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# The Cheeger constant as limit of Sobolev-type constants

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## Abstract

Let $\Omega$ be a bounded, smooth domain of $\mathbb{R}^N$, $N \geq 2$. For $1 < p < N$ and $0 < q(p) < p^* := \frac{Np}{N-p}$, let

$$\lambda_{p,q(p)} := \inf \left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x : u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |u|^{q(p)} \,\mathrm{d}x = 1 \right\}.$$

We prove that if $\lim_{p \to 1^+} q(p) = 1$, then $\lim_{p \to 1^+} \lambda_{p,q(p)} = h(\Omega)$, where $h(\Omega)$ denotes the Cheeger constant of $\Omega$. Moreover, we study the behavior of the positive solutions $w_{p,q(p)}$ to the Lane–Emden equation $-\operatorname{div}(|\nabla w|^{p-2} \nabla w) = |w|^{q-2} w$, as $p \to 1^+$.

**Keywords** Cheeger constant · p-Laplacian · Picone's inequality · Singular problem · Sobolev constants

**Mathematics Subject Classification** 35B40 · 35J92 · 49Q20

## 1 Introduction

Let $\Omega$ be a smooth, bounded domain of $\mathbb{R}^N$, $N \geq 2$. For $1 < p < N$ and $0 < q \leq p^* := \frac{Np}{N-p}$, let

$$\lambda_{p,q} := \inf \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_q = 1 \right\}, \tag{1.1}$$

where

$$\|u\|_r := \left( \int_\Omega |u|^r \,\mathrm{d}x \right)^{\frac{1}{r}}, \quad r > 0.$$

We recall that $\|\cdot\|_r$ is the standard norm of the Lebesgue space $L^r(\Omega)$ if $r \geq 1$, but it is not a norm if $0 < r < 1$.

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Note from (1.1) that

$$\lambda_{p,q} \leq \frac{\|\nabla u\|_p^p}{\|u\|_q^p} \quad \text{for all } u \in W_0^{1,p}(\Omega) \setminus \{0\} \tag{1.2}$$

since the above quotients are homogeneous.

When $0 < q < p^*$, the existence of a minimizer $u_{p,q}$ for the constrained minimization problem (1.1) follows from standard arguments of the Calculus of Variations. Moreover, $u_{p,q}$ is a weak solution to the Dirichlet problem for the $p$-Laplacian operator

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda_{p,q}\,|u|^{q-2}\,u \text{ in } & \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In consequence, $u_{p,q} > 0$ in $\Omega$ and $u_{p,q} \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. (We refer to Giacomoni et al. [9, Theorem 1(i)] for the regularity of $u_{p,q}$ when $0 < q < 1$, in which case (1.3) is singular.) These facts are well known when $1 \leq q < p^*$, since $\lambda_{p,q}$ is the best constant in the Sobolev (compact) embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

It is worth mentioning that $u_{p,q}$ is the only positive minimizer to (1.1) in the sublinear case: $0 < q < p$. However, this uniqueness property might fail in the superlinear, subcritical case: $p < q < p^*$. For examples and a discussion about this issue, we recommend the recent paper [4] by Brasco and Lindgren, where an important result is established for general smooth bounded domains: the uniqueness of the minimizer $u_{p,q}$ whenever $2 < p < q$ and $q$ is sufficiently close to $p$. We stress that such a uniqueness result for $1 < p < 2$ is not yet available in the literature.

When $q = p^*$, the infimum $\lambda_{p,p^*}$ cannot be attained in $W_0^{1,p}(\Omega)$ if $\Omega \neq \mathbb{R}^N$. Actually, $\lambda_{p,p^*}$ does not depend on $\Omega$ as it coincides with the well-known Sobolev constant $S_{N,p}$, that is:

$$\lambda_{p,p^*} = S_{N,p} := N\omega_N^{\frac{p}{N}}\left(\frac{N-p}{p-1}\right)^{p-1}\left(\frac{\Gamma(N/p)\Gamma(1+N-N/p)}{\Gamma(N)}\right)^{\frac{p}{N}}, \tag{1.4}$$

where $\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}\mathrm{d}s$ is the Gamma function and $\omega_N := \pi^{N/2}/\Gamma(1+N/2)$ is the $N$-dimensional Lebesgue volume of the unit ball of $\mathbb{R}^N$.

According to [2, Theorem 9] by Anello et al., for each fixed $p \in (1, N)$ the function

$$(0, p^*] \ni q \mapsto \lambda_{p,q}\,|\Omega|^{\frac{p}{q}}$$

is decreasing and absolutely continuous on compact sets of $(0, p^*]$. The same result, but for $q \in [1, p^*]$, had already been obtained by Ercole [8].

As for $q$ varying with $p$, Kawohl and Fridman proved in [10] that

$$\lim_{p\to 1^+} \lambda_{p,p} = h(\Omega) \tag{1.5}$$

where $h(\Omega)$ is the Cheeger constant of $\Omega$.

We recall that

$$h(\Omega) := \inf\left\{\frac{P(E)}{|E|} : E \subset \overline{\Omega} \text{ and } |E| > 0\right\},$$

where $P(E)$ stands for the perimeter of $E$ in $\mathbb{R}^N$ and $|E|$ stands for the $N$-dimensional Lebesgue volume of $E$.

The Cheeger problem consists of finding a subset $E \subset \overline{\Omega}$ such that

$$h(\Omega) = \frac{P(E)}{|E|}.$$

Such a subset $E$ is called Cheeger set of $\Omega$.

We notice from (1.4) that

$$\lim_{p \to 1^+} \lambda_{p,p^*} = N\omega_N^{\frac{1}{N}} = |\Omega|^{\frac{1}{N}}\, h(\Omega^\star) \tag{1.6}$$

where $\Omega^\star$ denotes the ball of $\mathbb{R}^N$ centered at the origin such that $|\Omega^\star| = |\Omega|$. The second equality in (1.6) is due to the fact that balls are Cheeger sets of themselves (i.e. they are calibrable). Hence, as $R = (|\Omega| / \omega_N)^{\frac{1}{N}}$ is the radius of $\Omega^\star$, one has $h(\Omega^\star) = N/R = N(\omega_N / |\Omega|)^{\frac{1}{N}}$. It is well known that $h(\Omega^\star) \le h(\Omega)$, the equality occurs if and only if $\Omega$ is a ball.

