\xi} w(t,x)$ solves the following Cauchy problem for a second-order O.D.E. with parameter ξ:

$$\begin{aligned} & D_t^2 \hat{w}(t,\xi) + (2\lambda(t)|\xi| c(t,\xi) - i c_0(t)) \, D_t \hat{w}(t,\xi) \\ & \quad - \left(\lambda(t)^2 |\xi|^2 a(t,\xi) - i\lambda'(t)|\xi| b(t,\xi)\right) \hat{w}(t,\xi) = -\hat{g}(t,\xi), \\ & \hat{w}(0,\xi) = \hat{w}_0(\xi), \quad \hat{w}_t(0,\xi) = \hat{w}_1(\xi), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} a(t,\xi) &= \sum_{i,j=1}^{n} a_{ij}(t) \frac{\xi_i \xi_j}{|\xi|^2}, \quad b(t,\xi) = -\sum_{j=1}^{n} b_j(t) \frac{\xi_j}{|\xi|}, \\ c(t,\xi) &= \sum_{j=1}^{n} c_j(t) \frac{\xi_j}{|\xi|}. \end{aligned} \tag{3.3}$$

It is clear that a unique solution $\hat{w}$ to (3.2) exists. Then we may conclude that a solution w to (3.1) exists and belongs to some Sobolev space. The scope of this section is to prove well-posedness for the linear Cauchy problem (3.1) in edge Sobolev spaces.

Introduce the number

$$Q_0 = -\frac{1}{2} + \sup_{\xi} \frac{|b(0,\xi) + c(0,\xi)|}{2\sqrt{c(0,\xi)^2 + a(0,\xi)}} \tag{3.4}$$

and fix $A_0 = Q_0 l_* / (l_* + 1) = \beta Q_0 l_*$.

Theorem 3.1. *Let s, $Q \in \mathbb{R}$, $s \geq 1$, $Q \geq Q_0$. Further let $w_0 \in H^{s+A}(\mathbb{R}^n)$, $w_1 \in H^{s+A-\beta}(\mathbb{R}^n)$, and $g \in H^{s-1,Q+1;\lambda}((0,T) \times \mathbb{R}^n)$, where $A = \beta Q l_*$. Then there is a solution $w \in H^{s,Q;\lambda}((0,T) \times \mathbb{R}^n)$ to* (3.1)*. Moreover, the solution w is unique in the space $H^{s,Q_0;\lambda}((0,T) \times \mathbb{R}^n)$.*

Remark 3.2. (a) The parameter A_0 describes the loss of regularity. The explicit representations of the solutions for special model operators in [15] and [21] show that the statement of the Theorem becomes false if $A < A_0$.

(b) Under the assumption that $w_0 \in H^{s+A}(\mathbb{R}^n)$, $w_1 \in H^{s+A-\beta}(\mathbb{R}^n)$ for some $A > A_0$ we obtain $w \in H^{s+A-A_0,Q_0;\lambda}((0,T) \times \mathbb{R}^n)$ provided that $g \in H^{s+A-A_0-1,Q_0+1;\lambda}((0,T) \times \mathbb{R}^n)$. If we merely have $g \in H^{s-1,Q+1;\lambda}((0,T) \times \mathbb{R}^n)$ (note that $H^{s+A-A_0-1,Q_0+1;\lambda}((0,T) \times \mathbb{R}^n) \subseteq H^{s-1,Q+1;\lambda}((0,T) \times \mathbb{R}^n)$), then we get the weaker conclusion $w \in H^{s,Q;\lambda}((0,T) \times \mathbb{R}^n)$ (note that $H^{s+A-A_0,Q_0;\lambda}((0,T) \times \mathbb{R}^n) \subseteq H^{s,Q;\lambda}((0,T) \times \mathbb{R}^n)$).

Let $H^{\infty}(\mathbb{R}^n) = \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$, $H^{s,\infty;\lambda}((0,T) \times \mathbb{R}^n) = \bigcap_{\delta \in \mathbb{R}} H^{s,\delta;\lambda}((0,T) \times \mathbb{R}^n)$, $H^{\infty,\infty;\lambda}((0,T) \times \mathbb{R}^n) = \bigcap_{s \in \mathbb{R}} H^{s,\infty;\lambda}((0,T) \times \mathbb{R}^n)$.

Corollary 3.3. (a) *Assume that $w_0 \in H^{\infty}(\mathbb{R}^n)$, $w_1 \in H^{\infty}(\mathbb{R}^n)$, and $g \in H^{s-1,\infty;\lambda}((0,T) \times \mathbb{R}^n)$ for some $s \geq 1$. Then $w \in H^{s,\infty;\lambda}((0,T) \times \mathbb{R}^n)$. In particular, $w \in H^{\infty,\infty;\lambda}((0,T) \times \mathbb{R}^n)$ if $g \in H^{\infty,\infty;\lambda}((0,T) \times \mathbb{R}^n)$.*

(b) *Assume that $w_0 = w_1 = 0$ and $g \in H_0^{s-1,Q+1;\lambda}([0,T) \times \mathbb{R}^n)$ for some $s \geq 1$, $Q \geq Q_0$. Then $w \in H_0^{s,Q;\lambda}([0,T) \times \mathbb{R}^n)$.*

From (a) we infer C^{∞} well-posedness for the linear Cauchy problem.

By interpolation, it suffices to prove Theorem 3.1 when $s \in \mathbb{N}_+$. In this case, Theorem 3.1 will follow by standard functional–analytic arguments if the following *a priori* estimate is established:

Proposition 3.4. *For each $s \in \mathbb{N}_+$, $Q \geq Q_0$, there is a constant $C_0 = C_0(s, Q)$ with the property that*

$$\begin{aligned} &\|w\|_{s,Q;T} \\ &\quad \leq C_0 \left(\|w_0\|_{H^{s+A}(\mathbb{R}^n)} + \|w_1\|_{H^{s+A-\beta}(\mathbb{R}^n)} + T \, \|g\|_{s-1,Q+1;T} \right) \end{aligned}$$

for all $0 < T \leq T_0$. The constant C_0 does not depend on T.

For the proof, we introduce

$$v(t,\xi) = \kappa^{(Q)}(\xi)^{-1} \hat{w}(t,\xi), \quad h(t,\xi) = \kappa^{(Q+1)}(\xi)^{-1} \hat{g}(t,\xi) \tag{3.5}$$

and split the (t, ξ) space into two zones: the pseudodifferential zone Z_{pd} and the hyperbolic zone Z_{hyp}:

$$Z_{\mathrm{pd}} = \{(t, \xi) \in [0, T_0] \times \mathbb{R}^n : t\langle\xi\rangle^{\beta} \leq 1\},$$
$$Z_{\mathrm{hyp}} = \{(t, \xi) \in [0, T_0] \times \mathbb{R}^n : t\langle\xi\rangle^{\beta} \geq 1\}.$$

Occasionally, we employ the equivalent description $Z_{\mathrm{pd}} = \{(t, \xi) : t \leq t_\xi\}$, $Z_{\mathrm{hyp}} = \{(t, \xi) : t \geq t_\xi\}$, where $t_\xi = \langle\xi\rangle^{-\beta}$. In terms of the functions v and h from (3.5), the borders of the two zones are given by $t = 1$.

3.1. Estimates in Z_{pd}

We start the proof of Proposition 3.4 with an estimate in Z_{pd}.

Lemma 3.5. *Let $\hat{w}$ be the solution to* (3.2) *and v, h the functions defined in* (3.5). *Then, for every $s \in \mathbb{N}_+$, there is a constant $C = C(s)$ such that*

$$\sum_{l=0}^{s} \Big(T'^{2l-1} \big\| \partial_t^l v(., \xi) \big\|^2_{L^2(0,T')} + T'^{2l} \big| (\partial_t^l v)(T', \xi) \big|^2 \Big)$$
$$\leq C(\langle\xi\rangle^{-\beta+2A} |\hat{w}_0(\xi)|^2 + T'^2 \langle\xi\rangle^{-3\beta+2A} |\hat{w}_1(\xi)|^2)$$
$$+ CT'^4 \sum_{l=0}^{s-1} T'^{2l-1} \langle\xi\rangle^{-2(1+\beta)} \big\| \partial_t^l h(., \xi) \big\|^2_{L^2(0,T')}$$

for all $0 < T' \leq 1$.

The proof is based on the following lemma the proof of which can be found in the appendix:

Lemma 3.6. *Let $a_0 = a_0(t), \ldots, a_{m-1} = a_{m-1}(t)$ be smooth functions and suppose that $f = f(t) \in H^{s-1}(0, T_0)$ for some $s \in \mathbb{N}_+$. Then the solution $y = y(t)$ to*

$$\partial_t^m y + a_{m-1}(t)\partial_t^{m-1} y + \cdots + a_0(t) y = f(t), \quad 0 \leq t \leq T \leq T_0, \tag{3.6}$$
$$\big(\partial_t^j y\big)(0) = y_{0j}, \quad j = 0, \ldots, m-1,$$

satisfies the estimate

$$\sum_{l=0}^{s+m-2} \Big(T^{2l-1} \big\| \partial_t^l y \big\|^2_{L^2(0,T)} + T^{2l} \big| (\partial_t^l y)(T) \big|^2 \Big) \tag{3.7}$$
$$\leq C \sum_{l=0}^{m-1} T^{2l} |y_{0l}|^2 + CT^{2m} \sum_{l=0}^{s-1} T^{2l-1} \big\| \partial_t^l f \big\|^2_{L^2(0,T)},$$

for every $0 < T \leq T_0$, *where the constant C depends only on T_0 and $\big\| a_j \big\|_{C^s([0,T_0])}$.*

Proof of Lemma 3.5. We apply $\kappa^{(Q)}(\xi)^{-1}$ to both sides of (3.2) and recall that $\kappa_\nu^{(\delta)}(\partial_t^k f(t)) = \nu^{-k\beta}\partial_t^k(\kappa_\nu^{(\delta)} f(t))$. Then we find

$$\begin{aligned}
&D_t^2 v(t,\xi) + \left(2\lambda(t)\frac{|\xi|}{\langle\xi\rangle}c(\langle\xi\rangle^{-\beta}t,\xi) - ic_0(\langle\xi\rangle^{-\beta}t,\xi)\right) D_t v(t,\xi) \\
&\quad - \left(\lambda(t)^2\frac{|\xi|^2}{\langle\xi\rangle^2}a(\langle\xi\rangle^{-\beta}t,\xi) - i\lambda'(t)\frac{|\xi|}{\langle\xi\rangle}b(\langle\xi\rangle^{-\beta}t,\xi)\right) v(t,\xi) \\
&= -\langle\xi\rangle^{-1-\beta}h(t,\xi).
\end{aligned}$$

This equation shows the effect of the group action $\kappa^{(Q)}(\xi)^{-1}$: the parameter $\xi \in \mathbb{R}^n$ has lost almost all of its influence, only terms of the form $|\xi|/\langle\xi\rangle$ or $\langle\xi\rangle^{-\beta}$ are still present.

