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Sasaki versus Kähler groups

D. Kotschick[1] · G. Placini[2]

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Abstract
We study fundamental groups of compact Sasaki manifolds and show that compared to Kähler groups, they exhibit rather different behaviour. This class of groups is not closed under taking direct products, and there is often an upper bound on the dimension of a Sasaki manifold realising a given group. The richest class of Sasaki groups arises in dimension 5.

Keywords Sasaki geometry · Sasaki group · Kähler group

Mathematics Subject Classification Primary 53C25; Secondary 57K50 · 57R17

1 Introduction

Sasaki geometry is often considered as the odd-dimensional analogue of Kähler geometry. In this paper we examine this analogy with regard to the fundamental group. It is well known now that Kähler groups, that is the fundamental groups of compact Kähler manifolds, form a very special subclass of of the class of all finitely presentable groups; cf. [2]. In their foundational monograph [9] on Sasaki geometry, Boyer and Galicki suggested that one study Sasaki groups, that is, fundamental groups of compact Sasaki manifolds, systematically. They noted that groups with virtually odd first Betti number cannot be Sasaki, and then suggested that perhaps the analogy with Kähler groups might stop there, cf. [9, p. 236]. Nevertheless, in the papers written on Sasaki groups since then, the point of view taken was usually that of pushing the analogy further, by proving that certain constraints known for Kähler groups also apply to Sasaki groups; see Chen [14], Biswas et al. [5, 6] and Kasuya [28–30].

In this paper we will also prove some results which extend the analogy between the Kähler and Sasaki cases. However, our main point is that Sasaki groups actually behave rather differently from Kähler groups. The reason that many results about Kähler groups extend to Sasaki groups is that within certain restricted classes of groups all Sasaki groups

✉ G. Placini
giovanni.placini@unica.it

D. Kotschick
dieter@math.lmu.de

1 Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 Munich, Germany

2 Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

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$$\begin{array}{ccccccccccc}
\mathcal{K}_1 & \subsetneq & \mathcal{K}_2 & = & \mathcal{K}_3 & \subset & \mathcal{K}_4 & \subset & \mathcal{K}_5 & \subset & \dots \\
\| & & \| & & \| & & \cup & & \cup & & \\
\mathcal{P}_1 & \subsetneq & \mathcal{P} & = & \mathcal{P} & = & \mathcal{P} & = & \mathcal{P} & = & \dots
\end{array}$$

Fig. 1 Kähler and projective groups

are in fact Kähler, and, therefore, within those classes of groups one does not encounter the important differences between the Kähler and Sasaki situations.

1.1 Kähler and projective groups

Let us denote by $\mathcal{K}_n$ the class of fundamental groups of closed Kähler manifolds of complex dimension n, and by $\mathcal{P}_n \subset \mathcal{K}_n$ the subclass of groups realised by smooth complex projective algebraic varieties. By taking products with $\mathbb{C}P^1$ one sees that there are natural inclusions $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. The latter inclusion is in fact an equality for $n \geq 2$ because of the Lefschetz hyperplane theorem. We therefore denote $\mathcal{P}_{n\geq 2}$ by $\mathcal{P}$. It is an open question whether the inclusion $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ is ever strict for $n > 1$.

For $n = 1$ we have $\mathcal{K}_1 = \mathcal{P}_1$, since every compact Riemann surface is an algebraic curve. These fundamental groups are just the orientable surface groups. A classical result of Kodaira says that $\mathcal{K}_2 = \mathcal{P}$, and a recent result of Claudon et al. [15] gives $\mathcal{K}_3 = \mathcal{P}$. Therefore, $\mathcal{K}_2 = \mathcal{K}_3$, a statement for which no direct proof is known, other than arguing that both sides equal $\mathcal{P}$.

We can summarise all this information in the diagram of inclusions depicted in Fig. 1.

The class of Kähler groups is the union of all the $\mathcal{K}_n$. There is a strict horizontal inclusion $\mathcal{K}_n \subsetneq \mathcal{K}_{n+1}$ for some $n \geq 3$ if and only if one[1] of the vertical inclusions is strict, meaning that there would be more Kähler groups than projective ones. If this were the case, then the smallest dimension in which a Kähler group could arise as the fundamental group of a closed Kähler manifold would be an interesting invariant of (non-projective) Kähler groups. Claudon, Höring and Lin[2] have conjectured that all Kähler groups are projective, cf. [15, Conjecture 1.1], which would mean that there are no vertical strict inclusions in this diagram.

Finally note that the classes of Kähler or projective groups are both closed under taking subgroups of finite index, and that they are also closed under direct products.

1.2 Sasaki and projective groups

Let us denote by $\mathcal{S}_n$ the class of fundamental groups of closed Sasaki manifolds of dimension n. Taking coverings shows that this is closed under passing to subgroups of finite index. However, there is no obvious way to realise direct products, since the Cartesian product of two Sasaki manifolds is of even dimension, and therefore certainly not Sasaki. This problem not only obstructs the realisation of direct products of groups, it also means that there is no straightforward way to get an inclusion of $\mathcal{S}_n$ into $\mathcal{S}_m$ for a larger m, be it $n + 2$ or some larger value.

It is known from work of Kasuya [29] that these issues cannot be resolved when $n = 3$. The Sasaki 3-manifolds with infinite fundamental groups are essentially the circle bundles

[1] And, therefore, infinitely many.

[2] And perhaps others as well.

with non-zero Euler classes over surfaces of positive genus, cf. Geiges [20]. They are not 1-formal, and, in fact, have non-trivial Massey triple products defined on H^1. Now Kasuya [29, Theorem 1.1] proved that Sasaki manifolds of dimension ≥ 5 are 1-formal, equivalently the Malcev algebra of a Sasaki group is quadratically presented. Therefore no infinite group in $\mathcal{S}_3$ will occur in $\mathcal{S}_n$ for any $n \geq 5$. The same conclusion holds for any direct product of groups that has an infinite factor from $\mathcal{S}_3$.

Kasuya's result suggests that groups in $\mathcal{S}_3$ are completely unlike high-dimensional Sasaki groups, and one might hope that once one discards the 3-dimensional case, things will be more or less uniform, as expected by analogy with the Kähler case. Of course, Kasuya's result about 1-formality further strengthens the analogy between Kähler groups and high-dimensional Sasaki groups.

Our first result is that in all odd dimensions there are Sasaki groups which do not occur in any larger dimension, so that the 3-dimensional case is actually not special in this regard.

Theorem 1 *Every $\mathcal{S}_n$ contains groups which are not contained in any $\mathcal{S}_m$ with $m > n$.*

This means that the problem caused by the absence of products in the Sasaki category cannot be resolved, and there is no inclusion between the $\mathcal{S}_n$ going up in dimension. However, we can adapt the Lefschetz hyperplane theorem to obtain an inclusion going down in dimension:

Theorem 2 *There is an inclusion $\mathcal{S}_n \subset \mathcal{S}_{n-2}$ for all $n \geq 7$.*

By the previous theorem, all these inclusions are strict. The combination of the two theorems shows that $\mathcal{S}_5$ is the union of all $\mathcal{S}_{n\geq 5}$, and that it is strictly larger than any union taken by leaving out $\mathcal{S}_5$. It is somewhat counterintuitive that the largest and most interesting class of fundamental groups occurs in a rather small dimension, in fact the smallest dimension beyond the exceptional $n = 3$.