Owing to (1.5), when $p \to 1^+$, the minimizer $u_{p,p}$ converges in $L^1(\Omega)$ (after passing to a subsequence) to a function $u_1$ whose the $t$-superlevel sets $E_t := \{x \in \Omega : u_1(x) > t\}$ are Cheeger sets, for almost every $t > 0$. As shown in [10], these properties are obtained from a variational version of the Cheeger problem in the $BV$ setting, which we briefly present in Sect. 2.

The approach of solving the Cheeger problem by a $p$-Laplacian approximation, as $p \to 1^+$, has been extended by Butazzo, Carlier and Comte in [6] to a slightly more general Cheeger problem where the volume and the perimeter are weighted by two positive weight functions. In that paper, after showing that such an approach does not provide a criterion for determining the maximal Cheeger set, they introduced an alternative approximation method in the $BV$ setting, based in concave penalizations, to select maximal Cheeger sets.

In this paper we suppose that $q$ varies with $p$ along a more general path, $q = q(p)$ for $p \in (1, p^*)$, and study the behavior of $\lambda_{p,q(p)}$ when $p \to 1^+$ and $q(p) \to 1$. Adapting an estimate from Ercole [8] (see Lemma 3.1 below) and making use of (1.5), we extend the results of Kawohl and Fridman [10]. Our main result, which will be proved in Sect. 3, is stated as follows.

**Theorem 1.1** *If* $0 < q(p) < p^*$ *and* $\lim_{p \to 1^+} q(p) = 1$, *then*

$$\lim_{p \to 1^+} \lambda_{p,q(p)} = h(\Omega), \tag{1.7}$$

$$\lim_{p \to 1^+} \left\| u_{p,q(p)} \right\|_1 = 1 \tag{1.8}$$

*and*

$$\lim_{p \to 1^+} \left\| u_{p,q(p)} \right\|_\infty^{q(p)-p} = 1. \tag{1.9}$$

*Moreover, any sequence* $\left(u_{p_n,q(p_n)}\right)$, *with* $p_n \to 1^+$, *admits a subsequence that converges in* $L^1(\Omega)$ *to a nonnegative function* $u \in L^1(\Omega) \cap L^\infty(\Omega)$ *such that:*

(a) $\|u\|_1 = 1$,

(b) $\dfrac{1}{|\Omega|} \le \|u\|_\infty \le \dfrac{h(\Omega)^N}{|\Omega|\, h(\Omega^\star)^N}$, *and*

(c) *for almost every* $t \ge 0$, *the* $t$*-superlevel set*

$$E_t := \{x \in \Omega : u(x) > t\}$$

*is a Cheeger set.*

Besides allowing $q(p) \to 1^-$, which embraces (1.3) in its singular form, our approach holds for every family of extremals $u_{p,q(p)}$ in the superlinear, subcritical case: $p < q(p) < p^*$.

It is simple to verify that the function

$$v_{p,q} := \lambda_{p,q}^{\frac{1}{q-p}} u_{p,q}, \tag{1.10}$$

is a positive weak solution to the Lane–Emden-type problem

$$\begin{cases} -\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) = |w|^{q-2} w & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.11}$$

The next corollary is stated to solutions to (1.11) in the form (1.10). It is an immediate consequence of (1.7) and (1.9) since

$$\left\|v_{p,q}\right\|_q^{q-p} = \lambda_{p,q}\left\|u_{p,q}\right\|_q^{q-p} = \lambda_{p,q} \quad \text{and} \quad \left\|v_{p,q}\right\|_\infty^{q-p} = \lambda_{p,q}\left\|u_{p,q}\right\|_\infty^{q-p}.$$

**Corollary 1.2** *If* $0 < q(p) < p^*$ *and* $\lim_{p\to 1^+} q(p) = 1$, *then*

$$\lim_{p\to 1^+}\left\|v_{p,q(p)}\right\|_{q(p)}^{q(p)-p} = h(\Omega) = \lim_{p\to 1^+}\left\|v_{p,q(p)}\right\|_\infty^{q(p)-p}.$$

In the particular case where $q(p) \equiv 1$, this result had already been obtained in [5] by Bueno and Ercole, without using (1.5).

As it is well known, (1.11) has a unique positive weak solution when $0 < q < p$, which is, of course, that given by (1.10). However, there are examples of smooth domains for which (1.11) has multiple positive weak solutions when $p < q < p^*$, which may be of the form (1.10) or not (see [4] and references therein). By the way, it is plain to check that

$$\lambda_{p,q} = \left\|v_{p,q}\right\|_q^{q-p} = \min\left\{\|w\|_q^{q-p} : w \text{ is a weak solution to (1.11)}\right\}. \tag{1.12}$$

Corollary 1.2 deals with the behavior of positive weak solutions to (1.11) that attains the minimum in (1.12). Aiming to cover a wider class of positive weak solutions $w_{p,q}$ to (1.11), including those satisfying $\left\|w_{p,q}\right\|_q^{q-p} > \lambda_{p,q}$, we provide the following stronger result, which will be proved in Sect. 4 by using Picone's inequality (see [1, 3]).

**Theorem 1.3** *Let* $w_{p,q(p)} \in W_0^{1,p}(\Omega)$ *be a positive weak solution to* (1.11)*, with* $p < q(p) < p^*$*. Then, either*

$$\limsup_{p\to 1^+}\left\|w_{p,q(p)}\right\|_\infty^{q(p)-p} = +\infty \tag{1.13}$$

*or*

$$\lim_{p\to 1^+}\left\|w_{p,q(p)}\right\|_{q(p)}^{q(p)-p} = h(\Omega) = \lim_{p\to 1^+}\left\|w_{p,q(p)}\right\|_\infty^{q(p)-p}.$$

The alternative (1.13) can be replaced with (see Remark 4.1)

$$\limsup_{p\to 1^+}\left\|w_{p,q(p)}\right\|_{q(p)}^{q(p)-p} = +\infty.$$

We believe that determining whether this alternative (or its equivalent version (1.13)) is actually possible is a very interesting open question that we plan to study in the near future.

## 2 The Cheeger problem in the *BV* setting

In this section, we assume that $\Omega$ is a Lipschitz bounded domain and collect some definitions, properties and basic results related to the variational version of the Cheeger problem in the $BV$ setting. For details, we refer to Carlier and Comte [7] and Parini [11].

