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Transference of bilinear multipliers on Lorentz spaces

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Abstract
We study DeLeeuw type transference theorems for multi-linear multiplier operators on the Lorentz spaces. To be detail, we show that, under some mild conditions on m, a bilinear multiplier operator $T_{m,1}(f, g)$ is bounded on the Lorentz space in $\mathbb{R}^n$ if and only if its periodic version $\widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})$ is bounded on the Lorentz space in the n-torus T^n uniformly on $\varepsilon > 0$. Most significantly, we prove that these two operators share the same operator norm. We also obtain the same results on their restriction versions and their maximal versions $T_m^*(f, g)$ and $\widetilde{T}_m^*(\widetilde{f}, \widetilde{g})$. The previous method by Kenig and Tomas to treat the sub-linear operator $T_m^*(f)$ is to linearize the operator and then invoke the duality argument. This approach seems complicated and difficult to be used when we study the sub-bilinear operator $T_m^*(f, g)$. Thus, we will use a simpler, but different method. Our results are substantial improvements and extensions of many known theorems.

Keywords Bilinear multipliers · Maximal operator · Transference · Restriction of multiplier · Lorentz spaces

Mathematics Subject Classification 42B15 · 42B20 · 42B25

1 Introduction

The classical multiplier operator on $\mathbb{R}^n$ is defined initially on $f \in S(\mathbb{R}^n)$ in the integral form

$$T_{m,\varepsilon}(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) m(\varepsilon\xi) \mathrm{e}^{i\langle \xi, x\rangle} \mathrm{d}\xi,$$

where $\varepsilon > 0$ and m is a function, which is called the multiplier of the operator. For the same m, the corresponding multiplier operator on the n torus T^n is defined via the Fourier series

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as

$$\widetilde{T}_{m,\varepsilon}(\widetilde{f})(x) = \sum_{k\in\mathbb{Z}^n} a_k m(\varepsilon k)\mathrm{e}^{i\langle k,x\rangle},$$

where we initially assume $\widetilde{f} \in C^{\infty}(T^n)$ so that $\widetilde{f}$ equals to its Fourier series

$$\widetilde{f}(x) = \sum_{k\in\mathbb{Z}^n} a_k \mathrm{e}^{i\langle k,x\rangle}.$$

It is well-known that

$$\lim_{\varepsilon\to 0} \left\| \widetilde{T}_{m,\varepsilon}(\widetilde{f}) - m(0)\widetilde{f} \right\|_{L^p(T^n)} = 0 \tag{1}$$

if and only if

$$\left\| \widetilde{T}_{m,\varepsilon}(\widetilde{f}) \right\|_{L^p(T^n)} \preceq \left\| \widetilde{f} \right\|_{L^p(T^n)} \tag{2}$$

uniformly on $\varepsilon > 0$. On the other hand, the famous DeLeeuw theorem [6] says that, under some mild conditions on m, $\widetilde{T}_{m,\varepsilon}$ is bounded on $L^p(T^n)$ uniformly on $\varepsilon > 0$ if and only if (see [1, 13])

$$\left\| T_{m,1}(f) \right\|_{L^p(\mathbb{R}^n)} \preceq \| f \|_{L^p(\mathbb{R}^n)}. \tag{3}$$

This result is quite significant since it shows that the classical convergence problem of the Fourier series is equivalent to an L^p boundedness of the corresponding operator on $\mathbb{R}^n$.

Further more, for the maximal operators

$$\widetilde{T}_m^*(\widetilde{f})(x) = \sup_{\varepsilon>0} \left| \widetilde{T}_{m,\varepsilon}(\widetilde{f})(x) \right|$$

and

$$T_m^*(f)(x) = \sup_{\varepsilon>0} \left| T_{m,\varepsilon}(f)(x) \right|,$$

Kenig and Tomas in [10] proved that T_m^* is bounded on $L^p(\mathbb{R}^n)$ if and only if $\widetilde{T}_m^*$ is bounded on $L^p(T^n)$.

DeLeeuw's theorem, as well as the result by Kenig and Tomas, have many extensions. Among numerous papers in this direction, the reader may see [8] for the extension of DeLeeuw's theorem on the Lorentz spaces $L^{p,q}$; see [3, 11] for the extension of DeLeeuw's theorem on the Hardy spaces H^p, $0 < p \leq 1$.

Now, we turn to study the bilinear multiplier operator on $\mathbb{R}^n$ defined by

$$T_{m,\varepsilon}(f,g)(x) = \int_{\mathbb{R}^{2n}} \widehat{f}(\xi_1)\widehat{g}(\xi_2) m(\varepsilon\xi_1, \varepsilon\xi_2)\mathrm{e}^{i\langle \xi_1+\xi_2,x\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

where $x, \xi_1, \xi_2 \in \mathbb{R}^n$. Again, in the definition, f and g are initially assumed to be Schwartz functions.

The corresponding bilinear multiplier operator on the n-torus T^n is defined as

$$\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x) = \sum_{k_1\in\mathbb{Z}^n}\sum_{k_2\in\mathbb{Z}^n} a_{k_1} b_{k_2} m(\varepsilon k_1, \varepsilon k_2)\mathrm{e}^{i\langle k_1+k_2,x\rangle},$$

where

$$\widetilde{f}(x) = \sum_{k_1\in\mathbb{Z}^n} a_{k_1}\mathrm{e}^{i\langle k_1,x\rangle},\ \widetilde{g}(x) = \sum_{k_2\in\mathbb{Z}^n} b_{k_2}\mathrm{e}^{i\langle k_2,x\rangle}$$

are assumed initially C^∞ functions on T^n, and $\varepsilon > 0$.

For simplicity of notation in our discussion, we denote

$$T_m = T_{m,1}, \quad \widetilde{T}_m = \widetilde{T}_{m,1}.$$

The study of Fourier analysis in multi-linear setting is very active in last two decades. Among many non-trivial extensions from the linear setting, we recall following two theorems, which are the first DeLeeuw type theorems on the multi-linear multiplier operators.

Theorem A [7] *Suppose that* $m \in L^\infty \cap C(\mathbb{R}^{2n})$, $1 \le p, q, r \le \infty$. *If*

$$\left\| \widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g}) \right\|_{L^p(T^n)} \le \widetilde{A} \left\| \widetilde{f} \right\|_{L^r(T^n)} \|\widetilde{g}\|_{L^q(T^n)}, \; 1/p = 1/r + 1/q,$$

for all $\widetilde{f}$ *and* $\widetilde{g}$ *uniformly on* $\varepsilon > 0$, *where* $\widetilde{A} > 0$, *then*

$$\left\| T_{m,1}(f, g) \right\|_{L^p(\mathbb{R}^n)} \le A \left\| f \right\|_{L^r(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}, \; 1/p = 1/r + 1/q,$$

for all f *and* g, *where* $0 < A \le \widetilde{A}$.

Theorem B [7] *Suppose that* $m \in L^\infty \cap C(\mathbb{R}^{2n})$, $1 \le p, q, r \le \infty$. *If*

$$\left\| \widetilde{T}_m^*(\widetilde{f}, \widetilde{g}) \right\|_{L^p(T^n)} \le \widetilde{B} \left\| \widetilde{f} \right\|_{L^r(T^n)} \|\widetilde{g}\|_{L^q(T^n)}, \; 1/p = 1/r + 1/q,$$

then

$$\left\| T_m^*(f, g) \right\|_{L^p(\mathbb{R}^n)} \le B \left\| f \right\|_{L^r(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}, \; 1/p = 1/r + 1/q,$$

where $0 < B \le \widetilde{B}$.

In Theorem B, the maximal operators is defined, same as the linear case, by

$$T_m^*(f, g)(x) = \sup_{\varepsilon > 0} \left| T_{m,\varepsilon}(f, g)(x) \right|$$

and

$$\widetilde{T}_m^*(\widetilde{f}, \widetilde{g})(x) = \sup_{\varepsilon > 0} \left| \widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})(x) \right|.$$

Inspired by [7], many research papers related to multi-linear DeLeeuw's theorem have appeared in the literature. For this information, the reader may check the citations on [7] in MathSciNet. In [2], Blasco and Villarroya extended Theorem A from the Lebesgue spaces to the Lorentz spaces. However, the result of Blasco and Villarroya is on the case $n = 1$, because of their methodology (see also [12]). Based on this observation, and we feel that it is interesting to have multi-linear DeLeeuw type theorems on Lorentz space for all dimensions n, the purpose of this article is to extend Theorems A and B to Lorentz spaces for all n. More importantly, our method allows us to show that the operator norms on Lorentz spaces of $T_{m,\varepsilon}$, T_m and $\widetilde{T}_{m,\varepsilon}$ are identically the same.

