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Uniform integrability in periodic homogenization of fully nonlinear elliptic equations

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Abstract
This paper is devoted to the study of uniform $W^{1,\frac{np}{n-p}}$- and $W^{2,p}$-estimates for periodic homogenization problems of fully nonlinear elliptic equations. We establish sharp, global, large-scale estimates under the Dirichlet boundary conditions. The main novelty of this paper can be found in the characterization of the size of the "effective" Hessian and gradient of viscosity solutions to homogenization problems. Moreover, the large-scale estimates work in a large class of non-convex problems. It should be stressed that our global estimates are new even for the standard problems without homogenization.

Keywords Uniform estimates · Periodic homogenization · Fully nonlinear equations · Integrability

Mathematics Subject Classification 35J57 · 35J60 · 35B27

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1 Introduction

This paper is devoted to the study of uniform integrability of the Hessian and gradient of viscosity solutions $u^\varepsilon \in C(\overline{\Omega})$ to fully nonlinear periodic homogenization problems, of the type

$$\begin{cases} F\left(D^2u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f & \text{in } \Omega, \\ u^\varepsilon = g & \text{on } \partial\Omega. \end{cases} \tag{1.0.1}$$

In [7], the size of the "Hessian" of a continuous function at a point is characterized by the smallest opening of touching convex and concave paraboloids at that point. Here we extend this concept by allowing the touching to take place in a neighborhood of size ε around the reference point, and denote this quantity by H_Ω^ε. Similarly, we characterize the size of the "gradient" by replacing paraboloids with cones, and denote it by G_Ω^ε. See Definition 2.2 for more precise definitions for these quantities. Especially given $u \in C(\Omega)$, we designed $H_\Omega^\varepsilon(u)$ ($G_\Omega^\varepsilon(u)$) in such a way that

$$\begin{aligned} &|\Delta_{\varepsilon e}^2 u(x)| \leq 2(H_\Omega^\varepsilon(u))(x), \quad \forall x \in \Omega_\varepsilon, \\ &(\text{resp., } |\Delta_{\varepsilon e} u(x)| \leq 2(G_\Omega^\varepsilon(u))(x), \quad \forall x \in \Omega_\varepsilon) \end{aligned} \tag{1.0.2}$$

where $\Delta_{\varepsilon e}^2 u(x) := (u(x+\varepsilon e) + u(x-\varepsilon e) - 2u(x))/\varepsilon^2$ (resp., $\Delta_{\varepsilon e} u(x) := (u(x+\varepsilon e) - u(x))/\varepsilon$) is the second (resp., first) ε-differential quotient in direction $e \in \partial B_1$; see Remark 2.3. From the above inequalities, the L^p-estimates of $H_\Omega^\varepsilon(u)$ ($G_\Omega^\varepsilon(u)$) yield the same estimates for $\Delta_{\varepsilon e}^2 u$ (resp. $\Delta_{\varepsilon e} u$) for all $e \in \partial B_1$. Thanks to this relation, denoting by u^ε the solution to our homogenization problem (1.0.1), the L^p-estimates of $H_\Omega^\varepsilon(u^\varepsilon)$ ($G_\Omega^\varepsilon(u^\varepsilon)$) can be understood as the so-called *large-scale* $W^{2,p}$-(resp., $W^{1,p}$-)estimates of u^ε.

As another important remark, from our definition, $H_\Omega^\varepsilon \to H_\Omega$ ($G_\Omega^\varepsilon \to G_\Omega$) as $\varepsilon \to 0$, where H_Ω (resp., G_Ω) controls the standard Hessian (resp., gradient). The quantity H_Ω is the classical one introduced in [7]. On the other hand, the quantity G_Ω for the gradient appeared here, and also independently in a very recent paper [25], for the first time.

The first main result of this paper is the uniform integrability of $H_\Omega^\varepsilon(u^\varepsilon)$; see Definition 2.5 for domains of $W^{2,p}$-type.

Theorem 1.1 ($W^{2,p}$-estimates) *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.4), $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in C(\Omega) \cap L^p(\Omega)$ for some finite $p > p_0$, $g \in C(\partial\Omega) \cap W^{2,p}(\Omega)$ and $u^\varepsilon \in C(\overline{\Omega})$ be a viscosity solution to (1.0.1). Suppose either of the following:*

(i) *Ω is a $W^{2,n}$-type domain, and $p_0 < p < n$;*
(ii) *Ω is a $W^{2,n+\sigma}$-type domain for some $\sigma > 0$, and $p = n$;*
(iii) *Ω is a $W^{2,p}$-type domain and $p > n$, all with size (δ, R).*

Then $H_\Omega^\varepsilon(u^\varepsilon) \in L^p(\Omega_\varepsilon)$, with $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, and

$$\left(\int_{\Omega_\varepsilon} (H_\Omega^\varepsilon(u^\varepsilon))^p dx\right)^{\frac{1}{p}} \leq C\left(\|u^\varepsilon\|_{L^\infty(\Omega)} + \|f + |D^2 g|\|_{L^p(\Omega)}\right),$$

where $C > 0$ depends only on n, λ, Λ, ψ, κ, σ, δ, R and p.

Let us provide some motivation for the assumption (2.0.4). Roughly speaking, the assumption says that the effective problem $\bar{F}(D^2 v) = 0$ has interior VMO-estimates for the Hessian of its (viscosity) solutions. Note that u^ε converges to its effective profile $\bar{u}$, only, uniformly.