The total variation of $u \in L^1(\Omega)$ is defined as

$$|Du|(\Omega) := \sup\left\{\int_\Omega u \operatorname{div}\varphi \mathrm{d}x : \varphi \in C_c^1(\Omega;\mathbb{R}^N) \quad \text{and} \quad \|\varphi\|_{L^\infty} \le 1\right\}.$$

The space $BV(\Omega)$ of the functions $u \in L^1(\Omega)$ of bounded variation in $\Omega$ (i.e. $|Du|(\Omega) < \infty$), endowed with the norm

$$\|u\|_{BV} := \|u\|_1 + |Du|(\Omega),$$

is a Banach space compactly embedded into $L^1(\Omega)$. Moreover, the functional $BV(\Omega) \ni u \mapsto |Du|(\Omega)$ is lower semicontinuous in $L^1(\Omega)$.

The Cheeger constant is also characterized as (see [11, Proposition 3.1])

$$h(\Omega) = \inf_{BV_0(\Omega)} \frac{|Du|(\mathbb{R}^N)}{\|u\|_1} \tag{2.1}$$

where

$$BV_0(\Omega) := \left\{u \in BV(\mathbb{R}^N) : \|u\|_1 > 0 \text{ and } u \equiv 0 \text{ in } \mathbb{R}^N \setminus \overline{\Omega}\right\}$$

and

$$|Du|(\mathbb{R}^N) = |Du|(\Omega) + \int_{\partial\Omega} |u| \,\mathrm{d}\mathcal{H}^{N-1}$$

($\mathcal{H}^{N-1}$ stands for the $(N-1)$-Hausdorff measure in $\mathbb{R}^N$).

**Proposition 2.1** ([7, Corollary 1(2)]) *Let $(u_n) \subset BV_0(\Omega)$ be such that $u_n \to u$ in $L^1(\mathbb{R}^N)$. Then,*

$$|Du|(\mathbb{R}^N) \le \liminf_{n\to\infty} |Du_n|(\mathbb{R}^N).$$

**Proposition 2.2** *Suppose that*

$$h(\Omega) = \frac{|Du|(\mathbb{R}^N)}{\|u\|_1}$$

*for some $u \in BV_0(\Omega)$. Then,*

$$E_t := \{x \in \Omega : u(x) > t\}$$

*is a Cheeger set for almost every $t \ge 0$.*

*Inversely, if $E \subset \overline{\Omega}$ is a Cheeger set of $\Omega$, then*

$$h(\Omega) = \frac{|D\chi_E|(\mathbb{R}^N)}{\|\chi_E\|_1}$$

*where $\chi_E$ stands for the characteristic function of $E$ in $\mathbb{R}^N$.*

***Proof*** Combining Coarea formula and Cavalieri's principle, we find

$$0 = |Du|(\mathbb{R}^N) - h(\Omega)\|u\|_1 = \int_0^\infty (P(E_t) - h(\Omega)|E_t|)\mathrm{d}t. \tag{2.2}$$

As $|E_t| > 0$ a.e. $t \geq 0$, we have that $P(E_t) - h(\Omega)|E_t| \geq 0$ a.e. $t \geq 0$. Therefore, it follows from (2.2) that

$$h(\Omega) = \frac{P(E_t)}{|E_t|} \quad \text{a.e.} \quad t \geq 0.$$

Now, if $E \subset \overline{\Omega}$ is a Cheeger set of $\Omega$, then $\chi_E \in BV_0(\Omega)$. As $P(E) = |D\chi_E|(\mathbb{R}^N)$ and $\|\chi_E\|_1 = |E|$, we have

$$h(\Omega) = \frac{P(E)}{|E|} = \frac{|D\chi_E|(\mathbb{R}^N)}{\|\chi_E\|_1}.$$

□

## 3 Proof of Theorem 1.1

We recall from the Introduction that $u_{p,q}$ (for $1 < p < N$ and $0 < q < p^*$) denotes the positive minimizer of the constrained minimization problem (1.1), so that $u_{p,q} \in W_0^{1,p}(\Omega)$,

$$u_{p,q} > 0 \text{ in } \Omega, \ \left\|u_{p,q}\right\|_q = 1, \ \lambda_{p,q} = \left\|\nabla u_{p,q}\right\|_p^p,$$

and $u_{p,q}$ is a weak solution to (1.3).

If $q = p$, the Dirichlet problem (1.3) is homogeneous and thus it can be recognized as an eigenvalue problem. In this setting, $\lambda_{p,p}$ is known as the first eigenvalue of the Dirichlet $p$-Laplacian. Actually, $\lambda_{p,p}$ is simple in the sense that the set of its corresponding eigenfunctions is generated by $u_{p,p}$, that is, $w \in W_0^{1,p}(\Omega)$ is a nontrivial weak solution to

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda_{p,p}|u|^{p-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

if and only if $w = ku_{p,p}$ for some $k \in \mathbb{R}\setminus\{0\}$.

In this section, we prove Theorem 1.1 by assuming that $\partial\Omega$ is smooth enough to ensure that $u_{p,q} \in C^1(\overline{\Omega})$. In consequence, $u_{p,q} \in BV_0(\Omega)$ (after extended as zero on $\mathbb{R}^N \setminus \overline{\Omega}$) and

$$\left|Du_{p,q}\right|(\mathbb{R}^N) = \left\|\nabla u_{p,q}\right\|_1,$$

since

$$\left|Du_{p,q}\right|(\Omega) = \left\|\nabla u_{p,q}\right\|_1 \quad \text{and} \quad \int_{\partial\Omega}\left|u_{p,q}\right|\mathrm{d}\mathcal{H}^{N-1} = 0.$$

The next result is adapted from Lemma 5 of Ercole [8] established there for $1 \leq q < p^*$.