To state our main results, we first recall the definition of Lorentz spaces. Let (X, μ) be a measure space. For a measurable function f, its distribution λ_f is defined by

$$\lambda_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\}.$$

The non-decreasing rearrangement of f, f_* is defined by

$$f_*(t) = \inf\left\{\alpha : \lambda_f(\alpha) \le t\right\}, t > 0.$$

The Lorentz space $L^{p,q}(X)$, $1 \le p, q \le \infty$, is the set of all measurable functions f on X satisfying

$$\|f\|_{L^{p,q}(X)} < \infty,$$

where

$$\|f\|_{L^{p,q}(X)} = \left\{ \frac{q}{p} \int_0^\infty \left[t^{1/p} f_*(t) \right]^q \frac{\mathrm{d}t}{t} \right\}^{1/q}, 1 \le q < \infty$$

and

$$\|f\|_{L^{p,\infty}(X)} = \sup_{t>0} t^{1/p} f_*(t).$$

In fact, $p = \infty$ only $q = \infty$ makes sense. It is well known (see [9])

$$\|f\|_{L^{\infty,\infty}(X)} = \|f\|_{L^\infty(X)},$$
$$\|f\|_{L^{p,p}(X)} = \|f\|_{L^p(X)}.$$

Define the triplets

$$\overrightarrow{p} = (p, p_1, p_2) \in \mathbb{R}_+^3, \overrightarrow{q} = (q, q_1, q_2) \in \mathbb{R}_+^3.$$

Let T be a bilinear operator

$$T : L^{p_1,q_1}(\mathbb{R}^n) \times L^{p_2,q_2}(\mathbb{R}^n) \to L^{p,q}(\mathbb{R}^n).$$

We define the operator norm

$$\|T\|_{\overrightarrow{p},\overrightarrow{q}} = \inf \left\{ c : \|T(f, g)\|_{L^{p,q}(\mathbb{R}^n)} \le c \right\},$$

where the infimum is taken over all Schwartz functions f and g satisfying

$$\|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} = 1, \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)} = 1.$$

Similarly, let $\widetilde{T}$ be a bilinear operator

$$\widetilde{T} : L^{p_1,q_1}(T^n) \times L^{p_2,q_2}(T^n) \to L^{p,q}(T^n).$$

We define the operator norm

$$\|\widetilde{T}\|_{\overrightarrow{p},\overrightarrow{q}} = \inf \left\{ c : \left\|\widetilde{T}(\widetilde{f}, \widetilde{g})\right\|_{L^{p,q}(T^n)} \le c \right\},$$

where the infimum is taken over all C^∞ functions $\widetilde{f}$ and $\widetilde{g}$ satisfying

$$\left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^n)} = 1, \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)} = 1.$$

We will establish the following two theorems.

Theorem 1 *Suppose that $m \in L^\infty \cap C(\mathbb{R}^{2n})$, $1 \le p, q, p_i, q_i \le \infty, i = 1, 2$ and $1/p = 1/p_1 + 1/p_2$. Then the following three statementsare equivalent.*

$$(a) \left\|T_{m,\varepsilon}(f, g)\right\|_{L^{p,q}(\mathbb{R}^n)} \le \left\|T_{m,\varepsilon}\right\|_{\overrightarrow{p},\overrightarrow{q}} \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)},$$

for all f and g.

$$(b) \left\|T_{m,1}(f, g)\right\|_{L^{p,q}(\mathbb{R}^n)} \le \left\|T_{m,1}\right\|_{\overrightarrow{p},\overrightarrow{q}} \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)},$$

for all f *and* g.

$$(c)\ \left\|\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})\right\|_{L^{p,q}(T^n)} \le \left\|\widetilde{T}_{m,\varepsilon}\right\|_{\overrightarrow{p},\overrightarrow{q}} \left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^n)} \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)},$$

for all $\widetilde{f}$ *and* $\widetilde{g}$ *uniformly on* $\varepsilon > 0$.

Moreover, we have

$$\left\|T_{m,\varepsilon}\right\|_{\overrightarrow{p},\overrightarrow{q}} = \left\|T_{m,1}\right\|_{\overrightarrow{p},\overrightarrow{q}} = \left\|\widetilde{T}_{m,\varepsilon}\right\|_{\overrightarrow{p},\overrightarrow{q}}.$$

Theorem 2 *Suppose that* $m \in L^\infty \cap C(\mathbb{R}^{2n})$, $1 \le p, q, p_i, q_i \le \infty$, $i = 1, 2$ *and* $1/p = 1/p_1 + 1/p_2$. *Then*

$$\left\|T_m^*(f, g)\right\|_{L^{p,q}(\mathbb{R}^n)} \le \left\|T_m^*\right\|_{\overrightarrow{p},\overrightarrow{q}} \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)},$$

for all f *and* g *if and only if*

$$\left\|\widetilde{T}_m^*(\widetilde{f},\widetilde{g})\right\|_{L^{p,q}(T^n)} \le \left\|\widetilde{T}_m^*\right\|_{\overrightarrow{p}\,\overrightarrow{q}} \left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^n)} \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)},$$

for all $\widetilde{f}$ *and* $\widetilde{g}$. *Moreover,*

$$\left\|T_m^*\right\|_{\overrightarrow{p},\overrightarrow{q}} = \left\|\widetilde{T}_m^*\right\|_{\overrightarrow{p},\overrightarrow{q}}.$$

Remark 1 The proof of Theorem 1 is based on refinements of the methods used in [7], together with some estimates involving analysis on measure. This method allows us to obtain Theorem 2 easily. In [10], in order to show the transference between $T_m^*(f)$ and $\widetilde{T}_m^*(\widetilde{f})$, Kenig and Tomas create linearizations of the maximal operators so that they are able to use the duality to complete the proof. However, in our case (Theorem 2) this method seems quite complicated and hard to be inherited. As a different method from those in [10], Theorem 2 however can be easily obtained as a consequence from the process in proving Theorem 1.

Remark 2 We state our theorems on bilinear multiplier operators. But our results are easily extended to multi-linear cases.

This paper is organized as follows. Sect. 2 we give some basic lemmas in order to prove main theorems. The proofs of Theorems 1 and 2 will be presented in Sect 3. In Sect 4, we give some extensions and discuss the transference between certain bilinear pseudo-differential operators and restrictions of bilinear multiplier operator. Finally we give some notes in Sect. 5. Throughout out this paper, the notation $A \preceq B$ means that there is a positive constant C independent of all essential variables such that $A \le CB$. Also we write $A \approx B$ to mean that there are two positive constants C_1 and C_2 independent of all essential variables such that $C_1 A \le B \le C_2 A$.

2 Basic lemmas

We need several lemmas. The first lemma was proved in [8]. For convenience to the reader, we give its proof here.

Lemma 1 *Suppose that* $\{f_n\}$ *is a sequence of nonnegative functions on the measure space* (X, μ) *and that* f *is a nonnegative function on the measure space* (Y, ν). *If* $\{a_n\}$ *is a positive sequence such that*

$$\lim_{n\to\infty} a_n \mu\{x \in X : f_n(x) > \alpha\} \ge \nu\{y \in Y : f(y) > \alpha\}$$

for all $\alpha > 0$, *then we have*

$$f_* (t) \le \liminf_{n\to\infty} (f_n)_* (t/a_n)$$

for all $t > 0$.

Proof Let

$$E_n (t/a_n) = \{\alpha > 0 : \mu \{x \in X : f_n (x) > \alpha\} \le t/a_n\}$$

and

$$E (t) = \{\alpha > 0 : \nu \{y \in Y : f (y) > \alpha\} \le t\}.$$

If $\beta \notin E (t)$ then

$$\nu \{y \in Y : f (y) > \beta\} > t.$$

By the assumption, there is an $N > 0$, such that if $n > N$ then

$$\mu \{x \in X : f_n (x) > \beta\} > t/a_n.$$

This says that $\beta \notin E_n (t/a_n)$. Thus we obtain the inclusion

$$E_n (t/a_n) \subset E (t)$$

for $n > N$. As a consequence we now obtain

$$f_*(t) = \inf E (t) \le \inf E_n (t/a_n) = (f_n)_* (t/a_n)$$

for $n > N$. The lemma is proved. □

The following lemma can be regarded as a Fatou Lemma on measure.

Lemma 2 *(A Fatou type lemma) Let* f_n *be a sequence of measurable functions.*

$$\mu \left\{x \in X : \liminf_{n\to\infty} |f_n(x)| > \alpha\right\} \le \liminf_{n\to\infty} \mu \{x \in X : |f_n(x)| > \alpha\}$$

for any $\alpha > 0$.

Proof It follows trivially from Fatou's lemma when applied to $g_n = \chi_{\{x\in X:|f_n(x)|>\alpha\}}$ and observing that $\liminf_{n\to\infty} g_n = \chi_{\{x\in X:\liminf_{n\to\infty}|f_n(x)|>\alpha\}}$. □

Let $[-\pi, \pi]^n = Q$ be the fundamental cube of T^n, that is

$$\int_{T^n} \widetilde{f} = \int_Q \widetilde{f}$$

for any integrable function on T^n. We let $\Psi \in S(\mathbb{R}^n)$ be a radial function and satisfy

$$\text{supp}\Psi \subset \Omega_K, 0 \le \Psi(x) \le 1, \text{and } \Psi(x) \equiv 1 \text{on } Q,$$

where

$$\Omega_K = [-\pi - 2\pi/K, \pi + 2\pi/K]^n.$$

for a large integer K. By this notation, we see

$$Q \subset \Omega_K,$$

and

$$\Omega_K \to Q \quad \text{as } K \to \infty.$$

For any positive integer N, denote $\Psi_{1/N}$ as the function such that

$$\Psi_{1/N}(x) = \Psi(x/N).$$

For this defined Ψ we have the following lemma.