This is too weak to ensure any closeness between their Hessian. Under the VMO-condition on $D^2\bar{u}$, however, $D^2\bar{u}$ satisfies a small BMO-condition at an intermediate scale. We observe that $\mathcal{P}_\pm(D^2(u^\varepsilon - \bar{u} - \varepsilon^2 w(\frac{\cdot}{\varepsilon}))) = o(1)$, at that scale, with w being an interior corrector. This is one of the key observations in Lemma 6.7, which is an approximation lemma for the interior $W^{2,p}$-estimates.

It should also be addressed that due to [16, Theorem 3.4], there is a large class of non-convex functionals satisfying (2.0.4). More specifically, the result implies the following: if there exists a functional $F_* : \mathcal{S}^n \times \mathbb{R}^n \to \mathbb{R}$, which is convex in the first argument and satisfies (2.0.1)–(2.0.3), such that $|(F - F_*)(P, y) - (F - F_*)(Q, y)| \leq \theta |P - Q|$ for all $P, Q \in \mathcal{S}^n$ and all $y \in \mathbb{R}^n$, for some small constant θ, then the effective functional $\bar{F}$ satisfies (2.0.4). It is also noteworthy that unless the governing functional is continuously differentiable [15], (2.0.4) is a strictly relaxed assumption than assuming that $\bar{F}(D^2\bar{u}) = 0$ has interior $C^{2,\alpha}$-estimates. For some further development in interior $W^{2,p}$-estimates for standard fully nonlinear problems, see e.g., [26].

We remark that throughout this paper, we do not assume continuous differentiability of $\bar{F}$ (or F). In fact, our result on the uniform L^p-estimate for $H^\varepsilon_\Omega(u^\varepsilon)$ only requires F to be continuous. Here we encounter another subtle issue that arises from the homogenization of L^p-viscosity solutions. It is worth mentioning that homogenization of viscosity solutions has not yet been justified for equations with measurable ingredients. For the moment, the author is not sure whether the measurable ingredients would be homogenized either. We use the continuity of F (as well as the datum f) to circumvent this issue.

The above estimates are sharp not only in terms of the regularity of the data, but also of the regularity of the boundary layer. The major challenge here arises from the fact that boundary flattening maps destroy the pattern of the rapid oscillation. For this reason, our analysis is quite different from, and in fact more complicated than, the argument for standard problems, c.f. [30].

As a matter of fact, the boundary estimates for the case $p_0 < p < n$ are even new in the context of standard problems. The analysis is based on the following sharp $W^{1,\frac{np}{n-p}}$-estimates up to the boundary, with $\frac{np}{n-p}$ being the critical Sobolev exponent.

Theorem 1.2 ($W^{1,\frac{np}{n-p}}$-estimates) *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying* (2.0.1)–(2.0.3)*, $\Omega \subset \mathbb{R}^n$ be a (δ, R)-Reifenberg flat, bounded domain, $f \in C(\Omega) \cap L^p(\Omega)$ for some $p_0 < p < n$, $g \in C(\partial\Omega) \cap W^{1,\frac{np}{n-p}}(\Omega)$ and $u^\varepsilon \in C(\overline{\Omega})$ be a viscosity solution to* (1.0.1)*. Then there exists a constant $\delta_0 \in (0, 1)$, depending only on n, λ, Λ, R and p, such that if $\delta \leq \delta_0$, then $G^\varepsilon_\Omega(u^\varepsilon) \in L^{\frac{np}{n-p}}(\Omega_\varepsilon)$, and*

$$\left(\int_{\Omega_\varepsilon} (G^\varepsilon_\Omega(u^\varepsilon))^{\frac{np}{n-p}} dx\right)^{\frac{1}{p}-\frac{1}{n}} \leq C\left(\|u^\varepsilon\|_{L^\infty(\Omega)} + \|f + |Dg|\|_{L^{\frac{np}{n-p}}(\Omega)}\right),$$

where C depends only on n, λ, Λ, δ_0, R and p.

To the best of author's knowledge, the closest result under the framework of standard problems is [11, Theorem 1.4], where interior gradient estimates are established in the Lorentz space. Their result shows that $f \in L^{p,\gamma}$ implies $|Du| \in L^{np/(n-p),\gamma}_{loc}$ for any $p \in (p_0, n)$ and any $\gamma > 0$. As for estimates for subcritical Sobolev exponents, the interior and boundary estimates (for C^2-domains and $C^{1,\alpha}$-data on boundaries) are obtained in [29] and respectively [30].

Our analysis is based on the decay estimate of the set of large "gradient", in the spirit of Caffarelli's approach in [7]. We present a parallel study for the set of large gradient by replacing the touching paraboloids of the Hessian with cones. The proof relies on the general maximum principle as well as an elementary observation that the slope of supporting hyperplanes for convex envelopes to viscosity solutions in the Pucci class can be universally bounded from below. Recently, in the framework of standard problems (without homogenization),

It is worthwhile to mention that the above estimates for the uniform integrability of the gradient are sharp in terms of the data, and that the domains are only required to be Reifenberg flat. Moreover, as a byproduct via the Sobolev embedding theorem, we obtain a uniform interior $C^{0,2-n/p}$-estimate, which is rather well-understood in the setting of linear homogenization problems [2] and was proved in the setting of standard fully nonlinear problems [27]. Nevertheless, our uniform $C^{0,2-n/p}$-estimates are new in the framework of fully nonlinear homogenization problems.

