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Homogenization of fundamental solutions for parabolic operators involving non-self-similar scales

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Abstract

We establish the asymptotic expansion of the fundamental solutions with precise error estimates for second-order parabolic operators

$$\partial_t - \operatorname{div}(A(x/\varepsilon, t/\varepsilon^{\ell})\nabla), \quad 0 < \varepsilon < 1,\ 0 < \ell < \infty,$$

in the case $\ell \neq 2$, where the spatial and temporal variables oscillate on non-self-similar scales and do not homogenize simultaneously. To achieve the goal, we explore the direct quantitative two-scale expansions for the aforementioned operators, which should be of some independent interests in quantitative homogenization of parabolic operators involving multiple scales. In the self-similar case $\ell = 2$, similar results have been obtained in Geng and Shen (Anal PDE 13(1): 147–170, 2020).

Keywords Periodic homogenization · Parabolic systems · Fundamental solutions

Mathematics Subject Classification 35B27

1 Introduction

We consider the asymptotic behavior of fundamental solutions to a family of second-order parabolic operators

$$\partial_t + \mathcal{L}_\varepsilon = \partial_t - \operatorname{div}(A(x/\varepsilon, t/\varepsilon^{\ell})\nabla) \qquad \text{in } \mathbb{R}^{d+1}, \tag{1.1}$$

where $0 < \varepsilon < 1, 0 < \ell < \infty$, the coefficient tensor $A = A(z, \tau) = (A_{ij}^{\alpha\beta}(z, \tau)), 1 \leq i, j \leq d, 1 \leq \alpha, \beta \leq m$, is real, bounded measurable and satisfy

- Ellipticity condition: there exists $\mu > 0$ such that

$$\mu|\xi|^2 \leq A_{ij}^{\alpha\beta}(z, \tau)\xi_i^\alpha \xi_j^\beta \leq \frac{1}{\mu}|\xi|^2 \tag{1.2}$$

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for any matrix $\xi = (\xi_i^\alpha) \in \mathbb{R}^{m\times d}$ and a.e. $(z,\tau) \in \mathbb{R}^{d+1}$.

- Periodicity condition: for any $(y,s) \in \mathbb{Z}^{d+1}$ and a.e. $(z,\tau) \in \mathbb{R}^{d+1}$,

$$A(z+y,\tau+s) = A(z,\tau)\ . \tag{1.3}$$

Under the conditions (1.2) and (1.3), it is well known that $\partial_t + \mathcal{L}_\varepsilon$ G-converges to $\partial_t + \mathcal{L}_0$, which has constant coefficients and depends on ℓ in three different cases: $0 < \ell < 2$, $\ell = 2$, and $2 < \ell < \infty$. See [1] and also Section 2 for the details.

Quantitative homogenization of (1.1) has aroused great interests in recent years. In the case $\ell = 2$, where the spatial and temporal variables oscillate on self-similar scales, the uniform Hölder and Lipschitz estimates in homogenization were derived by Geng and Shen in [7], while the two-scale expansions with precise error estimates for the operators were studied in different contexts in [8, 14, 17, 20, 21]. In particular, the asymptotic expansion of the fundamental solutions to (1.1) with sharp error estimates was established in [9].

In the case $\ell \neq 2$, the scales of the spatial and temporal variables do not coincide with the scales of the parabolic operators. As a result, the temporal and spatial variables do not homogenize simultaneously [1], and the quantitative homogenization theory for (1.1) is therefore much more intricate. Few results on the quantitative theory is known until very recently [10]. In [10], Geng and Shen developed an effective approach to study the quantitative homogenization theory of (1.1). One of the key ideas is to introduce a family of intermediate operators

$$\partial_t - \operatorname{div}\big(A(x/\varepsilon, t/(\lambda\varepsilon^2))\nabla\big), \quad \lambda > 0, \tag{1.4}$$

which converts the original operator to the one with self-similar scales (for each fixed λ). Then by the quantitative two-scale expansions for (1.4) and some intricate estimates on the corresponding (λ-dependent) correctors, they were able to establish the convergence rate and the large-scale estimates for the original operator (1.1).

In this paper, we shall explore the direct quantitative two-scale expansions of (1.1) in the non-self-similar case $\ell \neq 2$, and study the quantitative homogenization theory without introducing the intermediate operators (1.4). To fix the idea, we study the asymptotic expansion of the fundamental solution Γ_ε of (1.1). Our results, combined with those in [9] for the case $\ell = 2$, provide the whole view of asymptotic expansions of the fundamental solutions to (1.1).

Since the homogenized operator $\partial_t + \mathcal{L}_0$ has constant coefficients. It is well known that the matrix of fundamental solution $\Gamma_0(x,t;y,s)$ exists, and for any $x, y \in \mathbb{R}^d$, $-\infty < s < t < \infty$, and $M, N \geq 0$,

$$|\nabla_x^M \partial_t^N \Gamma_0(x,t;y,s)| \leq \frac{C}{(t-s)^{(d+M+2N)/2}} \exp\Big\{-\frac{\kappa|x-y|^2}{t-s}\Big\}, \tag{1.5}$$

where $\kappa > 0$ depends on μ, and C depends on d, m, μ, M, and N. See [2, 11] and also [3–5] for earlier works.

For the operator (1.1), in the case $m = 1$ Nash's theorem [15] implies that under the assumption (1.2) the local solutions to $(\partial_t+\mathcal{L}_\varepsilon)u_\varepsilon = 0$ are uniformly (in ε) Hölder continuous. For the case $m \geq 2$, the uniform Hölder estimate still holds if A satisfies (1.2), (1.3) and the assumption that $A \in \mathrm{VMO}_x$ (see Section 3 for the details). Here $A \in \mathrm{VMO}_x$ means that

$$\lim_{\varrho\to 0} \omega_\varrho(A) = 0, \tag{1.6}$$

where

$$\omega_\varrho(A) = \sup_{\substack{0<r<\varrho \\ (x,t)\in\mathbb{R}^{d+1}}} \fint_{t-r^2}^{t} \fint_{y\in B(x,r)} \fint_{z\in B(x,r)} \big|A(z,\tau) - A(y,\tau)\big| dz dy d\tau.$$

As a result, under the assumptions (1.2), (1.3), and also (1.6) if $m \geq 2$, the matrix of fundamental solutions Γ_ε of (1.1) exists and satisfies the Gaussian type estimate

$$|\Gamma_\varepsilon(x,t;y,s)| \leq \frac{C}{(t-s)^{d/2}} \exp\left\{ -\frac{\kappa|x-y|^2}{t-s} \right\} \tag{1.7}$$

for any $x, y \in \mathbb{R}^d$ and $-\infty < s < t < \infty$, with $\kappa > 0$ depending only on μ, and $C > 0$ depending only on d, m, μ and $\omega_\varrho(A)$ (if $m \geq 2$).

Theorem 1.1 *Assume $A(z,\tau)$ satisfies (1.2) and (1.3), and also (1.6) if $m \geq 2$. Also assume that $\|\partial_\tau A\|_\infty < \infty$ if $0 < \ell < 2$, and $\|\nabla^2 A\|_\infty < \infty$ if $2 < \ell < \infty$. Then for any $x, y \in \mathbb{R}^d$ and $-\infty < s < t < \infty$,*

$$\begin{aligned} &\big|\Gamma_\varepsilon(x,t;y,s) - \Gamma_0(x,t;y,s)\big| \\ &\leq \frac{C}{(t-s)^{\frac{d+1}{2}}} \exp\left\{ -\frac{\kappa|x-y|^2}{t-s} \right\} \cdot \begin{cases} \varepsilon^{\ell/2} + \varepsilon^{2-\ell}\sqrt{t-s} & \text{if } 0 < \ell < 2, \\ \varepsilon + \varepsilon^{\ell-2}\sqrt{t-s} & \text{if } 2 < \ell < \infty, \end{cases} \end{aligned} \tag{1.8}$$

where $\kappa > 0$ depends only on μ, and C depends only on d, m, μ, and $\omega_\varrho(A)$ (if $m \geq 2$), and also $\|\partial_\tau A\|_\infty$ (if $0 < \ell < 2$) and $\|\nabla^2 A\|_\infty$ (if $2 < \ell < \infty$).

Given a function $f(x,t)$ in $\mathbb{R}^{d+1}$, for $E \subseteq \mathbb{R}^{d+1}$ we define for $0 < \theta \leq 1$

$$\|f\|_{C^{\theta,0}(E)} = \sup\left\{ \frac{|f(x,t) - f(y,t)|}{|x-y|^\theta} : (x,t), (y,t) \in E \text{ and } x \neq y \right\}. \tag{1.9}$$

Let

$$\chi(z,\tau) = \begin{cases} \chi^\infty(z,\tau) & \text{if } 0 < \ell < 2, \\ \chi^0(z) & \text{if } 2 < \ell < \infty, \end{cases} \tag{1.10}$$

where χ^∞ and χ^0 are the correctors given by the cell problems (2.1) and (2.20) respectively.

Theorem 1.2 *Assume A satisfies (1.2) and (1.3), and*

$$|A(z,\tau) - A(z',\tau')| \leq h\left(|z-z'| + |\tau-\tau'|^{1/2}\right)^\theta \tag{1.11}$$

for any $(z,\tau), (z',\tau') \in \mathbb{R}^{d+1}$, where $h > 0$ and $\theta \in (0,1)$. Assume also that $\|\partial_\tau A\|_\infty < \infty$ if $0 < \ell < 2$, and $\|\nabla^2 A\|_{C^{\theta,0}} < \infty$ if $2 < \ell < \infty$. Then for any $x, y \in \mathbb{R}^d$ and $-\infty < s < t < \infty$,

$$\begin{aligned} &\Big|\frac{\partial}{\partial x_i}\Gamma_\varepsilon^{\alpha\beta}(x,t;y,s) - \frac{\partial}{\partial x_i}\big(\delta^{\alpha\gamma}x_j + \varepsilon\chi_j^{\alpha\gamma}(x/\varepsilon, t/\varepsilon^\ell)\big)\frac{\partial}{\partial x_j}\Gamma_0^{\gamma\beta}(x,t;y,s)\Big| \\ &\leq \frac{C\log\left(2+\varepsilon^{-1}\sqrt{t-s}\right)}{(t-s)^{\frac{d+2}{2}}} \exp\left\{ -\frac{\kappa|x-y|^2}{t-s} \right\} \cdot \begin{cases} \varepsilon^{\ell/2} + \varepsilon^{2-\ell}\sqrt{t-s} & \text{if } 0 < \ell < 2, \\ \varepsilon + \varepsilon^{\ell-2}\sqrt{t-s} & \text{if } 2 < \ell < \infty, \end{cases} \end{aligned} \tag{1.12}$$

where $1 \leq i, j \leq d$, $1 \leq \alpha, \beta, \gamma \leq m$, and δ is the Kronecker symbol. The constant κ depends only on μ, and C depends on d, m, μ, (h,θ) in (1.11), and also $\|\partial_\tau A\|_\infty$ (if $0 < \ell < 2$) and $\|\nabla^2 A\|_{C^{\theta,0}}$ (if $2 < \ell < \infty$).

Let $\widetilde{A}(z,\tau)=\big(\widetilde{A}_{ij}^{\alpha\beta}(z,\tau)\big)$ with $\widetilde{A}_{ij}^{\alpha\beta}(z,\tau)=A_{ji}^{\beta\alpha}(z,-\tau)$, and $\widetilde{\mathcal{L}}_\varepsilon=-\mathrm{div}(\widetilde{A}(x/\varepsilon,t/\varepsilon^\ell)\nabla)$. Let $\widetilde{\Gamma}_\varepsilon(x,t;y,s)=(\widetilde{\Gamma}_\varepsilon^{\alpha\beta}(x,t;y,s))$ be the fundamental matrix associated to the operator $\partial_t+\widetilde{\mathcal{L}}_\varepsilon$. It follows that $\Gamma_\varepsilon^{\beta\alpha}(x,t;y,s)=\widetilde{\Gamma}_\varepsilon^{\alpha\beta}(y,-s;x,-t)$. Since $\widetilde{A}(z,\tau)$ satisfies the same conditions as $A(z,\tau)$, (1.12) implies that

$$\Big|\frac{\partial}{\partial y_i}\widetilde{\Gamma}_\varepsilon^{\alpha\beta}(y,-s;x,-t)-\frac{\partial}{\partial y_i}\big(\delta^{\alpha\gamma}y_j+\varepsilon\widetilde{\chi}_j^{\alpha\gamma}(y/\varepsilon,-s/\varepsilon^\ell)\big)\frac{\partial}{\partial y_j}\widetilde{\Gamma}_0^{\gamma\beta}(y,-s;x,-t)\Big| \tag{1.13}$$

can be bounded by the right hand of (1.12), where $\widetilde{\chi}_j^{\alpha\gamma}(y/\varepsilon,-s/\varepsilon^\ell\big)$ is the corrector of $\partial_t+\widetilde{\mathcal{L}}_\varepsilon$. Therefore, we have

$$\begin{aligned}&\Big|\frac{\partial}{\partial y_i}\Gamma_\varepsilon^{\beta\alpha}(x,t;y,s)-\frac{\partial}{\partial y_i}\big(\delta^{\alpha\gamma}y_j+\varepsilon\widetilde{\chi}_j^{\alpha\gamma}(y/\varepsilon,-s/\varepsilon^\ell)\big)\frac{\partial}{\partial y_j}\Gamma_0^{\beta\gamma}(x,t;y,s)\Big|\\&\le\frac{C\log\big(2+\varepsilon^{-1}\sqrt{t-s}\big)}{(t-s)^{\frac{d+2}{2}}}\exp\Big\{-\frac{\kappa|x-y|^2}{t-s}\Big\}\cdot\begin{cases}\varepsilon^{\ell/2}+\varepsilon^{2-\ell}\sqrt{t-s} & \text{if } 0<\ell<2,\\ \varepsilon+\varepsilon^{\ell-2}\sqrt{t-s} & \text{if } 2<\ell<\infty.\end{cases}\end{aligned} \tag{1.14}$$

This, together with (1.12), allows us to derive the asymptotic expansion for $\nabla_x\nabla_y\Gamma_\varepsilon(x,t;y,s)$.

Theorem 1.3 *Assume A satisfies (1.2), (1.3) and (1.11). Assume also that $\|\partial_\tau A\|_\infty<\infty$ if $0<\ell<2$, and $\|\nabla^2A\|_{C^{\theta,0}}<\infty$ if $2<\ell<\infty$. Then for any $x,y\in\mathbb{R}^d$ and $-\infty<s<t<\infty$,*

$$\begin{aligned}&\Big|\frac{\partial^2}{\partial x_i\partial y_j}\Gamma_\varepsilon^{\alpha\beta}(x,t;y,s)-\frac{\partial}{\partial x_i}\big(\delta^{\alpha\gamma}x_k+\varepsilon\chi_k^{\alpha\gamma}(x/\varepsilon,t/\varepsilon^\ell)\big)\frac{\partial^2}{\partial x_k\partial y_l}\\&\quad\Gamma_0^{\gamma\zeta}(x,t;y,s)\frac{\partial}{\partial y_j}\big(\delta^{\beta\zeta}y_l+\varepsilon\widetilde{\chi}_l^{\beta\zeta}(y/\varepsilon,-s/\varepsilon^\ell)\big)\Big|\\&\le\frac{C\log\big(2+\varepsilon^{-1}\sqrt{t-s}\big)}{(t-s)^{\frac{d+3}{2}}}\exp\Big\{-\frac{\kappa|x-y|^2}{t-s}\Big\}\cdot\begin{cases}\varepsilon^{\ell/2}+\varepsilon^{2-\ell}\sqrt{t-s} & \text{if } 0<\ell<2,\\ \varepsilon+\varepsilon^{\ell-2}\sqrt{t-s} & \text{if } 2<\ell<\infty,\end{cases}\end{aligned} \tag{1.15}$$

where $1\le i,j,k,l\le d$, $1\le\alpha,\beta,\gamma,\zeta\le m$, and δ is the Kronecker symbol. The constant κ depends only on μ, and C depends on $d,m,\mu,(h,\theta)$ in (1.11), and also $\|\partial_\tau A\|_\infty$ (if $0<\ell<2$) and $\|\nabla^2A\|_{C^{\theta,0}}$ (if $2<\ell<\infty$).