We also show that projective groups are Sasaki in all dimensions > 3.

Theorem 3 *The class $\mathcal{S}_n$ of Sasaki groups of dimension n contains all projective groups for every $n \geq 5$.*

Without control on the dimension, this was proved previously by Chen [14, Proposition 1.2.]. His argument does not yield the most interesting case of $\mathcal{S}_5$. Note that by Theorem 1 all the inclusions $\mathcal{P} \subset \mathcal{S}_n$ in Theorem 3 are strict. For Sasaki groups the analog of the diagram in Fig. 1 is the diagram in Fig. 2, summarizing our discussion so far, and leaving $\mathcal{S}_3$ out in left field in splendid isolation.

In light of this diagram it seems interesting to investigate the intersection of all the $\mathcal{S}_n$ for $n \geq 5$. This intersection contains all the projective groups, and one might be tempted to conjecture that it equals $\mathcal{P}$. However, Theorem 3 is true not only for the fundamental groups of smooth complex projective algebraic varieties, but also for the potentially larger class of orbifold fundamental groups of cyclic polarised projective orbifolds in the sense of Ross and Thomas [40, Definition 2.7.], see Remark 13 below. Therefore, at least these groups are contained in all $\mathcal{S}_n$ for $n \geq 5$.

$$
\begin{array}{cccccccccc}
\mathcal{S}_3 & \mathcal{S}_5 & \supsetneq & \mathcal{S}_7 & \supsetneq & \mathcal{S}_9 & \supsetneq & \mathcal{S}_{11} & \supsetneq & \dots \\
 & \cup\!\!\!\!\neq & & \cup\!\!\!\!\neq & & \cup\!\!\!\!\neq & & \cup\!\!\!\!\neq & & \\
 & \mathcal{P} & = & \mathcal{P} & = & \mathcal{P} & = & \mathcal{P} & = & \dots
\end{array}
$$

Fig. 2 Sasaki and projective groups

1.3 Extending restrictions on Kähler groups to Sasaki groups

By Theorem 3 all projective groups are Sasaki, and, therefore, if all Kähler groups are projective, then all Kähler groups are Sasaki. However, there are many Sasaki groups which are not Kähler, for example all the ones constructed in the proof of Theorem 1 have this property, see Sect. 4. Nevertheless, within certain restricted classes of groups it turns out that all Sasaki groups are in fact projective. In this direction we will prove the following:

Theorem 4 *In the following classes of groups all Sasaki groups in $\mathcal{S}_5$ are projective:*

(1) *torsion-free groups with trivial centre,*
(2) *torsion-free hyperbolic groups,*
(3) *torsion-free Schreier groups,*
(4) *fundamental groups of non-positively curved closed manifolds of rank one,*
(5) *fundamental groups of compact locally symmetric spaces of non-compact type,*
(6) *fundamental groups of three-manifolds.*

This applies in particular to fundamental groups of manifolds of constant negative curvature, see Corollary 25. The philosophy behind Theorem 4 also explains other results that have been proved about Sasaki groups. For example, a posteriori, the recent result of Biswas and Mj [5] can be paraphrased as saying that Sasaki groups of deficiency at least 2 must be projective, compare [33]. For torsion-free groups this is an immediate consequence of Statement 3 in Theorem 4.

There are other restrictions on Kähler groups that can be extended to the Sasaki case, but that do not fit neatly into this philosophy. A case in point is the following Sasaki analog of the well known theorem of Johnson and Rees [27] about Kähler groups.

Theorem 5 *Let Γ_1 and Γ_2 be two groups. Assume $f_i : \Gamma_i \longrightarrow Q_i$ are non-trivial quotients with $|Q_i| = m_i < \infty$ for both $i = 1, 2$. Then the following two statements hold:*

(a) *$\Gamma_1 * \Gamma_2$ is not Sasaki,*
(b) *moreover, for any group H the product $(\Gamma_1 * \Gamma_2) \times H$ is not Sasaki.*

1.4 Structure of the paper

In Sect. 2 we introduce notation and recall some facts on the topology of Sasaki manifolds that are crucial for our arguments. Section 3 is dedicated to the relationship between Sasaki and projective groups, in particular it contains the proofs of Theorems 2 and 3. We also explain how the proof of Theorem 3 adapts to K-contact manifolds. In Sect. 4 we discuss Sasaki groups which are not projective, and not even Kähler, and which can only be realised as Sasaki groups on manifolds with explicit dimension bounds. This in particular proves Theorem 1. Finally Sect. 5 contains the proofs of Theorems 4 and 5, and some corollaries and applications.

2 Sasaki manifolds and the associated group extensions

We begin with some definitions and known results; for a more comprehensive treatment we refer to the monograph by Boyer and Galicki [9]. Unless otherwise stated, all manifolds are assumed to be smooth, closed and oriented.

A *K-contact structure* (M, η, ϕ) on a manifold M consists of a contact form η and an endomorphism ϕ of the tangent bundle $\mathrm{T}M$ satisfying the following properties:

- $\phi^2 = -\mathrm{Id} + R \otimes \eta$ where R is the Reeb vector field of η,
- $\phi_{|\mathcal{D}}$ is an almost complex structure compatible with the symplectic form $\mathrm{d}\eta$ on $\mathcal{D} = \ker \eta$,
- the Reeb vector field R is Killing with respect to the metric $g(\cdot,\cdot) = \mathrm{d}\eta(\phi\cdot,\cdot) + \eta(\cdot)\eta(\cdot)$.

Given such a structure one can consider the almost complex structure I on the Riemannian cone $\left(M \times (0,\infty), t^2 g + \mathrm{d}t^2\right)$ given by

- $I = \phi$ on $\mathcal{D} = \ker \eta$,
- $I(R) = t\partial_t$.

A *Sasaki structure* is a K-contact structure (M, η, ϕ) such that the associated almost complex structure I on the Riemannian cone is integrable.

An important example of a Sasaki (resp. K-contact) structure is obtained by the *Boothby–Wang fibration* M over a Kähler (resp. almost-Kähler) manifold (X, ω) with ω representing an integral class [7], that is, the principal S^1-bundle $\pi : M \longrightarrow X$ with Euler class $[\omega]$ and connection 1-form η such that $\pi^*(\omega) = \mathrm{d}\eta$.

