



Polyharmonic hypersurfaces into complex space forms

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Abstract

We characterize homogeneous hypersurfaces in complex space forms which arise as critical points of a higher order energy functional. As a consequence, we obtain existence and non-existence results for $\mathbb{C}\mathbb{P}^n$ and $\mathbb{C}\mathbb{H}^n$, respectively. Moreover, we study the stability of biharmonic hypersurfaces and compute the normal index for a large family of solutions.

Keywords r -Harmonic maps · Hopf hypersurfaces · Stability

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1 Introduction

Harmonic maps are critical points of the *energy functional*

$$E(\psi) = \frac{1}{2} \int_M |\mathrm{d}\psi|^2 \mathrm{d}V_g,$$

where $\psi : M \rightarrow N$ is a smooth map between two Riemannian manifolds (M, g) and (N, h) . Equivalently, ψ is harmonic if and only if it is a solution to the Euler–Lagrange equation associated with this functional, namely:

$$\tau(\psi) := \mathrm{Tr} \nabla \mathrm{d}\psi = 0.$$

The section $\tau(\psi) \in \psi^*TN$ is called the *tension field* of ψ . In particular, if ψ is an isometric immersion, it is harmonic if and only if $\psi(M)$ is a minimal submanifold of N . We refer the reader to the work of Eells and Lemaire [9, 10] for background on the theory of harmonic maps.

The study of higher order functionals has already been proposed in [9] as a generalization to the classic energy. In the last decades, this topic has raised the interest of many mathematicians, leading to intriguing results from both the analytic and the geometric perspective, see for instance [3–5, 19, 20]. If $r = 2s$ and $s \geq 1$, we define the r -energy functional as

$$E_{2s}(\psi) := \frac{1}{2} \int_M |\Delta^{s-1} \tau(\psi)|^2 \mathrm{d}V_g.$$

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If, on the other hand, $r = 2s + 1$, then

$$E_{2s+1}(\psi) := \frac{1}{2} \int_M \sum_{j=1}^n |\nabla_{e_j} \Delta^{s-1} \tau(\psi)|^2 dV_g,$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame on M and $\Delta = d^*d$ denotes the rough Laplacian acting on sections of ψ^*TN . A polyharmonic map of order r (in short, r -harmonic) is a critical point for the r -energy functional. In particular, if ψ is an isometric immersion, we say that M is an r -harmonic submanifold of N . The biharmonic case ($r = 2$) has shown itself to be of special interest, and we refer to [8, 11, 23] for an introduction to this field.

Note that any harmonic map is automatically polyharmonic of any order. If, on the contrary, a map is a critical point for the r -energy functional but not for the classic energy, we refer to it as a proper r -harmonic map. Specifically, an r -harmonic submanifold is said to be proper if it is not minimal.

In this manuscript, we consider Hopf hypersurfaces in a complex space form (N, h, J) . A hypersurface M of N is said to be Hopf if $-J\xi$ is an eigenvector for the shape operator of M , that is to say, $S(-J\xi) = -\alpha J\xi$ where ξ is a local choice of unit normal vector and $\alpha \in \mathbb{R}$. Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ or $\mathbb{C}P^n$ have been classified in [2, 14]: they are all open parts of homogeneous hypersurfaces.

More specifically, the main results that we present here are the following.

Theorem A *Let M be an open part of a homogeneous hypersurface in a complex space form $N^n(c)$, $n \geq 2$. Then M is proper r -harmonic for $r \geq 2$ if and only if*

$$\frac{4}{c}(\text{Tr } S^2)^2 - 2(n + 1)(\text{Tr } S^2) = (r - 2) \text{Tr } S(\text{Tr } S + 3\alpha).$$

All these hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ can be seen as tubes of radius $t \in I$ over a complex submanifold. This interpretation yields the following result, where $\{M_t\}_{t \in I}$ denotes the family of hypersurfaces of tubes of radius t over a fixed complex submanifold.

Theorem B *Let $\{M_t\}_{t \in I}$ be a family of hypersurfaces in $\mathbb{C}P^n$ as above. There exist two natural numbers r' and r'' such that*

- (1) *If $r > r'$, the family $\{M_t\}_{t \in I}$ contains at least two proper r -harmonic hypersurfaces.*
- (2) *If $r > r''$, the family $\{M_t\}_{t \in I}$ contains exactly four proper r -harmonic hypersurfaces for some suitable choice of the complex submanifold.*

Explicit bounds for r' and r'' are given. Moreover, we show that $\mathbb{C}H^n$ does not admit any polyharmonic hypersurface of this type.

For the biharmonic case, we recover the classification of proper biharmonic homogeneous real hypersurfaces in $\mathbb{C}P^n$ given in [24], these are tubes over a totally geodesic $\mathbb{C}P^{n-p}$ in $\mathbb{C}P^n$ of certain radii t_- or t_+ , and obtain the following result regarding their stability.

Theorem C *Every homogeneous and proper biharmonic hypersurface in $\mathbb{C}P^n$ is unstable. Moreover, there exists $C \equiv C(p) > 0$ such that if $n - p > C$ the normal index of the biharmonic tube over $\mathbb{C}P^{n-p}$ in $\mathbb{C}P^n$ of radius t_+ is exactly 1.*

The organization of the document is as follows. In Sect. 2 we introduce some basic notions on polyharmonic maps and Hopf hypersurfaces. In Sect. 3 we prove Theorem A. Theorem B and explicit upper bounds for r' and r'' are given in Sect. 4. The proof of Theorem C is given in Sect. 5.

2 Preliminaries

2.1 Polyharmonic maps

We introduce here some basic concepts on polyharmonic maps and the Laplace operator used throughout the manuscript.

The *r-tension field*, τ_r , is a higher-order analog of the tension field, in the sense that the system of partial differential equations $\tau_r(\psi) = 0$ characterizes polyharmonic maps of order r . The following equations depict an explicit formula for the *r-tension field*, see [16] for a reference. Here, $\psi : M \rightarrow N$ denotes a smooth map between two Riemannian manifolds (M, g) and (N, h) . In order to simplify the notation, we write τ_r instead of $\tau_r(\psi)$.

If $r = 2s, s \geq 1$, then

$$\begin{aligned} \tau_{2s} := & \Delta^{2s-1}\tau - R^N(\Delta^{2s-2}\tau, d\psi e_j)d\psi e_j \\ & + \sum_{l=1}^{s-1} [R^N(\Delta^{s-l-1}\tau, \nabla_j \Delta^{s+l-2}\tau)d\psi e_j \\ & - R^N(\nabla_j \Delta^{s-l-1}\tau, \Delta^{s+l-2}\tau)d\psi e_j]. \end{aligned} \tag{2.1}$$

On the other hand, if $r = 2s$,

$$\begin{aligned} \tau_{2s+1} := & \Delta^{2s}\tau - R^N(\Delta^{2s-1}\tau, d\psi e_j)d\psi e_j \\ & - \sum_{l=1}^{s-1} [R^N(\nabla_j \Delta^{s+l-1}\tau, \Delta^{s-l-1}\tau)d\psi e_j \\ & - R^N(\Delta^{s+l-1}\tau, \nabla_j \Delta^{s-l-1}\tau)d\psi e_j] \\ & - R^N(\nabla_j \Delta^{s-1}\tau, \Delta^{s-1}\tau)d\psi e_j. \end{aligned} \tag{2.2}$$

Recall that the rough Laplacian reads $\Delta := -\text{Tr}(\nabla\nabla - \nabla\nabla)$.