The proof of Theorem 1.1 follows the scheme of [9] (see [13] for the elliptic operators). Our main contribution is on the direct two-scale expansions for the operator (1.1). Two-scale expansions are essential in the study of quantitative homogenization theory. Here different from [10], we try to deal with the non-self similar scales directly, and introduce the two-scale expansions for (1.1) from the point view of reiterated homogenization [1]. Very recently, the second author developed the quantitative reiterated homogenization theory and established the convergence rate and uniform regularity estimates for the parabolic operators with several spatial and temporal scales [16].

Precisely speaking, in the cases $0<\ell<2$ and $2<\ell<\infty$, we introduce respectively the auxiliary functions

$$\begin{aligned}w_\varepsilon(x,t)&=u_\varepsilon(x,t)-u_0(x,t)-\varepsilon\chi^\infty(x/\varepsilon,t/\varepsilon^\ell)S_{\varepsilon^{\ell/2}}(\nabla u_0(x,t))\\&\quad-\varepsilon^\ell\mathcal{B}(t/\varepsilon^\ell)\nabla S_{\varepsilon^{\ell/2}}(\nabla u_0(x,t))-\varepsilon^2\nabla\mathfrak{B}(x/\varepsilon,t/\varepsilon^\ell)\nabla S_{\varepsilon^{\ell/2}}(\nabla u_0(x,t)),\end{aligned} \tag{1.16}$$

and

$$\begin{aligned}\widetilde{w}_\varepsilon(x,t) &= u_\varepsilon(x,t) - u_0(x,t) - \varepsilon\chi^0(x/\varepsilon)S_\varepsilon(\nabla u_0(x,t)) \\ &\quad - \varepsilon^{\ell-1}\nabla\Phi(x/\varepsilon, t/\varepsilon^\ell)S_\varepsilon(\nabla u_0(x,t)) - \varepsilon^\ell\Phi(x/\varepsilon, t/\varepsilon^\ell)\nabla S_\varepsilon(\nabla u_0(x,t)) \\ &\quad - \varepsilon^2\nabla\Upsilon(x/\varepsilon)\nabla S_\varepsilon(\nabla u_0(x,t)), \end{aligned} \tag{1.17}$$

to perform the two-scale expansions, where S_ε is the smoothing operator, χ^∞, χ^0 are the correctors, and $\mathcal{B}, \mathfrak{B}, \Phi, \Upsilon$ are understood as the flux correctors for (1.1). See Sect. 2 for the exact definitions and the properties. Compared with [10], the key ingredient in (1.16) and (1.17) is that the spatial and time variables are considered separately when we introduce the flux correctors. This is coherent with reiterated homogenization process, and allows us to overcome the problem brought by the non-self similar scales in the spatial and temporal variables, and show that

$$(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon \mathrm{div} F_\varepsilon \quad \text{and} \quad (\partial_t + \mathcal{L}_\varepsilon)\widetilde{w}_\varepsilon = \varepsilon \mathrm{div} \widetilde{F}_\varepsilon \tag{1.18}$$

with proper F_ε and $\widetilde{F}_\varepsilon$ depending only on u_0.

Prepared with the quantitative two-scale expansions, we then follow the ideas in [9] to consider the weighted functions

$$u_\varepsilon(x,t) = \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_\varepsilon(x,t;y,s) f(y,s) e^{-\psi(y)} dyds,$$
$$u_0(x,t) = \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_0(x,t;y,s) f(y,s) e^{-\psi(y)} dyds,$$

where $f \in C_c^\infty(Q_r(x_0,t_0))$ and ψ is a bounded Lipschitz function in $\mathbb{R}^d$, and establish proper bounds for $\|w_\varepsilon e^\psi\|_{L^2(\mathbb{R}^d)}$ and $\|\widetilde{w}_\varepsilon e^\psi\|_{L^2(\mathbb{R}^d)}$. This, together with the uniform L^∞ estimates for $\partial_t + \mathcal{L}_\varepsilon$, allows us to bound $\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))}$ by $\|f\|_{L^2(Q_r(y_0,s_0))}$. By duality this gives the weighted L^2 bound for $\Gamma_\varepsilon - \Gamma_0$ (see (6.4) and (6.6)). Finally, by utilizing the L^∞ estimates for the dual operator $\partial_t + \widetilde{\mathcal{L}}_\varepsilon$ and proper choice of the weight function, we get the desired estimate (1.8) and complete the proof of Theorem 1.1. The proof of Theorem 1.2 is based on Theorem 1.1 and the uniform Lipschitz estimate of w_ε and $\widetilde{w}_\varepsilon$. Theorem 1.3 is a direct consequence of Theorem 1.2 and (1.14).

2 Correctors and flux correctors

2.1 The case $0 < \ell < 2$

For $1 \le j \le d$, let $\chi_j^\infty = \chi_j^\infty(z,\tau)$ be the correctors given by the cell problem

$$\begin{cases} -\mathrm{div}(A\nabla\chi_j^\infty) = \mathrm{div}(A\nabla z_j) & \text{in } \mathbb{R}^{d+1}, \\ \chi_j^\infty = \chi_j^\infty(z,\tau) & \text{is 1-periodic in } (z,\tau), \\ \int_{\mathbb{T}^d} \chi_j^\infty(z,\tau)dz = 0, & \end{cases} \tag{2.1}$$

where $\mathbb{T}^d = [0,1)^d = \mathbb{R}^d/\mathbb{Z}^d$. By the energy estimate and Poincaré's inequality,

$$\int_{\mathbb{T}^d} (|\nabla\chi_j^\infty(z,\tau)|^2 + |\chi_j^\infty(z,\tau)|^2)dz \le C, \tag{2.2}$$

where C depends only on d and μ. Thanks to [1], the homogenized operator of $\partial_t + \mathcal{L}_\varepsilon$ is given by $\partial_t - \operatorname{div}(\widehat{A_\infty}\nabla)$ with $\widehat{A_\infty} = \int_{\mathbb{T}^{d+1}}(A + A\nabla\chi^\infty)dzd\tau$.

Lemma 2.1 *Suppose that $A(z,\tau)$ satisfies the conditions (1.2) and (1.3). If $m \geq 2$, we assume that*

$$\lim_{\varrho\to 0} \sup_{\substack{0<r<\varrho \\ x\in\mathbb{R}^d}} \fint_{B(x,r)} \Big| A(z,\tau) - \fint_{B(x,r)} A(z,\tau)dz \Big| dz = 0 \tag{2.3}$$

uniformly in τ. Then $\chi_j^\infty \in L^\infty(\mathbb{T}^{d+1})$.

Proof The result follows from the classical De Giorgi-Nash estimate in the case $m = 1$, and from the $W^{1,p}$ estimate for elliptic systems with VMO coefficients in the case $m \geq 2$ [12]. Moreover, we have for any $x \in \mathbb{R}^d$, and $0 < r < 1$,

$$\sup_{\tau\in\mathbb{R}} \left(\fint_{B(x,r)} |\nabla\chi_j^\infty(z,\tau)|^2 dz \right)^{1/2} \leq Cr^{\sigma-1}, \quad \text{where } \sigma \in (0,1). \tag{2.4}$$

□

Remark 2.1 If A satisfies (1.6) and $\|\partial_\tau A\|_\infty < \infty$. It is not difficult to see that A satisfies (2.3).

We now introduce the flux correctors.

Lemma 2.2 *Suppose that $A(z,\tau)$ satisfies the assumption of Lemma 2.1. There exists a unique 1-periodic (in (z,τ)) function $\mathfrak{B}(z,\tau)$ in $\mathbb{R}^{d+1}$ such that $\int_{\mathbb{T}^d} \mathfrak{B}(z,\tau)dz = 0$, $\mathfrak{B} \in L^\infty(0,1;H^2(\mathbb{T}^d))$, $\nabla\mathfrak{B}, \nabla^2\mathfrak{B} \in L^\infty(\mathbb{T}^{d+1})$, and*

$$\Delta_d\mathfrak{B}(z,\tau) = \chi^\infty(z,\tau) \quad \text{in } \mathbb{R}^{d+1}.$$

Moreover, if $\|\partial_\tau A\|_\infty < \infty$, then

$$\sup_{\tau\in\mathbb{R}} \{\|\partial_\tau\nabla\mathfrak{B}(z,\tau)\|_{L^p(\mathbb{T}^d)} + \|\partial_\tau\nabla^2\mathfrak{B}(z,\tau)\|_{L^p(\mathbb{T}^d)}\} \leq C \quad \text{for any } 1 \leq p < \infty. \tag{2.5}$$

Proof Since $\int_{\mathbb{T}^d} \chi^\infty(z,\tau)dz = 0$, for any $\tau \in \mathbb{R}$ there exists $\mathfrak{B}(\cdot,\tau) \in H^2(\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d} \mathfrak{B}(z,\tau)dz = 0$ and

$$\Delta_d\mathfrak{B}(z,\tau) = \chi^\infty(z,\tau) \quad \text{in} \quad \mathbb{R}^{d+1}. \tag{2.6}$$

Note that under the assumptions of Lemma 2.1, $\chi^\infty(z,\tau)$ is indeed Hölder continuous in z for any $\tau \in \mathbb{R}$. Standard Schauder estimates for (2.6) imply that

$$\sup_{\tau\in\mathbb{R}} \|\nabla\mathfrak{B}(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \leq C\Big\{ \sup_{\tau\in\mathbb{R}} \|\nabla\mathfrak{B}(\cdot,\tau)\|_{L^2(\mathbb{T}^d)} + \sup_{\tau\in\mathbb{R}} \|\chi^\infty(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \Big\} \leq C,$$
$$\sup_{\tau\in\mathbb{R}} \|\nabla^2\mathfrak{B}(\cdot,\tau)\|_{L^\infty(\mathbb{T}^d)} \leq C.$$

To prove (2.5), we differentiate the first equation in (2.1) to get

$$-\operatorname{div}(A\nabla\partial_\tau\chi_j^\infty) = \operatorname{div}(\partial_\tau A\nabla z_j) + \operatorname{div}(\partial_\tau A\nabla\chi_j^\infty) \qquad \text{in} \quad \mathbb{R}^{d+1}. \tag{2.7}$$

We claim that

$$\sup_{\tau\in\mathbb{R}} \|\partial_\tau\chi^\infty(z,\tau)\|_{L^q(\mathbb{T}^d)} \leq C \qquad \text{for any } 1 \leq q < \infty. \tag{2.8}$$

As a result, the desired estimate (2.5) follows directly from the $W^{2,p}$ estimate for

$$\Delta_d \partial_\tau \mathfrak{B}(z,\tau) = \partial_\tau \chi^\infty(z,\tau) \quad \text{in} \quad \mathbb{R}^{d+1}, \tag{2.9}$$

and Poincaré's inequality.

It remains to prove (2.8). In the case $m \geq 2$, since $\|\partial_\tau A\|_\infty < \infty$ and (1.6) holds, A satisfies (2.3). Standard $W^{1,p}$ estimate for (2.1) implies that $\sup_{\tau\in\mathbb{R}} \|\nabla \chi^\infty(z,\tau)\|_{L^p(\mathbb{T}^d)} \leq C$. This together with the $W^{2,p}$ estimate for (2.7) gives (2.8) directly.

In the case $m = 1$, we just assume A satisfies (1.2) and (1.3). To prove (2.8), for $B = B(x_0, R) \subseteq \mathbb{R}^d$, $0 < R < 1/8$, we decompose $\partial_\tau \chi_j^\infty$ as $\partial_\tau \chi_j^\infty = \partial_\tau \chi_j^{\infty,1} + \partial_\tau \chi_j^{\infty,2}$, where

$$-\operatorname{div}(A\nabla \partial_\tau \chi_j^{\infty,1}) = 0 \text{ in } \ B_R, \quad \text{and} \quad \partial_\tau \chi_j^{\infty,1} = \partial_\tau \chi_j^\infty \text{ on } \ \partial B_R, \tag{2.10}$$

and

$$\begin{cases} -\operatorname{div}(A\nabla \partial_\tau \chi_j^{\infty,2}) = \operatorname{div}(\partial_\tau A \nabla z_j) + \operatorname{div}(\partial_\tau A \nabla \chi_j^\infty) & \text{in } \ B_R, \\ \partial_\tau \chi_j^{\infty,2} = 0, \ \text{on } \ \partial B_R. \end{cases} \tag{2.11}$$

The De Giorgi-Nash estimate and energy estimate for (2.10) imply that

$$\begin{aligned} \|\partial_\tau \chi_j^{\infty,1}(\cdot,\tau)\|_{L^\infty(B_{R/2})} &\leq C\Big(⨍_{B_R} |\partial_\tau \chi_j^{\infty,1}(\cdot,\tau)|^2\Big)^{1/2} \\ &\leq C\Big\{\Big(⨍_{B_R} |\partial_\tau \chi_j^{\infty}(\cdot,\tau)|^2\Big)^{1/2} + R\Big(⨍_{B_R} |\nabla\partial_\tau \chi_j^{\infty}(\cdot,\tau)|^2\Big)^{1/2}\Big\} \end{aligned} \tag{2.12}$$

where standard energy estimates have been used in the last step. By Caccioppoli-type estimate for (2.7), and (2.4), we deduce that

$$\begin{aligned} R\Big(⨍_{B_R} |\nabla\partial_\tau \chi_j^{\infty}(\cdot,\tau)|^2\Big)^{1/2} &\leq C\Big\{\Big(⨍_{B_R} |\partial_\tau \chi_j^{\infty}(\cdot,\tau)|^2\Big)^{1/2} + R\Big(⨍_{B_R} |\nabla \chi_j^{\infty}(\cdot,\tau)|^2\Big)^{1/2} + R\Big\} \\ &\leq C\Big\{\Big(⨍_{B_R} |\partial_\tau \chi_j^{\infty}(\cdot,\tau)|^2\Big)^{1/2} + R^\sigma\Big\}, \end{aligned}$$

where $0 < \sigma < 1$. This combined with (2.12) implies that

$$\|\partial_\tau \chi_j^{\infty,1}(\cdot,\tau)\|_{L^\infty(B_{R/2})} \leq C\Big\{\Big(⨍_{B_R} |\partial_\tau \chi_j^{\infty}(\cdot,\tau)|^2\Big)^{1/2} + R^\sigma\Big\}, \tag{2.13}$$

On the other hand, by Poincaré's inequality, standard energy estimate for (2.11), and also (2.4),

$$\Big(⨍_{B_{R/2}} |\partial_\tau \chi_j^{\infty,2}(\cdot,\tau)|^2\Big)^{1/2} \leq CR\Big(⨍_{B_R} |\nabla\partial_\tau \chi_j^{\infty,2}(\cdot,\tau)|^2\Big)^{1/2} \leq C\big(1 + R^\sigma\big). \tag{2.14}$$

By using Theorem 4.2.3 in [19] with $F = \partial_\tau \chi_j^\infty(\cdot,\tau)$, $F_B = \partial_\tau \chi_j^{\infty,1}(\cdot,\tau)$, and $R_B = \partial_\tau \chi_j^{\infty,2}(\cdot,\tau)$, we get (2.8) from (2.13) and (2.14) immediately. The proof is complete. □

For $1 \leq i, j \leq d$, $1 \leq \alpha, \beta \leq m$, let

$$B_{ij}^{\alpha\beta}(z,\tau) = A_{ij}^{\alpha\beta}(z,\tau) + A_{ik}^{\alpha\gamma}(z,\tau)\frac{\partial}{\partial z_k}\chi_j^{\infty\gamma\beta}(z,\tau) - \widehat{A_{\infty ij}^{\alpha\beta}}. \tag{2.15}$$

It is obvious that B is 1-periodic in (z,τ), and $B \in L^\infty(0,1; L^2(\mathbb{T}^d))$. Define

$$\widehat{B}(\tau) = \int_{\mathbb{T}^d} B(z,\tau)dz. \tag{2.16}$$

Lemma 2.3 *Let $1 \le \alpha, \beta \le m$, $1 \le k, i, j \le d$. There exists a 1-periodic (in (z, τ)) function $\phi_{kij}^{\alpha\beta}(z, \tau) \in L^\infty(0, 1; H^1_{per}(\mathbb{T}^d))$, such that*

$$\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}, \quad \text{and} \quad B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau) = \frac{\partial}{\partial z_k}\phi_{kij}^{\alpha\beta}(z, \tau). \tag{2.17}$$

Furthermore, under the assumptions of Lemma 2.1, one has $\phi_{kij}^{\alpha\beta} \in L^\infty(\mathbb{T}^{d+1})$.

