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A sharp multiplier theorem for solvable extensions of Heisenberg and related groups

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Abstract

Let G be the semidirect product $N \rtimes \mathbb{R}$, where N is a stratified Lie group and $\mathbb{R}$ acts on N via automorphic dilations. Homogeneous left-invariant sub-Laplacians on N and $\mathbb{R}$ can be lifted to G, and their sum Δ is a left-invariant sub-Laplacian on G. In previous joint work of Ottazzi, Vallarino and the first-named author, a spectral multiplier theorem of Mihlin–Hörmander type was proved for Δ, showing that an operator of the form $F(\Delta)$ is of weak type $(1, 1)$ and bounded on $L^p(G)$ for all $p \in (1, \infty)$ provided F satisfies a scale-invariant smoothness condition of order $s > (Q + 1)/2$, where Q is the homogeneous dimension of N. Here we show that, if N is a group of Heisenberg type, or more generally a direct product of Métivier and abelian groups, then the smoothness condition can be pushed down to the sharp threshold $s > (d + 1)/2$, where d is the topological dimension of N. The proof is based on lifting to G weighted Plancherel estimates on N and exploits a relation between the functional calculi for Δ and analogous operators on semidirect extensions of Bessel–Kingman hypergroups.

Keywords Mihlin–Hörmander multiplier · Spectral multiplier · Sub-Laplacian · Solvable group · Hypergroup

Mathematics Subject Classification 22E30 · 42B15 · 42B20 · 43A22

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1 Introduction

Let N be a stratified Lie group of step r. In other words, N is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{n}$ is decomposed as a direct sum $\mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_r$ of nontrivial subspaces $\mathfrak{n}_j$, called layers, in such a way that $[\mathfrak{n}_1, \mathfrak{n}_j] = \mathfrak{n}_{j+1}$ for all $j = 1, \dots, r$, where $\mathfrak{n}_{r+1} = \{0\}$. As N is nilpotent and simply connected, the exponential map is a diffeomorphism between $\mathfrak{n}$ and N, and in exponential coordinates the Lebesgue measure on $\mathfrak{n}$ corresponds to the (left and right) Haar measure on N. We denote by $d = \sum_{j=1}^r \dim \mathfrak{n}_j$ and $Q = \sum_{j=1}^r j \dim \mathfrak{n}_j$ the topological and homogeneous dimensions of N. Let Δ_N be a homogeneous left-invariant sub-Laplacian on N, that is an operator of the form

$$\Delta_N = -\sum_{j=1}^{\dim \mathfrak{n}_1} X_j^2, \tag{1.1}$$

where the X_j are left-invariant vector fields forming a basis of the first layer $\mathfrak{n}_1$.

The stratified group N is naturally equipped with a one-parameter group of automorphic dilations $(e^{uD})_{u\in\mathbb{R}}$, where D is the derivation of $\mathfrak{n}$ which is j times the identity on $\mathfrak{n}_j$. We can then form the semidirect product $G = N \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on N by dilations; in other words, the product law on G is given by

$$(z, u) \cdot (\tilde{z}, \tilde{u}) = (z \cdot e^{uD}\tilde{z}, u + \tilde{u})$$

for all $(z, u), (\tilde{z}, \tilde{u}) \in G$. In these coordinates, the product measure $\mathrm{d}z\, \mathrm{d}u$ of the Haar measure on N and the Lebesgue measure on $\mathbb{R}$ is the right Haar measure on G, while the left Haar measure is given by $e^{-Qu}\, \mathrm{d}z\, \mathrm{d}u$; unless otherwise specified, we shall use the right Haar measure when integrating on G and in the definition of the Lebesgue spaces $L^p(G)$.

The vector fields ∂_u on $\mathbb{R}$ and X_j ($j = 1, \dots, \dim \mathfrak{n}_1$) on N lift to left-invariant vector fields on the semidirect product G, given by

$$X_j^\sharp = \begin{cases} \partial_u & \text{if } j = 0, \\ e^u X_j & \text{if } j = 1, \dots, \dim \mathfrak{n}_1, \end{cases} \tag{1.2}$$

and we can consider the corresponding left-invariant sub-Laplacian

$$\Delta = -\sum_{j=0}^{\dim \mathfrak{n}_1} (X_j^\sharp)^2 = -\partial_u^2 + e^{2u}\Delta_N \tag{1.3}$$

on G. This operator is essentially self-adjoint on $L^2(G)$; so, by the spectral theorem, the sub-Laplacian Δ has a Borel functional calculus, and for any bounded Borel function $F : [0, \infty) \to \mathbb{C}$ the operator $F(\Delta)$ is bounded on $L^2(G)$.

Here we are interested in investigating what additional conditions are to be required on the spectral multiplier F so that the operator $F(\Delta)$, initially defined on $L^2(G)$, extends to a bounded operator on $L^p(G)$ for some $p \neq 2$. More specifically, we look for relations between L^p-boundedness properties of $F(\Delta)$ and size and smoothness properties of F, such as Mihlin–Hörmander type estimates of the form

$$\|F(\Delta)\|_{L^p(G)\to L^p(G)} \lesssim_{p,s} \sup_{t>0} \|F(t\cdot)\chi\|_{L^2_s(\mathbb{R})} \tag{1.4}$$

for appropriate values of p and s, where $L^2_s(\mathbb{R})$ is the L^2 Sobolev space of order s on $\mathbb{R}$ and $\chi \in C_c^\infty(0, \infty)$ is a nontrivial cutoff. These are generalisations of the classical estimates for the Laplace operator on $\mathbb{R}^n$, for which the analogue of (1.4) holds for any $p \in (1, \infty)$ and $s > n/2$, as a consequence of the Mihlin–Hörmander theorem for Fourier multipliers [37, 59].

We point out that G is a solvable Lie group of exponential volume growth. In particular, the "standard machinery" providing L^p-bounds of Mihlin–Hörmander type for the functional calculus for sub-Laplacians on Lie groups of polynomial growth and more general doubling metric-measure spaces (see, e.g., [2, 13, 18, 29]) does not apply here; indeed, if we equip G with the Carnot–Carathéodory distance associated with the vector fields (1.2), then G is locally doubling, but not globally. Nevertheless, somewhat surprisingly, the sub-Laplacian Δ on G has a differentiable functional calculus on $L^p(G)$ [11, 22, 28, 68]; this should be contrasted with what happens for other exponential solvable Lie groups and sub-Laplacians, which may be of holomorphic L^p-type [9, 31]. More recently, by developing and extending previous results and ideas in [30, 32, 80], the authors of [54] proved a spectral multiplier theorem of Mihlin–Hörmander type for Δ, which can be stated as follows.

For a fixed cutoff $\chi \in C_c^\infty(0, \infty)$ with $\mathbb{1}_{[1/2,2]} \le \chi \le \mathbb{1}_{[1/4,4]}$, we define for $s \ge 0$ and a bounded Borel function $F : [0, \infty) \to \mathbb{C}$ the quantities

$$\|F\|_{0,s} = \sup_{0<t\le 1} \|F(t\cdot)\chi\|_{L^2_s(\mathbb{R})}, \qquad \|F\|_{\infty,s} = \sup_{t\ge 1} \|F(t\cdot)\chi\|_{L^2_s(\mathbb{R})}.$$

These are variants of the scale-invariant Sobolev norm in the right-hand side of (1.4), focusing on the "local part" and the "part at infinity" of the function F.

Theorem 1.1 ([32, 54]) *Suppose that both $s_0, s_\infty > 3/2$, and that moreover $s_\infty > (Q+1)/2$. If a bounded Borel function $F : [0, \infty) \to \mathbb{C}$ satisfies $\|F\|_{0,s_0} < \infty$ and $\|F\|_{\infty,s_\infty} < \infty$, then $F(\Delta)$ extends to an operator of weak type $(1, 1)$ and bounded on $L^p(G)$ for $p \in (1, \infty)$, bounded from $H^1(G)$ to $L^1(G)$ and from $L^\infty(G)$ to $BMO(G)$.*

We refer to [54, 81] for the precise definitions of the atomic Hardy space $H^1(G)$ and the dual space $BMO(G)$, which are adapted to the Calderón–Zygmund structure of G in the sense of [32].

We point out that the restriction $s_\infty > 3/2$ in the above statement is implied by $s_\infty > (Q+1)/2$ except when $Q = 1$, in which case N must be abelian. In general, the above result requires different orders of smoothness s_0 and s_∞ on the local part and the part at infinity of the multiplier F, related to the "pseudodimension" 3 and the "local doubling dimension" $Q + 1$ of G; here the pseudodimension plays the role that the "dimension at infinity" plays in [2] for groups of polynomial growth. In any case, Theorem 1.1 implies the validity of the estimate (1.4) for any $p \in (1, \infty)$ and $s > \max\{3, Q + 1\}/2$.

When N is abelian, i.e. N has step 1, then $N \cong \mathbb{R}^d$ and the above result is already contained in [32]. In that case, Δ_N is just the Laplacian on $\mathbb{R}^d$, and Δ is elliptic; moreover, $\mathfrak{n} = \mathfrak{n}_1$ and $Q = d$. A classical transplantation argument [42] then shows that the smoothness condition $s_\infty > (Q+1)/2$ is sharp, in the sense that the threshold $(Q+1)/2$, which in that case equals half the topological dimension $(d + 1)/2$ of G, cannot be replaced by a smaller quantity.

Instead, when N is not abelian, the operators Δ_N and Δ are not elliptic, and $Q > d \ge 3$. Here it is meaningful to ask about the sharpness of the condition $s_\infty > (Q + 1)/2$ in the previous theorem. Indeed, by the results of [53], we know that the threshold $(Q+1)/2$ cannot be replaced by anything smaller than $(d+1)/2$, but a gap remains between the two quantities. The main result of this paper shows that, at least for certain classes of nonabelian stratified groups N, the condition can indeed be pushed down to $s_\infty > (d + 1)/2$, thus leading to a sharp result, which improves Theorem 1.1 in this case.

We recall that the stratified group N is said to be a *Métivier group* [58] if N has step 2 and, for all $\mu \in \mathfrak{n}_2^* \setminus \{0\}$, the skewsymmetric bilinear form $\mu[\cdot,\cdot] : \mathfrak{n}_1 \times \mathfrak{n}_1 \to \mathbb{R}$ is nondegenerate. Heisenberg groups and, more generally, groups of Heisenberg type in the sense of Kaplan [41] are Métivier groups, but there exist also Métivier groups which are not of Heisenberg type [63]. Our improvement of Theorem 1.1 applies in particular to the case where N is a Métivier group.

Theorem 1.2 *Assume that the* 2*-step group* N *is a direct product of finitely many Métivier and abelian groups. Suppose that* $s_0 > 3/2$ *and* $s_\infty > (d+1)/2$. *If a bounded Borel function* $F : [0,\infty) \to \mathbb{C}$ *satisfies* $\|F\|_{0,s_0} < \infty$ *and* $\|F\|_{\infty,s_\infty} < \infty$, *then* $F(\Delta)$ *extends to an operator of weak type* $(1,1)$ *and bounded on* $L^p(G)$ *for* $p \in (1,\infty)$, *bounded from* $H^1(G)$ *to* $L^1(G)$ *and from* $L^\infty(G)$ *to* $BMO(G)$.

As a consequence, under the assumptions of this theorem, the estimate (1.4) holds true for all $p \in (1,\infty)$ and $s > (d+1)/2$. We point out once again that the threshold $(d+1)/2$, corresponding to half the topological dimension of G, is sharp. In this respect, this result can be considered as part of a programme aimed at determining the sharp threshold in Mihlin–Hörmander estimates for nonelliptic "Laplace-like" operators in a variety of settings; we refer to [6, 13, 14, 52, 53, 65] for a more extensive discussion and further references. The relevance of Theorem 1.2 in this context is that it appears to be the first sharp result of this type where the underlying manifold has exponential volume growth.

By a contraction argument (see, e.g., [50, Theorem 5.2]), Theorems 1.1 and 1.2 imply corresponding results for the direct product $\tilde{G} = N \times \mathbb{R}$, with smoothness conditions $s_0 = s_\infty > (Q+1)/2$ and $s_0 = s_\infty > (d+1)/2$ respectively. However, these results for $\tilde{G}$ are already available in the literature. Indeed, the direct product $\tilde{G}$ is a stratified group itself, and the results deduced by contraction from Theorems 1.1 and 1.2 correspond to the Christ–Mauceri–Meda theorem [8, 55] for homogeneous sub-Laplacians on stratified groups, and its improvement due to Hebisch [27] and Müller and Stein [65] for Heisenberg and related groups.

It is still an open problem whether the improvement to half the topological dimension in the Christ–Mauceri–Meda theorem is always possible for an arbitrary stratified group (see, e.g., the discussion in [51–53]). As a consequence, the question whether the additional assumption on N in Theorem 1.2 can be dropped altogether appears to be out of reach at this time, as the analogous and apparently easier question for the direct product $\tilde{G}$ is still open. Nevertheless, the technique developed in this paper can be used to improve Theorem 1.1 for a larger class of nonabelian stratified groups N than the one considered in Theorem 1.2, also including some groups of step higher than 2.

We refer to [48, Section 3] for the definition of h*-capacious* stratified group. Here we just recall that any stratified group N is 0-capacious, but, if N is Métivier, then it is also $(Q-d)$-capacious. Moreover, if N is the direct product of an h_1-capacious and an h_2-capacious group, then N is (h_1+h_2)-capacious. Furthermore, if N has step r and $\dim \mathfrak{n}_r = 1$, then N is 1-capacious [48, Proposition 3.9]. So the following result properly extends Theorem 1.2.

Theorem 1.3 *Assume that* N *is* h*-capacious for some* $h \in \mathbb{N}$. *Suppose that* $s_0, s_\infty > 3/2$ *and* $s_\infty > (Q-h+1)/2$. *If a bounded Borel function* $F : [0,\infty) \to \mathbb{C}$ *satisfies* $\|F\|_{0,s_0} < \infty$ *and* $\|F\|_{\infty,s_\infty} < \infty$, *then* $F(\Delta)$ *extends to an operator of weak type* $(1,1)$ *and bounded on* $L^p(G)$ *for* $p \in (1,\infty)$, *bounded from* $H^1(G)$ *to* $L^1(G)$ *and from* $L^\infty(G)$ *to* $BMO(G)$.

This result can be compared with that in [48, Corollary 6.1] for homogeneous sub-Laplacians on h-capacious stratified groups. The proof of Theorem 1.3 is a relatively straightforward modification of the proof of Theorem 1.2, but requires several adjustments and changes of notation. In order not to affect the clarity of the presentation, below we only discuss the details of the proof of Theorem 1.2.

Proof strategy

By duality and interpolation, each of the spectral multiplier theorems for Δ stated above reduces to endpoint estimates of the form

$$\begin{aligned} \|F(\Delta)\|_{L^1(G)\to L^{1,\infty}(G)} &\lesssim_\varepsilon \|F\|_{0,\varsigma_0+\varepsilon} + \|F\|_{\infty,\varsigma_\infty+\varepsilon}, \\ \|F(\Delta)\|_{H^1(G)\to L^1(G)} &\lesssim_\varepsilon \|F\|_{0,\varsigma_0+\varepsilon} + \|F\|_{\infty,\varsigma_\infty+\varepsilon} \end{aligned} \tag{1.5}$$

for all $\varepsilon > 0$, where $\varsigma_0, \varsigma_\infty \geq 0$ are appropriate thresholds. As in other works on the subject, by means of Calderón–Zygmund theory, the estimates (1.5) can essentially be reduced to L^1-estimates for the convolution kernels $K_{F(t\Delta)}$ of the operators $F(t\Delta)$, corresponding to rescaled versions of a multiplier F with compact support $\operatorname{supp} F \subseteq [-4, 4]$:

$$\sup_{0<t\leq 1} \|K_{F(t\Delta)}\|_{L^1(G)} \lesssim_\varepsilon \|F\|_{L^2_{\varsigma_0+\varepsilon}(\mathbb{R})}, \quad \sup_{t\geq 1} \|K_{F(t\Delta)}\|_{L^1(G)} \lesssim_\varepsilon \|F\|_{L^2_{\varsigma_\infty+\varepsilon}(\mathbb{R})}.$$

In turn, via a frequency decomposition, these can be deduced, at least when $\varsigma_0 \leq \varsigma_\infty$, from an estimate of the form

$$\|K_{F(\sqrt{\Delta})}\|_{L^1(G)} \lesssim r^{[\varsigma_\infty,\varsigma_0]} \left(\int_0^\infty |F(\lambda)|^2 \, \lambda^{[2\varsigma_0, 2\varsigma_\infty]} \frac{\mathrm{d}\lambda}{\lambda} \right)^{1/2} \tag{1.6}$$

for all $r > 0$ and $F \in \mathcal{E}_r$; here $\mathcal{E}_r$ is the set of the even Schwartz functions $F : \mathbb{R} \to \mathbb{C}$ whose Fourier supports are contained in $[-r, r]$, and we write

$$\lambda^{[a,b]} = \begin{cases} \lambda^a & \text{if } \lambda \leq 1, \\ \lambda^b & \text{if } \lambda \geq 1, \end{cases} \tag{1.7}$$

for any $a, b \in \mathbb{R}$ and $\lambda > 0$.

When $\varsigma_0 = 3/2$ and $\varsigma_\infty = (Q + 1)/2$, the estimate (1.6) can be proved by combining the Plancherel formula for the functional calculus for Δ, namely

$$\|K_{F(\sqrt{\Delta})}\|^2_{L^2(G)} \simeq \int_0^\infty |F(\lambda)|^2 \, \lambda^{[3,Q+1]} \frac{\mathrm{d}\lambda}{\lambda}, \tag{1.8}$$

and the finite propagation speed property for Δ, namely

$$\operatorname{supp} K_{F(\sqrt{\Delta})} \subseteq \overline{B_G}(0_G, r)$$

whenever $F \in \mathcal{E}_r$, where $\overline{B_G}(0_G, r)$ is the closed ball of radius r centred at the origin of G with respect to the Carnot–Carathéodory distance. Indeed, by finite propagation speed and the Cauchy–Schwarz inequality,

$$\|K_{F(\sqrt{\Delta})}\|_{L^1(G)} \leq |\overline{B_G}(0_G, r)|^{1/2} \|K_{F(\sqrt{\Delta})}\|_{L^2(G)}$$

when $F \in \mathcal{E}_r$, which in view of the Plancherel formula (1.8) gives (1.6) when $r \leq 1$, because $|\overline{B_G}(0_G, r)| \simeq r^{Q+1}$ in this case. This argument does not work as it is for $r \geq 1$, since $|\overline{B_G}(0_G, r)| \simeq \exp(Qr)$ for large r. However, one can fix this by introducing a suitable weight when applying the Cauchy–Schwarz inequality, in order to kill the exponential volume growth; finite propagation speed and radiality properties of $K_{F(\sqrt{\Delta})}$ can then be used, roughly speaking, to get rid of the extra weight in the L^2 norm, and obtain (1.6) for large r. This is broadly the approach used in [32, 54] to prove Theorem 1.1.

In order to improve the result, i.e., to push down the threshold ς_∞, here we introduce, also for small r, an appropriate weight $w = w(z)$ in the application of the Cauchy–Schwarz inequality:

$$\|K_{F(\sqrt{\Delta})}\|_{L^1(G)} \le \left(\int_{\overline{B_G}(0_G,r)} w(z)^{-2}\,dz\,du \right)^{1/2} \|w\,K_{F(\sqrt{\Delta})}\|_{L^2(G)}.$$

If w is chosen so that $\int_{\overline{B_G}(0_G,r)} w(z)^{-2}\,dz\,du \simeq r^{2\varsigma_\infty}$ for $r \le 1$, then the problem of obtaining (1.6) is essentially reduced, at least for small r, to the proof of a "weighted Plancherel estimate" of the form

$$\|w\,K_{F(\sqrt{\Delta})}\|^2_{L^2(G)} \lesssim \int_0^\infty |F(\lambda)|^2\,\lambda^{[3,2\varsigma_\infty]}\,\frac{\mathrm{d}\lambda}{\lambda}. \tag{1.9}$$

In the case $\varsigma_\infty = (\nu + 1)/2$ for some integer ν, in light of (1.8) we can equivalently rewrite the previous inequality as

$$\|w\,K_{F(\sqrt{\Delta})}\|_{L^2(G)} \lesssim \|K_{F(\sqrt{\Delta_\nu})}\|_{L^2(G_\nu)}, \tag{1.10}$$

where G_ν and Δ_ν are the analogues of G and Δ when N is replaced by $\mathbb{R}^\nu$. As w only depends on the variable z, the relation between the functional calculi of Δ and Δ_N allows one to reduce (1.10) to a similar estimate on the stratified group N:

$$\|w\,K_{F(\sqrt{\Delta_N})}\|_{L^2(N)} \lesssim \|K_{F(\sqrt{\Delta_{\mathbb{R}^\nu}})}\|_{L^2(\mathbb{R}^\nu)}, \tag{1.11}$$

or equivalently,

$$\|w\,K_{F(\sqrt{\Delta_N})}\|^2_{L^2(N)} \lesssim \int_0^\infty |F(\lambda)|^2\,\lambda^{\nu}\,\frac{\mathrm{d}\lambda}{\lambda}. \tag{1.12}$$

As it turns out, when N is a direct product of Métivier and abelian groups, the "weighted Plancherel estimate" (1.12) on N holds true for any $\nu \in (d, Q]$, with a suitable weight w depending on ν. Indeed, such an estimate is the fundamental ingredient used in [27] to sharpen the Christ–Mauceri–Meda theorem for this class of stratified groups. The intermediate steps (1.11) and (1.10) allow one to lift the weighted estimate to G and deduce (1.9) for $\varsigma_\infty = (\nu + 1)/2$, where $\nu \in (d, Q] \cap \mathbb{N}$.

In order to prove Theorem 1.2 with the sharp condition $s_\infty > (d + 1)/2$, in principle we would like to take $\varsigma_\infty = (d+1)/2$, that is $\nu = d$, but the weighted Plancherel estimate (1.12) on N fails in that case. As a workaround, we can take instead $\nu = d + \varepsilon$ for arbitrarily small $\varepsilon > 0$, as (1.12) then holds true, but this requires us to work with non-integer ν. The problem then becomes how to make sense of the intermediate steps (1.11) and (1.10) in the lifting argument leading to (1.9), as $\mathbb{R}^\nu$ and $G_\nu = \mathbb{R}^\nu \rtimes \mathbb{R}$ are not defined when ν is fractional.

The solution that we adopt here is to replace the Laplacian $\Delta_{\mathbb{R}^\nu}$ on $\mathbb{R}^\nu$ with the Bessel operator $L_\nu = -\partial_x^2 - (\nu - 1)x^{-1}\partial_x$ on the half-line $X_\nu = [0, \infty)$ equipped with the measure $x^{\nu-1}\,\mathrm{d}x$. When ν is an integer, the Bessel operator L_ν is just the radial part of the Laplacian on $\mathbb{R}^\nu$, but of course L_ν and X_ν make sense for fractional ν as well, and provide the following replacement for (1.11):

$$\|w\,K_{F(\sqrt{\Delta_N})}\|_{L^2(N)} \lesssim \|K_{F(\sqrt{L_\nu})}\|_{L^2(X_\nu)}. \tag{1.13}$$

Similarly, the lifted estimate (1.10) becomes meaningful for fractional ν by taking as G_ν the semidirect product $X_\nu \rtimes \mathbb{R}$, and as Δ_ν the operator $-\partial_u^2 + e^{2u} L_\nu$ thereon.