A proper biharmonic hypersurface M in N is said to be *normally stable* if the second variation of the bienergy functional is non-negative for any normal variation with compact support. Ou [22] showed that the second variation of the bienergy for a normal variation, $\delta^2 E(f\xi, f\xi) = Q(f)$, is given by

$$\begin{aligned} Q(f) = & \int_M [f(\text{Tr} S_t^2 - \text{Ric}^N(\xi, \xi)) - \Delta f]^2 + |f \nabla \text{Tr} S_t - 2(\text{Ric}^N(\xi))^T + 2 S_t(\nabla f)|^2 dV_g \\ & + \int_M f^2 \text{Tr} S_t [(\nabla_\xi^N \text{Ric}^N)(\xi, \xi) - 2 \text{Tr} R^N(\xi, \cdot, \nabla_\xi^N(\cdot), \xi)] - 4 f^2 \text{Tr} S_t \text{Tr} R^N(\xi, S_t(\cdot), \cdot, \xi) dV_g \end{aligned} \tag{2.3}$$

where ξ is a normal vector field along M .

The Laplace operator Δ has proved to be useful in the study of the second variation of biharmonic submanifolds, as shown in [18, 22]. For convenience, we recall some properties of this elliptic operator and its spectrum, we use [7] as a reference.

Let M be a compact and connected manifold. The set of eigenvalues for

$$\Delta f + \mu f = 0,$$

where $f \in C^2(M)$, consists of a sequence

$$0 = \mu_0 < \mu_1 < \dots < \mu_\ell \rightarrow +\infty,$$

and each associated eigenspace is finite-dimensional. Eigenspaces belonging to distinct eigenvalues are orthogonal in $L^2(M)$, and $L^2(M)$ is the direct sum of all the eigenspaces. Furthermore, each eigenfunction is in $C^\infty(M)$.

In addition, the following result turned out to be useful for our purposes

Theorem 2.1 [12, Theorem 2.1] *Suppose that M is a compact orientable hypersurface embedded in a compact Riemannian manifold N . If the Ricci curvature of N is bounded below by a positive constant k , then $2\mu_1 > k - \max_M |\text{Tr } S|$ where μ_1 is the first nonzero eigenvalue of the Laplacian of M .*

2.2 Hopf hypersurfaces

We give in this subsection a brief introduction to the theory of Hopf hypersurfaces. We use [6] as a reference.

Let $(N, \langle \cdot, \cdot \rangle, J)$ be a Kähler manifold and σ be a plane in the tangent space $T_p N$, $p \in N$. We write

$$K_p(\sigma) = R(X, Y, Y, X)$$

for the sectional curvature, where $\{X, Y\}$ is an orthonormal basis of σ . If σ is invariant by the almost complex structure J , then $K_p(\sigma)$ is called the *holomorphic sectional curvature* of σ . In particular, if $K(\sigma)$ is constant for all planes σ in $T_p N$ invariant by J and for all points $p \in N$, then N is called a *space of constant holomorphic sectional curvature*.

The Riemannian curvature tensor for a space of constant holomorphic sectional curvature $c \neq 0$ can be expressed as

$$\frac{4}{c}R(X, Y, Z) = \langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle X, JZ \rangle JY - \langle Y, JZ \rangle JX + 2\langle X, JY \rangle JZ. \tag{2.4}$$

The sectional curvature for any plane σ in $T_p N$ spanned by two orthonormal vectors X, Y reads

$$K(\sigma) = \frac{c}{4}(1 + 3\langle X, JY \rangle^2).$$

The complex projective space endowed with the Fubini–Study metric or the complex hyperbolic space with the Bergman metric are examples of spaces with constant holomorphic sectional curvature. Moreover, any simply connected complete $2n$ -dimensional Kähler manifold of constant holomorphic sectional curvature c is holomorphically isometric to $\mathbb{C}\mathbb{P}^n(c)$, \mathbb{C}^n or $\mathbb{C}\mathbb{H}^n(c)$ depending if $c > 0$, $c = 0$ or $c < 0$, respectively. We refer the reader to [15] for an overview of the general aspects of complex geometry.

In what follows, we use N^n to denote either $\mathbb{C}\mathbb{H}^n$ or $\mathbb{C}\mathbb{P}^n$. Let M be an $(2n - k)$ -dimensional Riemannian manifold, $k < 2n$, and let $f : M \rightarrow N^n$ be a Riemannian immersion. Write BM for the bundle of unit normal vectors to $f(M)$ in N^n . We define the *tube of radius $t > 0$ over M* , denoted by M_t , as the image of the map $f_t : BM \rightarrow N^n$ defined by

$$f_t(p, \xi) = \exp_p(t\xi).$$

Note that given any $p \in M$, there is always a neighborhood U of p in M such that for all $t > 0$ sufficiently small, the restriction of f_t to BU is an immersion onto an $(2n - 1)$ -dimensional manifold.

Let now $M \subset N^n$ be a hypersurface and ξ a local choice of unit normal vector. We say that M is a *Hopf hypersurface* if

$$W = -J\xi$$

Table 1 Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}\mathbb{H}^n(-4)$

Type	Focal submanifold	Principal curvatures	Multiplicities
A_0	Horosphere in $\mathbb{C}\mathbb{H}^n$	$\alpha = 2$ $\lambda_1 = 1$	$m_\alpha = 1$ $m_1 = 2n - 2$
A_1	Totally geodesic $\mathbb{C}\mathbb{H}^{n-1}$	$\alpha = 2 \coth 2t$ $\lambda_1 = \tanh t$	$m_\alpha = 1$ $m_1 = 2n - 2$
A_1	$\mathbb{C}\mathbb{H}^0$	$\alpha = 2 \coth 2t$ $\lambda_1 = \coth t$	$m_\alpha = 1$ $m_1 = 2n - 2$
A_2	Totally geodesic $\mathbb{C}\mathbb{H}^k$ $1 \leq k \leq n - 2$	$\alpha = 2 \coth 2t$ $\lambda_1 = \coth t$ $\lambda_2 = \tanh t$	$m_\alpha = 1$ $m_1 = 2(n - k - 1)$ $m_2 = 2k$
B^*	Totally geodesic $\mathbb{R}\mathbb{H}^n$	$\alpha = 2 \tanh 2t$ $\lambda_1 = \coth t$ $\lambda_2 = \tanh t$	$m_\alpha = 1$ $m_1 = n - 1$ $m_2 = n - 1$

*The radius $t = \frac{1}{2} \ln(2 + \sqrt{3})$ is not allowed

is a principal vector for the shape operator S , that is, $S W = \alpha W$ where $\alpha \in \mathbb{R}$ is called the *Hopf principal curvature*. We will refer to W as the *structure vector*. In a Hopf hypersurface M of $\mathbb{C}\mathbb{H}^n$ or $\mathbb{C}\mathbb{P}^n$, the Hopf principal curvature remains constant [6, Theorem 6.16].