Lemma 3 *Let m be bounded and continuous. For $C^\infty(T^n)$ functions*

$$\widetilde{f}(x) = \sum_{k_1 \in \mathbb{Z}^n} a_{k_1} e^{i\langle k_1, x\rangle}$$

and

$$\widetilde{g}(x) = \sum_{k_2 \in \mathbb{Z}^n} b_{k_2} e^{i\langle k_2, x\rangle},$$

the error function

$$E_{N,\varepsilon}(\widetilde{f}, \widetilde{g})(x) = \Psi(x/N)^2 \widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})(x) - T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})(x)$$

satisfies

$$\lim_{N\to\infty} E_{N,\varepsilon}(\widetilde{f}, \widetilde{g})(x) = 0$$

uniformly on $x \in \mathbb{R}^n$, for fixed $\varepsilon > 0$.

Proof The detail of the proof to Lemma 3 is contained in the proof of Theorem 3. (one also can see [7]). □

Lemma 4 *Let $X = \mathbb{R}^n$. For $\varepsilon > 0$ define*

$$f_\varepsilon(x) = f(\varepsilon x).$$

Then

$$\|f_\varepsilon\|_{L^{p,q}(\mathbb{R}^n)} = \varepsilon^{-n/p} \|f\|_{L^{p,q}(\mathbb{R}^n)}.$$

Proof By an easy scaling argument. □

3 Proof of main theorem

3.1 Proof of Theorem 1

First we show that (b) implies (a). By changing variables, we see that

$$\begin{aligned} T_{m,\varepsilon}(f, g)(x) &= \int_{\mathbb{R}^{2n}} \widehat{f_\varepsilon}(\xi_1)\widehat{g_\varepsilon}(\xi_2) m(\xi_1, \xi_2) \mathrm{e}^{i\langle \xi_1+\xi_2, \frac{x}{\varepsilon}\langle} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\ &= T_m(f_\varepsilon, g_\varepsilon)\left(\frac{x}{\varepsilon}\right), \end{aligned}$$

where

$$f_\varepsilon(x) = f(\varepsilon x), \, g_\varepsilon(x) = g(\varepsilon x).$$

Since

$$1/p = 1/p_1 + 1/p_2,$$

by Lemma 4 and the assumption, we have that

$$\begin{aligned}\left\|T_{m,\varepsilon}(f,g)\right\|_{L^{p,q}(\mathbb{R}^n)} &= \varepsilon^{n/p}\left\|T_m(f_\varepsilon,g_\varepsilon)\right\|_{L^{p,q}(\mathbb{R}^n)}\\ &\le \varepsilon^{n/p}\left\|T_m\right\|_{\vec{p},\vec{q}}\left\|f_\varepsilon\right\|_{L^{p_1,q_1}(\mathbb{R}^n)}\left\|g_\varepsilon\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}\\ &= \left\|T_m\right\|_{\vec{p},\vec{q}}\left\|f\right\|_{L^{p_1,q_1}(\mathbb{R}^n)}\left\|g\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}.\end{aligned}$$

This clearly shows that, for any $\varepsilon > 0$,

$$\left\|T_{m,\varepsilon}\right\|_{\vec{p},\vec{q}} \le \left\|T_m\right\|_{\vec{p},\vec{q}}. \tag{4}$$

To show that (a) implies (b), we observe

$$T_m(f,g)(x) = T_{m,\varepsilon}(f_{\varepsilon^{-1}}, g_{\varepsilon^{-1}})(\varepsilon x).$$

So, by Lemma 4,

$$\begin{aligned}\left\|T_m(f,g)\right\|_{L^{p,q}(\mathbb{R}^n)} &= \varepsilon^{-n/p}\left\|T_{m,\varepsilon}(f_{\varepsilon^{-1}},g_{\varepsilon^{-1}})\right\|_{L^{p,q}(\mathbb{R}^n)}\\ &\le \left\|T_{m,\varepsilon}\right\|_{\vec{p},\vec{q}}\varepsilon^{-n/p}\left\|f_{\varepsilon^{-1}}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)}\left\|g_{\varepsilon^{-1}}\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}\\ &= \left\|T_{m,\varepsilon}\right\|_{\vec{p},\vec{q}}\left\|f\right\|_{L^{p_1,q_1}(\mathbb{R}^n)}\left\|g\right\|_{L^{p_2,q_2}(\mathbb{R}^n)},\end{aligned}$$

which yields

$$\left\|T_m\right\|_{\vec{p},\vec{q}} \le \left\|T_{m,\varepsilon}\right\|_{\vec{p},\vec{q}} \tag{5}$$

for any $\varepsilon > 0$.

(4) and (5) give

$$\left\|T_m\right\|_{\vec{p},\vec{q}} = \left\|T_{m,\varepsilon}\right\|_{\vec{p},\vec{q}} \tag{6}$$

for any $\varepsilon > 0$.

We continue to prove that (a) implies (c). Using a density argument we may consider $\widetilde{f}, \widetilde{g} \in C^\infty(T^n)$. Note that $\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x)$ is a periodic function. For any $\alpha > 0$, and fixed $\varepsilon > 0$,

$$\begin{aligned}&\left|\left\{x \in Q : \left|\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x)\right| > \alpha\right\}\right|\\ &\quad = N^{-n}\left|\left\{x \in NQ : \left|\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x)\right| > \alpha\right\}\right|.\end{aligned}$$

Since $\Psi\left(\frac{x}{N}\right) \equiv 1$ if $x \in NQ$, using Lemma 3, we may write

$$\begin{aligned}&\left|\left\{x \in Q : \left|\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x)\right| > \alpha\right\}\right|\\ &\quad = N^{-n}\left|\left\{x \in NQ : \left|\Psi\left(\frac{x}{N}\right)^2 \widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x)\right| > \alpha\right\}\right|\\ &\quad \le N^{-n}\left|\left\{x \in NQ : \left|T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})(x)\right| > \theta\alpha\right\}\right|\\ &\qquad + N^{-n}\left|\left\{x \in NQ : \left|E_{N,\varepsilon}(\widetilde{f},\widetilde{g})(x)\right| > (1-\theta)\alpha\right\}\right|,\end{aligned}$$

where θ is a fixed small number in the interval (0, 1).

By Lemma 3, we choose sufficiently large N such that

$$N^{-n}\left|\left\{x \in NQ : \left|E_{N,\varepsilon}(\widetilde{f},\widetilde{g})(x)\right| > (1-\theta)\alpha\right\}\right| = 0.$$

It further yields

$$\begin{aligned}&\left|\left\{x \in Q : \left|\widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})(x)\right| > \alpha\right\}\right| \\ &\quad \le \lim_{N\to\infty} N^{-n} \left|\left\{x \in NQ : \left|T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})(x)\right| > \theta\alpha\right\}\right|.\end{aligned}$$

By Lemma 1, we obtain that

$$\left(\widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})\right)_*(t) \le \lim_{N\to\infty} \left(\theta^{-1} T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})\right)_*(N^n t).$$

Without loss of generality, here we assume that the limit on the right side of the inequality above exists. If $1 \le q < \infty$, then we have that

$$\begin{aligned}&\left\|\widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})\right\|_{L^{p,q}(T^n)} \\ &\le \lim_{N\to\infty} \theta^{-1} \left[\frac{q}{p}\int_0^\infty \left(t^{1/p}\left(T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})\right)_*(N^n t)\right)^q \frac{\mathrm{d}t}{t}\right]^{1/q} \\ &= \theta^{-1} \lim_{N\to\infty} N^{-n/p} \left[\frac{q}{p}\int_0^\infty \left(t^{1/p}\left(T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})\right)_*(t)\right)^q \frac{\mathrm{d}t}{t}\right]^{1/q}.\end{aligned}$$

Here

$$\begin{aligned}&\left[\frac{q}{p}\int_0^\infty \left(t^{1/p}\left(T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})\right)_*(t)\right)^q \frac{\mathrm{d}t}{t}\right]^{1/q} \\ &\quad = \left\|T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})\right\|_{L^{p,q}(\mathbb{R}^n)} \\ &\quad \le \|T_m\|_{\vec{p},\vec{q}} \left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)} \left\|\Psi_{1/N}\widetilde{g}\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}.\end{aligned}$$

If $q = \infty$, then we have that

$$\begin{aligned}&\left\|\widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})\right\|_{L^{p,\infty}(T^n)} \\ &\le \sup_{t>0} t^{1/p} \lim_{N\to\infty} \left(\theta^{-1} T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})\right)_*(N^n t) \\ &\le \theta^{-1} \lim_{N\to\infty} \sup_{t>0} t^{1/p} T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})_*(N^n t) \\ &= \theta^{-1} \lim_{N\to\infty} N^{-n/p} \sup_{t>0} t^{1/p} T_{m,\varepsilon}(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})_*(t) \\ &\le \|T_m\|_{\vec{p},\vec{q}}\, \theta^{-1} \lim_{N\to\infty} N^{-n/p} \left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)} \left\|\Psi_{1/N}\widetilde{g}\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, for all $1 \le q \le \infty$ we obtain

$$\begin{aligned}&\left\|\widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})\right\|_{L^{p,q}(T^n)} \\ &\le \|T_m\|_{\vec{p},\vec{q}}\, \theta^{-1} \lim_{N\to\infty} N^{-n/p} \left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)} \left\|\Psi_{1/N}\widetilde{g}\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}.\end{aligned} \tag{7}$$

In the inequality in (7), we need to further estimate $\left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)}$ and $\left\|\Psi_{1/N}\widetilde{g}\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}$.