A contact form is called *regular* (respectively *quasi-regular*, *irregular*) if its Reeb foliation is such. Rukimbira [41] proved that any irregular Sasaki structure can be deformed to a quasi-regular one. Moreover, the geometry of quasi-regular structures is described by the following (cf. [9, Theorem 7.5.1]):

Theorem 6 (Structure Theorem) *Let (M, η, ϕ) be a quasi-regular Sasaki structure and let $|X|$ be the space of leaves of the Reeb folation. Then $|X|$ carries the structure of a projective orbifold X with an integral Kähler class $[\omega] \in H^2_{orb}(X;\mathbb{Z})$, and $\pi : M \longrightarrow X$ is the principal S^1-orbibundle with connection 1-form η such that $\pi^*\omega = \mathrm{d}\eta$. Moreover, if η is regular then $\pi : M \longrightarrow X$ is a principal S^1-bundle over a smooth projective manifold.*

The orbifold homotopy, homology and cohomology groups are defined using Haefliger's classifying space [22], cf. [9, Section 4.3].

Since the leaf holonomy groups of the Reeb foliation are always cyclic, the orbifold X is a cyclic orbifold. In particular, it is normal and $\mathbb{Q}$-factorial. Moreover, because M is a smooth manifold, the cohomology class of ω being integral means that X is a polarised orbifold in the sense of Ross and Thomas [40, Definition 2.7]. Throughout this paper, the only orbifolds we consider are these cyclic polarised orbifolds. Starting from one of these orbifolds, one has the following converse to the structure theorem (cf. [9, Theorem 7.5.2]), arising from the orbifold version of the Boothby–Wang construction.

Theorem 7 *Let X be a cyclic projective orbifold equipped with a polarisation defined by an integral Kähler class $[\omega] \in H^2_{orb}(X;\mathbb{Z})$. Then the total space of the principal S^1-orbibundle $\pi : M \longrightarrow X$ with Euler class $[\omega]$ is a manifold that can be equipped with a quasi-regular Sasaki structure such that $\pi^*\omega = \mathrm{d}\eta$, where η is the contact form.*

In this particular case the total space of the orbibundle is actually smooth, because we started with a polarised cyclic orbifold in the sense of Ross and Thomas [40, Definition 2.7], which means in particular that all local uniformizing groups inject into S^1, so that smoothness of the total space follows from [9, Lemma 4.2.8].

The Structure Theorem 6 together with Rukimbira's result [41] implies that the fundamental group of a Sasaki manifold M always fits into the long exact sequence of homotopy groups associated to the principal S^1-orbibundle $\pi : M \to X$, namely:

$$\ldots \longrightarrow \pi_2^{orb}(X) \xrightarrow{\partial} \pi_1(S^1) \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(X) \longrightarrow 0.$$

Thus $\pi_1(M)$ fits into a short exact sequence of the following type:

$$0 \longrightarrow C = \mathrm{coker}\partial \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(X) \longrightarrow 0. \tag{1}$$

The group C is a – possibly trivial – cyclic subgroup of the centre of $\pi_1(M)$ determined by the Euler class $e = [\omega] \in H^2_{orb}(X; \mathbb{Z})$ since the map ∂ factors as

$$\begin{array}{ccc} \pi_2^{orb}(X) & \xrightarrow{\partial} & \mathbb{Z} \cong \pi_1(S^1) \\ & \searrow^{\psi} & \uparrow \langle -; e\rangle \\ & & H_2^{orb}(X; \mathbb{Z}) \end{array} \tag{2}$$

where $\langle -; e\rangle$ is the evaluation of the Euler class and ψ is the Hurewicz homomorphism; see for instance [24].

Remark 8 Let $\mathrm{B}\pi_1^{orb}(X)$ be the classifying space constructed from the orbifold classifying space $\mathrm{B}X$ by attaching cells of dimension larger than 2. The inclusion map $\iota\colon \mathrm{B}X \longrightarrow \mathrm{B}\pi_1^{orb}(X)$ induces an isomorphism

$$\iota^*\colon H^1(\mathrm{B}\pi_1^{orb}(X); \mathbb{Z}) \longrightarrow H^1_{orb}(X; \mathbb{Z})$$

and an injective homomorphism

$$\iota^*\colon H^2(\mathrm{B}\pi_1^{orb}(X); \mathbb{Z}) \hookrightarrow H^2_{orb}(X; \mathbb{Z}).$$

In the case where $C \cong \mathbb{Z}$ in (1) one can identify the Euler class $e \in H^2_{orb}(X; \mathbb{Z})$ of the principal S^1-orbibundle $p\colon M \longrightarrow X$ with the characteristic class $c \in H^2(\pi_1^{orb}(X); \mathbb{Z})$ of the central extension via the pullback of the inclusion $\iota\colon \mathrm{B}X \longrightarrow \mathrm{B}\pi_1^{orb}(X)$.

Let us now give some more details about the extension (1). In particular, we want to relate the orbifold fundamental group $\pi_1^{orb}(X)$ to a genuine projective group. Note that the map $p\colon BX \longrightarrow X$ from the orbifold classifying space to the underlying topological space induces a surjective map p_* at the level of fundamental groups. Moreover, the kernel of p_* is normally generated by loops around the irreducible divisors D contained in the singular set of X. These loops represent torsion elements of order m, the ramification index of D. Therefore, the kernel K of the map $p_*\colon \pi_1^{orb}(X) \longrightarrow \pi_1(X)$ is generated by (possibly infinitely many) torsion elements. Now the cohomology of K with coefficients in $\mathbb{R}$ is trivial in all positive degrees because $\mathbb{R}$ is m-divisible for all m. Thus the Lyndon–Hochschild–Serre spectral sequence of $K \subset \pi_1^{orb}(X)$ yields an isomorphism $H^*(\pi_1^{orb}(X); \mathbb{R}) \cong H^*(\pi_1(X); \mathbb{R})$. Moreover, X admits a resolution of singularities which does not change the fundamental group by a result of Kollár, see [31, Theorem 7.5.2]. Thus the real cohomology ring of $\pi_1^{orb}(X)$ is that of a projective group. Notice that whenever $C \neq \mathbb{Z}$ we have an isomorphism $H^*(\pi_1^{orb}(X); \mathbb{R}) \cong H^*(\pi_1(M); \mathbb{R})$. In this instance $\pi_1(M)$ itself has the real cohomology ring of the projective group $\pi_1(X)$. We summarize this discussion in a lemma for future reference.

Lemma 9 *For any quasi-regular structure $\pi : M \longrightarrow X$ on a Sasaki manifold M one has the diagram*

$$\begin{array}{ccccccccc} & & & & & & K & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & \Gamma & \xrightarrow{\pi_*} & \pi_1^{orb}(X) & \longrightarrow & 0 \\ & & & & & & \downarrow p_* & & \\ & & & & & & \pi_1(X) & & \end{array} \tag{3}$$

where $\Gamma = \pi_1(M)$ and C is cyclic. Moreover, $\pi_1(X)$ is a projective group and the kernel K of p_* is generated by torsion elements so that $H^*(\pi_1^{orb}(X);\mathbb{R}) \cong H^*(\pi_1(X);\mathbb{R})$. If in addition $C \neq \mathbb{Z}$, then $H^*(\pi_1(X);\mathbb{R}) \cong H^*(\Gamma;\mathbb{R})$.

We will also need the following.