One of the technical problems in implementing this strategy is that the half-line X_ν is not a Lie group; however, it is a hypergroup [3, 40] (more precisely, X_ν is known as a Bessel–Kingman hypergroup), so there is a convolution structure in terms of which the convolution kernels $K_{F(\sqrt{L_\nu})}$ are defined. Additionally, X_ν has a group of automorphic dilations, so the semidirect product $G_\nu = X_\nu \rtimes \mathbb{R}$ can be made sense of in terms of hypergroup theory [35, 84], and again the operators in the functional calculus for $\Delta_\nu = -\partial_u^2 + e^{2u} L_\nu$ are convolution operators on G_ν. Thus, one of the tasks that we undertake here is to develop the theory of the operators L_ν and Δ_ν, mirroring the classical one for $\Delta_{\mathbb{R}^\nu}$ and $-\partial_u^2 + e^{2u}\Delta_{\mathbb{R}^\nu}$ when ν is an integer, and establish all the properties that we need in order to run the aforementioned argument when ν is fractional (e.g., finite propagation speed, explicit formula for the control distance, radiality of convolution kernels in the functional calculus, Plancherel formula...). A fundamental tool for this is the theory developed in [22], which explicitly relates the heat semigroups generated by L_ν and Δ_ν, thus providing the link between their functional calculi that justifies the lifting of (1.13) to (1.10) and eventually yields the required estimates for the proof of Theorem 1.2.

Structure of the paper

In Sect. 2 we recall the main properties of the stratified groups N, their solvable extensions $G = N \rtimes \mathbb{R}$, and the sub-Laplacians thereon. We also state and prove a suitable version of the weighted Plancherel estimate on N, extending those available in the literature, and show that the weight appearing in that estimate has the correct integrability properties on G.

Sections 3 and 4 are devoted to the discussion of the hypergroups X_ν and G_ν, and the corresponding operators L_ν and Δ_ν. Of course, there is a vast literature on the Bessel operator L_ν, so we only recall some of the main properties, in connection with the Hankel transform on X_ν. In comparison, not so many results appear to be available for the operator Δ_ν on the semidirect product hypergroup G_ν, so we spend some time to derive the required properties. One of the issues we need to deal with is the fact that X_ν and G_ν are manifolds with boundary, and indeed boundary conditions play a role (at least for ν small) in the choice of self-adjoint extensions of L_ν and Δ_ν, which in turn determine their functional calculi.

By using the aforementioned hypergroup structures, in Sect. 5 we lift to G the weighted Plancherel estimate on N obtained in Sect. 2 and prove our main result, Theorem 1.2.

Finally, we devote an Appendix (Sect. 6) to recalling some terminology and results about self-adjoint extensions of divergence-form differential operators with Dirichlet and Neumann boundary conditions, as well as the finite propagation speed property for the associated wave equation, using the “first-order approach” from [56].

Some remarks and open questions

The technique developed here hinges on lifting weighted Plancherel estimates on N of the form (1.12) to the semidirect product $G = N \rtimes \mathbb{R}$. As already mentioned, similar estimates were originally obtained in [27, 65] as a tool to sharpen the Christ–Mauceri–Meda multiplier theorem on N. However, not all stratified groups N are amenable to this approach. More recently [49, 51] a different approach was developed, which applies to other classes of stratified groups and is based on variants of (1.12) where, roughly speaking, the right-hand side may also contain derivatives of F. This alternative approach can be used on a number of 2-step stratified groups N, other than direct products of Métivier and abelian groups, to push down the condition in the Christ–Mauceri–Meda theorem to half the topological dimension. For those stratified groups N, one may expect that an analogous improvement of Theorem 1.1 can be obtained on the corresponding semidirect product $G = N \rtimes \mathbb{R}$, but the approach presented here does not directly apply and new ideas would appear to be needed.

Another natural question is the validity of L^p estimates of Miyachi–Peral type [60, 70] for the wave equation associated with the sub-Laplacian Δ on G; indeed sharp estimates of this type would likely imply via subordination [62] a sharp multiplier theorem for Δ. Such estimates on G are known [66] in the case N is abelian (see also the extension in [67] for an elliptic Laplacian on Damek–Ricci spaces), but the case of nonabelian N and nonelliptic Δ appears to be wide open. For the homogeneous sub-Laplacian Δ_N on N, sharp Miyachi–Peral estimates are known when N is of Heisenberg type [64], and one may wonder whether similar estimates also hold for Δ on the corresponding solvable extension G.

Notation

$\mathbb{N}$ denotes the set of natural numbers, including 0. We write $\mathbb{R}_+$ and $\mathring{\mathbb{R}}_+$ for the closed and open half-lines $[0, \infty)$ and $(0, \infty)$. For two nonnegative quantities A and B, we write $A \lesssim B$ to denote that there exists a constant $C \in \mathring{\mathbb{R}}_+$ such that $A \leq CB$. We also write $A \simeq B$ for the conjunction of $A \lesssim B$ and $B \lesssim A$. Subscripted variants such as $\lesssim_p$ or $\simeq_p$ indicate that the constant may depend on the parameter p. Finally, we write $\mathbb{1}_S$ for the characteristic function of a set S.

2 Stratified Lie groups and their solvable extensions

2.1 Weighted Plancherel estimate for products of Métivier groups

In this section our aim is to justify the weighted Plancherel estimate (1.12) that is at the core of our argument. Versions of this estimate can be found elsewhere in the literature (see, e.g., [27, Lemma 1.4] or [47, Proposition 3.5]), but they are stated and proved under the assumption that the multiplier F is supported in a fixed compact subset of $\mathring{\mathbb{R}}_+$. The proof that we present here shows that, in fact, this restriction on the support of F can be dropped for certain homogeneous weights w in the left-hand side of (1.12), provided one chooses the appropriate homogeneity degree of the measure in the right-hand side. This version of the estimate without support restrictions turns out to be crucial in our proof of Theorem 1.2.

Let N be a 2-step stratified Lie group. So the Lie algebra $\mathfrak{n}$ of N is the direct sum of two layers, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. If we denote by d_1 and d_2 the dimensions of $\mathfrak{n}_1$ and $\mathfrak{n}_2$, then $d = d_1 + d_2$ and $Q = d_1 + 2d_2$ are the topological and homogeneous dimensions of N. Through the exponential map we can identify N with $\mathfrak{n}$; via this identification, Lebesgue measure on $\mathfrak{n}$ is a left and right Haar measure on N. The same identification allows us to define the Schwartz class $\mathcal{S}(N)$ of rapidly decaying smooth functions on N.

We choose a system $\{X_j\}_{j=1}^{d_1}$ of left-invariant vector fields on N which form a basis of $\mathfrak{n}_1$, and let Δ_N as in (1.1) be the corresponding homogeneous sub-Laplacian on N. Since Δ_N with domain $C_c^\infty(N)$ is essentially self-adjoint on $L^2(N)$, a Borel functional calculus for Δ_N is defined via the spectral theorem. For any bounded Borel function $F : \mathbb{R}_+ \to \mathbb{C}$, the L^2-bounded operator $F(\Delta_N)$ is left-invariant, so, by the Schwartz kernel theorem, there exists a convolution kernel $K_{F(\Delta_N)} \in \mathcal{S}'(N)$ such that

$$F(\Delta_N)f = f * K_{F(\Delta_N)}, \qquad f \in \mathcal{S}(N).$$

As is well-known (see, e.g., [12, 57, 75] or Proposition 6.3 below), Δ_N satisfies the finite propagation speed property:

$$\operatorname{supp} K_{\cos(t\sqrt{\Delta})} \subseteq \overline{B_N}(0_N, |t|), \qquad t \in \mathbb{R}, \tag{2.1}$$

where $\overline{B_N}(z, r)$ is the closed ball of centre z and radius r relative to the Carnot–Carathéodory distance associated with Δ_N, and 0_N is the identity element of N. Furthermore, a Plancherel formula holds for Δ_N:

$$\|K_{F(\Delta)}\|^2_{L^2(N)} \simeq \int_0^\infty |F(\lambda)|^2 \, \lambda^{Q/2} \frac{\mathrm{d}\lambda}{\lambda} \tag{2.2}$$

(see, e.g., [8, 16, 39]). This corresponds to the unweighted version of (1.12), and holds without any assumptions on N.

Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{n}_1$ which makes $\{X_j\}_{j=1}^{d_1}$ an orthonormal basis. For all $\mu \in \mathfrak{n}_2^*$ we define the skewsymmetric linear operator $J_\mu : \mathfrak{n}_1 \to \mathfrak{n}_1$ by

$$\langle J_\mu x, y \rangle = \mu[x, y] \qquad \forall x, y \in \mathfrak{n}_1.$$

Let us choose linear coordinates on $\mathfrak{n}_1$ and $\mathfrak{n}_2$. So we shall write $z = (z', z'') \in N \simeq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, and also $\mu \in \mathfrak{n}_2^* \simeq \mathbb{R}^{d_2}$. Observe that $|J_\mu z'|^2$ is a polynomial in (z', μ), separately homogeneous of order 2 in both z' and μ; the norm on $\mathfrak{n}_1$ appearing in the expression $|J_\mu z'|^2$ is the one induced by the inner product $\langle \cdot, \cdot \rangle$.

Let $Z'' = -i\nabla_{z''} = (-i\partial_{z''_1}, \dots, -i\partial_{z''_{d_2}})$ be the vector of second-layer derivatives in N, and $P = (z'_1, \dots, z'_{d_1})$ be the vector of multiplication operators by z'_j. The components of P and Z'' together form a strongly commuting system of self-adjoint operators on $L^2(N)$ (cf. [48, pp. 1229–1230]). Since $|J_\mu z'|^2$ is a polynomial, we consider $|J_{Z''} z'|^2 := |J_{Z''} P|^2$ as a differential operator on N of order 2 with polynomial coefficients.

Lemma 2.1 *Let $f \in \mathcal{S}(N)$ be such that $f * K_{e^{-t\Delta_N}} = K_{e^{-t\Delta_N}} * f$ for all $t > 0$. Then, for all $k \in \mathbb{N}$,*

$$\| |J_{Z''} P|^{2k} f \|_{L^2(N)} \lesssim_k \|\Delta_N^k f\|_{L^2(N)}.$$

Proof For every smooth differential operator D on N we define D° by

$$(Df)^* = D^\circ f^*,$$

where $f^*(z) = \overline{f(z^{-1})}$ denotes the involution on N. If D is left-invariant, then D° is right-invariant, and vice versa. By [48, Lemma 3.4] we have

$$|J_{Z''} z'|^2 = \sum_{j=1}^{d_1} (X_j + X_j^\circ)^2. \tag{2.3}$$

As the above expression contains a mixture of left- and right-invariant differential operators, it is convenient to analyse it by means of a lifting to the direct product $N \times N$; this idea has already been exploited in [33, 48].

The Lie algebra of the direct product $N \times N$ is canonically identified with $\mathfrak{n} \oplus \mathfrak{n}$. In particular, we can think of $N \times N$ as a 2-step group too, with first and second layers given by $\mathfrak{n}_1 \oplus \mathfrak{n}_1$ and $\mathfrak{n}_2 \oplus \mathfrak{n}_2$. Let ξ be the unitary representation of $N \times N$ on $L^2(N)$ defined by

$$\xi(x, y) f(z) = f(y^{-1} z x).$$

for all $x, y, z \in N$ and $f : N \to \mathbb{C}$. If D is a left-invariant differential operator on $L^2(N \times N)$, then $\mathrm{d}\xi(D)$ is a smooth differential operator on N. Moreover,

$$\mathrm{d}\xi(D^\bullet) = \mathrm{d}\xi(D)^\circ$$

(see [48, p. 1228]), where $D \mapsto D^\bullet$ is the correspondence on the algebra of left-invariant differential operators on $N \times N$ defined as the unique conjugate-linear automorphism extending the Lie algebra automorphism $(X, Y) \mapsto (Y, X)$ of $\mathfrak{n} \oplus \mathfrak{n}$.

Now, notice that $X_j = \mathrm{d}\xi(\tilde{X}_j)$, $X_j^\circ = \mathrm{d}\xi(\tilde{X}_j^\bullet)$, where $\tilde{X}_j = X_j \otimes \mathrm{id}$ is the lifting of X_j to $N \times N$, and $\tilde{X}_j$, $\tilde{X}_j^\bullet$ are left-invariant 1-homogeneous vector fields on the 2-step group $N \times N$. In particular, by (2.3), for all $k \in \mathbb{N}$,

$$|J_{Z''} P|^{2k} = \mathrm{d}\xi(D_k), \tag{2.4}$$

where

$$D_k = \left(\sum_{j=1}^{d_1} (\tilde{X}_j + \tilde{X}_j^\bullet)^2 \right)^k$$

is a left-invariant $2k$-homogeneous differential operator on $N \times N$.

Define $A = \frac{1}{2}(\Delta_N + \Delta_N^\circ)$. Notice that $A = \mathrm{d}\xi(\tilde{A})$, where

$$\tilde{A} = \frac{1}{2}(\tilde{\Delta}_N + \tilde{\Delta}_N^\bullet) = \frac{1}{2}(\Delta_N \otimes \mathrm{id} + \mathrm{id} \otimes \Delta_N)$$

is a 2-homogeneous sub-Laplacian on the 2-step group $N \times N$. In particular,

$$A^k = \mathrm{d}\xi(\tilde{A}^k), \tag{2.5}$$

where $\tilde{A}^k$ is a $2k$-homogeneous positive Rockland operator on $N \times N$.

Hence, by the Helffer–Nourrigat theorem [34] (see also [79, Theorem 2.5] and [33, p. 32]), from (2.4) and (2.5) we deduce, for any $k \in \mathbb{N}$, the estimate

$$\| |J_{Z''} P|^{2k} f \|_{L^2(N)} \lesssim_k \| A^k f \|_{L^2(N)}, \qquad f \in \mathcal{S}(N).$$

Finally, by [48, Lemma 3.2], for any $f \in \mathcal{S}(N)$ commuting with heat kernels associated with Δ_N, we have $A^k f = \Delta_N^k f$. □

Notice that Δ_N and the components of Z'' are also a system of strongly commuting self-adjoint differential operators on $L^2(N)$. Actually, they form a *homogeneous weighted subcoercive system*, or a *Rockland system* (see [47, Theorem 5.2] and [4, Theorem 3.5]). In particular, by [47, Corollary 3.3], Δ_N and Z'' admit a joint functional calculus on $L^2(N)$ and, for any bounded Borel function $H : \mathbb{R} \times \mathbb{R}^{d_2} \to \mathbb{C}$, the $L^2(N)$-bounded operator $H(\Delta_N, Z'')$ is left-invariant. Moreover, there exists a Plancherel measure σ for this joint functional calculus (see [47, Theorem 3.10]), that is, a regular Borel measure on $\mathbb{R} \times \mathbb{R}^{d_2}$, supported on the joint spectrum (so in particular $\operatorname{supp} \sigma \subseteq \mathbb{R}_+ \times \mathbb{R}^{d_2}$), such that

$$\| K_{H(\Delta_N, Z'')} \|_{L^2(N)} = \int_{\mathbb{R} \times \mathbb{R}^{d_2}} |H(\lambda, \mu)|^2 \, \mathrm{d}\sigma(\lambda, \mu). \tag{2.6}$$

As Δ_N and the components of Z'' are 2-homogeneous, [47, Proposition 5.1] yields that, for all $t > 0$,

$$K_{H(t^2 \Delta_N, t^2 Z'')}(z', z'') = t^{-Q} K_{H(\Delta_N, Z'')}(t^{-1} z', t^{-2} z'').$$

Consequently,

$$\int_{\mathbb{R}\times\mathbb{R}^{d_2}} H(t^2\lambda, t^2\mu)\,\mathrm{d}\sigma(\lambda,\mu) = t^{-Q}\int_{\mathbb{R}\times\mathbb{R}^{d_2}} H(\lambda,\mu)\,\mathrm{d}\sigma(\lambda,\mu). \tag{2.7}$$

Remark 2.2 If $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^{d_2})$ is nonnegative, then the push-forward via the projection $(\lambda,\mu) \mapsto \lambda$ of the measure $\varphi(\mu)\,\mathrm{d}\sigma(\lambda,\mu)$ is a regular Borel measure $\sigma_\varphi(\lambda)$ on $\mathbb{R}$, supported on $\mathbb{R}_+$ (see [47, Lemma 3.7]). Moreover, if φ is homogeneous of degree $\omega \in \mathbb{R}$, then for any $F \in C_c(\mathbb{R})$ and $t > 0$ we have

$$\begin{aligned}\int_0^\infty F(t^2\lambda)\,\mathrm{d}\sigma_\varphi(\lambda) &= \int_{\mathbb{R}\times\mathbb{R}^{d_2}} F(t^2\lambda)\varphi(\mu)\,\mathrm{d}\sigma(\lambda,\mu)\\ &= t^{-Q}\int_{\mathbb{R}\times\mathbb{R}^{d_2}} F(\lambda)\varphi(t^{-2}\mu)\,\mathrm{d}\sigma(\lambda,\mu)\\ &= t^{-Q-2\omega}\int_0^\infty F(\lambda)\,\mathrm{d}\sigma_\varphi(\lambda),\end{aligned}$$

where the last equality follows from (2.7). Thus, the measure σ_φ is homogeneous and there exists a positive constant C_φ such that

$$\mathrm{d}\sigma_\varphi(\lambda) = C_\varphi \lambda^{(Q+2\omega)/2}\frac{\mathrm{d}\lambda}{\lambda}.$$

In the case $\varphi \equiv 1$ we have $\omega = 0$, showing that (2.6) is consistent with (2.2).

We remark that, by a multivariate extension of Hulanicki's theorem [38] (see, e.g., [46, Proposition 4.2.1]), if $H \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{d_2})$, then $K_{H(\Delta_N, Z'')} \in \mathcal{S}(N)$; so any partial (Euclidean) Fourier transform of $K_{H(\Delta_N,Z'')}$ is also a Schwartz function.

Lemma 2.3 *If* $\gamma \geq 0$ *and* $H \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^{d_2})$*, then*

$$\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}^{d_2}} ||J_\mu z'|^{2\gamma}\hat{K}_{H(\Delta_N,Z'')}(z',\mu)|^2\,\mathrm{d}z'\,\mathrm{d}\mu \lesssim_\gamma \int_{\mathbb{R}\times\mathbb{R}^{d_2}} |H(\lambda,\mu)|^2|\lambda|^\gamma\,\mathrm{d}\sigma(\lambda,\mu),$$

where $\hat{K}_{H(\Delta_N,Z'')}(z',\mu)$ *is the partial Fourier transform in* z'' *of* $K_{H(\Delta_N,Z'')}(z',z'')$.

Proof Clearly, $K_{H(\Delta_N,Z'')}$ commutes with the heat kernels $K_{e^{-t\Delta_N}}$, as the operators $H(\Delta_N, Z'')$ and $e^{-t\Delta_N}$ are in the joint functional calculus of Δ_N and Z''. Thus, assuming firstly that $\gamma \in \mathbb{N}$, the Plancherel theorem in $\mathbb{R}^{d_2}$ and Lemma 2.1 give

$$\begin{aligned}\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}^{d_2}} ||J_\mu z'|^{2\gamma}\hat{K}_{H(\Delta_N,Z'')}(z',\mu)|^2\,\mathrm{d}z'\,\mathrm{d}\mu &\simeq \int_N ||J_{Z''}z'|^{2\gamma}K_{H(\Delta_N,Z'')}(z)|^2\,\mathrm{d}z\\ &\lesssim_\gamma \int_N |\Delta_N^\gamma K_{H(\Delta_N,Z'')}(z)|^2\,\mathrm{d}z\\ &= \int_{\mathbb{R}\times\mathbb{R}^{d_2}} |H(\lambda,\mu)|^2|\lambda|^{2\gamma}\,\mathrm{d}\sigma(\lambda,\mu).\end{aligned}$$

By interpolation [77] we extend the obtained inequality to non-integer γ. □

From now on we assume that N satisfies the assumption of Theorem 1.2, that is, N is the direct product

$$N = N^{(0)} \times N^{(1)} \times \cdots \times N^{(\ell)} \tag{2.8}$$

of an abelian group $N^{(0)}$ and finitely many Métivier groups $N^{(j)}$, $j = 1, \dots, \ell$, for some $\ell \in \mathbb{N}\setminus\{0\}$. The factor $N^{(j)}$ is a 2-step group for each $j = 1, \dots, \ell$, so $\mathfrak{n}^{(j)} = \mathfrak{n}_1^{(j)} \oplus \mathfrak{n}_2^{(j)}$, and we set $d_i^{(j)} = \dim \mathfrak{n}_i^{(j)}$ for $i = 1, 2$. Since $N^{(0)}$ is abelian, it is a 1-step group, so $\mathfrak{n}^{(0)} = \mathfrak{n}_1^{(0)}$, and we just set $d^{(0)} = \dim \mathfrak{n}^{(0)}$. We emphasise that with a proper interpretation one can allow $d^{(0)} = 0$ and consider $N = N^{(1)} \times \cdots \times N^{(\ell)}$.

As N is a direct product, the first and second layers of $\mathfrak{n}$ are given by

$$\mathfrak{n}_1 = \mathfrak{n}^{(0)} \oplus \mathfrak{n}_1^{(1)} \oplus \cdots \oplus \mathfrak{n}_1^{(\ell)}, \qquad \mathfrak{n}_2 = \mathfrak{n}_2^{(1)} \oplus \cdots \oplus \mathfrak{n}_2^{(\ell)}. \tag{2.9}$$

In particular, if we define $\vec{d}_i = (d_i^{(1)}, \dots, d_i^{(\ell)})$ and $|\vec{d}_i| = \sum_{j=1}^{\ell} d_i^{(j)}$ for $i = 1, 2$, then the dimensions of the two layers of $\mathfrak{n}$ are $d_1 = d^{(0)} + |\vec{d}_1|$ and $d_2 = |\vec{d}_2|$. Consequently, the homogeneous and topological dimensions of N are $Q = d^{(0)} + |\vec{d}_1| + 2|\vec{d}_2|$ and $d = d^{(0)} + |\vec{d}_1| + |\vec{d}_2|$.

We choose linear coordinates for all the layers $\mathfrak{n}_i^{(j)}$, so we can identify $N^{(0)} \simeq \mathbb{R}^{d^{(0)}}$ and $N^{(j)} \simeq \mathbb{R}^{d_1^{(j)}} \times \mathbb{R}^{d_2^{(j)}}$ as manifolds, for $1 \leq j \leq \ell$, and correspondingly $N \simeq \mathbb{R}^{d^{(0)}} \times \mathbb{R}^{\vec{d}_1} \times \mathbb{R}^{\vec{d}_2}$, where $\mathbb{R}^{\vec{d}_i} := \mathbb{R}^{d_i^{(1)}} \times \cdots \times \mathbb{R}^{d_i^{(\ell)}}$.

We emphasise that the vector fields X_j on N that define the sub-Laplacian Δ_N need not be compatible with the direct product decomposition. In other words, we do not require the sub-Laplacian Δ_N to be the sum of sub-Laplacians on the factors N_j, nor do we require the decomposition of $\mathfrak{n}_1$ in (2.9) to be an orthogonal decomposition with respect to the inner product $\langle \cdot, \cdot \rangle$ associated with Δ_N.

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^\ell$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_\ell$ for its length. We also use the symbols $\prec, \preccurlyeq, \succ, \succcurlyeq$ to denote componentwise inequalities between multi-indices: for instance we write $\alpha \preccurlyeq \beta$ for $\alpha, \beta \in \mathbb{R}^\ell$ whenever $\alpha_j \leq \beta_j$ for all $j = 1, \dots, \ell$. Also, we denote $\vec{0} = (0, \dots, 0) \in \mathbb{R}^\ell$. Unless stated otherwise, all the multi-indices we use have ℓ components.