Clearly we only need to estimate $\left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)}$, since two estimates share the same idea.

By the support condition of $\Psi_{1/N}$, we have

$$\begin{aligned}
&\lambda_{\Psi_{1/N}\widetilde{f}}(\alpha) \\
&\quad= \left|\left\{x \in [-N\pi - 2N\pi/K, N\pi + 2N\pi/K]^n : \left|\Psi_{1/N}(x)\,\widetilde{f}(x)\right| > \alpha\right\}\right| \\
&\quad= \left|\left\{x \in [-N\pi, N\pi]^n : \left|\widetilde{f}(x)\right| > \alpha\right\}\right| \\
&\qquad+ \left|\left\{x \in [-N\pi - 2N\pi/K, -N\pi]^n : \left|\Psi_{1/N}(x)\,\widetilde{f}(x)\right| > \alpha\right\}\right| \\
&\qquad+ \left|\left\{x \in [N\pi, N\pi + 2N\pi/K]^n : \left|\Psi_{1/N}(x)\,\widetilde{f}(x)\right| > \alpha\right\}\right|.
\end{aligned}$$

First,

$$\begin{aligned}
\left|\left\{x \in [-N\pi, N\pi]^n : \left|\widetilde{f}(x)\right| > \alpha\right\}\right| &= N^n \left|\left\{x \in [-\pi, \pi]^n : \left|\widetilde{f}(x)\right| > \alpha\right\}\right| \\
&= N^n \lambda_{\widetilde{f}}(\alpha).
\end{aligned}$$

Also, we choose N for which $\frac{N}{K}$ are positive integers. It yields that

$$\begin{aligned}
&\left|\left\{x \in [-N\pi - 2N\pi/K, -N\pi]^n : \left|\Psi_{1/N}(x)\,\widetilde{f}(x)\right| > \alpha\right\}\right| \\
&\quad\le \left|\left\{x \in [-N\pi - 2N\pi/K, -N\pi]^n : \left|\widetilde{f}(x)\right| > \alpha\right\}\right| \\
&\quad= \left(\frac{N}{K}\right)^n \lambda_{\widetilde{f}}(\alpha).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|\left\{x \in [N\pi, N\pi + 2N\pi/K]^n : \left|\Psi_{1/N}(x)\,\widetilde{f}(x)\right| > \alpha\right\}\right| \\
&\quad\le \left(\frac{N}{K}\right)^n \lambda_{\widetilde{f}}(\alpha).
\end{aligned}$$

By this computation, we obtain that

$$\left(\Psi_{1/N}\widetilde{f}\right)_*(t) \le \left(\widetilde{f}\right)_*\left(tN^{-n}(1 + 2k^{-n})^{-1}\right).$$

We first assume $1 \le q_1 < \infty$. In this case, by the definition,

$$\begin{aligned}
\left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)} &\le \left[\frac{q_1}{p_1}\int_0^\infty \left(t^{1/p_1}\left(\widetilde{f}\right)_*(tN^{-n}(1 + 2k^{-n})^{-1})\right)^{q_1} \frac{\mathrm{d}t}{t}\right]^{1/q_1} \\
&= N^{n/p_1}\left(1 + 2K^{-n}\right)^{1/p_1}\left[\frac{q_1}{p_1}\int_0^\infty \left(t^{1/p_1}(\widetilde{f})_*(t)\right)^{q_1} \frac{\mathrm{d}t}{t}\right]^{1/q_1} \\
&= N^{n/p_1}\left(1 + 2K^{-n}\right)^{1/p_1}\left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^n)}.
\end{aligned}$$

Next, if $q_1 = \infty$ then

$$\begin{aligned}
\left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,\infty}(\mathbb{R}^n)} &\le \sup_{t>0} t^{1/p_1}\left(\widetilde{f}\right)_*\left(tN^{-n}(1 + 2k^{-n})^{-1}\right) \\
&\le N^{n/p_1}\left(1 + 2K^{-n}\right)^{1/p_1}\sup_{t>0} t^{1/p_1}\left(\widetilde{f}\right)_*(t) \\
&= N^{n/p_1}\left(1 + 2K^{-n}\right)^{1/p_1}\left\|\widetilde{f}\right\|_{L^{p_1,\infty}(T^n)}.
\end{aligned}$$

We now obtain that

$$\left\|\Psi_{1/N}\widetilde{f}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)} \le N^{n/p_1}\left(1 + 2K^{-n}\right)^{1/p_1}\left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^n)} \tag{8}$$

for all $1 \le q_1 \le \infty$, and similarly,

$$\left\| \Psi_{1/N} \widetilde{g} \right\|_{L^{p_2,q_2}(\mathbb{R}^n)} \le N^{n/p_2} \left(1 + 2K^{-n}\right)^{1/p_2} \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)} \tag{9}$$

for all $1 \le q_2 \le \infty$.

Combining (7), (8), (9), we finally obtain that

$$\begin{aligned} \left\| \widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g}) \right\|_{L^{p,q}(T^n)} &\le \theta^{-1} \|T_m\|_{\overrightarrow{p}, \overrightarrow{q}} \lim_{N \to \infty} \left(1 + 2K^{-n}\right)^{1/p_1} \\ &\times \left(1 + 2K^{-n}\right)^{1/p_2} \left\| \widetilde{f} \right\|_{L^{p_1,q_1}(T^n)} \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)} . \end{aligned}$$

By letting first $K \to \infty$ then $\theta \to 1$, we obtain that

$$\left\| \widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g}) \right\|_{L^{p,q}(T^n)} \le \|T_m\|_{\overrightarrow{p}, \overrightarrow{q}} \left\| \widetilde{f} \right\|_{L^{p_1,q_1}(T^n)} \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)} ,$$

which clearly yields

$$\left\| \widetilde{T}_{m,\varepsilon} \right\|_{\overrightarrow{p}, \overrightarrow{q}} \le \|T_m\|_{\overrightarrow{p}, \overrightarrow{q}} . \tag{10}$$

Next, we will show that (c) implies (a). By a density argument, we may assume f, $g \in C_c^\infty(\mathbb{R}^n)$. We obtain their dilation-periodic versions

$$\widetilde{f}^\varepsilon(x) = \varepsilon^{-n} \sum_{k \in \mathbb{Z}^n} f\left(\frac{x + 2\pi k}{\varepsilon}\right),$$

$$\widetilde{g}^\varepsilon(x) = \varepsilon^{-n} \sum_{k \in \mathbb{Z}^n} g\left(\frac{x + 2\pi k}{\varepsilon}\right).$$

By the Poisson summation formula (see [13]),

$$\widetilde{f}^\varepsilon(x) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(\varepsilon k) \mathrm{e}^{i\langle x,k\rangle},$$

$$\widetilde{g}^\varepsilon(x) = \sum_{k \in \mathbb{Z}^n} \widehat{g}(\varepsilon k) \mathrm{e}^{i\langle x,k\rangle}.$$

Let η be the characteristic function of Q. We now claim that for each $x \in \mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} \varepsilon^{2n} \widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^\varepsilon, \widetilde{g}^\varepsilon\right)(\varepsilon x) = T_m(f, g)(x).$$

In fact,

$$\varepsilon^{2n} \widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^\varepsilon, \widetilde{g}^\varepsilon\right)(\varepsilon x) = \varepsilon^{2n} \sum_{k_1 \in \mathbb{Z}^n} \sum_{k_2 \in \mathbb{Z}^n} \widehat{f}(\varepsilon k_1) \widehat{g}(\varepsilon k_2) m(\varepsilon k_1, \varepsilon k_2) \mathrm{e}^{i\langle \varepsilon x, k_1 + k_2\rangle}$$

is a Riemann sum of

$$\int_{\mathbb{R}^{2n}} \widehat{f}(\xi_1) \widehat{g}(\xi_2) m(\xi_1, \xi_2) \mathrm{e}^{i\langle x, \xi_1 + \xi_2\rangle} \mathrm{d}\xi_1 \mathrm{d}\xi_2 = T_m(f, g)(x).$$

We choose $\{\varepsilon\}$ as a discrete sequence going to 0. By Lemma 2, for any $\alpha > 0$,

$$\begin{aligned} &\left|\left\{x \in \mathbb{R}^n : |T_m(f, g)(x)| > \alpha\right\}\right| \\ &\le \lim_{\varepsilon \to 0} \left|\left\{x \in \mathbb{R}^n : \left|\varepsilon^{2n} \widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^\varepsilon, \widetilde{g}^\varepsilon\right)(\varepsilon x) \eta(\varepsilon x)\right| > \alpha\right\}\right| \\ &= \lim_{\varepsilon \to 0} \varepsilon^{-n} \left|\left\{x \in \mathbb{R}^n : \left|\varepsilon^{2n} \widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^\varepsilon, \widetilde{g}^\varepsilon\right)(x) \eta(x)\right| > \alpha\right\}\right| . \end{aligned}$$