Lemma 10 *Let $\pi: M \longrightarrow X$ be the principal orbibundle associated to a quasi-regular Sasaki structure. Then there is a non-degenerate skew-symmetric bilinear pairing*

$$H^1(\pi_1^{orb}(X);\mathbb{R}) \times H^1(\pi_1^{orb}(X);\mathbb{R}) \longrightarrow \mathbb{R} \tag{4}$$

which factors through the cup product

$$H^1(\pi_1^{orb}(X);\mathbb{R}) \times H^1(\pi_1^{orb}(X);\mathbb{R}) \stackrel{\cup}{\longrightarrow} H^2(\pi_1^{orb}(X);\mathbb{R}).$$

Proof In the quasi-regular case the basic cohomology ring $H^*_B(\mathcal{F};\mathbb{R})$ of the Reeb fibration, see [9, Section 7.2], coincides with the orbifold cohomology ring with real coefficients $H^*_{orb}(X;\mathbb{R})$. Moreover, the Hard Lefschetz Theorem holds for the basic cohomology of a Sasaki manifold, see [18] and [9, Theorem 7.2.9]. The claim then follows by composing the non-degenerate bilinear map coming from the transverse Hard Lefschetz Theorem with the map $\iota^*\colon H^*(B\pi_1^{orb}(X)) \longrightarrow H^*_{orb}(X)$ given in Remark 8. □

Remark 11 This lemma is crucial for the proof of Theorem 5. In [11] a different version of the Hard Lefschetz Theorem is proved for Sasaki manifolds. However, the non-degenerate bilinear pairing constructed in [11] does not factor through the cup product in group cohomology, and is therefore not suitable for our purposes.

3 Sasaki groups from projective groups

In this section we prove Theorems 2 and 3, and give some variations on the latter.

3.1 The Lefschetz hyperplane theorem for Sasaki manifolds

In algebraic geometry, the Lefschetz hyperplane theorem is a statement relating the homotopy groups of a complex projective variety to those of a generic hyperplane section. In the Sasaki context we consider quasi-regular structures defining orbifold bundles over projective orbifolds, and then take hyperplane sections of the base orbifold and restrict the orbifold bundle to such a hyperplane section. The resulting statements can be formulated for higher homotopy groups as well, but they are easier in that case, and, possibly, less useful. So we will stick to the discussion of fundamental groups and just prove Theorem 2.

Let Γ be the fundamental group of a Sasaki manifold M of dimension $2n+1 \geq 7$. We may assume that the Sasaki structure is quasi-regular, and obtain the associated orbifold fibration $\pi: M \longrightarrow X$ with X a projective orbifold of complex dimension n. Now let $Y \subset X$ be a generic hyperplane section. We denote the inclusion of Y in X by ι, and we let $N \subset M$ be the preimage of Y under π, i.e. the total space of the orbifold circle bundle restricted from X to Y.

Since X arises as the quotient of the quasi-regular Sasaki structure on M, the local uniformizing groups of X are cyclic and inject in S^1. Restricting to the suborbifold $Y \subset X$, we have the same local uniformizing groups with the same injections into S^1. In other words, the restriction of the polarisation of X to Y is again a polarisation in the sense of Ross and Thomas [40, Definition 2.7]. Therefore Theorem 7 applies, and N is a smooth Sasaki manifold, rather than just an orbifold.

This situation gives rise to the following commutative diagram of orbifold homotopy exact sequences:

$$\begin{array}{ccccccccc} \pi_2^{orb}(Y) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(N) & \longrightarrow & \pi_1^{orb}(Y) & \longrightarrow & 1 \\ \downarrow \iota_* & & \| & & \downarrow \iota_* & & \downarrow \iota_* & & \\ \pi_2^{orb}(X) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1^{orb}(X) & \longrightarrow & 1\,. \end{array}$$

By the Lefschetz hyperplane theorem for orbifolds or Deligne–Mumford stacks [23, Corollary 2.7] the assumption $n \geq 3$ implies that the vertical inclusion-induced map

$$\iota_* \colon \pi_i^{orb}(Y) \longrightarrow \pi_i^{orb}(X)$$

is an isomorphism for $i = 1$ and a surjection for $i = 2$. Therefore, this diagram shows, by the usual diagram chase, that $\iota_* \colon \pi_1(N) \longrightarrow \pi_1(M)$ is also an isomorphism. Now N is a Sasaki manifold of real dimension two less than the dimension of M having the same fundamental group. This completes the proof of Theorem 2.

Remark 12 It was pointed out to us by J. Kollár that the isomorphism $\pi_1^{orb}(Y) \longrightarrow \pi_1^{orb}(X)$ can also be deduced from more classical Lefschetz theorems for quasi-projective varieties. Indeed, if $D \subset X$ is the singular locus of the orbifold X, one can apply Theorem 1.1.3 (ii) of Hamm–Lê [25] to see that $X \backslash D$ is obtained from its hyperplane section by attaching cells of dimension ≥ 3. Therefore, they have the same fundamental group. Then the orbifold fundamental groups of X and Y are obtained from these fundamental groups of the complements of the singular loci by introducing the same relations on both sides, so the orbifold fundamental groups are the same.

3.2 The Boothby–Wang construction with control on the fundamental group

We now prove Theorem 3. Let Γ be a projective group. Taking products with $\mathbb{C}P^1$ in one direction, and using the Lefschetz hyperplane theorem in the other, one sees that $\Gamma = \pi_1(X)$ where X can be taken as a smooth complex projective variety of (real) dimension $2n$ for any $n \geq 2$. Then X is equipped with an integral Kähler class $[\omega]$ so that the corresponding line bundle is ample. The blow-up of X at a point, topologically $X\#\overline{\mathbb{C}P^n}$, can be endowed with the integral Kähler class $k[\omega] - E$ where E is the Poincaré dual of the exceptional divisor D and $k \in \mathbb{N}$ is large enough, so that the corresponding line bundle is ample on the blowup.

Consider now the Boothby–Wang fibration M over $X\#\overline{\mathbb{C}P^n}$ with Euler class $e = k[\omega] - E$ and the associated long exact sequence

$$\cdots \longrightarrow \pi_2(X\#\overline{\mathbb{C}P^n}) \stackrel{\partial}{\longrightarrow} \pi_1(S^1) \longrightarrow \pi_1(M) \longrightarrow \Gamma \longrightarrow 0.$$

The exceptional divisor $D \cong \mathbb{C}P^{n-1}$ contributes a non-torsion spherical class to $H_2(X\#\overline{\mathbb{C}P^n}; \mathbb{Z})$ on which the Euler class evaluates as ± 1. Thus the map ∂ is surjective by (2), that is, we get the desired isomorphism $\pi_1(M) \cong \Gamma$. This completes the proof of Theorem 3.

Remark 13 This proof generalizes to the orbifold setting in the following way. Instead of starting with a smooth projective variety, we may start with any cyclic projective orbifold with an integral polarisation. We first blow up a smooth point and modify the polarisation to one on the blowup which evaluates as ± 1 on a projective line in the exceptional divisor. Then we apply Theorem 7 to this polarisation on the blowup to obtain a Sasaki manifold whose fundamental group is the orbifold fundamental group of the cyclic orbifold we started with.