We easily find that $v(0,\xi) = \langle\xi\rangle^{-\beta/2+A}\hat{w}_0(\xi)$, $v_t(0,\xi) = \langle\xi\rangle^{-3\beta/2+A}\hat{w}_1(\xi)$. An application of Lemma 3.6 concludes the proof. □

3.2. *Estimates in* Z_{hyp}

The goal of this subsection is to prove the following estimate:

Lemma 3.7. *Let $\hat{w}$ be the solution to* (3.2) *and v, h be the functions defined by* (3.5). *Then, for every $s \in \mathbb{N}_+$, there is a constant C such that*

$$\begin{aligned}
\sum_{l=0}^{s}\langle\xi\rangle^{2s}\int_1^{\langle\xi\rangle^\beta T}\left|\lambda(t)^{-Q-l}\partial_t^l v(t,\xi)\right|^2 dt &\le CT\sum_{l=0}^{s}\langle\xi\rangle^{2(s+\beta/2)}|(\partial_t^l v)(1,\xi)|^2 \\
&\quad + CT^2\sum_{l=0}^{s-1}\langle\xi\rangle^{2(s-1)}\int_1^{\langle\xi\rangle^\beta T}\left|\lambda(t)^{-Q-1-l}\partial_t^l h(t,\xi)\right|^2 dt,
\end{aligned}$$

provided that $\langle\xi\rangle^\beta T \ge 1$, i.e., $(T,\xi) \in Z_{\text{hyp}}$.

The proof will be given after we have found pointwise estimates of $\hat{w}(t,\xi)$. We introduce the vector $W(t,\xi) = {}^t\left(\lambda(t)|\xi|\hat{w}(t,\xi), D_t\hat{w}(t,\xi)\right)$ and obtain the first-order system

$$\begin{aligned}
D_t W(t,\xi) &= A(t,\xi)W(t,\xi) + G(t,\xi), \\
A(t,\xi) &= \begin{pmatrix} 0 & 1 \\ a(t,\xi) & -2c(t,\xi)\end{pmatrix}\lambda(t)|\xi| - i\begin{pmatrix} \frac{\lambda'(t)}{\lambda(t)} & 0 \\ \frac{\lambda'(t)}{\lambda(t)}b(t,\xi) & -c_0(t)\end{pmatrix},
\end{aligned}$$

where $a(t,\xi)$, $b(t,\xi)$, $c(t,\xi)$ are given by (3.3) and $G(t,\xi) = {}^t(0, -\hat{g}(t,\xi))$. If $X(t,t',\xi)$ denotes the fundamental matrix, i.e.,

$$D_t X(t,t',\xi) = A(t,\xi)X(t,t',\xi), \qquad X(t',t',\xi) = I,$$

then $W(t,\xi) = X(t,t',\xi)W(t',\xi) + i\int_{t'}^t X(t,t'',\xi)G(t'',\xi)dt''$. This immediately gives estimates of $|W(t,\xi)|$ if estimates of $X(t,t',\xi)$ have been found. For the investigations of higher-order derivatives $D_t^l W(t,\xi)$, we define

$$W_l(t,\xi) = (\lambda(t)\langle\xi\rangle)^{-l-1} D_t^l W(t,\xi), \quad G_l(t,\xi) = (\lambda(t)\langle\xi\rangle)^{-l-1} D_t^l G(t,\xi),$$
$$X_m(t,t',\xi) = X(t,t',\xi)\left(\frac{\lambda(t')}{\lambda(t)}\right)^m,$$

and observe that

$$\begin{aligned} D_t W_l &= \left(A - (l+1)\frac{\lambda'}{\lambda}\right) W_l + \tilde{G}_l \\ &= \left(A - (l+1)\frac{\lambda'}{\lambda}\right) W_l + G_l + \sum_{m=1}^{l} \binom{l}{m} (\lambda(t)\langle\xi\rangle)^{-m} (D_t^m A) W_{l-m}, \end{aligned}$$

hence

$$W_l(t,\xi) = X_{l+1}(t,t',\xi) W_l(t',\xi) + i\int_{t'}^{t} X_{l+1}(t,t'',\xi)\tilde{G}_l(t'',\xi)\,dt''. \tag{3.8}$$

Lemma 3.8. *There are some* (*large*) *constants* $c, C > 0$ *such that*

$$\left\| X(t,t',\xi) \right\| \le C\left(\frac{\lambda(t)}{\lambda(t')}\right)^{Q_0+1}, \quad ct_\xi \le t' \le t \le T_0,$$

holds for all $\xi \in \mathbb{R}^n$, *where* $Q_0 \le Q$ *is given by* (3.4).

Proof. See [7]. ☐

From this and $\left\| X(t,t',\xi) \right\| \le \exp(\int_{t'}^{t} \left\| A(t'',\xi) \right\| dt'')$ we obtain $\left\| X(t,t',\xi) \right\| \le C(\lambda(t)/\lambda(t'))^{Q+1}$ for arbitrary $t_\xi \le t' \le t \le T_0$.

Next we derive the following estimate:

Lemma 3.9. *For* $(t,\xi) \in Z_{\mathrm{hyp}}$ *and each* $l \in \mathbb{N}$,

$$\frac{|W_l(t,\xi)|}{\lambda(t)^Q} \le C_l \sum_{m=0}^{l} \left(\frac{|W_m(t_\xi,\xi)|}{\lambda(t_\xi)^Q} + \int_{t_\xi}^{t} \frac{|G_m(t',\xi)|}{\lambda(t')^Q}\,dt' \right). \tag{3.9}$$

Proof. This is true for $l = 0$, compare with (3.8). Now let $l \ge 1$ and assume (3.9) for $l-1$. From (3.8) and $l \ge 1$ it follows that

$$|W_l(t,\xi)| \le C\left(\frac{\lambda(t)}{\lambda(t_\xi)}\right)^Q |W_l(t_\xi,\xi)| + C\int_{t_\xi}^{t} \left(\frac{\lambda(t)}{\lambda(t')}\right)^{Q-1} |\tilde{G}_l(t',\xi)|\,dt'.$$

For the estimate of the integral, we recall that

$$(\lambda(t')\langle\xi\rangle)^{-m} \left\| D_{t'}^m A(t',\xi) \right\| \le C(\lambda(t')\langle\xi\rangle)^{-m}\lambda(t')\langle\xi\rangle t'^{-m} \le C\frac{\lambda'(t')}{\lambda(t')}.$$

Then we conclude that

$$\lambda(t)^{-Q}|W_l(t,\xi)| \le C\lambda(t_\xi)^{-Q}|W_l(t_\xi,\xi)| + C\int_{t_\xi}^{t}\lambda(t')^{-Q}|G_l(t',\xi)|\,dt' + C\sum_{m=1}^{l}\int_{t_\xi}^{t}\frac{\lambda'(t')}{\lambda(t)}\lambda(t')^{-Q}|W_{l-m}(t',\xi)|\,dt'.$$

The induction assumption gives

$$\frac{\lambda'(t')}{\lambda(t)}\frac{|W_{l-m}(t',\xi)|}{\lambda(t')^Q} \le C\frac{\lambda'(t')}{\lambda(t')}\sum_{k=0}^{l-m}\left(\frac{|W_k(t_\xi,\xi)|}{\lambda(t_\xi)^Q} + \int_{t_\xi}^{t'}\frac{|G_k(t'',\xi)|}{\lambda(t'')^Q}\,dt''\right).$$

Partial integration shows (3.9). □

Let us formulate the statement of Lemma 3.9 in another way:

Lemma 3.10. *If $s \in \mathbb{N}_+$, $\xi \in \mathbb{R}^n$ and $t \ge t_\xi$, then*

$$\sum_{l=0}^{s}\langle\xi\rangle^{s-l}\lambda(t)^{-Q-l}|\partial_t^l\hat{w}(t,\xi)| \le C\sum_{l=0}^{s}\langle\xi\rangle^{s-l}\lambda(t_\xi)^{-Q-l}|(\partial_t^l\hat{w})(t_\xi,\xi)| + C\sum_{l=0}^{s-1}\langle\xi\rangle^{s-1-l}\int_{t_\xi}^{t}\lambda(t')^{-Q-1-l}|\partial_t^l\hat{g}(t',\xi)|\,dt'.$$

Proof of Lemma 3.7. From the identities $\partial_t(\kappa_\nu^{(\delta)}f(t)) = \nu^\beta\kappa_\nu^{(\delta)}(\partial_t f(t))$ and $\lambda(t)^\alpha\kappa_\nu^{(\delta)}f(t) = \nu^{-\alpha\beta l_*}\kappa_\nu^{(\delta)}(\lambda(t)^\alpha f(t))$ we obtain

$$\lambda(t)^{-Q-l}\partial_t^l\big(\kappa_\nu^{(\delta)}f(t)\big) = \nu^{l+A}\kappa_\nu^{(\delta)}\big(\lambda(t)^{-Q-l}\partial_t^l f(t)\big).$$

Hence it follows that

$$\sum_{l=0}^{s}\langle\xi\rangle^{s-l}\lambda(t_\xi)^{-Q-l}|(\partial_t^l\hat{w})(t_\xi,\xi)| = \sum_{l=0}^{s}\langle\xi\rangle^{s+\beta/2}|(\partial_t^l v)(1,\xi)|.$$