By Lemma 1, without loss of generality, here we assume that the limit on the right side of the inequality below exists. We have that

$$\begin{aligned}&(T_m(f,g))_*(t)\\&\quad\le \lim_{\varepsilon\to 0}\left(\inf\left\{\alpha>0:\left|\left\{x\in\mathbb{R}^n:\left|\varepsilon^{2n}\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)(\varepsilon x)\eta(\varepsilon x)\right|>\alpha\right\}\right|<t\right\}\right).\end{aligned}$$

But

$$\begin{aligned}&\inf\left\{\alpha>0:\left|\left\{x\in\mathbb{R}^n:\left|\varepsilon^{2n}\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)(\varepsilon x)\eta(\varepsilon x)\right|>\alpha\right\}\right|<t\right\}\\&\quad=\inf\left\{\alpha>0:\left|\left\{x\in\mathbb{R}^n:\left|\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)(x)\eta(x)\right|>\varepsilon^{-2n}\alpha\right\}\right|<\varepsilon^n t\right\}\\&\quad=\varepsilon^{2n}\inf\left\{\beta>0:\left|\left\{x\in Q:\left|\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)(x)\right|>\beta\right\}\right|<\varepsilon^n t\right\}.\end{aligned}$$

Thus we obtain

$$(T_m(f,g))_*(t)\le \lim_{\varepsilon\to 0}\varepsilon^{2n}\left(\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)\right)_*\left(\varepsilon^n t\right). \tag{11}$$

If $q\neq\infty$, from (11) and the definition we see that

$$\begin{aligned}\|T_m(f,g)\|_{L^{p,q}(\mathbb{R}^n)}&\le \lim_{\varepsilon\to 0}\left\{\int_0^\infty\left(t^{1/p}\varepsilon^{2n}\left(\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)\right)_*\left(\varepsilon^n t\right)\right)^q\frac{\mathrm{d}t}{t}\right\}^{1/q}\\&=\lim_{\varepsilon\to 0}\varepsilon^{-n/p}\varepsilon^{2n}\left\{\int_0^\infty\left(t^{1/p}\left(\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)\right)_*(t)\right)^q\frac{\mathrm{d}t}{t}\right\}^{1/q}\\&\le \lim_{\varepsilon\to 0}\varepsilon^{-n/p}\varepsilon^{2n}\left\|\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)\right\|_{L^{p,q}(T^n)}\\&\le \lim_{\varepsilon\to 0}\varepsilon^{-n/p}\varepsilon^{2n}\left\|\widetilde{T}_{m,\varepsilon}\right\|_{\overrightarrow{p},\overrightarrow{q}}\left\|\widetilde{f}^{\varepsilon}\right\|_{L^{p_1,q_1}(T^n)}\left\|\widetilde{g}^{\varepsilon}\right\|_{L^{p_2,q_2}(T^n)}.\end{aligned}$$

If $q=\infty$, from (11) and the definition we obtain that

$$\begin{aligned}\|T_m(f,g)\|_{L^{p,\infty}(\mathbb{R}^n)}&\le \sup_{t>0}t^{1/p}\lim_{\varepsilon\to 0}\varepsilon^{2n}\left(\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)\right)_*\left(\varepsilon^n t\right)\\&\le \lim_{\varepsilon\to 0}\varepsilon^{2n}\sup_{t>0}t^{1/p}\left(\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)\right)_*\left(\varepsilon^n t\right)\\&=\lim_{\varepsilon\to 0}\varepsilon^{2n}\varepsilon^{-n/p}\sup_{t>0}t^{1/p}\left(\widetilde{T}_{m,\varepsilon}\left(\widetilde{f}^{\varepsilon},\widetilde{g}^{\varepsilon}\right)\right)_*(t)\\&\le \lim_{\varepsilon\to 0}\varepsilon^{-n/p}\varepsilon^{2n}\left\|\widetilde{T}_{m,\varepsilon}\right\|_{\overrightarrow{p},\overrightarrow{q}}\left\|\widetilde{f}^{\varepsilon}\right\|_{L^{p_1,q_1}(T^n)}\left\|\widetilde{g}^{\varepsilon}\right\|_{L^{p_2,q_2}(T^n)}.\end{aligned}$$

We notice that since $f\in C_c^\infty(\mathbb{R}^n)$,

$$\widetilde{f}^{\varepsilon}(x)=\varepsilon^{-n}f\left(\frac{x}{\varepsilon}\right)$$

if $x\in Q$ and ε is sufficiently small. Therefore

$$\lambda_{\widetilde{f}^{\varepsilon}}(\alpha)=\left|\left\{x\in Q:\left|f\left(\frac{x}{\varepsilon}\right)\right|>\alpha\varepsilon^n\right\}\right|$$

and

$$\left(\widetilde{f}^{\varepsilon}\right)_*(t)=\left(\varepsilon^{-n}f\right)_*(t\varepsilon^{-n}).$$

An easy computation gives that when $q_1\neq\infty$,

$$\left\|\widetilde{f}^{\varepsilon}\right\|_{L^{p_1,q_1}(T^n)}=\left\{\frac{q_1}{p_1}\int_0^\infty\left(t^{1/p_1}\left(\varepsilon^{-n}f\right)_*(t\varepsilon^{-n})\right)^{q_1}\frac{\mathrm{d}t}{t}\right\}^{1/q_1}$$

$$= \varepsilon^{-n} \varepsilon^{n/p_1} \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)}$$

and when $q_1 = \infty$,

$$\begin{aligned} \left\| \widetilde{f}^{\varepsilon} \right\|_{L^{p_1,\infty}(T^n)} &= \sup_{t>0} t^{1/p_1} \left(\varepsilon^{-n} f\right)_* (t\varepsilon^{-n}) \\ &= \varepsilon^{n/p_1} \sup_{t>0} t^{1/p_1} \left(\varepsilon^{-n} f\right)_* (t) \\ &= \varepsilon^{-n} \varepsilon^{n/p_1} \|f\|_{L^{p_1,\infty}(\mathbb{R}^n)} . \end{aligned}$$

Similarly, for any $1 \leq q_1 \leq \infty$,

$$\left\| \widetilde{g}^{\varepsilon} \right\|_{L^{p_2,q_2}(T^n)} = \varepsilon^{-n} \varepsilon^{n/p_2} \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)} .$$

Finally,

$$\varepsilon^{-n/p} \varepsilon^{2n} \left\| \widetilde{g}^{\varepsilon} \right\|_{L^{p_2,q_2}(T^n)} \left\| \widetilde{f}^{\varepsilon} \right\|_{L^{p_1,q_1}(T^n)} = \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)} .$$

Combining all estimates, we complete the proof. The process of the proof clearly yields

$$\|T_m\|_{\vec{p},\vec{q}} \leq \left\| \widetilde{T}_{m,\varepsilon} \right\|_{\vec{p},\vec{q}} \tag{12}$$

Particularly, combining (10), (12) and (6), we have

$$\left\| T_{m,\varepsilon} \right\|_{\vec{p},\vec{q}} = \|T_m\|_{\vec{p},\vec{q}} = \left\| \widetilde{T}_{m,\varepsilon} \right\|_{\vec{p},\vec{q}}$$

for all $\varepsilon > 0$.

3.2 Proof of Theorem 2

The proof of Theorem 2 follows the same idea used in the proof of Theorem 1. We consider

$$\widetilde{T}^*_{m,R}(\widetilde{f}, \widetilde{g})(x) = \sup_{1/R \leq \varepsilon \leq R} \left| \widetilde{T}_{m,\varepsilon}(\widetilde{f}, \widetilde{g})(x) \right| .$$

Since

$$\lim_{R\to\infty} \widetilde{T}^*_{m,R}(\widetilde{f}, \widetilde{g})(x) = \widetilde{T}^*_m(\widetilde{f}, \widetilde{g})(x)$$

monotonically, we have

$$\left\| \widetilde{T}^*_m(\widetilde{f}, \widetilde{g}) \right\|_{L^{p,q}(T^n)} = \lim_{R\to\infty} \left\| \widetilde{T}^*_{m,R}(\widetilde{f}, \widetilde{g}) \right\|_{L^{p,q}(T^n)} .$$

To prove $\left\| T^*_m \right\|_{\vec{p},\vec{q}} \geq \left\| \widetilde{T}^*_m \right\|_{\vec{p},\vec{q}}$, it needs to show

$$\left\| \widetilde{T}^*_{m,R}(\widetilde{f}, \widetilde{g}) \right\|_{L^{p,q}(T^n)} \leq \left\| T^*_m \right\|_{\vec{p},\vec{q}} \left\| \widetilde{f} \right\|_{L^{p_1,q_1}(T^n)} \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)}$$

uniformly on R. The proof is the same as before. The only place that we need to pay attention is that when we apply Lemma 3, we observe

$$\lim_{N\to\infty} \sup_{1/R \leq \varepsilon \leq R} E_{N,\varepsilon}(\widetilde{f}, \widetilde{g})(x) = 0$$

uniformly on $x \in \mathbb{R}^n$.