The above structural assumption on N yields the following crucial estimate (cf., e.g., [48, Proposition 3.9]).

Lemma 2.4 *For all $z' = (z'_0, z'_1 \dots, z'_\ell) \in \mathfrak{n}_1$, $\mu = (\mu_1, \dots, \mu_\ell) \in \mathfrak{n}_2$,*

$$|J_\mu z'| \gtrsim \sum_{j=1}^{\ell} |\mu_j||z'_j|. \tag{2.10}$$

In particular, for any $\alpha \succeq \vec{0}$,

$$|J_\mu z'|^{|\alpha|} \gtrsim_\alpha \prod_{j=1}^{\ell} (|\mu_j||z'_j|)^{\alpha_j}. \tag{2.11}$$

Moreover,

$$\vec{d}_2 \prec \vec{d}_1. \tag{2.12}$$

Proof As N is a direct product,

$$\langle J_\mu x, y \rangle = \mu[x, y] = \sum_{j=1}^{\ell} \mu_j [x_j, y_j]$$

for all $x = (x_0, x_1, \dots, x_\ell), y = (y_0, y_1, \dots, y_\ell) \in \mathfrak{n}_1$, $\mu = (\mu_1, \dots, \mu_\ell) \in \mathfrak{n}_2^*$. Consequently,

$$|J_\mu x| \simeq |\mu[x, \cdot]| \simeq \sum_{j=1}^{\ell} |\mu_j[x_j, \cdot]| \gtrsim \sum_{j=1}^{\ell} |\mu_j||x_j|, \tag{2.13}$$

where $\mu[x, \cdot]$ and $\mu_j[x_j, \cdot]$ are thought of as elements of $\mathfrak{n}_1^*$ and $(\mathfrak{n}_1^{(j)})^*$ respectively, and in the last inequality of (2.13) we used that each of the N_j is Métivier. This proves (2.10), and (2.11) follows immediately.

Finally, the (well-known) inequality $d_2^{(j)} < d_1^{(j)}$ between the dimensions of the layers of a Métivier group is due to the fact that, for any nonzero $x \in \mathfrak{n}_1^{(j)}$, the linear map $[x, \cdot] : \mathfrak{n}_1^{(j)} \to \mathfrak{n}_2^{(j)}$ is surjective (due to the Métivier condition), but not injective, as its kernel contains x. □

By combining Lemmas 2.3 and 2.4, we can finally obtain the desired weighted Plancherel estimate on N.

Proposition 2.5 *For* $\vec{0} \preccurlyeq \alpha \prec \vec{d_2}$ *and all* $F \in \mathcal{S}(\mathbb{R})$,

$$\int_N |K_{F(\Delta_N)}(z)|^2 \prod_{j=1}^{\ell} |z_j'|^{\alpha_j} \,\mathrm{d}z \lesssim_\alpha \int_0^\infty |F(\lambda)|^2 \lambda^{(Q-|\alpha|)/2} \frac{\mathrm{d}\lambda}{\lambda}.$$

Proof For any $H \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^{\vec{d_2}})$ and $\alpha \succcurlyeq \vec{0}$, Lemma 2.3 and (2.11) yield

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^{\vec{d_2}}} |H(\lambda, \mu)|^2 |\lambda|^{|\alpha|/2} \,\mathrm{d}\sigma(\lambda, \mu) \\
&\gtrsim_\alpha \int_{\mathbb{R}^{d^{(0)}}\times\mathbb{R}^{\vec{d_1}}} \int_{\mathbb{R}^{\vec{d_2}}} |\hat{K}_{H(\Delta_N, Z'')}(z', \mu)|^2 \prod_{j=1}^{\ell} (|\mu_j||z_j'|)^{\alpha_j} \,\mathrm{d}\mu \,\mathrm{d}z' \\
&= \int_N |Z_1''|^{\alpha_1} \dots |Z_\ell''|^{\alpha_\ell} |K_{H(\Delta_N, Z'')}(z)|^2 \prod_{j=1}^{\ell} |z_j'|^{\alpha_j} \,\mathrm{d}z \\
&= \int_N |K_{|Z_1''|^{\alpha_1}\dots|Z_\ell''|^{\alpha_\ell} H(\Delta_N, Z'')}(z)|^2 \prod_{j=1}^{\ell} |z_j'|^{\alpha_j} \,\mathrm{d}z.
\end{aligned}$$

Let $F \in \mathcal{S}(\mathbb{R})$. We apply the above for $H(\lambda, \mu) = F(\lambda)\chi(\mu) \prod_{j=1}^{\ell} |\mu_j|^{-\alpha_j/2}$ where $\chi(\mu) = \prod_{j=1}^{\ell} \chi_j(\mu_j)$ for some $\chi_j \in C_c^\infty(\mathbb{R}^{d_2^{(j)}} \setminus \{0\})$, obtaining

$$\begin{aligned}
&\int_N |K_{F(\Delta_N)\chi(Z'')}(z)|^2 \prod_{j=1}^{\ell} |z_j'|^{\alpha_j} \,\mathrm{d}z \\
&\lesssim_\alpha \|\chi\|_{L^\infty}^2 \int_{\mathbb{R}\times\mathbb{R}^{\vec{d_2}}} |F(\lambda)|^2 |\lambda|^{|\alpha|/2} \prod_{j=1}^{\ell} |\mu_j|^{-\alpha_j} \,\mathrm{d}\sigma(\lambda, \mu).
\end{aligned} \tag{2.14}$$

Now we apply Remark 2.2 to $\varphi(\mu) = \prod_{j=1}^{\ell} |\mu_j|^{-\alpha_j}$ and denote the corresponding measure σ_φ by σ_α. We restrict $\alpha \prec \vec{d_2}$, so that φ is locally integrable. Since φ is homogeneous of degree $-|\alpha|$, we obtain $\sigma_\alpha(\lambda) = C_\alpha \lambda^{(Q-2|\alpha|)/2} \frac{\mathrm{d}\lambda}{\lambda}$.

Hence, (2.14) gives

$$\int_N |K_{F(\Delta_N)\chi(Z'')}(z)|^2 \prod_{j=1}^{\ell} |z'_j|^{\alpha_j} \,\mathrm{d}z \lesssim_\alpha \|\chi\|_{L^\infty}^2 \int_0^\infty |F(\lambda)|^2 \lambda^{(Q-|\alpha|)/2} \,\frac{\mathrm{d}\lambda}{\lambda}. \qquad (2.15)$$

Now we choose a sequence $\chi^{(n)} = \chi_1^{(n)} \otimes \cdots \otimes \chi_\ell^{(n)}$ of smooth cutoffs such that $0 \leq \chi^{(n)} \leq 1$ and χ_n converges monotonically to 1 on $\mathbb{R}^{\vec{d}_2} \setminus A$, where

$$A := (\{0\} \times \mathbb{R}^{d_2^{(2)}} \times \cdots \times \mathbb{R}^{d_2^{(\ell)}}) \cup \cdots \cup (\mathbb{R}^{d_2^{(1)}} \times \cdots \times \mathbb{R}^{d_2^{(\ell-1)}} \times \{0\}).$$

We claim that $\sigma(\mathbb{R}_+ \times A) = 0$. Indeed, for $b > a > 0$ we have

$$\sigma((a,b) \times A) = \|K_{\mathbb{1}_{(a,b)\times A}(\Delta_N, Z'')}\|_{L^2(N)} = \|\mathbb{1}_{(a,b)}(\Delta_N)\mathbb{1}_A(Z'')\|_{L^1(N)\to L^2(N)} = 0,$$

because $\mathbb{1}_A(Z'')$ vanishes: in the chosen coordinates on N, the operator $\mathbb{1}_A(Z'')$ is just the Euclidean Fourier multiplier operator on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with symbol $\mathbb{1}_{\mathbb{R}^{d_1} \times A}$, which vanishes almost everywhere. Hence, $F(\lambda)\chi^{(n)}(\mu) \to F(\lambda)$ in $L^2(\mathrm{d}\sigma)$, by the dominated convergence theorem. Thus $K_{F(\Delta_N)\chi_n^{(n)}(Z'')} \to K_{F(\Delta_N)}$ in $L^2(N)$ by the Plancherel formula (2.6), and, up to extracting a subsequence, also almost everywhere on N.

Finally, by Fatou's lemma and (2.15) we arrive at

$$\begin{aligned} \int_N |K_{F(\Delta_N)}(z)|^2 \prod_{j=1}^{\ell} |z'_j|^{\alpha_j} \,\mathrm{d}z &\leq \liminf_{n\to\infty} \int_N |K_{F(\Delta_N)\chi^{(n)}(Z'')}(z)|^2 \prod_{j=1}^{\ell} |z'_j|^{\alpha_j} \,\mathrm{d}z \\ &\lesssim_\alpha \int_0^\infty |F(\lambda)|^2 \lambda^{(Q-|\alpha|)/2} \,\frac{\mathrm{d}\lambda}{\lambda}, \end{aligned}$$

as desired. □

2.2 The semidirect product extension

As in the introduction, we set $G = N \rtimes \mathbb{R}$, where $\mathbb{R}$ acts on N via automorphic dilations. The left-invariant vector fields X_j, $j = 1\ldots, d_1$, forming a basis of the first layer of N, and the standard basis $X_0 = \partial_u$ of the Lie algebra of $\mathbb{R}$, can be lifted, as in (1.2), to left-invariant vector fields $\{X_j^\sharp\}_{j=0}^{d_1}$ on G, which generate the Lie algebra of G. As in (1.3), let Δ be the corresponding left-invariant sub-Laplacian on G.

The group G and the sub-Laplacian Δ are among those considered in [54], to which we refer for a more extensive discussion. Here we just recall that, much as in the case of Δ_N discussed earlier, the operator Δ on the domain $C_c^\infty(G)$ is essentially self-adjoint on $L^2(G)$, and for any bounded Borel function $F : \mathbb{R}_+ \to \mathbb{C}$ we define $F(\Delta)$ via the spectral theorem. Since it is left-invariant, by the Schwartz kernel theorem there exists a (possibly distributional) convolution kernel $K_{F(\Delta)}$ such that

$$F(\Delta)f = f * K_{F(\Delta)}, \qquad f \in C_c^\infty(G).$$

Moreover, a Plancherel formula holds for Δ [54, Corollary 4.6], which says that

$$\|K_{F(\Delta)}\|_{L^2(G)}^2 \simeq \int_0^\infty |F(\lambda)|^2 \,\lambda^{[3/2,(Q+1)/2]} \,\frac{\mathrm{d}\lambda}{\lambda},$$

where we use the notation (1.7). Next, if $|z|_N$ denotes the Carnot–Carathéodory distance of $z \in N$ from the identity element 0_N, then, by [54, Proposition 2.7], the distance of $(z,u) \in G$

from the identity 0_G on G is given by

$$|(z,u)|_G = \operatorname{arccosh}\left(\cosh u + \frac{|z|_N^2}{2e^u}\right). \tag{2.16}$$

Furthermore, we have finite propagation speed for Δ:

$$\operatorname{supp} K_{\cos(t\sqrt{\Delta})} \subseteq \overline{B_G}(0_G, |t|), \qquad t \in \mathbb{R}, \tag{2.17}$$

where $\overline{B_G}(0_G, r)$ is the closed ball of radius r centred at the identity of G with respect to the Carnot–Carathéodory distance.

All the above results hold without any assumptions on the 2-step group N. We now restrict to the case where N is a product of abelian and Métivier groups as in (2.8). Recall the notation $z = (z', z'') \in N$, $z' = (z'_0, z'_1, \dots, z'_\ell)$ from Sect. 2.1. The following results are variations of [54, Proposition 2.9], where we also include the reciprocals of the weights appearing in Proposition 2.5.

Lemma 2.6 *Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable functions, and $\vec{0} \preccurlyeq \alpha \prec \vec{d}_1$. Then*

$$\begin{aligned}&\int_G f(|(z,u)|_G) g(|z|_N) \prod_{j=1}^{\ell} |z'_j|^{-\alpha_j} \,\mathrm{d}z\,\mathrm{d}u\\ &= C_{N,\alpha} \int_{\mathbb{R}} \int_0^\infty f(\operatorname{arccosh}(\cosh u + e^{-u} t/2)) g(\sqrt{t})\, t^{\frac{Q-|\alpha|}{2}} \,\frac{\mathrm{d}t}{t}\,\mathrm{d}u\end{aligned}$$

for a suitable constant $C_{N,\alpha} \in \mathring{\mathbb{R}}_+$.

Proof Let $S = \{z \in N : |z|_N = 1\}$. Then there exists a Borel measure τ on S such that

$$\int_N h(z)\,\mathrm{d}z = \int_0^\infty \int_S h(t\omega)\, t^Q \,\frac{\mathrm{d}t}{t}\,\mathrm{d}\tau(\omega) \tag{2.18}$$

(see, e.g., [19, Proposition 1.15]). Hence, by (2.16),

$$\begin{aligned}&\int_G f(|(z,u)|_G) g(|z|_N) \prod_{j=1}^{\ell} |z'_j|^{-\alpha_j} \,\mathrm{d}z\,\mathrm{d}u\\ &= C_N \int_{\mathbb{R}} \int_S \int_0^\infty f(\operatorname{arccosh}(\cosh u + e^{-u}|t|^2/2)) g(t) t^Q \prod_{j=1}^{\ell} (t|\omega'_j|)^{-\alpha_j} \,\frac{\mathrm{d}t}{t}\,\mathrm{d}\tau(\omega)\,\mathrm{d}u.\end{aligned} \tag{2.19}$$

Since $\alpha \prec \vec{d}_1$, the function $h(z) = \mathbb{1}_{[1,2]}(|z|_N) \prod_{j=1}^{\ell} |z'_j|^{-\alpha}$ is integrable on N, so from (2.18) we also deduce that $\int_S \prod_{j=1}^{\ell} |\omega'_j|^{-\alpha_j}\,\mathrm{d}\tau(\omega) < \infty$. A simple change of variables in the right-hand side of (2.19) then completes the proof. □

Corollary 2.7 *Let $\vec{0} \preccurlyeq \alpha \prec \vec{d}_1$. If $r \in (0, 1]$, then*

$$\int_{B_G(0_G, r)} \prod_{j=1}^{\ell} |z'_j|^{-\alpha_j} \,\mathrm{d}z\,\mathrm{d}u \lesssim_\alpha r^{Q-|\alpha|+1}.$$

If $r \geq 1$, *then*

$$\int_{B_G(0_G,r)} (1+|z|_N^{Q-|\alpha|})^{-1} \prod_{j=1}^{\ell} |z'_j|^{-\alpha_j} \,\mathrm{d}z\,\mathrm{d}u \lesssim_\alpha r^2.$$

Proof In the first case we apply Lemma 2.6 with $f = \mathbb{1}_{[0,r]}$ and $g \equiv 1$:

$$\begin{aligned}
&\int_{B_G(0_G,r)} \prod_{j=1}^{\ell} |z'_j|^{-\alpha_j} \,\mathrm{d}z\,\mathrm{d}u \\
&\simeq_\alpha \int_{\mathbb{R}} \int_0^\infty \mathbb{1}_{[0,r]}(\operatorname{arccosh}(\cosh u + e^{-u}t/2))\, t^{\frac{Q-|\alpha|}{2}} \frac{\mathrm{d}t}{t}\,\mathrm{d}u \\
&= \int_{-r}^{r} \int_0^{2e^u(\cosh r - \cosh u)} t^{\frac{Q-|\alpha|}{2}} \frac{\mathrm{d}t}{t}\,\mathrm{d}u \\
&\leq \int_{-r}^{r} \int_0^{8e^u(r+u)(r-u)} t^{\frac{Q-|\alpha|}{2}} \frac{\mathrm{d}t}{t}\,\mathrm{d}u \\
&\simeq_\alpha \int_0^{2r} ((2r-v)v)^{\frac{Q-|\alpha|}{2}} \,\mathrm{d}v \simeq_\alpha r^{Q-|\alpha|+1},
\end{aligned}$$

where we used that $|u| \leq r \leq 1$ in the domain of integration.

In the second case we again use Lemma 2.6, this time with $f(x) = \mathbb{1}_{[0,r]}(x)$ and $g(x) = (1+x^{Q-|\alpha|})^{-1}$. Thus,

$$\begin{aligned}
&\int_{B_G(0_G,r)} \frac{\prod_{j=1}^{\ell} |z'_j|^{-\alpha_j}}{1+|z|_N^{Q-|\alpha|}} \,\mathrm{d}z\,\mathrm{d}u \\
&\simeq_\alpha \int_{\mathbb{R}} \int_0^\infty \mathbb{1}_{[0,r]}(\operatorname{arccosh}(\cosh u + e^{-u}t/2)) \frac{t^{\frac{Q-|\alpha|}{2}}}{1+t^{\frac{Q-|\alpha|}{2}}} \frac{\mathrm{d}t}{t}\,\mathrm{d}u \\
&\leq \int_{-r}^{r} \int_0^{2e^{u+r}} \frac{t^{\frac{Q-|\alpha|}{2}}}{1+t^{\frac{Q-|\alpha|}{2}}} \frac{\mathrm{d}t}{t}\,\mathrm{d}u \\
&\leq \int_{-r}^{r} \int_0^{1} t^{\frac{Q-|\alpha|}{2}} \frac{\mathrm{d}t}{t}\,\mathrm{d}u + \int_{-r}^{r} \int_1^{2e^{u+r}} \frac{\mathrm{d}t}{t}\,\mathrm{d}u \\
&\simeq_\alpha \int_{-r}^{r} (1+u+r)\,\mathrm{d}u \simeq r^2,
\end{aligned}$$

as $r \geq 1$. □

3 The Bessel–Kingman hypergroup

3.1 Bessel–Kingman hypergroup and Hankel transform

We gather here some known facts about the Hankel convolution and the Hankel transform. Further details can be found, e.g., in [23–25, 36, 78] and references therein.

Let $\nu \geq 1$ and denote $X_\nu = (\mathbb{R}_+, \mu_\nu)$, where $\mathrm{d}\mu_\nu(x) = x^{\nu-1}\,\mathrm{d}x$. The *Hankel convolution* of suitable functions $f, g : X_\nu \to \mathbb{C}$ is given by

$$f *_\nu g(x) = \int_0^\infty \tau_\nu^{[x]} f(y)\, g(y)\,\mathrm{d}\mu_\nu(y) = \int_0^\infty f(y)\, \tau_\nu^{[x]} g(y)\,\mathrm{d}\mu_\nu(y), \tag{3.1}$$

where $\tau_\nu^{[x]} f$ is the *Hankel translation* of f,

$$\begin{aligned}\tau_\nu^{[x]} f(y) &= \tau_\nu^{[y]} f(x) \\ &= \begin{cases} \mathrm{B}(\frac{\nu-1}{2}, \frac{1}{2})^{-1} \int_0^\pi f(\sqrt{x^2+y^2-2xy\cos\omega}) \sin^{\nu-2}\omega\,\mathrm{d}\omega & \text{if } \nu > 1, \\ \frac{1}{2}[f(x+y) + f(|x-y|)] & \text{if } \nu = 1, \end{cases}\end{aligned}$$

for all $x, y \in X_\nu$, and B is the Beta function. The Hankel translations are normalised so that if $f \equiv 1$, then $\tau_\nu^{[x]} f \equiv 1$. Moreover, the $\tau_\nu^{[x]}$ commute pairwise, are self-adjoint on $L^2(X_\nu)$ and contractions on $L^p(X_\nu)$, $p \in [1, \infty]$, for all $x \geq 0$.

If ν is an integer, then the Hankel convolution $*_\nu$ corresponds to the restriction of the Euclidean convolution $*$ on $\mathbb{R}^\nu$ to radial functions; more precisely, if $C_\nu = 2\pi^{\nu/2}/\Gamma(\nu/2)$ denotes the surface measure of the unit sphere in $\mathbb{R}^\nu$, then

$$f *_\nu g(|z|) = C_\nu^{-1} F * G(z), \tag{3.2}$$

where $F, G : \mathbb{R}^\nu \to \mathbb{C}$ are radial and $F(z) = f(|z|)$, $G(z) = g(|z|)$. Nonetheless, the convolution $*_\nu$ is well defined for non-integer ν as well.

For an arbitrary $\nu \geq 1$, the convolution $*_\nu$ is commutative, associative and satisfies Young's inequality, that is

$$\|f *_\nu g\|_{L^r(X_\nu)} \leq \|f\|_{L^p(X_\nu)} \|g\|_{L^q(X_\nu)}, \tag{3.3}$$

where $p, q, r \in [1, \infty]$ and $1 + 1/r = 1/p + 1/q$ [36, Theorem 2b].

The Hankel convolution can be extended to measures; as usual, here we identify $f \in L^1(X_\nu)$ with the measure $f\mathrm{d}\mu_\nu$. In particular, if δ_x denotes the Dirac delta measure at $x \in X_\nu$, then $\delta_x *_\nu \delta_y$ is the probability measure on X_ν given by

$$\delta_x *_\nu \delta_y(A) = \begin{cases} \mathrm{B}(\frac{\nu-1}{2}, \frac{1}{2})^{-1} \int_0^\pi \mathbb{1}_A(\sqrt{x^2+y^2-2xy\cos\omega}) \sin^{\nu-2}\omega\,\mathrm{d}\omega & \text{if } \nu > 1, \\ \frac{1}{2}(\delta_{x+y} + \delta_{|x-y|})(A) & \text{if } \nu = 1, \end{cases} \tag{3.4}$$

for all Borel subsets A of X_ν, and has support in $[|x-y|, x+y]$. Moreover, Hankel translations can be expressed as convolution operators:

$$\tau_\nu^{[x]} f = \delta_x *_\nu f.$$

Equipped with this convolution structure, X_ν is a *hypergroup* [3, 40], known as a *Bessel–Kingman hypergroup* [43]. It is a commutative hypergroup, whose unit element is δ_0, whereas the involution is the identity (i.e. X_ν is a hermitian hypergroup) and the Haar measure is μ_ν. The hypergroup X_ν is a particular case of a *Sturm–Liouville hypergroup* [85], and more specifically a *Chébli–Trimèche hypergroup* [1, 7], associated with the Bessel operator L_ν we discuss in Sect. 3.2.

The *(modified) Hankel transform* H_ν on X_ν is given by

$$H_\nu f(x) = \int_0^\infty f(y)\, j_x^\nu(y)\,\mathrm{d}\mu_\nu(y), \qquad x \geq 0, \qquad f \in L^1(X_\nu), \tag{3.5}$$

where

$$j_x^\nu(y) = j_y^\nu(x) := j^\nu(xy), \qquad j^\nu(t) := \kappa_\nu \frac{J_{\nu/2-1}(t)}{t^{\nu/2-1}}, \qquad \kappa_\nu := 2^{\nu/2-1}\Gamma(\nu/2) \tag{3.6}$$

and J_s denotes the Bessel function of the first kind and order s. We point out that j^ν is an even analytic function, satisfying

$$|j^\nu(t)| \le 1 \qquad \forall t \in \mathbb{R}, \tag{3.7}$$

as $\nu \ge 1$ [17, eq. (10.14.4)].