Proof By (2.16), $\int_{\mathbb{T}^d}\big(B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau)\big)dz = 0$. As a result, for any $\tau \in \mathbb{R}$ there exists a 1-periodic (in (z, τ)) function $f_{ij}^{\alpha\beta}(\cdot, \tau) \in H^2(\mathbb{T}^d)$ such that $\int_{\mathbb{T}^d} f_{ij}^{\alpha\beta}(z, \tau)dz = 0$ and

$$\Delta_d f_{ij}^{\alpha\beta}(z, \tau) = B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau) \quad \text{in} \quad \mathbb{R}^d. \tag{2.18}$$

Define

$$\phi_{kij}^{\alpha\beta}(z, \tau) = \frac{\partial}{\partial z_k} f_{ij}^{\alpha\beta}(z, \tau) - \frac{\partial}{\partial z_i} f_{kj}^{\alpha\beta}(z, \tau). \tag{2.19}$$

It is obviously that $\phi_{kij}^{\alpha\beta} \in L^\infty(0, 1; H^1(\mathbb{T}^d))$ and $\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}$. Moreover by (2.1) and (2.15),

$$\frac{\partial}{\partial z_i}\big(B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau)\big) = 0.$$

It follows from (2.18) that $\frac{\partial}{\partial z_i} f_{ij}^{\alpha\beta}(z, \tau)$ is a 1-periodic harmonic function, and thus a constant. Therefore,

$$\frac{\partial}{\partial z_k}\phi_{kij}^{\alpha\beta}(z, \tau) = \Delta_d f_{ij}^{\alpha\beta}(z, \tau) - \frac{\partial^2}{\partial z_k \partial z_i} f_{kj}^{\alpha\beta}(z, \tau) = B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau).$$

To show $\phi_{kij}^{\alpha\beta} \in L^\infty(\mathbb{T}^{d+1})$, we note that by (2.4) for any $x \in \mathbb{R}^d$ and $0 < r < 1$,

$$\sup_{\tau\in\mathbb{R}} \int_{B(x,r)} |B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau)|^2 \le Cr^{d+2\sigma-2}$$

for some $\sigma \in (0, 1)$. By Hölder's inequality,

$$\sup_{\tau\in\mathbb{R}} \int_{B(x,r)} |B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau)| \le Cr^{d+\sigma-1}.$$

This implies that

$$\begin{aligned}
&\sup_{\tau\in\mathbb{R}} \sup_{x\in\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau)|}{|x - z|^{d-1}} dz \\
&\quad \le C \sup_{\tau\in\mathbb{R}} \sum_{j=1}^{\infty} 2^{j(d-1)} \int_{|x-z|\sim 2^{-j}} |B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau)| dz \le C.
\end{aligned}$$

In view of (2.18), we can use the fundamental solution of Δ_d to deduce that

$$\|\nabla f_{ij}^{\alpha\beta}\|_{L^\infty(\mathbb{T}^{d+1})} \le C \sup_{\tau\in\mathbb{R}} \|f_{ij}^{\alpha\beta}(z, \tau)\|_{L^2(\mathbb{T}^d)} + \sup_{\tau\in\mathbb{R}} \sup_{x\in\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|B_{ij}^{\alpha\beta}(z, \tau) - \widehat{B}_{ij}^{\alpha\beta}(\tau)|}{|x - z|^{d-1}} dz \le C.$$

It follows from (2.19) that $\phi_{kij}^{\alpha\beta} \in L^\infty(\mathbb{T}^{d+1})$. The proof is complete. □

Lemma 2.4 *Let $\widehat{B}(\tau)$ be defined as in (2.16). There exists a 1-periodic function $\mathcal{B}(\tau) \in L^\infty(\mathbb{R})$ such that $\partial_\tau \mathcal{B}(\tau) = \widehat{B}(\tau)$.*

Proof Since $\int_0^1 \widehat{B}(\tau)d\tau = 0$, $\mathcal{B}(\tau) = \int_0^\tau \widehat{B}(\tau)d\tau$ is the desired function. □

Remark 2.2 Under the assumption (1.11), $\nabla_z\chi^\infty \in C^{\theta,\theta/2}(\mathbb{T}^{d+1})$, i.e.,

$$\sup_{\substack{(\tau_1,z_1),(\tau_2,z_2)\in\mathbb{T}^{d+1}\\(\tau_1,z_1)\neq(\tau_2,z_2)}} \frac{|\nabla_z\chi^\infty(z_1,\tau_1) - \nabla_z\chi^\infty(z_2,\tau_2)|}{(|z_1-z_2|+|\tau_1-\tau_2|^{1/2})^\theta} \le C.$$

By the definition, $\mathcal{B}(\tau)$ is Hölder continuous in τ. On the other hand, by (2.6) and (2.19), we know that $\nabla_z\mathfrak{B}$, $\nabla_z^2\mathfrak{B}$, ϕ and $\nabla_z\phi$ are also Hölder continuous. Moreover, if $\|\partial_\tau A\|_\infty < \infty$, (2.7) implies that $\partial_\tau\chi^\infty \in C^{\theta,0}(\mathbb{T}^{d+1})$ for some $0<\theta<1$, i.e.,

$$\sup_{\tau\in\mathbb{R}} \sup_{z_1,z_2\in\mathbb{T}^d, z_1\neq z_2} \frac{|\partial_\tau\chi^\infty(z_1,\tau) - \partial_\tau\chi^\infty(z_2,\tau)|}{|z_1-z_2|^\theta} \le C,$$

which by (2.9) implies that $\partial_\tau\nabla\mathfrak{B}(z,\tau) \in C^{\theta,0}(\mathbb{T}^{d+1})$. These regularity results would be used in the proof of Theorems 1.2 and 1.3.

2.2 The case $2 < \ell < \infty$

For $1 \le j \le d$, let $\chi_j^0 = \chi_j^0(z)$ be the correctors given by the cell problem

$$\begin{cases} -\mathrm{div}(\bar{A}\nabla\chi_j^0) = \mathrm{div}(\bar{A}\nabla z_j) & \text{in } \mathbb{R}^d,\\ \chi_j^0 = \chi_j^0(z) & \text{is 1-periodic in } z,\\ \int_{\mathbb{T}^d}\chi_j^0(z)dy = 0, \end{cases} \tag{2.20}$$

where $\bar{A} = \bar{A}(z) = \int_0^1 A(z,\tau)d\tau$. By the energy estimates and Poincaré's inequality,

$$\int_{\mathbb{T}^d} \left(|\nabla\chi_j^0(z)|^2 + |\chi_j^0(z)|^2\right)dz \le C, \tag{2.21}$$

where C depends only on d and μ. Thanks to [1], the homogenized operator of $\partial_t + \mathcal{L}_\varepsilon$ is given by $\partial_t - \mathrm{div}(\widehat{A_0}\nabla)$ with $\widehat{A_0} = \int_{\mathbb{T}^d}(\bar{A} + \bar{A}\nabla\chi^0)dz$.

Similar to Lemmas 2.1 and 2.2, it is not difficult to prove the following two lemmas.

Lemma 2.5 *Suppose that $A(z,\tau)$ satisfies conditions (1.2) and (1.3). If $m \ge 2$, we assume that $\bar{A} \in$ VMO, i.e.,*

$$\lim_{\delta\to 0} \sup_{\substack{0<r<\delta\\x\in\mathbb{R}^d}} ⨍_{B(x,r)} \left|\bar{A}(z) - ⨍_{B(x,r)}\bar{A}(z)dz\right|dz = 0.$$

Then $\chi_j^0 \in L^\infty(\mathbb{T}^d)$.

Lemma 2.6 *Let $\chi^0(z)$ be defined as in (2.20). There exists a 1-periodic function $\Upsilon(z) \in H^2(\mathbb{T}^d)$ such that $\Delta_d\Upsilon(z) = \chi^0(z)$. Moreover, assume that $\bar{A} \in$ VMO for $m \ge 2$. Then*

$$\|\nabla\Upsilon\|_{L^\infty(\mathbb{T}^d)} \le C \text{ and } \|\nabla^2\Upsilon\|_{L^\infty(\mathbb{T}^d)} \le C. \tag{2.22}$$

Let

$$B_0(z,\tau) = A(z,\tau) + A(z,\tau)\nabla_z \chi^0(z) - \widehat{A_0}, \quad \text{and} \quad \widetilde{B_0}(z) = \int_0^1 B_0(z,\tau)d\tau. \tag{2.23}$$

Lemma 2.7 *For $1 \le \alpha, \beta \le m$ and $1 \le k, i, j \le d$, there exist functions $\Psi_{kij}^{\alpha\beta}(z)$ in $H^1_{per}(\mathbb{T}^d)$ such that*

$$\Psi_{kij}^{\alpha\beta} = -\Psi_{ikj}^{\alpha\beta} \quad \text{and} \quad \widetilde{B_0}_{ij}^{\alpha\beta} = \frac{\partial}{\partial z_k}\Psi_{kij}^{\alpha\beta}. \tag{2.24}$$

Furthermore, under the assumptions of Lemma 2.5 we have $\Psi_{kij}^{\alpha\beta} \in L^\infty(\mathbb{T}^d)$.

Proof Note that

$$\int_{\mathbb{T}^d} \widetilde{B_0}_{ij}^{\alpha\beta}(z)dz = 0 \quad \text{and} \quad \frac{\partial}{\partial z_i}\widetilde{B_0}_{ij}^{\alpha\beta} = 0.$$

The proof is almost the same as that for Lemma 2.3. □

Lemma 2.8 *There exist 1-periodic functions $\Phi(z,\tau)$ in $\mathbb{R}^{d+1}$ such that $\partial_\tau \Phi(z,\tau) = B_0(z,\tau) - \widetilde{B_0}(z)$. Moreover, assume that $\|\nabla_z^2 A\|_\infty < \infty$. Then $\Phi, \nabla\Phi \in L^\infty(\mathbb{T}^{d+1})$ and $\Phi \in L^\infty(0,1; W^{2,p}(\mathbb{T}^d))$ for any $1 \le p < \infty$.*

Proof Let $\Phi(z,\tau) = \int_0^\tau \big(B_0(z,s) - \widetilde{B_0}(z)\big)ds$. We have $\partial_\tau \Phi(z,\tau) = B_0(z,\tau) - \widetilde{B_0}(z)$. If $\|\nabla_z^2 A\|_\infty < \infty$, standard Schauder theory for the system (2.20) implies that

$$\|\nabla^2\chi^0\|_{L^\infty(\mathbb{T}^d)} + \|\nabla\chi^0\|_{L^\infty(\mathbb{T}^d)} + \|\chi^0\|_{L^\infty(\mathbb{T}^d)} \le C. \tag{2.25}$$

By the definition of $\Phi(z,\tau)$, B_0 and $\widetilde{B_0}$, we have

$$\|\Phi\|_{L^\infty(\mathbb{T}^{d+1})} \le C\|B_0\|_{L^\infty(\mathbb{T}^{d+1})} + C\|\widetilde{B_0}\|_{L^\infty(\mathbb{T}^d)} \le C + C\|\nabla\chi^0\|_{L^\infty(\mathbb{T}^d)} \le C,$$

and

$$\|\nabla\Phi\|_{L^\infty(\mathbb{T}^{d+1})} \le C + C\|\nabla^2\chi^0\|_{L^\infty(\mathbb{T}^d)} + C\|\nabla\chi^0\|_{L^\infty(\mathbb{T}^d)} \le C,$$

where (2.25) is used for the last step.

Finally, note that

$$\begin{aligned}-\operatorname{div}\big(\bar{A}\nabla(\partial_{z_i}\partial_{z_k}\chi_j^0)\big) &= \operatorname{div}(\partial_{z_i}\partial_{z_k}\bar{A}\nabla z_j) + \operatorname{div}\big(\partial_{z_i}\bar{A}\nabla(\partial_{z_k}\chi_j^0)\big)\\ &\quad + \operatorname{div}\big(\partial_{z_k}\bar{A}\nabla(\partial_{z_i}\chi_j^0)\big) + \operatorname{div}\big(\partial_{z_i}\partial_{z_k}\bar{A}\nabla\chi_j^0\big)\end{aligned}$$

in $\mathbb{R}^d$. Standard $W^{1,p}$ estimate implies that $\|\nabla^2\chi^0\|_{W^{1,p}(\mathbb{T}^d)} \le C$ for any $1 \le p < \infty$, which, together with the definition of Φ, implies that

$$\sup_{\tau\in\mathbb{R}} \|\Phi(\cdot,\tau)\|_{W^{2,p}(\mathbb{T}^d)} \le C.$$

The proof is complete. □

Remark 2.3 If A satisfies (1.11), it is easy to see that $\nabla\chi^0, \nabla^2\Upsilon, \Psi, \nabla\Psi \in C^\theta(\mathbb{T}^d)$. Moreover, assume that $\|\nabla^2 A\|_{C^{\theta,0}} < \infty$, then (2.20) implies that $\nabla^3\chi^0 \in C^\theta(\mathbb{T}^d)$. As a result, by the definition of $\Phi(z,\tau)$ we know that $\nabla^2\Phi \in C^{\theta,0}(\mathbb{T}^{d+1})$.

3 Existence of fundamental solutions

In this part, we provide the uniform regularity estimates for the operator $\partial_t + \mathcal{L}_\varepsilon$, and also the existence and the size estimates for the fundamental solutions.

Theorem 3.1 *Assume that $A = A(z, \tau)$ satisfies (1.2), (1.3), and (1.6) if $m \geq 2$. Let u_ε be the weak solution to $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = div\ f$ in $Q_{2r} = Q_{2r}(x_0, t_0)$ with $f \in L^p(Q_{2r})$, $p > d+2$. Then for $\theta = 1 - (d+2)/p$,*

$$\|u_\varepsilon\|_{C^{\theta,\theta/2}(Q_r)} \leq Cr^{1-\theta}\Big\{\frac{1}{r}\Big(\fint_{Q_{2r}} |u_\varepsilon|^2\Big)^{1/2} + \Big(\fint_{Q_{2r}} |f|^p\Big)^{1/p}\Big\}, \tag{3.1}$$

where C depends on d, m, p, μ and $\omega_\varrho(A)$ in (1.6) (if $m \geq 2$). In particular,

$$\|u_\varepsilon\|_{L^\infty(Q_r)} \leq C\Big\{\Big(\fint_{Q_{2r}} |u_\varepsilon|^2\Big)^{1/2} + r\Big(\fint_{Q_{2r}} |f|^p\Big)^{1/p}\Big\}. \tag{3.2}$$

Proof It suffices to prove (3.1), since (3.2) is a direct consequence of (3.1). For the scalar case $m = 1$, the estimate follows from the well-known De Giorgi-Nash estimates.