To prove $\left\| T^*_m \right\|_{\vec{p},\vec{q}} \leq \left\| \widetilde{T}^*_m \right\|_{\vec{p},\vec{q}}$, we follow the same proof as that in Theorem 1, and notice that

$$\varepsilon^{2n} \widetilde{T}_{m,\varepsilon,\delta} \left(\widetilde{f}^{\varepsilon}, \widetilde{g}^{\varepsilon} \right) (\varepsilon x)$$

$$= \varepsilon^{2n} \sum_{k_1 \in \mathbb{Z}^n} \sum_{k_2 \in \mathbb{Z}^n} \widehat{f}(\varepsilon k_1)\widehat{g}(\varepsilon k_2) m(\varepsilon\delta k_1, \varepsilon\delta k_2) \mathrm{e}^{i\langle \varepsilon x, k_1+k_2\rangle}$$

is a Riemann sum of

$$\int_{\mathbb{R}^{2n}} \widehat{f}(\xi_1)\widehat{g}(\xi_2) m(\delta\xi_1, \delta\xi_2) \mathrm{e}^{i\langle x, \xi_1+\xi_2\rangle} \mathrm{d}\xi_1 \mathrm{d}\xi_2 = T_{m,\delta}(f, g)(x)$$

for any $\delta > 0$. We leave the details to the interested reader.

4 Pseudo-differential operators and restriction of bilinear multiplier

4.1 Transference of pseudo-differential operators

Following the linear case [14], we may consider the bilinear pseudo-differential operators

$$\widetilde{S}_m(\widetilde{f}, \widetilde{g})(x) = \sum_{k_1 \in \mathbb{Z}^n} \sum_{k_2 \in \mathbb{Z}^n} a_{k_1} b_{k_2} m(x, k_1, k_2) \mathrm{e}^{i\langle k_1+k_2, x\rangle},$$

$$S_m(f, g)(x) = \int_{\mathbb{R}^{2n}} \widehat{f}(\xi_1)\widehat{g}(\xi_2) m(x, \xi_1, \xi_2) \mathrm{e}^{i\langle \xi_1+\xi_2, x\rangle} \mathrm{d}\xi_1 \mathrm{d}\xi_2,$$

where $m(x, \xi_1, \xi_2)$ satisfies

$$m(x + 2\pi, \xi_1, \xi_2) = m(x, \xi_1, \xi_2).$$

Theorem 3 *Let* $m(x, \xi_1, \xi_2) \in L^\infty \cap C\left(\mathbb{R}^{2n}\right)$ *uniformly on* x, $1 \le p, q, p_i, q_i \le \infty, i = 1, 2$. *If*

$$\|S_m(f, g)\|_{L^{p,q}(\mathbb{R}^n)} \le A \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)}, \ 1/p = 1/p_1 + 1/p_2,$$

for all f *and* g, *then*

$$\left\|\widetilde{S}_m(\widetilde{f}, \widetilde{g})\right\|_{L^{p,q}(T^n)} \le A \left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^n)} \|\widetilde{g}\|_{L^{p_2,q_2}(T^n)}, \ 1/p = 1/p_1 + 1/p_2,$$

for all $\widetilde{f}$ *and* $\widetilde{g}$.

Proof The idea is consistent with the proof that (a) deduces (c) in Theorem 1, and only the following error function needs to be considered:

$$E_N(\widetilde{f}, \widetilde{g})(x) = \Psi(x/N)^2 \widetilde{S}_m(\widetilde{f}, \widetilde{g})(x) - S_m(\Psi_{1/N}\widetilde{f}, \Psi_{1/N}\widetilde{g})(x).$$

Recalling

$$\Psi(x/N)^2 \widetilde{S}_m(\widetilde{f}, \widetilde{g})(x) = \Psi(x/N)^2 \sum_{k_1 \in \mathbb{Z}^n} \sum_{k_2 \in \mathbb{Z}^n} a_{k_1} b_{k_2} m(x, k_1, k_2) \mathrm{e}^{i\langle k_1+k_2, x\rangle},$$

where $\{a_{k_1}\}$, $\{b_{k_2}\}$ are the sets of Fourier coefficients of f, g respectively, and they all decay rapidly to 0. First, we notice

$$\begin{aligned}\Psi(x/N)^2\widetilde{S}_m(\widetilde{f},\widetilde{g})(x) &= \sum_{k_1\in\mathbb{Z}^n}\sum_{k_2\in\mathbb{Z}^n} a_{k_1}b_{k_2}\mathrm{e}^{i\langle k_1+k_2,x\rangle}N^{2n}\\ &\quad\times\int_{\mathbb{R}^{2n}} m(x,k_1,k_2)\widehat{\Psi}(N\xi_1)\widehat{\Psi}(N\xi_2)\mathrm{e}^{i\langle\xi_1+\xi_2,x\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2\\ &= \sum_{k_1\in\mathbb{Z}^n}\sum_{k_2\in\mathbb{Z}^n} a_{k_1}b_{k_2}\mathrm{e}^{i\langle k_1+k_2,x\rangle}\\ &\quad\times\int_{\mathbb{R}^{2n}} m(x,k_1,k_2)\widehat{\Psi}(\xi_1)\widehat{\Psi}(\xi_2)\mathrm{e}^{i\langle\xi_1+\xi_2,x/N\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2.\end{aligned}$$

On the other hand, we recall that

$$\begin{aligned}&S_m(\Psi_{1/N}\widetilde{f},\Psi_{1/N}\widetilde{g})(x)\\ &\quad=\int_{\mathbb{R}^{2n}} m(x,\xi_1,\xi_2)(\Psi_{1/N}\widetilde{f})\widehat{\ }(\xi_1)(\Psi_{1/N}\widetilde{g})\widehat{\ }(\xi_2)\mathrm{e}^{i\langle\xi_1+\xi_2,x\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned}$$

where we easily compute

$$\begin{aligned}(\Psi_{1/N}\widetilde{f})\widehat{\ }(\xi_1) &= \sum_{k_1\in\mathbb{Z}^n} a_{k_1}N^n\widehat{\Psi}(N(\xi_1-k_1)),\\ (\Psi_{1/N}\widetilde{g})\widehat{\ }(\xi_2) &= \sum_{k_2\in\mathbb{Z}^n} b_{k_2}N^n\widehat{\Psi}(N(\xi_2-k_2)).\end{aligned}$$

It is easy to check

$$\begin{aligned}&S_m(\Psi_{1/N}\widetilde{f},\Psi_{1/N}\widetilde{g})(x)\\ &\quad=\sum_{k_1\in\mathbb{Z}^n}\sum_{k_2\in\mathbb{Z}^n} a_{k_1}b_{k_2}\mathrm{e}^{i\langle k_1+k_2,x\rangle}\\ &\quad\times\int_{\mathbb{R}^{2n}} m(x,k_1+\frac{\xi_1}{N},k_2+\frac{\xi_2}{N})\widehat{\Psi}(\xi_1)\widehat{\Psi}(\xi_2)\mathrm{e}^{i\langle\xi_1+\xi_2,\frac{x}{N}\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2.\end{aligned}$$

Thus

$$\begin{aligned}&\left|E_N(\widetilde{f},\widetilde{g})(x)\right|\\ &\quad\le\int_{\mathbb{R}^{2n}}\sum_{k_1\in\mathbb{Z}^n}\sum_{k_2\in\mathbb{Z}^n}\left|a_{k_1}\right|\left|b_{k_2}\right|\mu_{k_1,k_2}\left(x,\frac{\xi_1}{N},\frac{\xi_2}{N}\right)\left|\widehat{\Psi}(\xi_1)\right|\left|\widehat{\Psi}(\xi_2)\right|\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned}$$

where

$$\mu_{k_1,k_2}(x,\frac{\xi_1}{N},\frac{\xi_2}{N})=\left|m\left(x,k_1+\frac{\xi_1}{N},k_2+\frac{\xi_2}{N}\right)-m(x,k_1,k_2)\right|.$$

By the dominated convergence theorem,

$$\lim_{N\to\infty}E_N(\widetilde{f},\widetilde{g})(x)=0$$

uniformly on $x\in\mathbb{R}^n$. The theorem is proved. □

4.2 Restriction of bilinear multiplier

The purpose of this subsection is to establish transference and restriction of bilinear multiplier to subspaces. These results are similar to those of DeLeeuw [6] for Fourier multipliers. The study of such transplantations was initiated by DeLeeuw [6], see also Calderón [4] and Coifman and Weiss [5]. Recall

$$T_{m,\varepsilon}(f,g)(x)=\int_{\mathbb{R}^{2n}}\widehat{f}(\xi_1)\widehat{g}(\xi_2)m(\varepsilon\xi_1,\varepsilon\xi_2)\mathrm{e}^{i\langle\xi_1+\xi_2,x\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2,$$
$$\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x)=\sum_{k_1\in\mathbb{Z}^n}\sum_{k_2\in\mathbb{Z}^n}a_{k_1}b_{k_2}m(\varepsilon k_1,\varepsilon k_2)\mathrm{e}^{i\langle k_1+k_2,x\rangle}.$$

Let d be an integer in the interval $[1,n)$. Write $\xi_i=(\xi_i^{(d)},\xi_i^{(n-d)})$, $(i=1,2)$, where $\xi_i^{(d)}$ is the d-vector of first d components of ξ_i and $\xi_i^{(n-d)}$ is the $(n-d)$-vector of last $n-d$ components of ξ_i. Similarly, we write

$$k_i=(k_i^{(d)},k_i^{(n-d)}),\ i=1,2,$$

where $k_i^{(d)}\in\mathbb{Z}^d$ and $k_i^{(n-d)}\in\mathbb{Z}^{n-d}$.