Utilizing the identity $\lambda(t)^{-l}\partial_t^l(\kappa_\nu^{(\delta)}f(t)) = \nu^l\kappa_\nu^{(\delta)}(\lambda(t)^{-l}\partial_t^l f(t))$ and squaring the inequality of Lemma 3.10, we obtain

$$\sum_{l=0}^{s}\langle\xi\rangle^{2s}\lambda(t)^{-2Q}\left|\kappa^{(Q)}(\xi)\lambda(t)^{-l}\partial_t^l v(t,\xi)\right|^2 \le C\sum_{l=0}^{s}\langle\xi\rangle^{2s+\beta}|(\partial_t^l v)(1,\xi)|^2 + C\sum_{l=0}^{s-1}\langle\xi\rangle^{2(s-1)}t\int_{t_\xi}^{t}\lambda(t')^{-2(Q+1)}\left|\kappa^{(Q+1)}(\xi)\lambda(t')^{-l}\partial_t^l h(t',\xi)\right|^2 dt'.$$

Integrating over (t_ξ, T) and employing

$$\int_{t_\xi}^{t}\lambda(t')^{-2Q}\left|\kappa^{(Q)}(\xi)f(t')\right|^2 dt' = \int_1^{\langle\xi\rangle^\beta t}\lambda(t')^{-2Q}|f(t')|^2\,dt'$$

yield the assertion of Lemma 3.7. □

3.3. Estimates in edge Sobolev spaces

In this part, we patch the inequalities of Lemmas 3.5 and 3.7 together in order to prove Proposition 3.4.

Proof of Proposition 3.4. The norm $\|w\|_{s,Q;T}$ can be written in the equivalent form

$$\|w\|^2_{s,Q;T} \sim \sum_{l=0}^{s} T^{2l-1} \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s} \int_0^{\min(1,\langle\xi\rangle^\beta T)} \left|\partial_t^l v(t,\xi)\right|^2 \, dt\, d\xi$$
$$+ \sum_{l=0}^{s} T^{2l-1} \int_{\mathbb{R}^n_\xi} \langle\xi\rangle^{2s} \int_{\min(1,\langle\xi\rangle^\beta T)}^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-l}\partial_t^l v(t,\xi)\right|^2 \, dt\, d\xi.$$

A similar representation holds for $\|g\|_{s-1,Q+1;T}$. We set $T' = \min(1, \langle\xi\rangle^\beta T)$ and make use of Lemma 3.5. Then it follows that

$$T^{2l-1} \left\|\partial_t^l v(.,\xi)\right\|^2_{L^2(0,T')}$$
$$\le C\left(\frac{T}{T'}\right)^{2l-1} (\langle\xi\rangle^{-\beta+2A}|\hat{w}_0(\xi)|^2 + T'^2\langle\xi\rangle^{-3\beta+2A}|\hat{w}_1(\xi)|^2)$$
$$+ CT'^4 \sum_{r=0}^{l-1} T'^{2r-2l} T^{2l-1} \langle\xi\rangle^{-2(1+\beta)} \left\|\partial_t^r h(.,\xi)\right\|^2_{L^2(0,T')},$$

for all $1 \le l \le s$. Consequently,

$$\sum_{l=0}^{s} T^{2l-1} \left\|\partial_t^l v(.,\xi)\right\|^2_{L^2(0,T')}$$
$$\le C(\langle\xi\rangle^{-\beta+2A}|\hat{w}_0(\xi)|^2 + T'^2\langle\xi\rangle^{-3\beta+2A}|\hat{w}_1(\xi)|^2) \sum_{l=0}^{s} \left(\frac{T}{T'}\right)^{2l-1}$$
$$+ CT'^4 \sum_{r=0}^{s-1} T'^{2r-1} \langle\xi\rangle^{-2(1+\beta)} \left\|\partial_t^r h(.,\xi)\right\|^2_{L^2(0,T')} \sum_{l=r+1}^{s-1} \left(\frac{T}{T'}\right)^{2l-1}$$
$$\le C_{T_0}(\langle\xi\rangle^{-\beta+2A}|\hat{w}_0(\xi)|^2 + T'^2\langle\xi\rangle^{-3\beta+2A}|\hat{w}_1(\xi)|^2)\langle\xi\rangle^\beta$$
$$+ C_{T_0} T^2 T'^2 \sum_{r=0}^{s-1} T^{2r-1} \langle\xi\rangle^{-2(1+\beta)} \left\|\partial_t^r h(.,\xi)\right\|^2_{L^2(0,T')}.$$

By Lemma 3.7, we obtain

$$T^{2l-1} \int_1^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-l}\partial_t^l v(t,\xi)\right|^2 dt \le CT^{2l} \sum_{r=0}^{l} \langle\xi\rangle^\beta |(\partial_t^r v)(1,\xi)|^2$$
$$+ CT^2 \sum_{r=0}^{l-1} T^{2l-1} \langle\xi\rangle^{-2} \int_1^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-1-r}\partial_t^r h(t,\xi)\right|^2 dt, \tag{3.10}$$

for all $1 \le l \le s$, and

$$T^{-1} \int_1^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q} v(t,\xi)\right|^2 \, dt \le C \sum_{r=0}^{1} \langle\xi\rangle^\beta |(\partial_t^r v)(1,\xi)|^2 + CT\langle\xi\rangle^{-2} \int_1^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-1} h(t,\xi)\right|^2 \, dt. \tag{3.11}$$

Summing up (3.10) for $l = 1, \dots, s$ and (3.11) yields

$$\begin{aligned}
&\sum_{l=1}^{s} T^{2l-1} \int_1^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-l} \partial_t^l v(t,\xi)\right|^2 \, dt \\
&\quad \le C_{T_0} \sum_{r=0}^{s} T^{2r} \langle\xi\rangle^\beta |(\partial_t^r v)(1,\xi)|^2 \\
&\qquad + C_{T_0} T^2 \sum_{r=0}^{s-1} T^{2r+1} \langle\xi\rangle^{-2} \int_1^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-1-r} \partial_t^r h(t,\xi)\right|^2 \, dt.
\end{aligned}$$

Employing this technique (of picking one term and summing up) a third time, we deduce from Lemma 3.5 that

$$\begin{aligned}
\sum_{r=1}^{s} T^{2r} \langle\xi\rangle^\beta |(\partial_t^r v)(1,\xi)|^2 &\le C_{T_0} (\langle\xi\rangle^{2A} |\hat{w}_0(\xi)|^2 + T^2 \langle\xi\rangle^{-2\beta+2A} |\hat{w}_1(\xi)|^2) \\
&\quad + C_{T_0} T^3 \sum_{r=0}^{s-1} T^{2r-1} \langle\xi\rangle^{-2(1+\beta)} \left\|\partial_t^r h(.,\xi)\right\|_{L^2(0,1)}^2 .
\end{aligned}$$

Finally, Lemma 3.5 shows that

$$\begin{aligned}
\sum_{r=0}^{1} \langle\xi\rangle^\beta |(\partial_t^r v)(1,\xi)|^2 &\le C(\langle\xi\rangle^{2A} |\hat{w}_0(\xi)|^2 + \langle\xi\rangle^{-2\beta+2A} |\hat{w}_1(\xi)|^2) \\
&\quad + CT\langle\xi\rangle^{-2} \|h(.,\xi)\|_{L^2(0,1)}^2 ,
\end{aligned}$$

where we have used $\langle\xi\rangle^{-\beta} \le T$. Taking into account all estimates obtained so far, we find

$$\begin{aligned}
&\sum_{l=0}^{s} T^{2l-1} \left(\left\|\partial_t^l v(.,\xi)\right\|_{L^2(0,T')}^2 + \int_{T'}^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-l} \partial_t^l v(t,\xi)\right|^2 dt \right) \\
&\quad \le C(\langle\xi\rangle^{2A} |\hat{w}_0(\xi)|^2 + \langle\xi\rangle^{-2\beta+2A} |\hat{w}_1(\xi)|^2) \\
&\qquad + CT^2 \sum_{l=0}^{s-1} T^{2l-1} \langle\xi\rangle^{-2} \left\|\partial_t^l h(.,\xi)\right\|_{L^2(0,T')}^2 \\
&\qquad + CT^2 \sum_{l=0}^{s-1} T^{2l-1} \langle\xi\rangle^{-2} \int_{T'}^{\langle\xi\rangle^\beta T} \left|\lambda(t)^{-Q-1-l} \partial_t^l h(t,\xi)\right|^2 dt.
\end{aligned}$$

Multiplying by $\langle\xi\rangle^{2s}$ and integrating the resulting expressions over $\mathbb{R}^n$ with respect to ξ completes the proof. □

4. The algebra property

In this section, we show first that edge Sobolev spaces $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ are algebras under certain conditions on s and δ. Then we easily conclude that superposition operators which are formed by entire functions map these edge Sobolev spaces into themselves.

Proposition 4.1. *Assume that $s+\delta\geq 0$. We suppose that $s\in\mathbb{N}$ and $\min\{s, s+\beta\delta l_*\}>(n+2)/2$. Then $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ is an algebra under pointwise multiplication for any $0<T\leq T_0$. In other words, we have*

$$\|uv\|_{s,\delta;T}\leq C\,\|u\|_{s,\delta;T}\,\|v\|_{s,\delta;T}\,,$$

for $u,v\in H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$. Moreover, the constant C is independent of $0<T\leq T_0$.

Corollary 4.2. *Let $f=f(u)$ be an entire function with $f(0)=0$, i.e., $f(u)=\sum_{j=1}^{\infty}f_ju^j$ for all $u\in\mathbb{R}$. Then, under the assumptions of Proposition 4.1, there is, for each $R>0$, a constant $C_1(R)$ with the property that*

$$\|f(u)\|_{s,\delta;T}\leq C_1(R)\,\|u\|_{s,\delta;T}\,,$$
$$\|f(u)-f(v)\|_{s,\delta;T}\leq C_1(R)\,\|u-v\|_{s,\delta;T}\,,$$

provided that $u,v\in H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ and $\|u\|_{s,\delta;T}\leq R$, $\|v\|_{s,\delta;T}\leq R$.