**Lemma 3.1** *Let $u \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ be a positive weak solution to the Dirichlet problem*

$$\begin{cases} -div\left(|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{q-2}u & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{3.2}$$

*with $0 \leq q < p^*$ and $\lambda > 0$. If $\sigma \geq 1$, then*

$$C_{\lambda,\sigma,q}\|u\|_\infty^{\frac{N(p-q)+p\sigma}{p}} \leq \|u\|_\sigma^\sigma, \tag{3.3}$$

*where*

$$C_{\lambda,\sigma,q} := \left(\frac{\lambda_{p,p^*}}{\lambda}\right)^{\frac{N}{p}} \left(\frac{p}{p+N(p-1)}\right)^{N+1} I_{\sigma,q}$$

*and*

$$I_{\sigma,q} := \begin{cases} \sigma \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{\frac{N(1-q)}{p}+\sigma-1} \mathrm{d}\tau & \text{if } 0 \le q < 1 \\ \sigma \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{\sigma-1} \mathrm{d}\tau & \text{if } 1 \le q < p^*. \end{cases}$$

***Proof*** For each $0 < t < \|u\|_\infty$, let us define

$$(u-t)_+ := \max\{u-t, 0\} \quad \text{and} \quad A_t := \{x \in \Omega : u(x) > t\}.$$

As $(u-t)_+ \in W_0^{1,p}(\Omega)$ and $u$ is a positive weak solution to (3.2), we have

$$\begin{aligned} \int_{A_t} |\nabla u|^p \, \mathrm{d}x &= \lambda \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (u-t)_+ \mathrm{d}x \\ &= \lambda \int_\Omega u^{q-1}(u-t)_+ \mathrm{d}x = \lambda \int_{A_t} u^{q-1}(u-t)\mathrm{d}x. \end{aligned} \tag{3.4}$$

We also have

$$\left(\int_{A_t} (u-t)\mathrm{d}x\right)^p \le |A_t|^{p-\frac{p}{p^*}} \left(\int_{A_t} (u-t)^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}} \le \frac{|A_t|^{p-\frac{p}{p^*}}}{\lambda_{p,p^*}} \int_{A_t} |\nabla u|^p \, \mathrm{d}x, \tag{3.5}$$

where we have used Hölder's inequality and (1.2). Note that

$$\lambda_{p,p^*} \le \frac{\|\nabla(u-t)_+\|_p^p}{\|(u-t)_+\|_{p^*}^p} = \frac{\displaystyle\int_{A_t} |\nabla u|^p \, \mathrm{d}x}{\left(\displaystyle\int_{A_t} (u-t)^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}}}.$$

We divide the remaining of the proof in two cases.
**Case 1.** $0 \le q < 1$. As

$$\int_{A_t} u^{q-1}(u-t)\mathrm{d}x \le t^{q-1} \int_{A_t} (\|u\|_\infty - t)\mathrm{d}x, \tag{3.6}$$

we obtain from (3.4) the estimate

$$\int_{A_t} |\nabla u|^p \, \mathrm{d}x \le \lambda t^{q-1}(\|u\|_\infty - t)\,|A_t|. \tag{3.7}$$

Combining (3.7) and (3.5), we obtain the inequalities

$$\lambda_{p,p^*} |A_t|^{-p+\frac{p}{p^*}} \left(\int_{A_t} (u-t)\mathrm{d}x\right)^p \le \int_{A_t} |\nabla u|^p \, \mathrm{d}x \le \lambda t^{q-1}(\|u\|_\infty - t)\,|A_t|$$

which lead to

$$\frac{\lambda_{p,p^*} t^{1-q}}{\lambda(\|u\|_\infty - t)} \left(\int_{A_t} (u-t)\mathrm{d}x\right)^p \le |A_t|^{p(1-\frac{1}{p^*}+\frac{1}{p})} = |A_t|^{p(\frac{N+1}{N})}. \tag{3.8}$$

Now, let us define the function

$$g(t) := \int_{A_t} (u-t)\mathrm{d}x.$$

It is simple to verify that

$$g(t) = \int_t^{\|u\|_\infty} |A_s| \, \mathrm{d}s,$$

so that

$$g'(t) = -|A_t|.$$

Then, (3.8) can be rewritten as

$$\left(\frac{\lambda_{p,p^*}}{\lambda}\right)^{\frac{N}{p(N+1)}} \left(\frac{t^{1-q}}{\|u\|_\infty - t}\right)^{\frac{N}{p(N+1)}} \leq -g'(t)g(t)^{-\frac{N}{N+1}}. \tag{3.9}$$

Integration of the right-hand side of (3.9) over $[t, \|u\|_\infty]$ yields

$$\begin{aligned} -\int_t^{\|u\|_\infty} g'(s)g(s)^{-\frac{N}{N+1}} \mathrm{d}s &= (N+1)g(t)^{\frac{1}{N+1}} - (N+1)g(\|u\|_\infty)^{\frac{1}{N+1}} \\ &\leq (N+1)g(t)^{\frac{1}{N+1}} \end{aligned} \tag{3.10}$$

whereas integration of the function at the left-hand side of (3.9) over $[t, \|u\|_\infty]$ yields

$$\begin{aligned} \int_t^{\|u\|_\infty} \left(\frac{s^{1-q}}{\|u\|_\infty - s}\right)^{\frac{N}{p(N+1)}} \mathrm{d}s &\geq t^{\frac{N(1-q)}{p(N+1)}} \int_t^{\|u\|_\infty} (\|u\|_\infty - s)^{-\frac{N}{p(N+1)}} \mathrm{d}s \\ &= \frac{p(N+1)}{p+N(p-1)} t^{\frac{N(1-q)}{p(N+1)}} (\|u\|_\infty - t)^{\frac{p+N(p-1)}{p(N+1)}}. \end{aligned} \tag{3.11}$$

Thus, after integrating (3.9) we obtain from (3.10) and (3.11) the inequality

$$\left(\frac{\lambda_{p,p^*}}{\lambda}\right)^{\frac{N}{p}} \left(\frac{p}{p+N(p-1)}\right)^{N+1} t^{\frac{N(1-q)}{p}} (\|u\|_\infty - t)^{\frac{p+N(p-1)}{p}} \leq g(t). \tag{3.12}$$

As $g(t) \leq (\|u\|_\infty - t)\, |A_t|$, it follows from (3.12) that

$$\left(\frac{\lambda_{p,p^*}}{\lambda}\right)^{\frac{N}{p}} \left(\frac{p}{p+N(p-1)}\right)^{N+1} t^{\frac{N(1-q)}{p}} (\|u\|_\infty - t)^{\frac{N(p-1)}{p}} \leq |A_t|.$$