For integer ν the Hankel transform is related to the Fourier transform: indeed, if $\nu \in \mathbb{N}$, then, for radial functions $F : \mathbb{R}^\nu \to \mathbb{C}$, denoting $F(z) = f(|z|)$, we have

$$H_\nu f(|\xi|) = C_\nu^{-1} \mathcal{F}F(\xi),$$

where C_ν is as in (3.2) and $\mathcal{F}F$ stands for the Fourier transform of F in $\mathbb{R}^\nu$,

$$\mathcal{F}F(\xi) = \int_{\mathbb{R}^\nu} F(z)\, e^{-iz\cdot\xi}\, \mathrm{d}z.$$

We point out that many works in the literature use the order $\nu/2-1$ of the Bessel function in (3.5) in place of ν as a parameter for the corresponding Hankel transform and Bessel–Kingman hypergroup; as in [20], we prefer our choice of the parameter as it is immediately linked to the dimension of the related Euclidean space when ν is integer.

For an arbitrary $\nu \ge 1$, by (3.7) it follows that the Hankel transform maps $L^1(X_\nu)$ into $C \cap L^\infty(X_\nu)$. Moreover, the functions j_x^ν satisfy

$$\tau_\nu^{[y]} j_x^\nu(z) = j_x^\nu(y)\, j_x^\nu(z) \tag{3.8}$$

(cf. [83, Section 11.41]), whence one deduces that

$$H_\nu(\tau_\nu^{[z]} f) = j_z^\nu H_\nu f, \qquad H_\nu(j_z^\nu f) = \tau_\nu^{[z]} H_\nu f. \tag{3.9}$$

In other words, the Hankel transform intertwines Hankel translations with certain multiplication operators, which we can think of as analogues of modulations.

Let $\mathcal{S}_e(\mathbb{R}_+)$ be the space of restrictions of even Schwartz functions on $\mathbb{R}$ to $\mathbb{R}_+$. Then $\mathcal{S}_e(\mathbb{R}_+)$ is closed under Hankel translations and convolution, and the Hankel transform maps $\mathcal{S}_e(\mathbb{R}_+)$ onto itself (cf. [10, 76, 78]). Moreover, the Hankel transform extends to an isomorphism $H_\nu : L^2(X_\nu) \to L^2(X_\nu)$ satisfying

$$H_\nu^{-1} = \kappa_\nu^{-2} H_\nu \tag{3.10}$$

(cf. [83, Section 14.4]) and such that Parseval's theorem holds, namely

$$\int_0^\infty H_\nu f(x)\, \overline{H_\nu g(x)}\, \mathrm{d}\mu_\nu(x) = \kappa_\nu^2 \int_0^\infty f(y)\, \overline{g(y)}\, \mathrm{d}\mu_\nu(y)$$

for all $f, g \in L^2(X_\nu)$ (cf. [45]). In particular,

$$\|H_\nu f\|_{L^2(X_\nu)} = \kappa_\nu \|f\|_{L^2(X_\nu)}. \tag{3.11}$$

By interpolating this fact with the boundedness of $H_\nu : L^1(X_\nu) \to L^\infty(X_\nu)$ we get the Hausdorff–Young inequality

$$\|H_\nu f\|_{L^{p'}(X_\nu)} \lesssim_{\nu,p} \|f\|_{L^p(X_\nu)}, \qquad f \in L^p(X_\nu), \tag{3.12}$$

where $p \in [1, 2]$ and $1/p + 1/p' = 1$.

Observe that, by (3.8),

$$H_\nu(f *_\nu g) = H_\nu f \cdot H_\nu g \tag{3.13}$$

for $f, g \in L^1(X_\nu)$ [36, Theorem 2d]. In fact, by (3.3) and (3.12) this can be extended to the case where $f \in L^p(X_\nu)$ and $g \in L^q(X_\nu)$, provided $p, q \in [1, 2]$ satisfy $1/p + 1/q \in [3/2, 2]$.

3.2 The Bessel operator

Let $\mathring{X}_\nu = (\mathring{\mathbb{R}}_+, \mu_\nu)$ be the interior of X_ν. We now introduce the *Bessel operator* L_ν, that is the second order differential operator with smooth coefficients on $\mathring{X}_\nu$ given by

$$L_\nu = -\partial_x^2 - \frac{\nu - 1}{x}\partial_x = D_\nu^+ D_\nu,$$

where $D_\nu = \partial_x$ and D_ν^+ is its formal adjoint with respect to μ_ν. Notice that, although D_ν does not depend on ν, the adjoint does. If $\nu \in \mathbb{N}$, then L_ν is the radial part of the standard Laplacian in $\mathbb{R}^\nu$.

As the Bessel operator $L_\nu = D_\nu^+ D_\nu$ is a smooth divergence-form second-order differential operator on the manifold $\mathring{X}_\nu$, we can discuss its self-adjoint extensions on $L^2(\mathring{X}_\nu)$ with the language introduced in the Appendix, to which we refer for the definitions of Dirichlet and Neumann domains used here. On the other hand, as X_ν and $\mathring{X}_\nu$ differ by a null set, we shall identify $L^2(\mathring{X}_\nu) = L^2(X_\nu)$, and consider L_ν as an unbounded operator on $L^2(X_\nu)$.

Observe that L_ν is symmetric and positive on the domain $\mathcal{S}_e(\mathbb{R}_+)$, which L_ν maps into itself. Moreover, it can be checked [17, eq. (10.13.4)] that

$$L_\nu j_x^\nu = x^2 j_x^\nu.$$

Thus, for $f \in \mathcal{S}_e(\mathbb{R}_+)$,

$$H_\nu(L_\nu f)(x) = x^2 H_\nu f(x). \tag{3.14}$$

This easily allows us to define a self-adjoint extension of L_ν that is compatible with the Hankel transform.

Lemma 3.1 *The operator L_ν is essentially self-adjoint on $\mathcal{S}_e(\mathbb{R}_+)$, and the identity (3.14) holds for any f in the domain of the self-adjoint extension and almost all $x \in X_\nu$. Moreover, this self-adjoint extension is equal to the operator $D_\nu^+ D_\nu$ with Neumann domain.*

This result is certainly known to experts. We include some details of the proof as they will be useful in Sect. 4.2 below, when discussing the “lifting” of the Bessel operator to the semidirect product hypergroup.

Proof The formula (3.14) shows that H_ν intertwines L_ν on the domain $\mathcal{S}_e(\mathbb{R}_+)$ with the multiplication operator by $M(x) = x^2$. The latter is self-adjoint on the natural domain

$$\{f \in L^2(X_\nu) : Mf \in L^2(X_\nu)\},$$

and $\mathcal{S}_e(\mathbb{R}_+)$ is clearly dense in this domain with respect to the graph norm. By the invariance of $\mathcal{S}_e(\mathbb{R}_+)$ with respect to H_ν, we conclude that L_ν is essentially self-adjoint on $\mathcal{S}_e(\mathbb{R}_+)$, and that the intertwining relation (3.14) remains true on the domain of the self-adjoint extension.

In light of the definition (6.4) of the Neumann domain $\mathfrak{D}_{\mathrm{Neu}}(D_\nu^+ D_\nu)$, in order to justify the remaining part of the statement, it suffices to verify that, if $f \in \mathcal{S}_e(\mathbb{R}_+)$, then $D_\nu f \in$

$\mathfrak{D}_{\min}(D_\nu^+)$. As $D_\nu f = f' \in \mathcal{S}_o(\mathbb{R}_+)$, the space of restrictions of odd Schwartz functions on $\mathbb{R}$ to $\mathbb{R}_+$, it is enough to show that any element of $g \in \mathcal{S}_o(\mathbb{R}_+)$ can be approximated in the graph norm of D_ν^+ by a sequence of functions in $C_c^\infty(\mathring{\mathbb{R}}_+)$.

Let $g \in \mathcal{S}_o(\mathbb{R}_+)$. Then $D_\nu^+ g(x) = -g'(x) - (\nu - 1)g(x)/x$, and both $g'(x)$ and $g(x)/x$ are in $\mathcal{S}_e(\mathbb{R}_+)$. Take smooth functions $\psi, \phi : \mathbb{R}_+ \to [0, 1]$ such that $\psi(x) = 1$ for $x \geq 2$, $\psi(x) = 0$ for $x \leq 1$, $\phi(x) = 1$ for $x \leq 2$, and $\phi(x) = 0$ for $x \geq 4$. We set $\psi_n(x) = \psi(nx)$, $\phi_n(x) = \phi(x/n)$, and define the approximating sequence $g_n = g\psi_n\phi_n \in C_c^\infty(\mathring{\mathbb{R}}_+)$.

Clearly, $g_n \to g$ and $g_n(x)/x \to g(x)/x$ in $L^2(X_\nu)$, by dominated convergence. Moreover,

$$g_n'(x) = \psi(nx)\phi(x/n)g'(x) + [nx\psi'(nx)]\phi(x/n)g(x)/x + \psi(nx)\phi'(x/n)g(x)/n.$$

Again, the first summand tends to g' in $L^2(X_\nu)$ by dominated convergence, while the second and third summands tend to 0; for the second summand, we use that $g(x)/x$ is in $L^2(X_\nu)$, while the multiplying factor is in $L^\infty(X_\nu)$ uniformly in n and vanishes outside $[1/n, 2/n]$. Thus $g_n' \to g'$ in $L^2(X_\nu)$, which combined with the previous observations gives that $D_\nu^+ g_n \to D_\nu^+ g$ in $L^2(X_\nu)$ and $g_n \to g$ in $\mathfrak{D}_{\max}(D_\nu^+)$. □

Remark 3.2 Lemma 3.1 remains true if $\mathcal{S}_e(\mathbb{R}_+)$ is replaced by the space $\mathcal{D}_e(\mathbb{R}_+)$ of the restrictions to $\mathbb{R}_+$ of even, smooth, and compactly supported functions on $\mathbb{R}$; indeed, by using appropriate cutoffs, $\mathcal{D}_e(\mathbb{R}_+)$ is easily seen to be dense in $\mathcal{S}_e(\mathbb{R}_+)$ in the Fréchet structure of the latter, so also in the graph norm of L_ν. On the other hand, on the domain $C_c^\infty(\mathring{\mathbb{R}}_+)$ the operator L_ν is essentially self-adjoint if and only if $\nu \geq 4$, see [72, p. 161]; mind that, in contrast to those of $\mathcal{D}_e(\mathbb{R}_+)$, elements of $C_c^\infty(\mathring{\mathbb{R}}_+)$ must vanish near 0. See also the discussion in [26, Section 2].

From now on, we write L_ν for the self-adjoint extension of the Bessel operator discussed in Lemma 3.1. From (3.9) and (3.14) it follows immediately that L_ν commutes with Hankel translations:

$$\tau_\nu^{[y]} L_\nu f = L_\nu \tau_\nu^{[y]} f \tag{3.15}$$

for any f in the domain of L_ν, i.e., L_ν is translation-invariant on X_ν.

As L_ν is self-adjoint and nonnegative, by the spectral theorem we can define, for any bounded Borel function $F : \mathbb{R}_+ \to \mathbb{C}$, the bounded operator $F(L_\nu)$ on $L^2(X_\nu)$, which, in light of (3.14), satisfies

$$H_\nu(F(L_\nu)g)(x) = F(x^2)H_\nu g(x)$$

for all $g \in L^2(X_\nu)$. Again, from (3.9) we deduce that

$$\tau_\nu^{[y]} F(L_\nu) = F(L_\nu)\tau_\nu^{[y]}, \tag{3.16}$$

so $F(L_\nu)$ is translation-invariant on X_ν.

If additionally $x \mapsto F(x^2)$ is in $L^2(X_\nu)$, equivalently $F \in L^2(\mathbb{R}_+, \lambda^{\nu/2} \frac{d\lambda}{\lambda})$, then, by (3.13), $F(L_\nu)$ is a $*_\nu$-convolution operator,

$$F(L_\nu)g = g *_\nu K_{F(L_\nu)}, \qquad K_{F(L_\nu)} = H_\nu^{-1}(x \mapsto F(x^2)). \tag{3.17}$$

Notice that (3.11) implies a Plancherel formula for these kernels:

$$\|K_{F(L_\nu)}\|^2_{L^2(X_\nu)} = \frac{1}{2\kappa_\nu^2} \int_0^\infty |F(\lambda)|^2 \lambda^{\nu/2} \frac{d\lambda}{\lambda}. \tag{3.18}$$

Now we consider the heat semigroup $\{e^{-tL_\nu}\}_{t>0}$ associated with L_ν. In the following statement we record a few basic facts about it.

Lemma 3.3 *The heat semigroup associated with L_ν is a family of $*_\nu$-convolution operators, where the corresponding kernels are given by*

$$K_{e^{-tL_\nu}}(x) = \frac{2}{\Gamma(\nu/2)(4t)^{\nu/2}} \exp\left(-\frac{x^2}{4t}\right). \tag{3.19}$$

Moreover, $\|K_{e^{-tL_\nu}}\|_{L^1(X_\nu)} = 1$ *for all* $t > 0$*. Consequently, the operators* $e^{-t\Delta_\nu}$ *are contractions on* $L^p(X_\nu)$*,* $p \in [1, \infty]$*, for all* $t > 0$*.*

Proof The formula (3.19) follows from (3.17), (3.10) and [17, eq. (10.22.51)]. Integration of this formula immediately gives $\|K_{e^{-tL_\nu}}\|_{L^1(X_\nu)} = 1$, and the L^p-contraction property then follows by Young's inequality (3.3) for $*_\nu$. □

We conclude with a few remarks on the dilation structure on X_ν and the related homogeneity of L_ν. Denote $\delta_\alpha g(x) = g(\alpha x)$ for $\alpha > 0$. As $\mathrm{d}\mu_\nu(x) = x^{\nu-1}\,\mathrm{d}x$, these dilations are multiples of isometries on $L^p(X_\nu)$ for any $p \in [1, \infty]$. It is straightforward to verify that the Bessel operator L_ν is 2-homogeneous, i.e.,

$$L_\nu(\delta_\alpha g) = \alpha^2 \delta_\alpha (L_\nu g), \qquad \alpha > 0, \tag{3.20}$$

for all g in the domain of L_ν. Thus, by spectral calculus, for any bounded Borel functions $F : \mathbb{R}_+ \to \mathbb{C}$,

$$F(L_\nu)\delta_\alpha = \delta_\alpha F(\alpha^2 L_\nu), \qquad \alpha > 0. \tag{3.21}$$

4 The semidirect product hypergroup

4.1 Extension of the Bessel–Kingman hypergroup

For any $u \in \mathbb{R}$, the dilation $\gamma_u : x \mapsto e^u x$ is an automorphism of the Bessel–Kingman hypergroup X_ν, in the sense that, by (3.4),

$$\delta_{\gamma_u(x)} *_\nu \delta_{\gamma_u}(y) = \gamma_u(\delta_x *_\nu \delta_y)$$

for all $x, y \in X_\nu$, and the right-hand side is the push-forward of the measure $\delta_x *_\nu \delta_y$ via γ_u. In addition, $\gamma_{u+u'} = \gamma_u \gamma_{u'}$ and γ_0 is the identity; in other words, $u \mapsto \gamma_u$ is an action of the group $\mathbb{R}$ by automorphisms on the hypergroup X_ν. Through this action we can define the *semidirect product* hypergroup $G_\nu := X_\nu \rtimes \mathbb{R}$ (see [35, Proposition 4.1 and Section 4.1, Example 4] and also [84, Definition 3.1]).

Convolution of measures on G_ν is defined by setting

$$\delta_{(x,u)} \diamond_\nu \delta_{(y,v)} = (\delta_x *_\nu \delta_{e^u y}) \otimes \delta_{u+v}$$

for all $(x, u), (y, v) \in G_\nu$. The semidirect product G_ν is a noncommutative hypergroup, with unit element $(0, 0)$ and involution given by $(x, u)^- = (e^{-u}x, -u)$. The measures $e^{-\nu u}\,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u$ and $\mathrm{d}\mu_\nu(x)\,\mathrm{d}u$ are the left and right Haar measures, and the modular function is

$$m(x, u) := e^{-\nu u}; \tag{4.1}$$

notice that the left and right Haar measures are one the push-forward of the other via the involution on G_ν.

Unless otherwise specified, we shall use Lebesgue spaces $L^p(G_\nu)$ on G_ν defined in terms of the right Haar measure. This must be kept in mind when comparing the results below with the literature on hypergroups, as most references define Lebesgue spaces with respect to the left Haar measure.

We define left and right translations of functions f on G_ν by

$$\ell_{(x,u)} f(y,v) = r_{(y,v)} f(x,u) = \int_{G_\nu} f \, \mathrm{d}(\delta_{(x,u)} \diamond_\nu \delta_{(y,v)}) \tag{4.2}$$

for all $(x,u), (y,v) \in G_\nu$; more explicitly,

$$\begin{aligned} \ell_{(x,u)} f(y,v) &= \tau_\nu^{[x]} f(e^u y, u+v), \\ r_{(x,u)} f(y,v) &= \tau_\nu^{[e^v x]} f(y, u+v), \end{aligned} \tag{4.3}$$

where the Hankel translation acts on the first variable of f. With these definitions, convolution of functions on G_ν is given by

$$\begin{aligned} f \diamond_\nu g(x,u) &= \int_{G_\nu} f((y,v)^-) r_{(x,u)} g(y,v) \, \mathrm{d}\mu_\nu(y) \, \mathrm{d}v \\ &= \int_{G_\nu} \ell_{(x,u)} f((y,v)^-) g(y,v) \, \mathrm{d}\mu_\nu(y) \, \mathrm{d}v. \end{aligned} \tag{4.4}$$

From the theory of hypergroups we deduce a number of properties of $\diamond_\nu$, such as Young's inequality.

Lemma 4.1 *For all* $p, q, r \in [1, \infty]$ *satisfying* $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

$$\|(f m^{-1/q'}) \diamond_\nu g\|_{L^r(G_\nu)} \leq \|f\|_{L^p(G_\nu)} \|g\|_{L^q(G_\nu)}, \tag{4.5}$$

where $1/q + 1/q' = 1$. *In particular, for any* $p \in [1, \infty]$,

$$\|f \diamond_\nu g\|_{L^p(G_\nu)} \leq \|f\|_{L^p(G_\nu)} \|g\|_{L^1(G_\nu)}. \tag{4.6}$$

Proof We follow the proof of the analogous result for groups (see, e.g, [44, Lemma 2.1]). From [40, Theorem 6.2E] we deduce

$$\|f \diamond_\nu g\|_{L^\infty(G_\nu)} \leq \|f^-\|_{L^{q'}(G_\nu)} \|g\|_{L^q(G_\nu)} = \|f m^{1/q'}\|_{L^{q'}(G_\nu)} \|g\|_{L^q(G_\nu)},$$

where $f^-(x,u) := f((x,u)^-)$; this inequality can be rewritten as

$$\|(f m^{-1/q'}) \diamond_\nu g\|_{L^\infty(G_\nu)} \leq \|f\|_{L^{q'}(G_\nu)} \|g\|_{L^q(G_\nu)}. \tag{4.7}$$

From [40, Theorems 5.3C and 6.2C] we also deduce

$$\|(f m^{1/q}) \diamond_\nu (g m^{1/q})\|_{L^q(G_\nu)} = \|(f \diamond_\nu g) m^{1/q}\|_{L^q(G_\nu)} \leq \|f m\|_{L^1(G_\nu)} \|g m^{1/q}\|_{L^q(G_\nu)},$$

which can be equivalently rewritten as

$$\|(f m^{-1/q'}) \diamond_\nu g\|_{L^q(G_\nu)} \leq \|f\|_{L^1(G_\nu)} \|g\|_{L^q(G_\nu)}. \tag{4.8}$$

Interpolation of (4.7) and (4.8) gives (4.5), and taking $q = 1$ gives (4.6). □

We also have the following mapping properties of translations on L^p spaces.

Lemma 4.2 *Let* $p \in [1, \infty]$ *and* $f \in L^p(G_\nu)$. *Then:*

(i) $\|r_{(x,u)} f\|_{L^p(G_\nu)} \leq \|f\|_{L^p(G_\nu)}$ *for any* $(x,u) \in G_\nu$;

(ii) $\|\ell_{(x,u)} f\|_{L^p(G_\nu)} \leq m(x,u)^{1/p} \|f\|_{L^p(G_\nu)}$ *for any* $(x,u) \in G_\nu$.

In addition, if $p < \infty$*, or if* $p = \infty$ *and* $f \in C_0(G_\nu)$*, then:*

(iii) $\|r_{(x,u)} f - r_{(\bar{x},\bar{u})} f\|_{L^p(G_\nu)} \to 0$ *as* $(x,u) \to (\bar{x},\bar{u})$ *in* G_ν*;*
(iv) $\|\ell_{(x,u)} f - \ell_{(\bar{x},\bar{u})} f\|_{L^p(G_\nu)} \to 0$ *as* $(x,u) \to (\bar{x},\bar{u})$ *in* G_ν.

Proof Notice that $\ell_{(x,u)} f = \delta_{(x,u)^-} \diamond_\nu f$ and $r_{(x,u)} f = f \diamond_\nu \delta_{(x,u)^-}$ [40, Section 4.2].

The bounds (i) and (ii) are consequences of the invariance properties of the right Haar measure under translations; specifically, (i) can be found in [40, Theorem 6.2B], while (ii) can be deduced from the former by using the fact that $\ell_{(x,u)} f = (r_{(x,u)^-} f^-)^-$ [40, Lemma 4.2K], and also $r_{(x,u)}(m^\alpha f) = m(x,u)^\alpha \, m^\alpha \, r_{(x,u)} f$ for any $\alpha \in \mathbb{C}$, due to the properties of the modular function [40, Theorem 5.3C].

Finally, the continuity property (iii) follows from [40, Lemmas 2.2B, 4.2F and 5.4H], and by involution we also deduce the case $p = \infty$ of (iv); the case $p < \infty$ of (iv) then follows by a density argument, as in the proof of [40, Lemma 5.4H]. □

4.2 Lifting of the Bessel operator

Write $\mathring{G}_\nu = (\mathring{\mathbb{R}}_+ \times \mathbb{R}, \mathrm{d}\mu_\nu(x)\,\mathrm{d}u)$ for the interior of G_ν. On the manifold $\mathring{G}_\nu$ we introduce the following differential operator with smooth coefficients:

$$\Delta_\nu = -\partial_u^2 + e^{2u} L_\nu = \nabla_\nu^+ \nabla_\nu, \tag{4.9}$$

where

$$\nabla_\nu = \begin{pmatrix} \partial_u \\ e^u \partial_x \end{pmatrix} \tag{4.10}$$

and ∇_ν^+ denotes the formal adjoint of ∇_ν on $\mathring{G}_\nu$. Much as in Sect. 3.2, the operator ∇_ν does not depend on ν, but its formal adjoint ∇_ν^+ does. As Δ_ν is a smooth divergence-form second-order differential operator on the manifold $\mathring{G}_\nu$, the discussion of the Appendix applies to it as well. Furthermore, since G_ν and $\mathring{G}_\nu$ differ by a set of measure zero, we can think of Δ_ν as an unbounded operator on $L^2(G_\nu)$. We define Δ_ν initially on $\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$, i.e. the space of finite linear combinations of tensor products $f \otimes g$, where $f \in \mathcal{S}_e(\mathbb{R}_+)$, $g \in C_c^\infty(\mathbb{R})$.

Lemma 4.3 *The operator* Δ_ν *is essentially self-adjoint on* $\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$*. Moreover, the self-adjoint extension is equal to* $\nabla_\nu^+ \nabla_\nu$ *with Neumann domain.*

Proof To justify the essential self-adjointness, we proceed in a way similar to [14, Section 3.2]. By [71, pp. 256–257] the operator Δ_ν with domain $\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$ is essentially self-adjoint if and only if $\mathrm{Im}((\Delta_\nu + i)|_{\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})})^\perp = \{0\}$. In other words, it is sufficient to show that, if $f \in L^2(G_\nu)$ satisfies

$$\int_{G_\nu} f(x,u)\, \overline{(\Delta_\nu + i)(\varphi_1 \otimes \varphi_2)(x,u)}\, \mathrm{d}\mu_\nu(x)\,\mathrm{d}u = 0$$

for all $\varphi_1 \in \mathcal{S}_e(\mathbb{R}_+)$ and $\varphi_2 \in C_c^\infty(\mathbb{R})$, then $f = 0$ almost everywhere.

