Annali di Matematica 180, 429–440 (2002)

DOI (Digital Object Identifier) 10.1007/s102310100020

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Existence and multiplicity of solutions for a nonlinear Neumann problem

Received: October 30, 1999
Published online: December 19, 2001 –

Abstract. We consider a Neumann problem of the type $-\varepsilon\Delta u + F'(u(x)) = 0$ in an open bounded subset Ω of $\mathbf{R}^n$, where F is a real function which has exactly k maximum points.

Using Morse theory we find that, for ε suitably small, there are at least $2k$ nontrivial solutions of the problem and we give some qualitative information about them.

Mathematics Subject Classification (2000). 35J65; 58E05

Key words. PDE – critical points – Morse theory

1. Introduction

Let us consider the problem

$$(P) \quad \begin{cases} u \in C^2(\overline{\Omega}) \\ -\varepsilon\Delta u + F'(u(x)) = 0 \text{ in } \Omega, \\ \partial u/\partial n = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is an open bounded domain with sufficiently regular boundary, $n \geq 2$ and $\varepsilon > 0$.

The aim of this paper is to study the existence and multiplicity of solutions to (P) under some structure assumptions on the real function $F \in C^2(\mathbf{R})$. In particular we are interested in the case in which F has k maximum points.

We observe that this kind of problem has been intensively studied recently. Here we just quote Modica, Mortola and Passaseo [12], [13], [15] who studied the problem in the case that F is an even function with only one maximum point $(k = 1)$.

One of the motivations for studying (P) is that it can be used as a mathematical model for some phase transition problems arising from mathematical physics (see the introduction of [12]).

Here we consider problem (P) under the following fairly general assumptions:

(i) $\lim_{t\to\pm\infty} F(t) = +\infty$;

(ii) $\exists\ a, b \geq 0$ such that $|F'(t)| \leq a|t|^{p-1} + b\ \forall t \in \mathbf{R}$, with $p \in\ ]2, 2^*[$ and $2^* = 2n/(n-2)$ if $n \geq 3$, while $p > 2$ if $n = 2$;

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(iii) F has exactly k maximum points and it does not have any critical points apart from its maximum or minimum points;
(iv) $F'' \neq 0$ in the critical points of F.

We now briefly comment on our technique. It is clear that solutions to (P) are critical points of the functional

$$J_\varepsilon(u) = \varepsilon/2 \int_\Omega |\nabla u|^2 \; + \int_\Omega F\,(u\,(x))\,dx$$

defined on the Sobolev space $H^1(\Omega)$, so it is natural to follow a variational approach.

Moreover (P) has $2k+1$ trivial solutions, corresponding to the functions u_a constantly equal to a in Ω, where a is one of the k maximum points or of the $k+1$ minimum points of F.

In order to estimate from below the total number of critical points, we will use Morse theory, exploiting algebraically the Morse relations. Here we find a technical complication due to the fact that the critical points of J_ε may be degenerate (see Definition 2.1). This difficulty can be overcome by using a generalized Morse theory recently developed by Benci and Giannoni [3] which works for a larger class of functionals and permits us to prove the following:

Theorem. *If ε is suitably small and it does not belong to a numerable set A, then there are at least $2k$ nontrivial critical points of J_ε, if counted with their own multiplicity.*

This result appears as Theorem 3.1 in Sect. 3, while the notion of multiplicity is explained in Definition 2.9 and Remark 2.11.

Moreover we obtain some qualitative information about the solutions to (P). Namely in Theorem 3.2 we prove that, as $\varepsilon \to 0$, all critical points u of J_ε having a fixed Morse index tend to "concentrate" outside a neighborhood of the maximum points of F.

2. Preliminary lemmas

In this section we recall some definitions and give the formulation of the problem. Finally we provide some useful estimates that will be used in Sect. 3.

Since F satisfies (ii), standard computations show that J_ε is of class C^1 on $H^1(\Omega)$. Moreover

$$dJ_\varepsilon(u)(v) = \varepsilon \int_\Omega (\nabla u/\nabla v) + \int_\Omega F'(u(x))v(x)dx$$

for each $u, v \in H^1(\Omega)$.

Now we recall some standard facts about classical Morse theory.

Definition 2.1. *A point $u \in H^1(\Omega)$ is said to be* **critical** *for J_ε if*

$$dJ_\varepsilon(u) = 0.$$

We will denote by K_{J_ε} the set of these points.

A real number c is a critical value of J_ε if

$$\{u \in K_{J_\varepsilon}/J_\varepsilon(u) = c\} \neq \emptyset.$$

Moreover, denoting by $H^{1'}(\Omega)$ the dual space of $H^1(\Omega)$, we say that J_ε verifies the **Palais–Smale condition** *(P.S. for short), if any sequence $(u_n)_{n\in\mathbf{N}} \subset H^1(\Omega)$ such that*

$$\lim_{n\to\infty} J_\varepsilon(u_n) = c \in \mathbf{R}$$

and

$$\lim_{n\to\infty} dJ_\varepsilon(u_n) = 0 \ \ in \ \ H^{1'}(\Omega)$$

has a subsequence which converges to some $u \in H^1(\Omega)$.