Now for a multiplier $m(\xi_1,\xi_2)$ on $\mathbb{R}^n\times\mathbb{R}^n$, we define its restriction m' on $\mathbb{R}^d\times\mathbb{R}^d$ by

$$m'(\xi_1^{(d)},\xi_2^{(d)})=m(\xi_1^{(d)},c_1,\xi_2^{(d)},c_2)$$

for any fixed $c_1,c_2\in\mathbb{R}^{n-d}$. We have the following two theorems.

Theorem 4 *Suppose that* $m\in L^\infty\cap C(\mathbb{R}^{2n})$, $1\le p,q,p_i,q_i\le\infty$, $i=1,2$. *If*

$$\left\|T_{m,\varepsilon}(f,g)\right\|_{L^{p,q}(\mathbb{R}^n)}\le C\,\|f\|_{L^{p_1,q_1}(\mathbb{R}^n)}\,\|g\|_{L^{p_2,q_2}(\mathbb{R}^n)},\ 1/p=1/p_1+1/p_2,$$

for all $f\in L^{p_1,q_1}(\mathbb{R}^n)$ *and* $g\in L^{p_2,q_2}(\mathbb{R}^n)$, *then*

$$\left\|T_{m',\varepsilon}(f,g)\right\|_{L^{p,q}(\mathbb{R}^d)}\le C'\,\|f\|_{L^{p_1,q_1}(\mathbb{R}^d)}\,\|g\|_{L^{p_2,q_2}(\mathbb{R}^d)},\ 1/p=1/p_1+1/p_2,$$

for all $f\in L^{p_1,q_1}(\mathbb{R}^d)$ *and* $g\in L^{p_2,q_2}(\mathbb{R}^d)$.

Proof By Theorem 1, we assume $\varepsilon=1$. We define another multiplier m_{c_1,c_2} by

$$m_{c_1,c_2}(\xi_1,\xi_2)=m(\xi_1^{(d)},\xi_1^{(n-d)}+c_1,\xi_2^{(d)},\xi_2^{(n-d)}+c_2).$$

By definition and changing variables, we see that

$$\begin{aligned}
&T_{m_{c_1,c_2}}(f,g)(x)\\
&\quad=\int_{\mathbb{R}^{2n}}m(\xi_1^{(d)},\xi_1^{(n-d)}+c_1,\xi_2^{(d)},\xi_2^{(n-d)}+c_2)\widehat{f}(\xi_1)\widehat{g}(\xi_2)\mathrm{e}^{i\langle\xi_1+\xi_2,x\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2\\
&\quad=\int_{\mathbb{R}^{2n}}m(\xi_1,\xi_2)\widehat{f}(\xi_1^{(d)},\xi_1^{(n-d)}-c_1)\widehat{g}(\xi_2^{(d)},\xi_2^{(n-d)}-c_2)\\
&\qquad\times\mathrm{e}^{i\langle\xi_1+\xi_2,x\rangle}\mathrm{e}^{-i\langle c_1+c_2,x_2\rangle}\mathrm{d}\xi_1\mathrm{d}\xi_2\\
&\quad=\mathrm{e}^{-i\langle c_1+c_2,x_2\rangle}T_m(f_{c_1},g_{c_2})(x),
\end{aligned}$$

where

$$f_{c_1}(x_1,x_2)=f(x_1,x_2)\mathrm{e}^{i\langle c_1,x_2\rangle},$$
$$g_{c_2}(x_1,x_2)=g(x_1,x_2)\mathrm{e}^{i\langle c_2,x_2\rangle}.$$

Thus,

$$\begin{aligned}\left\| T_{m_{c_1,c_2}}(f,g)\right\|_{L^{p,q}(\mathbb{R}^n)} &= \left\| T_m(f_{c_1},g_{c_2})\right\|_{L^{p,q}(\mathbb{R}^n)},\\ \|f\|_{L^{p_1,q_1}(\mathbb{R}^n)} &= \left\| f_{c_1}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)},\\ \|g\|_{L^{p_2,q_2}(\mathbb{R}^n)} &= \left\| g_{c_2}\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}.\end{aligned}$$

By the assumption, we have now

$$\begin{aligned}\left\| T_{m_{c_1,c_2}}(f,g)\right\|_{L^{p,q}(\mathbb{R}^n)} &\le C\left\| f_{c_1}\right\|_{L^{p_1,q_1}(\mathbb{R}^n)}\left\| g_{c_2}\right\|_{L^{p_2,q_2}(\mathbb{R}^n)}\\ &= C\|f\|_{L^{p_1,q_1}(\mathbb{R}^n)}\|g\|_{L^{p_1,q_1}(\mathbb{R}^n)}.\end{aligned}$$

By rescaling, it is trivial to check

$$\left\| T_{m_{c_1,c_2,\varepsilon}}\right\|_{\overrightarrow{p},\overrightarrow{q}} = \left\| T_{m_{c_1,c_2}}\right\|_{\overrightarrow{p},\overrightarrow{q}}$$

for all $\varepsilon > 0$.

So by the transference result in Theorem 1, we further have

$$\left\| \widetilde{T}_{m_{c_1,c_2,\varepsilon}}(\widetilde{f},\widetilde{g})\right\|_{L^{p,q}(T^n)} \le C\left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^n)}\|\widetilde{g}\|_{L^{p_2,q_2}(T^n)}$$

uniformly on $\varepsilon > 0$. For any

$$\widetilde{f}(x_1)\in L^{p_1,q_1}(T^d)\cap C^\infty(T^d),\ \widetilde{g}(x_1)\in L^{p_2,q_2}(T^d)\cap C^\infty(T^d),$$

write

$$\begin{aligned}\widetilde{f}(x_1) &= \sum_{k_1^{(d)}\in\mathbb{Z}^d} a'_{k_1^{(d)}}\mathrm{e}^{i\langle k_1^{(d)},x_1\rangle},\\ \widetilde{g}(x_1) &= \sum_{k_2^{(d)}\in\mathbb{Z}^d} b'_{k_2^{(d)}}\mathrm{e}^{i\langle k_2^{(d)},x_1\rangle}.\end{aligned}$$

We define

$$F(x) = F(x_1,x_2) = (\widetilde{f}\otimes 1)(x_1,x_2) = \widetilde{f}(x_1)$$

and

$$G(x) = G(x_1,x_2) = (\widetilde{g}\otimes 1)(x_1,x_2) = \widetilde{g}(x_1),$$

where $x = (x_1,x_2)\in T^n$, $x_1\in T^d$, $x_2\in T^{n-d}$.

Clearly

$$(F,G)\in L^{p_1,q_1}(T^n)\times L^{p_2,q_2}(T^n)$$

and

$$\begin{aligned}\|F\|_{L^{p_1,q_1}(T^n)} &= \left|T^{n-d}\right|^{1/p_1}\left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T^d)},\\ \|G\|_{L^{p_2,q_2}(T^n)} &= \left|T^{n-d}\right|^{1/p_2}\|\widetilde{g}\|_{L^{p_2,q_2}(T^d)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\widetilde{T}_{m_{c_1,c_2},\varepsilon}(F,G)(x_1,x_2) &= \sum_{k_2^{(d)}\in\mathbb{Z}^d}\sum_{k_1^{(d)}\in\mathbb{Z}^d} a'_{k_1^{(d)}} b'_{k_2^{(d)}} \left|T^{n-d}\right|^2 \\ &\quad \times m_{c_1,c_2}(\varepsilon\xi_1^{(d)},0,\varepsilon\xi_2^{(d)},0)\mathrm{e}^{i\langle k_1^{(d)},x_1\rangle}\mathrm{e}^{i\langle k_2^{(d)},x_1\rangle} \\ &= \sum_{k_2^{(d)}\in\mathbb{Z}^d}\sum_{k_1^{(d)}\in\mathbb{Z}^d} a'_{k_1^{(d)}} b'_{k_2^{(d)}} \left|T^{n-d}\right|^2 \\ &\quad \times m(\varepsilon\xi_1^{(d)},c_1,\varepsilon\xi_2^{(d)},c_2)\mathrm{e}^{i\langle k_1^{(d)}+k_2^{(d)},x_1\rangle} \\ &= \left|T^{n-d}\right|^2 \widetilde{T}_{m',\varepsilon}(\widetilde{f},\widetilde{g})(x_1).\end{aligned}$$