We now consider the case $m \geq 2$. If $2 < \ell < \infty$, we let $\lambda = \varepsilon^2/\varepsilon^\ell > 1$, and $t = \lambda^{-1}[\lambda]t'$. By setting $v_\varepsilon(x, t') = u_\varepsilon(x, \lambda^{-1}[\lambda]t')$, where $[\lambda]$ denotes the integer part of λ, we get

$$A(x/\varepsilon, t/\varepsilon^\ell) = A(x/\varepsilon, \lambda t/\varepsilon^2) = A(x/\varepsilon, [\lambda]t'/\varepsilon^2).$$

Let $A^\sharp(z, \tau) = \lambda^{-1}[\lambda]A(z, [\lambda]\tau)$. Then $A^\sharp(z, \tau)$ is periodic in (z, τ), and

$$\partial_{t'} v_\varepsilon - \text{div}(A^\sharp(x/\varepsilon, t'/\varepsilon^2)\nabla v_\varepsilon) = \lambda^{-1}[\lambda]\text{div} f.$$

By the uniform Hölder estimates in periodic homogenization of parabolic equations [7],

$$\|v_\varepsilon\|_{C^{\theta,\theta/2}(Q_r)} \leq Cr^{1-\theta}\Big\{\frac{1}{r}\Big(\fint_{Q_{2r}} |v_\varepsilon|^2\Big)^{1/2} + \Big(\fint_{Q_{2r}} |f|^p\Big)^{1/p}\Big\}. \tag{3.3}$$

This by changing variables gives (3.1).

Likewise, if $0 < \ell < 2$ we set $\lambda = \varepsilon^\ell/\varepsilon^2 > 1$ and $x = (\sqrt{\lambda})^{-1}[\sqrt{\lambda}]x'$. Let $\widetilde{v}_\varepsilon(x', t) = u_\varepsilon((\sqrt{\lambda})^{-1}[\sqrt{\lambda}]x', t)$. It follows that

$$A(x/\varepsilon, t/\varepsilon^\ell) = A\big(\sqrt{\lambda}x/\varepsilon^{\ell/2}, t/\varepsilon^\ell\big) = A\big([\sqrt{\lambda}]x'/\varepsilon^{\ell/2}, t/\varepsilon^\ell\big),$$

and

$$\partial_t \widetilde{v}_\varepsilon - \text{div}(\widetilde{A}^\sharp(x'/\varepsilon^{\ell/2}, t/\varepsilon^\ell)\nabla\widetilde{v}_\varepsilon) = \text{div}\widetilde{f},$$

where $\widetilde{A}^\sharp(z, \tau) = \lambda[\sqrt{\lambda}]^{-2}A([\sqrt{\lambda}]z, \tau)$ and $\widetilde{f}(x', t) = \sqrt{\lambda}[\sqrt{\lambda}]^{-1}f((\sqrt{\lambda})^{-1}[\sqrt{\lambda}]x', t)$. Since $\widetilde{A}^\sharp(z, \tau)$ is periodic in (z, τ). The uniform Hölder estimates in periodic homogenization of parabolic equations implies that $\widetilde{v}_\varepsilon$ satisfies the estimate (3.3). This gives (3.1) by a changing of variables. The proof is complete. □

As we have mentioned, the uniform Hölder estimate (3.2) implies that the matrix of fundamental solutions for $\partial_t + \mathcal{L}_\varepsilon$ exists and satisfies the Gaussian type estimate (1.7).

By using the same reperiodization argument above, it not difficult to prove the following uniform Lipschitz estimate (see [6, 18]), which provides further estimates on the upper bounds for $\nabla_x\Gamma_\varepsilon(x, t; y, s)$, $\nabla_y\Gamma_\varepsilon(x, t; y, s)$, and $\nabla_x\nabla_y\Gamma_\varepsilon(x, t; y, s)$.

Theorem 3.2 *Assume that $A = A(z, \tau)$ satisfies (1.2), (1.3), and the Hölder continuity condition (1.11). Let u_ε be the weak solutions of $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = F$ in $Q_{2r} = Q_{2r}(x_0, t_0)$, where $0 < r < \infty$ and $F \in L^p(Q_{2r})$, $p > d + 2$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(Q_r)} \le C\Big\{\Big(\frac{1}{r}\fint_{Q_{2r}} |u_\varepsilon|^2\Big)^{1/2} + r\Big(\fint_{Q_{2r}} |F|^p\Big)^{1/p}\Big\}, \tag{3.4}$$

where C depends on d, m, p, μ and (h, θ) in (1.11).

Theorem 3.3 *Suppose A satisfies the assumptions in Theorem 3.2. Then for any $x, y \in \mathbb{R}^d$, $-\infty < s < t < \infty$,*

$$|\nabla_x \Gamma_\varepsilon(x, t; y, s)| + |\nabla_y \Gamma_\varepsilon(x, t; y, s)| \le \frac{C}{(t-s)^{(d+1)/2}} \exp\Big\{-\frac{\kappa|x-y|^2}{t-s}\Big\}, \tag{3.5}$$

$$|\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)| \le \frac{C}{(t-s)^{(d+2)/2}} \exp\Big\{-\frac{\kappa|x-y|^2}{t-s}\Big\}, \tag{3.6}$$

where κ depends only on μ, and C depends on $d, m, \mu, (h, \theta)$ in (1.11).

Proof The estimates follow directly from (3.4) and (1.7). We refer readers to Theorem 2.7 in [9] for the details. □

4 Two-scale expansions

We now perform the two-scale expansions for the operator $\partial_t + \mathcal{L}_\varepsilon$ in the case $\ell \ne 2$. For fixed $\varphi \in C_c^\infty(B(0, 1))$ such that $\varphi \ge 0$ and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. We define the smoothing operator

$$S_\delta(f)(x, t) = \int_{\mathbb{R}^d} \varphi_\delta(y) f(x-y, t)\, dy, \tag{4.1}$$

where $\varphi_\delta(x) = \frac{1}{\delta^d}\varphi(x/\delta)$. The following two Lemmas have been proved in [9].

Lemma 4.1 *Let $g(z, \tau)$ be a 1-periodic function in (z, τ) in $\mathbb{R}^{d+1}$, and $\psi = \psi(x)$ be a bounded Lipschitz function in $\mathbb{R}^d$. Then for $1 \le p < \infty$, $0 < \varepsilon \le \delta < 1$ and $k = 0, 1$,*

$$\|e^\psi g^\varepsilon S_\delta(\nabla^k f)\|_{L^p(\mathbb{R}^{d+1})} \le C\delta^{-k} e^{\delta\|\nabla\psi\|_\infty} \sup_{\tau\in\mathbb{R}} \|g(\tau)\|_{L^p(\mathbb{T}^d)} \|e^\psi f\|_{L^p(\mathbb{R}^{d+1})}, \tag{4.2}$$

where $g^\varepsilon(x, t) = g(x/\varepsilon, t/\varepsilon^\ell)$ and C depends only on d and p. Likewise,

$$\|e^\psi g^\varepsilon S_\delta(\nabla^k f)\|_{L^p(\Omega\times(T_0,T_1))} \le C\delta^{-k} e^{\delta\|\nabla\psi\|_\infty} \sup_{\tau\in\mathbb{R}} \|g(\tau)\|_{L^p(\mathbb{T}^d)} \|e^\psi f\|_{L^p(T_0,T_1;L^p(\Omega_\delta))} \tag{4.3}$$

for any $\Omega \subseteq \mathbb{R}^d$ and $(T_0, T_1) \subseteq \mathbb{R}$, where Ω_δ is given by

$$\Omega_\delta = \{x \in \mathbb{R}^d : dist(x, \Omega) < \delta\}. \tag{4.4}$$

Lemma 4.2 *Let $\psi = \psi(x)$ be a bounded Lipschitz function in $\mathbb{R}^d$. Then for any $1 \le p < \infty$,*

$$\int_{T_0}^{T_1}\int_\Omega \big|e^\psi(S_\delta(\nabla f) - \nabla f)\big|^p dxdt \le C\delta^p e^{\delta p\|\nabla\psi\|_\infty} \int_{T_0}^{T_1}\int_{\Omega_\delta} \big|e^\psi \nabla^2 f\big|^p dxdt, \tag{4.5}$$

where C depends only on d and p.

We now perform the two-scale expansions for $\partial_t + \mathcal{L}_\varepsilon = \partial_t - \operatorname{div}(A(x/\varepsilon, t/\varepsilon^\ell)\nabla)$. Assume

$$(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 \quad \text{in} \quad \Omega \times (T_0, T_1), \tag{4.6}$$

where $\partial_t + \mathcal{L}_0$ is the homogenized operator associated with $\partial_t + \mathcal{L}_\varepsilon$. Recall that

$$\mathcal{L}_0 = \begin{cases} -\operatorname{div}(\widehat{A_\infty}\nabla) & \text{if } 0 < \ell < 2, \\ -\operatorname{div}(\widehat{A_0}\nabla) & \text{if } 2 < \ell < \infty. \end{cases} \tag{4.7}$$

Let S_δ be defined as in (4.1). For $0 < \ell < 2$, we consider the two-scale expansion

$$\begin{aligned} w_{\varepsilon,\delta}(x,t) &= u_\varepsilon(x,t) - u_0(x,t) - \varepsilon\chi^\infty(x/\varepsilon, t/\varepsilon^\ell)S_\delta(\nabla u_0(x,t)) \\ &\quad - \varepsilon^\ell \mathcal{B}(t/\varepsilon^\ell)\nabla S_\delta(\nabla u_0(x,t)) - \varepsilon^2\nabla\mathfrak{B}(x/\varepsilon, t/\varepsilon^\ell)\nabla S_\delta(\nabla u_0(x,t)), \end{aligned} \tag{4.8}$$

where χ^∞ is the corrector, and $\mathcal{B}$, $\mathfrak{B}$ are the flux correctors defined in Sect. 2.1. For $2 < \ell < \infty$, we consider the two-scale expansion

$$\begin{aligned} \widetilde{w}_\varepsilon(x,t) &= u_\varepsilon(x,t) - u_0(x,t) - \varepsilon\chi^0(x/\varepsilon)S_\varepsilon(\nabla u_0(x,t)) \\ &\quad - \varepsilon^{\ell-1}\nabla\Phi(x/\varepsilon, t/\varepsilon^\ell)S_\varepsilon(\nabla u_0(x,t)) - \varepsilon^\ell\Phi(x/\varepsilon, t/\varepsilon^\ell)\nabla S_\varepsilon(\nabla u_0(x,t)) \\ &\quad - \varepsilon^2\nabla\Upsilon(x/\varepsilon)\nabla S_\varepsilon(\nabla u_0(x,t)), \end{aligned} \tag{4.9}$$

where χ^0 is the corrector, and Φ, Υ are the flux correctors defined in Sect. 2.2. For convenience, hereafter we shall use f^ε to denote $f(x/\varepsilon, t/\varepsilon^\ell)$. Particularly $f^\varepsilon = f(x/\varepsilon)$ if f is independent of t, and $f^\varepsilon = f(t/\varepsilon^\ell)$ if f is independent of x. For example, $\mathcal{B}^\varepsilon = \mathcal{B}(t/\varepsilon^\ell)$ and $\mathfrak{B}^\varepsilon = \mathfrak{B}(x/\varepsilon, t/\varepsilon^\ell)$.

Lemma 4.3 *Let u_ε and u_0 satisfy (4.6), and $w_{\varepsilon,\delta}$ be defined as in (4.8). Then we have*

$$(\partial_t + \mathcal{L}_\varepsilon)w_{\varepsilon,\delta} = \varepsilon \mathit{div}(F_{\varepsilon,\delta}) \quad \mathit{in}\ \Omega \times (T_0, T_1), \tag{4.10}$$

where

$$\begin{aligned} F_{\varepsilon,\delta} &= \varepsilon^{-1}(A^\varepsilon - \widehat{A_\infty})(\nabla u_0 - S_\delta(\nabla u_0)) - \phi^\varepsilon\nabla S_\delta(\nabla u_0) - \varepsilon^{\ell-1}\mathcal{B}^\varepsilon\partial_t S_\delta(\nabla u_0) \\ &\quad + \varepsilon^{\ell-1}A^\varepsilon\mathcal{B}^\varepsilon\nabla^2 S_\delta(\nabla u_0) + A^\varepsilon(\chi^\infty)^\varepsilon\nabla S_\delta(\nabla u_0) - \varepsilon(\nabla\mathfrak{B})^\varepsilon\partial_t S_\delta(\nabla u_0) \\ &\quad - \varepsilon^{1-\ell}(\partial_\tau\nabla\mathfrak{B})^\varepsilon S_\delta(\nabla u_0) + A^\varepsilon(\nabla^2\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0) + \varepsilon A^\varepsilon(\nabla\mathfrak{B})^\varepsilon\nabla^2 S_\delta(\nabla u_0). \end{aligned} \tag{4.11}$$

Proof By direct calculations, we deduce that

$$\begin{aligned} (\partial_t + \mathcal{L}_\varepsilon)w_{\varepsilon,\delta} &= (\mathcal{L}_0 - \mathcal{L}_\varepsilon)u_0 - (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon(\chi^\infty)^\varepsilon S_\delta(\nabla u_0)\big\} \\ &\quad - (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon^\ell\mathcal{B}^\varepsilon\nabla S_\delta(\nabla u_0)\big\} - (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon^2(\nabla\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)\big\} \\ &= -\operatorname{div}\big\{(\widehat{A_\infty} - A^\varepsilon)\big(\nabla u_0 - S_\delta(\nabla u_0)\big)\big\} - \operatorname{div}\big\{(\widehat{A_\infty} - A^\varepsilon)S_\delta(\nabla u_0)\big\} \\ &\quad + \operatorname{div}\big\{A^\varepsilon(\nabla\chi^\infty)^\varepsilon S_\delta(\nabla u_0)\big\} + \varepsilon\operatorname{div}\big\{A^\varepsilon(\chi^\infty)^\varepsilon\nabla S_\delta(\nabla u_0)\big\} \\ &\quad - \partial_t\big\{\varepsilon(\chi^\infty)^\varepsilon S_\delta(\nabla u_0)\big\} - (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon^\ell\mathcal{B}^\varepsilon\nabla S_\delta(\nabla u_0)\big\} \\ &\quad - (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon^2(\nabla\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)\big\}. \end{aligned} \tag{4.12}$$

In view of Lemma 2.2, we have

$$\begin{aligned} \partial_t\big\{\varepsilon(\chi^\infty)^\varepsilon S_\delta(\nabla u_0)\big\} &= \varepsilon^2\partial_t\operatorname{div}\big\{(\nabla\mathfrak{B})^\varepsilon S_\delta(\nabla u_0)\big\} - \varepsilon^2\partial_t\big\{(\nabla\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)\big\} \\ &= \varepsilon^2\operatorname{div}\big\{(\nabla\mathfrak{B})^\varepsilon\partial_t S_\delta(\nabla u_0)\big\} + \varepsilon^{2-\ell}\operatorname{div}\big\{(\partial_\tau\nabla\mathfrak{B})^\varepsilon S_\delta(\nabla u_0)\big\} \\ &\quad - \varepsilon^2\partial_t\big\{(\nabla\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)\big\}. \end{aligned} \tag{4.13}$$