If $u \in K_{J_\varepsilon}$ and if $d^2J_\varepsilon(u)$ exists, the **Morse index** *of u is the maximal dimension of a subspace of $H^1(\Omega)$ on which $d^2J_\varepsilon(u)$ is negative definite. It is denoted by $m(u)$.*

The **nullity** *of u is the dimension of the kernel of $d^2J_\varepsilon(u)$ (i.e. the subspace consisting of all v such that $d^2J_\varepsilon(u)(v,w) = 0$, for all $w \in H^1(\Omega)$).*

The **large Morse index** *is the sum of the Morse index and the nullity, and it will be denoted by $m^*(u)$.*

A critical point u is called **nondegenerate** *if its nullity is 0, otherwise it is called* **degenerate**.

Some elementary results are now proved. The first one gives an a priori estimate on the values assumed by critical points of J_ε.

Lemma 2.2. *Let γ and δ respectively denote the smallest and the greatest of the $k+1$ minimum points of F. For each fixed $u \in K_{J_\varepsilon}$, $u(x) \in [\gamma, \delta]$ almost everywhere in Ω.*

Proof. Let $G \in C^1(\mathbf{R}, \mathbf{R})$ be a function such that $G(t) = 0$ for each $t \leq \delta$ and $0 < G'(t) \leq M$ for each $t > \delta$, where $M > 0$.

If $u \in K_{J_\varepsilon}$, then $v(x) = G(u(x)) \in H^1(\Omega)$ and we have that

$$\varepsilon \int_\Omega G'(u)|\nabla u|^2 + \int_\Omega F'(u(x))G(u(x))\,dx = dJ_\varepsilon(u)(v) = 0.$$

As the sum of these two positive quantities is 0, both of them have to vanish. So, observing that $F'(t)G(t) > 0$ when $t > \delta$, u must be less or equal to δ almost everywhere in Ω.

Analogously it can be showed also that $u \geq \gamma$ almost everywhere in Ω, so the assert is completely proved.

The next lemma contains a regularity result for solutions to (P).

Lemma 2.3. *Let $u \in H^1(\Omega)$ be a critical point of J_ε, then u is a classical solution to (P), i.e.,*

$$\begin{cases} u \in C^2(\overline{\Omega}) & \\ -\Delta u + 1/\varepsilon F'(u(x)) = 0 & in\ \Omega, \\ \partial u/\partial n = 0 & on\ \partial\Omega. \end{cases}$$

Proof. We start by recalling a standard regularity result holding under our assumptions (see [7, Theorem 2.4.2.7]).
If $r > 1$ and $f \in L^r(\Omega)$, there exists a unique $v \in W^{2,r}(\Omega)$ which is solution to

$$\begin{cases} -\Delta v + v = f & \text{in } \Omega, \\ \gamma(\partial v/\partial n) = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\gamma : W^{1,r}(\Omega) \longrightarrow W^{1-1/r,r}(\partial\Omega)$ is the trace operator.

We give the proof relative to the case $n \geq 3$, the case $n = 2$ being analogous and actually simpler.

From Lemma 2.2, and continuity of F', it follows that $f(x) = -1/\varepsilon F'(u(x)) + u(x) \in L^\infty(\Omega)$, so that, by the previous result and the fact that u is a critical point of J_ε, we get that $u \in W^{2,r}(\Omega)$ for every $r < +\infty$. Moreover, by Sobolev's embedding theorem, $u \in C^1(\overline{\Omega})$ (actually $u \in C^{1,\alpha}(\overline{\Omega})$ for every $\alpha < 1$) and we can conclude that $f \in C^1(\overline{\Omega})$. In particular $f \in L^2(\Omega)$, thus $u \in W^{2,2}(\Omega)$ and, taking $v \in C_0^\infty(\Omega) \subset H^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega (-\varepsilon\Delta u + F'(u(x)))v(x)dx &= \varepsilon \int_\Omega (\nabla u/\nabla v) + \int_\Omega F'(u(x))v(x)dx \\ &= dJ_\varepsilon(u)(v) = 0 \end{aligned}$$

so that $-\varepsilon\Delta u + F'(u(x)) = 0$ almost everywhere in Ω.

Now we know that

- F' is a C^1 function,
- $\gamma \leq u \leq \delta$, a.e. in Ω,
- $\Delta u = 1/\varepsilon F'(u(x))$, a.e. in Ω,
- $\partial\Omega$ is sufficiently regular.