Remark 1.3 A uniform L^p-estimate for the *full* Hessian (gradient), $H_\Omega(u^\varepsilon)$ (resp., $G_\Omega(u^\varepsilon)$), can be obtained under suitable hypotheses that ensure the regularity in *small-scales*. The passage from Theorem 1.1 (resp., 1.2) to the full estimates is by now standard, whence is omitted in this paper.

Let us briefly summarize the recent development of uniform estimates in the homogenization theory. Needless to say, the study has gained its interests due to a series of papers by Avellaneda and Lin. In particular, $W^{1,p}$-estimates are established in [3] for linear divergence-type equations, based on the study of Green functions. Later in [9], Caffarelli and Peral proved $W^{1,p}$-estimates for nonlinear divergence-type equations, via Calderón-Zygmund cube decomposition argument. In [24], Melcher and Schweizer proved the estimates via a more direct approach, based on the observation that ε-difference quotients solve the same class of equations. We would also like to mention [19], where uniform integrability estimates are established for nonlinear systems in divergence form. More recently, Byun and Jang proved, in [4], uniform $W^{1,p}$-estimates for linear divergence-type systems, under small BMO-condition on the periodically oscillating operators, up to Reifenberg flat domains. Some sharp "large-scale" estimates for linear divergence-type equations, without any regularity assumption on the governing operators can be found in [28]. All the above results are concerned with periodic homogenization of solutions to either interior problems or Dirichlet problems. As for Neumann problems, some important sharp estimates can be found in [17]. There is also a large amount of literature concerning uniform pointwise estimates for random homogenization, for which we would like to refer readers to a recent book [1] and the references therein.

Most of the aforementioned works are concerned with weak solutions to divergence-type problems. Uniform estimates for viscosity solutions to non-divergence type equations was done only recently in a collaboration [18] by Lee and the author, where pointwise $C^{1,\alpha}$- and $C^{1,1}$-estimates are proved for a class of non-convex functionals. The uniform integrability estimates established in this paper are new, even for linear equations in non-divergence form.

The paper is organized as follows. In the next section, we collect the notation, main assumptions and some preliminaries. Section 3 is devoted to several technical tools used in the subsequent analysis, yet of their own independent interests. In Sect. 4, we study universal decay estimates for the set of large Hessian and gradient that will play an important role in the subsequent analysis. In Sect. 5.1, we establish the uniform $W^{1,\frac{np}{n-p}}$-estimates for both interior (Theorem 5.1) and boundaries (Theorem 6.1). Finally, in Sect. 6.2, we prove the uniform $W^{2,p}$-estimates, whose proof is again divided into the case of interior (Theorem 5.4) and of boundaries (Theorem 6.5).

2 Preliminaries

We shall denote by $B_r(x)$ the n-dimensional ball centered at x with radius r, and by $Q_r(x)$ the n-dimensional cube centered at x with side-length r. By $S_\delta(\nu)$ we denote the slab centered at the origin with width δ in direction ν, i.e., $S_\delta(\nu) = \{x \in \mathbb{R}^n : |x\cdot\nu| < \delta\}$. By $H_t(\nu)$ we denote the half-space in direction ν with the lowest level being t, i.e., $H_\delta(\nu) = \{x \in \mathbb{R}^n : x \cdot \nu > t\}$. Also we shall write $H_0(\nu)$ simply by $H(\nu)$. Moreover, by $\mathcal{S}^n$ we denote the space of all symmetric $(n \times n)$-matrices.

Throughout this paper, λ and Λ will be fixed as some positive constants, with $\lambda \le \Lambda$, and will also denote the lower and respectively upper ellipticity bound for the governing functional. By p_0 we shall denote Escauriaza's constant such that the generalized maximum principle holds for all $p > p_0$; note $\frac{n}{2} < p_0 < n$ and it depends only on n and the ellipticity bounds, λ and Λ. For more details, we refer readers to [12] and [8]. In addition, we denote by $\mathcal{P}_-$ and $\mathcal{P}_+$ the Pucci minimal and respectively maximal functional on $\mathcal{S}^n$, associated with ellipticity bounds λ and Λ, such that

$$\mathcal{P}_-(P) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i, \quad \mathcal{P}_+(P) = -\mathcal{P}_-(-P),$$

where e_i is the i-th eigenvalue of P.

Let $\psi : (0,\infty) \to (0,\infty)$ be a nondecreasing, strictly concave function such that $\psi(0+) = 0$, and Ω be a domain in $\mathbb{R}^n$. We say $g \in BMO_\psi(\Omega)$, if $g \in L^1(\Omega)$ and

$$[g]_{BMO_\psi(\Omega)} = \sup_{B\subset\Omega} \frac{1}{|B|\psi(\text{rad } B)} \int_B |g-(g)_B|dx < \infty,$$

where the supremum is taken over all balls $B \subset \Omega$. Given any $g \in L^1(\mathbb{R}^n)$ with $g \ge 0$, $M(g)$ denotes the maximal function of g, i.e.,

$$M(g)(x) = \sup_B \frac{1}{|B|} \int_B g(y)dy,$$

where the supremum is taken over all balls B containing x. Given any $g \in L^1_{loc}(\mathbb{R}^n)$, $I_\alpha(g)$ denotes the Riesz potential of g, i.e.,

$$I_\alpha(g)(x) = c_\alpha \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}} dy,$$

with c_α being a suitable normalization constant.