In particular, Hopf hypersurfaces for which all their principal curvatures are constant have been classified by Kimura [2] in $\mathbb{C}\mathbb{H}^n$ and by Berndt [14] in $\mathbb{C}\mathbb{P}^n$. They are all tubes of a certain radius, as shown in Table 1 for $\mathbb{C}\mathbb{H}^n$ and Table 2 for $\mathbb{C}\mathbb{P}^n$.

The following property turned out to be useful for our purposes.

Lemma 2.2 [6, Lemma 8.1] *Let M be a Hopf hypersurface with constant principal curvatures in a complex space form N^n . For all eigenvalues λ, μ which are not the Hopf principal curvature, we have*

$$\nabla_X Y \perp T_\lambda \text{ if } X \in T_\lambda, Y \in T_\mu, \lambda \neq \mu,$$

where T_λ, T_μ are the corresponding eigendistributions and ∇ is the Levi-Civita connection on M .

In what follows we write ‘‘Hopf hypersurface’’ instead of ‘‘Hopf hypersurface with constant principal curvatures’’ since we will not deal with the general case.

2.3 Quartic polynomials

We found it convenient to briefly recall here some general properties of polynomials and their roots.

A general quartic equation over \mathbb{R} is any equation of the form

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \tag{2.5}$$

where $a_4, a_3, a_2, a_1, a_0 \in \mathbb{R}$ and $a_4 \neq 0$. If we divide every term by a_4 and apply the change of variable

$$y = x - \frac{a_3}{4a_4},$$

Table 2 Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}\mathbb{P}^n(4)$

Type	Focal submanifold	Principal curvatures	Multiplicities
A_1	Totally geodesic $\mathbb{C}\mathbb{P}^{n-1}$	$\alpha = 2 \cot 2t$ $\lambda_1 = -\tan t$	$m_\alpha = 1$ $m_1 = 2n - 2$
A_2	Totally geodesic $\mathbb{C}\mathbb{P}^k$ $1 \leq k \leq n - 2$	$\alpha = 2 \cot 2t$ $\lambda_1 = \cot t$ $\lambda_2 = -\tan t$	$m_\alpha = 1$ $m_1 = 2(n - k - 1)$ $m_2 = 2k$
B	Totally geodesic $\mathbb{R}\mathbb{P}^n$	$\alpha = 2 \tan 2t$ $\lambda_1 = -\cot t$ $\lambda_2 = \tan t$	$m_\alpha = 1$ $m_1 = n - 1$ $m_2 = n - 1$
C	Segre embedding $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^k \hookrightarrow \mathbb{C}\mathbb{P}^n,$ $k \geq 2, n = 2k + 1$	$\alpha = 2 \cot 2t$ $\lambda_1 = \cot(t - \frac{\pi}{4})$ $\lambda_2 = \cot(t - \frac{3\pi}{4})$ $\lambda_3 = \cot(t - \frac{\pi}{2})$ $\lambda_4 = \cot(t)$	$m_\alpha = 1$ $m_1 = 2$ $m_2 = 2$ $m_3 = n - 3$ $m_4 = n - 3$
D	Plücker embedding $G_2(\mathbb{C}^5) \hookrightarrow \mathbb{C}\mathbb{P}^9$	$\alpha = 2 \cot 2t$ $\lambda_1 = \cot(t - \frac{\pi}{4})$ $\lambda_2 = \cot(t - \frac{3\pi}{4})$ $\lambda_3 = \cot(t - \frac{\pi}{2})$ $\lambda_4 = \cot(t)$	$m_\alpha = 1$ $m_1 = 4$ $m_2 = 4$ $m_3 = 4$ $m_4 = 4$
E	Half spin embedding $SO(10)/U(5) \hookrightarrow \mathbb{C}\mathbb{P}^{15}$	$\alpha = 2 \cot 2t$ $\lambda_1 = \cot(t - \frac{\pi}{4})$ $\lambda_2 = \cot(t - \frac{3\pi}{4})$ $\lambda_3 = \cot(t - \frac{\pi}{2})$ $\lambda_4 = \cot(t)$	$m_\alpha = 1$ $m_1 = 6$ $m_2 = 6$ $m_3 = 8$ $m_4 = 8$

then (2.5) reads

$$y^4 + p_2y^2 + p_1y + p_0 = 0$$

where

$$p_2 = \frac{8a_4a_2 - 3b_3^2}{8a_4^2}, \quad p_1 = \frac{a_3^3 - 4a_4a_3a_2 + 8a_4^2a_1}{8a_4^3},$$

$$p_0 = \frac{16a_4a_3^2a_2 - 64a_4^2a_3a_1 - 3a_3^4 + 256a_4^3a_0}{256a_4^4}.$$

In particular, if $p_1 = 0$ then (2.5) can be reduced to a biquadratic equation and solved by

$$y_{1,2} = \pm \sqrt{\frac{-p_2 + \sqrt{p_2^2 - 4p_0}}{2}}, \quad y_{3,4} = \pm \sqrt{\frac{-p_2 - \sqrt{p_2^2 - 4p_0}}{2}}.$$

We will make use of the so-called *Cauchy bound* for the roots of a polynomial (see, for example [17, Chapter VII]), which states that an upper bound of the absolute value on the

roots of (2.5) is given by

$$1 + \max \left\{ \left| \frac{a_3}{a_4} \right|, \left| \frac{a_2}{a_4} \right|, \left| \frac{a_1}{a_4} \right|, \left| \frac{a_0}{a_4} \right| \right\}.$$

3 Polyharmonic equation

The goal of this section is to obtain the r -harmonic equation for the isometric immersion of a Hopf hypersurface into $N^n(c)$. We write

$$i_t : M_t \hookrightarrow N^n(c)$$

for the inclusion of the hypersurface M_t arising as the tube of radius t over the corresponding complex submanifold. We use e_0 for the local unit normal vector field to M_t in $N^n(c)$ such that if we flow in the direction of e_0 , the radius of the tube increases. Every object in M_t is written with the subscript t , such as the shape operator S_t or the Hopf principal curvature α_t .

Theorem 3.1 *A Hopf hypersurface M_t in $N^n(c)$ is r -harmonic, $r \geq 2$, if and only if it is minimal or*

$$\frac{4}{c}(\text{Tr } S_t^2)^2 - 2(n + 1)(\text{Tr } S_t^2) - (r - 2)(\text{Tr } S_t)^2 - 3\alpha_t(r - 2) \text{Tr } S_t = 0. \tag{3.1}$$

In order to prove the theorem above, we first state a couple of auxiliary identities. The first lemma below shows that for any $m \in \mathbb{N}$ the tension field $\tau(i_t)$ is an eigenvector for the m^{th} -iterated rough Laplacian.