So, using a regularity results like in [4, Theorem 1.4.27], we deduce that $u \in C^2(\overline{\Omega})$.

Remark 2.4. From previous lemmas it follows that, replacing F with F_0 such that $F_0 = F$ on $[\gamma, \delta]$, solutions to (P) do not change. So there is no loss of generality in supposing

(ii′) $\exists\, \overline{a}, \overline{b} \geq 0$ such that $|F''(t)| \leq \overline{a}|t|^{p-2} + \overline{b}\ \ \forall t \in \mathbf{R}$

where, as in assumption (ii), $p \in\,]2, 2^*[$ and $2^* = 2n/(n-2)$ if $n \geq 3$, while $p > 2$ if $n = 2$. It is evident that assumption (ii′) is stronger than (ii).
So J_ε becomes a C^2 functional and

$$d^2 J_\varepsilon(u)(v, w) = \varepsilon \int_\Omega (\nabla v/\nabla w) + \int_\Omega F''(u(x))v(x)w(x)\, dx, \tag{1}$$

for all $u, v, w \in H^1(\Omega)$.

In the following, $0 = \lambda_0 < \lambda_1 \leq \cdots \lambda_i \leq \cdots$ will denote the eigenvalues of $-\Delta$ on Ω with Neumann boundary conditions, while u_a will denote a function which is constantly equal to a in Ω.

The next lemma deals with the Morse index of trivial critical points of J_ε.

Lemma 2.5. 1. *If c is a minimum point of F, then $m(u_c) = m^*(u_c) = 0$.*

2. *If d is a maximum point of F and $\varepsilon \neq -F''(d)/\lambda_i$ for each $i \geq 1$, then u_d is nondegenerate.*
3. *If $\varepsilon \in \left]-F''(d)/\lambda_i,\ -F''(d)/\lambda_{i-1}\right[$, where $i \geq 2$, then $m(u_d) = m^*(u_d) = i$.*
   If $\varepsilon \in \left]-F''(d)/\lambda_1,\ +\infty\right[$, then $m(u_d) = m^*(u_d) = 1$.

Proof. 1. By assumption (iv) and the previous remark $d^2 J_\varepsilon(u_c)$ is positive definite, so that assertion follows from Definition 2.1.

2. Let $(e_i)_{i\in\mathbf{N}}$ be the orthonormal basis for $L^2(\Omega)$ such that e_i is the eigenfunction relative to λ_i :

$$\begin{cases} e_i \in H^1(\Omega) \cap C^\infty(\Omega) & \\ -\Delta e_i = \lambda_i e_i & \text{in } \Omega, \\ \partial e_i/\partial n = 0 & \text{on } \partial\Omega. \end{cases}$$

Now let u_d be degenerate, so that, by (1), there exists $v \neq 0 \in H^1(\Omega)$ such that

$$\varepsilon \int_\Omega (\nabla v/\nabla w)\ + F''(d) \int_\Omega v(x) w(x)\, dx = d^2 J_\varepsilon(u_d)(v, w) = 0 \quad \forall w \in H^1(\Omega).$$

This means that v solves the problem

$$\begin{cases} -\varepsilon\Delta v + F''(d) v = 0 \text{ in } \Omega, \\ \ \partial v/\partial n = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases}$$

In other words $-F''(d)/\varepsilon$ is an eigenvalue of $-\Delta$ on Ω with Neumann conditions, thus u_d is nondegenerate if and only if these eigenvalues are avoided, i.e. if ε is different from $-F''(d)/\lambda_i$ for each $i \geq 1$.

3. Now letting e_j be one of the previous eigenfunctions, we have

$$d^2 J_\varepsilon(u_d)(e_j, e_j) = \varepsilon \int_\Omega |\nabla e_j|^2 + F''(d) \int_\Omega e_j^2(x)\, dx = \varepsilon\lambda_j + F''(d).$$

For $j = 0$, we have that $\lambda_0 = 0$ and $d^2 J_\varepsilon(u_d)(e_0, e_0) = F''(d) < 0$, thus, surely $m(u_d) \geq 1$.

If $i \geq 2$, we observe that $\lambda_{i-1} > 0$ and if $\varepsilon < -F''(d)/\lambda_{i-1}$, then $\varepsilon\lambda_j + F''(d) < 0$ for every $j < i$. This fact, together with $d^2 J_\varepsilon(u_d)(e_s, e_t) = 0$ whenever $s \neq t$, implies that $d^2 J_\varepsilon(u_d)$ is negative definite on $\bigoplus_{0\leq j\leq i-1} \mathbf{R} e_j$, so that

$$m(u_d) \geq i. \tag{2}$$

In order to establish the desired equality, we observe that $d^2 J_\varepsilon(u_d)(e_j, e_j) > 0$, if $\underline{i \geq 1,\ \varepsilon} > -F''(d)/\lambda_i$ and $j \geq i$, so that $d^2 J_\varepsilon(u_d)$ is positive definite on $\overline{\bigoplus_{j\geq i} \mathbf{R} e_j}$. Consequently the assertion follows from (2), being $H^1(\Omega) = \overline{\bigoplus_{j\in N} \mathbf{R} e_j}$.