For definiteness, we shall assume that $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ is a functional satisfying, for any $P, Q \in \mathcal{S}^n$ and any $y \in \mathbb{R}^n$,

$$\mathcal{P}_-(P-Q) \le F(P,y) - F(Q,y) \le \mathcal{P}_+(P-Q), \tag{2.0.1}$$

$$F(P, y+k) = F(P,y), \tag{2.0.2}$$

$$F(0,y) = 0. \tag{2.0.3}$$

Under the first two assumptions, there exists a unique functional $\bar{F} : \mathcal{S}^n \to \mathbb{R}$ (the so-called effective functional), according to [13, Theorem 3.1], such that any limit $\bar{u} \in C(\Omega)$ of the sequence of viscosity solutions $u^\varepsilon \in C(\Omega)$ to $F(D^2u^\varepsilon, \frac{\cdot}{\varepsilon}) = f$ in Ω, with $f \in C(\Omega)$ and $\varepsilon > 0$, under locally uniform convergence as $\varepsilon \to 0$ is a viscosity solution to $\bar{F}(D^2\bar{u}) = f$ in Ω. We shall suppose that

$$\bar{F}(D^2 v) = 0 \text{ has interior } W^{2,BMO_\psi}\text{-estimates with constant } \kappa. \tag{2.0.4}$$

where $\psi : (0,\infty) \to (0,\infty)$ is a concave, non-decreasing function satisfying $\psi(0+) = 0$ and $\kappa > 0$ is a fixed constant. More specifically, by (2.0.4), we indicate the following: given any ball $B_R \subset \mathbb{R}^n$ and any function $v_0 \in C(\partial B_R)$, there exists a viscosity solution $v \in C(\overline{B_R}) \cap W^{2,1}(B_R)$ to $\bar{F}(D^2 v) = 0$ in B_R, $v = v_0$ on ∂B_R such that for any $r \in (0, R)$,

$$\frac{1}{(R-r)^n}\int_{B_r} |D^2 v|\,dx + [D^2 v]_{BMO_\psi(B_r)} \le \frac{\kappa}{(R-r)^2}\|v_0\|_{L^\infty(\partial B_R)}.$$

Let us collect by now standard results regarding periodic homogenization for fully nonlinear problems in the following lemma.

Lemma 2.1 [13, Lemma 3.1-–3.2] *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying* (2.0.1) *and* (2.0.2). *Then there exists a unique functional $\bar{F} : \mathcal{S}^n \to \mathbb{R}$, satisfying* (2.0.1)*, such that for each $P \in \mathcal{S}^n$, $\bar{F}(P)$ is the unique constant for which there exists a viscosity solution to*

$$\begin{cases} F(D^2 w + P, \cdot) = \bar{F}(P) & \text{in } \mathbb{R}^n, \\ w(y+k) = w(y) & \text{for all } y \in \mathbb{R}^n, k \in \mathbb{Z}^n. \end{cases}$$

In particular, if $w \in C(\mathbb{R}^n)$ is a viscosity solution to this problem, $w \in C^\alpha(\mathbb{R}^n)$ and

$$\|w - w(0)\|_{C^\alpha(\mathbb{R}^n)} + |\bar{F}(P)| \le c(|P| + \|F(P,\cdot)\|_{L^\infty(\mathbb{R}^n)}),$$

where $c > 0$ and $\alpha \in (0,1)$ depend only on n, λ and Λ.

Next, we introduce the set of large "Hessian", as well as the set of large "gradient", with room for errors of order ε^2 and respectively ε. These sets will play the main role throughout this paper, as our primary goal is to establish "large-scale" estimates. The set of large Hessian without any room for error has played a central role in the $W^{2,p}$-theory for standard fully nonlinear problems. Nevertheless, the set of large gradient seems to appear in this paper for the first time, in the literature. Needless to say, the sets with room for errors are entirely new, as far as the author is concerned. It should be stressed that one can generalize this concept to homogenization problems under various oscillating structures, such as quasi-periodic, almost-periodic or random environment.

Definition 2.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\varepsilon \ge 0$, $t > 0$ and $u \in C(\Omega)$ be given. Let $A_t^\varepsilon(u,\Omega)$ (and $L_t^\varepsilon(u,\Omega)$) be defined as a subset of Ω such that $x_0 \in \Omega\backslash A_t^\varepsilon(u,\Omega)$ (resp., $\Omega\backslash L_t^\varepsilon(u,\Omega)$) if and only if there exists a linear polynomial ℓ (resp., a constant a) for which $|u(x) - \ell(x)| \le \frac{t}{2}(|x-x_0|^2 + \varepsilon^2)$ (resp., $|u(x) - a| \le t(|x - x_0| + \varepsilon)$) for all $x \in \Omega$. Denote by $A_t(u,\Omega)$ (and $L_t(u,\Omega)$) the set $A_t^0(u,\Omega)$ (resp., $L_t^0(u,\Omega)$). Let $H_\Omega^\varepsilon(u)$ (resp., $G_\Omega^\varepsilon(u)$) : $\Omega \to \mathbb{R} \cup \{\pm\infty\}$ be defined as

$$H_\Omega^\varepsilon(u)(x) = \inf\{t > 0 : x \in A_t^\varepsilon(u,\Omega)\},$$
$$(\text{resp., } G_\Omega^\varepsilon(u)(x) = \inf\{t > 0 : x \in L_t^\varepsilon(u,\Omega)\}),$$

and denote by $H_\Omega(u)$ (and $G_\Omega(u)$) the function $H_\Omega^0(u)$ (resp., $G_\Omega^0(u)$).