Now, for a given $\sigma \geq 1$, we multiply the latter inequality by $\sigma t^{\sigma-1}$ and integrate over $[0, \|u\|_\infty]$ to get (3.3) after noticing that

$$\sigma \int_0^{\|u\|_\infty} |A_t|\, t^{\sigma-1} \mathrm{d}t = \int_\Omega u^\sigma \mathrm{d}x,$$

and that the change of variable $t = \|u\|_\infty \tau$ yields

$$\int_0^{\|u\|_\infty} t^{\frac{N(1-q)}{p}+\sigma-1} (\|u\|_\infty - t)^{\frac{N(p-1)}{p}} \mathrm{d}t = \|u\|_\infty^{\frac{N(p-q)}{p}+\sigma} \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{\frac{N(1-q)}{p}+\sigma-1} \mathrm{d}\tau.$$

**Case 2.** $1 \leq q < p^*$. The factor $t^{q-1}$ in (3.6) can be replaced with $\|u\|_\infty^{q-1}$, so that (3.9) and (3.11) become

$$\left(\frac{\lambda_{p,p^*}}{\lambda}\right)^{\frac{N}{p(N+1)}} \left(\frac{\|u\|_\infty^{1-q}}{\|u\|_\infty - t}\right)^{\frac{N}{p(N+1)}} \leq -g'(t)g(t)^{-\frac{N}{N+1}}$$

and

$$\int_t^{\|u\|_\infty} \left( \frac{\|u\|_\infty^{1-q}}{\|u\|_\infty - s} \right)^{\frac{N}{p(N+1)}} \mathrm{d}s = \frac{p(N+1)}{p+N(p-1)} \|u\|_\infty^{\frac{N(1-q)}{p(N+1)}} (\|u\|_\infty - t)^{\frac{p+N(p-1)}{p(N+1)}},$$

respectively. Hence, we obtain from (3.10) that

$$\left( \frac{\lambda_{p,p^*}}{\lambda} \right)^{\frac{N}{p(N+1)}} \frac{p(N+1)}{p+N(p-1)} \|u\|_\infty^{\frac{N(1-q)}{p(N+1)}} (\|u\|_\infty - t)^{\frac{p+N(p-1)}{p(N+1)}} \le (N+1) g(t)^{\frac{1}{N+1}}.$$

Then, using that $g(t) \le (\|u\|_\infty - t)\,|A_t|$, the latter inequality leads to

$$\left( \frac{\lambda_{p,p^*}}{\lambda} \right)^{\frac{N}{p}} \left( \frac{p}{p+N(p-1)} \right)^{N+1} \|u\|_\infty^{\frac{N(1-q)}{p}} (\|u\|_\infty - t)^{\frac{N(p-1)}{p}} \le |A_t|. \tag{3.13}$$

Multiplying (3.13) by $\sigma t^{\sigma-1}$ and integrating over $[0, \|u\|_\infty]$, we arrive at (3.3) with

$$I_{\sigma,q} = \sigma \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{\sigma-1} \mathrm{d}\tau.$$

□

***Remark 3.2*** The estimate (3.3) can be rewritten as

$$C_{\lambda,\sigma,q} \|u\|_\infty^{\frac{N}{p^*}(p^*-q)+(\sigma-q)} \le \|u\|_\sigma^\sigma.$$

In the sequel, $e_p$ denotes the $L^\infty$-normalized minimizer corresponding to $\lambda_{p,p}$, that is:

$$e_p := \frac{u_{p,p}}{\|u_{p,p}\|_\infty}. \tag{3.14}$$

As $e_p$ is also a positive weak solution to the homogeneous Dirichlet problem (3.1), Lemma 3.1 applied to $e_p$, with $q = p$, $\sigma = 1$ and $\lambda = \lambda_{p,p}$, yields

$$\left( \frac{\lambda_{p,p^*}}{\lambda_{p,p}} \right)^{\frac{N}{p}} \left( \frac{p}{p+N(p-1)} \right)^{N+1} \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \mathrm{d}\tau \le \|e_p\|_1.$$

Hence, we have

$$0 < |\Omega| \left( \frac{h(\Omega^\star)}{h(\Omega)} \right)^N \le \liminf_{p\to 1^+} \|e_p\|_1, \tag{3.15}$$

since

$$\lim_{p\to 1^+} \left( \frac{\lambda_{p,p^*}}{\lambda_{p,p}} \right)^{\frac{N}{p}} = |\Omega| \left( \frac{h(\Omega^\star)}{h(\Omega)} \right)^N$$

and

$$\lim_{p\to 1^+} \left( \frac{p}{p+N(p-1)} \right)^{N+1} = \lim_{p\to 1^+} \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \mathrm{d}\tau = 1. \tag{3.16}$$

**Lemma 3.3** *If $q_n \to 1$ and $p_n \to 1^+$, then (up to a subsequence) $e_{p_n}$ converges in $L^1(\Omega)$ to a function $e$. Moreover,*

$$\lim_{n\to\infty} \int_\Omega e_{p_n}^{q_n} \mathrm{d}x = \lim_{n\to\infty} \int_\Omega e_{p_n}^{p_n} \mathrm{d}x = \|e\|_1 > 0. \tag{3.17}$$

***Proof*** We have $\left\|e_p\right\|_1 \leq \left\|e_p\right\|_\infty |\Omega| = |\Omega|$ and, by Hölder inequality,

$$\left\|\nabla e_p\right\|_1 \leq \left\|\nabla e_p\right\|_p |\Omega|^{1-\frac{1}{p}} = \lambda_{p,p}^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}}.$$

Hence, it follows from (1.5) that the family $\left(e_p\right)$ is uniformly bounded in $BV(\Omega)$. Therefore, owing to the compactness of the embedding $BV(\Omega) \hookrightarrow L^1(\Omega)$, we can assume that (up to a subsequence) $e_{p_n}$ converges to a function $e$ in $L^1(\Omega)$ and also pointwise almost everywhere in $\Omega$. In view of (3.15), the convergence in $L^1(\Omega)$ shows that $\|e\|_1 > 0$. As the nonnegative functions $e_{p_n}^{q_n}$ and $e_{p_n}^{p_n}$ are dominated by 1, the convergence a.e. in $\Omega$ leads to the equalities in (3.17). □