On the other hand, note that

$$
\begin{aligned}
&-\operatorname{div}\big\{(\widehat{A_\infty}-A^\varepsilon)S_\delta(\nabla u_0)\big\}+\operatorname{div}\big\{A^\varepsilon(\nabla\chi^\infty)^\varepsilon S_\delta(\nabla u_0)\big\}\\
&\quad=\operatorname{div}\big\{(B^\varepsilon-\widehat{B}^\varepsilon)S_\delta(\nabla u_0)\big\}+\operatorname{div}\big\{\widehat{B}^\varepsilon S_\delta(\nabla u_0)\big\},
\end{aligned}
$$

where B and $\widehat{B}$ are defined by (2.15) and (2.16). By Lemmas 2.3 and 2.4, we deduce that

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}\Big\{(B_{ij}^\varepsilon-\widehat{B}_{ij}^\varepsilon)S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}+\frac{\partial}{\partial x_i}\Big\{\widehat{B}_{ij}^\varepsilon S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&\quad=\varepsilon\frac{\partial^2}{\partial x_i\partial x_k}\Big\{\phi_{kij}^{\alpha\beta}(x/\varepsilon,t/\varepsilon^\ell)S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}-\varepsilon\frac{\partial}{\partial x_i}\Big\{\phi_{kij}^{\alpha\beta}(x/\varepsilon,t/\varepsilon^\ell)\frac{\partial}{\partial x_k}S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&\qquad+\varepsilon^\ell\partial_t\frac{\partial}{\partial x_i}\Big\{\mathcal{B}_{ij}^{\alpha\beta}(t/\varepsilon^\ell)S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}-\varepsilon^\ell\frac{\partial}{\partial x_i}\Big\{\mathcal{B}_{ij}^{\alpha\beta}(t/\varepsilon^\ell)\partial_t S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&\quad=-\varepsilon\frac{\partial}{\partial x_i}\Big\{\phi_{kij}^{\alpha\beta}(x/\varepsilon,t/\varepsilon^\ell)\frac{\partial}{\partial x_k}S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}+\varepsilon^\ell\partial_t\Big\{\mathcal{B}_{ij}^{\alpha\beta}(t/\varepsilon^\ell)\frac{\partial}{\partial x_i}S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&\qquad-\varepsilon^\ell\frac{\partial}{\partial x_i}\Big\{\mathcal{B}_{ij}^{\alpha\beta}(t/\varepsilon^\ell)\partial_t S_\delta\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\},
\end{aligned}
\tag{4.14}
$$

where we have used the skew-symmetry of ϕ for the last step.

Finally, note that

$$
-(\partial_t+\mathcal{L}_\varepsilon)\big\{\varepsilon^\ell\mathcal{B}^\varepsilon\nabla S_\delta(\nabla u_0)\big\}=-\varepsilon^\ell\partial_t\big\{\mathcal{B}^\varepsilon\nabla S_\delta(\nabla u_0)\big\}+\varepsilon^\ell\operatorname{div}\big\{A^\varepsilon\mathcal{B}^\varepsilon\nabla^2 S_\delta(\nabla u_0)\big\},
\tag{4.15}
$$

and

$$
\begin{aligned}
(\partial_t+\mathcal{L}_\varepsilon)\big\{\varepsilon^2(\nabla\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)\big\}&=\varepsilon^2\partial_t\big\{(\nabla\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)\big\}-\varepsilon\operatorname{div}\big\{A^\varepsilon(\nabla^2\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)\big\}\\
&\quad-\varepsilon^2\operatorname{div}\big\{A^\varepsilon(\nabla\mathfrak{B})^\varepsilon\nabla^2 S_\delta(\nabla u_0)\big\}.
\end{aligned}
\tag{4.16}
$$

By taking (4.13)–(4.16) into (4.12), one gets (4.10) immediately. □

Lemma 4.4 *Let u_ε and u_0 be given by (4.6). Let $\widetilde{w}_\varepsilon$ be defined by (4.9). Then we have*

$$
(\partial_t+\mathcal{L}_\varepsilon)\widetilde{w}_\varepsilon=\varepsilon\,div(\widetilde{F}_\varepsilon)\quad in\ \Omega\times(T_0,T_1),
\tag{4.17}
$$

where

$$
\begin{aligned}
\widetilde{F}_\varepsilon&=\varepsilon^{-1}(A^\varepsilon-\widehat{A_0})\big(\nabla u_0-S_\varepsilon(\nabla u_0)\big)+A^\varepsilon(\chi^0)^\varepsilon\nabla S_\varepsilon(\nabla u_0)\\
&\quad-\varepsilon^{\ell-1}\Phi^\varepsilon\partial_t S_\varepsilon(\nabla u_0)-\Psi^\varepsilon\nabla S_\varepsilon(\nabla u_0)\\
&\quad+\varepsilon^{\ell-3}A^\varepsilon(\nabla^2\Phi)^\varepsilon S_\varepsilon(\nabla u_0)+2\varepsilon^{\ell-2}A^\varepsilon(\nabla\Phi)^\varepsilon\nabla S_\varepsilon(\nabla u_0)\\
&\quad+\varepsilon^{\ell-1}A^\varepsilon\Phi^\varepsilon\nabla^2 S_\varepsilon(\nabla u_0)-\varepsilon(\nabla\Upsilon)^\varepsilon\partial_t S_\varepsilon(\nabla u_0)\\
&\quad+A^\varepsilon(\nabla^2\Upsilon)^\varepsilon\nabla S_\varepsilon(\nabla u_0)+\varepsilon A^\varepsilon(\nabla\Upsilon)^\varepsilon\nabla^2 S_\varepsilon(\nabla u_0).
\end{aligned}
\tag{4.18}
$$

Proof In view of (4.6), we deduce that

$$
\begin{aligned}
(\partial_t+\mathcal{L}_\varepsilon)\widetilde{w}_\varepsilon&=(\mathcal{L}_0-\mathcal{L}_\varepsilon)u_0\\
&\quad-(\partial_t+\mathcal{L}_\varepsilon)\big\{\varepsilon(\chi^0)^\varepsilon S_\varepsilon(\nabla u_0)\big\}-(\partial_t+\mathcal{L}_\varepsilon)\big\{\varepsilon^{\ell-1}(\nabla\Phi)^\varepsilon S_\varepsilon(\nabla u_0)\big\}\\
&\quad-(\partial_t+\mathcal{L}_\varepsilon)\big\{\varepsilon^\ell\Phi^\varepsilon\nabla S_\varepsilon(\nabla u_0)\big\}-(\partial_t+\mathcal{L}_\varepsilon)\big\{\varepsilon^2(\nabla\Upsilon)^\varepsilon\nabla S_\varepsilon(\nabla u_0)\big\}\\
&=-\operatorname{div}\big\{(\widehat{A_0}-A^\varepsilon)\big(\nabla u_0-S_\varepsilon(\nabla u_0)\big)\big\}-\operatorname{div}\big\{(\widehat{A_0}-A^\varepsilon)S_\varepsilon(\nabla u_0)\big\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \operatorname{div}\big\{A^\varepsilon(\nabla\chi^0)^\varepsilon S_\varepsilon(\nabla u_0)\big\} + \varepsilon\operatorname{div}\big\{A^\varepsilon(\chi^0)^\varepsilon \nabla S_\varepsilon(\nabla u_0)\big\}\\
&- \partial_t\big\{\varepsilon(\chi^0)^\varepsilon S_\varepsilon(\nabla u_0)\big\} - (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon^{\ell-1}(\nabla\Phi)^\varepsilon S_\varepsilon(\nabla u_0)\big\}\\
&- (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon^{\ell}\Phi^\varepsilon \nabla S_\varepsilon(\nabla u_0)\big\} - (\partial_t + \mathcal{L}_\varepsilon)\big\{\varepsilon^{2}(\nabla\Upsilon)^\varepsilon \nabla S_\varepsilon(\nabla u_0)\big\}. \qquad (4.19)
\end{aligned}
$$

Thanks to (2.23), we have

$$
\begin{aligned}
&- \operatorname{div}\big\{(\widehat{A_0} - A^\varepsilon) S_\varepsilon(\nabla u_0)\big\} + \operatorname{div}\big\{A^\varepsilon(\nabla\chi^0)^\varepsilon S_\varepsilon(\nabla u_0)\big\}\\
&= \operatorname{div}\big\{(B_0^\varepsilon - \widetilde{B_0}^\varepsilon) S_\varepsilon(\nabla u_0)\big\} + \operatorname{div}\big\{\widetilde{B_0}^\varepsilon S_\varepsilon(\nabla u_0)\big\}.
\end{aligned}
$$

By Lemma 2.8 and (2.24), we deduce that

$$
\begin{aligned}
&\frac{\partial}{\partial x_i}\Big\{\big((B_0^\varepsilon)_{ij} - (\widetilde{B_0}^\varepsilon)_{ij}\big) S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\} + \frac{\partial}{\partial x_i}\Big\{(\widetilde{B_0}^\varepsilon)_{ij} S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&= \varepsilon^\ell \partial_t \frac{\partial}{\partial x_i}\Big\{\Phi_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^\ell) S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\} - \varepsilon^\ell \frac{\partial}{\partial x_i}\Big\{\Phi_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^\ell)\partial_t S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&\quad + \varepsilon \frac{\partial^2}{\partial x_i \partial x_k}\Big\{\Psi_{kij}^{\alpha\beta}(x/\varepsilon) S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\} - \varepsilon \frac{\partial}{\partial x_i}\Big\{\Psi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_k} S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&= \varepsilon^{\ell-1}\partial_t\Big\{\Big(\frac{\partial}{\partial x_i}\Phi_{ij}^{\alpha\beta}\Big)(x/\varepsilon, t/\varepsilon^\ell) S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\} + \varepsilon^\ell\partial_t\Big\{\Phi_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^\ell)\frac{\partial}{\partial x_i} S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}\\
&\quad - \varepsilon^\ell \frac{\partial}{\partial x_i}\Big\{\Phi_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^\ell)\partial_t S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\} - \varepsilon\frac{\partial}{\partial x_i}\Big\{\Psi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_k} S_\varepsilon\Big(\frac{\partial u_0}{\partial x_j}\Big)\Big\}, \qquad (4.20)
\end{aligned}
$$

where the skew-symmetry of Ψ is used in the last step. On the other hand, by Lemma 2.6,

$$
\begin{aligned}
-\partial_t\big\{\varepsilon(\chi^0)^\varepsilon S_\varepsilon(\nabla u_0)\big\} &= -\varepsilon^2\partial_t \operatorname{div}\big\{(\nabla\Upsilon)^\varepsilon S_\varepsilon(\nabla u_0)\big\} + \varepsilon^2\partial_t\big\{(\nabla\Upsilon)^\varepsilon \nabla S_\varepsilon(\nabla u_0)\big\}\\
&= -\varepsilon^2 \operatorname{div}\big\{(\nabla\Upsilon)^\varepsilon \partial_t S_\varepsilon(\nabla u_0)\big\} + \varepsilon^2\partial_t\big\{(\nabla\Upsilon)^\varepsilon \nabla S_\varepsilon(\nabla u_0)\big\}. \qquad (4.21)
\end{aligned}
$$

By taking (4.20) and (4.21) into (4.19), and some direct calculations, one gets (4.17) immediately. □

Theorem 4.1 *Assume $A(z,\tau)$ satisfies (1.2) and (1.3), and also (1.6) if $m \geq 2$. Also assume that $\|\partial_\tau A\|_\infty < \infty$ if $0 < \ell < 2$, and $\|\nabla^2 A\|_\infty < \infty$ if $2 < \ell < \infty$. Let $F_{\varepsilon,\delta}$ be given by (4.11), and $\widetilde{F}_\varepsilon$ be given by (4.18). Let $\psi = \psi(x)$ be a bounded Lipschitz function in $\mathbb{R}^d$. Then for any $1 \leq p < \infty$, we have*

$$
\begin{aligned}
&\int_{T_0}^{T_1}\int_\Omega \big|e^\psi F_{\varepsilon,\delta}\big|^p dxdt\\
&\quad \leq C\big(\delta^{2p}\varepsilon^{-p} + \delta^p + \varepsilon^{p(\ell-1)} + \varepsilon^p\big)\delta^{-p} e^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\big|e^\psi \nabla^2 u_0\big|^p dxdt\\
&\qquad + C\big(\varepsilon^{p(\ell-1)} + \varepsilon^p\big)\delta^{-p} e^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\big|e^\psi \partial_t u_0\big|^p dxdt\\
&\qquad + C\varepsilon^{p(1-\ell)} e^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\big|e^\psi \nabla u_0\big|^p dxdt, \qquad (4.22)
\end{aligned}
$$

and

$$
\int_{T_0}^{T_1}\int_\Omega \big|e^\psi \widetilde{F}_\varepsilon\big|^p dxdt \leq C\big(1 + \varepsilon^{p(\ell-2)}\big)e^{\varepsilon p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\varepsilon}\big\{\big|e^\psi\nabla^2 u_0\big|^p + |e^\psi \partial_t u_0|^p\big\}dxdt
$$

$$+\varepsilon^{p(\ell-3)}e^{\varepsilon p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\varepsilon}\left|e^\psi\nabla u_0\right|^p dxdt, \tag{4.23}$$

where Ω_δ is defined as in (4.4), and C depends only on d, m, p, μ, and $\omega_\varrho(A)$ (if $m\geq 2$).

Proof It suffices to prove (4.22), as the proof of (4.23) is almost the same by using the estimates for χ^0, Υ, Ψ, and Φ. As a consequence of (4.11), we note that

$$\begin{aligned}
&\int_{T_0}^{T_1}\int_\Omega\left|e^\psi F_{\varepsilon,\delta}\right|^p dxdt\\
&\leq C\varepsilon^{-p}\int_{T_0}^{T_1}\int_\Omega|\nabla u_0-S_\delta(\nabla u_0)|^p e^{p\psi}dxdt+C\int_{T_0}^{T_1}\int_\Omega|\phi^\varepsilon\nabla S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&\quad+C\varepsilon^{p(\ell-1)}\int_{T_0}^{T_1}\int_\Omega|\mathcal{B}^\varepsilon\partial_t S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&\quad+C\varepsilon^{p(\ell-1)}\int_{T_0}^{T_1}\int_\Omega|\mathcal{B}^\varepsilon\nabla^2 S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&\quad+C\int_{T_0}^{T_1}\int_\Omega|(\chi^\infty)^\varepsilon\nabla S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&\quad+C\varepsilon^{p}\int_{T_0}^{T_1}\int_\Omega|(\nabla\mathfrak{B})^\varepsilon\partial_t S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&\quad+C\varepsilon^{p(1-\ell)}\int_{T_0}^{T_1}\int_\Omega|(\partial_\tau\nabla\mathfrak{B})^\varepsilon S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&\quad+C\int_{T_0}^{T_1}\int_\Omega|(\nabla^2\mathfrak{B})^\varepsilon\nabla S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&\quad+C\varepsilon^{p}\int_{T_0}^{T_1}\int_\Omega|(\nabla\mathfrak{B})^\varepsilon\nabla^2 S_\delta(\nabla u_0)|^p e^{p\psi}dxdt\\
&=I_1+\ldots+I_9,
\end{aligned}\tag{4.24}$$

where C depends only on d and μ. By Lemma 4.2, we obtain that

$$I_1\leq C\delta^p\varepsilon^{-p}e^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\left|e^\psi\nabla^2u_0\right|^p dxdt.$$

By (4.3) and the estimates for χ^∞, $\mathfrak{B}$, ϕ in Lemmas 2.1, 2.2, and 2.3, we may deduce that

$$I_2+I_5+I_8\leq Ce^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\left|e^\psi\nabla^2u_0\right|^p dxdt.$$

On the other hand, by (4.3) and the estimates of $\mathfrak{B}$, $\mathcal{B}$ in Lemmas 2.2 and 2.4, we get

$$\begin{aligned}
I_3+I_4+I_7&\leq C\varepsilon^{p(\ell-1)}\delta^{-p}e^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\left(\left|e^\psi\nabla^2u_0\right|^p+|e^\psi\partial_tu_0|^p\right)dxdt\\
&\quad+C\varepsilon^{p(1-\ell)}e^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\left|e^\psi\nabla u_0\right|^p dxdt,
\end{aligned}$$

and also

$$I_6+I_9\leq C\varepsilon^p\delta^{-p}e^{\delta p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\delta}\left(\left|e^\psi\nabla^2u_0\right|^p+|e^\psi\partial_tu_0|^p\right)dxdt.$$

By taking the estimates for I_1-I_9 into (4.24), one gets (4.22) immediately. The proof is complete. □

5 Weighted estimates

Let Γ_0 be the matrix of fundamental solutions of the homogenized operator $\partial_t + \mathcal{L}_0$ in $\mathbb{R}^{d+1}$ and ψ be a bounded Lipschitz function in $\mathbb{R}^d$. Given $f \in C_0^\infty(\mathbb{R}^{d+1})$, let

$$u_0(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^d} \Gamma_0(x,t;y,s) f(y,s) e^{-\psi(y)} dy ds. \tag{5.1}$$

Then $(\partial_t + \mathcal{L}_0)u_0 = e^{-\psi} f$ in $\mathbb{R}^{d+1}$. The following two lemmas concerning on the weighted estimates of $\partial_t + \mathcal{L}_0$ and $\partial_t + \mathcal{L}_\varepsilon$ have been proved in [9].