We remark that H_Ω^ε (and G_Ω^ε) controls the second (resp. first) ε-differential quotients of u^ε. Since the proof is essentially the same (and in fact shorter) as that of [6, Lemma 2.1], we shall only present the statement.

Remark 2.3 Let $u \in C(\Omega)$ and $\varepsilon > 0$ be given, and choose any $x_0 \in \Omega$ with dist $(x_0, \partial\Omega) > \varepsilon$. Suppose that $H_\Omega^\varepsilon(u)(x_0) \le t$ for some $t > 0$. Then

$$u^\varepsilon(x_0 \pm \varepsilon e) - u(x_0) \le \pm\varepsilon b \cdot e + t\varepsilon^2, \quad \forall e \in \partial B_1.$$

Hence, $\Delta^2_{\varepsilon e} u(x_0) \le 2t$. Taking infimum over all $t > 0$ for which $H^\varepsilon_\Omega(u)(x_0) \le t$, we obtain

$$\Delta^2_{\varepsilon e} u(x_0) \le 2H^\varepsilon_\Omega(u)(x_0).$$

Similarly, one can prove the reverse inequality, which proves (1.0.2) for H^ε_Ω. In the same way, we can also verify (1.0.2) for G^ε_Ω.

The following is the definition for the Reifenberg flat sets, which will appear in uniform boundary $W^{1,\frac{np}{n-p}}$-estimates.

Definition 2.4 Let Ω be a domain, and U be a neighborhood of a point at $\partial\Omega$. Set $\partial\Omega \cap U$ is said to be (δ, R)-Reifenberg flat from exterior (or interior), if for any $x_0 \in \partial\Omega \cap U$ and any $r \in (0, R]$, $\Omega \cap B_r(x_0) \subset \{x : (x - x_0) \cdot \nu_{x_0} > -\delta r\}$ (resp., $B_r(x_0) \setminus \Omega \subset \{x : (x - x_0) \cdot \nu_{x_0} < -\delta r\}$) for some unit vector $\nu_{x_0,r}$; here $\nu_{x_0,r}$ may vary upon both x_0 and r. The set $\partial\Omega \cap U$ is said to be (δ, R)-Reifenberg flat, if it is (δ, R)-Reifenberg flat from both exterior and interior. The domain Ω is said to be (δ, R)-Reifenberg flat, if $\partial\Omega \cap B_R(x_0)$ is (δ, R)-Reifenberg flat for each $x_0 \in \partial\Omega$.

Next, we define domains of $W^{2,p}$-type.

Definition 2.5 Let $p > 1$ be a constant, Ω be a domain, and U be a neighborhood of a point at $\partial\Omega$. Set $\partial\Omega \cap U$ is said to be of $W^{2,p}$-type with size κ, if there exists a neighborhood $V \subset \mathbb{R}^n$ and a diffeomorphism $\Phi \in C^1(U; V) \cap W^{2,p}(U; V)$ such that $\Phi(\Omega \cap U) = H(e_n) \cap V$, $\Phi(\partial\Omega \cap U) = H(e_n) \cap V$, $\operatorname{osc}_U D\Phi \le \delta$ and $\|D^2\Phi\|_{L^p(U)} \le \delta$. The domain Ω is said to be of $W^{2,p}$-type with size (κ, R), if $\partial\Omega \cap B_R(x_0)$ is of $W^{2,p}$-type with size κ for each $x_0 \in \partial\Omega$.

We shall also need some covering lemmas. As for the analysis for interior estimates, we shall use the classical Calderón-Zygmund cube decomposition lemma, c.f. [14, Section 9.2] and [6, Lemma 4.1]:

Lemma 2.6 (Calderón-Zygmund cube decomposition) *Let $A \subset Q_1$ be a measurable set such that $|A| \le \eta$ for some $\eta \in (0, 1)$. Then there exists a finite collection $\mathcal{F}$ of cubes from the dyadic subdivision of Q_1, such that $|A \cap Q| > \eta|Q|$ for all $Q \in \mathcal{F}$, and $|A \cap \tilde{Q}| \le \eta|\tilde{Q}|$ for the predecessor of Q.*

Let B be a measurable set such that $A \subset B \subset Q_1$. If $\tilde{Q} \subset B$ for the predecessor $\tilde{Q}$ of any dyadic cube Q satisfying $|Q \cap A| > \eta|Q|$, then $|A| \le \eta|B|$.

As for the boundary estimates, we shall utilize the Vitali-type covering lemma for Reifenberg flat domains [5, Theorem 2.8]. We present its statement for the reader's convenience.