The aim of next lemmas is to establish some useful topological properties of J_ε.

Lemma 2.6. *J_ε is coercive.*

Proof. Let $(u_n)_{n\in\mathbf{N}}$ be a sequence of elements in $H^1(\Omega)$ such that $\|u_n\|_{H^1(\Omega)} \to \infty$. If $\|\nabla u_n\|_{L^2(\Omega)} \to \infty$, as F is bounded from below, then necessarily $J_\varepsilon(u_n) \to \infty$. In the other case, up to subsequences, $\|u_n\|_{L^2(\Omega)} \to \infty$. Now, reasoning as in Remark 2.4, there is no loss of generality if we replace F with another function which is equal to F on $[\gamma, \delta]$. In particular, fixed $\widetilde{a} > \max\{|\gamma|, |\delta|\}$, we can suppose that

$$\exists \widetilde{b} > 0 \text{ s.t.} \quad F(t) \geq \widetilde{b}|t|^2 \ \ \forall t \notin [-\widetilde{a}, \widetilde{a}].$$

Denoting by Ω_n the set $\{x \in \Omega / |u_n(x)| > \widetilde{a}\}$, from

$$\int_{\Omega\setminus\Omega_n} |u_n(x)|^2 dx \leq \widetilde{a}^2 |\Omega|$$

it follows that

$$\int_{\Omega_n} |u_n(x)|^2 dx \to \infty,$$

and this, together with

$$J_\varepsilon(u_n) \geq \int_{\Omega_n} F(u_n(x)) dx - k \geq \widetilde{b} \int_{\Omega_n} |u_n(x)|^2 dx - k,$$

$k \geq 0$ being a suitable constant, gives J_ε is coercive.

Lemma 2.7. *J_ε verifies P.S. and*

$$J_\varepsilon(u) = \varepsilon/2 < u, u >_{H^1(\Omega)} + \Psi(u) \quad \forall u \in H^1(\Omega),$$

where Ψ' is completely continuous.

Proof. Let $(u_n) \subset H^1(\Omega)$ be a sequence such that $J_\varepsilon(u_n) \to c$ and $dJ_\varepsilon(u_n) \to 0$. We need to find a subsequence of (u_n) which converges in $H^1(\Omega)$.

As J_ε is coercive, (u_n) is bounded, so that a subsequence (u_{k_n}) which weakly converges to an element $\overline{u} \in H^1(\Omega)$ exists.

Now consider the function $h(t) = F(t) - \varepsilon/2\, t^2$, we know by assumption (ii) that

$$\exists\, a_1, b_1 \geq 0 \quad \text{s.t.} \qquad |h'(t)| \leq a_1 |t|^{p-1} + b_1, \tag{3}$$

where $2 < p < 2^* = 2n/(n-2)$ if $n \geq 3$, while $2 < p$ if $n = 2$.

The functional $\Psi = u \in H^1(\Omega) \mapsto \int_\Omega h(u(x))\, dx \in \mathbf{R}$ is differentiable, Ψ' being the Nemytskii operator relative to h', so that

$$\Psi'(u)(v) = \int_\Omega h'(u(x)) v(x)\, dx \qquad \forall u, v \in H^1(\Omega).$$

Thus J_ε and its differential can be written in the following way:

$$J_\varepsilon(u) = \varepsilon/2 < u, u >_{H^1(\Omega)} + \Psi(u)$$

$$dJ_\varepsilon = \varepsilon\mathcal{L} + \Psi',$$

where $\mathcal{L} : H^1(\Omega) \longrightarrow H^{1'}(\Omega)$ is the Riesz isomorphism, hence

$$\lim_{n\to\infty}(\varepsilon\mathcal{L}(u_n) + \Psi'(u_n)) = \lim_{n\to\infty} dJ_\varepsilon(u_n) = 0.$$

By (3), $\Psi' : H^1(\Omega) \longrightarrow H^{1'}(\Omega)$ is completely continuous, so that $\Psi'(u_{k_n})$ strongly converges to $\Psi'(\overline{u})$. Thus $\mathcal{L}(u_{k_n})$ converges in $H^{1'}(\Omega)$ and, $\mathcal{L}$ being an isomorphism, also (u_{k_n}) strongly converges in $H^1(\Omega)$.

Lemma 2.8. *K_{J_ε} is a compact set.*

Proof. Let us show first that K_{J_ε} is a bounded set of $H^1(\Omega)$.