The proof is split into several lemmas and makes heavy use of so-called weight functions.

Definition 4.3. A function $\alpha\colon\mathbb{R}^n_\xi\to[c,\infty)$ $(c>0)$ is called a *weight function* if α is a continuous, monotonically increasing function of $|\xi|$ with the property that $\alpha(2\xi)\leq C\alpha(\xi)$ holds for all $\xi\in\mathbb{R}^n$.

Lemma 4.4. *Suppose that α,β,γ are weight functions with*

$$C_0^2=\sup_{\xi\in\mathbb{R}^n}\int_{\mathbb{R}^n_\eta}\frac{\alpha(\xi)^2}{\beta(\eta)^2\gamma(\xi-\eta)^2}d\eta<\infty. \tag{4.1}$$

If u,v are functions with $\beta(\xi)\hat{u}(\xi)\in L^2(\mathbb{R}^n)$ and $\gamma(\xi)\hat{v}(\xi)\in L^2(\mathbb{R}^n)$, then $\alpha(\xi)(uv)\widehat{\ }(\xi)\in L^2(\mathbb{R}^n)$ and

$$\|\alpha(\xi)(uv)\widehat{\ }(\xi)\|_{L^2(\mathbb{R}^n)}\leq C_0\left\|\beta(\xi)\hat{u}(\xi)\right\|_{L^2(\mathbb{R}^n)}\left\|\gamma(\xi)\hat{v}(\xi)\right\|_{L^2(\mathbb{R}^n)}.$$

Proof. Choose some arbitrary $w\in L^2(\mathbb{R}^n)$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n_\xi}\alpha(\xi)(uv)\widehat{\ }(\xi)\hat{w}(\xi)\,d\xi\right| &\leq\left\{\int_{\mathbb{R}^n_\xi}|\hat{w}(\xi)|^2\int_{\mathbb{R}^n_\eta}\frac{\alpha(\xi)^2}{\beta(\eta)^2\gamma(\xi-\eta)^2}\,d\eta\,d\xi\right\}^{1/2}\\ &\quad\times\left\{\int_{\mathbb{R}^n_\eta}\beta(\eta)^2|\hat{u}(\eta)|^2\int_{\mathbb{R}^n_\xi}\gamma(\xi-\eta)^2|\hat{v}(\xi-\eta)|^2\,d\xi\,d\eta\right\}^{1/2}\\ &\leq C_0\|\hat{w}\|_{L^2(\mathbb{R}^n)}\left\|\beta(\xi)\hat{u}(\xi)\right\|_{L^2(\mathbb{R}^n)}\left\|\gamma(\xi)\hat{v}(\xi)\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Applying the Riesz representation theorem concludes the proof. □

The following lemma gives a sufficient condition for (4.1):

Lemma 4.5. *Suppose that* α, β, γ *are weight functions. Furthermore, assume that*

$$\int_{\mathbb{R}^n_\eta} \frac{\alpha(\eta)^2}{\beta(\eta)^2\gamma(\eta)^2}\, d\eta < \infty,$$

$$\sup_{\xi\in\mathbb{R}^n} \left(\frac{\alpha(\xi)^2}{\beta(\xi)^2} \int_{|\eta|\le 2|\xi|} \frac{d\eta}{\gamma(\eta)^2} + \frac{\alpha(\xi)^2}{\gamma(\xi)^2} \int_{|\eta|\le 2|\xi|} \frac{d\eta}{\beta(\eta)^2} \right) < \infty.$$

Then (4.1) *holds.*

Proof. For each $\xi \in \mathbb{R}^n$, we split $\mathbb{R}^n_\eta$ into three parts:

$$\begin{aligned} A(\xi) &= \{\eta\colon |\eta| \ge 2|\xi|\}, \\ B(\xi) &= \{\eta\colon |\eta| \le 2|\xi|, |\xi-\eta| \le |\eta|\}, \\ C(\xi) &= \{\eta\colon |\eta| \le 2|\xi|, |\xi-\eta| \ge |\eta|\}. \end{aligned}$$

Case 1: $\eta \in A(\xi)$. We have $|\xi-\eta| \ge |\eta|/2$, hence $\gamma(\xi-\eta) \ge \gamma(\eta/2) \ge C\gamma(\eta)$. From this and $\alpha(\xi) \le \alpha(\eta)$, it follows that

$$\int_{A(\xi)} \frac{\alpha(\xi)^2}{\beta(\eta)^2\gamma(\xi-\eta)^2}\, d\eta \le C.$$

Case 2: $\eta \in B(\xi)$. It holds $|\xi| \le |\xi-\eta|+|\eta| \le 2|\eta|$, hence $\beta(\eta) \ge \beta(\xi/2) \ge C\beta(\xi)$. Then we obtain

$$\int_{B(\xi)} \frac{\alpha(\xi)^2}{\beta(\eta)^2\gamma(\xi-\eta)^2}\, d\eta \le C\frac{\alpha(\xi)^2}{\beta(\xi)^2} \int_{|\zeta|\le 2|\xi|} \frac{d\zeta}{\gamma(\zeta)^2} \le C.$$

Case 3: $\eta \in C(\xi)$. In this case, we have $|\xi| \le |\xi-\eta|+|\eta| \le 2|\xi-\eta|$, consequently, $\gamma(\xi-\eta) \ge \gamma(\xi/2) \ge C\gamma(\xi)$ which implies

$$\int_{C(\xi)} \frac{\alpha(\xi)^2}{\beta(\eta)^2\gamma(\xi-\eta)^2}\, d\eta \le C\frac{\alpha(\xi)^2}{\gamma(\xi)^2} \int_{|\eta|\le 2|\xi|} \frac{d\eta}{\beta(\eta)^2} \le C.$$

The proof is complete. □

In a certain case, a more precise estimate than that of Lemma 4.4 is required:

Lemma 4.6. *Let* $u, v \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *be functions with* $\alpha(\xi)\hat{u}(\xi) \in L^2(\mathbb{R}^n)$, $\alpha(\xi)\hat{v}(\xi) \in L^2(\mathbb{R}^n)$, *where* α *is a weight function. If*

$$C_0^2 = \sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{R}^n_\eta} \frac{(\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))^2\langle\eta\rangle^2}{\alpha(\eta)^2\alpha(\xi-\eta)^2}\, d\eta < \infty, \tag{4.2}$$

then

$$\begin{aligned} \|\alpha(\xi)\widehat{(uv)}(\xi)\|_{L^2(\mathbb{R}^n)} &\le \|u(x)\|_{L^\infty(\mathbb{R}^n)}\|\alpha(\xi)\hat{v}(\xi)\|_{L^2(\mathbb{R}^n)} \\ &\quad + C_0\|\alpha(\xi)\hat{u}(\xi)/\langle\xi\rangle\|_{L^2(\mathbb{R}^n)}\|\alpha(\xi)\hat{v}(\xi)\|_{L^2(\mathbb{R}^n)} \\ &\quad + \|v(x)\|_{L^\infty(\mathbb{R}^n)}\|\alpha(\xi)\hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Proof. We can decompose the term $\alpha(\xi)(uv)\widehat{\ }(\xi)$ as

$$\int_{\mathbb{R}^n_\eta} \alpha(\xi)\hat{u}(\eta)\hat{v}(\xi-\eta)\,d\eta = \int_{\mathbb{R}^n_\eta} \alpha(\eta)\hat{u}(\eta)\hat{v}(\xi-\eta)\,d\eta + \int_{\mathbb{R}^n_\eta} \alpha(\xi-\eta)\hat{u}(\eta)\hat{v}(\xi-\eta)\,d\eta + \int_{\mathbb{R}^n_\eta} (\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))\hat{u}(\eta)\hat{v}(\xi-\eta)\,d\eta.$$

From $\int_{\mathbb{R}^n_\eta} \alpha(\eta)\hat{u}(\eta)\hat{v}(\xi-\eta)\,d\eta = F_{x\to\xi}(v(x)F_{\xi\to x}(\alpha(\xi)\hat{u}(\xi)))(\xi)$ and Plancherel's theorem we get

$$\left\| \int_{\mathbb{R}^n_\eta} \alpha(\eta)\hat{u}(\eta)\hat{v}(\xi-\eta)\,d\eta \right\|_{L^2(\mathbb{R}^n)} \le \|v(x)\|_{L^\infty(\mathbb{R}^n)} \left\|\alpha(\xi)\hat{u}(\xi)\right\|_{L^2(\mathbb{R}^n)}.$$

Now choose some arbitrary $w \in L^2(\mathbb{R}^n)$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_\eta} (\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))\hat{u}(\eta)\hat{v}(\xi-\eta)\,d\eta\,\hat{w}(\xi)\,d\xi \right| \\
&\le \left\{ \int_{\mathbb{R}^n_\xi} |\hat{w}(\xi)|^2 \int_{\mathbb{R}^n_\eta} \frac{(\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))^2\langle\eta\rangle^2}{\alpha(\eta)^2\alpha(\xi-\eta)^2}\,d\eta\,d\xi \right\}^{1/2} \\
&\quad\times \left\{ \int_{\mathbb{R}^n_\eta} \frac{\alpha(\eta)^2}{\langle\eta\rangle^2}|\hat{u}(\eta)|^2 \int_{\mathbb{R}^n_\xi} \alpha(\xi-\eta)^2|\hat{v}(\xi-\eta)|^2\,d\xi\,d\eta \right\}^{1/2} \\
&\le C_0 \left\|\hat{w}\right\|_{L^2(\mathbb{R}^n)} \left\|\alpha(\xi)\hat{u}(\xi)/\langle\xi\rangle\right\|_{L^2(\mathbb{R}^n)} \left\|\alpha(\xi)\hat{v}(\xi)/\langle\xi\rangle\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Applying the Riesz representation theorem concludes the proof. □

Now we present a sufficient condition for (4.2):

Lemma 4.7. *Suppose that* $\alpha = \alpha(\xi)$ *is a weight function with*

$$|\alpha(\xi)-\alpha(\eta)| \le C|\xi-\eta|\frac{\alpha(\xi)}{\langle\xi\rangle} \quad \forall |\xi-\eta| \le \frac{1}{2}|\xi|, \tag{4.3}$$

$$\int_{\mathbb{R}^n_\eta} \frac{\langle\eta\rangle^2}{\alpha(\eta)^2}d\eta < \infty. \tag{4.4}$$