**Lemma 3.4** *If* $0 < q(p) < p^*$ *and* $\lim_{p\to 1^+} q(p) = 1$, *then*

$$\limsup_{p\to 1^+} \lambda_{p,q(p)} \leq h(\Omega) \tag{3.18}$$

*and*

$$\limsup_{p\to 1^+} \left\|u_{p,q(p)}\right\|_1 \leq 1. \tag{3.19}$$

***Proof*** Let us take $p_n \to 1^+$ such that

$$\lim_{n\to\infty} \lambda_{p_n,q(p_n)} = L := \limsup_{p\to 1^+} \lambda_{p,q(p)}.$$

Using (1.2) for $\lambda_{p_n,q(p_n)}$ and the definition of $e_{p_n}$, we have that

$$\lambda_{p_n,q(p_n)} \leq \frac{\left\|\nabla e_{p_n}\right\|_{p_n}^{p_n}}{\left\|e_{p_n}\right\|_{q(p_n)}^{p_n}} = \lambda_{p_n,p_n} \left(\frac{\left\|e_{p_n}\right\|_{p_n}}{\left\|e_{p_n}\right\|_{q(p_n)}}\right)^{p_n}.$$

Hence, we can apply Lemma 3.3 to get (3.18) from (1.5), since

$$L = \lim_{n\to\infty} \lambda_{p_n,q(p_n)} \leq \lim_{n\to\infty} \lambda_{p_n,p_n} \frac{\lim_{n\to\infty} \left\|e_{p_n}\right\|_{p_n}}{\lim_{n\to\infty} \left\|e_{p_n}\right\|_{q(p_n)}} = \lim_{n\to\infty} \lambda_{p_n,p_n} = h(\Omega).$$

Using Hölder's inequality and exploiting (1.2) with respect to $\lambda_{p,p}$, we obtain

$$\left\|u_{p,q(p)}\right\|_1 \leq \left\|u_{p,q(p)}\right\|_p |\Omega|^{1-\frac{1}{p}} \leq \lambda_{p,p}^{-\frac{1}{p}} \left\|\nabla u_{p,q(p)}\right\|_p |\Omega|^{1-\frac{1}{p}} = \lambda_{p,p}^{-\frac{1}{p}} \lambda_{p,q(p)}^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}}. \tag{3.20}$$

Hence, (3.19) follows from (1.5) and (3.18). □

**Lemma 3.5** *If* $0 < q(p) < p^*$ *and* $\lim_{p\to 1^+} q(p) = 1$, *then*

$$\frac{1}{|\Omega|} \leq \liminf_{p\to 1^+} \left\|u_{p,q(p)}\right\|_\infty \quad \text{and} \quad \limsup_{p\to 1^+} \left\|u_{p,q(p)}\right\|_\infty \leq \frac{h(\Omega)^N}{|\Omega|\, h(\Omega^\star)^N}. \tag{3.21}$$

***Proof*** The first estimate in (3.21) is immediate since

$$1 = \left\|u_{p,q(p)}\right\|_{q(p)}^{q(p)} \leq \left\|u_{p,q(p)}\right\|_\infty^{q(p)} |\Omega|.$$

According to Remark 3.2, we have that

$$C_p \left\|u_{p,q(p)}\right\|_\infty^{\frac{N(p^*-q(p))}{p^*}} \leq \left\|u_{p,q(p)}\right\|_{q(p)}^{q(p)} = 1 \tag{3.22}$$

where

$$C_p := \left(\frac{\lambda_{p,p^*}}{\lambda_{p,q(p)}}\right)^{\frac{N}{p}} \left(\frac{p}{p+N(p-1)}\right)^{N+1} I_p$$

and

$$I_p := \begin{cases} q(p) \int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{\frac{N(1-q(p))}{p}+q(p)-1} \mathrm{d}\tau & \text{if} \quad 0 \le q(p) < 1 \\ q(p) \displaystyle\int_0^1 (1-\tau)^{\frac{N(p-1)}{p}} \tau^{q(p)-1} \mathrm{d}\tau & \text{if} \quad 1 \le q(p) < p^*. \end{cases}$$

It follows from (3.22) that

$$\left\| u_{p,q(p)} \right\|_\infty \le C_p^{-\frac{p^*}{N(p^*-q(p))}}.$$

As

$$\lim_{p\to 1^+} \frac{p^*}{N(p^*-q(p))} = 1 = \lim_{p\to 1^+} \left(\frac{p+N(p-1)}{p}\right)^{N+1} = \lim_{p\to 1^+} I_p$$

and

$$C_p^{-1} = \left(\frac{\lambda_{p,q(p)}}{\lambda_{p,p^*}}\right)^{\frac{N}{p}} \left(\frac{p+N(p-1)}{p}\right)^{N+1} I_p^{-1},$$

we obtain the second estimate in (3.21) from (1.6) and (3.18). □

***Proof of Theorem 1.1*** Of course, (1.9) follows directly from (3.21).

Let us prove (1.8). If $0 < q(p) < 1$, then Hölder's inequality yields

$$1 = \left\| u_{p,q(p)} \right\|_{q(p)}^{q(p)} \le \left\| u_{p,q(p)} \right\|_1 |\Omega|^{1-q(p)}$$

so that

$$1 = \lim_{p\to 1^+} \frac{1}{|\Omega|^{1-q(p)}} \le \liminf_{p\to 1^+} \left\| u_{p,q(p)} \right\|_1.$$

As for $1 \le q(p) < p^*$, we first note from (3.21) that

$$\lim_{p\to 1^+} \left\| u_{p,q(p)} \right\|_\infty^{q(p)-1} = 1.$$

Then, taking into account that

$$1 = \left\| u_{p,q(p)} \right\|_{q(p)}^{q(p)} \le \left\| u_{p,q(p)} \right\|_\infty^{q(p)-1} \left\| u_{p,q(p)} \right\|_1,$$

we get

$$1 = \lim_{p\to 1^+} \frac{1}{\left\| u_{p,q(p)} \right\|_\infty^{q(p)-1}} \le \liminf_{p\to 1^+} \left\| u_{p,q(p)} \right\|_1.$$

We have thus proved the estimate

$$1 \le \liminf_{p\to 1^+} \left\| u_{p,q(p)} \right\|_1$$

which, in view of (3.19), leads us to (1.8).

Exploiting (1.2) with respect to $\lambda_{p,p}$ again (see (3.20)), we obtain from (1.5) and (1.8) that

$$h(\Omega) = \lim_{p\to 1^+}\left(\lambda_{p,p}^{-1}\,|\Omega|^{1-p}\,\|u_{p,q(p)}\|_1^p\right) \le \liminf_{p\to 1^+}\lambda_{p,q(p)}.$$

Bearing in mind (3.18), this proves (1.7).