If u is a critical point of J_ε, we have

$$\varepsilon\int_\Omega |\nabla u|^2 + \int_\Omega F'(u(x))u(x)\,dx = dJ_\varepsilon(u)(u) = 0.$$

The second term of the sum is uniformly bounded with respect to u by Lemma 2.2, consequently the first one too. Still using Lemma 2.2, we have that also $\int_\Omega |u(x)|^2 dx$ is uniformly bounded and so

$$\exists\, M > 0 \text{ s.t. } ||u||_{H^1(\Omega)} \leq M \;\forall\, u \in K_{J_\varepsilon}.$$

Now since J_ε verifies P.S., this shows that K_{J_ε} is compact.

Now our aim is to show a multiplicity result for the solutions to (P) via Morse theory. More precisely we want to prove that, choosing ε suitably small, there are $2k$ nontrivial solutions to (P). We shall do it by computing the Morse index of the trivial solutions and proving the existence of critical points of J_ε which have a different Morse index.

As J_ε could have degenerate critical points, classical Morse theory cannot be applied to it. However we can use a generalized Morse theory due to Benci and Giannoni (see [3]) which works with a larger class of functionals. In our case we will denote by $\mathcal{F}(H^1(\Omega))$ this class of functionals on $H^1(\Omega)$.

We note that in [3] (see Example 5.2) it is established that, denoting by $(\ ,\ )_V$ the inner product of a Hilbert space V, if $\Psi \in C^1(V)$ is a function whose gradient is completely continuous and if $f(x) = (x, x)_V + \Psi(x)$ satisfies P.S. and is bounded, then $f \in \mathcal{F}(V)$.

The result still holds if we replace the hypothesis that f is bounded with the one that K_f is bounded. Hence our functional J_ε belongs to $\mathcal{F}(H^1(\Omega))$, by Lemmas 2.7 and 2.8.

Moreover, as J_ε is bounded from below, from [3, Theorem 5.9] we have the Morse equality

$$i_\lambda(K_{J_\varepsilon}) = P_\lambda(H^1(\Omega)) + (1+\lambda)Q_\lambda, \tag{4}$$

where

- $i_\lambda(K_{J_\varepsilon})$ is the Morse index of K_{J_ε}, i.e. a formal series in one variable λ with coefficients in $\mathbf{N} \cup \{+\infty\}$ that in this generalized theory took the place of the Morse polynomial in the classical theory;
- $P_\lambda(H^1(\Omega))$ is the Poincare polynomial of $H^1(\Omega)$ (with $\mathbf{Z}_2$ as field of coefficients), so that, $H^1(\Omega)$ being contractible,

$$P_\lambda(H^1(\Omega)) = 1; \tag{5}$$

- Q_λ is a formal series in λ with coefficients in $\mathbf{N} \cup \{+\infty\}$.

From the general theory we have the following definition and proposition:

Definition 2.9. *If K is an isolated subset of K_{J_ε} (i.e. if there exists an open set ω such that $K_{J_\varepsilon} \cap \omega = K$), then it makes sense to consider the Morse Index $i_\lambda(K)$ of K and the number $i_1(K)$ is called the* **multiplicity** *of K.*

Proposition 2.10. *If u is a nondegenerate critical point of J_ε, then $\{u\}$ is isolated and $i_\lambda(\{u\}) = \lambda^{m(u)}$.*

Moreover if $K_1, K_2 \subset K_{J_\varepsilon}$ are disjointed isolated and compact sets, then $i_\lambda(K_1 \cup K_2) = i_\lambda(K_1) + i_\lambda(K_2)$.

A proof of the previous proposition is given in [3, Theorem 5.8].

Remark 2.11. Let us remark that if K is a critical set made only by nondegenerate critical points and $i_\lambda(K) = \sum_{h\in\mathbf{N}} a_h\lambda^h$, then each a_h is exactly the number of elements of K whose Morse index is h. Instead, in the general case we are considering, the multiplicity of an eventually degenerate critical point (or critical set) computes the number of nondegenerate critical points in which the point (or set) is "solved" by the approximation method that permits it to pass from the Morse polynomial to the Morse index. In particular if K is an isolated critical set and $i_\lambda(K) = \sum_{h\in\mathbf{N}} a_h\lambda^h$, then each a_h computes the number of nondegenerate critical points of Morse index h corresponding to K and it is called the number of critical points of Morse index h of K in the sense of multiplicity.

Moreover Theorem 5.10 of [3] says that in our hypothesis:

$$a_h \neq 0 \;\Rightarrow\; \exists u \in K \text{ s.t. } m(u) \leq h \leq m^*(u). \tag{6}$$

3. Results

Let $c_1, \ldots, c_{k+1}$ be the minimum points and $d_1, \ldots, d_k$ be the maximum points of F, ordered in such a way that $-F''(d_1) \leq \cdots - F''(d_k)$.