Then (4.2) *holds.*

Proof. For each $\xi \in \mathbb{R}^n$, we split $\mathbb{R}^n_\eta$ into four parts:

$$\begin{aligned}
A(\xi) &= \{\eta\colon |\eta| \ge 2|\xi|\}, \\
B(\xi) &= \{\eta\colon |\eta| \le 2|\xi|, |\xi-\eta| \le |\eta|/2\}, \\
C(\xi) &= \{\eta\colon |\eta| \le 2|\xi|, |\xi-\eta| \ge 2|\eta|\}, \\
D(\xi) &= \{\eta\colon |\eta| \le 2|\xi|, |\eta|/2 \le |\xi-\eta| \le 2|\eta|\}.
\end{aligned}$$

For the sake of brevity, we introduce the notation

$$\delta(\xi,\eta)=\frac{(\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))^2\langle\eta\rangle^2}{\alpha(\eta)^2\alpha(\xi-\eta)^2}.$$

Case 1: $\eta\in A(\xi)$. We have $|\xi-\eta|\le 3|\eta|/2$ and $|\eta|\le 2|\xi-\eta|$, hence

$$(\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))^2\le C\alpha(\eta)^2\le C\frac{\alpha(\eta)^2\langle\xi-\eta\rangle^2}{\langle\eta\rangle^2},$$

which implies $\delta(\xi,\eta)\le C\langle\xi-\eta\rangle^2/\alpha(\xi-\eta)^2$.

Case 2: $\eta\in B(\xi)$. By (4.3) we deduce that

$$\begin{aligned}(\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))^2&\le 2(\alpha(\xi)-\alpha(\eta))^2+2\alpha(\xi-\eta)^2\\&\le C\langle\xi-\eta\rangle^2\frac{\alpha(\eta)^2}{\langle\eta\rangle^2}+2\alpha(\xi-\eta)^2,\end{aligned}$$

and, consequently, $\delta(\xi,\eta)\le C\langle\xi-\eta\rangle^2/\alpha(\xi-\eta)^2+C\langle\eta\rangle^2/\alpha(\eta)^2$.

Case 3: $\eta\in C(\xi)$. Applying (4.3) again, we get

$$\begin{aligned}(\alpha(\xi)-\alpha(\eta)-\alpha(\xi-\eta))^2&\le 2(\alpha(\xi)-\alpha(\xi-\eta))^2+2\alpha(\eta)^2\\&\le C\langle\eta\rangle^2\frac{\alpha(\xi-\eta)^2}{\langle\xi-\eta\rangle^2}+C\alpha(\eta)^2\frac{\langle\xi-\eta\rangle^2}{\langle\eta\rangle^2}\\&\le C\alpha(\xi-\eta)^2+C\alpha(\eta)^2\frac{\langle\xi-\eta\rangle^2}{\langle\eta\rangle^2}.\end{aligned}$$

Then we obtain $\delta(\xi,\eta)\le C\langle\eta\rangle^2/\alpha(\eta)^2+C\langle\xi-\eta\rangle^2/\alpha(\xi-\eta)^2$.

Case 4: $\eta\in D(\xi)$. Here we have $|\xi|\le|\xi-\eta|+|\eta|\le 3|\eta|$, hence

$$(\alpha(\xi)-\alpha(\xi-\eta)-\alpha(\eta))^2\le C\alpha(\eta)^2\le C\alpha(\eta)^2\frac{\langle\xi-\eta\rangle^2}{\langle\eta\rangle^2}.$$

Proceeding as in Case 1 we find $\delta(\xi,\eta)\le C\langle\xi-\eta\rangle^2/\alpha(\xi-\eta)^2$.

From (4.4) we obtain $\int_{\mathbb{R}^n_\eta}\delta(\xi,\eta)\,d\eta\le C$ uniformly in ξ. □

Definition 4.8. Let $\{\vartheta_l=\vartheta_l(t,\xi)\}_{l=0}^s$ be a family of weight functions depending on the parameter $t\in[0,T_0]$. Then we define the norm

$$\|u\|_{s,T}^2=\sum_{l=0}^{s}T^{2l-1}\int_0^T\left\|\vartheta_l(t,\xi)\partial_t^l\hat{u}(t,\xi)\right\|^2_{L^2(\mathbb{R}^n)}\,dt.$$

Lemma 4.9. *Assume that the family of weight functions* $\{\vartheta_0,\dots,\vartheta_s\}$ *satisfies the following conditions:*

$$\sup_{[0,T]}\int_{\mathbb{R}^n_\eta}\frac{\langle\eta\rangle^2}{\vartheta_0(t,\eta)^2}\,d\eta<\infty, \tag{4.5}$$

$$\exists\,\varepsilon>0:\quad \langle\xi\rangle^{n/2+\varepsilon}\le C\vartheta_1(t,\xi)\quad\forall(t,\xi)\in[0,T_0]\times\mathbb{R}^n, \tag{4.6}$$

$$c|\partial_t \vartheta_0(t,\xi)|/\langle\xi\rangle \le \vartheta_0(t,\xi) \le C\vartheta_1(t,\xi)\langle\xi\rangle \quad \forall(t,\xi), \tag{4.7}$$

$$|\partial_t \vartheta_{l+1}(t,\xi)| + \vartheta_{l+1}(t,\xi) \le C\vartheta_l(t,\xi), \quad l \le s-1, \quad \forall(t,\xi), \tag{4.8}$$

$$|\vartheta_0(t,\xi) - \vartheta_0(t,\eta)| \le C|\xi-\eta| \frac{\vartheta_0(t,\xi)}{\langle\xi\rangle} \quad \forall t, \quad \forall|\xi-\eta| \le |\xi|/2, \tag{4.9}$$

$$\sup_{[0,T]} \int_{\mathbb{R}^n_\eta} \frac{\vartheta_l(t,\eta)^2}{\vartheta_{l-k}(t,\eta)^2 \vartheta_{k+1}(t,\eta)^2} \, d\eta < \infty, \quad l \ge k+1, \tag{4.10}$$

$$\sup_{[0,T]\times\mathbb{R}^n_\xi} \frac{\vartheta_l(t,\xi)^2}{\vartheta_{l-k}(t,\xi)^2} \int_{|\eta|\le 2|\xi|} \frac{d\eta}{\vartheta_{k+1}(t,\eta)^2} < \infty, \quad l \ge k+1. \tag{4.11}$$

Then there is a constant C_0 (*independent of* T) *such that*

$$\|uv\|_{s,T} \le C_0 \|u\|_{s,T} \|v\|_{s,T} \, .$$

Proof. Obviously,

$$\|uv\|^2_{s,T} \le C \sum_{l=0}^{s} \sum_{k=0}^{l} T^{2l-1} \left\| \vartheta_l(t,\xi) \big((\partial_t^k u)(\partial_t^{l-k} v) \big)\widehat{\ }(t,\xi) \right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} .$$

First we consider the terms with $l \ge 1$. Without loss of generality we may assume that $k+1 \le l$ (otherwise we change the roles of k and $l-k$ in the sequel). Due to (4.10), (4.11) we may apply the Lemmas 4.4 and 4.5 in the following way:

$$\begin{aligned} &T^{2l-1} \int_0^T \left\| \vartheta_l(t,\xi) \big((\partial_t^k u)(\partial_t^{l-k} v) \big)\widehat{\ }(t,\xi) \right\|^2_{L^2(\mathbb{R}^n)} dt \\ &\le C T^{2l-1} \int_0^T \left\| \vartheta_{k+1}(t,\xi) \partial_t^k \hat{u}(t,\xi) \right\|^2_{L^2(\mathbb{R}^n)} \left\| \vartheta_{l-k}(t,\xi) \partial_t^{l-k} \hat{v}(t,\xi) \right\|^2_{L^2(\mathbb{R}^n)} dt \\ &\le C T^{2k} \left\| \vartheta_{k+1} \partial_t^k \hat{u} \right\|^2_{L^\infty([0,T],L^2(\mathbb{R}^n))} T^{2(l-k)-1} \left\| \vartheta_{l-k} \partial_t^{l-k} \hat{v} \right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} . \end{aligned}$$

We have the embedding $W^1_2(0,T) \subset L^\infty(0,T)$. More precisely,

$$\|g\|_{L^\infty(0,T)} \le C T^{-1} \|g\|_{L^2(0,T)} + C T \|\partial_t g\|_{L^2(0,T)} . \tag{4.12}$$

Then (4.8) yields

$$\begin{aligned} T^{2k} \left\| \vartheta_{k+1} \partial_t^k \hat{u} \right\|^2_{L^\infty([0,T],L^2(\mathbb{R}^n))} &\le C T^{2k-1} \left\| \vartheta_{k+1} \partial_t^k \hat{u} \right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\quad + C T^{2k+1} \left\| (\partial_t \vartheta_{k+1}) \partial_t^k \hat{u} \right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\quad + C T^{2k+1} \left\| \vartheta_{k+1} \partial_t^{k+1} \hat{u} \right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\le C \|u\|^2_{s,T} . \end{aligned}$$

Now it remains to consider the term with $l = 0$. According to Lemmata 4.6 and 4.7, we have

$$\begin{aligned} T^{-1} \|\vartheta_0(t,\xi)(uv)\widehat{\ }(t,\xi)\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} &\\ \le T^{-1} \|u(t,x)\|^2_{L^\infty([0,T]\times\mathbb{R}^n)} \|\vartheta_0\hat{v}\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} &\\ + CT^{-1}\|\vartheta_0(t,\xi)\hat{u}(t,\xi)/\langle\xi\rangle\|^2_{L^\infty([0,T],L^2(\mathbb{R}^n))}\|\vartheta_0\hat{v}\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} &\\ + T^{-1}\|v(t,x)\|^2_{L^\infty([0,T]\times\mathbb{R}^n)}\|\vartheta_0\hat{u}\|^2_{L^2([0,T],L^2(\mathbb{R}^n))}. & \end{aligned}$$