Lemma 5.1 *Let u_0 be given by (5.1), where $f(x,t) = 0$ for $t \leq s_0$. Then we have*

$$\int_{s_0}^{t} \int_{\mathbb{R}^d} |e^{\psi} \nabla u_0|^2 dx dt \leq C(t - s_0) e^{\kappa_1 (t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t} \int_{\mathbb{R}^d} |f|^2 dx dt, \tag{5.2}$$

$$\int_{s_0}^{t} \int_{\mathbb{R}^d} |e^{\psi} (|\nabla^2 u_0| + |\partial_t u_0|)|^2 dx dt \leq C e^{\kappa (t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t} \int_{\mathbb{R}^d} |f|^2 dx dt \tag{5.3}$$

for any $s_0 < t < \infty$, where $\kappa, \kappa_1 > 0$ depend only on μ, and C depends only on μ and d.

Lemma 5.2 *Assume that*

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon) w_\varepsilon = \varepsilon div(F_\varepsilon) & in\ \mathbb{R}^d \times (s_0, \infty), \\ w_\varepsilon = 0 & on\ \mathbb{R}^d \times (t = s_0). \end{cases} \tag{5.4}$$

Let ψ be a bounded Lipschitz function in $\mathbb{R}^d$. Then for any $t > s_0$,

$$\int_{\mathbb{R}^d} |w_\varepsilon(x,t)|^2 e^{2\psi(x)} dx \leq C\varepsilon^2 e^{\kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t} \int_{\mathbb{R}^d} |F_\varepsilon(x,s)|^2 e^{2\psi(x)} dx ds, \tag{5.5}$$

where $\kappa > 0$, and $C > 0$ depends only on μ.

Theorem 5.1 *Assume that $u_\varepsilon \in L^2((-\infty,T); H^1(\mathbb{R}^d))$ and $u_0 \in L^2((-\infty,T); H^2(\mathbb{R}^d))$, $T \in \mathbb{R}$, satisfy*

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon) u_\varepsilon = (\partial_t + \mathcal{L}_0) u_0 & in\ \mathbb{R}^{d+1}, \\ u_\varepsilon(x,t) = u_0(x,t) = 0 & for\ \ t \leq s_0. \end{cases}$$

Let ψ be a bounded Lipschitz function in $\mathbb{R}^d$. Let $w_{\varepsilon,\delta}$ be given by (4.8) with $\delta = \varepsilon^{\ell/2}$. Then we have for any $t > s_0$,

$$\begin{aligned} &\int_{\mathbb{R}^d} |w_{\varepsilon,\delta}(x,t)|^2 e^{2\psi(x)} dx \\ &\leq C\varepsilon^{\ell} e^{2\varepsilon^{\ell/2}\|\nabla\psi\|_\infty + \kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t} \int_{\mathbb{R}^d} \left(\left|e^{\psi}\nabla^2 u_0\right|^2 + \left|e^{\psi}\partial_t u_0\right|^2\right) dx ds \\ &\quad + C\varepsilon^{4-2\ell} e^{2\varepsilon^{\ell/2}\|\nabla\psi\|_\infty + \kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t} \int_{\mathbb{R}^d} \left|e^{\psi}\nabla u_0\right|^2 dx ds, \end{aligned} \tag{5.6}$$

where $\kappa > 0$ depends only on μ, and $C > 0$ depends only on μ and d. Likewise, let $\widetilde{w}_\varepsilon$ be given by (4.9), we have for any $t > s_0$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\widetilde{w}_\varepsilon(x,t)|^2 e^{2\psi(x)} dx \\
&\quad \le C\varepsilon^2 e^{2\varepsilon\|\nabla\psi\|_\infty + \kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} \left(\left|e^\psi \nabla^2 u_0\right|^2 + \left|e^\psi \partial_t u_0\right|^2\right) dxds \\
&\qquad + C\varepsilon^{2\ell-4} e^{2\varepsilon\|\nabla\psi\|_\infty + \kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} \left|e^\psi \nabla u_0\right|^2 dxds,
\end{aligned} \tag{5.7}$$

where $\kappa > 0$ depends only on μ, and $C > 0$ depends only on μ and d.

Proof Since $w_{\varepsilon,\delta}$ satisfies (4.10) and $w_{\varepsilon,\delta} = 0$ on $\mathbb{R}^d \times (t = s_0)$. By taking $w_\varepsilon = w_{\varepsilon,\delta}$ and $F_\varepsilon = F_{\varepsilon,\delta}$ in (5.5), we obtain that

$$\int_{\mathbb{R}^d} |w_{\varepsilon,\delta}(x,t)|^2 e^{2\psi(x)} dx \le C\varepsilon^2 e^{\kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} |F_{\varepsilon,\delta}(x,s)|^2 e^{2\psi(x)} dxds,$$

which, together with (4.22) in the case $p = 2$ and $\delta = \varepsilon^{\ell/2}$, gives (5.6). Similarly, taking $w_\varepsilon = \widetilde{w}_\varepsilon$ and $F_\varepsilon = \widetilde{F}_\varepsilon$ in (5.5), it yields

$$\int_{\mathbb{R}^d} |\widetilde{w}_\varepsilon(x,t)|^2 e^{2\psi(x)} dx \le C\varepsilon^2 e^{\kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^t \int_{\mathbb{R}^d} |\widetilde{F}_\varepsilon(x,s)|^2 e^{2\psi(x)} dxds,$$

which, combined with (4.23) with $p = 2$, gives (5.7). □

Next, we consider the weighed L^∞ estimates.

Theorem 5.2 *Assume that A satisfies (1.2), (1.3), and $A \in \mathrm{VMO}_x$ if $m \ge 2$. Also assume that $\|\partial_\tau A\|_\infty < \infty$ if $0 < \ell < 2$, and $\|\nabla^2 A\|_\infty < \infty$ if $2 < \ell < \infty$. Let $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0$ in $B(x_0, 3r) \times (t_0 - 4r^2, t_0)$ with $\varepsilon + \varepsilon^{\ell/2} \le r < \infty$. Let ψ be a bounded Lipschitz function in $\mathbb{R}^d$.*

- *For the case $0 < \ell < 2$, we have*

$$\begin{aligned}
&\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))} \\
&\quad \le Ce^{3r\|\nabla\psi\|_\infty} \left(\fint_{Q_{2r}(x_0,t_0)} |e^\psi(u_\varepsilon - u_0)|^2\right)^{1/2} \\
&\qquad + C\varepsilon^{\ell/2} r e^{3r\|\nabla\psi\|_\infty} \|e^\psi(|\nabla^2 u_0| + |\partial_t u_0|)\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))} \\
&\qquad + C\varepsilon^{2-\ell} r e^{3r\|\nabla\psi\|_\infty} \|e^\psi \nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))} \\
&\qquad + C\varepsilon e^{3r\|\nabla\psi\|_\infty} \|e^\psi \nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))},
\end{aligned} \tag{5.8}$$

  where $C > 0$ depends only on d, μ, m, $\omega_\varrho(A)$ in (1.6) (if $m \ge 2$), and $\|\partial_\tau A\|_\infty$.
- *For the case $2 < \ell < \infty$, we have*

$$\begin{aligned}
&\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))} \\
&\quad \le Ce^{3r\|\nabla\psi\|_\infty} \left(\fint_{Q_{2r}(x_0,t_0)} |e^\psi(u_\varepsilon - u_0)|^2\right)^{1/2} \\
&\qquad + C\varepsilon r e^{3r\|\nabla\psi\|_\infty} \|e^\psi(|\nabla^2 u_0| + |\partial_t u_0|)\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))} \\
&\qquad + C\varepsilon^{\ell-2} r e^{3r\|\nabla\psi\|_\infty} \|e^\psi \nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))} \\
&\qquad + C\varepsilon e^{3r\|\nabla\psi\|_\infty} \|e^\psi \nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))},
\end{aligned} \tag{5.9}$$

  where $C > 0$ depends only on d, μ, m, and $\|\nabla^2 A\|_\infty$.

Proof Let $w_{\varepsilon,\delta}$ be defined by (4.8). By Lemma 4.3, we have $(\partial_t + \mathcal{L}_\varepsilon) w_{\varepsilon,\delta} = \varepsilon \text{div}(F_{\varepsilon,\delta})$ in $Q_{2r}(x_0, t_0)$. It follows from Theorem 3.1 that

$$\|w_{\varepsilon,\delta}\|_{L^\infty(Q_r(x_0,t_0))} \le C\Big\{\Big(\fint_{Q_{2r}(x_0,t_0)} |w_{\varepsilon,\delta}|^2\Big)^{1/2} + \varepsilon r\Big(\fint_{Q_{2r}(x_0,t_0)} |F_{\varepsilon,\delta}|^p\Big)^{1/p}\Big\},$$

where $p > d+2$. This gives

$$\begin{aligned}\|u_\varepsilon - u_0\|_{L^\infty(Q_r(x_0,t_0))} &\le C\Big(\fint_{Q_{2r}(x_0,t_0)} |u_\varepsilon - u_0|^2\Big)^{1/2} + C\varepsilon r\Big(\fint_{Q_{2r}(x_0,t_0)} |F_{\varepsilon,\delta}|^p\Big)^{1/p}\\ &\quad + C\varepsilon \|S_\delta(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))} + C(\varepsilon^\ell + \varepsilon^2)\|S_\delta(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))},\end{aligned}$$

where we have used the boundedness of χ^∞, $\mathcal{B}$ and $\nabla\mathfrak{B}$ in Lemmas 2.1, 2.4 and 2.2, respectively. Using $|\psi(x) - \psi(y)| \le 2r\|\nabla\psi\|_\infty$ for $x, y \in B(x_0, 2r)$, we obtain

$$\begin{aligned}&\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))}\\ &\le Ce^{2r\|\nabla\psi\|_\infty}\Big(\fint_{Q_{2r}(x_0,t_0)} |e^\psi(u_\varepsilon - u_0)|^2\Big)^{1/2} + C\varepsilon r e^{2r\|\nabla\psi\|_\infty}\Big(\fint_{Q_{2r}(x_0,t_0)} |e^\psi F_{\varepsilon,\delta}|^p\Big)^{1/p}\\ &\quad + C\varepsilon e^{2r\|\nabla\psi\|_\infty}\|e^\psi S_\delta(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))} + C\varepsilon^\ell e^{2r\|\nabla\psi\|_\infty}\|e^\psi S_\delta(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))}.\end{aligned}$$

By setting $\delta = \varepsilon^{\ell/2}$, and using the assumption $\varepsilon^{\ell/2} \le r$ and Theorem 4.1, we derive (5.8) immediately.

The proof of (5.9) is completely parallel by using Lemma 4.4 and the estimates of χ^0, Φ, and $\nabla\Upsilon$. We therefore omit the details for concision. □

6 Proof of the main results

We are now ready to provide the proofs of Theorems 1.1 to 1.3.

Proof of Theorem 1.1 We follow the scheme of [9]. For fixed $x_0, y_0 \in \mathbb{R}^d$ and $s_0 < t_0$, it suffices to consider the case $\varepsilon + \varepsilon^{\ell/2} < r = \sqrt{t_0 - s_0}/100$, since otherwise the estimate (1.8) follows directly from (1.7). For $f \in C_0^\infty(Q_r(y_0, s_0))$, let

$$\begin{aligned} u_0(x,t) &= \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_0(x,t;y,s) f(y,s) e^{-\psi(y)} dy ds,\\ u_\varepsilon(x,t) &= \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_\varepsilon(x,t;y,s) f(y,s) e^{-\psi(y)} dy ds. \end{aligned} \tag{6.1}$$

Then $(\partial_t + \mathcal{L}_0)u_0 = (\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = e^{-\psi} f$ in $\mathbb{R}^{d+1}$ and $u_\varepsilon(x,t) = u_0(x,t) = 0$ if $t \le s_0 - r^2$.