Take f as above, and define $g(x,u) = H_\nu(f(\cdot,u))(x)$. By (3.14) and since the Hankel transform is, up to a constant, an isometry on $L^2(X_\nu)$, and also a bijection on $\mathcal{S}_e(\mathbb{R}_+)$, the function g is in $L^2(G_\nu)$ and satisfies

$$\int_{G_\nu} g(x,u)\, \overline{(-\partial_u^2 + e^{2u} x^2 + i)\widetilde{\varphi_1}(x)\varphi_2(u)}\, \mathrm{d}\mu_\nu(x)\,\mathrm{d}u = 0 \tag{4.11}$$

for all $\widetilde{\varphi_1} \in \mathcal{S}_e(\mathbb{R}_+)$, $\varphi_2 \in C_c^\infty(\mathbb{R})$ (here $\widetilde{\varphi_1} = H_\nu \varphi_1$). The task is to show that $g = 0$ almost everywhere.

Fix $\varphi_2 \in C_c^\infty(\mathbb{R})$, and define

$$h(x) = (1+x^2)^{-1} \int_{\mathbb{R}} g(x,u)\, \overline{(-\partial_u^2 + e^{2u}x^2 + i)\varphi_2(u)}\, \mathrm{d}u, \qquad x \in \mathbb{R}_+.$$

Since $g \in L^2(G_\nu)$ and $\varphi_2 \in C_c^\infty(\mathbb{R})$, it is easily checked that $h \in L^2(X_\nu)$. Moreover, by (4.11),

$$\int_0^\infty h(x)\, \overline{(i+x^2)\widetilde{\varphi_1}(x)}\, \mathrm{d}\mu_\nu(x) = 0, \qquad \widetilde{\varphi_1} \in \mathcal{S}_e(\mathbb{R}_+).$$

Since $(i+x^2)\widetilde{\varphi_1}(x) = H_\nu((i+L_\nu)\varphi_1)(x)$ and H_ν is a multiple of an isometry on $L^2(X_\nu)$, the latter condition can be rewritten as

$$\langle H_\nu h, (i+L_\nu)\varphi_1 \rangle_{L^2(X_\nu)} = 0, \qquad \varphi_1 \in \mathcal{S}_e(\mathbb{R}_+).$$

The essential self-adjointness of L_ν on $\mathcal{S}_e(\mathbb{R}_+)$, discussed in Lemma 3.1, then implies that $H_\nu h = 0$ almost everywhere. Consequently, $h = 0$ almost everywhere.

As φ_2 in the definition of h was arbitrary, this means that, for all $\varphi_2 \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} g(x,u)\, \overline{(i+T_x)\varphi_2(u)}\, \mathrm{d}u = 0 \tag{4.12}$$

for almost all $x \in \mathbb{R}_+$, where $T_x := -\partial_u^2 + x^2 e^{2u}$ is a Schrödinger operator with nonnegative smooth potential on $\mathbb{R}$. It is known that, as an operator on $L^2(\mathbb{R})$, the Schrödinger operator T_x is essentially self-adjoint on $C_c^\infty(\mathbb{R})$.

Let $M \subseteq C_c^\infty(\mathbb{R})$ be countable and dense, and set $V = \operatorname{span} M$. Thus, (4.12) holds for all $\varphi_2 \in V$ and $x \in \mathbb{R}_+ \setminus E$, where E is a Lebesgue null set independent of φ_2. So, for all $x \in \mathbb{R}_+ \setminus E$, we have $g(x,\cdot) \in \operatorname{Im}((i+T_x)|_V)^\perp$. Since each T_x is essentially self-adjoint on V, we conclude that, for all $x \in \mathbb{R}_+ \setminus E$, we have $g(x,\cdot) = 0$ almost everywhere on $\mathbb{R}$. This simply means that g vanishes almost everywhere on G_ν, thus proving the essential self-adjointness of Δ_ν.

It remains to prove that the self-adjoint extension coincides with the operator $\nabla_\nu^+ \nabla_\nu$ with Neumann domain, in the sense of (6.4). As ∇_ν maps $\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$ into $(\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})) \oplus (\mathcal{S}_o(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R}))$, it is sufficient to show that

$$(\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})) \oplus (\mathcal{S}_o(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})) \subseteq \mathfrak{D}_{\min}(\nabla_\nu^+).$$

On the other hand, as

$$\nabla_\nu^+ = \begin{pmatrix} -\partial_u & e^u D_\nu^+ \end{pmatrix} = \begin{pmatrix} -1 \otimes \partial_u & D_\nu^+ \otimes e^u \end{pmatrix},$$

and both ∂_u and e^u preserve $C_c^\infty(\mathbb{R})$, this reduces to showing that any element of $\mathcal{S}_e(\mathbb{R}_+)$ can be approximated in $L^2(X_\nu)$ by a sequence in $C_c^\infty(\mathring{\mathbb{R}}_+)$, and also that any element of $\mathcal{S}_o(\mathbb{R}_+)$ can be approximated by a sequence in $C_c^\infty(\mathring{\mathbb{R}}_+)$ in the graph norm of D_ν^+. The first fact is a triviality, while the second one is discussed in the proof of Lemma 3.1. □

From now on, when writing Δ_ν we always mean the self-adjoint extension. Notice that (3.15) and (3.20), together with (4.3) and (4.9), imply that Δ_ν is left-invariant, namely

$$\ell_{(y,v)} \Delta_\nu f = \Delta_\nu \ell_{(y,v)} f \qquad \forall (y,v) \in G_\nu, \tag{4.13}$$

for all f in the domain of Δ_ν; this is easily checked first for $f \in \mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$, and then arguing by density. By the spectral theorem, from (4.13) we deduce the left-invariance of any operator in the functional calculus:

$$\ell_{(y,v)} F(\Delta_\nu) = F(\Delta_\nu)\ell_{(y,v)} \qquad \forall (y, v) \in G_\nu, \tag{4.14}$$

for all bounded Borel functions $F : \mathbb{R}_+ \to \mathbb{C}$; this should be compared with (3.16) and (3.21). As we shall see, the $F(\Delta_\nu)$ are actually right $\diamond_\nu$-convolution operators.

4.3 Heat semigroup and Plancherel formula

We consider the heat semigroup $\{e^{-t\Delta_\nu}\}_{t>0}$ associated with Δ_ν. We shall make use of [22, Theorem 2.1], where the relation between heat kernels associated with certain operators, in our case L_ν, and their lifted versions, in our case Δ_ν, is established.

Proposition 4.4 *The heat semigroup associated with Δ_ν is given by a family of $\diamond_\nu$-convolution operators, namely*

$$e^{-t\Delta_\nu} f = f \diamond_\nu K_{e^{-t\Delta_\nu}} \tag{4.15}$$

for all $t > 0$, where the kernels are given by

$$K_{e^{-t\Delta_\nu}}(x, u) = \frac{2}{\Gamma(\nu/2)} \int_0^\infty \Psi_t(\xi)\, (2\xi e^u)^{-\nu/2} \exp\left(-\frac{\cosh u}{\xi} - \frac{x^2}{2\xi e^u}\right) \mathrm{d}\xi, \tag{4.16}$$

and

$$\Psi_t(\xi) = \frac{e^{\frac{\pi^2}{4t}}}{\xi^2\sqrt{4\pi^3 t}} \int_0^\infty \sinh(\theta) \sin\left(\frac{\pi\theta}{2t}\right) \exp\left(-\frac{\theta^2}{4t} - \frac{\cosh\theta}{\xi}\right) \mathrm{d}\theta, \qquad \xi > 0. \tag{4.17}$$

Moreover, for all $t > 0$, $K_{e^{-t\Delta_\nu}} \geq 0$ on G_ν, and $\|K_{e^{-t\Delta_\nu}}\|_{L^1(G_\nu)} = 1$. Thus, the $e^{-t\Delta_\nu}$ are positivity-preserving contractions on $L^p(G_\nu)$, $p \in [1, \infty]$, for all $t > 0$.

Proof To justify the formula (4.16), we want to apply the theory of [22]. Specifically, the operator Δ_ν here corresponds to the self-adjoint operator T of [22, Section 2], where L in [22] is taken to be our L_ν; with this choice, the domain of T in [22] clearly contains $\mathcal{S}_e(\mathbb{R}_+) \otimes C_c^\infty(\mathbb{R})$, and as the latter is a core for Δ_ν, by Lemma 4.3, the two operators are indeed the same. So, by [22, Theorem 2.1],

$$e^{-t\Delta_\nu} f(x, u) = \left(\int_{\mathbb{R}} \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh(u-v)}{\xi}\right) e^{-\xi e^{u+v} L_\nu/2} f_v \,\mathrm{d}\xi\, \mathrm{d}v\right)(x), \tag{4.18}$$

for all $f \in L^1 \cap L^2(G_\nu)$ and almost all $(x, u) \in G_\nu$, where we use the notation $f_v(\cdot) = f(\cdot, v)$. Notice that, for any $t > 0$, the function Ψ_t satisfies the bound

$$|\Psi_t(\xi)| \lesssim_t \xi^{-2} \qquad \forall \xi > 0, \tag{4.19}$$

see [22, Theorem 2.1]. By (4.19) and (3.17) we can rewrite (4.18) as

$$
\begin{aligned}
&e^{-t\Delta_\nu} f(x,u) \\
&\quad = \int_{\mathbb{R}} \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh(u-v)}{\xi}\right) \left(f_v *_\nu K_{e^{-\xi e^{u+v} L_\nu/2}}\right)(x) \, \mathrm{d}\xi \, \mathrm{d}v \\
&\quad = \int_{\mathbb{R}} f_v *_\nu \left(\int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh(u-v)}{\xi}\right) K_{e^{-\xi e^{u+v} L_\nu/2}}(\cdot) \, \mathrm{d}\xi \right)(x) \, \mathrm{d}v.
\end{aligned}
\tag{4.20}
$$

By Lemma 3.3 the inner integral is

$$
\begin{aligned}
&\int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh(u-v)}{\xi}\right) K_{e^{-\xi e^{u+v} L_\nu/2}}(y) \, \mathrm{d}\xi \\
&\quad = \frac{2}{\Gamma(\nu/2)} \int_0^\infty \Psi_t(\xi) \, (2\xi e^{u+v})^{-\nu/2} \exp\left(-\frac{\cosh(u-v)}{\xi} - \frac{y^2}{2\xi e^{u+v}}\right) \mathrm{d}\xi \\
&\quad = e^{-\nu v} K_{e^{-t\Delta_\nu}}(e^{-v} y, u-v),
\end{aligned}
$$

where $K_{e^{-t\Delta_\nu}}$ is the function defined in (4.16). Plugging this into (4.20) and expanding the Hankel convolution as in (3.1) gives

$$
\begin{aligned}
e^{-t\Delta_\nu} f(x,u) &= \int_{G_\nu} \tau_\nu^{[x]} f(y,v) K_{e^{-t\Delta_\nu}}(e^{-v} y, u-v) e^{-\nu v} \, \mathrm{d}\mu_\nu(y) \, \mathrm{d}v \\
&= \int_{G_\nu} \tau_\nu^{[x]} f(e^{u-v} y, u-v) K_{e^{-t\Delta_\nu}}(y,v) \, \mathrm{d}\mu_\nu(y) \, \mathrm{d}v \\
&= \int_{G_\nu} \ell_{(x,u)} f((y,v)^-) K_{e^{-t\Delta_\nu}}(y,v) \, \mathrm{d}\mu_\nu(y) \, \mathrm{d}v \\
&= f \diamond_\nu K_{e^{-t\Delta_\nu}}(x,u),
\end{aligned}
$$

by (4.3) and (4.4). This confirms that $K_{e^{-t\Delta_\nu}}$ given by (4.16) is indeed the $\diamond_\nu$-convolution kernel of $e^{-t\Delta_\nu}$.

Now we move on to justifying that $K_{e^{-t\Delta_\nu}} \geq 0$ for all $t > 0$. By (4.19) it is easily checked that the integral in (4.16) is absolutely convergent and $K_{e^{-t\Delta_\nu}} \in C \cap L^1(G_\nu)$ for any $t > 0$. Moreover, from (4.15), (4.4) and (4.2) we deduce that $e^{-t\Delta_\nu}$ is an integral operator on G_ν,

$$
e^{-t\Delta_\nu} f(x,u) = \int_{G_\nu} K_t((x,u),(y,v)) \, f(y,v) \, \mathrm{d}\mu_\nu(y) \, \mathrm{d}v,
$$

with integral kernel K_t given by

$$
\begin{aligned}
K_t((x,u),(y,v)) &= e^{-\nu v} r_{(x,u)} K_{e^{-t\Delta_\nu}}((y,v)^-) \\
&= e^{-\nu v} \ell_{(y,v)^-} K_{e^{-t\Delta_\nu}}(x,u)
\end{aligned}
\tag{4.21}
$$

(see also [22, Theorem 2.3]). In particular, by Lemma 4.2, $K_t(\cdot,(y,u)) \to K_{e^{-t\Delta_\nu}}$ in $L^1(G_\nu)$ as $(y,u) \to (0,0)$; thus, in order to prove that $K_{e^{-t\Delta_\nu}} \geq 0$, it is enough to show that $K_t \geq 0$ almost everywhere on $G_\nu \times G_\nu$, that is, that $e^{-t\Delta_\nu}$ is positivity-preserving.

Now, by [15, Theorem 1.3.2], the fact that the heat semigroup $e^{-t\Delta_\nu}$ is positivity-preserving is equivalent to the property that, for any real-valued f in the domain $\mathfrak{D}(\sqrt{\Delta_\nu})$ of $\sqrt{\Delta_\nu}$, there holds $|f| \in \mathfrak{D}(\sqrt{\Delta_\nu})$ and

$$
\langle \sqrt{\Delta_\nu} |f|, \sqrt{\Delta_\nu} |f| \rangle \leq \langle \sqrt{\Delta_\nu} f, \sqrt{\Delta_\nu} f \rangle.
\tag{4.22}
$$

By Lemma 4.3, here Δ_ν is the operator $\nabla_\nu^+ \nabla_\nu$ with Neumann domain (see (6.4)), that is, $\Delta_\nu = \nabla_\nu^* \nabla_\nu$ where ∇_ν is given the maximal domain. Thus,

$$\mathfrak{D}(\sqrt{\Delta_\nu}) = \mathfrak{D}_{\max}(\nabla_\nu) = \{f \in L^2(G_\nu) : |\nabla_\nu f| \in L^2(G_\nu)\}$$

and, for all $f \in \mathfrak{D}(\sqrt{\Delta_\nu})$,

$$\langle \nabla_\nu f, \nabla_\nu f \rangle = \langle \sqrt{\Delta_\nu} f, \sqrt{\Delta_\nu} f \rangle$$

(cf. the proof of [72, Theorem X.25]). Moreover, if $f \in \mathfrak{D}(\sqrt{\Delta_\nu})$ is real-valued, then $f, |\nabla_\nu f| \in L^1_{\mathrm{loc}}(G_\nu)$, so by applying [21, Lemma 7.6] we obtain that $\nabla_\nu |f| = \operatorname{sign}(f) \nabla_\nu f$. Hence, $|f| \in \mathfrak{D}(\sqrt{\Delta_\nu})$ and (4.22) is satisfied.

We are left with calculating the $L^1(G_\nu)$ norm of the heat kernels. Observe that

$$\begin{aligned} &\|K_{e^{-t\Delta_\nu}}\|_{L^1(G_\nu)} \\ &= \int_{\mathbb{R}} \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi}\right) \frac{2}{\Gamma(\nu/2)} \int_0^\infty \left(\frac{x^2}{2\xi e^u}\right)^{\nu/2} \exp\left(-\frac{x^2}{2\xi e^u}\right) \frac{\mathrm{d}x}{x} \,\mathrm{d}\xi \,\mathrm{d}u \\ &= \int_{\mathbb{R}} \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi}\right) \mathrm{d}\xi \,\mathrm{d}u, \end{aligned}$$

since the heat kernels $K_{e^{-sL_\nu}}$ have $L^1(X_\nu)$-norm equal to 1 for any $s > 0$. Furthermore, by (4.17), the above expression can be rewritten as

$$\frac{e^{\frac{\pi^2}{4t}}}{\sqrt{4\pi^3 t}} \int_{\mathbb{R}} \int_0^\infty \sinh(\theta) \sin\left(\frac{\pi\theta}{2t}\right) e^{-\frac{\theta^2}{4t}} \int_0^\infty \xi^{-2} \exp\left(-\frac{\cosh\theta + \cosh u}{\xi}\right) \mathrm{d}\xi \,\mathrm{d}\theta \,\mathrm{d}u.$$

Notice that the integral over ξ equals $(\cosh\theta + \cosh u)^{-1}$. Moreover,

$$\int_{\mathbb{R}} \frac{\mathrm{d}u}{\cosh\theta + \cosh u} = \frac{2\theta}{\sinh\theta}.$$

Hence, combining the above we arrive at

$$\begin{aligned} \int_{\mathbb{R}} \int_0^\infty K_{e^{-t\Delta_\nu}}(x,u) \,\mathrm{d}\mu_\nu(x) \,\mathrm{d}u &= \frac{e^{\frac{\pi^2}{4t}}}{\sqrt{\pi^3 t}} \int_0^\infty \theta \sin\left(\frac{\pi\theta}{2t}\right) e^{-\frac{\theta^2}{4t}} \,\mathrm{d}\theta \\ &= \frac{e^{\frac{\pi^2}{4t}}}{\sqrt{4\pi t}} \int_{\mathbb{R}} \cos\left(\frac{\pi\theta}{2t}\right) e^{-\frac{\theta^2}{4t}} \,\mathrm{d}\theta, \end{aligned}$$

which is equal to 1 by the formula for the Fourier transform of a Gaussian.

Therefore, $\|K_{e^{-t\Delta_\nu}}\|_{L^1(G_\nu)} = 1$ for all $t > 0$. Consequently, by Young's convolution inequality (4.6) we obtain that the $e^{-t\Delta_\nu}$ are contractions on $L^p(G_\nu)$ for any $p \in [1, \infty]$. □

The heat kernel formula of Proposition 4.4 is our starting point to derive a number of properties of the functional calculus for Δ_ν, including the fact that it is given by convolution operators, and the validity of a Plancherel formula relating the L^2 norm of the convolution kernel with a suitable L^2 norm of the corresponding spectral multiplier.

We start with establishing a basic property, which is related to the left-invariance (4.14) of the operators in the functional calculus (cf. [69, Theorem 5]). Recall that m is the modular function on G_ν, given by (4.1).

Lemma 4.5 *Let $F : \mathbb{R}_+ \to \mathbb{C}$ be a bounded Borel function and $f \in L^1(G_\nu)$ be such that $fm^{1/2} \in L^1(G_\nu)$. Then*

$$F(\Delta_\nu)(f \diamond_\nu g) = f \diamond_\nu F(\Delta_\nu)g, \qquad g \in L^2(G_\nu).$$

Proof Fix f as above. Associativity of $\diamond_\nu$ on $L^1(G_\nu)$ and Young's inequality (Lemma 4.1), together with a density argument, show that

$$(f \diamond_\nu g) \diamond_\nu h = f \diamond_\nu (g \diamond_\nu h)$$

for all $g \in L^2(G_\nu)$ and all $h \in L^1(G_\nu)$. By Proposition 4.4 we can apply this to $h = K_{e^{-\Delta_\nu}}$ and obtain

$$e^{-\Delta_\nu}(f \diamond_\nu g) = f \diamond_\nu e^{-\Delta_\nu} g, \qquad g \in L^2(G_\nu).$$

As the left-convolution operator $g \mapsto f \diamond_\nu g$ is L^2-bounded, by the spectral calculus for $e^{-\Delta_\nu}$ we get that, for all bounded Borel functions $G : \mathbb{R}_+ \to \mathbb{C}$,

$$G(e^{-\Delta_\nu})(f \diamond_\nu g) = f \diamond_\nu G(e^{-\Delta_\nu})g, \qquad g \in L^2(G_\nu).$$

Finally, taking $G(\lambda) = F(-\log \lambda)$ finishes the proof. □

For $t > 0$ and $u \in \mathbb{R}$ we define the function $M_{t,u} : \mathbb{R}_+ \to \mathbb{R}$ by

$$M_t(\lambda, u) := M_{t,u}(\lambda) = \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi} - \frac{\xi \lambda e^u}{2}\right) \mathrm{d}\xi \tag{4.23}$$

for all $\lambda > 0$, where Ψ_t is defined in (4.17).

We emphasize that $M_{t,u}$ is independent of ν, which is of paramount importance. This fact was already exploited in the literature, for instance in [30, 54].

Lemma 4.6 *For all $t > 0$ and $u \in \mathbb{R}$, the function $M_{t,u}$ is bounded. Moreover, $M_{t,u} \in L^2(\mathbb{R}_+, \lambda^{\alpha/2} \frac{\mathrm{d}\lambda}{\lambda})$ for all $\alpha > 0$.*

Proof By (4.19) we have

$$\|M_{t,u}\|_\infty \lesssim_t \int_0^\infty \xi^{-2} e^{-1/\xi} \, \mathrm{d}\xi = 1,$$

and also

$$\|M_{t,u}\|^2_{L^2(\mathbb{R}_+, \lambda^{\alpha/2} \frac{\mathrm{d}\lambda}{\lambda})} \lesssim \int_0^\infty \left(\int_0^\infty \xi^{-2} \exp\left(-\frac{\cosh u}{\xi} - \frac{\xi \lambda e^u}{2}\right) \mathrm{d}\xi \right)^2 \lambda^{\alpha/2} \frac{\mathrm{d}\lambda}{\lambda}.$$

We split the integration over λ onto the intervals $(0, 1)$ and $(1, \infty)$. In the former case we have

$$\int_0^1 \left(\int_0^\infty \xi^{-2} \exp\left(-\frac{\cosh u}{\xi} - \frac{\xi \lambda e^u}{2}\right) \mathrm{d}\xi \right)^2 \lambda^{\alpha/2} \frac{\mathrm{d}\lambda}{\lambda} \lesssim_\alpha \left(\int_0^\infty \xi^{-2} e^{-1/\xi} \, \mathrm{d}\xi \right)^2 = 1.$$

On the other hand, for the remaining interval we have

$$\begin{aligned} &\int_1^\infty \left(\int_0^\infty \xi^{-2} \exp\left(-\frac{\cosh u}{\xi} - \frac{\xi \lambda e^u}{2}\right) \mathrm{d}\xi \right)^2 \lambda^{\alpha/2} \frac{\mathrm{d}\lambda}{\lambda} \\ &\lesssim_\alpha \int_1^\infty \left(\int_0^\infty \xi^{-2} (\xi \lambda e^u)^{-(\alpha+2)/4} \exp\left(-\frac{\cosh u}{\xi}\right) \mathrm{d}\xi \right)^2 \lambda^{\alpha/2} \frac{\mathrm{d}\lambda}{\lambda} \\ &\lesssim \left(\int_0^\infty \left(\frac{\eta}{e^u \cosh u} \right)^{(\alpha+2)/4} e^{-\eta} (\cosh u)^{-1} \, \mathrm{d}\eta \right)^2 \lesssim 1. \end{aligned}$$

Combining the above finishes the proof. ◻

The functions $M_{t,u}$ encode the relation between the heat propagators on X_ν and L_ν, already exploited in the proof of Proposition 4.4.