The embedding $H^{n/2+\varepsilon}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, an argument similar to that of (4.12), and (4.6), and (4.8) imply

$$\begin{aligned} \|u(t,x)\|^2_{L^\infty([0,T]\times\mathbb{R}^n)} &\le CT\|\langle\xi\rangle^{n/2+\varepsilon}\partial_t\hat{u}(t,\xi)\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\quad + CT^{-1}\left\|\langle\xi\rangle^{n/2+\varepsilon}\hat{u}(t,\xi)\right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\le C\left(T\|\vartheta_1\partial_t\hat{u}\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} + T^{-1}\|\vartheta_0\hat{u}\|^2_{L^2([0,T],L^2(\mathbb{R}^n))}\right). \end{aligned}$$

Exploiting (4.7) we deduce, in a similar manner, that

$$\begin{aligned} &\left\|\vartheta_0(t,\xi)\hat{u}(t,\xi)/\langle\xi\rangle\right\|^2_{L^\infty([0,T],L^2(\mathbb{R}^n))} \\ &\quad\le CT^{-1}\left\|\vartheta_0(t,\xi)\hat{u}(t,\xi)/\langle\xi\rangle\right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\qquad + CT\left\|(\partial_t\vartheta_0(t,\xi))\hat{u}(t,\xi)/\langle\xi\rangle\right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\qquad + CT\left\|\vartheta_0(t,\xi)\partial_t\hat{u}(t,\xi)/\langle\xi\rangle\right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} \\ &\quad\le C\left(T\left\|\vartheta_1\partial_t\hat{u}\right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))} + T^{-1}\left\|\vartheta_0\hat{u}\right\|^2_{L^2([0,T],L^2(\mathbb{R}^n))}\right). \end{aligned}$$

The proof is complete. □

Proof of Proposition 4.1. The norm $\|.\|_{s,\delta;T}$ of the space $H^{s,\delta;\lambda}((0,T)\times\mathbb{R}^n)$ is equivalent to the norm $\|.\|_{s,T}$ from Definition 4.8, where

$$\vartheta_l(t,\xi) = \begin{cases} \langle\xi\rangle^{s-l}\lambda(t_\xi)^{-\delta-l} & : 0 \le t \le t_\xi = \langle\xi\rangle^{-\beta}, \\ \langle\xi\rangle^{s-l}\lambda(t)^{-\delta-l} & : t_\xi \le t \le T. \end{cases}$$

It is easy to check that these functions are continuous, increasing in $|\xi|$ (since $s+\delta \ge 0$) and that $\vartheta_l(t,2\xi) \le C\vartheta_l(t,\xi)$ for all (t,ξ) and all l. Hence they are weight functions in the sense of Definition 4.3. The proof is complete provided we show that $\vartheta_0,\ldots,\vartheta_s$ satisfy the conditions (4.5)–(4.11).

If $\delta \ge 0$, then

$$\int_{\mathbb{R}^n_\eta} \frac{\langle\eta\rangle^2}{\vartheta_0(t,\eta)^2}\,d\eta \le \lambda(T)^{2\delta}\int_{\mathbb{R}^n_\eta}\langle\eta\rangle^{-2s+2}\,d\eta < \infty,$$

since $s > (n+2)/2$. If $\delta \le 0$, then $s > (n+2)/2 + \beta|\delta|l_*$, hence

$$\int_{\mathbb{R}^n_\eta} \frac{\langle\eta\rangle^2}{\vartheta_0(t,\eta)^2}\,d\eta \le \int_{\mathbb{R}^n_\eta}\langle\eta\rangle^{-2s+2}\lambda(t_\eta)^{2\delta}\,d\eta < \infty.$$

This proves (4.5), and (4.6) can be considered similarly. We observe that

$$\frac{\vartheta_0(t,\xi)}{\vartheta_1(t,\xi)} \le \lambda(T)\langle\xi\rangle, \quad \frac{\vartheta_{l+1}(t,\xi)}{\vartheta_l(t,\xi)} \le t_\xi \le C, \quad \frac{|\partial_t\vartheta_l(t,\xi)|}{\vartheta_l(t,\xi)} \le Ct_\xi^{-1},$$

which yield (4.7) and (4.8). Now we prove (4.9). Fix t, ξ, η. The derivative of the function $\vartheta_0(t, \xi + \varrho(\eta - \xi)) \colon [0, 1] \to \mathbb{R}_+$ has at most two jumps, say at ϱ_1, ϱ_2 with $0 \le \varrho_1 < \varrho_2 \le 1$. We write

$$\begin{aligned}\vartheta_0(t,\xi) - \vartheta_0(t,\eta) &= (\vartheta_0(t,\xi) - \vartheta_0(t,\xi+\varrho_1(\eta-\xi))) \\ &\quad + (\vartheta_0(t,\xi+\varrho_1(\eta-\xi)) - \vartheta_0(t,\xi+\varrho_2(\eta-\xi))) \\ &\quad + (\vartheta_0(t,\xi+\varrho_2(\eta-\xi)) - \vartheta_0(t,\eta)),\end{aligned}$$

and apply Hadamard's formula to each term on the right. We obtain, for example,

$$\begin{aligned}|\vartheta_0(t,\xi) - \vartheta_0(t,\xi+\varrho_1(\eta-\xi))| &\le \int_0^{\varrho_1} |\nabla\vartheta_0(t,\xi+\varrho(\eta-\xi))| \cdot |\xi-\eta| \, d\varrho \\ \le C\int_0^{\varrho_1} \frac{\vartheta_0(t,\xi+\varrho(\eta-\xi))}{\langle\xi+\varrho(\eta-\xi)\rangle} \, d\varrho \cdot |\xi-\eta| &\le C\frac{\vartheta_0(t,\xi)}{\langle\xi\rangle}|\xi-\eta|,\end{aligned}$$

since $|\xi - \eta| \le |\xi|/2$. Then (4.9) follows. By (4.7) we deduce that

$$\frac{\vartheta_l(t,\eta)}{\vartheta_{l-k}(t,\eta)\vartheta_{k+1}(t,\eta)} = \frac{1}{\vartheta_1(t,\eta)} \le \frac{C\langle\eta\rangle}{\vartheta_0(t,\eta)}.$$

Then (4.10) follows from (4.5) immediately. It remains to show (4.11). Suppose that $(t, 2\xi) \in Z_{\mathrm{pd}}$. Then we have

$$\begin{aligned}\frac{\vartheta_l(t,\xi)^2}{\vartheta_{l-k}(t,\xi)^2}\int_{|\eta|\le 2|\xi|}\frac{d\eta}{\vartheta_{k+1}(t,\eta)^2} &= (\langle\xi\rangle\lambda(t_\xi))^{-2k}\int_{|\eta|\le 2|\xi|}\frac{\lambda(t_\eta)^{2(\delta+k+1)}}{\langle\eta\rangle^{2(s-k-1)}}\,d\eta \\ &\le C\langle\xi\rangle^{-2k\beta}\int_{|\eta|\le 2|\xi|}\langle\eta\rangle^{-2(s-k-1+\beta(\delta+k+1)l_*)+n-1}d|\eta| \\ &\le C\langle\xi\rangle^{-2k\beta-2(s-k-1+\beta(\delta+k+1)l_*)+n+\varepsilon} = C\langle\xi\rangle^{-2s+n+2-2\beta(\delta+1)l_*+\varepsilon}.\end{aligned}$$

Due to our assumptions, the last exponent is negative. Now assume that $(t, \xi) \in Z_{\mathrm{hyp}}$, which is equivalent to $t^{-(l_*+1)} \le \langle\xi\rangle$. Then a short calculation reveals

$$\begin{aligned}&\frac{\vartheta_l(t,\xi)^2}{\vartheta_{l-k}(t,\xi)^2}\int_{|\eta|\le 2|\xi|}\frac{d\eta}{\vartheta_{k+1}(t,\eta)^2} \\ &\quad = (\langle\xi\rangle\lambda(t))^{-2k}\int_{|\eta|\le t^{-(l_*+1)}}\frac{\lambda(t_\eta)^{2(\delta+k+1)}\,d\eta}{\langle\eta\rangle^{2(s-k-1)}} \\ &\qquad + (\langle\xi\rangle\lambda(t))^{-2k}\int_{t^{-(l_*+1)}\le|\eta|\le|2\xi|}\frac{\lambda(t)^{2(\delta+k+1)}\,d\eta}{\langle\eta\rangle^{2(s-k-1)}}\end{aligned}$$

$$
\begin{aligned}
&\leq C(\langle\xi\rangle\lambda(t))^{-2k}\int_{|\eta|\leq t^{-(l_*+1)}}\langle\eta\rangle^{-2(s-k-1+\beta(\delta+k+1)l_*)+n-1}\,d|\eta| \\
&\quad + C\langle\xi\rangle^{-2k}\lambda(t)^{2(\delta+1)}\int_{|\eta|\leq 2|\xi|}\langle\eta\rangle^{-2(s-k-1)+n-1}\,d|\eta| \\
&\leq C\langle\xi\rangle^{-2k}\lambda(t)^{-2k}t^{-(l_*+1)(-2(s-k-1+\beta(\delta+k+1)l_*)+n+\varepsilon)} \\
&\quad + C\lambda(t)^{2(\delta+1)}\langle\xi\rangle^{-2s+n+2+\varepsilon} \\
&= C(\langle\xi\rangle\Lambda(t))^{-2k}\Lambda(t)^{2s-n-2+2\beta(\delta+1)l_*-\varepsilon} + C\vartheta_1(t,\xi)^{-2}\langle\xi\rangle^{n+\varepsilon} \\
&\leq C,
\end{aligned}
$$

where we have used (4.6). The remaining case of $(t,\xi)\in Z_{\mathrm{pd}}$, $(t,2\xi)\in Z_{\mathrm{hyp}}$ can be considered similarly. The proof is complete. □

5. The semilinear Cauchy problem

First we prove a local in time well-posedness result for the semilinear Cauchy problem.