Another variation on the proof of Theorem 3 is given by the following:

Proposition 14 *Every finitely presentable group is the fundamental group of a closed K-contact manifold of dimension* $2n+1$ *for any* $n \geq 2$.

Without the control on the dimension, this result appears in [6, Theorem 5.2]. The proof given there is different from ours, and does not prove the case $n = 2$. The following argument implements a suggestion of [1, Remark 2.2].

Proof Fix an $n \geq 2$ and a finitely presentable group Γ. By a celebrated theorem of Gompf [21] there exists a closed symplectic $2n$-manifold Y such that $\pi_1(Y) = \Gamma$. Since non-degeneracy is an open condition in the space of closed forms, there exists a symplectic form on Y representing a rational class in cohomology. After multiplication with a large integer we may assume that Y is equipped with a symplectic form ω representing a primitive integral class $[\omega]$.

After possibly replacing Y by its symplectic blow-up at a point, there exists a spherical class in $H_2(Y; \mathbb{Z})$ on which $[\omega]$ evaluates as ± 1. Therefore, the Boothby–Wang fibration over Y with Euler class $[\omega]$ is a compact $(2n+1)$-dimensional K-contact manifold N with $\pi_1(N) = \Gamma$. □

4 Some non-Kähler Sasaki groups

In this section we construct interesting Sasaki groups which are not Kähler, and use some of them to prove Theorem 1.

Since $\mathcal{S}_5$ is the richest class of Sasaki groups, it makes sense to start in dimension 5.

Proposition 15 *Let* X *be an aspherical algebraic surface, and* $\pi \colon M \longrightarrow X$ *the Sasaki* 5*-manifold obtained by the Boothby–Wang construction using as Euler class* $[\omega]$ *the first Chern class of an ample line bundle on* X*. If* X *has complex Albanese dimension* 2*, then* $\pi_1(M)$ *is not a Kähler group.*

Proof Suppose Y is a compact Kähler manifold with fundamental group $\pi_1(Y) = \pi_1(M)$. Consider the composition

$$Y \xrightarrow{c_Y} B\pi_1(Y) = M \xrightarrow{\pi} X,$$

where c_Y is the classifying map of the universal covering of Y. Both maps induce isomorphisms on $H^1(-; \mathbb{R})$. However, any four-fold cup product of classes in $H^1(X; \mathbb{R})$ lands in $H^4(X; \mathbb{R})$, which is spanned by $[\omega]^2$. Since $[\omega]$ is killed by π^*, so is $[\omega]^2$. Therefore, the cup-length of degree one classes on Y is strictly less than 4. This means that the image of Y under its Albanese map $\alpha_Y \colon Y \longrightarrow Alb(Y)$ is a curve D, cf. [2, p. 23]. A standard argument then implies that D is smooth, and the Albanese map has connected fibres, compare [19, p. 289]. The Albanese image D is necessarily of genus ≥ 2, since the first Betti number of X, and therefore of Y, is at least 4.

Since the Albanese map has connected fibres, it induces a surjective homomorphism

$$(\alpha_Y)_* \colon \pi_1(Y) \longrightarrow \pi_1(D).$$

However, as D is of genus $g \geq 2$, its fundamental group has trivial centre, and so this homomorphism factors through the quotient map $\pi_*: \pi_1(Y) = \pi_1(M) \longrightarrow \pi_1(X)$ as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(Y) & \xrightarrow{\pi_*} & \pi_1(X) & \longrightarrow & 1. \\ & & & & {\scriptstyle (\alpha_Y)_*}\downarrow & \swarrow & & & \\ & & & & \pi_1(D) & & & & \end{array} \tag{5}$$

Since both $(\alpha_Y)_*$ and π_* induce isomorphisms on $H^1(-; \mathbb{R})$, so does the diagonal map on the bottom right. This means that $H^1(X; \mathbb{R})$ comes from the curve D, and so the cuplength is 2, and not 4. This contradicts the assumption that X has real Albanese dimension 4, and this contradiction finally shows that Y does not exist, and $\pi_1(M)$ is not a Kähler group. □

There are many examples where this result applies.

Example 16 If we take X to be an Abelian surface, we can arrange $\pi_1(M)$ to be the five-dimensional integral Heisenberg group. That this is not a Kähler group was first proved by Carlson and Toledo [13] using a completely different argument.

Remark 17 Carlson and Toledo [13] pointed out that the integral Heisenberg groups of dimension at least 5 are known to be 1-formal, in the sense of having quadratically presented Malcev algebra, so there is no obstruction at that level to them being Kähler. Indeed, even if this had not been known thirty years ago, it now follows via Kasuya's result [29, Theorem 1.1] from the fact that they are all Sasaki groups in dimension ≥ 5. The claim in [6, Section 3.1] that these groups are not 1-formal is based on the use of a different notion of 1-formality, cf. [29, Remark 3.8].

Example 18 We can also take for X any Cartesian product of curves of positive genus, not necessarily of genus one. Then $\pi_1(M)$ is no longer nilpotent as in the previous example.

Example 19 Proposition 15 applies whenever X is a Kodaira fibration whose Albanese image is not a curve. Almost all examples of Kodaira fibrations constructed explicitly do have this property, compare the discussion in [10] and the references given there.

Finally, we can also apply Proposition 15 to ball quotients of complex dimension 2 whose Albanese image is also of dimension 2. However, in this case we can prove more, in that we can allow ball quotients of arbitrary dimension, and we can dispense with the assumption about the Albanese dimension.

Let $G \subset PU(n, 1)$ be a torsion-free cocompact lattice in the isometry group of the n-dimensional complex ball $\mathbb{C}H^n$ equipped with the Bergman metric. Then $X = \mathbb{C}H^n/G$ is a smooth projective variety, since the Kähler class $[\omega]$ of the Bergman metric is the first Chern class of an ample line bundle. Let $\pi : M \longrightarrow X$ be the principal circle bundle with Euler class $[\omega]$. Its total space carries a Sasaki structure obtained by the Boothby–Wang construction. Since X is aspherical, the fundamental group $\Gamma = \pi_1(M)$ is the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow G \longrightarrow 1 \tag{6}$$

with extension class $[\omega]$.

The following statement is known to some experts, cf. for example [39], but since a quick and clean proof is not readily available in the literature, we give one here.

Proposition 20 *The extension Γ is not a Kähler group.*

Proof In case that $n = 1$, the manifold $M = B\Gamma$ has positive first Betti number, but the cup product from H^1 to H^2 vanishes identically. Therefore, Γ cannot be Kähler by the Hard Lefschetz Theorem.