Lemma 3.2 *For every $m \in \mathbb{N}$, the following identity holds:*

$$\Delta^m \tau(i_t) = (\text{Tr } S_t^2)^m \tau(i_t).$$

Proof We proceed by induction on m . Take an orthonormal basis $\{e_i\}_{i=1}^{2n-1}$ of $T_p M_t$ and extend it to a local orthonormal frame in a neighborhood of p . Since $\Delta^0 \tau \equiv \tau$, by definition of the tension field we get:

$$\begin{aligned} \tau(i_t) &= \sum_{i=1}^{2n-1} \nabla_{e_i} di_t e_i - di_t \nabla_{e_i} e_i = \sum_{i=1}^{2n-1} \nabla_{e_i} di_t e_i - \sum_{j=1}^{2n-1} \langle \nabla_{e_i} e_i, e_j \rangle di_t e_j \\ &= \sum_{i=1}^{2n-1} \langle \nabla_{e_i} di_t e_i, e_0 \rangle e_0 + \sum_{j=1}^{2n-1} [\langle \nabla_{e_i} di_t e_i, di_t e_j \rangle - \langle \nabla_{e_i} e_i, e_j \rangle] di_t e_j = \text{Tr } S_t e_0, \end{aligned}$$

where in the last identity we used the fact that i_t is an isometric embedding and applied the Gauss formula. Assume that the property holds for an arbitrary m , then

$$\Delta^{m+1} \tau = \Delta(\beta_m(t)e_0),$$

where, in order to simplify the notation, we write $\beta_m(t) = \text{Tr } S_t (\text{Tr } S_t^2)^m$. Note that since the principal curvatures are constant on M_t , then β_m is constant on M_t . By definition of the rough Laplacian, we obtain:

$$\Delta^{m+1} \tau = \sum_{i=1}^{2n-1} \underbrace{\nabla_{\nabla_{e_i} e_i} \beta_m e_0}_{(A)} - \underbrace{\nabla_{e_i} \nabla_{e_i} \beta_m e_0}_{(B)}.$$

Let us start with the term (A). We have

$$\sum_{i=1}^{2n-1} \nabla_{\nabla_{e_i} e_i} \beta_m e_0 = \beta_m \sum_{i,j=1}^{2n-1} \langle \nabla_{e_i} e_i, e_j \rangle \nabla_{e_j} e_0 = -\beta_m \sum_{i,j=1}^{2n-1} \langle \nabla_{e_i} e_i, e_j \rangle S_t(di_t e_j), \tag{3.2}$$

and therefore

$$\sum_{i=1}^{2n-1} \langle \nabla_{\nabla_{e_i} e_i} \beta_m e_0, e_0 \rangle = 0. \tag{3.3}$$

For part (B) we get

$$\sum_{i=1}^{2n-1} \nabla_{e_i} \nabla_{e_i} \beta_m e_0 = \beta_m \sum_{i=1}^{2n-1} \nabla_{e_i} \nabla_{e_i} e_0 = -\beta_m \sum_{i=1}^{2n-1} \nabla_{e_i} S_t(di_t e_i), \tag{3.4}$$

thus

$$\sum_{i=1}^{2n-1} \langle \nabla_{e_i} \nabla_{e_i} \beta_m e_0, e_0 \rangle = -\beta_m \sum_{i=1}^{2n-1} \langle \nabla_{e_i} S_t(di_t e_i), e_0 \rangle = -\beta_m \text{Tr} S_t^2. \tag{3.5}$$

Once we prove that the tangential part vanishes, Eqs. (3.3) and (3.5) give the result. Joining (3.2) and (3.4), we have that

$$(\Delta^m \tau(i_t))^{\text{tan}} = \beta_m \sum_{i=1}^{2n-1} \left(\nabla_{e_i} S_t(di_t e_i) - \sum_{j=1}^{2n-1} \langle \nabla_{e_i} e_i, e_j \rangle S_t(di_t e_j) \right)^{\text{tan}}. \tag{3.6}$$

This expression does not depend on the choice of the basis, since the normal part does neither. Take then an orthonormal basis $\{E_i\}_{i=1}^{2n-1}$ of $T_p M_t$ such that $\{di_t E_1 = W, di_t E_2, \dots, di_t E_{2n-1}\}$ forms an orthonormal basis of principal vectors of $T_{i_t(p)} i_t(M_t)$ with respective principal curvatures $\{\lambda_1 = \alpha, \lambda_2, \dots, \lambda_{2n-1}\}$. Extend $\{E_i\}_{i=1}^{2n-1}$ to a local orthonormal frame, then expression (3.6) reads

$$(\Delta^m \tau(i_t))^{\text{tan}} = \beta_m \sum_{i=1}^{2n-1} \left(\lambda_i \nabla_{E_i} di_t E_i - \sum_{j=1}^{2n-1} \lambda_j \langle \nabla_{E_i} E_i, E_j \rangle di_t E_j \right)^{\text{tan}},$$

where we used the property that the eigenvalues are constant on $i_t(M_t)$. Equivalently,

$$\begin{aligned} (\Delta^m \tau(i_t))^{\text{tan}} &= \beta_m \sum_{i,j=1}^{2n-1} [\lambda_i \langle \nabla_{E_i} di_t E_i, di_t E_j \rangle - \lambda_j \langle \nabla_{E_i} E_i, E_j \rangle] di_t E_j \\ &= \beta_m \sum_{i,j=1}^{2n-1} [\lambda_i - \lambda_j] \langle \nabla_{E_i} di_t E_i, di_t E_j \rangle di_t E_j \end{aligned}$$

by the Gauss formula.

It is clear that the terms with $\lambda_i = \lambda_j$ vanish. Moreover, Lemma 2.2 ensures that if $\lambda_i \neq \lambda_j$, and none of them are the Hopf curvature, then $\langle \nabla_{E_i} di_t E_i, di_t E_j \rangle = 0$. Hence, only those terms involving the Hopf curvature, α , are left. In this case,

$$\langle \nabla_{E_1} W, di_t E_j \rangle = -\langle \nabla_{E_1} J e_0, di_t E_j \rangle = -\langle J \nabla_{E_1} e_0, di_t E_j \rangle = \alpha \langle J W, di_t E_j \rangle = 0,$$

and

$$\langle \nabla_{E_i} di_t E_i, W \rangle = -\langle di_t E_i, \nabla_{E_i} W \rangle = \langle di_t E_i, \nabla_{E_i} J e_0 \rangle = -\lambda_i \langle di_t E_i, J di_t E_i \rangle = 0.$$

This is, the tangential part vanishes, as we wanted to show. □

Lemma 3.3 *For any $\ell, m \in \mathbb{N}$, the following identities hold:*

$$\sum_{j=1}^{2n-1} R(\Delta^m \tau, di_t e_j, di_t e_j, e_0) = \frac{c}{4} 2(n+1) \beta_m(t),$$

$$\sum_{j=1}^{2n-1} R(\Delta^m \tau, \nabla_{e_j} \Delta^\ell \tau, di_t e_j, e_0) = -\frac{c}{4} \beta_m(t) \beta_\ell(t) [\text{Tr } S_t + 3\alpha_t],$$

where $\beta_\ell(t) = \text{Tr } S_t (\text{Tr } S_t^2)^\ell$, $\ell \in \mathbb{N}$, and $\{e_i\}_{i=1}^{2n-1}$ is any local orthonormal frame on M_t .