We first consider the case $0 < \ell < 2$. Let $w_{\varepsilon,\delta}$ be defined by (4.8) with $\delta = \varepsilon^{\ell/2}$. It follows from (5.2), (5.3) and (5.6) that

$$\begin{aligned}&\int_{\mathbb{R}^d} |w_{\varepsilon,\delta}(x,t)|^2 e^{2\psi(x)} dx\\ &\le C\varepsilon^\ell e^{2\varepsilon^{\ell/2}\|\nabla\psi\|_\infty + 2\kappa(t-s_0+r^2)\|\nabla\psi\|_\infty^2} \int_{s_0-r^2}^t \int_{\mathbb{R}^d} |f|^2 dx ds\\ &\quad + C(t - s_0 + r^2)\varepsilon^{4-2\ell} e^{2\varepsilon^{\ell/2}\|\nabla\psi\|_\infty + 2\kappa(t-s_0+r^2)\|\nabla\psi\|_\infty^2} \int_{s_0-r^2}^t \int_{\mathbb{R}^d} |f|^2 dx ds \end{aligned} \tag{6.2}$$

for any $t > s_0 - r^2$. By (5.8) and the definition of $w_{\varepsilon,\delta}$, we obtain that

$$\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))}$$

$$
\begin{aligned}
&\le Ce^{3r\|\nabla\psi\|_\infty}\Big(\fint_{Q_{2r}(x_0,t_0)}|e^\psi w_{\varepsilon,\delta}|^2\Big)^{1/2}\\
&\quad + C\varepsilon^{\ell/2} r e^{3r\|\nabla\psi\|_\infty}\|e^\psi(|\nabla^2 u_0|+|\partial_t u_0|)\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))}\\
&\quad + C\varepsilon^{2-\ell} r e^{3r\|\nabla\psi\|_\infty}\|e^\psi\nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))}\\
&\quad + C\varepsilon e^{3r\|\nabla\psi\|_\infty}\|e^\psi\nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-4r^2,t_0))}.
\end{aligned}
\tag{6.3}
$$

Since $f\in C_0^\infty(Q_r(y_0,s_0))$, (1.5) implies that

$$
\begin{aligned}
&|\nabla^2 u_0(x,t)|+|\partial_t u_0(x,t)|+r^{-1}|\nabla u_0(x,t)|\\
&\quad\le C\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\fint_{Q_r(y_0,s_0)}|fe^{-\psi}|dyds
\end{aligned}
$$

for any $x\in B(x_0,3r)$ and $|t-t_0|\le 4r^2$, where $\kappa>0$ depends only on μ. Therefore, by taking (6.2) into (6.3) we deduce that

$$
\begin{aligned}
&\|e^\psi(u_\varepsilon-u_0)\|_{L^\infty(Q_r(x_0,t_0))}\\
&\quad\le C(\varepsilon^{\ell/2}r+\varepsilon^{2-\ell}r^2)e^{cr\|\nabla\psi\|_\infty+cr^2\|\nabla\psi\|_\infty^2}\Big(\fint_{Q_r(y_0,s_0)}|f|^2\Big)^{1/2}\\
&\qquad + C\varepsilon^{\ell/2} r e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\Big(\fint_{Q_r(y_0,s_0)}|f|^2\Big)^{1/2}\\
&\qquad + C\varepsilon^{2-\ell} r^2 e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\Big(\fint_{Q_r(y_0,s_0)}|f|^2\Big)^{1/2}.
\end{aligned}
$$

By duality we find that for any $(x,t)\in Q_r(x_0,t_0)$,

$$
\begin{aligned}
&\Big(\fint_{Q_r(y_0,s_0)}|e^{\psi(x)-\psi(y)}(\Gamma_\varepsilon(x,t;y,s)-\Gamma_0(x,t;y,s))|^2dyds\Big)^{1/2}\\
&\quad\le C(\varepsilon^{\ell/2}r^{-d-1}+\varepsilon^{2-\ell}r^{-d})e^{cr\|\nabla\psi\|_\infty+cr^2\|\nabla\psi\|_\infty^2}\\
&\qquad + C\varepsilon^{\ell/2}r^{-d-1}e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\\
&\qquad + C\varepsilon^{2-\ell}r^{-d}e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}.
\end{aligned}
\tag{6.4}
$$

Let $\widetilde{A}=\widetilde{A}(z,\tau)=A^*(z,-\tau)$ and $\widetilde{\mathcal{L}}_\varepsilon=-\mathrm{div}(\widetilde{A}\nabla)$. Let $v_\varepsilon(y,s)=\Gamma_\varepsilon(x_0,t_0;y,-s)$ and $v_0(y,s)=\Gamma_0(x_0,t_0;y,-s)$. Then

$$
(\partial_t+\widetilde{\mathcal{L}}_\varepsilon)v_\varepsilon=(\partial_t+\widetilde{\mathcal{L}}_0)v_0=0\quad\text{in } B(y_0,3r)\times(-s_0-4r^2,-s_0),
$$

where $\partial_t+\widetilde{\mathcal{L}}_0$ be the homogenized operator of $\partial_t+\widetilde{\mathcal{L}}_\varepsilon$. Since $\widetilde{A}$ satisfies the the same conditions as $A(z,\tau)$, we apply Theorem 5.2 with ψ replaced by $-\psi$ to obtain that

$$
\begin{aligned}
&|e^{\psi(x_0)-\psi(y_0)}(v_\varepsilon(y_0,-s_0)-v_0(y_0,-s_0))|\\
&\quad\le\|e^{\psi(x_0)-\psi(y)}(v_\varepsilon(y,-s)-v_0(y,-s))\|_{L^\infty(Q_r(y_0,-s_0))}\\
&\quad\le Ce^{3r\|\nabla\psi\|_\infty}\Big(\fint_{Q_{r/2}(y_0,-s_0)}|e^{\psi(x_0)-\psi(y)}(v_\varepsilon-v_0)|^2dyds\Big)^{1/2}\\
&\qquad + C\varepsilon^{\ell/2}re^{3r\|\nabla\psi\|_\infty}\|e^{\psi(x_0)-\psi(y)}(|\nabla^2 v_0|+|\partial_t v_0|)\|_{L^\infty(B(y_0,3r)\times(-s_0-4r^2,-s_0))}\\
&\qquad + C\varepsilon^{2-\ell}re^{3r\|\nabla\psi\|_\infty}\|e^{\psi(x_0)-\psi(y)}\nabla v_0\|_{L^\infty(B(y_0,3r)\times(-s_0-4r^2,-s_0))}\\
&\qquad + C\varepsilon e^{3r\|\nabla\psi\|_\infty}\|e^{\psi(x_0)-\psi(y)}\nabla v_0\|_{L^\infty(B(y_0,3r)\times(-s_0-4r^2,-s_0))}
\end{aligned}
$$

$$
\begin{aligned}
&\leq Ce^{3r\|\nabla\psi\|_\infty}\Big(\fint_{Q_r(y_0,s_0)}|e^{\psi(x_0)-\psi(y)}(\Gamma_\varepsilon(x_0,t_0;y,s)-\Gamma_0(x_0,t_0;y,s))|^2dyds\Big)^{1/2}\\
&\quad + C\varepsilon^{\ell/2}r^{-d-1}e^{cr\|\nabla\psi\|_\infty}e^{\psi(x_0)-\psi(y_0)}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\\
&\quad + C\varepsilon^{2-\ell}r^{-d}e^{cr\|\nabla\psi\|_\infty}e^{\psi(x_0)-\psi(y_0)}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\\
&\leq C(\varepsilon^{\ell/2}r^{-d-1}+\varepsilon^{2-\ell}r^{-d})e^{cr\|\nabla\psi\|_\infty+cr^2\|\nabla\psi\|^2_\infty}\\
&\quad + C\varepsilon^{\ell/2}r^{-d-1}e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\\
&\quad + C\varepsilon^{2-\ell}r^{-d}e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\},
\end{aligned}\tag{6.5}
$$

where we have used (1.5) for the third step and (6.4) for the last step.

Finally, to derive (1.8) we follow the ideas of [2, 11] (see also [9]) to set $\psi(y)=\gamma\psi_0(|y-y_0|)$ with $\psi_0(\rho)=\rho$ if $\rho\leq|x_0-y_0|$ and $\psi_0(\rho)=|x_0-y_0|$ if $\rho>|x_0-y_0|$, where $\gamma\geq 0$ is to be determined later. Note that $\|\nabla\psi\|_\infty=\gamma$ and $\psi(x_0)-\psi(y_0)=\gamma|x_0-y_0|$. By (6.5), we have

$$
\begin{aligned}
&|\Gamma_\varepsilon(x_0,t_0;y,s)-\Gamma_0(x_0,t_0;y,s))|\\
&\quad\leq C(\varepsilon^{\ell/2}r^{-d-1}+\varepsilon^{2-\ell}r^{-d})e^{-\gamma|x_0-y_0|}e^{c_0(\gamma\sqrt{t_0-s_0}+(t_0-s_0)\gamma^2)}\\
&\qquad + C\varepsilon^{\ell/2}r^{-d-1}e^{-\gamma|x_0-y_0|}e^{c_0(\sqrt{t_0-s_0}+|x_0-y_0|)\gamma}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\\
&\qquad + C\varepsilon^{2-\ell}r^{-d}e^{-\gamma|x_0-y_0|}e^{c_0(\sqrt{t_0-s_0}+|x_0-y_0|)\gamma}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\},
\end{aligned}
$$

where c_0 depends at most on μ. If $|x_0-y_0|\leq 2c_0\sqrt{t_0-s_0}$, we choose $\gamma=0$. As a result,

$$
\begin{aligned}
|\Gamma_\varepsilon(x_0,t_0;y,s)-\Gamma_0(x_0,t_0;y,s))|&\leq C\varepsilon^{\ell/2}(t_0-s_0)^{\frac{-d-1}{2}}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}\\
&\quad + C\varepsilon^{2-\ell}(t_0-s_0)^{-\frac{d}{2}}\exp\Big\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\Big\}.
\end{aligned}
$$

If $|x_0-y_0|>2c_0\sqrt{t_0-s_0}$, we choose $\gamma=\sigma|x_0-y_0|/(t_0-s_0)$ with $\sigma\leq\min\{\frac{1}{4}c_0^{-1},1/2(c_0+1/2)^{-1}\kappa\}$. As a result, we get

$$
-\gamma|x_0-y_0|+c_0\gamma\sqrt{t_0-s_0}+c_0(t_0-s_0)\gamma^2\leq\frac{-\sigma|x_0-y_0|^2}{4(t_0-s_0)},
$$

and

$$
\begin{aligned}
c_0\gamma|x_0-y_0|+c_0\gamma\sqrt{t_0-s_0}-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}&\leq\Big\{\frac{1}{2}\sigma+c_0\sigma-\kappa\Big\}\frac{|x_0-y_0|^2}{t_0-s_0}\\
&\leq\frac{-\kappa|x_0-y_0|^2}{2(t_0-s_0)}.
\end{aligned}
$$

Recall that $r=\sqrt{t_0-s_0}/100$. We have thus proved that

$$
|\Gamma_\varepsilon(x_0,t_0;y,s)-\Gamma_0(x_0,t_0;y,s))|\leq C\varepsilon^{\ell/2}(t_0-s_0)^{\frac{-d-1}{2}}\exp\Big\{-\frac{\kappa_1|x_0-y_0|^2}{t_0-s_0}\Big\}
$$

$$+ C\varepsilon^{2-\ell}(t_0 - s_0)^{-\frac{d}{2}} \exp\Big\{ - \frac{\kappa_1 |x_0 - y_0|^2}{t_0 - s_0}\Big\}.$$

This gives the estimate (1.8) for $0 < \ell < 2$.

Next we consider the case $2 < \ell < \infty$. Let $\widetilde{w}_\varepsilon$ be defined by (4.9). By (5.2), (5.3) and (5.7),

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\widetilde{w}_\varepsilon(x,t)|^2 e^{2\psi(x)} dx \\
&\le C\varepsilon^2 e^{2\varepsilon\|\nabla\psi\|_\infty + 2\kappa(t-s_0+r^2)\|\nabla\psi\|_\infty^2} \int_{s_0-r^2}^t \int_{\mathbb{R}^d} |f|^2 dxdt \\
&\quad + C(t - s_0 + r^2)\varepsilon^{2\ell-4} e^{2\varepsilon\|\nabla\psi\|_\infty + 2\kappa(t-s_0+r^2)\|\nabla\psi\|_\infty^2} \int_{s_0-r^2}^t \int_{\mathbb{R}^d} |f|^2 dxdt
\end{aligned}$$

for any $t > s_0$. By using (1.5) and (5.9) we can deduce that

$$\begin{aligned}
&\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))} \\
&\le C(\varepsilon r + \varepsilon^{\ell-2} r^2) e^{cr\|\nabla\psi\|_\infty + cr^2\|\nabla\psi\|_\infty^2} \Big(\fint_{Q_r(y_0,s_0)} |f|^2 \Big)^{1/2} \\
&\quad + C\varepsilon r e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty} \exp\Big\{ - \frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\Big\} \Big(\fint_{Q_r(y_0,s_0)} |f|^2 \Big)^{1/2} \\
&\quad + C\varepsilon^{\ell-2} r^2 e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty} \exp\Big\{ - \frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\Big\} \Big(\fint_{Q_r(y_0,s_0)} |f|^2 \Big)^{1/2}.
\end{aligned}$$

This, by duality, implies that

$$\begin{aligned}
&\Big(\fint_{Q_r(y_0,s_0)} |e^{\psi(x)-\psi(y)}(\Gamma_\varepsilon(x,t;y,s) - \Gamma_0(x,t;y,s))|^2 dyds \Big)^{1/2} \\
&\le C(\varepsilon r^{-d-1} + \varepsilon^{\ell-2} r^{-d}) e^{cr\|\nabla\psi\|_\infty + r^2\|\nabla\psi\|_\infty^2} \\
&\quad + C(\varepsilon r^{-d-1} + \varepsilon^{\ell-2} r^{-d}) e^{c(r+|x_0-y_0|)\|\nabla\psi\|_\infty} \exp\Big\{ - \frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\Big\}
\end{aligned} \tag{6.6}$$

for any $(x,t) \in Q_r(x_0,t_0)$. Then we perform the same argument as in the case $0 < \ell < 2$ to consider the dual operators and chose proper function ψ and finally to deduce that

$$\begin{aligned}
|\Gamma_\varepsilon(x_0,t_0;y,s) - \Gamma_0(x_0,t_0;y,s))| &\le C\varepsilon(t_0 - s_0)^{\frac{-d-1}{2}} \exp\Big\{ - \frac{\kappa_1|x_0 - y_0|^2}{t_0 - s_0}\Big\} \\
&\quad + C\varepsilon^{\ell-2}(t_0 - s_0)^{-\frac{d}{2}} \exp\Big\{ - \frac{\kappa_1|x_0 - y_0|^2}{t_0 - s_0}\Big\},
\end{aligned}$$

which is exactly (1.8) in the case $2 < \ell < \infty$. The proof is complete. □

To prove Theorem 1.2, we shall use the following Lipschitz estimate.

Theorem 6.1 *Assume that A satisfies conditions (1.2), (1.3) and (1.11). We further assume that $\|\partial_\tau A\|_\infty < \infty$ if $0 < \ell < 2$, and $\|\nabla^2 A\|_{C^{\theta,0}} < \infty$ if $2 < \ell < \infty$. Suppose that $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0$ in $Q_{2r}(x_0,t_0)$ for some $(x_0,t_0) \in \mathbb{R}^{d+1}$ and $\varepsilon + \varepsilon^{\ell/2} \le r < \infty$.*

- *For the case $0 < \ell < 2$, we have*

$$\begin{aligned}
&\|\nabla u_\varepsilon - \nabla u_0 - \nabla\chi^\infty \nabla u_0\|_{L^\infty(Q_r(x_0,t_0))} \\
&\le r^{-1} \Big(\fint_{Q_{2r}(x_0,t_0)} |u_\varepsilon - u_0|^2 \Big)^{1/2} + \varepsilon r^{-1} \|\nabla u_0\|_{L^\infty(Q_{2r}(x_0,t_0))} + \varepsilon^{\ell/2} \|\nabla^2 u_0\|_{L^\infty(Q_{2r}(x_0,t_0))}
\end{aligned}$$

$$+ C\varepsilon \ln(r\varepsilon^{-1}+2)\| |\nabla^2 u_0| + \varepsilon^{\ell-1}|\partial_t \nabla u_0| + \varepsilon^{\ell-1}|\nabla^3 u_0| + \varepsilon^{1-\ell}|\nabla u_0| \|_{L^\infty(Q_{2r}(x_0,t_0))}$$
$$+ C\varepsilon^{1+\theta}\| |\nabla^2 u_0| + \varepsilon^{\ell-1}|\partial_t \nabla u_0| + \varepsilon^{\ell-1}|\nabla^3 u_0| + \varepsilon^{1-\ell}|\nabla u_0| \|_{C^{\theta,0}(Q_{2r}(x_0,t_0))}, \tag{6.7}$$

where C depends on d, m, μ, (h,θ) in (1.11), and $\|\partial_\tau A\|_{C^{\theta,0}}$

- *For the case $2 < \ell < \infty$, we have*

$$\|\nabla u_\varepsilon - \nabla u_0 - \nabla\chi^0 \nabla u_0\|_{L^\infty(Q_r(x_0,t_0))}$$
$$\le Cr^{-1}\Big(\fint_{Q_{2r}(x_0,t_0)} |u_\varepsilon - u_0|^2\Big)^{1/2} + C\varepsilon r^{-1}\|\nabla u_0\|_{L^\infty(Q_{2r}(x_0,t_0))}$$
$$+ C\varepsilon \ln(r\varepsilon^{-1}+2)\| |\nabla^2 u_0| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0| + \varepsilon^{\ell-3}|\nabla u_0| \|_{L^\infty(Q_{2r}(x_0,t_0))}$$
$$+ C\varepsilon^{1+\theta}\| |\nabla^2 u_0| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0| + \varepsilon^{\ell-3}|\nabla u_0| \|_{C^{\theta,0}(Q_{2r}(x_0,t_0))}, \tag{6.8}$$

where C depends on d, m, μ, (h,θ) in (1.11) and $\|\nabla^2 A\|_{C^{\theta,0}}$.