Theorem 5.1. *Let* $s\in\mathbb{N}$ *and assume that* $\min\{s, s+\beta Q_0 l_*\} > (n+2)/2$, *where* Q_0 *is the number from* (3.4). *Let* $Q\geq Q_0$ *and* $A=\beta Q l_*$. *Then, for* $u_0\in H^{s+A}(\mathbb{R}^n)$, $u_1\in H^{s+A-\beta}(\mathbb{R}^n)$, *there is a number* $T>0$ *with the property that a solution* $u\in H^{s,Q;\lambda}((0,T)\times\mathbb{R}^n)$ *to the Cauchy problem* (1.1) *exists. This solution* u *is unique in the space* $H^{s,Q_0;\lambda}((0,T)\times\mathbb{R}^n)$.

Proof. Uniqueness follows from the basic energy estimate and the local Lipschitz continuity of the map $H^{s,Q;\lambda}((0,T)\times\mathbb{R}^n)\to H^{s,Q;\lambda}((0,T)\times\mathbb{R}^n)$, $u\mapsto f(u)$; see Proposition 3.4 and Corollary 4.2. To get existence, let

$$R = 2C_0\left(\|u_0\|_{H^{s+A}(\mathbb{R}^n)} + \|u_1\|_{H^{s+A-\beta}(\mathbb{R}^n)} + 1\right),$$

and choose $0<T\leq T_0$ such that

$$
\begin{aligned}
&C_0\left(\|u_0\|_{H^{s+A}(\mathbb{R}^n)} + \|u_1\|_{H^{s+A-\beta}(\mathbb{R}^n)} + TC_1(R)R\right)\leq R, \\
&TC_0C_1(R)\leq 1/2,
\end{aligned}
$$

where C_0 and $C_1(R)$ are the constants from Proposition 3.4 and Corollary 4.2. Recall that the constant of the embedding $H^{s,Q;\lambda}((0,T)\times\mathbb{R}^n)\subset H^{s-1,Q+1;\lambda}((0,T)\times\mathbb{R}^n)$ is (uniformly in $0<T\leq T_0$) bounded by 1. Fix the closed ball,

$$B=\left\{u\in H^{s,Q;\lambda}((0,T)\times\mathbb{R}^n)\colon\ \|u\|_{s,Q;T}\leq R\right\},$$

and observe that the map $\mathcal{A}\colon w\mapsto u$, $Lu=f(w)$, $u(0,x)=u_0(x)$, $u_t(0,x)=u_1(x)$ maps B into itself, according to Proposition 3.4 and Corollary 4.2. Moreover,

$$\|\mathcal{A}(w)-\mathcal{A}(w')\|_{s,Q;T}\leq\frac{1}{2}\|w-w'\|_{s,Q;T}$$

such that Banach's fixed point theorem applies to yield the existence of a unique fixed point $u\in B$ of $\mathcal{A}$ which is then a solution to (1.1). □

Remark 5.2. The same proof yields local in time well-posedness in $H^{s,Q_0;\lambda}((0,T) \times \mathbb{R}^n)$ for semilinear equations of the form

$$Lu = f\big(u, \partial_t u, t^{l_*}\partial_{x_1}u, \dots, t^{l_*}\partial_{x_n}u\big),$$

where f is an entire function on $\mathbb{R}^{n+2}$ satisfying $f(0,\dots,0) \equiv 0$ provided that $s-1 > (n+2)/2$.

Eventually we state a result concerning the propagation of mild singularities.

Theorem 5.3. *Let s satisfy the assumptions of* Theorem 5.1. *Assume $u_0 \in H^{s+\beta Q_0 l_*}(\mathbb{R}^n)$, $u_1 \in H^{s+\beta Q_0 l_*-\beta}(\mathbb{R}^n)$, where Q_0 is given by* (3.4). *Then the unique solutions $u, v \in H^{s,Q_0;\lambda}((0,T)\times\mathbb{R}^n)$ to* (1.1) *and* (1.2) *satisfy*

$$u - v \in H^{s+\beta,Q_0;\lambda}((0,T)\times\mathbb{R}^n).$$

Proof. Corollary 4.2 implies $f(u) \in H^{s,Q_0;\lambda}((0,T)\times\mathbb{R}^n)$. From Lemma 2.14 we deduce that $f(u) \in H^{s-1+\beta,Q_0+1;\lambda}((0,T)\times\mathbb{R}^n)$. The function $w(t,x) = (u-v)(t,x)$ solves $Lw = f(u)$ and has vanishing initial data. An application of Theorem 3.1 concludes the proof. □

Example 5.4. Consider Qi's operator L from (1.8). Then $l_* = 1$, $\beta = 1/2$, and $Q_0 = 2m$. Theorems 3.1, 5.1, and 5.3 state that the solutions u, v to (1.1), (1.2) satisfy

$$u,\ v \in H^{s,2m;\lambda}((0,T)\times\mathbb{R}), \qquad u - v \in H^{s+1/2,2m;\lambda}((0,T)\times\mathbb{R}),$$

provided that $u_0 \in H^{s+m}(\mathbb{R})$, $u_1 \in H^{s+m-1/2}(\mathbb{R})$. Proposition 2.8 then implies

$$u,\ v \in H^s_{\mathrm{loc}}((0,T)\times\mathbb{R}), \qquad u - v \in H^{s+1/2}_{\mathrm{loc}}((0,T)\times\mathbb{R}).$$

We find that the strongest singularities of u coincide with the singularities of v. The latter can be looked up in (1.9) in the case $u_1 \equiv 0$.

A. Appendix

A.1. End of Proof of Lemma 2.14

The proof that (2.8) is implied by (2.9) is divided into several lemmata.

For $s \geq 0$, $\delta \in \mathbb{R}$, we introduce the spaces

$$\begin{aligned} E_0(s,\delta,T) &= \mathcal{W}^s\big(\mathbb{R}^n; \big\{H^s(\mathbb{R}_+), \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big\}\big)\big|_{(0,T)\times\mathbb{R}^n}, \\ E_1(s,\delta,T) & \\ &= \mathcal{W}^s\big(\mathbb{R}^n; \big\{\mathcal{H}^{0,\delta l_*}(\mathbb{R}_+) \cap \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+), \{\kappa_\nu^{(\delta)}\}_{\nu>0}\big\}\big)\big|_{(0,T)\times\mathbb{R}^n}. \end{aligned}$$

Lemma A.1. *Let $s \geq 0$, $\delta \in \mathbb{R}$, and $0 < a_1 < a_0 < T$. Then, for some function u on $(0, T) \times \mathbb{R}^n$, we have $u \in H^{s,\delta;\lambda}((0, T) \times \mathbb{R}^n)$, if and only if,*

$$u(t, x) = u_0(t, x) + u_1(t, x),$$

where $u_i \in E_i(s, \delta, T)$, $i = 0, 1$, and

$$\operatorname{supp} \hat{u}_0 \subseteq \{(t, \xi)\colon t\langle\xi\rangle^\beta \leq a_0\}, \quad \operatorname{supp} \hat{u}_1 \subseteq \{(t, \xi)\colon t\langle\xi\rangle^\beta \geq a_1\}.$$

Proof. It suffices to set

$$\hat{u}_0(t, \xi) = \omega(t\langle\xi\rangle^\beta)\hat{u}(t, \xi), \quad \hat{u}_1(t, \xi) = (1 - \omega(t\langle\xi\rangle^\beta))\hat{u}(t, \xi),$$

where $\omega \in C^\infty([0, T], \mathbb{R})$ satisfies $\omega(t) = 1$ for $0 \leq t \leq a_1$ and $\omega(t) = 0$ for $a_0 \leq t \leq T$. □

Next we provide a characterization of the spaces $E_i(s, \delta, T)$, $i = 0, 1$:

Lemma A.2. *For $s \geq 0$, $\delta \in \mathbb{R}$, and $T > 0$,*

$$E_0(s, \delta, T) = H^{s+\beta\delta l_*}(\mathbb{R}^n, H^0(0, T)) \cap H^{\beta(s+\delta)l_*}(\mathbb{R}^n, H^s(0, T)).$$

Proof. This follows from a direct manipulation using properties like $H^s_0(\overline{\mathbb{R}}_+) = \mathcal{H}^{0,0}(\mathbb{R}_+) \cap \mathcal{H}^{s,s}(\mathbb{R}_+)$ in Example 2.1, and $\|\kappa_\nu v\|_{\mathcal{H}^{s,\gamma(l_*+1)+\delta l_*}(\mathbb{R}_+)} = \nu^\gamma \|v\|_{\mathcal{H}^{s,\gamma(l_*+1)+\delta l_*}(\mathbb{R}_+)}$ for $s \in \mathbb{R}$, $\nu > 0$. □

Lemma A.3. *For $s \geq 0$, $\delta \in \mathbb{R}$, and $T > 0$,*

$$\begin{aligned} &E_1(s, \delta, T) \\ &\quad = H^s\big(\mathbb{R}^n, \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\big|_{(0,T)}\big) \cap H^0\big(\mathbb{R}^n, \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)\big|_{(0,T)}\big). \end{aligned}$$

Proof. This has been shown in the proof of Lemma 2.5. □

Lemma A.4. *Let $s,\ s' \geq 0$, $\delta,\ \delta' \in \mathbb{R}$ satisfy* (2.9). *Then*

$$E_0(s, \delta, T) \subseteq E_0(s', \delta', T), \quad E_1(s, \delta, T) \subseteq E_1(s', \delta', T), \tag{A.1}$$

with continuous embeddings.

Proof. (a) For $E_0(s, \delta, T) \subseteq E_0(s', \delta', T)$, note that

$$\begin{aligned} &H^{s+\beta\delta l_*}(\mathbb{R}^n, H^0(0, T)) \cap H^{\beta(s+\delta)l_*}(\mathbb{R}^n, H^s(0, T)) \\ &\quad \subseteq H^{s-\beta s'+\beta\delta l_*}(\mathbb{R}^n, H^{s'}(0, T)) \subseteq H^{\beta(s'+\delta')l_*}(\mathbb{R}^n, H^{s'}(0, T)) \end{aligned}$$

by interpolation and $s - \beta s' + \beta\delta l_* \geq \beta(s' + \delta')l_*$.