We now assume $n \geq 2$ and argue by contradiction. So suppose that Y is a compact Kähler manifold with fundamental group Γ. Consider the classifying map of its universal covering composed with the projection from $M = B\Gamma$ to $X = BG$:

$$Y \xrightarrow{c_Y} M \xrightarrow{\pi} X.$$

Since X is negatively curved, the composition $\pi \circ c_Y$ is homotopic to a harmonic map $h \colon Y \longrightarrow X$ by the Eells–Sampson theorem. Note that on $\pi_1(Y)$ the induced map h_* is just the projection $\Gamma \longrightarrow G$ with kernel $\mathbb{Z}$. The assumption $n \geq 2$ implies that h_* cannot factor through a surface group. We now invoke the classification of harmonic maps from compact Kähler manifolds to ball quotients due to Carlson and Toledo [12, Theorem 7.2]. This says that either the rank of the differential Dh is everywhere ≤ 2, in which case h has image a closed geodesic or factors through a Riemann surface, or it is holomorphic (after perhaps conjugating the complex structure on X). The first case is not possible because of what we know about h_* at the fundamental group level, and so we conclude that h is indeed holomorphic. Since it is non-constant, its image is a positive-dimensional analytic subvariety of X, on which the appropriate power of the Kähler class $[\omega]$ is positive. This means $h^*[\omega] \neq 0$, contradicting the fact that $h^* = c_Y^* \circ \pi^*$, and $[\omega] \in \ker(\pi^*)$ by the Gysin sequence. This contradiction proves the claim. □

We can now prove Theorem 1.

Proof of Theorem 1 For every n, there do exist torsion-free cocompact lattices G in the isometry group of the n-dimensional complex ball $\mathbb{C}H^n$ equipped with the Bergman metric; see for example Borel [8]. As above, we let X be the corresponding ball quotient with fundamental group G, and $\pi : M \longrightarrow X$ the Sasaki manifold which is the total space of the circle bundle with Euler class $[\omega]$. Then the fundamental group of M is the Sasaki group described by the extension (6). To prove Theorem 1 we show that if N is any compact Sasaki manifold whose fundamental group is the group $\Gamma = \pi_1(M)$, then $\dim(N) \leq 2n + 1$.

As usual we may assume the Sasaki structure on N to be quasi-regular, and consider the quotient map $N \longrightarrow Z$, where Z is a projective orbifold. We obtain the corresponding exact sequence

$$1 \longrightarrow C \longrightarrow \Gamma \longrightarrow \pi_1^{orb}(Z) \longrightarrow 1. \tag{7}$$

Since Γ is torsion-free, C cannot be non-trivial and finite. Moreover, if C were trivial, then $\Gamma = \pi_1^{orb}(Z)$, and using torsion-freeness again, this would show that Γ is projective, contradicting Proposition 20. Thus C must be infinite cyclic. Since the centre $C(\Gamma)$ of Γ is infinite cyclic, consisting of the copy of $\mathbb{Z}$ on the left in (6), we conclude that C is a finite index subgroup of $C(\Gamma)$.

Assume first that $C = C(\Gamma)$. Then $\pi_1^{orb}(Z) = G$, and since G is torsion-free, we conclude $\pi_1(Z) = G$. Moreover, Remark 8 shows that the circle bundle $N \longrightarrow Z$ is the pullback of $M \longrightarrow X$ under the classifying map $c_Z \colon Z \longrightarrow X$. On X, the $(n+1)^{\text{st}}$ power of the Euler class vanishes for dimension reasons, but this Euler class pulls back to a Kähler class on Z. Therefore, $\dim_{\mathbb{C}}(Z) \leq n$, which implies $\dim(N) \leq 2n + 1$.

If $C \subset C(\Gamma)$ has index $k > 1$, then we divide (6) by C, and obtain

$$1 \longrightarrow C(\Gamma)/C = \mathbb{Z}_k \longrightarrow \Gamma/C = \pi_1^{orb}(Z) \longrightarrow G \longrightarrow 1.$$

Again the extension class of (7) in $H^2(\pi_1^{orb}(Z);\mathbb{Z}) \subset H^2(Z;\mathbb{Z})$ is a pullback from $BG = X$, showing that its $(n+1)^{\text{st}}$ power vanishes. Since it is a Kähler class, we obtain the same dimension bound as before. This completes the proof. □

Remark 21 For $n = 1$, the above proof shows that the fundamental groups of 3-manifolds which are circle bundles with non-zero Euler classes over surfaces of genus ≥ 2 are not Sasaki groups in dimensions ≥ 3. The same argument also applies to circle bundles over the torus. As mentioned in the introduction, this conclusion also follows from Kasuya's result about 1-formality [29, Theorem 1.1]. In the proof of Theorem 4 below we will generalize the statement from these circle bundles to all 3-manifolds with infinite fundamental groups.

5 Further restrictions on Sasaki groups

In this section we prove Theorems 4 and 5, and we discuss some applications and variations.

5.1 About Theorem 5

For the proof we use the following lemma.

Lemma 22 ([32]) *Let Γ_1 and Γ_2 be two groups. Assume $f_i\colon \Gamma_i \longrightarrow Q_i$ is a non-trivial quotient with kernel K_i and $|Q_i| = m_i < \infty$ for $i = 1, 2$. Then the free product $\Gamma_1 * \Gamma_2$ admits a finite index subgroup with odd first Betti number.*

Proof Consider the following composition

$$\Gamma_1 * \Gamma_2 \xrightarrow{\pi_{ab}} \Gamma_1 \times \Gamma_2 \xrightarrow{f_1 \times f_2} Q_1 \times Q_2.$$

By the Kurosh subgroup theorem, the kernel of this homomorphism has the form $F_m * K$ where F_m is the free group on $m = (m_1 - 1)(m_2 - 1)$ generators and K is a free product of subgroups isomorphic to the K_i. Now let $f\colon F_m \longrightarrow Q$ be a finite quotient with $|Q| = d$. Extend f trivially on K to get a homomorphism $\bar{f}\colon F_m * K \longrightarrow Q$. Then the kernel of $\bar{f}$ has the form $F_n * K * \cdots * K$ where $n = 1 + d(m-1)$ and K appears d many times. Thus, $\ker(\bar{f})$ is a finite index subgroup in $\Gamma_1 * \Gamma_2$ and

$$\mathrm{b}_1(\ker(\bar{f})) = n + d\mathrm{b}_1(K) = 1 + d(m - 1 + \mathrm{b}_1(K)).$$

By picking d even we get a finite index subgroup of $\Gamma_1 * \Gamma_2$ with odd first Betti number. □

We are now ready to prove Theorem 5. Clearly part a follows directly from Lemma 22, so we only have to prove part b. Set $\Gamma = (\Gamma_1 * \Gamma_2) \times H$. The proof is divided into two cases.

5.1.1 Case 1: $\mathrm{b}_1(H)$ is even

By Lemma 22, there exists a finite index subgroup $\Delta \subset \Gamma_1 * \Gamma_2$ with $\mathrm{b}_1(\Delta)$ odd. Hence, the group $\Delta \times H$ is a finite index subgroup of Γ with odd first Betti number. Thus Γ cannot be Sasaki.