Lemma 4.7 *For any $t > 0$ and $u \in \mathbb{R}$ there holds*

$$K_{e^{-t\Delta_\nu}}(x, u) = K_{M_{t,u}(L_\nu)}(x) \qquad \text{for a.a. } x \in \mathbb{R}_+.$$

Proof Fix $t > 0$ and $u \in \mathbb{R}$. By (3.17) and Lemma 4.6 we have

$$K_{M_{t,u}(L_\nu)}(x) = \kappa_\nu^{-2} \int_0^\infty \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi} - \frac{\xi e^u y^2}{2}\right) j_x^\nu(y) \, \mathrm{d}\xi \, \mathrm{d}\mu_\nu(y).$$

Because of (4.19) we can apply Fubini's theorem and Lemma 3.3 to obtain

$$\begin{aligned} K_{M_{t,u}(L_\nu)}(x) &= \int_0^\infty \Psi_t(\xi) \exp\left(-\frac{\cosh u}{\xi}\right) H_\nu^{-1}\left(y \mapsto e^{-\xi e^u y^2/2}\right)(x) \, \mathrm{d}\xi \\ &= \frac{2}{\Gamma(\nu/2)} \int_0^\infty \Psi_t(\xi)(2\xi e^u)^{-\nu/2} \exp\left(-\frac{\cosh u}{\xi}\right) \exp\left(-\frac{x^2}{2\xi e^u}\right) \mathrm{d}\xi, \end{aligned}$$

and this is equal to $K_{e^{-t\Delta_\nu}}(x, u)$, see (4.16). ◻

Remark 4.8 Lemma 4.7 establishes in the context of the hypergroup G_ν the analogue of the relation between the heat kernels on a stratified group N and the semidirect product group $G = N \rtimes \mathbb{R}$, which is already known in the literature. In particular, for $n \in \mathbb{N} \setminus \{0\}$, let $\Delta_{\mathbb{R}^n}$ be the classical Laplacian on $\mathbb{R}^n$ and $\widetilde{\Delta}_{\mathbb{R}^n} = -\partial_u^2 + e^{2u} \Delta_{\mathbb{R}^n}$ the corresponding left-invariant Laplacian on $\mathbb{R}^n \rtimes \mathbb{R}$. By [54, Proposition 4.3],

$$K_{e^{-t\widetilde{\Delta}_{\mathbb{R}^n}}}(x, u) = K_{M_{t,u}(\Delta_{\mathbb{R}^n})}(x). \tag{4.24}$$

Moreover, the Plancherel measures associated with $\Delta_{\mathbb{R}^n}$ and $\widetilde{\Delta}_{\mathbb{R}^n}$ can be explicitly calculated (cf. [54, pp. 388–389]) and, for all bounded Borel functions $F : \mathbb{R}_+ \to \mathbb{C}$,

$$\begin{aligned} \int_{\mathbb{R}^n} |K_{F(\Delta_{\mathbb{R}^n})}(z)|^2 \, \mathrm{d}z &\simeq_n \int_0^\infty |F(\lambda)|^2 \lambda^{n/2} \frac{\mathrm{d}\lambda}{\lambda}, \\ \int_{\mathbb{R}} \int_{\mathbb{R}^n} |K_{F(\widetilde{\Delta}_{\mathbb{R}^n})}(z, u)|^2 \, \mathrm{d}z \, \mathrm{d}u &\simeq_n \int_0^\infty |F(\lambda)|^2 \lambda^{[3/2,(n+1)/2]} \frac{\mathrm{d}\lambda}{\lambda}, \end{aligned}$$

where we use the notation (1.7). Of course, the first formula is analogous to the Plancherel formula (3.18) for L_ν. As we shall see below, an analogue of the second formula holds for Δ_ν.

Let $\mathcal{J}$ be the set of all finite linear combinations of decaying exponentials $\lambda \mapsto e^{-t\lambda}$, $\lambda \in \mathbb{R}_+$, for some $t > 0$, as in [54, p. 389]. The Stone–Weierstrass theorem yields that $\mathcal{J}$ is uniformly dense in $C_0(\mathbb{R}_+)$.

Define the linear operator $\Phi : \mathcal{J} \to \bigcap_{\alpha>0} L^2(\mathbb{R}_+ \times \mathbb{R}, \lambda^\alpha \frac{\mathrm{d}\lambda}{\lambda} \, \mathrm{d}u)$ by setting

$$\Phi(e^{-t\cdot}) = M_t$$

for all $t > 0$, where M_t is given in (4.23). We denote $(\Phi F)_u(\lambda) = \Phi F(\lambda, u)$.

Lemma 4.9 *For all $\alpha \in [1, \infty)$, the operator Φ extends to a bounded operator from $L^2(\mathbb{R}_+, \lambda^{[3/2,(\alpha+1)/2]} \frac{\mathrm{d}\lambda}{\lambda})$ to $L^2(\mathbb{R}_+ \times \mathbb{R}, \lambda^{\alpha/2} \frac{\mathrm{d}\lambda}{\lambda} \, \mathrm{d}u)$.*

Proof This is already known for $\alpha \in \mathbb{N}$, and was implicitly used in [54]. For the reader's convenience we provide a short justification.

Fix $\alpha \in \mathbb{N}$, $\alpha \geq 1$. By Remark 4.8, using (4.24) and linearity, we deduce that, for all $F \in \mathcal{J}$,

$$K_{F(\widetilde{\Delta}_{\mathbb{R}^\alpha})}(z,u) = K_{(\Phi F)_u(\Delta_{\mathbb{R}^\alpha})}(z),$$

and therefore

$$\begin{aligned}\int_{\mathbb{R}} \int_0^\infty |\Phi F(\lambda,u)|^2 \lambda^{\alpha/2} \,\frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{d}u &\simeq \int_{\mathbb{R}} \int_{\mathbb{R}^\alpha} |K_{(\Phi F)_u(\Delta_{\mathbb{R}^\alpha})}(z)|^2 \,\mathrm{d}z\,\mathrm{d}u \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^\alpha} |K_{F(\widetilde{\Delta}_{\mathbb{R}^\alpha})}(z)|^2 \,\mathrm{d}z\,\mathrm{d}u \\ &\simeq \int_0^\infty |F(\lambda)|^2 \lambda^{[3/2,(\alpha+1)/2]} \,\frac{\mathrm{d}\lambda}{\lambda}.\end{aligned}$$

Hence, if α is an integer, then Φ extends to a bounded operator between the spaces $L^2(\mathbb{R}_+, \lambda^{[3/2,(\alpha+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$ and $L^2(\mathbb{R}_+ \times \mathbb{R}, \lambda^{\alpha/2}\,\frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{d}u)$.

The result for fractional α then follows by Stein's interpolation theorem for weighted L^2 spaces [77]. □

Corollary 4.10 *If $F \in \mathcal{J}$, then $F(\Delta_\nu)$ is a right $\diamond_\nu$-convolution operator, with convolution kernel $K_{F(\Delta_\nu)} \in L^1 \cap L^2(G_\nu)$. Moreover,*

$$\|K_{F(\Delta_\nu)}\|_{L^2(G_\nu)} \lesssim \|F\|_{L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})}.$$

In particular, the heat kernels associated with Δ_ν belong to $L^2(G_\nu)$.

Proof By Proposition 4.4 and linearity it is clear that, for all $F \in \mathcal{J}$, the operator $F(\Delta_\nu)$ is a convolution operator, with convolution kernel $K_{F(\Delta_\nu)} \in L^1(G_\nu)$. Moreover, again by linearity and Lemma 4.7, we deduce that

$$K_{F(\Delta_\nu)}(x,u) = K_{(\Phi F)_u(L_\nu)}(x). \tag{4.25}$$

So, by (3.18), and Lemma 4.9,

$$\begin{aligned}\int_{G_\nu} |K_{F(\Delta_\nu)}(x,u)|^2 \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u &= \int_{\mathbb{R}} \int_0^\infty |K_{(\Phi F)_u(L_\nu)}(x)|^2 \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u \\ &\simeq \int_{\mathbb{R}} \int_0^\infty |(\Phi F)_u(\lambda)|^2 \lambda^{\nu/2} \,\frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{d}u \\ &\lesssim \int_0^\infty |F(\lambda)|^2 \lambda^{[3/2,(\nu+1)/2]} \,\frac{\mathrm{d}\lambda}{\lambda},\end{aligned}$$

as desired. □

We now want to extend Corollary 4.10 to a larger class of multipliers F. To this purpose, it is useful to establish some density properties of the class $\mathcal{J}$.

Lemma 4.11 *Let $F \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$ be bounded.*

(i) There exists a uniformly bounded sequence $F_n \in C_c^\infty(\mathring{\mathbb{R}}_+)$ converging to F almost everywhere and in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$.

(ii) If additionally $F \in C_0(\mathbb{R}_+)$, then there exists a sequence $F_n \in \mathcal{J}$ converging to F uniformly and in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$.

Proof We start with (i). Given any $F \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda})$, we can approximate it first by the functions $F\mathbb{1}_{[1/k,k]}$, $k \in \mathbb{N}\setminus\{0\}$, which have compact support in $\mathring{\mathbb{R}}_+$ and are dominated by $|F|$; in turn, each of these compactly supported functions can be approximated via mollifiers by $C_c^\infty(\mathring{\mathbb{R}}_+)$ functions. Then a diagonal argument gives a sequence in $C_c^\infty(\mathring{\mathbb{R}}_+)$ converging to F in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda})$ and, up to extracting a subsequence, also almost everywhere. If additionally F is bounded, then each approximant is also bounded by $\|F\|_{L^\infty}$.

For (ii) it suffices to justify that $\mathcal{J}$ is dense in the Banach space

$$W = C_0(\mathbb{R}_+) \cap L^2\left(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda}\right),$$
$$\|\cdot\|_W = \|\cdot\|_{L^\infty} + \|\cdot\|_{L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda})}.$$

Clearly, $C_c(\mathbb{R}_+)$ is dense in W; this is easily seen by using compactly supported cutoffs and dominated convergence. Thus, to conclude, it is enough to verify that the closure of $\mathcal{J}$ in W contains $C_c(\mathbb{R}_+)$. Much as in the proof of [47, Lemma 3.13], let $G \in C_c(\mathbb{R}_+)$ and set $g(x) := G(x)e^x \in C_c(\mathbb{R}_+)$. By the Stone–Weierstrass theorem we can find $g_k \in \mathcal{J}$ converging uniformly to g. Thus, $G_k(x) := g_k(x)e^{-x} \in \mathcal{J}$ satisfy

$$|G_k(x) - G(x)| = e^{-x}|g_k(x) - g(x)| \le e^{-x}\|g_k - g\|_{L^\infty}$$

so G_k converges uniformly to G. Moreover, since $|G_k(x)| \lesssim e^{-x}$, Lebesgue's dominated convergence theorem implies that G_k converges to G also in $L^2(\mathbb{R}_+, (\lambda^{3/2} + \lambda^{(\nu+1)/2}) \frac{d\lambda}{\lambda})$. Hence, $C_c(\mathbb{R}_+)$ is contained in the closure of $\mathcal{J}$ in W. □

We can finally obtain the existence of the convolution kernel for a larger class of operators in the calculus for Δ_ν, and establish the existence of the associated Plancherel measure.

Proposition 4.12 *For all bounded Borel functions $F : \mathbb{R}_+ \to \mathbb{C}$ that belong to $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda})$, the operator $F(\Delta_\nu)$ is a $\diamond_\nu$-convolution operator and the corresponding kernel $K_{F(\Delta_\nu)}$ is in $L^2(G_\nu)$. Moreover, there exists a regular Borel measure σ_ν, called the* Plancherel measure *associated with Δ_ν, such that*

$$\begin{aligned}\int_{\mathbb{R}}\int_0^\infty |K_{F(\Delta_\nu)}(x,u)|^2 \, d\mu_\nu(x)\, du &= \int_0^\infty |F(\lambda)|^2 \, d\sigma_\nu(\lambda) \\ &\lesssim \int_0^\infty |F(\lambda)|^2 \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda}\end{aligned} \tag{4.26}$$

for all such F. Additionally, the null sets for σ_ν and for the spectral measure associated with Δ_ν are the same.

Proof Firstly, let $F \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda})$ be also in $C_0(\mathbb{R}_+)$. By Lemma 4.11 (ii) we find a sequence $F_n \in \mathcal{J}$ converging to F in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda})$ and uniformly. By the spectral theorem, $F_n(\Delta_\nu) \to F(\Delta_\nu)$ in the operator norm on $L^2(G_\nu)$. On the other hand, by Corollary 4.10, the operator mapping $G \in \mathcal{J}$ to $K_{G(\Delta_\nu)} \in L^2(G_\nu)$ extends to a bounded operator

$$T : L^2\left(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{d\lambda}{\lambda}\right) \to L^2(G_\nu).$$

Thus, $K_{F_n(\Delta_\nu)} \to TF$ in $L^2(G_\nu)$. Combining the above, for all $f \in C_c(G_\nu)$, we see that $F_n(\Delta_\nu)f$ converges to $F(\Delta_\nu)f$ in $L^2(G_\nu)$, and also, by Young's inequality, $F_n(\Delta_\nu)f$ converges to $f \diamond_\nu TF$ in $L^2(G_\nu)$. This shows that $F(\Delta_\nu)$ is indeed a $\diamond_\nu$-convolution operator with kernel $K_{F(\Delta_\nu)} := TF \in L^2(G_\nu)$.

In order to construct the Plancherel measure we follow the approach in the proof of [74, Lemma 1]. Let E_ν be the spectral measure associated with Δ_ν, by the spectral theorem. For any Borel function $G : \mathbb{R}_+ \to \mathbb{C}$ the operator

$$G(\Delta_\nu) = \int_0^\infty G(\lambda)\,\mathrm{d}E_\nu(\lambda)$$

is a closed operator on $L^2(G_\nu)$, which is bounded whenever G is bounded. Moreover, $\langle \mathrm{d}E_\nu f, f\rangle$ is a regular Borel measure on $\mathbb{R}_+$ for any $f \in L^2(G_\nu)$, and

$$\|G(\Delta_\nu) f\|^2_{L^2(G_\nu)} = \int_0^\infty |G(\lambda)|^2 \langle \mathrm{d}E_\nu(\lambda) f, f\rangle, \tag{4.27}$$

where the right-hand side is finite if and only if f is in the domain of $G(\Delta_\nu)$.

Now take $F \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda}) \cap C_0(\mathbb{R}_+)$. Notice that, for any $f \in C_c(G_\nu)$, Lemma 4.5 gives

$$e^{-\Delta_\nu} F(\Delta_\nu) f = e^{-\Delta_\nu}(f \diamond_\nu K_{F(\Delta_\nu)}) = f \diamond_\nu (e^{-\Delta_\nu} K_{F(\Delta_\nu)}).$$

Clearly, $e^{-\Delta_\nu}$ and $F(\Delta_\nu)$ commute, so we also get

$$e^{-\Delta_\nu} F(\Delta_\nu) f = F(\Delta_\nu) e^{-\Delta_\nu} f = f \diamond_\nu (F(\Delta_\nu) K_{e^{-\Delta_\nu}}).$$

Thus,

$$e^{-\Delta_\nu} K_{F(\Delta_\nu)} = F(\Delta_\nu) K_{e^{-\Delta_\nu}}. \tag{4.28}$$

Combining (4.27) and (4.28), we deduce that

$$\begin{aligned} \int_{G_\nu} |K_{F(\Delta_\nu)}(x,u)|^2 \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u &= \int_0^\infty e^{2\lambda} \langle \mathrm{d}E_\nu(\lambda) e^{-\Delta_\nu} K_{F(\Delta_\nu)}, e^{-\Delta_\nu} K_{F(\Delta_\nu)}\rangle \\ &= \int_0^\infty e^{2\lambda} \langle \mathrm{d}E_\nu(\lambda) F(\Delta_\nu) K_{e^{-\Delta_\nu}}, F(\Delta_\nu) K_{e^{-\Delta_\nu}}\rangle \\ &= \int_0^\infty |F(\lambda)|^2 e^{2\lambda} \langle \mathrm{d}E_\nu(\lambda) K_{e^{-\Delta_\nu}}, K_{e^{-\Delta_\nu}}\rangle. \end{aligned}$$

This gives the formula for the Plancherel measure:

$$\mathrm{d}\sigma_\nu(\lambda) = e^{2\lambda} \langle \mathrm{d}E_\nu(\lambda) K_{e^{-\Delta_\nu}}, K_{e^{-\Delta_\nu}}\rangle. \tag{4.29}$$

With this definition of the measure σ_ν, the equality in (4.26) is proved under the additional assumption that $F \in C_0(\mathbb{R}_+)$.

Now we establish the inequality in (4.26). For any $G \in \mathcal{J}$, by Corollary 4.10,

$$\begin{aligned} \int_0^\infty |G(\lambda)|^2 \,\mathrm{d}\sigma_\nu(\lambda) &= \int_{\mathbb{R}} \int_0^\infty |K_{G(\Delta_\nu)}(x,u)|^2 \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u \\ &\lesssim \int_0^\infty |G(\lambda)|^2 \,\lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda}. \end{aligned}$$

Since $\mathcal{J}$ is dense in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$ we obtain the inequality in (4.26) for all $G \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$. In particular, we deduce that σ_ν is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$ and its density is bounded by a multiple of $\lambda^{[1/2,(\nu-1)/2]}$.

Now we prove that the null sets for σ_ν and the spectral measure are the same. By the definition (4.29) of σ_ν, the only nontrivial implication is that if for $A \subseteq \mathbb{R}_+$ we have $\sigma_\nu(A) = 0$, then also $\mathbb{1}_A(\Delta_\nu) = E_\nu(A) = 0$.

Fix such a set A. Let D be a countable dense subset of $L^2(G_\nu)$. Since σ_ν and the measures $\langle \mathrm{d}E_\nu f, f\rangle$ for $f \in D$ are all regular Borel measures, we can find a sequence of compact sets K_n and open sets U_n such that $K_n \subseteq K_{n+1} \subseteq \cdots \subseteq A \subseteq \cdots \subseteq U_{n+1} \subseteq U_n$ and also $\sigma_\nu(U_n \setminus K_n) \to 0$ and $\langle E_\nu(U_n\setminus K_n) f, f\rangle \to 0$, $f \in D$. Let $\varphi_n \in C_c(\mathbb{R})$ be such that $0 \le \varphi_n \le 1$, $\operatorname{supp}\varphi_n \subseteq U_n$ and $\varphi_{n|_{K_n}} \equiv 1$. Then, $\varphi_n \to \mathbb{1}_A$ almost everywhere with respect to σ_ν and each $\langle \mathrm{d}E_\nu f, f\rangle$, $f \in D$.

Since $0 \le \varphi_n \le \mathbb{1}_{U_n}$ and $\sigma_\nu(U_n)$ tends to 0, we also have that $\varphi_n \to 0$ in $L^2(\mathrm{d}\sigma_\nu)$. As $\varphi_n \in C_c(\mathbb{R}_+)$, we can apply (4.26) and deduce that $K_{\varphi_n(\Delta_\nu)} \to 0$ in $L^2(G_\nu)$; since the φ_n are uniformly bounded, by Young's convolution inequality (Lemma 4.1) and a density argument we obtain that $\varphi_n(\Delta_\nu) \to 0$ in the strong operator topology on $L^2(G_\nu)$. On the other hand, for any $f \in D$,

$$\|\varphi_n(\Delta_\nu) f - \mathbb{1}_A(\Delta_\nu) f\|_{L^2(G_\nu)} = \int_0^\infty |\varphi_n(\lambda) - \mathbb{1}_A(\lambda)|^2 \langle \mathrm{d}E_\nu(\lambda) f, f\rangle \to 0,$$

by the dominated convergence theorem, because $\varphi_n \to \mathbb{1}_A$ almost everywhere with respect to $\langle \mathrm{d}E_\nu f, f\rangle$, and moreover $|\varphi_n - \mathbb{1}_A| \le 1$ and $\langle \mathrm{d}E_\nu f, f\rangle$ is a finite measure. From the density of D we deduce that $\varphi_n(\Delta_\nu)$ converges to $\mathbb{1}_A(\Delta_\nu)$ in the strong operator topology. Hence, by the uniqueness of the limit, $\mathbb{1}_A(\Delta_\nu) = 0$ as an operator, so A is a null set for the spectral measure.

It remains to prove the existence of the convolution kernel and the equality in (4.26) for general F, namely for a bounded function $F \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$. We can apply Lemma 4.11 (i) to find a uniformly bounded sequence of functions $F_n \in C_c^\infty(\mathring{\mathbb{R}}_+)$ converging to F almost everywhere (with respect to Lebesgue measure) and in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$. Since σ_ν is absolutely continuous with respect to Lebesgue measure, and the spectral measure has the same null sets, we obtain that $F_n(\Delta_\nu)$ converges to $F(\Delta_\nu)$ in the strong operator topology. Much as above, the kernels $K_{F_n(\Delta_\nu)}$ converge in $L^2(G_\nu)$, and the limit is the $\diamond_\nu$-convolution kernel of $F(\Delta_\nu)$. Finally, a density argument proves the equality in (4.26) for F. □

The next result is a generalization of Lemma 4.7.

Lemma 4.13 *Let F be bounded and in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$. Then*

$$K_{F(\Delta_\nu)}(x,u) = K_{(\Phi F)_u(L_\nu)}(x)$$

for almost all $(x,u) \in G_\nu$.

Proof By (3.18) and Lemma 4.9, for any $H \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$,

$$\begin{aligned}&\int_{G_\nu} |K_{(\Phi H)_u(L_\nu)}(x)|^2 \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u \\ &\quad\simeq \int_{\mathbb{R}}\int_0^\infty |(\Phi H)_u(\lambda)|^2 \lambda^{\nu/2}\,\frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{d}u \lesssim \|H\|^2_{L^2(\mathbb{R}_+,\lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})}\end{aligned}$$

So, the correspondence T mapping H to the function $(x,u) \mapsto K_{(\Phi H)_u(L_\nu)}(x)$ is a bounded operator $T : L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda}) \to L^2(G_\nu)$.

Fix F as in the statement. Let $F_n \in \mathcal{J}$ be a sequence of functions converging to F in $L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]}\,\frac{\mathrm{d}\lambda}{\lambda})$. Proposition 4.12 implies that $K_{F_n(\Delta_\nu)}$ converges to $K_{F(\Delta_\nu)}$ in $L^2(G_\nu)$. At the same time, the boundedness of T gives that $K_{(\Phi F_n)_u(L_\nu)}(x)$ converges to $K_{(\Phi F)_u(L_\nu)}(x)$ in $L^2(G_\nu)$. Hence, up to extracting a subsequence,

$$K_{F_n(\Delta_\nu)}(x,u) \to K_{F(\Delta_\nu)}(x,u) \quad \text{and} \quad K_{(\Phi F_n)_u(L_\nu)}(x) \to K_{(\Phi F)_u(L_\nu)}(x) \tag{4.30}$$

almost everywhere on G_ν. As $K_{F_{n_k}(\Delta_\nu)}(x,u) = K_{(\Phi F_{n_k})_u(L_\nu)}(x)$ by (4.25), we conclude that $K_{F(\Delta_\nu)}(x,u) = K_{(\Phi F)_u(L_\nu)}(x)$ almost everywhere. □

4.4 Riemannian distance and finite propagation speed

Recall that $\Delta_\nu = \nabla_\nu^+ \nabla_\nu$, where ∇_ν is given in (4.10). We equip the manifold $\mathring{G}_\nu$ with the Riemannian structure which makes the vector fields ∂_u and $e^u \partial_x$ an orthonormal frame. As these are the vector fields appearing in (4.10) as the components of ∇_ν, the Riemannian distance ϱ on $\mathring{G}_\nu$ is the control distance associated with ∇_ν in the sense of the Appendix (see Remark 6.2).