Lemma 2.7 (Vitali-type covering [5, Theorem 2.8]) *Let $\Omega \subset \mathbb{R}^n$ be a domain such that $\partial\Omega \cap B_1$ is $(\delta, 1)$-Reifenberg flat, with some $\delta \in (0, \frac{1}{8})$, and contains the origin. Let $D \subset E \subset \Omega \cap B_1$ be two measurable sets. Suppose that $|D| \le \eta|B_1|$, for some $\eta \in (0, 1)$, and that for any ball $B \subset B_1$ whose center lies in $\overline{\Omega} \cap B_1$ and radius is at most 1, $|D \cap B| > \eta|B|$ implies that $\Omega \cap B \subset E$. Then $|D| \le (10/(1 - \delta))^n \eta|E|$.*

3 Some technical tools

Let us begin with an assertion that the difference between a viscosity solution and a viscosity sub- or super-solution belongs to the Pucci class in the viscosity sense. It is particularly important that one of two must be a solution. This assertion might be known for some experts. Still, we intend to present a proof because the assertion is not as simple as it sounds, apart from the fact that the author was not able to find a proof in the literature. It should be stressed that the assertion is yet to be known if we replace viscosity solution with viscosity super- or sub-solution (depending on what it is compared with).

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $F \in C(\mathcal{S}^n \times \Omega)$ be a functional satisfying $\mathcal{P}_-(P-Q) \leq F(P,x) - F(Q,x) \leq \mathcal{P}_+(P-Q)$ for all $P, Q \in \mathcal{S}^n$ and $x \in \Omega$. Let $u, v \in C(\Omega)$ be such that $F(D^2u,\cdot) = f$ in Ω and $F(D^2v,\cdot) \geq g$ in Ω in the viscosity sense, for some $f, g \in C(\Omega)$. Then $\mathcal{P}_+(D^2(u-v)) \leq f - g$ in Ω in the viscosity sense.*

Proof Fix $\delta > 0$, and denote $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. Clearly, $u, v \in C(\overline{\Omega_\delta})$, and Ω_δ satisfies the uniform exterior sphere condition with radius at most δ^{-1}. Given any pair (τ, σ) of real parameters such that $0 < \sigma < \tau < \delta$, let $v_\tau : \Omega \to \mathbb{R}$ be the sup-inf convolution of v over Ω; i.e.,

$$v_\tau(x) = \inf_{z\in\Omega} \sup_{y\in\Omega} \left(v(y) - \frac{|z-y|^2}{2\tau} + \frac{|x-z|^2}{2\sigma} \right).$$

Such a regularization is by now considered standard. Among other important properties, we shall use the following, which can be found in [20, thm] that $v_\tau \to v$ uniformly in $\overline{\Omega_\delta}$, $v_\tau \in W^{2,\infty}(\Omega)$, $|Dv_\tau| \leq c_\delta \tau^{-1/2}$ in Ω, with $c_\delta > 0$ being a constant depending only on the sup-norm and the modulus of continuity of v over $\overline{\Omega_\delta}$, and

$$F_\tau(D^2 v_\tau, \cdot) \geq g_\tau \quad \text{a.e. in } \Omega_\tau,$$

where $F_\tau : \mathcal{S}^n \times \Omega_\tau \to \mathbb{R}$ and $g_\tau : \Omega_\tau \to \mathbb{R}$ are defined by $F_\tau(P,x) = F(P, x - (\tau - \sigma)Dv_\tau(x))$ and $g_\tau(x) = g(x - (\tau-\sigma)Dv_\tau(x))$. In particular, since $\tau|Dv_\tau| \to 0$ uniformly on Ω, $F_\tau \to F$ locally uniformly in $\mathcal{S}^n \times \overline{\Omega_\delta}$ and $g_\tau \to g$ uniformly on $\overline{\Omega_\delta}$.

Consider an auxiliary Dirichlet boundary value problem,

$$\begin{cases} F_\tau(D^2u_\tau, \cdot) = f & \text{in } \Omega_\delta \\ u_\tau = u & \text{on } \partial\Omega_\delta. \end{cases}$$

Since $F_\tau \in C(\mathcal{S}^n \times \Omega_\delta)$ is uniformly elliptic, $f \in C(\Omega_\delta)$, $u \in C(\partial\Omega_\delta)$ and Ω_δ satisfies the uniform exterior sphere condition, there exists a unique viscosity solution $u_\tau \in C(\overline{\Omega_\delta})$ to this problem, according to e.g., [10, Theorem 4.1]. Moreover, as the radius for the uniform exterior sphere condition for Ω_δ is independent of τ, it follows from the global regularity of viscosity solutions [6, Proposition 4.14] that $\{u_\tau : 0 < \tau < \delta\}$ is a uniformly bounded and equicontinuous family on $\overline{\Omega_\delta}$. However, since $F_\tau \to F$ locally uniformly on $\mathcal{S}^n \times \Omega_\delta$, by the stability [6, Proposition 4.11] and the comparison principle [10, Theorem 3.3] for viscosity solutions, one can easily deduce that $u_\tau \to u$ uniformly on $\overline{\Omega_\delta}$ as $\tau \to 0$.

On the other hand, as $v_\tau \in W^{2,\infty}(\Omega)$ and $F_\tau(D^2v_\tau, \cdot) \geq g_\tau$ a.e. in Ω_δ, we can compute that

$$\mathcal{P}_-(D^2(u_\tau - v_\tau)) \leq F_\tau(D^2u_\tau, \cdot) - F_\tau(D^2v_\tau, \cdot) \leq f - g_\tau \quad \text{in } \Omega_\delta,$$

in the L^∞-viscosity sense, but then in the usual (C-)viscosity sense as $f - g_\tau \in C(\Omega_\delta)$; here, we refer to [8] the notion of L^∞-viscosity solutions. Now letting $\tau \to 0$, and recalling that $u_\tau \to u$, $v_\tau \to v$ and $g_\tau \to g$ uniformly on $\overline{\Omega_\delta}$, we may conclude from the stability theory again that

$$\mathcal{P}_-(D^2(u-v)) \leq f - g \quad \text{in } \Omega_\delta.$$

As $\delta > 0$ was an arbitrary constant, the assertion of the lemma follows by sending $\delta \to 0$. □

Let us close this section with a few results that are essentially due to [18], but extended so as to be adoptable in our subsequent analysis. We shall start with an interior L^∞-approximation of viscosity solutions to periodic homogenization problems by those to the corresponding effective problems. The assertion is a slight generalization of [18, Lemma 3.1, 4.2], which was established for bounded data. Here we shall extend the result to L^p-integrable data.