In order to complete the proof, let us take $p_n \to 1^+$ and set

$$q_n := q(p_n) \text{ and } u_n := u_{p_n,q_n}.$$

Then, $\lambda_{p_n,q_n} = \|\nabla u_n\|_{p_n}^{p_n}$, $\|u_n\|_{q_n} = 1$, and $\lim_{n\to\infty} q_n = 1$. Moreover, it follows from (1.8) that

$$\lim_{n\to\infty}\|u_n\|_1 = 1. \tag{3.23}$$

We note that

$$|Du_n|(\mathbb{R}^N) = |Du_n|(\Omega) = \|\nabla u_n\|_1 \le \|\nabla u_n\|_{p_n}\,|\Omega|^{1-\frac{1}{p_n}} = \lambda_{p_n,q_n}^{\frac{1}{p_n}}\,|\Omega|^{1-\frac{1}{p_n}}.$$

Hence, (1.7) implies that

$$\limsup_{n\to\infty}|Du_n|(\mathbb{R}^N) \le \lim_{n\to\infty}\lambda_{p_n,q_n}^{\frac{1}{p_n}}\,|\Omega|^{1-\frac{1}{p_n}} = h(\Omega). \tag{3.24}$$

We conclude from (3.23) and (3.24) that the sequence $(u_n)$ is bounded in $BV(\Omega)$. Thus, by the compactness of the embedding $BV(\Omega) \hookrightarrow L^1(\Omega)$ we can assume (up to passing to a subsequence) that $u_n \to u$, in $L^1(\Omega)$ and also pointwise almost everywhere in $\Omega$. Extending $u_n$ as zero on $\mathbb{R}^N \setminus \overline{\Omega}$, we have that $u_n$ converges in $L^1(\mathbb{R}^N)$ to $u$ extended as zero on $\mathbb{R}^N \setminus \overline{\Omega}$.

Owing to (3.23), we have $\|u\|_1 = 1$, which confirms item (a) and also implies that $u \in BV_0(\Omega)$. Hence, it follows from (2.1) that

$$h(\Omega) \le \frac{|Du|(\mathbb{R}^N)}{\|u\|_1} = |Du|(\mathbb{R}^N).$$

Moreover, Proposition 2.1 and (3.24) yield

$$|Du|(\mathbb{R}^N) \le \liminf_{n\to\infty}|Du_n|(\mathbb{R}^N) \le h(\Omega),$$

showing that $|Du|(\mathbb{R}^N) = h(\Omega)$. Then, item (c) is consequence of Proposition 2.2.

Now, let us prove item (b). Let us fix $r > 1$ and $\epsilon > 0$. As $q_n \to 1$, we have that $q_n < r$ for all $n \ge n_0$ and some $n_0 \in \mathbb{N}$. Moreover, owing to the second estimate in (3.21) we can also assume that

$$\|u_n\|_\infty \le \frac{h(\Omega)^N}{|\Omega|\,h(\Omega^\star)^N} + \epsilon, \text{ for all } n \ge n_0. \tag{3.25}$$

By Hölder's inequality, we have

$$1 = \|u_n\|_{q_n} \le \|u_n\|_r\,|\Omega|^{\frac{1}{q_n}-\frac{1}{r}},$$

so that

$$|\Omega|^{\frac{1}{r}-\frac{1}{q_n}} \le \|u_n\|_r, \text{ for all } n \ge n_0. \tag{3.26}$$

We also have

$$\|u_n\|_r^r \le \|u_n\|_\infty^{r-1}\,\|u_n\|_1 \le \left(\frac{h(\Omega)^N}{|\Omega|\,h(\Omega^\star)^N} + \epsilon\right)^{r-1}\|u_n\|_1, \quad \text{for all } n \ge n_0. \tag{3.27}$$

Convergence dominated theorem and (3.25) imply that $u_n \to u$ in $L^r(\Omega)$. Hence, (3.26) and (3.27) imply that

$$|\Omega|^{\frac{1}{r}-1} \leq \|u\|_r \leq \left(\frac{h(\Omega)^N}{|\Omega|\, h(\Omega^\star)^N} + \epsilon\right)^{\frac{r-1}{r}} \|u\|_1^{\frac{1}{r}} = \left(\frac{h(\Omega)^N}{|\Omega|\, h(\Omega^\star)^N} + \epsilon\right)^{\frac{r-1}{r}}. \tag{3.28}$$

As $r$ and $\epsilon$ are arbitrarily fixed, (3.28) implies that $u \in L^\infty(\Omega)$ and

$$|\Omega|^{-1} \leq \|u\|_\infty \leq \frac{h(\Omega)^N}{|\Omega|\, h(\Omega^\star)^N}.$$

□

As mentioned in the Introduction, right after Corollary 1.2, Bueno and Ercole proved in [5] that

$$\lim_{p\to 1^+} \left\|v_{p,1}\right\|_1^{1-p} = h(\Omega) = \left\|v_{p,1}\right\|_\infty^{1-p}.$$

As $\lambda_{p,1} = \left\|v_{p,1}\right\|_1^{1-p}$, a fact that was not noticed in [5], the first equality above leads directly to

$$\lim_{p\to 1^+} \lambda_{p,1} = h(\Omega), \tag{3.29}$$

which is (1.7) in the case where $q(p) \equiv 1$. Thus, (3.29) combined with (1.5) and the monotonicity of the function $q \mapsto \lambda_{p,q} |\Omega|^{\frac{p}{q}}$ also produces (1.7) for $q(p) \in (1, p)$. However, this combination does not lead to the same result for $q(p) \in (0, 1) \cup (p, p^*)$ as, for example, $q(p) = p^\beta$ with $\beta < 0$ or $\beta > 1$ (and $p$ close to $1^+$). Our approach combining (1.5) with Lemma 3.1 provides a unified proof to (1.7) as well as allows us to estimate the limit function $u$.

## 4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3, by applying Picone's inequality to $w_{p,q(p)}$ and $e_p$, where $e_p$ is the first eigenfunction defined in (3.14).