We emphasise that ∇_ν does not depend on ν, so also the Riemannian structure on G_ν and the corresponding distance do not. This is confirmed by the following statement, where we obtain an explicit formula for the Riemannian distance on $\mathring{G}_\nu$, analogous to the formula (2.16) for the sub-Riemannian distance on the group G.

Proposition 4.14 *For any $\nu \geq 1$, the Riemannian distance on $\mathring{G}_\nu$ is given by*

$$\varrho((x,u),(x',u')) = \operatorname{arccosh}\left(\cosh(u-u') + \frac{|x-x'|^2}{2e^{u+u'}}\right). \tag{4.31}$$

The same expression also gives a distance on the whole G_ν.

Proof When $N = \mathbb{R}^d$ is abelian and Δ_N is the standard Laplacian on $\mathbb{R}^d$, then the semidirect product $G = \mathbb{R}^d \rtimes \mathbb{R}$ discussed in the introduction, equipped with the Riemannian metric which makes the vector fields (1.2) an orthonormal frame, is a realisation of the real hyperbolic space of dimension $d+1$. It is well known (see, e.g., [54, Proposition 2.7]) that the Riemannian distance between two points $(x,u), (x',u') \in \mathbb{R}^d \rtimes \mathbb{R}$ is given by the right-hand side of (4.31).

Take now $d = 1$. Then $\mathring{G}_\nu$ and G_ν can be thought of as an open subset and a closed subset of the hyperbolic plane $\mathbb{R} \rtimes \mathbb{R}$. So, certainly the expression (4.31) defines a distance on each of G_ν and $\mathring{G}_\nu$. Moreover, the vector fields ∂_u and $e^u \partial_x$ on $\mathring{G}_\nu$ are the restrictions to $\mathring{G}_\nu$ of the vector fields (1.2) on $\mathbb{R} \rtimes \mathbb{R}$. Thus, the Riemannian metric tensor on $\mathring{G}_\nu$ is just the restriction of the corresponding tensor on the hyperbolic plane, and the length of a curve γ in the Riemannian manifold $\mathring{G}_\nu$ is the same as the length of γ thought of as a curve in the hyperbolic plane.

Recall that the hyperbolic plane is a complete Riemannian manifold, so the Riemannian distance between two points of the plane is the length of the unique length-minimising curve joining those points. Thus, in order to conclude that the formula (4.31) also gives the Riemannian distance on $\mathring{G}_\nu$, it is enough to show that $\mathring{G}_\nu$ is geodesically convex in the hyperbolic plane.

This convexity property is easily seen if one works with the half-plane model $\{(x,a) : x \in \mathbb{R},\ a \in \mathring{\mathbb{R}}_+\}$ of the hyperbolic plane, corresponding to the change of variables $a = e^u$. Via this change of variables, $\mathring{G}_\nu$ corresponds to the open quadrant $\{(x,a) : x, a \in \mathring{\mathbb{R}}_+\}$. As geodesics in the half-plane model are just segments of lines or circles perpendicular to the boundary $\{a = 0\}$ (see, e.g., [5, Theorem 9.3]), it is clear that geodesics joining two points in $\mathring{G}_\nu$ are entirely contained in $\mathring{G}_\nu$ (see Fig. 1). □

From now on, we equip G_ν with the distance ϱ of Proposition 4.14. Moreover, for all $(x,u) \in G_\nu$, we set

$$|(x,u)|_{G_\nu} := \varrho((0,0),(x,u)) = \operatorname{arccosh}(\cosh(u) + e^{-u}x^2/2),$$

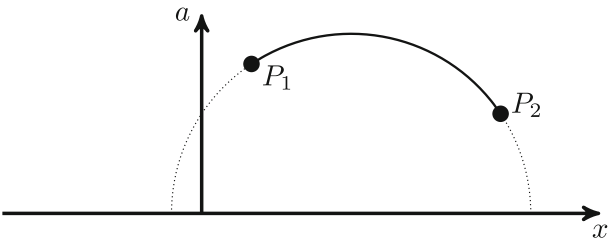

Fig. 1 The geodesic between P_1 and P_2

and denote $B_{G_\nu}(0, r) = \{(x, u) \in G_\nu : |(x, u)|_{G_\nu} < r\}$ for all $r > 0$. A function on G_ν will be called *radial* if it has the form $f(|\cdot|_{G_\nu})$, i.e., if it depends only on the distance of its argument from the origin $(0, 0)$ of G_ν.

The following proposition concerning integration of radial functions on G_ν is a simple modification of [54, Propositions 2.8 and 2.9].

Proposition 4.15 *For any Borel function $f : \mathbb{R}_+ \to \mathbb{R}_+$ there holds*

$$\begin{aligned}&\int_{G_\nu} f(|(x, u)|_{G_\nu})\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}u \\ &= \int_{G_\nu} f(|(x, u)|_{G_\nu})\, e^{-\nu u}\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}u = c_\nu \int_0^\infty f(r)(\sinh r)^\nu\, \mathrm{d}r,\end{aligned} \tag{4.32}$$

where $c_\nu = 2^{\nu-1}\mathrm{B}(\nu/2, \nu/2)$. Moreover,

$$\int_{G_\nu} f(|(x, u)|_{G_\nu}) e^{-u\nu} x^\nu\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}u \lesssim_\nu \int_{G_\nu} f(|(x, u)|_{G\nu})\, |(x, u)|_{G_\nu}\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}u. \tag{4.33}$$

Radial functions arise naturally when considering operators in the functional calculus for Δ_ν, up to a twist with the modular function m of (4.1).

Corollary 4.16 *For all bounded functions $F \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{\mathrm{d}\lambda}{\lambda})$, the function $m^{-1/2} K_{F(\Delta_\nu)}$ is radial on G_ν. Moreover, for all $r > 0$,*

$$\int_{B_{G_\nu}(0,r)} |K_{F(\sqrt{\Delta_\nu})}(x, u)|^2 x^\nu\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}u \lesssim_\nu r \int_{B_{G_\nu}(0,r)} |K_{F(\sqrt{\Delta_\nu})}(x, u)|^2\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}u.$$

Proof Notice that we can rewrite (4.16) as

$$e^{\nu u/2} K_{e^{-t\Delta_\nu}}(x, u) = \frac{2}{\Gamma(\nu/2)} \int_0^\infty \Psi_t(\xi)(2\xi)^{-\nu/2} \exp\left(-\frac{\cosh |(x, u)|_{G_\nu}}{\xi}\right) \mathrm{d}\xi.$$

This means that $m^{-1/2} K_{e^{-t\Delta_\nu}}$ is radial on G_ν for all $t > 0$. By linearity, this implies the radiality of $m^{-1/2} K_{F(\Delta_\nu)}$ for all $F \in \mathcal{J}$, and a density argument, as in (4.30), extends the result to all bounded $F \in L^2(\mathbb{R}_+, \lambda^{[3/2,(\nu+1)/2]} \frac{\mathrm{d}\lambda}{\lambda})$.

Now, let $r \geq 0$. By applying (4.33) and (4.32) to the radial function

$$f_r(x, u) = e^{\nu u}\, |K_{F(\Delta_\nu)}(x, u)|^2\, \mathbb{1}_{B_{G_\nu}(0,r)}(x, u)$$

we get

$$\begin{aligned}\int_{B_{G_\nu}(0,r)} |K_{F(\sqrt{\Delta_\nu})}(x,u)|^2 x^\nu \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u &= \int_{G_\nu} f_r(x,u) e^{-\nu u} x^\nu \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u \\ &\lesssim_\nu \int_{G_\nu} f_r(x,u)\,|(x,u)|_{G_\nu} \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u \\ &\le r \int_{G_\nu} f_r(x,u)\,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u \\ &= r \int_{G_\nu} f_r(x,u)\, e^{-\nu u}\,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u \\ &= r \int_{B_{G_\nu}(0,r)} |K_{F(\sqrt{\Delta_\nu})}(x,y)|^2 \,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u,\end{aligned}$$

as desired. □

Now we justify that Δ_ν has the finite propagation speed property. As in the introduction, for any $r > 0$, we set

$$\mathcal{E}_r = \{F \in \mathcal{S}(\mathbb{R}) : F \text{ even, } \operatorname{supp} \hat{F} \subseteq [-r,r]\}. \tag{4.34}$$

Lemma 4.17 *The family of operators* $\{\cos(t\sqrt{\Delta_\nu})\}_{t\in\mathbb{R}}$ *has finite propagation speed with respect to the distance* ϱ*, that is,*

$$\operatorname{supp}(\cos(t\sqrt{\Delta_\nu})f) \subseteq \{(x,u) \in G_\nu : \varrho((x,u), \operatorname{supp} f) \le |t|\} \qquad \forall t \in \mathbb{R} \tag{4.35}$$

for all $f \in L^2(G_\nu)$*. Moreover, for all* $r > 0$*,*

$$\operatorname{supp} K_{F(\Delta_\nu)} \subseteq \overline{B_{G_\nu}}(0,r) \qquad \forall F \in \mathcal{E}_r.$$

Proof In light of Lemma 4.3, we can apply Proposition 6.3 to the operator $\Delta_\nu = \nabla^+\nabla$ with Neumann domain on the manifold $\mathring{G}_\nu$, and obtain the finite propagation speed property (4.35). To be precise, Proposition 6.3 gives the analogue of (4.35) with $\mathring{G}_\nu$ in place of G_ν, and supports interpreted accordingly. However, $\mathring{G}_\nu$ is a dense open subset with full measure in G_ν; so, for any $f \in L^2(G_\nu) = L^2(\mathring{G}_\nu)$, the support of f in G_ν is the closure in G_ν of its support in $\mathring{G}_\nu$, and (4.35) follows as stated.

Now fix an $F \in \mathcal{E}_r(\mathbb{R})$ for some $r > 0$. By the Fourier inversion formula (cf. [13, Lemma 2.1]),

$$F(\sqrt{\Delta_\nu}) = \frac{1}{2\pi}\int_{-r}^{r} \hat{F}(t)\cos(t\sqrt{\Delta_\nu})\,\mathrm{d}t, \tag{4.36}$$

whence, by (4.35),

$$\operatorname{supp} F(\sqrt{\Delta_\nu})f \subseteq \{z \in G_\nu : \varrho(z, \operatorname{supp} f) \le r\}, \qquad f \in L^2(G_\nu). \tag{4.37}$$

Since $F(\sqrt{\Delta_\nu})$ is a $\diamond_\nu$-convolution operator, it is also an integral operator on G_ν, whose integral kernel K, much as in (4.21), is given by

$$K((x,u),(y,v)) = e^{-\nu v}\ell_{(y,v)^-} K_{F(\sqrt{\Delta_\nu})}(x,u). \tag{4.38}$$

Now observe that (4.37) can be equivalently restated as

$$\operatorname{supp} K \subseteq \{((x,u),(y,v)) \in G_\nu \times G_\nu : \varrho((x,u),(y,v)) \le r\}.$$

Let us write $K_{(y,v)}(x,u) = K((x,u),(y,v))$. Thus, for any fixed $\varepsilon > 0$,

$$\operatorname{supp} K_{(y,v)} \subseteq \overline{B_{G_\nu}}(0, r+\varepsilon) \qquad \text{for almost all } (y,v) \in B_{G_\nu}(0,\varepsilon).$$

Now, by (4.38) and Lemma 4.2, it easily follows that

$$K_{(y,v)} \to K_{(0,0)} = K_{F(\sqrt{\Delta_\nu})} \text{ in } L^2(G_\nu) \text{ as } (y,v) \to (0,0),$$

whence we deduce that $\operatorname{supp} K_{(0,0)} \subseteq \overline{B_{G_\nu}}(0, r+\varepsilon)$ for any $\varepsilon > 0$, and therefore $\operatorname{supp} K_{(0,0)} \subseteq \overline{B_{G_\nu}}(0, r)$, as desired. □

5 The multiplier theorem

Here we revert to the setting of Sect. 2. So N is a 2-step stratified group as in (2.8), that is, a direct product of an abelian group $N^{(0)}$ and Métivier groups $N^{(1)}, \dots, N^{(\ell)}$, and $G = N \rtimes \mathbb{R}$ is its semidirect product extension. Recall that $\vec{d}_1$ and $\vec{d}_2$ are the vectors of dimensions of the first layers and second layers of the Métivier groups $N^{(j)}$, $j = 1, \dots, \ell$, while $d^{(0)}$ is the dimension of $N^{(0)}$; in particular, $d = d^{(0)} + |\vec{d}_1| + |\vec{d}_2|$ and $Q = d^{(0)} + |\vec{d}_1| + 2|\vec{d}_2|$ are the topological and homogeneous dimensions of N.

We aim at "lifting" the weighted estimate on N contained in Proposition 2.5 to the semidirect product G. To this purpose, we first compare weighted norms for convolution kernels in the calculus for Δ_N to analogous norms in the calculus of the Bessel operator L_ν, for appropriate choices of ν.

Proposition 5.1 *Let $\vec{0} \preccurlyeq \alpha \prec \vec{d}_2$ and $\beta \geq 0$. For any $F \in \mathcal{S}(\mathbb{R})$,*

$$\int_N |K_{F(\Delta_N)}(z)|^2 |z|_N^\beta \prod_{j=1}^{\ell} |z_j'|^{\alpha_j} \,\mathrm{d}z \lesssim_\alpha \int_0^\infty |K_{F(L_{Q-|\alpha|})}(x)|^2 x^\beta \,\mathrm{d}\mu_{Q-|\alpha|}(x).$$

Proof The proof strongly relies on ideas from [73].

Fix $\vec{0} \preccurlyeq \alpha \prec \vec{d}_2$, $\beta \geq 0$ and $F \in \mathcal{S}(\mathbb{R})$. In view of (3.18), the case $\beta = 0$ is already covered by Proposition 2.5, so we may assume $\beta > 0$. Denote $\nu = Q - |\alpha|$; observe that $\nu > Q - |\vec{d}_2| = d \geq 3$, as N is a nonabelian 2-step group.

Define $G \in \mathcal{S}_e(\mathbb{R}_+)$ by $G(\lambda) = F(\lambda^2)$, and, for $r > 0$,

$$G_r := H_\nu(\mathbb{1}_{[0,r)} H_\nu^{-1} G), \qquad G^r := H_\nu(\mathbb{1}_{[r,\infty)} H_\nu^{-1} G).$$

Thus, $G = G_r + G^r$. Moreover, by (3.17),

$$K_{G_r(\sqrt{L_\nu})} = K_{G(\sqrt{L_\nu})} \mathbb{1}_{[0,r)}, \qquad K_{G^r(\sqrt{L_\nu})} = K_{G(\sqrt{L_\nu})} \mathbb{1}_{[r,\infty)}.$$

By (3.6) and [17, eq. (10.9.4)],

$$j_x^\nu(y) = \frac{1}{\sqrt{\pi}} \int_{-1}^{1} (1-\xi^2)^{\nu/2-3/2} e^{ixy\xi} \,\mathrm{d}\xi, \qquad x, y \in \mathbb{R}.$$

Hence, if $\mathcal{F}$ denotes the Euclidean Fourier transform on $\mathbb{R}$,

$$\mathcal{F}(j_x^\nu)(\xi) = \frac{2\sqrt{\pi}}{x} \left(1 - \left|\frac{\xi}{x}\right|^2\right)_+^{(\nu-3)/2}. \tag{5.1}$$

Denote by $\widetilde{G}_r$ the even extension of G_r to $\mathbb{R}$; namely, by (3.5),

$$\widetilde{G}_r(y) = \int_0^r H_\nu^{-1} G(x)\, j_x^\nu(y)\, \mathrm{d}\mu_\nu(x)$$

for all $y \in \mathbb{R}$ (recall that j_x^ν is even). As $\nu > 3$, by (5.1) we get

$$\mathcal{F}(\widetilde{G}_r)(\xi) = 2\sqrt{\pi} \int_0^r H_\nu^{-1} G(x) \left(1 - \left|\frac{\xi}{x}\right|^2\right)_+^{(\nu-3)/2} x^{\nu-2}\, \mathrm{d}x;$$

notice that $|\xi| \le x \le r$ in the above integral, otherwise the integrand vanishes. Thus, $\operatorname{supp} \mathcal{F}(\widetilde{G}_r) \subseteq [-r, r]$.

Recall from (2.1) that Δ_N has finite propagation speed. Much as in (4.36), from $\operatorname{supp} \mathcal{F}(\widetilde{G}_r) \subseteq [-r, r]$ we then deduce that

$$\operatorname{supp} K_{G_r(\sqrt{\Delta_N})} \subseteq \{z \in N : |z|_N \le r\}.$$

Consequently, for $|z|_N > r$ we have $K_{G^r(\sqrt{\Delta_N})}(z) = K_{G(\sqrt{\Delta_N})}(z)$. Hence,

$$\begin{aligned}
&\int_N |K_{G(\sqrt{\Delta_N})}(z)|^2 |z|_N^\beta \prod_{j=1}^{\ell} |z_j'|^{\alpha_j}\, \mathrm{d}z \\
&= \int_0^\infty \beta r^{\beta-1} \int_{|z|_N > r} |K_{G(\sqrt{\Delta_N})}(z)|^2 \prod_{j=1}^{\ell} |z_j'|^{\alpha_j}\, \mathrm{d}z\, \mathrm{d}r \\
&\le \int_0^\infty \beta r^{\beta-1} \int_N |K_{G^r(\sqrt{\Delta_N})}(z)|^2 \prod_{j=1}^{\ell} |z_j'|^{\alpha_j}\, \mathrm{d}z\, \mathrm{d}r.
\end{aligned}$$

Recall that $\nu = Q - |\alpha|$. We apply Proposition 2.5 and (3.18) to obtain

$$\begin{aligned}
&\int_0^\infty \beta r^{\beta-1} \int_N |K_{G^r(\sqrt{\Delta_N})}(z)|^2 \prod_{j=1}^{\ell} |z_j'|^{\alpha_j}\, \mathrm{d}z\, \mathrm{d}r \\
&\lesssim \int_0^\infty \beta r^{\beta-1} \int_0^\infty |K_{G^r(\sqrt{L_\nu})}(x)|^2\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}r \\
&= \int_0^\infty \beta r^{\beta-1} \int_r^\infty |K_{G(\sqrt{L_\nu})}(x)|^2\, \mathrm{d}\mu_\nu(x)\, \mathrm{d}r \\
&= \int_0^\infty |K_{G(\sqrt{L_\nu})}(x)|^2 x^\beta\, \mathrm{d}\mu_\nu(x).
\end{aligned}$$

Combining the above finishes the proof. □

We can now lift the previous inequality to G and G_ν.

Corollary 5.2 *Let* $\vec{0} \preccurlyeq \alpha \prec \vec{d}_2$ *and* $\beta \ge 0$. *For any* $F \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
&\int_G |K_{F(\Delta)}(z,u)|^2 |z|_N^\beta \prod_{j=1}^{\ell} |z_j'|^{\alpha_j}\, \mathrm{d}z\, \mathrm{d}u \\
&\lesssim \int_{G_{Q-|\alpha|}} |K_{F(\Delta_{Q-|\alpha|})}(x,u)|^2 x^\beta\, \mathrm{d}\mu_{Q-|\alpha|}(x)\, \mathrm{d}u. \qquad (5.2)
\end{aligned}$$

Proof Observe that Lemma 4.13 implies

$$\begin{aligned}&\int_{G_{Q-|\alpha|}} |K_{F(\Delta_{Q-|\alpha|})}(x,u)|^2 x^\beta \,\mathrm{d}\mu_{Q-|\alpha|}(x)\,\mathrm{d}u\\ &\quad= \int_{\mathbb{R}}\int_0^\infty |K_{(\Phi F)_u(L_{Q-|\alpha|})}(x)|^2 x^\beta \,\mathrm{d}\mu_{Q-|\alpha|}(x)\,\mathrm{d}u. \qquad (5.3)\end{aligned}$$

Moreover, an analogue of Lemma 4.13 is true for Δ and Δ_N in place of Δ_ν and L_ν:

$$K_{F(\Delta)}(z,u) = K_{(\Phi F)_u(\Delta_N)}(z)$$

for almost all $(z,u) \in G$; this can be proved much in the same way as in Lemma 4.13, and is implicitly used in [54, Corollary 4.5]. This gives

$$\begin{aligned}&\int_{G_Q} |K_{F(\Delta)}(z,u)|^2 |z|_N^\beta \prod_{j=1}^{\ell} |z'_j|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u\\ &\quad= \int_{\mathbb{R}}\int_N |K_{(\Phi F)_u(\Delta_N)}(z)|^2 |z|_N^\beta \prod_{j=1}^{\ell} |z'_j|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u. \qquad (5.4)\end{aligned}$$

Now, Proposition 5.1 allows us to compare the right-hand sides of (5.3) and (5.4) and deduce the desired inequality. □

In the case $\beta = 0$, the right-hand side of (5.2) can be turned into a weighted L^2-norm of F by Proposition 4.12, thus yielding the weighted Plancherel estimate (1.9) on G discussed in the introduction.

We now combine the previous weighted estimate on G together with finite propagation speed to deduce the following $L^1 \to L^2$ bound, corresponding to (1.6). This improves [54, Proposition 5.1], where the case $\nu = Q$ of the following bound is proved; of course, this improvement depends on our assumptions on N, which are more restrictive than those in [54].

Recall from (4.34) the definition of $\mathcal{E}_r$.

Proposition 5.3 *Let* $\nu \in (d, Q]$. *Let* $F \in \mathcal{E}_r$ *for some* $r > 0$. *Then,*

$$\int_G |K_{F(\sqrt{\Delta})}(z,u)|\,\mathrm{d}z\,\mathrm{d}u \lesssim_\nu r^{[(\nu+1)/2,3/2]} \left(\int_0^\infty |F(\lambda)|^2 \lambda^{[3,\nu+1]} \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/2}.$$

Proof As $\nu \in (d, Q]$, we can choose α so that $\vec{0} \preccurlyeq \alpha \prec \vec{d}_2$ and $\nu = Q - |\alpha|$. Recall from (2.17) that Δ satisfies finite propagation speed, thus $\operatorname{supp} K_{F(\Delta)} \subseteq \overline{B_G}(0_G, r)$.