Lemma 3.2 (Due to [18, Lemma 3.1, 4.2]) *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.3), $f \in C(B_R) \cap L^p(B_R)$ for some $p > p_0$ and some $R > 0$, $u^\varepsilon \in C(B_R)$ be a viscosity solution to*

$$F\left(D^2 u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f \quad \text{in } B_R,$$

for some $\varepsilon > 0$. Let $r \in (0, R)$ be given. Then for each $\eta > 0$, one can find some $\varepsilon_\eta > 0$, depending only on n, λ, Λ and η, such that if $0 < \varepsilon < r\varepsilon_\eta$ and $0 < r < R$, then there exists a viscosity solution $\bar{u} \in C(\overline{B_r})$ to

$$\bar{F}(D^2 \bar{u}) = 0 \quad \text{in } B_r,$$

for which

$$\|\bar{u}\|_{L^\infty(B_r)} + \frac{\|u^\varepsilon - \bar{u}\|_{L^\infty(B_r)}}{\eta} \leq \frac{C}{(R-r)^\alpha}\left(\|u^\varepsilon\|_{L^\infty(B_R)} + R^{2-\frac{n}{p}}\|f\|_{L^p(B_R)}\right),$$

where $\alpha \in (0, 1)$ depends only on n, λ and Λ, and $C > 1$ may depend further on p.

Proof The assertion for $f = 0$ is a direct consequence of [18, Lemma 3.1, 4.2]. As for the general case, we consider an auxiliary boundary value problem,

$$\begin{cases} F\left(D^2 \hat{u}^\varepsilon, \frac{\cdot}{\varepsilon}\right) = 0 & \text{in } B_R, \\ \hat{u}^\varepsilon = u^\varepsilon & \text{on } \partial B_R, \end{cases}$$

which admits a unique viscosity solution. By the general maximum principle, one may easily construct a barrier function to verify that

$$\|u^\varepsilon - \hat{u}^\varepsilon\|_{L^\infty(B_R)} \leq c R^{2-\frac{n}{p}}\|f\|_{L^p(B_R)},$$

for some $c > 0$ depending only on n, λ, Λ and p. This combined with the assertion with $f = 0$ yields the conclusion. □

With the above lemma at hand, we can extend the uniform pointwise $C^{1,\alpha}$-estimates for fully nonlinear homogenization problems, established in [18], to a more general setting.

Lemma 3.3 (Due to [18, Theorem 4.1]) *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.3), $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in C(\Omega) \cap L^p(\Omega)$ for some $p > p_0$, and $u^\varepsilon \in C(\Omega)$ be a viscosity solution to*

$$F\left(D^2 u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f \quad \text{in } \Omega,$$

for some $\varepsilon > 0$. Suppose that $\bar{F}(D^2 v) = 0$ admits an interior $C^{1,\bar{\alpha}}$-estimates with constant κ, for some $\bar{\alpha} \in (0, 1)$ and $\kappa > 0$. Let $\alpha \in (0, \bar{\alpha})$ be given. Given any $x_0 \in \Omega_\varepsilon$ for which

$I_{(1-\alpha)p}(|f|^p\chi_\Omega)(x_0) < \infty$, *there exists a linear polynomial* $\ell^\varepsilon_{x_0}$ *such that for any* $x \in \Omega$,

$$|D\ell^\varepsilon_{x_0}| + \sup_{x\in\Omega} \frac{|(u^\varepsilon - \ell^\varepsilon_{x_0})(x)|}{|x-x_0|^{1+\alpha} + \varepsilon^{1+\alpha}} \leq C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega)}}{\text{dist}\,(x_0,\partial\Omega)^{1+\alpha}} + ((I_{(1-\alpha)p}(|f|^p\chi_\Omega)(x_0))^{\frac{1}{p}}\right),$$

where $C > 0$ *depends only on* n, λ, Λ, κ, $\bar{\alpha}$, α, p *and* diam (Ω).

Proof This assertion is proved for the case $f \in L^\infty$ in [18, Theorem 4.1]. However, the same proof works equally well for any point $x_0 \in \Omega$ featuring $I_{(1-\alpha)p}(|f|^p\chi_\Omega)(x_0) < \infty$, since this implies that

$$\sup_{r>0} r^{(1-\alpha)p-n} \int_{B_r(x_0)\cap\Omega} |f|^p\,dx < \infty.$$

With the latter observation, the iteration argument in [18, Lemma 4.3] works, without any notable modification, once we invoke Lemma 3.2 as the approximation lemma, in place of [18, Lemma 4.2], in the proof there. Let us remark that the iteration technique for standard problems is by now understood as standard, c.f. [29, Remark 2.5]. Therefore, we shall not repeat the detail here. □

The following lemma is a uniform boundary $C^{1,\alpha}$-estimates, which extends [18, Theorem 5.1] to L^p-integrable datum, and $C^{1,\alpha}$-domains.