5.1.2 Case 2: $b_1(H)$ is odd

In this case $b_1(\Gamma_1 * \Gamma_2) > 0$. Then we can assume that the first Betti number of Γ_1 is positive, and so Γ_1 has finite quotients of arbitrarily large order. The first step in the proof of Lemma 22 provides a finite index subgroup of $\Gamma_1 \times \Gamma_2$ of the form $F_m * G$. Moreover, the rank m of F_m can be chosen to be arbitrarily large by using suitable quotients of Γ_1 (and some fixed quotient of Γ_2). Since the class of Sasaki groups is closed under taking finite index subgroups, we can assume $\Gamma = (F_m * G) \times H$ with $m > b_1(H)$.

Let M be a compact Sasaki manifold with $\pi_1(M) = \Gamma$. Consider a quasi-regular Sasaki structure $\pi : M \longrightarrow X$ and let

$$0 \longrightarrow C \longrightarrow \Gamma \longrightarrow \pi_1^{orb}(X) \longrightarrow 0$$

be the associated central extension. Since C is mapped to the centre of Γ, it must be mapped into H. It follows that

$$\pi_1^{orb}(X) = (F_m * G) \times (H/C).$$

Now $H^1(\pi_1^{orb}(X); \mathbb{R})$ is endowed with a skew-symmetric non-degenerate bilinear pairing by Lemma 10. Moreover, this factors through the cup product, i.e.

$$H^1(\pi_1^{orb}(X); \mathbb{R}) \times H^1(\pi_1^{orb}(X); \mathbb{R}) \stackrel{\cup}{\longrightarrow} H^2(\pi_1^{orb}(X); \mathbb{R}) \longrightarrow \mathbb{R}\,. \tag{8}$$

Therefore, for this pairing $H^1(F_m)$ is an isotropic subspace in $H^1(\pi_1^{orb}(X)) \cong H^1(F_m) \oplus H^1(G) \oplus H^1(H/C)$ which is orthogonal to $H^1(G)$. Since $b_1(H/C) = b_1(H)$, the inequality $m > b_1(H) = b_1(H/C)$ contradicts the non-degeneracy of the skew-symmetric pairing. This contradiction proves that Γ is not a Sasaki group. We have now completed the proof of Theorem 5.

5.2 About Theorem 4

We now go through the different cases in Theorem 4 giving the proofs, and, in some cases, further applications.

Statement 1

Triviality of the centre means that C has to be trivial. Torsion-freeness then implies that the kernel K in (3) is trivial, and so Γ is isomorphic to the projective group $\pi_1(X)$. This completes the proof in this case.

Napier and Ramachandran [38] proved that the Thompson group F and its generalisations $F_{n,\infty}$ and F_n are not Kähler. Since these groups are torsion-free with trivial centre, we obtain:

Corollary 23 *The Thompson group F and its generalisations $F_{n,\infty}$ and F_n are not Sasaki groups.*

Statement 2

Since Sasaki groups have even first Betti numbers, an infinite Sasaki group cannot be cyclic. Therefore we may assume that our hyperbolic group is not virtually cyclic. Then by [35, Lemma 3.5] its centre is finite. By torsion–freeness the centre is trivial, and so Statement 1 applies.

Statement 3

Recall from [16, Definition 3.1] that a group is Schreier if every finitely generated normal subgroup is either finite or of finite index. For a Sasaki group Γ being Schreier means that C cannot be infinite cyclic. For it it were so, then it would have to be of finite index, so that Γ would be virtually infinite cyclic. Once C cannot be infinite, torsion-freeness again proves that Γ is isomorphic to the projective group $\pi_1(X)$.

Llosa Isenrich [36, Corollary 3.2.9] proved that any torsion-free Schreier and Kähler group with virtually positive first Betti number is an orientable surface group of genus ≥ 2. Thus we obtain:

Corollary 24 *Any torsion-free Sasaki and Schreier group with virtually positive first Betti number is an orientable surface group of genus* ≥ 2.

Statement 4

Let N be a closed Riemannian manifold of non-positive curvature with fundamental group $\pi_1(N) = \Gamma$ a Sasaki group. By the Cartan–Hadamard theorem, Γ is torsion-free. Since Γ is Sasaki it cannot be cyclic, so we may assume that N is of dimension ≥ 2.

The notion of rank for an abstract group was introduced by Ballmann and Eberlein [4], who proved that for the fundamental groups of closed manifolds of non-positive sectional curvature the rank agrees with the geometric rank of the Riemannian metric defined via spaces of parallel Jacobi fields. The rank is additive under Cartesian products of manifolds respectively direct products of groups, and is invariant under passage to finite coverings respectively to finite index subgroups. The assumption that N be of rank one therefore implies that N is locally irreducible and that Γ is irreducible (any direct factor of Γ would be infinite because the group is torsion-free).

The irreducibility of N and the assumption $\dim N \geq 2$ imply that N has no Euclidean local de Rham factor, so that by the result of Eberlein [17, p. 210f], the centre of Γ is trivial. Therefore Statement 1 applies to give the proof in this case.

Of course, if the sectional curvature of N is strictly negative, then the triviality of the centre follows directly from Preissmann's theorem. However, there are many more manifolds of non-positive curvature and rank one than there are negatively curved manifolds.

Since the fundamental groups of closed real hyperbolic manifolds of dimension ≥ 3 are known not to be Kähler by a result of Carlson and Toledo [12, Theorem 8.1], we conclude:

Corollary 25 *No fundamental group of a closed real hyperbolic manifold of dimension* ≥ 3 *is Sasaki.*

With the added assumption or arithmeticity, this also appears in the statement of [6, Proposition 5.4]. However, the proof given there, while using some of the same arguments we use, seems elliptical and not quite to the point.

Statement 5

For N to be a locally symmetric space of non-compact type means that its universal covering $\tilde{N}$ is a globally symmetric space without compact or Euclidean factors in its de Rham decomposition. Thus $\pi_1(N)$ is torsion-free. The absence of a Euclidean de Rham factor again implies triviality of the centre via the result of Eberlein [17, p. 210f]. Therefore Statement 1 applies.

Statement 6

Suppose that $\Gamma \in \mathcal{S}_5$ is the fundamental group of a 3-manifold N. Since all finite groups are projective by a classical result of Serre, cf. [2, p. 6], we only have to consider infinite groups Γ.

By a result of Jaco [26], finite presentability of Γ implies that we may take N to be compact, possibly with non-empty boundary. Since this does not change the fundamental group, we cap off any spherical boundary components by balls.

Next, we may assume N to be prime. For if it were not prime, its fundamental group Γ would be a non-trivial free product. Since 3-manifold groups are residually finite,[3] Lemma 22 would show that Γ has virtually odd first Betti number, which is impossible since Γ is Sasaki.

Moreover, being Sasaki, Γ cannot be virtually cyclic, and so the prime manifold N is irreducible by the sphere theorem, cf. [37]. This in particular implies that N is aspherical and Γ is torsion-free. Thus, if the centre of Γ is trivial, the conclusion follows from Statement 1.