Proof For the case $0 < \ell < 2$, we define

$$w_\varepsilon(x,t) = u_\varepsilon(x,t) - u_0(x,t) - \varepsilon\chi^\infty(x/\varepsilon, t/\varepsilon^\ell)\nabla u_0(x,t)$$
$$- \varepsilon^\ell \mathcal{B}(t/\varepsilon^\ell)\nabla^2 u_0(x,t) - \varepsilon^2 \nabla\mathfrak{B}(x/\varepsilon, t/\varepsilon^\ell)\nabla^2 u_0(x,t).$$

By Lemma 4.3, we have $(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon \mathrm{div}(F_\varepsilon)$ in $Q_{2r}(x_0,t_0)$, where F_ε is defined by (4.11) with S_δ replaced by the identity operator. Let $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$ such that $0 \le \varphi \le 1$ and

$$\varphi = 1 \quad \text{in} \quad Q_{3r/2}(x_0,t_0), \quad \varphi = 0 \quad \text{if } |x-x_0| \ge \frac{7}{4}r \text{ or } t < t_0 - \Big(\frac{7}{4}r\Big)^2,$$
$$|\nabla\varphi| \le Cr^{-1}, \quad |\nabla^2\varphi| + |\partial_t\varphi| \le Cr^{-2}. \tag{6.9}$$

Note that

$$(\partial_t + \mathcal{L}_\varepsilon)(\varphi w_\varepsilon) = (\partial_t\varphi)w_\varepsilon + \mathrm{div}(\varepsilon\varphi F_\varepsilon - A^\varepsilon\nabla\varphi w_\varepsilon) - (\varepsilon F_\varepsilon + A^\varepsilon \nabla w_\varepsilon)\nabla\varphi.$$

For any $(x,t) \in Q_r(x_0,t_0)$, we have

$$w_\varepsilon(x,t) = \int_{-\infty}^t \int_{\mathbb{R}^d} \Gamma_\varepsilon(x,t;y,s)\big\{(\partial_s\varphi)w_\varepsilon - \varepsilon F_\varepsilon(\nabla\varphi) - A^\varepsilon\nabla w_\varepsilon(\nabla\varphi)\big\}dyds$$
$$- \int_{-\infty}^t \int_{\mathbb{R}^d} \nabla_y\Gamma_\varepsilon(x,t;y,s)\big\{\varepsilon\varphi F_\varepsilon - A^\varepsilon(\nabla\varphi)w_\varepsilon\big\}dyds$$
$$\doteq I_1 + I_2,$$

where $I_2 = -\varepsilon\int_{-\infty}^t \int_{\mathbb{R}^d} \nabla_y\Gamma_\varepsilon(x,t;y,s)\varphi(y,s)F_\varepsilon(y,s)dyds$. By Theorem 3.3, we deduce that

$$|\nabla I_1(x,t)| \le C\int_{-\infty}^t \int_{\mathbb{R}^d} |\nabla_x\Gamma_\varepsilon(x,t;y,s)|\big\{|\partial_s\varphi||w_\varepsilon| + \varepsilon|F_\varepsilon||\nabla\varphi| + |\nabla w_\varepsilon||\nabla\varphi|\big\}dyds$$
$$+ \int_{-\infty}^t \int_{\mathbb{R}^d} |\nabla_x\nabla_y\Gamma_\varepsilon(x,t;y,s)||\nabla\varphi||w_\varepsilon|dyds$$
$$\le C\Big\{\frac{1}{r}\fint_{Q_{7r/4}(x_0,t_0)}|w_\varepsilon| + \varepsilon\fint_{Q_{2r}(x_0,t_0)}|F_\varepsilon| + \fint_{Q_{7r/4}(x_0,t_0)}|\nabla w_\varepsilon|\Big\}$$
$$\le C\Big\{\frac{1}{r}\Big(\fint_{Q_{2r}(x_0,t_0)}|w_\varepsilon|^2\Big)^{1/2} + \varepsilon\Big(\fint_{Q_{2r}(x_0,t_0)}|F_\varepsilon|^2\Big)^{1/2}\Big\}$$

$$
\begin{aligned}
&\leq C\Big\{\frac{1}{r}\Big(\fint_{Q_{2r}(x_0,t_0)}|u_\varepsilon-u_0|^2\Big)^{1/2}+\frac{\varepsilon}{r}\|\nabla u_0\|_{L^\infty(Q_{2r}(x_0,t_0))}\\
&\quad+\Big(\frac{\varepsilon^2}{r}+\frac{\varepsilon^\ell}{r}\Big)\|\nabla^2u_0\|_{L^\infty(Q_{2r}(x_0,t_0))}+\varepsilon\Big(\fint_{Q_{2r}(x_0,t_0)}|F_\varepsilon|^2\Big)^{1/2}\Big\},
\end{aligned}
$$

where we have used Caccioppoli's inequality for the third step. By the boundedness of ϕ, $\nabla\mathfrak{B}$, $\nabla^2\mathfrak{B}$ and $\partial_\tau\nabla\mathfrak{B}$ (see Lemmas 2.3 and 2.2, and Remark 2.2), we know that

$$
|F_\varepsilon|\leq C\big\{|\nabla^2u_0|+(\varepsilon^{\ell-1}+\varepsilon)|\partial_t\nabla u_0|+(\varepsilon+\varepsilon^{\ell-1})|\nabla^3u_0|+\varepsilon^{1-\ell}|\nabla u_0|\big\}, \tag{6.10}
$$

which implies that $\|\nabla I_1\|_{L^\infty(Q_r(x_0,t_0))}$ can be bounded by the right-hand side of (6.7).

To estimate I_2, we note that

$$
\begin{aligned}
I_2(x,t)=&-\varepsilon\int_{-\infty}^t\int_{\mathbb{R}^d}\nabla_y\{\Gamma_\varepsilon(x,t;y,s)\varphi(y,s)\}(F_\varepsilon(y,s)-F_\varepsilon(x,s))dyds\\
&+\varepsilon\int_{-\infty}^t\int_{\mathbb{R}^d}\Gamma_\varepsilon(x,t;y,s)\nabla_y\varphi(y,s)F_\varepsilon(y,s)dyds.
\end{aligned}
$$

Then for any $(x,t)\in Q_r(x_0,t_0)$,

$$
\begin{aligned}
|\nabla I_2(x,t)|&\leq\varepsilon\int_{Q_{2r}(x_0,t_0)}|\nabla_x\nabla_y\{\Gamma_\varepsilon(x,t;y,s)\varphi(y,s)\}||F_\varepsilon(y,s)-F_\varepsilon(x,s)|dyds\\
&\quad+\varepsilon\int_{Q_{2r}(x_0,t_0)}|\nabla_x\Gamma_\varepsilon(x,t;y,s)||\nabla_y\varphi(y,s)||F_\varepsilon(y,s)|dyds\\
&\leq C\varepsilon\int_{Q_{2r}(x_0,t_0)}\frac{|F_\varepsilon(y,s)-F_\varepsilon(x,s)|}{(|x-y|+|t-s|^{1/2})^{d+2}}dyds+C\varepsilon\fint_{Q_{2r}(x_0,t_0)}|F_\varepsilon|\\
&\leq C\varepsilon\int_{Q_{2r}(x_0,t_0)\setminus Q_\varepsilon(x,t)}\frac{|F_\varepsilon(y,s)-F_\varepsilon(x,s)|}{(|x-y|+|t-s|^{1/2})^{d+2}}dyds\\
&\quad+C\varepsilon\int_{Q_\varepsilon(x,t)}\frac{|F_\varepsilon(y,s)-F_\varepsilon(x,s)|}{(|x-y|+|t-s|^{1/2})^{d+2}}dyds+C\varepsilon\fint_{Q_{2r}(x_0,t_0)}|F_\varepsilon|\\
&\leq C\varepsilon\ln(\varepsilon^{-1}r+2)\|F_\varepsilon\|_{L^\infty(Q_{2r}(x_0,t_0))}+C\varepsilon^{1+\theta}\|F_\varepsilon\|_{C^{\theta,0}(Q_{2r}(x_0,t_0))}.
\end{aligned}
$$

In view of (6.10) and the regularity of ϕ, $\mathfrak{B}$, we obtain that

$$
\begin{aligned}
&|\nabla I_2(x,t)|\\
&\quad\leq C\varepsilon\ln(\varepsilon^{-1}r+2)\||\nabla^2u_0|+\varepsilon^{\ell-1}|\partial_t\nabla u_0|+\varepsilon^{\ell-1}|\nabla^3u_0|+\varepsilon^{1-\ell}|\nabla u_0|\|_{L^\infty(Q_{2r}(x_0,t_0))}\\
&\qquad+C\varepsilon^{1+\theta}\||\nabla^2u_0|+\varepsilon^{\ell-1}|\partial_t\nabla u_0|+\varepsilon^{\ell-1}|\nabla^3u_0|+\varepsilon^{1-\ell}|\nabla u_0|\|_{C^{\theta,0}(Q_{2r}(x_0,t_0))}.
\end{aligned}
$$

Thus $\|\nabla I_2\|_{L^\infty(Q_r(x_0,t_0))}$ is bounded by the right-hand side of (6.7). Since

$$
\|\nabla w_\varepsilon-\{\nabla u_\varepsilon-\nabla u_0-(\nabla\chi^\infty)^\varepsilon\nabla u_0\}\|_{L^\infty(Q_r(x_0,t_0))}\leq\varepsilon\||\nabla^2u_0|+\varepsilon^{\ell-1}|\nabla^3u_0|\|_{L^\infty(Q_r(x_0,t_0))},
$$

one gets (6.7) immediately.

For the case $2<\ell<\infty$, we consider the function

$$
\begin{aligned}
\widetilde{w}_\varepsilon(x,t)=&\,u_\varepsilon(x,t)-u_0(x,t)-\varepsilon\chi^0(x/\varepsilon)\nabla u_0(x,t)-\varepsilon^{\ell-1}\nabla\Phi(x/\varepsilon,t/\varepsilon^\ell)\nabla u_0(x,t)\\
&-\varepsilon^\ell\Phi(x/\varepsilon,t/\varepsilon^\ell)\nabla^2u_0(x,t)-\varepsilon^2\nabla\Upsilon(x/\varepsilon)\nabla^2u_0(x,t).
\end{aligned}
$$

By performing similar analysis as above, it is not difficult to derive (6.8). Let us omit the details. □

Proof of Theorem 1.2 For fixed $x_0, y_0 \in \mathbb{R}^d$ and $s_0 < t_0$, it suffices to consider the case $\varepsilon + \varepsilon^{\ell/2} < r = (t_0 - s_0)^{1/2}/100$, since otherwise the estimate follows directly from (3.5). Note that $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = 0$ in $Q_{4r}(x_0, t_0)$. We apply Theorem 6.1 to the functions $u_\varepsilon(x,t) = \Gamma_\varepsilon(x,t;y_0,s_0)$ and $u_0(x,t) = \Gamma_0(x,t;y_0,s_0)$ in $Q_{2r}(x_0,t_0)$. For the case $0 < \ell < 2$, it follows from (6.7) that

$$
\begin{aligned}
&|\nabla_x \Gamma_\varepsilon(x,t;y_0,s_0) - \nabla_x \Gamma_0(x,t;y_0,s_0) - \nabla\chi^\infty(x/\varepsilon, t/\varepsilon^\ell)\nabla_x\Gamma_0(x,t;y_0,s_0)| \\
&\quad \le \frac{1}{r}\Big(\fint_{Q_{2r}(x_0,t_0)} |\Gamma_\varepsilon - \Gamma_0|^2\Big)^{1/2} + \frac{\varepsilon}{r}\|\nabla\Gamma_0\|_{L^\infty(Q_{2r}(x_0,t_0))} + \varepsilon^{\ell/2}\|\nabla^2\Gamma_0\|_{L^\infty(Q_{2r}(x_0,t_0))} \\
&\qquad + C\varepsilon\ln(r\varepsilon^{-1}+2)\||\nabla^2\Gamma_0| + \varepsilon^{\ell-1}|\partial_t\nabla\Gamma_0| + \varepsilon^{\ell-1}|\nabla^3\Gamma_0| + \varepsilon^{1-\ell}|\nabla\Gamma_0|\|_{L^\infty(Q_{2r}(x_0,t_0))} \\
&\qquad + C\varepsilon^{1+\theta}\||\nabla^2\Gamma_0| + \varepsilon^{\ell-1}|\partial_t\nabla\Gamma_0| + \varepsilon^{\ell-1}|\nabla^3\Gamma_0| + \varepsilon^{1-\ell}|\nabla\Gamma_0|\|_{C^{\theta,0}(Q_{2r}(x_0,t_0))}.
\end{aligned}
$$

Since $\varepsilon^{\ell/2} \le r$, by (1.5) and (1.8), we deduce that

$$
\begin{aligned}
&\Big|\nabla_x\Gamma_\varepsilon(x,t;y,s) - \nabla_x\Gamma_0(x,t;y,s) - \nabla\chi^\infty(x/\varepsilon,t/\varepsilon^\ell)\nabla_x\Gamma_0(x,t;y,s)\Big| \\
&\quad \le \frac{C}{(t-s)^{\frac{d+2}{2}}}\ln(2+\sqrt{t-s}/\varepsilon)\exp\Big\{-\frac{\kappa|x-y|^2}{t-s}\Big\}\cdot\big(\varepsilon^{\ell/2} + \varepsilon^{2-\ell}\sqrt{t-s}\big).
\end{aligned}
$$

For the case $2 < \ell < \infty$, we use (1.5) and (1.8) to bound the right-hand side of (6.8) to get the desired estimate. The proof is complete. □

Proof of Theorem 1.3 For fixed $x_0, y_0 \in \mathbb{R}^d$ and $s_0 < t_0$, we may assume that $\varepsilon + \varepsilon^{\ell/2} < r = (t_0 - s_0)^{1/2}/100$. For otherwise, the estimate (1.15) follows directly from (3.6). For fixed $1 \le j \le d$, $1 \le \beta \le m$, let

$$
\begin{aligned}
u_\varepsilon^\alpha(x,t) &= \frac{\partial}{\partial y_j}\{\Gamma_\varepsilon^{\alpha\beta}\}(x,t;y_0,s_0), \\
u_0^\alpha(x,t) &= \frac{\partial}{\partial y_k}\{\Gamma_0^{\alpha\zeta}\}(x,t;y,s)\frac{\partial}{\partial y_j}\big(\delta^{\beta\zeta}y_k + \varepsilon\widetilde{\chi}_k^{\beta\zeta}(y/\varepsilon, -s/\varepsilon^\ell)\big),
\end{aligned}
$$

where $\widetilde{\chi}_j^{\alpha\gamma}(y/\varepsilon, -s/\varepsilon^\ell)$ is the corrector of $\partial_t + \widetilde{\mathcal{L}}_\varepsilon$. Note that $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = 0$ in $Q_{4r}(x_0,t_0)$. We can apply Theorem 6.1 to $u_\varepsilon = (u_\varepsilon^\alpha)$ and $u_0 = (u_0^\alpha)$ in $Q_{2r}(x_0,t_0)$, and then use (1.5) as well as (1.14) to derive the desired estimate. □

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