From now on we will assume that

$$\varepsilon \neq -F''(d_j)/\lambda_i \quad \forall j = 1, \ldots k \quad \forall i \geq 1. \tag{7}$$

We recall the statement of the theorem already announced in the introduction.

Theorem 3.1. *If ε is suitably small and it does not belong to a numerable set A, then there are at least $2k$ nontrivial critical points of J_ε (if counted with their own multiplicity).*

Proof. Let K denote the set of the nontrivial critical points of J_ε, A be the (numerable) set made by $\varepsilon > 0$ which do not verify assumption (7) and let $\varepsilon \in]0, +\infty[/A$. By Lemma 2.5, each trivial critical point of J_ε is nondegenerate, hence isolated. So K_{J_ε} can be decomposed in the following isolated critical sets

$$K_{J_\varepsilon} = K \cup \{u_{c_1}\} \cup \ \dots \ \{u_{c_{k+1}}\} \cup \{u_{d_1}\} \cup \ \dots \ \{u_{d_k}\}$$

and, by Proposition 2.10, Morse equality (4) together with (5) gives

$$i_\lambda(K) + \lambda^{m(u_{c_1})} + \dots \lambda^{m(u_{c_{k+1}})} + \lambda^{m(u_{d_1})} + \dots \lambda^{m(u_{d_k})} = 1 + (1+\lambda)Q_\lambda. \quad (8)$$

Lemma 2.5 assures that $m(u_{c_1}) = \dots \ m(u_{c_{k+1}}) = 0$. So writing $i_\lambda(K) = \sum_{h\in\mathbf{N}} a_h\lambda^h$ and $Q_\lambda = \sum_{h\in\mathbf{N}} b_h\lambda^h$, equation (8) becomes

$$\sum_{h\in\mathbf{N}} a_h\lambda^h + k + 1 + \lambda^{m(u_{d_1})} + \dots \lambda^{m(u_{d_k})} = 1 + (1+\lambda)\sum_{h\in\mathbf{N}} b_h\lambda^h. \quad (9)$$

If we assume $\varepsilon < -F''(d_1)/\lambda_1$, this yields $m(u_{d_j}) \geq 2$ for each $j = 1, \ \dots, k$.

By equaling the coefficients of the same degree in (9) we get

$$a_1 = b_0 + b_1 \geq b_0 = k + a_0 \geq k, \quad (10)$$

so there are at least k critical points (if counted with their multiplicity) whose Morse index is 1.

In order to conclude the proof, we are going to show that for each $j = 1, \ \dots, k$ there is a nontrivial critical point of index $m(u_{d_j}) + 1$ or $m(u_{d_j}) - 1$, and they are all different from each other.

Firstly we show it in the case in which $F''(d_1), \ \dots, F''(d_k)$ are different from each other. From [1, Theorem 14.6] we know that the number of eigenvalues of $-\nabla$ on Ω with Neumann boundary conditions which are less or equal to $\lambda \in \mathbf{R}$ is asymptotically equal to $C\lambda^{n/2}$, where C is a constant depending only from n and the measure of Ω.

Consequently, choosing ε sufficiently small, we have that for each $j = 1, \ \dots, k-1$ the interval $]-F''(d_j)/\varepsilon, \ -F''(d_{j+1})/\varepsilon[$ contains at least 2 of these eigenvalues $(\lambda_i)_{i\in\mathbf{N}}$.

As we have seen in Lemma 2.5, $m(u_{d_j})$ is the minimal number i such that

$$\varepsilon\lambda_i + F''(d_j) > 0 \quad \Leftrightarrow \quad \lambda_i > -F''(d_j)/\varepsilon,$$

thus

$$m(u_{d_{j+1}}) \geq m(u_{d_j}) + 2 \quad \forall j = 1, \ \dots, k-1. \quad (11)$$

For $j = 1$ we know from (9) that $b_{m(u_{d_1})} \neq 0$ or $b_{m(u_{d_1})-1} \neq 0$, thus $a_{m(u_{d_1})+1} \neq 0$ or $a_{m(u_{d_1})-1} \neq 0$, respectively. This means that there is a critical point of index $m(u_{d_1}) + 1$ or $m(u_{d_1}) - 1$. Moreover it is surely different from the u_{c_i} and from the k nontrivial critical points already counted.

Now, assuming the assertion is true for j, let us show it is also true for $j+1$.

In fact $\lambda^{m(u_{d_{j+1}})} \neq 0$ yields $b_{m(u_{d_{j+1}})} \neq 0$ or $b_{m(u_{d_{j+1}})-1} \neq 0$, hence there exists a critical point of J_ε whose Morse index is $m(u_{d_{j+1}})+1$ or $m(u_{d_{j+1}})-1$ respectively and by (11) it is neither a trivial critical point nor a nontrivial critical point already counted.