Proof Note that both terms are independent of the choice of the orthonormal basis. Take then $\{E_i\}_{i=1}^{2n-1}$ an orthonormal basis of $T_p M_t$ such that $\{di_t E_1 = W, di_t E_2, \dots, di_t E_{2n-1}\}$ is an orthonormal basis of principal vectors in $T_{i_t(p)} i_t(M_t)$, and extend it to a local orthonormal frame in M_t . Using formula (2.4) for the curvature tensor of a complex space form we obtain

$$\frac{4}{c} R(e_0, W)W = e_0 + 3\langle e_0, JW \rangle JW = 4e_0.$$

On the other hand, if $j \neq 2$,

$$\frac{4}{c} R(e_0, di_t E_j) di_t E_j = e_0 + 3\langle e_0, J di_t E_j \rangle J di_t E_j = e_0,$$

where in the last identity we used that $\langle e_0, J di_t E_j \rangle = \langle W, di_t E_j \rangle = 0$. Then the result follows from a direct computation after noting that $\nabla_{E_j} \Delta^m \tau = -\beta_m \lambda_j di_t E_j$. □

Proof (Proof of Theorem 3.1) By Eq. (2.1), Lemmas (3.2) and (3.3) we get

$$\tau_{2s}(i_t) = (\text{Tr } S_t^2)^{2s-3} [(\text{Tr } S_t^2)^2 - 2(n+1) \frac{c}{4} (\text{Tr } S_t^2) - 2(s-1) \frac{c}{4} (\text{Tr } S_t)^2 - 6 \frac{c}{4} \alpha_t (s-1) \text{Tr } S_t] \tau(i_t).$$

If the hypersurface is minimal, then $\text{Tr } S_t = 0$, so $\tau_{2s}(i_t)$ vanishes. Note that $\text{Tr } S_t^2 = 0$ would also imply $\text{Tr } S_t = 0$. In any other case, we obtain the equation stated in the theorem. A similar computation applies for the $(2s + 1)$ -tension field (2.2). □

Corollary 3.4 *There are no proper r -harmonic Hopf hypersurfaces in $\mathbb{C}\mathbb{H}^n(c)$.*

Proof By Theorem 3.1 and since $c < 0$, we have that

$$\frac{4}{c} (\text{Tr } S_t^2)^2 - 2(n+1) (\text{Tr } S_t^2) - (r-2) (\text{Tr } S_t)^2 - 3\alpha_t (r-2) \text{Tr } S_t < 0$$

due to the fact that the principal curvatures of any hypersurface listed in Table 1 stay positive for $t > 0$. □

4 Existence of proper polyharmonic hypersurfaces in $\mathbb{C}\mathbb{P}^n$

Let M_t be a Hopf hypersurface in $\mathbb{C}\mathbb{P}^n(c)$. Note that the eigenvalues of the shape operator scale with a factor of \sqrt{c} , so Eq. (3.1) is invariant with respect to the choice of the parameter c . That is to say, the existence of r -harmonic hypersurfaces is not affected by the holomorphic sectional curvature. We will then take $c = 4$ to be consistent with Table 2, so the equation obtained in the last section reads:

$$(\text{Tr } S_t^2)^2 - 2(n+1) (\text{Tr } S_t^2) - (r-2) (\text{Tr } S_t)^2 - 3\alpha_t (r-2) \text{Tr } S_t = 0. \tag{4.1}$$

Table 3 Polynomials characterizing r -harmonicity

Type	Variable	Coefficients	
A_1	$x = \sin^2 t$	$a_4 = 4(n^2 + 3n)r - 8(n - 1)$ $a_3 = -2(2n^2 + 11n + 3)r$ $+ 4(n^2 + 3n - 4)$ $a_2 = 10(n + 1)r - 2(3n - 5)$	$a_1 = -4r - 2(n + 1)$ $a_0 = 1$
A_2	$x = \cos^2 t$	$a_4 = 4(n^2 + 3n)r - 8(n - 1)$ $a_3 = -2(2n^2 + (4k + 11)n + 6k + 3)r$ $+ 4(n^2 - (2k - 3)n - 4)$ $a_2 = 2((4k + 5)n + 2k^2 + 11k + 5)r$ $+ 2((2k - 3)n + 4k^2 + 4k + 5)$	$a_1 = -2(2k^2 + 5k + 2)r$ $- 2((2k + 1)n + (2k + 1)^2)$ $a_0 = (2k + 1)^2$
B	$x = \cos^2 2t$	$a_4 = n(n + 3)r - 2(n - 1)$ $a_3 = -(n^2 + 8n + 3)r + 4n^2 + 2n - 10$ $a_2 = (5n + 7)r - 2(5n - 11)$	$a_1 = -4r + 2(n - 7)$ $a_0 = 4$
C	$x = \cos^2 2t$	$a_4 = n(n + 3)r - 2(n - 1)$ $a_3 = -(n^2 + 7n + 6)r + 4(n^2 - 3n - 4)$ $a_2 = 2(2n + 5)r - 2(3n - 41)$	$a_1 = -4r + 4(n - 17)$ $a_0 = 16$
D	$x = \cos^2 2t$	$a_4 = 27r - 4$ $a_3 = -48r + 11$ $a_2 = 25r + 46$	$a_1 = -4r + 44$ $a_0 = 16$
E	$x = \cos^2 2t$	$a_4 = 135r - 14$ $a_3 = -234r + 100$ $a_2 = 117r + 184$	$a_1 = -18r - 180$ $a_0 = 72$

By Theorem (3.1), a lengthy computation shows that there is a one-to-one correspondence between r -harmonic Hopf hypersurfaces and roots of the polynomial $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where the coefficients a_4, a_3, a_2, a_1, a_0 and the variable x are given in Table 3.

Remark 4.1 For the cases of type $C, D,$ and $E,$ it is convenient to write everything in terms of $2t$ instead of $t.$ To this end, we found the following trigonometric identities useful:

$$\sum_{k=0}^3 \cot(t - k\frac{\pi}{4}) = 4 \cot 4t, \quad \sum_{k=0}^3 \cot^2(t - k\frac{\pi}{4}) = 12 + 16 \cot^2 4t.$$

Note that if $n = 1,$ and we plug the Hopf principal curvature of a type A_1 Hopf hypersurface ($\alpha_t = 2 \cot 2t$) in Eq. (4.1), one easily sees that the tube around a point of radius t is a proper r -harmonic curve if and only if

$$\sin^2 2t = r^{-1}.$$

Since $\mathbb{C}P^1$ is just the Riemann sphere, this agrees with [21, Theorem 1.1]. From now on, unless otherwise indicated, we will assume $n \geq 2.$

We point out that, in general, there are two values of t which will be of special interest since they make the left-hand side of (4.1) independent of $r:$ these are the solutions for

$$\text{Tr } S_t = 0,$$

which yields minimal hypersurfaces, and

$$\text{Tr } S_t + 3\alpha_t = 0.$$

Theorem 4.2 *Let M_t , $0 < t < \frac{\pi}{2}$, be a family of type A_1 Hopf hypersurfaces in $\mathbb{C}\mathbb{P}^n$, $n \geq 2$. Then:*

- (1) *The family M_t contains at least two proper r -harmonic hypersurfaces for $r \geq 2$.*
- (2) *If $r \geq 2n + 13$, the family M_t contains exactly four proper r -harmonic hypersurfaces.*

Proof The solution for $\text{Tr } S_t = 0$ is given by

$$t_0 = \arctan\left(\frac{1}{2n-1}\right)^{\frac{1}{2}}.$$

Define then $x_0 = \sin^2 t_0 = (2n)^{-1}$. After simplifying, we get that

$$2n^4 P_{A_1}(x_0) = -(n-1)(2n-1)^2 < 0,$$

where $P_{A_1}(x)$ is defined in Table 3. Thus, by Theorem 3.1, every zero x^* of $P_{A_1}(x)$ lying in the interval $(0, 1)$ corresponds to a proper r -harmonic tube of radius $t^* = \arcsin \sqrt{x^*}$ over a totally geodesic $\mathbb{C}\mathbb{P}^{n-1}$. Since $P_{A_1}(0) = 1$, $P_{A_1}(1) = (2n-1)^2$ and $P_{A_1}(x_0) < 0$ with $0 < x_0 < 1$, we obtain 4.2 (1).