We begin with the case $r \leq 1$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}&\|K_{F(\sqrt{\Delta})}\|_{L^1(G)}\\ &\quad\leq \left(\int_{B_G(0_G,r)} \prod_{j=1}^{\ell} |z'_j|^{-\alpha_j}\,\mathrm{d}z\,\mathrm{d}u\right)^{1/2} \left(\int_G |K_{F(\Delta)}(z,u)|^2 \prod_{j=1}^{\ell} |z'_j|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u\right)^{1/2}.\end{aligned}$$

Recall that $\alpha \prec \vec{d}_2 \prec \vec{d}_1$ by (2.12). Hence, Corollary 2.7, Corollary 5.2 with $\beta = 0$, and Proposition 4.12 with $\nu = Q - |\alpha|$ yield

$$\|K_{F(\sqrt{\Delta})}\|_{L^1(G)} \lesssim_\nu r^{(\nu+1)/2} \left(\int_0^\infty |F(\lambda)|^2 \lambda^{[3,\nu+1]} \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/2}.$$

For $r > 1$, notice that, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\left(\int_G |K_{F(\sqrt{\Delta})}(z,u)|\,\mathrm{d}z\,\mathrm{d}u\right)^2 \le \int_{B_G(0_G,r)} &(1+|z|_N^{Q-|\alpha|})^{-1}\prod_{j=1}^{\ell}|z_j'|^{-\alpha_j}\,\mathrm{d}z\,\mathrm{d}u\\ &\times\left(\int_G |K_{F(\sqrt{\Delta})}(z,u)|^2\prod_{j=1}^{\ell}|z_j'|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u\right.\\ &\left.+\int_G |K_{F(\sqrt{\Delta})}(z,u)|^2|z|_N^{Q-|\alpha|}\prod_{j=1}^{\ell}|z_j'|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u\right).\end{aligned}$$

Now, as above,

$$\int_G |K_{F(\sqrt{\Delta})}(z,u)|^2\prod_{j=1}^{\ell}|z_j'|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u \lesssim_\nu \int_0^\infty |F(\lambda)|^2\,\lambda^{[3,\nu+1]}\,\frac{\mathrm{d}\lambda}{\lambda}.$$

Moreover, by Corollary 2.7,

$$\int_{B_G(0,r)} (1+|z|_N^{Q-|\alpha|})^{-1}\prod_{j=1}^{\ell}|z_j'|^{-\alpha_j}\,\mathrm{d}z\,\mathrm{d}u \lesssim_\nu r^2.$$

Further, by Corollary 5.2 with $\beta = Q-|\alpha| = \nu$,

$$\int_G |K_{F(\sqrt{\Delta})}(z,u)|^2|z|_N^{Q-|\alpha|}\prod_{j=1}^{\ell}|z_j'|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u \lesssim_\nu \int_{G_\nu} |K_{F(\Delta_\nu)}(x,u)|^2x^\nu\,\mathrm{d}\mu_\nu(x)\,\mathrm{d}u.$$

By Lemma 4.17 we have $\operatorname{supp} K_{F(\Delta_\nu)} \subseteq \overline{B_{G_\nu}}(0,r)$. Thus, Corollary 4.16 and Proposition 4.12 imply

$$\int_G |K_{F(\sqrt{\Delta})}(z,u)|^2|z|_N^{Q-|\alpha|}\prod_{j=1}^{\ell}|z_j'|^{\alpha_j}\,\mathrm{d}z\,\mathrm{d}u \lesssim_\nu r\int_0^\infty |F(\lambda)|^2\,\lambda^{[3,\nu+1]}\,\frac{\mathrm{d}\lambda}{\lambda},$$

and the desired estimate follows. □

We can now obtain the following improvement of [54, Propositions 5.3 and 5.5].

Proposition 5.4 *Let $F \in L^2(\mathbb{R})$ be supported in $[-4,4]$. Then*

$$\sup_{z\in G}\int_G |K_{F(t\Delta)}(y,z)|\,(1+t^{-1/2}\varrho(y,z))^{\varepsilon}\,\mathrm{d}y \lesssim_{s,\varepsilon} \|F\|_{L^2_s}$$

for all $\varepsilon \ge 0$, and $s,t > 0$ satisfying one of the following conditions:

- *$t \ge 1$ and $s > 3/2+\varepsilon$;*
- *$t \le 1$ and $s > d/2+\varepsilon$.*

Moreover,

$$\int_G |K_{F(t\Delta)}(x,y) - K_{F(t\Delta)}(x,z)|\,\mathrm{d}x \lesssim_s t^{-1/2}\varrho(y,z)\|F\|_{L^2_s}$$

for all $y,z \in G$ and $s,t > 0$ satisfying the above condition with $\varepsilon = 0$.

Proof The case $t \geq 1$ is already contained in [54]. For $t \leq 1$, we choose $\nu \in (d, Q]$ so that $s > \nu/2 + \varepsilon$; then the proof runs exactly in the same way as in [54], with Proposition 5.3 applied in place of [54, Proposition 5.1].

We point out that the functions f_ℓ resulting from the frequency decomposition of $f(\lambda) = F(\lambda^2)$ given by [54, Lemma 5.2] are in the Schwartz class: indeed, from the original proof in [30, Lemma (1.3)] we see that $\hat{f}_\ell = \chi_\ell \hat{f}$ for appropriate cutoffs $\chi_\ell \in C_c^\infty(\mathbb{R})$, and $\hat{f}$ is analytic because f is compactly supported. So the Schwartz class assumption, implicit in the definition of $\mathcal{E}_r$, on the spectral multiplier in Proposition 5.3 here is not an obstacle. □

Based on the above we can conclude the proof of our main result.

Proof of Theorem 1.2 The proof goes just as in [54, proof of Theorem 1.1], using Proposition 5.4 in place of [54, Propositions 5.3 and 5.5], and relies on the Calderón–Zygmund theory developed in [32, 54]. □

6 Appendix: Boundary conditions and finite propagation speed

In this section we recall some basic terminology and results related to self-adjoint extensions of smooth second-order differential operators in divergence form on smooth manifolds, which are used throughout the paper. While our main application is to second-order operators acting on scalar functions, their expression in divergence form naturally leads to considering first-order operators, such as gradient and divergence, which act on vector-valued functions, or more general sections of vector bundles. We find it therefore more natural to work directly in the setup of operators between spaces of sections of vector bundles.

As we shall see, this allows us to discuss in a unified and relatively simple way a number of results about self-adjointness and domains, as well as present a general derivation of the finite propagation speed property under Dirichlet or Neumann conditions based on the first-order approach of [56].

We introduce some of the setup and notation from [12], to which we refer for additional details. Let M be a smooth manifold (without boundary) equipped with a smooth measure. Fix two smooth vector bundles $\mathscr{E}, \mathscr{F}$ on M equipped with fibre inner products. We use notation such as $C^\infty(\mathscr{E})$, $C_c^\infty(\mathscr{E})$, $L^2(\mathscr{E})$ to denote the spaces of sections of $\mathscr{E}$ which are smooth, smooth and compactly supported, square integrable. Let $\nabla : C^\infty(\mathscr{E}) \to C^\infty(\mathscr{F})$ be a first-order differential operator, and $\nabla^+ : C^\infty(\mathscr{F}) \to C^\infty(\mathscr{E})$ be its formal adjoint. As usual, we can extend ∇ and ∇^+ to spaces of distributions.

We define the *maximal domain* and the *minimal domain* for ∇ on $L^2(\mathscr{E})$:

$$\mathfrak{D}_{\max}(\nabla) = \{f \in L^2(\mathscr{E}) : \nabla f \in L^2(\mathscr{F})\}, \qquad \mathfrak{D}_{\min}(\nabla) = \overline{C_c^\infty(\mathscr{E})}^{\mathfrak{D}_{\max}(\nabla)}, \tag{6.1}$$

where $\mathfrak{D}_{\max}(\nabla)$ is equipped with the graph norm of ∇. By a standard mollification technique (see, e.g., [12, Propositions 5.5 and 6.1]), one can see that

$$\mathfrak{D}_{\min}(\nabla) = \overline{\mathfrak{D}_{\max,c}(\nabla)}^{\mathfrak{D}_{\max}(\nabla)}, \tag{6.2}$$

where $\mathfrak{D}_{\max,c}(\nabla)$ is the set of compactly supported elements of $\mathfrak{D}_{\max}(\nabla)$. Analogous considerations apply to the formal adjoint ∇^+. Moreover, formal and Hilbert space adjoints are related as follows:

$$(\nabla|_{\mathfrak{D}_{\min}(\nabla)})^* = \nabla^+|_{\mathfrak{D}_{\max}(\nabla^+)} \qquad \text{and} \qquad (\nabla|_{\mathfrak{D}_{\max}(\nabla)})^* = \nabla^+|_{\mathfrak{D}_{\min}(\nabla^+)}. \tag{6.3}$$

We are interested in the second-order divergence-form operator $\nabla^+\nabla$ associated with ∇. We define the *Dirichlet domain* and the *Neumann domain* for $\nabla^+\nabla$ as

$$\begin{aligned} \mathfrak{D}_{\mathrm{Dir}}(\nabla^+\nabla) &= \{f \in \mathfrak{D}_{\min}(\nabla) : \nabla f \in \mathfrak{D}_{\max}(\nabla^+)\}, \\ \mathfrak{D}_{\mathrm{Neu}}(\nabla^+\nabla) &= \{f \in \mathfrak{D}_{\max}(\nabla) : \nabla f \in \mathfrak{D}_{\min}(\nabla^+)\}. \end{aligned} \tag{6.4}$$

In light of (6.3) and [72, Theorem X.25], both $\nabla^+\nabla|_{\mathfrak{D}_{\mathrm{Dir}}(\nabla^+\nabla)}$ and $\nabla^+\nabla|_{\mathfrak{D}_{\mathrm{Neu}}(\nabla^+\nabla)}$ are nonnegative self-adjoint operators on $L^2(\mathscr{E})$. As $(\nabla^+)^+ = \nabla$, analogous considerations apply to the operator $\nabla\nabla^+$.

Remark 6.1 The Dirichlet domain in (6.4) is the same as the domain of the Friedrichs extension of $\nabla^+\nabla|_{C_c^\infty(\mathscr{E})}$ (cf. [72, Theorems X.23 and X.25]). Of course, when $\nabla^+\nabla|_{C_c^\infty(\mathscr{E})}$ is essentially self-adjoint, the Dirichlet and Neumann domains are the same; however, here we are also interested in the case where essential self-adjointness may fail.

The *control distance* ϱ_∇ associated with ∇ can be defined by (cf. [12, p. 175])

$$\varrho_\nabla(x, y) = \inf \left\{ \int_0^1 P_\nabla^*(\gamma'(t))\, \mathrm{d}t : \gamma \in \mathrm{AC}([0, 1]; M),\ \gamma(0) = x,\ \gamma(1) = y \right\}$$

for all $x, y \in M$, where we write $\mathrm{AC}([0, 1]; M)$ for the set of absolutely continuous curves in M with domain $[0, 1]$, and P_Δ for the fibre seminorm on T^*M associated with ∇, namely,

$$P_\nabla(\xi) = |\sigma_1(\nabla)(\xi)|_{\mathrm{op}}, \qquad \xi \in T^*M,$$

while $\sigma_1(\nabla) \in C^\infty(\mathrm{Hom}(T^*M, \mathrm{Hom}(\mathscr{E}, \mathscr{F})))$ is the symbol of ∇, and P_∇^* denotes the extended fibre norm on TM dual to P_∇ (see [12, p. 153] for details).

Remark 6.2 As discussed in [12, Section 8.5], this definition of the control distance ϱ_∇ includes, for appropriate choices of ∇, that of Riemannian and sub-Riemannian distances on M. In particular, assume that the bundles $\mathscr{E}$ and $\mathscr{F}$ are the trivial bundles of ranks 1 and r, while ∇ has the form

$$\nabla = \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}$$

for a system $X_1, \dots, X_r$ of linearly independent smooth real vector fields on M. If $r = \dim M$ and g is the Riemannian metric tensor on M that makes $X_1, \dots, X_r$ an orthonormal frame, then $P_\nabla^*(v) = \sqrt{g(v, v)}$ for all $v \in TM$, and ϱ_∇ is the Riemannian distance on M induced by g. More generally, if $r \leq \dim M$ and the vector fields $X_1, \dots, X_r$ are bracket-generating, then ϱ_∇ is the Carnot–Carathéodory distance associated to the system of vector fields (cf. also [61, 82]). In each of these cases, the control distance ϱ_∇ induces on M the manifold topology (on this, see also [12, Proposition 4.23]).

We say that a family $\{W_t\}_{t\in\mathbb{R}}$ of bounded operators on $L^2(\mathscr{E})$ has *finite propagation speed* with respect to ϱ_∇ if

$$\operatorname{supp} W_t f \subseteq \{x \in M : \varrho_\nabla(x, \operatorname{supp} f) \leq |t|\} \qquad \forall f \in L^2(\mathscr{E}),\ t \in \mathbb{R}. \tag{6.5}$$

We are especially interested in the case $W_t = \cos(t\sqrt{\nabla^+\nabla})$, corresponding to the *wave propagator* associated to $\nabla^+\nabla$; in this case, when (6.5) holds, we also say that $\nabla^+\nabla$ satisfies the finite propagation speed property.

Of course, the definition of the cosine family $\{\cos(t\sqrt{\nabla^+\nabla})\}_{t\in\mathbb{R}}$ in terms of spectral calculus on $L^2(\mathscr{E})$ becomes meaningful only once a self-adjoint extension of $\nabla^+\nabla$ has been chosen, and indeed the validity of the finite propagation speed property may depend on this choice. This is easily seen, e.g., by taking $\nabla = -\partial_x$ on the interval $M = (0, 1)$ with Lebesgue measure, and considering $\nabla^+\nabla = -\partial_x^2$ with periodic boundary conditions: here finite propagation speed is violated, because, roughly speaking, due to periodicity, a wave that crosses one endpoint of the interval immediately re-enters from the other endpoint, effectively travelling at infinite speed (cf. [72, Section X.1, Example 1]).

Possibly to avoid such problems, several results on finite propagation speed in the literature are either stated under some completeness assumption on the underlying (sub-)Riemannian manifold (see, e.g., [57] or the examples in [75, Sections 4–8]), effectively preventing a solution with compactly supported initial datum from reaching the boundary in finite time, or proved only for a restricted time interval depending on the datum (see, e.g., [12, Section 7]).

In contrast, the general approach of [56] can be readily used to deduce finite propagation speed for any second-order divergence form operator, for appropriate choices of the self-adjoint extension, without any completeness assumptions on the manifold. (In fact, the results of [56] go well beyond our setup, as they do not require self-adjointness or smooth coefficients, and also apply to operators on L^p for $p \neq 2$.) The examples presented in [56, Section 5] only consider elliptic differential operators; however, as the next statement shows, ellipticity can be replaced by a weaker topological assumption on the control distance, which holds in greater generality, including for sub-elliptic operators (see Remark 6.2).

Proposition 6.3 *Assume that the control distance associated with ∇ induces the manifold topology on M. Consider $\nabla^+\nabla$ with either Neumann domain or Dirichlet domain. Then $\{\cos(t\sqrt{\nabla^+\nabla})\}_{t\in\mathbb{R}}$ has finite propagation speed with respect to the control distance.*

Proof We are going to deduce the finite propagation speed property from [56, Theorem 3.1]. To this purpose, we shall write, roughly speaking, the second-order operator $\nabla^+\nabla$ as the square of a first-order differential operator.

More precisely, we define a "matrix operator" $\not{D} : C^\infty(\mathscr{E} \oplus \mathscr{F}) \to C^\infty(\mathscr{E} \oplus \mathscr{F})$ by

$$\not{D} = \begin{pmatrix} 0 & \nabla^+ \\ \nabla & 0 \end{pmatrix}.$$

Clearly $\not{D}$ is formally self-adjoint, i.e. $\not{D}^+ = \not{D}$, and moreover

$$\not{D}^2 = \begin{pmatrix} \nabla^+\nabla & 0 \\ 0 & \nabla\nabla^+ \end{pmatrix}.$$

We now construct an appropriate self-adjoint extension of $\not{D}$. Let

$$\mathfrak{D}_{\mathrm{mix}}(\not{D}) = \mathfrak{D}_{\max}(\nabla) \oplus \mathfrak{D}_{\min}(\nabla^+). \tag{6.6}$$

By (6.3) we obtain

$$(\not{D}|_{\mathfrak{D}_{\mathrm{mix}}(\not{D})})^* = \begin{pmatrix} 0 & \nabla^+|_{\mathfrak{D}_{\min}(\nabla^+)} \\ \nabla_{\mathfrak{D}_{\max}(\nabla)} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & \nabla^+|_{\mathfrak{D}_{\min}(\nabla^+)} \\ \nabla_{\mathfrak{D}_{\max}(\nabla)} & 0 \end{pmatrix}$$
$$= \not{D}|_{\mathfrak{D}_{\mathrm{mix}}(\not{D})}.$$

Thus, $\not{D}|_{\mathfrak{D}_{\mathrm{mix}}(\not{D})}$ is self-adjoint. Consequently, the square operator $\not{D}^2|_{\mathfrak{D}_{\mathrm{mix}}(\not{D}^2)}$ is self-adjoint as well (see [72, Theorem X.25]), where

$$\begin{aligned}
&\mathfrak{D}_{\mathrm{mix}}(\not{D}^2)\\
&:= \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{D}_{\mathrm{mix}}(\not{D}) : \not{D} \begin{pmatrix} f \\ g \end{pmatrix} \in \mathfrak{D}_{\mathrm{mix}}(\not{D}) \right\}\\
&= \{f \in \mathfrak{D}_{\max}(\nabla) : \nabla f \in \mathfrak{D}_{\min}(\nabla^+)\} \oplus \{g \in \mathfrak{D}_{\min}(\nabla^+) : \nabla^+ g \in \mathfrak{D}_{\max}(\nabla)\}\\
&= \mathfrak{D}_{\mathrm{Neu}}(\nabla^+\nabla) \oplus \mathfrak{D}_{\mathrm{Dir}}(\nabla\nabla^+),
\end{aligned}$$

by (6.6), (6.4) and the definition of $\not{D}$. Hence,

$$\not{D}^2|_{\mathfrak{D}_{\mathrm{mix}}(\not{D}^2)} = \begin{pmatrix} \nabla^+\nabla|_{\mathfrak{D}_{\mathrm{Neu}}(\nabla^+\nabla)} & 0 \\ 0 & \nabla\nabla^+|_{\mathfrak{D}_{\mathrm{Dir}}(\nabla\nabla^+)} \end{pmatrix},$$

and consequently

$$\cos\left(t\not{D}|_{\mathfrak{D}_{\mathrm{mix}}(\not{D})}\right) = \begin{pmatrix} \cos\left(t\sqrt{\nabla^+\nabla|_{\mathfrak{D}_{\mathrm{Neu}}(\nabla^+\nabla)}}\right) & 0 \\ 0 & \cos\left(t\sqrt{\nabla\nabla^+|_{\mathfrak{D}_{\mathrm{Dir}}(\nabla\nabla^+)}}\right) \end{pmatrix}, \tag{6.7}$$

since $\cos(t\cdot)$ is an even analytic function.

We want to apply [56, Theorem 3.1] to the operator $\not{D}|_{\mathfrak{D}_{\mathrm{mix}}(\not{D})}$ on $L^2(\mathscr{E} \oplus \mathscr{F})$, where the underlying manifold M is equipped with the distance ϱ_∇. As $\not{D}|_{\mathfrak{D}_{\mathrm{mix}}(\not{D})}$ is self-adjoint, it generates a C_0 group $(e^{it\not{D}|_{\mathfrak{D}_{\mathrm{mix}}(\not{D})}})_{t\in\mathbb{R}}$ of unitary operators on $L^2(\mathscr{E} \oplus \mathscr{F})$, so the first assumption of the theorem is satisfied.

We now check the second assumption of [56, Theorem 3.1]. Let $\mathrm{Lip}(M)$ be the space of bounded real-valued ϱ_∇-Lipschitz functions; in other words, $\eta \in \mathrm{Lip}(M)$ if and only if $\eta \in L^\infty(M; \mathbb{R})$ and

$$\|\eta\|_{\mathrm{Lip}} := \sup_{x,y\in M,\, x\neq y} \frac{|\eta(x) - \eta(y)|}{\varrho_\nabla(x, y)} < \infty.$$

Since ϱ_∇ is varietal, i.e. ϱ_∇ induces the topology on the manifold M, by [12, Propositions 5.2 and 5.4] we have the equivalent characterisation

$$\begin{aligned}
\mathrm{Lip}(M) &= \{\eta \in L^\infty(M; \mathbb{R}) : \nabla^\sigma \eta \in L^\infty(\mathrm{Hom}(\mathscr{E}, \mathscr{F}))\},\\
\|\eta\|_{\mathrm{Lip}} &= \|\nabla^\sigma \eta\|_{L^\infty} \qquad \forall \eta \in \mathrm{Lip}(M),
\end{aligned} \tag{6.8}$$

where ∇^σ denotes the "symbol operator" (see [12, Section 2.2]), that is, the first-order differential operator that appears in the Leibniz rule for ∇:

$$\nabla(\eta f) = (\nabla^\sigma \eta) f + \eta \nabla f, \tag{6.9}$$

where η is a scalar-valued function and f is a section of $\mathscr{E}$ with suitable differentiability and integrability properties [12, Proposition 3.7]. By (6.1), (6.8) and (6.9), we deduce that

$$\begin{aligned}
\mathrm{Lip}(M) \cdot \mathfrak{D}_{\max}(\nabla) &\subseteq \mathfrak{D}_{\max}(\nabla),\\
\mathrm{Lip}(M) \cdot \mathfrak{D}_{\max,c}(\nabla) &\subseteq \mathfrak{D}_{\max,c}(\nabla),
\end{aligned}$$

whence, by (6.2), we also get

$$\mathrm{Lip}(M) \cdot \mathfrak{D}_{\min}(\nabla) \subseteq \mathfrak{D}_{\min}(\nabla).$$

Since the control distances associated to the three operators ∇, ∇^+, $\not{D}$ are the same (see [12, p. 181]), the corresponding Lipschitz space $\mathrm{Lip}(M)$ is the same, so similar inclusions are true for ∇^+. Thus, by (6.6) we obtain

$$\mathrm{Lip}(M) \cdot \mathfrak{D}_{\mathrm{mix}}(\slashed{D}) \subseteq \mathfrak{D}_{\mathrm{mix}}(\slashed{D}).$$

In particular, for all $\eta \in \mathrm{Lip}(M)$ and $f \in \mathfrak{D}_{\mathrm{mix}}(\slashed{D})$, we have $\eta f \in \mathfrak{D}_{\mathrm{mix}}(\slashed{D})$; moreover, by (6.9) and (6.8),

$$\|[\eta, \slashed{D}]f\|_{L^2} = \|(\slashed{D}^\sigma \eta) f\|_{L^2} \leq \|\slashed{D}^\sigma \eta\|_{L^\infty} \|f\|_{L^2} = \|\eta\|_{\mathrm{Lip}} \|f\|_{L^2},$$

and finally $[\eta, [\eta, \slashed{D}]] = [\eta, \slashed{D}^\sigma \eta] = 0$, as the scalar multiplication operator η commutes with the matrix multiplication operator $\slashed{D}^\sigma \eta$.

Hence, we can apply [56, Theorem 3.1] to $\slashed{D}$ on the domain $\mathfrak{D}_{\mathrm{mix}}(\slashed{D})$, and deduce finite propagation speed for the one-parameter group $\{e^{it\slashed{D}|_{\mathfrak{D}_{\mathrm{mix}}(\slashed{D})}}\}_{t\in\mathbb{R}}$. As $e^{it\slashed{D}} + e^{-it\slashed{D}} = 2\cos(t\slashed{D})$, by (6.7) we obtain finite propagation speed for the cosine families $\left\{\cos\left(t\sqrt{\nabla^+\nabla|_{\mathfrak{D}_{\mathrm{Neu}}(\nabla^+\nabla)}}\right)\right\}_{t\in\mathbb{R}}$ and $\left\{\cos\left(t\sqrt{\nabla\nabla^+|_{\mathfrak{D}_{\mathrm{Dir}}(\nabla\nabla^+)}}\right)\right\}_{t\in\mathbb{R}}$. This gives the desired result for $\nabla^+\nabla$ with Neumann domain; by exchanging the roles of ∇ and ∇^+ in the above argument, we also obtain the result for $\nabla^+\nabla$ with Dirichlet domain. ◻

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