In the other case, if $-F''(d_j)$ are not all different each other, then we may have, for example, that

$$\exists\, j, l \quad \text{s.t.} \quad -F''(d_{j-1}) < -F''(d_j) = \cdots = -F''(d_{j+l}) < -F''(d_{j+l+1}).$$

Hence, reasoning as in the previous case, $b_{m(u_{d_j})} + b_{m(u_{d_j})-1} \geq l+1$ and there are at least $l+1$ critical points (if counted with their multiplicity) whose Morse index belongs to $\{m(u_{d_j}) - 1, m(u_{d_j}) + 1\}$.

The next theorem yields qualitative information about critical points of J_ε.

Let $\alpha \in]0, -F''(d_1)[$, $l > 0$, Q_l be an open n-dimensional hypercube of side l.

Moreover, let $0 < a_1(l) < a_2(l) \leq a_3(l) \leq \cdots$ be the eigenvalues of $-\Delta$ on Q_l with Dirichlet boundary conditions, namely,
$a_1(l) = n\pi^2/l^2$
$a_2(l) = a_3(l) = \ldots = a_{n+1}(l) = (n+3)\pi^2/l^2$
$a_{n+2}(l) = a_{n+3}(l) = \ldots = a_{n+1+n(n-1)/2}(l) = (n+6)\,\pi^2/l^2$
$\ldots$.
For each $u \in K_{J_\varepsilon}$, let $\Gamma_\alpha(u)$ be the following set:

$$\Gamma_\alpha(u) = \left\{x \in \Omega \;/\; F''(u(x)) < -\alpha\right\},$$

whose direct image under u is a neighborhood of the maximum points of F. Finally let $N(u,l)$ be the greatest number of disjointed open hypercubes of side l which can be contained in $\Gamma_\alpha(u)$.

Theorem 3.2. *Following the previous notations, if $\varepsilon < \alpha/a_j(l)$, then $N(u,l) \leq m(u)/j$.*

Proof. Let Q_l be an hypercube of side $l > 0$ contained in $\Gamma_\alpha(u)$. For all $i \in \mathbf{N}$ we denote by v_i the eigenfunction of $-\Delta$ corresponding to the eigenvalue $a_i(l)$, v_i being 0 out of Q_l.

We shall prove that if $\varepsilon < \alpha/a_j(l)$, then $d^2J_\varepsilon(u)$ is negative definite on v_i for all $i \leq j$. Indeed $F''(u(x)) \leq -\alpha$ for each $x \in Q_l \subset \Gamma_\alpha(u)$, thus

$$d^2J_\varepsilon(u)(v_i, v_i) = \varepsilon \int_{Q_l} |\nabla v_i|^2 + \int_{Q_l} F''(u(x))\, v_i^2(x)\, dx \leq$$

$$\leq \varepsilon a_i(l) - \alpha \leq \varepsilon a_j(l) - \alpha < 0 \qquad \forall i \leq j.$$

If we set $k = N(u,l)$, there are k disjointed hypercubes $Q_l^1, \ldots, Q_l^k$ contained in $\Gamma_\alpha(u)$. For each $h = 1, \ldots, k$ let $v_1^h, \ldots, v_j^h$ be the j eigenfunctions of $-\Delta$ on Q_l^h relative to $a_1(l), \ldots, a_j(l)$, respectively.

The functions $v_1^1, v_2^1, \ldots, v_j^1, v_1^2, \ldots, v_j^2, \ldots, v_j^k$ are linearly independent. Indeed, for each $h = 1, \ldots, k$, $v_1^h, \ldots, v_j^h$ are linearly independent both in

$L^2\left(Q_l^h\right)$ and in $H_0^1\left(Q_l^h\right)$, hence in $H^1(\Omega)$ too. On the other side, if $h_1 \neq h_2 \in \{1, \ldots, k\}$, then $v_{i_1}^{h_1}$ and $v_{i_2}^{h_2}$ are linearly independent as they have disjointed supports, so that $B = \bigoplus_{\substack{1 \le h \le k \\ 1 \le i \le j}} \mathbf{R} v_i^h$ has dimension $jk = jN(u, l)$.

In order to show that $d^2 J_\varepsilon(u)$ is negative definite on B, we remark that each $w \in B$ admits the following representation:

$$w = \sum_{h=1}^{k} \sum_{i=1}^{j} c_i^h v_i^h,$$

and, observing that if $h_1 \neq h_2$, then the supports of $\sum c_i^{h_1} v_i^{h_1}$ and $\sum c_i^{h_2} v_i^{h_2}$ are disjointed, it is clear that

$$d^2 J_\varepsilon(u) \left(\sum_{i=1}^{j} c_i^{h_1} v_i^{h_1}, \sum_{i=1}^{j} c_i^{h_2} v_i^{h_2} \right) = 0,$$

so that

$$d^2 J_\varepsilon(u)(w, w) = \sum_{h=1}^{k} d^2 J_\varepsilon(u) \left(\sum_{i=1}^{j} c_i^h v_i^h, \sum_{i=1}^{j} c_i^h v_i^h \right). \tag{12}$$