For 4.2 (2), the equation $\text{Tr } S_t + 3\alpha_t = 0$ is solved by

$$t_2 = \arctan\left(\frac{2}{n+1}\right)^{\frac{1}{2}},$$

so it is natural to define $x_2 = \sin^2 t_2 = 2(n+3)^{-1}$, obtaining

$$(n+3)^4 P_{A_1}(x_2) = -(3n^2 + 2n + 11)(n+7)(n-1) < 0.$$

The goal now is to define x_1 in such a way that we have $0 < x_0 < x_1 < x_2 < 1$ with $P_{A_1}(0), P_{A_1}(x_1), P_{A_1}(1) > 0$ and $P_{A_1}(x_0), P_{A_1}(x_2) < 0$. By continuity, this would yield 4.2 (2).

Take $x_1 = x_0 + (nr)^{-1}$, and write

$$Q(r) := 2n^4 r^4 P_{A_1}(x_1) = b_4 r^4 + b_3 r^3 + b_2 r^2 + b_1 r + b_0$$

where

$$\begin{aligned} [I]b_4 &= (2n-1)(n^2-1), & b_1 &= 8(n^3+4n^2-5n+4), \\ b_3 &= -2(2n^4+n^3-2n^2+7n-4), & b_0 &= -16(n-1), \\ b_2 &= -4(2n^3-7n^2+9n-6). \end{aligned}$$

Since $Q(r)$ is a fourth-degree polynomial with $b_4 > 0$, it suffices to find a bound for its largest root. A classic analysis shows that for $n \geq 4$:

$$\max \left\{ \left| \frac{b_0}{b_4} \right|, \left| \frac{b_1}{b_4} \right|, \left| \frac{b_2}{b_4} \right|, \left| \frac{b_3}{b_4} \right| \right\} = \left| \frac{b_0}{b_4} \right| < 2(n+3).$$

For $n = 2$ and $n = 3$, $\left| \frac{b_1}{b_4} \right|$ dominates, but since this fraction is bounded by 16 for every $n \geq 2$, we can modify the former bound so that it also covers these cases. For instance,

$$\max \left\{ \left| \frac{b_0}{b_4} \right|, \left| \frac{b_1}{b_4} \right|, \left| \frac{b_2}{b_4} \right|, \left| \frac{b_3}{b_4} \right| \right\} < 2(n+6).$$

Hence, using the Cauchy bound, we obtain that condition $r \geq 2n + 13$ suffices to ensure $P_{A_1}(x_1) < 0$, obtaining the result. □

Theorem 4.3 *Let M_t , $0 < t < \frac{\pi}{4}$, be a family of type B Hopf hypersurfaces in $\mathbb{C}P^n$, $n \geq 2$. Then:*

- (1) *If $r \geq \min\{6001, 12n^2 + 16n - 19\}$, the family M_t contains at least two proper r -harmonic hypersurfaces.*
- (2) *If $r \geq \max\{6001, 12n^2 + 16n - 19\}$, the family M_t contains exactly four proper r -harmonic hypersurfaces.*

Proof In this case, we have that $P_B(0) = 4$ and $P_B(1) = 4(n - 1)^2$. The equation $\text{Tr } S_t = 0$ is solved by

$$t_1 = \frac{1}{2} \arctan(\sqrt{n - 1}),$$

and if we plug $x_1 = \cos^2 2t_1 = n^{-1}$ in P_B we see that

$$n^4 P_B(x_1) = 2(3n - 1)(n - 1)^3 > 0.$$

Thus, this is not enough to ensure the existence of proper r -harmonic hypersurfaces. To solve this problem, define

$$x_0 = 2r^{-1} \quad \text{and} \quad x_2 = 1 - 5r^{-1}.$$

Then we have $P_B(0), P_B(x_1), P_B(1) > 0$ and

$$\begin{aligned} P_B(x_0) &= -4 + O(r^{-1}), \\ P_B(x_2) &= -(n - 1)^2 + O(r^{-1}) \end{aligned}$$

as $r \rightarrow \infty$. Hence, a similar argument as in the proof of the previous theorem using the Cauchy bound shows that, to ensure $P_B(x_0) < 0$, it is enough to ask for

$$r \geq 12n^2 + 16n - 19.$$

On the other hand, for $P_B(x_0) < 0$ to hold it suffices to take

$$r \geq 126 + \frac{125(23n + 1)}{(n - 1)^2}.$$

Since the right hand side of the inequality above is bounded by 6001, the result follows. \square

Theorem 4.4 *Let M_t , $0 < t < \frac{\pi}{4}$, be a family of type C Hopf hypersurfaces in $\mathbb{C}P^n$, $n \geq 5$. Then:*

- (1) *If $r \geq 300$, the family M_t contains at least two proper r -harmonic hypersurfaces.*
- (2) *If $4r \geq 1125n^2 + 375n - 1996$, the family M_t contains exactly four proper r -harmonic hypersurfaces.*

Proof Note that $P_C(0) = 16$ and $P_C(1) = 4(n - 2)^2$. Define

$$x_0 = 5r^{-1}, \quad x_1 = 2n^{-1} \quad \text{and} \quad x_2 = 1 - 4r^{-1}.$$

Since

$$\begin{aligned} P_C(x_0) &= -4 + O(r^{-1}), \\ P_C(x_2) &= -12(n - 2) + O(r^{-1}) \end{aligned}$$

as $r \rightarrow \infty$, and

$$n^4 P_C(x_1) = 8(3n - 1)(n - 1)(n - 2)^2,$$

the result comes after using the Cauchy bound as we did before. \square

Theorem 4.5 Let $M_t, 0 < t < \frac{\pi}{4}$, be a family of type D Hopf hypersurfaces in $\mathbb{C}\mathbb{P}^9$. Then:

- (1) If $r \geq 32$, the family M_t contains at least two proper r -harmonic hypersurfaces.
- (2) If $r \geq 89$, the family M_t contains exactly four proper r -harmonic hypersurfaces.