***Proof of Theorem 1.3*** As

$$\lambda_{p,q(p)} \leq \frac{\left\|\nabla w_{p,q(p)}\right\|_p^p}{\left\|w_{p,q(p)}\right\|_{q(p)}^p} = \left\|w_{p,q(p)}\right\|_{q(p)}^{q(p)-p} \leq \left\|w_{p,q(p)}\right\|_\infty^{q(p)-p} |\Omega|^{\frac{q(p)-p}{q(p)}},$$

we obtain from (1.7) that

$$h(\Omega) \leq \liminf_{p\to 1^+} \left\|w_{p,q(p)}\right\|_{q(p)}^{q(p)-p} \leq \liminf_{p\to 1^+} \left\|w_{p,q(p)}\right\|_\infty^{q(p)-p}. \tag{4.1}$$

Applying Picone's inequality and using that $w_{p,q(p)}$ is a weak solution to (1.11), we find

$$\begin{aligned}\lambda_{p,p} \int_\Omega e_p^p \mathrm{d}x &= \int_\Omega \left|\nabla e_p\right|^p \mathrm{d}x \\ &\geq \int_\Omega \left|\nabla w_{p,q(p)}\right|^{p-2} \nabla w_{p,q(p)} \cdot \nabla\left(\frac{e_p^p}{w_{p,q(p)}^{p-1}}\right) \mathrm{d}x \\ &= \int_\Omega w_{p,q(p)}^{q(p)-1} \frac{e_p^p}{w_{p,q(p)}^{p-1}} \mathrm{d}x = \int_\Omega w_{p,q(p)}^{q(p)-p} e_p^p \mathrm{d}x.\end{aligned}$$

Hence,

$$\|w_{p,q(p)}\|_\infty^{q(p)-p}\int_\Omega W_p^{q(p)-p}e_p^p\mathrm{d}x\le\lambda_{p,p}\int_\Omega e_p^p\mathrm{d}x \tag{4.2}$$

where

$$W_p:=\frac{w_{p,q(p)}}{\|w_{p,q(p)}\|_\infty}.$$

Now, let us assume that

$$L:=\limsup_{p\to1^+}\|w_{p,q(p)}\|_\infty^{q(p)-p}<\infty.$$

Using again that $w_{p,q(p)}$ is a weak solution to (1.11), we have from Lemma 3.1, with $\lambda=\sigma=1$, that

$$C_p\|w_{p,q(p)}\|_\infty^{\frac{N(p-q(p))}{p}}\|w_{p,q(p)}\|_\infty\le\|w_{p,q(p)}\|_1 \tag{4.3}$$

where

$$C_p:=\lambda_{p,p^*}^{\frac{N}{p}}\left(\frac{p}{p+N(p-1)}\right)^{N+1}\sigma\int_0^1(1-\tau)^{\frac{N(p-1)}{p}}\mathrm{d}\tau.$$

It follows from (4.3) that

$$\frac{C_p}{\|w_{p,q(p)}\|_\infty^{\frac{N(q(p)-p)}{p}}}\le\|W_p\|_1,$$

so that (1.6) and (3.16) yield

$$0<\frac{|\Omega|\,h(\Omega^\star)^N}{L^N}\le\liminf_{p\to1^+}\|W_p\|_1. \tag{4.4}$$

We also have that

$$\|\nabla w_{p,q(p)}\|_1\le|\Omega|^{1-\frac{1}{p}}\|\nabla w_{p,q(p)}\|_p=|\Omega|^{1-\frac{1}{p}}\|w_{p,q(p)}\|_{q(p)}^{\frac{q(p)}{p}}$$

so that

$$|DW_p|(\Omega)=\|\nabla W_p\|_1\le|\Omega|^{1-\frac{1}{p}}\|w_{p,q(p)}\|_\infty^{\frac{q(p)-p}{p}}\|W_p\|_{q(p)}^{\frac{q(p)}{p}}.$$

Hence, as $\|W_p\|_{q(p)}^{\frac{q(p)}{p}}\le|\Omega|^{\frac{1}{p}}$ and $\|W_p\|_1\le|\Omega|$, we conclude that the family $(W_p)$ is uniformly bounded in $BV(\Omega)$.

Now, let $p_n\to1^+$ be such that

$$\lim_{n\to\infty}\|w_{p_n,q(p_n)}\|_\infty^{q(p_n)-p_n}=L.$$

Owing to the compactness of $BV(\Omega)\hookrightarrow L^1(\Omega)$, we can assume (passing to subsequences, if necessary) that $W_{p_n}\to W$ in $L^1(\Omega)$ and also pointwise almost everywhere in $\Omega$. It follows from (4.4) that $W>0$ a.e. in $\Omega$ and this implies that $W_{p_n}^{q(p_n)-p_n}\to1$ pointwise almost everywhere in $\Omega$. As $\left\|W_{p_n}^{q(p_n)-p_n}e_{p_n}^{p_n}\right\|_\infty\le1$, dominated convergence theorem and Lemma 3.3 guarantee that

$$\lim_{n\to\infty}\int_\Omega W_{p_n}^{q(p_n)-p_n}e_{p_n}^{p_n}\mathrm{d}x=\|e\|_1>0.$$

Hence, (4.2) and (1.5) yield

$$L \|e\|_1 \leq h(\Omega) \|e\|_1 ,$$

so that $L \leq h(\Omega)$. Combining this inequality with (4.1), we conclude that

$$h(\Omega) = \lim_{p\to 1^+} \left\| w_{p,q(p)} \right\|_{q(p)}^{q(p)-p} = \lim_{p\to 1^+} \left\| w_{p,q(p)} \right\|_{\infty}^{q(p)-p} .$$

□

***Remark 4.1*** One can derive from Remark 3.2 that if $0 < q(p) < p^*$, then

$$\liminf_{p\to 1^+} \left\| w_{p,q(p)} \right\|_{q(p)}^{q(p)-p} = \liminf_{p\to 1^+} \left\| w_{p,q(p)} \right\|_{\infty}^{q(p)-p}$$

and

$$\limsup_{p\to 1^+} \left\| w_{p,q(p)} \right\|_{q(p)}^{q(p)-p} = \limsup_{p\to 1^+} \left\| w_{p,q(p)} \right\|_{\infty}^{q(p)-p} .$$

Thus, the alternative (1.13) in the statement of Theorem 1.3 can be replaced with

$$\limsup_{p\to 1^+} \left\| w_{p,q(p)} \right\|_{q(p)}^{q(p)-p} = +\infty.$$

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