If the centre of Γ is not trivial, then N is Seifert fibered, cf. [3, Theorem 2.5.5]. After passing to a suitable finite covering, respectively a finite index subgroup of Γ, we may then assume that N is a circle bundle over a compact orientable surface S. If S has non-empty boundary, then the Euler class of the circle bundle is trivial, and the total space N has odd first Betti number, which is impossible for a Sasaki group Γ. So S must be closed, and the Euler class of the circle bundle $N \longrightarrow S$ must generate $H^2(S)$. Since Γ is not virtually cyclic, S has positive first Betti number $b_1(S) = b_1(N)$. As explained in Remark 21, these groups can actually not be Sasaki in dimensions ≥ 5.

This completes the proof of Statement 6. Combining this statement with the result of [34] yields the following conclusion:

Corollary 26 *Any infinite group in $\mathcal{S}_5$ that is also the fundamental group of a 3-manifold is an oriented surface group.*

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[3] Compare [3, p. 60].

References

1. A'Campo, N., Kotschick, D.: Contact structures, foliations, and the fundamental group. Bull. Lond. Math. Soc. **26**, 102–106 (1994)
2. Amorós, J., Burger, M., Corlette, K., Kotschick, D., Toledo, D.: Fundamental Groups of Compact Kähler Manifolds. Mathematical Surveys and Monographs, vol. 44. American Mathematical Society, Providence (1996)
3. Aschenbrenner, M., Friedl, S., Wilton, H.: 3-Manifold Groups. EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich (2015)
4. Ballmann, W., Eberlein, P.: Fundamental groups of manifolds of nonpositive curvature. J. Differ. Geom. **25**, 1–22 (1987)
5. Biswas, I., Mj, M.: One-relator Sasakian groups. Colloq. Math. **167**, 159–170 (2022)
6. Biswas, I., Fernández, M., Muñoz, V., Tralle, A.: On formality of Sasakian manifolds. J. Topol. **9**, 161–180 (2016)
7. Boothby, W.M., Wang, H.C.: On contact manifolds. Ann. Math. (2) **68**, 721–734 (1958)
8. Borel, A.: Compact Clifford–Klein forms of symmetric spaces. Topology **2**, 111–122 (1963)
9. Boyer, C., Galicki, K.: Sasakian Geometry. Oxford Mathematical Monographs, Oxford University Press, Oxford (2008)
10. Bregman, C.: On Kodaira firbations with invariant cohomology. Geom. Topol. **25**, 2385–2404 (2021)
11. Cappelletti-Montano, B., De Nicola, A., Yudin, I.: Hard Lefschetz theorem for Sasakian manifolds. J. Differ. Geom. **101**(1), 47–66 (2015)
12. Carlson, J.A., Toledo, D.: Harmonic mappings of Kähler manifolds to locally symmetric spaces. Publ. Math. I.H.E.S. **69**, 173–201 (1989)
13. Carlson, J.A., Toledo, D.: Quadratic presentations and nilpotent Kähler groups. J. Geom. Anal. **5**(3), 359–377 (1995)
14. Chen, X.: On the fundamental groups of compact Sasakian manifolds. Math. Res. Lett. **20**(1), 27–39 (2013)
15. Claudon, B., Höring, A., Lin, H.-Y.: The fundamental group of compact Kähler threefolds. Geom. Topol. **23**, 3233–3271 (2019)
16. de la Harpe, P., Kotschick, D.: Presentability by products for some classes of groups. J. Topol. Anal. **9**, 27–49 (2017)
17. Eberlein, P.: Euclidean de Rham factor of a lattice of nonpositive curvature. J. Differ. Geom. **18**, 209–220 (1983)
18. El Kacimi-Alaoui, A.: Opérateurs transversalement elliptiques sur un feuilletage Riemannien et applications. Compos. Math. **73**(1), 57–106 (1990)
19. Friedman, R.: Algebraic Surfaces and Holomorphic Vector Bundles. Springer, New York (1998)
20. Geiges, H.: Normal contact structures on 3-manifolds. Tohoku Math. J. (2) **49**(3), 415–422 (1997)
21. Gompf, R.E.: A new construction of symplectic manifolds. Ann. Math. (2) **142**(3), 527–595 (1995)
22. Haefliger, A.: Groupoïdes d'holonomie et classifiants, Transversal structure of foliations (Toulouse, 1982). Astérisque No. **116**, 70–97 (1984)
23. Halpern-Leistner, D.: Lefschetz hyperplane theorem for stacks. Preprint arxiv:1008.0891 [math.DG] (2010)
24. Hamilton, M.J.D.: Inequivalent contact structures on Boothby–Wang five-manifolds. Math. Z. **274**(3–4), 719–743 (2013)
25. Hamm, H., Tráng, L.D.: Lefschetz theorems on quasiprojective varieties. Bull. Soc. Math. Fr. **113**, 123–142 (1985)
26. Jaco, W.H.: Finitely presented subgroups of three-manifold groups. Invent. Math. **13**, 335–346 (1971)
27. Johnson, F.E.A., Rees, E.G.: On the fundamental group of a complex algebraic manifold. Bull. Lond. Math. Soc. **19**, 463–466 (1987)
28. Kasuya, H.: Cohomologies of Sasakian groups and Sasakian solvmanifolds. Ann. Mat. Pura Appl. (4) **195**(5), 1713–1719 (2016)
29. Kasuya, H.: Mixed Hodge structures and Sullivan's minimal models of Sasakian manifolds. Ann. Inst. Fourier (Grenoble) **67**(6), 2533–2546 (2017)
30. Kasuya, H.: Almost formality and deformations of representations of the fundamental groups of Sasakian manifolds. Ann. Mat. Pura Appl. **202**, 1793–1801 (2023)
31. Kollár, J.: Shafarevich maps and plurigenera of algebraic varieties. Invent. Math. **113**(1), 177–215 (1993)
32. Kotschick, D.: Three-manifolds and Kähler groups. Ann. Inst. Fourier (Grenoble) **62**(3), 1081–1090 (2012)
33. Kotschick, D.: The deficiencies of Kähler groups. J. Topol. **5**(3), 639–650 (2012)
34. Kotschick, D.: Kählerian three-manifold groups. Math. Res. Lett. **20**(3), 521–525 (2013)

35. Kotschick, D., Löh, C.: Fundamental classes not representable by products. J. Lond. Math. Soc. **79**, 545–561 (2009)
36. Llosa Isenrich, C.: Kähler Groups and Geometric Group Theory. D.Phil. Thesis, Oxford (2017)
37. Milnor, J.: A unique decomposition theorem for 3-manifolds. Am. J. Math. **84**, 1–7 (1962)
38. Napier, T., Ramachandran, M.: Thompson's Group F is not Kähler. Contemporary Mathematics, vol. 394. American Mathematical Society, Providence (2006)
39. Reznikov, A.: The structure of Kähler groups. I. Second Cohomology. In: Motives, Polylogarithms and Hodge Theory, Part II (Irvine, CA, 1998). International Press Lecture Series 3, II. International Press, Somerville (2002)
40. Ross, J., Thomas, R.: Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics. J. Differ. Geom. **88**, 109–159 (2011)
41. Rukimbira, P.: Chern–Hamilton's conjecture and K-contactness. Houston J. Math. **21**(4), 709–718 (1995)