Moreover if $i_1 \neq i_2$, then $\int_\Omega (\nabla v_{i_1}^h / \nabla v_{i_2}^h) = a_{i_1} \int_{Q_l^h} v_{i_1}^h(x) v_{i_2}^h(x)\, dx = 0$. So, denoting by $w^h = \sum_{i=1}^{j} c_i^h v_i^h$, for each $h = 1, \ldots, k$, we have

$$\begin{aligned} d^2 J_\varepsilon(u)(w^h, w^h) &= \varepsilon \int_\Omega \left(\nabla w^h / \nabla w^h \right) + \int_\Omega F''(u(x))(w^h(x))^2\, dx \le \\ \le \sum_{i=1}^{j} \left(\varepsilon (c_i^h)^2 \int_{Q_l^h} |\nabla v_i^h|^2 \right) &- \alpha \int_{Q_l^h} (w^h(x))^2\, dx = \sum_{i=1}^{j} (c_i^h)^2 (\varepsilon a_i(l) - \alpha) < 0. \end{aligned} \tag{13}$$

As the sum of k negative quantities is negative, by (12) and (13), $d^2 J_\varepsilon(u)$ is negative definite on B .

Thus, according to the definition of Morse index, $m(u) \ge \dim B = jN(u, l)$, so finally

$$N(u, l) \le m(u)/j.$$

Corollary 3.3. *If $\varepsilon < -F''(d_1)/\lambda_1$, then there exists $u_{1,\varepsilon} \in K_{J_\varepsilon}$ such that*

$$m(u_{1,\varepsilon}) \le 1 \le m^*(u_{1,\varepsilon}).$$

Moreover there are no hypercubes of side $l > \pi\sqrt{(n+3)\varepsilon/\alpha}$ contained in $\Gamma_\alpha(u_{1,\varepsilon})$.

Proof. The existence of $u_{1,\varepsilon}$ is assured by (6) and (10). Moreover $l > \pi\sqrt{(n+3)\varepsilon/\alpha}$ means that $\varepsilon < \alpha/a_2(l)$. Thus, by the previous theorem,

$$N(u_{1,\varepsilon}, l) \leq m(u_{1,\varepsilon})/2 \leq 1/2,$$

hence $N(u_{1,\varepsilon}, l) = 0$.

Acknowledgements. I wish to express my thanks to Professor Vieri Benci for suggesting the problem and for many stimulating conversations on the subject.

References

1. Agmon, S.: Elliptic boundary value problems. Princeton, New Jersey: Van nostrand Company 1965
2. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
3. Benci, V., Giannoni, F.: Morse Theory for C^1 functionals and Conley Blocks. Topological Methods in Nonlinear Analysis **4**, 365–398 (1994)
4. Berger, M.: Nonlinearity and Functional Analysis. London: Academic Press 1977
5. Brezis, H.: Analisi Funzionale. Liguori 1986
6. Dold, A.: Lectures on Algebraic Topology. Berlin-Heidelberg-New York: Springer 1972
7. Grisvard, P.: Elliptic problems in nonsmooth domains. Marshfield: Pitman Publishing 1985
8. Greenberg, M.J., Harper, J.R.: Algebraic Topology – a first course. Reedwood City: Addison Wesley 1981
9. Klee, V.: Some topological properties of convex sets. Trans. Amer. Math. Soc. **78**, 30–45 (1955)
10. Marino, A., Prodi, G.: Metodi perturbativi nella teoria di Morse. Boll. UMI (4)-**11** Suppl. Fasc. 3, 1–32 (1975)
11. Milnor, J.: Morse Theory. Ann. Math. Stud. **51** Princeton University Press (1963)
12. Modica, L.: The Gradient Theory of Phase Transitions and the Minimal Interface Criterion. Arch. Rational Mech. Anal. **98**, 123–142 (1986)
13. Modica, L., Mortola, S.: The Γ-convergence of some functionals. Istit. Matem., "L. Tonelli"-Univ. Pisa, preprint no. 77-7 (1977)
14. Palais, R.: Morse Theory on Hilbert Manifolds. Topology **2**, 299–340 (1963)
15. Passaseo, D.: Multiplicity of critical points for some functionals related to the minimal surfaces problem. Calc. Var. and PDE **6**(2), 105–121 (1998)
16. Rabinowitz, P.H.: Minimax methods in critical point theory with applications to Differential Equations. CBSM, Reg. Conf. Ser. Math. **65**, Amer. Math. Soc. (1984)
17. Vannella, G.: Some qualitative properties of the solutions of an elliptic equation via Morse Theory. Topological Methods in Nonlinear Analysis **9**, 297–312 (1997)