(b) For $E_1(s, \delta, T) \subseteq E_1(s', \delta', T)$, note that

$$H^0\big(\mathbb{R}^n, \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)\big|_{(0,T)}\big) \subseteq H^0\big(\mathbb{R}^n, \mathcal{H}^{s',s'(l_*+1)+\delta' l_*}(\mathbb{R}_+)\big|_{(0,T)}\big)$$

because of $s + \beta\delta l_* \geq s' + \beta\delta' l_*$, and

$$\begin{aligned} &H^s\big(\mathbb{R}^n, \mathcal{H}^{0,\delta l_*}(\mathbb{R}_+)\big|_{(0,T)}\big) \cap H^0\big(\mathbb{R}^n, \mathcal{H}^{s,s(l_*+1)+\delta l_*}(\mathbb{R}_+)\big|_{(0,T)}\big) \\ &\quad \subseteq H^{s'}\big(\mathbb{R}^n, \mathcal{H}^{s-s',(s-s')(l_*+1)+\delta l_*}(\mathbb{R}_+)\big|_{(0,T)}\big) \\ &\quad \subseteq H^{s'}\big(\mathbb{R}^n, \mathcal{H}^{0,\delta' l_*}(\mathbb{R}_+)\big|_{(0,T)}\big) \end{aligned}$$

by interpolation and $(s - s')(l_* + 1) + \delta l_* \geq \delta' l_*$. Also note that, for the spaces $\mathcal{H}^{s,\delta}(\mathbb{R}_+)\big|_{(0,T)}$, interpolation jointly in s, δ is possible, as seen by (2.1) and the three lines theorem of complex interpolation theory. □

In view of Lemma A.1, (A.1) completes the proof of Lemma 2.14.

A.2. *Proof of Lemma 3.6*

We introduce the vectors

$$Y = {}^t\big(y, T\partial_t y, \ldots, T^{m-1}\partial_t^{m-1} y\big), \quad F = {}^t(0, \ldots, 0, T^{m-1} f),$$

and obtain the first-order system $\partial_t Y(t) = A(t)Y(t) + F(t)$, where $A = A(t)$ is some $m \times m$ matrix with

$$\begin{aligned} \|A(t)\| &\leq C_0\big(T^{-1} + \max_j |T^{m-1-j} a_j(t)|\big), \\ \big\|\partial_t^k A(t)\big\| &\leq C_k \max_j |T^{m-1-j}\partial_t^k a_j(t)|, \quad k \geq 1. \end{aligned}$$

If $X = X(t, t')$ denotes the fundamental matrix, $\partial_t X(t, t') = A(t)X(t, t')$, $X(t', t') = I$, then $Y(t) = X(t, 0)Y(0) + \int_0^t X(t, t')F(t')\, dt'$. It is well-known that $\big\|X(t, t')\big\| \leq \exp(\int_{t'}^t \big\|A(t'')\big\|\, dt'') \leq C$, where C does not depend on T, since $0 \leq t' \leq t \leq T \leq T_0$. We differentiate our first-order system r times and obtain $(\partial_t - A(t))(\partial_t^r Y) = F_r(t)$ with some F_r containing derivatives of Y up to the order $r - 1$. Then we find that

$$|\partial_t^r Y(t)| \leq C|(\partial_t^r Y)(0)| + C\sum_{l=0}^{r-1} \int_0^t |\partial_t^l Y(t')|\, dt' + C\int_0^t |\partial_t^r F(t')|\, dt'.$$

Multiplying with T^r and summing over $r = 0, \ldots, s-1$ gives

$$\begin{aligned} \sum_{r=0}^{s-1} T^r|\partial_t^r Y(t)| \leq{}& C\sum_{r=0}^{s-1} T^r|(\partial_t^r Y)(0)| \\ &+ C\sum_{r=0}^{s-1} T^r \int_0^t |\partial_t^r F(t')|\, dt' + C_{T_0}\sum_{r=0}^{s-2} T^{r+1}\int_0^t |\partial_t^r Y(t')|\, dt'. \end{aligned}$$

Then Gronwall's inequality leads to

$$\sum_{r=0}^{s-1} T^r|\partial_t^r Y(t)| \leq C\sum_{r=0}^{s-1} T^r \left(|(\partial_t^r Y)(0)| + \int_0^t |\partial_t^r F(t')|\, dt'\right).$$

Taking squares, applying the Cauchy–Schwarz inequality, and integrating over $(0, T)$ give

$$\sum_{r-0}^{s-1} T^{2r-1} \left\| \partial_t^r Y \right\|_{L^2(0,T)}^2$$
$$\leq C \sum_{r=0}^{s-1} T^{2r} |(\partial_t^r Y)(0)|^2 + CT^2 \sum_{r=0}^{s-1} T^{2r-1} \left\| \partial_t^r F \right\|_{L^2(0,T)}^2 .$$

Now it remains to estimate the values $|(\partial_t^r Y)(0)|$. Repeated application of (3.6) shows

$$|(\partial_t^{r+m} y)(0)| \leq C \sum_{l=0}^{m-1} |y_{0l}| + C \sum_{l=0}^{r} |(\partial_t^l f)(0)|, \quad r \geq 0.$$

From this, and $\|g\|_{L^\infty(0,T)}^2 \leq CT^{-1} \|g\|_{L^2(0,T)}^2 + CT \|\partial_t g\|_{L^2(0,T)}^2$, we deduce that

$$\sum_{r=0}^{s-1} T^{2r} |(\partial_t^r Y)(0)|^2 = \sum_{r=0}^{s-1} \sum_{l=0}^{m-1} T^{2(r+l)} |(\partial_t^{r+l} y)(0)|^2$$
$$\leq C_{T_0} \sum_{j=0}^{m-1} T^{2j} |y_{0j}|^2 + C_{T_0} \sum_{l=0}^{s-2} T^{2(l+m)} |(\partial_t^l f)(0)|^2$$
$$\leq C \sum_{j=0}^{m-1} T^{2j} |y_{0j}|^2 + C \sum_{l=0}^{s-1} T^{2(l+m)-1} \left\| \partial_t^l f \right\|_{L^2(0,T)}^2 .$$

This proves one part of (3.7); the other part can be proved similarly.

Acknowledgements. The work of this article was partly done during a stay by the authors at Freiberg University of Mining and Technology in January 2000. The authors would like to thank Freiberg University for its hospitality, and Prof. Reissig and Prof. Kajitani for useful discussions and comments.

References

1. Aleksandrian, G.: Parametrix and propagation of the wave front of a solution to a Cauchy problem for a model hyperbolic equation. Izv. Akad. Nauk Arm. SSR **19**, 219–232 (1984)
2. Amano, K., Nakamura, G.: Branching of singularities for degenerate hyperbolic operators. Publ. Res. Inst. Math. Sci. **20**, 225–275 (1984)
3. Colombini, F., Jannelli, E., Spagnolo, S.: Well–posedness in the Gevrey classes of the Cauchy problem for a non–strictly hyperbolic equation with coefficients depending on time. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10**, 291–312 (1983)
4. Colombini, F., Spagnolo, S.: An example of a weakly hyperbolic Cauchy problem not well posed in C^∞. Acta Math. **148**, 243–253 (1982)
5. Dreher, M.: Local solutions to quasilinear weakly hyperbolic differential equations. Ph.D. Thesis, Technische Universität Bergakademie Freiberg, Freiberg 1999

6. Dreher, M.: Weakly hyperbolic equations, Sobolev spaces of variable order, and propagation of singularities. To appear in Osaka J. Math.
7. Dreher, M., Reissig, M.: Propagation of mild singularities for semilinear weakly hyperbolic differential equations. J. Analyse Math. **82**, 233–266 (2000)
8. Egorov, Y., Schulze, B.-W.: Pseudo-Differential Operators, Singularities, Applications. Oper. Theory Adv. Appl. **93**, Basel: Birkhäuser 1997
9. Hörmander, L.: Linear Partial Differential Operators. Grundlehren Math. Wiss. **116**, Berlin: Springer 1969
10. Ivrii, V.Y., Petkov, V.M.: Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed. Russian Math. Surveys **29**, 1–70 (1974)
11. Kajitani, K.: Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes. Hokkaido Math. J. **12**, 434–460 (1983)
12. Kajitani, K., Yagdjian, K.: Quasilinear hyperbolic operators with the characteristics of variable multiplicity. Tsukuba J. Math. **22**, 49–85 (1998)
13. Levendorskii, S.: Degenerate Elliptic Equations. Math. Appl. (Soviet Ser.) **258**, Dordrecht: Kluwer Acad. Publ. 1993
14. Oleinik, O.A.: On the Cauchy problem for weakly hyperbolic equations. Comm. Pure Appl. Math. **23**, 569–586 (1970)
15. Qi, M.-Y.: On the Cauchy problem for a class of hyperbolic equations with initial data on the parabolic degenerating line. Acta Math. Sinica **8**, 521–529 (1958)
16. Rabinovich, V., Schulze, B.-W., Tarkhanov, N.: Boundary value problems in cuspidal wedges. Preprint 98/24, Institut für Mathematik, Potsdam, Universität Potsdam 1998
17. Reissig, M.: Weakly hyperbolic equations with time degeneracy in Sobolev spaces. Abstract Appl. Anal. **2**, 239–256 (1997)
18. Schulze, B.-W.: Boundary Value Problems and Singular Pseudo-Differential Operators. Wiley Ser. Pure Appl. Math., Chichester: J. Wiley 1998
19. Schulze, B.-W., Tarkhanov, N.: Ellipticity and parametrices on manifolds with cuspidal edges. Geometric aspects of partial differential equations (Roskilde, 1998), Contemp. Math. **242**, 217–255, Providence, RI: Amer. Math. Soc. 1999
20. Shinkai, K.: Stokes multipliers and a weakly hyperbolic operator. Comm. Partial Differential Equations **16**, 667–682 (1991)
21. Taniguchi, K., Tozaki, Y.: A hyperbolic equation with double characteristics which has a solution with branching singularities. Math. Japon. **25**, 279–300 (1980)
22. Yagdjian, K.: The Cauchy Problem for Hyperbolic Operators. Multiple Characteristics. Micro–Local Approach, Math. Topics **12**, Berlin: Akademie Verlag 1997