Proof Note that $P_D(0) = 16$ and $P_D(1) = 25$. Define

$$x_0 = 5r^{-1}, \quad x_1 = \frac{4}{9} \quad \text{and} \quad x_2 = 1 - 3r^{-1}.$$

Since $P_D(x_1) > 0$, the only conditions we need to ensure are $P_D(x_0) < 0$ and $P_D(x_2) < 0$. A simple computation shows

$$L(r) := \frac{d^2}{dr^2} r^4 P_D(Z_0) = -48r^2 + 2430r - 9700.$$

Hence, we know that if $r \geq 47$ then the function $L(r)$ is concave on r . As $P_D(88) > 0 > P_D(89)$, we obtain that $P_D(x_0) < 0$ if $r \geq 89$. A similar argument applies to the other condition. □

Theorem 4.6 Let $M_t, 0 < t < \frac{\pi}{4}$, be a family of type E Hopf hypersurfaces in $\mathbb{C}\mathbb{P}^{15}$. Then:

- (1) If $r \geq 27$, the family M_t contains at least two proper r -harmonic hypersurfaces.
- (2) If $r \geq 100$, the family M_t contains exactly four proper r -harmonic hypersurfaces.

Proof The proof follows in the same way as in the previous theorem after defining

$$x_0 = 5r^{-1}, \quad x_1 = \frac{2}{5} \quad \text{and} \quad x_2 = 1 - 4r^{-1}.$$

□

Only the case of type A_2 hypersurfaces is left. Even though this is the richest case, the dependence on one additional parameter k leads to some extra difficulties. In [1], the author constructed two uncountable families of explicit harmonic selfmaps by translation of these hypersurfaces in the normal direction.

Theorem 4.7 Let $M_t, 0 < t < \frac{\pi}{2}$, be a family of type A_2 Hopf hypersurfaces in $\mathbb{C}\mathbb{P}^n$, $n \geq 2$, $1 \leq k \leq n - 2$. Then:

- (1) The family M_t contains at least two proper r -harmonic hypersurfaces for $r \geq 2$. If, in particular, $2k = n - 1$, these hypersurfaces are tubes of radius t over a totally geodesic $\mathbb{C}\mathbb{P}^{\frac{n-1}{2}}$, with t given by

$$\cos 4t = \frac{n\sqrt{\omega} - 2(2n - 1)(n + 1)}{n(n + 3)r - 2(n - 1)}$$

where

$$\omega = (n + 3)^2 r^2 - 8n(n + 3)r + 16(n^2 + 2n - 2).$$

- (2) Let

$$k_1 := \frac{5n^2 - 4n + 2 - n\sqrt{13n^2 - 8n + 4}}{4(n - 1)}.$$

If $k < k_1$ and

$$r \geq 4(22k^4 + 85k^3 + 123k^2 + 54k + 8)k^2,$$

the family M_t contains exactly four proper r -harmonic hypersurfaces.

(3) *Let*

$$k_2 := \frac{n\sqrt{13n^2 - 8n + 4} - n^2 - 4n + 2}{4(n - 1)}.$$

If $k > k_2$ and

$$r \geq 4(6k^4 + 19k^3 + 39k^2 + 8k + 2)(2k + 1)k,$$

the family M_r contains exactly four proper r -harmonic hypersurfaces.

Proof If $2k = n - 1$, equation $P_{A_2}(x) = 0$ reads

$$4[n(n + 3)r - 2(n - 1)]x^4 - 8[n(n + 3)r - 2(n - 1)]x^3 + [5n(n + 3)r + 4(n^2 - 2n + 2)]x^2 - [n(n + 3)r + 4n^2]x + n^2 = 0. \tag{4.2}$$

Note that, in this case, the relation $a_3^3 - 4a_4a_3a_2 + 8a_4^2a_1 = 0$ holds for the coefficients of $P_{A_2}(x)$. Thus (4.2) can be reduced to a biquadratic equation, obtaining that the real zeros are given by

$$x_{\pm} = \frac{1}{2} \pm \frac{1}{2}[2n(n + 3)r - 4(n - 1)]^{-\frac{1}{2}}[n(n + 3)r - 4(n^2 + n - 1) + n\sqrt{\omega}]^{\frac{1}{2}}.$$

Fix now $k \neq \frac{n-1}{2}$. Note that $P_{A_2}(0) = (2k + 1)^2$ and $P_{A_2}(1) = (2k - 2n + 1)^2$, so if we find $x_1 \in (0, 1)$ such that $P_{A_2}(x_1) < 0$, we will have ensured the existence of at least two r -harmonic hypersurfaces. The equation $\text{Tr } S_r = 0$ is solved by

$$t^* = \arctan \left(\frac{2n-2k-1}{2k+1} \right)^{\frac{1}{2}}.$$

Plugging $x^* = \cos^2 t^* = (2k + 1)(2n)^{-1}$ in P_{A_2} we obtain

$$2n^4 P_{A_2}(x^*) = -(n - 1)(2n - 2k - 1)^2(2k + 1)^2 < 0,$$

from where 4.7 (1) follows.

Assume now that $k < k_1$, where k_1 is defined as stated in the theorem, and define

$$x_0 = x^*, \quad x_1 = x^* + r^{-1} \quad \text{and} \quad x_2 = 1 - r^{-1}.$$

A direct computation shows that

$$2n^4 P_{A_2}(x_1) = (2k + 1)(2n - 2k - 1)\eta_1(n, k) + O(r^{-1})$$

$$P_{A_2}(x_2) = -3(2n - 2k - 1) + O(r^{-1})$$

as $n, r \rightarrow \infty$, where

$$\eta_1(n, k) = 4(n - 1)k^2 - (10n^2 - 8n + 4)k + 3n^3 - 5n^2 + 3n - 1,$$

which satisfies $\eta_1(n, k) > 0$ since $k < k_1$.

On the other hand, if $k > k_2$, take

$$x_0 = r^{-1}, \quad x_1 = x^* - r^{-1} \quad \text{and} \quad x_2 = x^*.$$

Hence

$$P_{A_2}(x_0) = -3(2k + 1) + O(r^{-1})$$

$$2n^4 P_{A_2}(x_1) = \eta_2(n, k)(2n - 2k + 1)(2k + 1) + O(r^{-1})$$

as $n, r \rightarrow \infty$, where

$$\eta_2(n, k) = 4(n - 1)k^2 + 2(n^2 + 4n - 2)k - 3n^3 + n^2 + 3n - 1,$$

which also satisfies $\eta_2(n, k) > 0$ since $k > k_2$.

In this case the Cauchy bound leads to unmanageable expressions. We avoid this by using a more direct method, paying the price of a worse bound. Since the strategy is similar for each condition, we only write here the argument to ensure condition $P_{A_2}(1 - r^{-1}) < 0$ when $k < k_1$. We have that

$$Q(r) := r^4 P_{A_2}(1 - r^{-1}) = -3(2n - 2k - 1)r^4 - 2(2k^2 - (2k + 13)n + 11k + 5)r^3 + 4(2k^2 - (3k + 11)n + 5k + 4)r^2 + 8((k + 4)n - 2)r - 8(n - 1).$$

Since, in any case, $3(2n - 2k - 1) > 1$, a very conservative bound can be given by

$$Q(r) < -r^4 + (2k + 13)nr^3 + 4(2k^2 + 5k + 4)r^2 + 8n(k + 4)r < -r^4 + (18n^2 + 65n + 16)r^3$$

since $r \geq 2$ and $k < n$. Hence, it suffices to take

$$r > 18n^2 + 65n + 16.$$

A similar computation holds for the rest of the conditions. □

5 Biharmonic Hopf hypersurfaces

In this section we focus on the biharmonic case. Note that Eq. (4.1) for the case $r = 2$ reduces to

$$\text{Tr } S_t^2 = 2(n + 1). \tag{5.1}$$

With this equation, we recover the classification of biharmonic Hopf hypersurfaces in $\mathbb{C}P^n$, first studied in [13] and corrected later in [24].