Now

$$\begin{aligned}\lambda_{\widetilde{T}_{m_{c_1,c_2},\varepsilon}(F,G)}(\alpha) &= \left|\left\{(x_1,x_2)\in T^n : \left|\widetilde{T}_{m_{c_1,c_2},\varepsilon}(F,G)((x_1,x_2))\right| > \alpha\right\}\right| \\ &= \int_{T^{n-d}}\int_{\left\{x_1\in T^d:\left|\widetilde{T}_{m',\varepsilon}(\widetilde{f},\widetilde{g})(x_1)\right|>\alpha/\left|T^{n-d}\right|^2\right\}} dx_1dx_2 \\ &= \left|T^{n-d}\right| \lambda_{\widetilde{T}_{m',\varepsilon}(\widetilde{f},\widetilde{g})}(\alpha/\left|T^{n-d}\right|^2).\end{aligned}$$

Thus,

$$\left\|\widetilde{T}_{m_{c_1,c_2},\varepsilon}(F,G)\right\|_{L^{p,q}(T^n)} = \left|T^{n-d}\right|^{1/p+2} \left\|\widetilde{T}_{m',\varepsilon}(\widetilde{f},\widetilde{g})\right\|_{L^{p,q}(T^d)}.$$

By Theorem 1, we have

$$\begin{aligned}\left\|\widetilde{T}_{m',\varepsilon}(\widetilde{f},\widetilde{g})\right\|_{L^{p,q}(T^d)} &\le C\left|T^{n-d}\right|^{-(1/p+2)} \|F\|_{L^{p_1,q_1}(T^n)} \|G\|_{L^{p_2,q_2}(T^n)} \\ &= C\left|T^{n-d}\right|^{-2} \|\widetilde{f}\|_{L^{p_1,q_1}(T^d)} \|\widetilde{g}\|_{L^{p_2,q_2}(T^d)}\end{aligned}$$

uniformly on $\varepsilon > 0$.

Finally by the transference result in Theorem 1 again, we obtain

$$\left\|T_{m',\varepsilon}(f,g)\right\|_{L^{p,q}(\mathbb{R}^d)} \le C\left|T^{n-d}\right|^{-2} \|f\|_{L^{p_1,q_1}(\mathbb{R}^d)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^d)},$$

where

$$1/p = 1/p_1 + 1/p_2.$$

The proof is done. □

Recall

$$\begin{aligned}T_m^*(f,g)(x) &= \sup_{\varepsilon>0}\left|T_{m,\varepsilon}(f,g)(x)\right|, \\ \widetilde{T}_m^*(\widetilde{f},\widetilde{g})(x) &= \sup_{\varepsilon>0}\left|\widetilde{T}_{m,\varepsilon}(\widetilde{f},\widetilde{g})(x)\right|.\end{aligned}$$

Now we define a restriction to $\mathbb{R}^d\times\mathbb{R}^d$ of the multiplier $m(\xi_1,\xi_2)$ on $\mathbb{R}^n\times\mathbb{R}^n$ by

$$m^0(\xi_1^{(d)},\xi_2^{(d)}) = m(\xi_1^{(d)},0,\xi_2^{(d)},0).$$

Theorem 5 *Suppose that* $m \in L^{\infty} \cap C(\mathbb{R}^{2n})$, $1 \le p, q, p_i, q_i \le \infty, i = 1, 2.$ *If*

$$\left\| T_m^*(f, g) \right\|_{L^{p,q}(\mathbb{R}^n)} \le C \left\| f \right\|_{L^{p_1,q_1}(\mathbb{R}^n)} \left\| g \right\|_{L^{p_2,q_2}(\mathbb{R}^n)}, 1/p = 1/p_1 + 1/p_2,$$

for all $f \in L^{p_1,q_1}(\mathbb{R}^n)$ *and* $g \in L^{p_2,q_2}(\mathbb{R}^n)$, *then*

$$\left\| T_{m^0}^*(f, g) \right\|_{L^{p,q}(\mathbb{R}^d)} \le C' \left\| f \right\|_{L^{p_1,q_1}(\mathbb{R}^d)} \left\| g \right\|_{L^{p_2,q_2}(\mathbb{R}^d)}, 1/p = 1/p_1 + 1/p_2,$$

for all $f \in L^{p_1,q_1}(\mathbb{R}^d)$ *and* $g \in L^{p_2,q_2}(\mathbb{R}^d)$.

Proof The proof of Theorem 5 follows the same idea used in the proof of Theorem 4. □

5 Some notes

Recall that in Theorems 1 and 2 we assume $m \in L^{\infty}(\mathbb{R}^{2n}) \cap C(\mathbb{R}^{2n})$. Actually, this condition $m \in C(\mathbb{R}^{2n})$ can be relaxed. In [10], Kenig and Tomas assume that m is regulated, which means every point of $\mathbb{R}^n$ is a Lebesgue point of m [6]. Clearly, we can define the regulated condition on $m(\xi_1, \xi_2)$ and use this condition instead of $m \in C(\mathbb{R}^{2n})$ to ensure that the transference can be completed from T^n to $\mathbb{R}^n$. However, to prove the transference can be completed from $\mathbb{R}^n$ to T^n, we only need the condition

$$\frac{1}{t^{2n}} \int_{|\xi_1| \le t} \int_{|\xi_2| \le t} |m(\varepsilon k_1 + \xi_1, \varepsilon k_2 + \xi_2) - m(\varepsilon k_1, \varepsilon k_2)| \mathrm{d}\xi_1 \mathrm{d}\xi_2 = o(1),$$

if $t \to 0^+$, for all $\{\varepsilon k_1, \varepsilon k_2\}_{(k_1,k_2) \in \mathbb{Z}^n \times \mathbb{Z}^n}$. This means that we only need that all points in $\{\varepsilon k_1, \varepsilon k_2\}_{(k_1,k_2) \in \mathbb{Z}^n \times \mathbb{Z}^n, \varepsilon > 0}$ are Lebesgue points of $m(\xi_1, \xi_2)$.
In the proof of $\left\| T_m^* \right\|_{\vec{p}, \vec{q}} \ge \left\| \widetilde{T}_m^* \right\|_{\vec{p}, \vec{q}}$, the condition on m is used to show

$$\lim_{N \to \infty} E_N(\widetilde{f}, \widetilde{g})(x) = 0. \tag{13}$$

In the proof $\left\| T_m^* \right\|_{\vec{p}, \vec{q}} \le \left\| \widetilde{T}_m^* \right\|_{\vec{p}, \vec{q}}$, the condition on m is used to show that

$$\varepsilon^{2n} \sum_{k_1 \in \mathbb{Z}^n} \sum_{k_2 \in \mathbb{Z}^n} \widehat{f}(\varepsilon k_1) \widehat{g}(\varepsilon k_2) m(\varepsilon k_1, \varepsilon k_2) \mathrm{e}^{i \langle \varepsilon x, k_1 + k_2 \rangle} \tag{14}$$

is a Riemann sum of

$$\int_{\mathbb{R}^{2n}} \widehat{f}(\xi_1) \widehat{g}(\xi_2) m(\xi_1, \xi_2) \mathrm{e}^{i \langle x, \xi_1 + \xi_2 \rangle} \mathrm{d}\xi_1 \mathrm{d}\xi_2. \tag{15}$$

We look the bilinear Hilbert transform

$$H(f, g)(x) = i \int_{\mathbb{R}^2} \widehat{f}(\xi_1) \widehat{g}(\xi_2) sgn(\xi_1 - \xi_2) \mathrm{e}^{i \langle x, \xi_1 + \xi_2 \rangle} \mathrm{d}\xi_1 \mathrm{d}\xi_2,$$

and its periodic version

$$\widetilde{H}(\widetilde{f}, \widetilde{g})(x) = i \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} a_{k_1} b_{k_2} sgn(k_1 - k_2) \mathrm{e}^{i \langle k_1 + k_2, x \rangle}.$$

It was proved in [7] that the symbol sgn $(\xi_1 - \xi_2)$ ensures that (13) holds. Also, it is clear that sgn $(\xi_1 - \xi_2)$ makes that (14) is a Riemann sum of (15). Also, we observe sgn $(\varepsilon\xi_1 - \varepsilon\xi_2) = sgn$ $(\xi_1 - \xi_2)$. Therefore, we have the following result.

Theorem 6 *For* $1 \le p, q, p_i, q_i \le \infty, i = 1, 2,$

$$\|H(f,g)\|_{L^{p,q}(\mathbb{R})} \le \|H\|_{\overrightarrow{p},\overrightarrow{q}} \|f\|_{L^{p_1,q_1}(\mathbb{R})} \|g\|_{L^{p_2,q_2}(\mathbb{R})}, \; 1/p = 1/p_1 + 1/p_2,$$

for all f *and* g *if and only if*

$$\left\|\widetilde{H}(\widetilde{f},\widetilde{g})\right\|_{L^{p,q}(T)} \le \left\|\widetilde{H}\right\|_{\overrightarrow{p},\overrightarrow{q}} \left\|\widetilde{f}\right\|_{L^{p_1,q_1}(T)} \|\widetilde{g}\|_{L^{p_2,q_2}(T)}, \; 1/p = 1/p_1 + 1/p_2,$$

for all $\widetilde{f}$ *and* $\widetilde{g}$.

Moreover,

$$\|H\|_{\overrightarrow{p},\overrightarrow{q}} = \left\|\widetilde{H}\right\|_{\overrightarrow{p},\overrightarrow{q}}.$$

Remark 3 We may further consider the transference of bilinear multiplier between Lorentz spaces of $\mathbb{R}^n$ and $\mathbb{Z}^n$.

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