Theorem 5.1 [24, Theorem 2.4] *A Hopf hypersurface in $\mathbb{C}P^n$ is proper biharmonic if and only if it is an open part of a tube over a totally geodesic $\mathbb{C}P^{n-p}$ in $\mathbb{C}P^n$, $1 \leq p \leq n - 1$, with radius t given by*

$$\cos^2 t_{\pm} = \frac{3(n + 1) - 2p \pm \sqrt{n^2 + 6n - 4(n + 1)p + 4p^2 + 5}}{4(n + 1)}.$$

Regarding stability, we obtain the following result.

Theorem 5.2 *Every homogeneous and proper biharmonic hypersurface in $\mathbb{C}P^n$ is unstable. Moreover, there exists $C \equiv C(p) > 0$ such that if $n - p > C$ the normal index of the biharmonic tube over $\mathbb{C}P^{n-p}$ in $\mathbb{C}P^n$ of radius t_+ is exactly 1.*

Proof Write M_t for the biharmonic tube over $\mathbb{C}P^{n-p}$ in $\mathbb{C}P^n$. Since $\mathbb{C}P^n$ endowed with the Fubini–Study metric is an Einstein manifold with $\text{Ric}(\cdot, \cdot) = 2(n + 1)g(\cdot, \cdot)$, we obtain the following relations:

$$\text{Ric}(\xi, \xi) = 2(n + 1), \quad (\text{Ric}(\xi))^\top = 0, \quad (\nabla_\xi \text{Ric})(\xi, \xi) = 0.$$

Moreover, for any local orthonormal frame $\{e_i\}_{i=0}^{2n-1}$ on $\mathbb{C}P^n$ we have

$$\text{Tr } R(\xi, \cdot, \nabla_\xi(\cdot), \xi) = \sum_{i,j=0}^{2n-1} \langle \nabla_\xi e_i, e_j \rangle R(\xi, e_i, e_j, \xi) = \sum_{i=0}^{2n-1} \langle \nabla_\xi e_i, e_i \rangle R(\xi, e_i, e_i, \xi) = 0.$$

Then, since

$$\text{Tr } R(\xi, S_t(\cdot), \cdot, \xi) = \text{Tr } S_t + 3\alpha_t$$

and $\text{Tr } S_t$ is constant on M_t , Eqs. (5.1) and (2.3) yield

$$Q(f) = \int_M [(\Delta f)^2 + 4|S_t(\nabla f)|^2 - 4f^2 \text{Tr } S_t(\text{Tr } S_t + 3\alpha_t)]dV_g.$$

By the Sturm-Liouville’s decomposition we have that

$$C^\infty(M_t) = \oplus_{i=0}^\infty E_{\mu_i}$$

in L^2 , where E_{μ_i} denotes the eigenspace of the Laplacian on M_t with respect to the eigenvalue μ_i . For a detailed discussion on the orthogonal decomposition of eigenspaces of the Laplacian on compact manifolds see [7, Theorem III.9.1].

We see that for any constant function f_0 we have $Q(f_0) < 0$. Since $\mu_0 = 0$ has multiplicity 1, the index is at least 1 and therefore M_t is unstable. For any the other eigenfunction f note that

$$\int_M 4|S_t(\nabla f)|^2 dV_g \geq 4\lambda_{\min}^2 \int_M |\nabla f|^2 dV_g = -4\lambda_{\min}^2 \int_M f \Delta f dV_g,$$

where λ_{\min}^2 represents the minimum between all squared principal curvatures. With this

$$Q(f) \geq \int_M [(\Delta f)^2 - 4\lambda_{\min}^2 f \Delta f - 4f^2 \text{Tr } S_t(\text{Tr } S_t + 3\alpha_t)]dV_g.$$

Since in our case we have limited knowledge about the spectrum of the Laplacian on M_t , we make use of the following bound for the first eigenvalue:

$$\mu_1 \geq (n + 1) - \frac{1}{2}|\text{Tr } S_t|.$$

This is just an application for our particular case of Theorem 2.1. Hence, if we denote by f_1 the eigenfunction corresponding to the eigenvalue μ_1 , we have

$$Q(f_1) \geq \int_M [(\mu_1)^2 + 4\lambda_{\min}^2 \mu_1 - 4 \text{Tr } S_t(\text{Tr } S_t + 3\alpha_t)]f^2 dV_g \geq \int_M [((n + 1) - \frac{1}{2}|\text{Tr } S_t|)^2 + 4\lambda_{\min}^2((n + 1) - \frac{1}{2}|\text{Tr } S_t|) - 4 \text{Tr } S_t(\text{Tr } S_t + 3\alpha_t)]f^2 dV_g.$$

After rearranging, we see that for $Q(f_1) > 0$ to hold, it is enough to have

$$(n + 1)(4\lambda_{\min}^2 + n + 1) > \frac{15}{4}(\text{Tr } S_t)^2 + (2\lambda_{\min}^2 + n + 1)|\text{Tr } S_t| + 12\alpha_t \text{Tr } S_t. \tag{5.2}$$

Now, if μ_ℓ denotes the ℓ^{th} -eigenvalue, and since $\mu_\ell > \mu_1$, it follows that

$$(\mu_\ell)^2 + 4\lambda_{\min}^2 \mu_\ell - 4 \text{Tr } S_t(\text{Tr } S_t + 3\alpha_t) > (\mu_1)^2 + 4\lambda_{\min}^2 \mu_1 - 4 \text{Tr } S_t(\text{Tr } S_t + 3\alpha_t),$$

so to check if $Q(f_\ell) > 0$ for any $\ell \geq 1$, it is enough to ensure (5.2).

Note now that for the values t_+ given in 5.1, the following asymptotic relations hold:

$$\begin{aligned} 4 \cot^2 2t_+ &= \frac{2n}{2p-1} + o(n) & \cot^2 t_+ &= \frac{2n}{2p-1} + o(n) \\ \tan^2 t_+ &= \frac{2p-1}{2n} + o(n^{-1}) & \text{Tr } S_{t_+} &= \frac{2\sqrt{4p-2}}{\sqrt{n}} + o(n^{-\frac{1}{2}}) \end{aligned}$$

as $n \rightarrow \infty$. Therefore, there exists $n_1 > p$ such that $\lambda_{\min}^2 = \tan t_+$ for $n > n_1$. Using this, we get

$$(n + 1)(4\lambda_{\min}^2 + n + 1) = n^2 + o(n^2)$$

and

$$\frac{15}{4}(\text{Tr } S_t)^2 + (2\lambda_{\min}^2 + n + 1)|\text{Tr } S_t| + 12\alpha_t \text{Tr } S_t = 48 + 2\sqrt{4p-2}\sqrt{n} + o(\sqrt{n})$$

as $n \rightarrow \infty$, obtaining the result. □

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