Lemma 3.4 (Due to [18, Theorem 5.1]) *Let* $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ *be a functional satisfying* (2.0.1)–(2.0.3), $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $U \subset \mathbb{R}^n$ *be a neighborhood of a point on* $\partial\Omega$ *such that* $\partial\Omega\cap U$ *is a* $C^{1,\alpha}$*-graph, whose norm is bounded by* κ*, for some* $\alpha \in (0,1)$ *and some* $\kappa > 0$, $f \in C(\Omega) \cap L^p(\Omega\cap U)$ *for some* $p > n$, $g \in C^{1,\alpha}(\partial\Omega\cap U)$, *and* $u^\varepsilon \in C(\overline{\Omega}\cap U)$ *be a viscosity solution to*

$$\begin{cases} F\left(D^2u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f & \text{in } \Omega\cap U,\\ u^\varepsilon = g & \text{on } \partial\Omega\cap U,\end{cases}$$

for some $\varepsilon > 0$. *Set* $\alpha_p = \min\{\alpha, 1-\frac{n}{p}\}$. *Given any* $x_0 \in \partial\Omega\cap U_\varepsilon$, *there exists a linear polynomial* $\ell^\varepsilon_{x_0}$ *such that for any* $x \in \Omega\cap U$,

$$|D\ell^\varepsilon_{x_0}| + \sup_{x\in\Omega} \frac{|(u^\varepsilon - \ell^\varepsilon_{x_0})(x)|}{|x-x_0|^{1+\alpha_p} + \varepsilon^{1+\alpha_p}} \leq C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega\cap U)}}{\text{dist}\,(x_0,\partial U)^{1+\alpha_p}} + \|f\|_{L^p(\Omega\cap U)} + \|g\|_{C^{1,\alpha}(\partial\Omega\cap U)}\right),$$

where $C > 0$ *depends only on* n, λ, Λ, κ, α, p *and* diam (U).

Remark 3.5 The reason that we state the above lemma for the case $p > n$ is only because the other case, i.e., $p_0 < p \leq n$, holds in a much general setting, namely for standard problems in the Pucci class, c.f. [21, Theorem 1.6]; of course, in the latter case, one needs to replace α_p with α and $\|f\|_{L^p(\Omega\cap U)}$ with $((I_{(1-\alpha)p}(|f|^p\chi_{\Omega\cap U})(x_0))^{1/p}$, as in the case of Lemma 3.3.

Proof of Lemma 3.4 With a similar modification shown in the proof of Lemma 3.2, we can extend $f \in L^\infty(\Omega\cap U)$, required in [18, Lemma 5.2 and 5.3], to $L^p(\Omega\cap U)$. To extend the lemmas to $\partial\Omega\cap U \in C^{1,\alpha}$ (from $C^{1,1}$), we may combine the compactness argument in [23, Lemma 3.1] with [18, Lemma 3.1, 5.2]. We skip the detail. □

4 Universal decay estimates

This section is devoted to a global, universal decay estimate of the measure of the set for large "gradient" and "Hessian" of viscosity solutions to fully nonlinear equations, up to Reifenberg flat boundaries. As surprising as it may sound, our estimates would not see the boundary value of solutions, as long as the solutions are bounded up to the boundaries. Roughly speaking, this is because of the fact that the boundary layer, as a Reifenberg flat set, can be trapped in between a thin slab, which already has small measure and thus can be neglected. Of course, at a cost, the decay rate we establish here could be extremely slow, yet universal. Let us remark that such a global estimate is hinted in [30], which proves global universal decay estimate for Hessians, up to flat boundaries.

4.1 Set of large gradient

Let us begin with estimates for the set of large gradient. Throughout this section, given any $u \in C(\Omega)$ and $t > 0$, $\underline{L}_t(u, \Omega)$ is the subset of Ω such that $x_0 \in \Omega \setminus \underline{L}_t(u, \Omega)$ if and only if there exists a constant a for which $u(x) \geq a - t|x - x_0|$ for all $x \in \Omega$. Clearly, $L_t(u, \Omega) = \underline{L}_t(u, \Omega) \cap \underline{(} - u, \Omega)$, with $L_t(u, \Omega)$ as in Definition 2.2.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded, (δ, R)-Reifenberg flat domain with $R \in (0, 1]$, $f \in L^p(\Omega)$ be given, with some $p > p_0$, and $u \in C(\Omega) \cap L^\infty(\Omega)$ be an L^p-viscosity solution to $\mathcal{P}_-(D^2u) \leq f$ in Ω. Then for any $t > 0$,*

$$|\underline{L}_t(u, \Omega)| \leq \frac{C|\Omega|}{R^\mu t^\mu}(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)})^\mu,$$

where $C > 1$, $\delta > 0$ and $\mu > 0$ depend only on n, λ, Λ and p.

We shall split our analysis into two parts, each concerning interior and respectively boundary layer. Let us begin with the interior case first. As the analysis below will be of local character, we shall confine ourselves to the case $\Omega = B_{4\sqrt{n}}$ and $\Omega' = Q_1$. The following lemma is the gradient-counterpart of [6, Lemma 7.5].

Lemma 4.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $B_{4\sqrt{n}} \subset \Omega$, $u \in C(\Omega)$ be an L^p-viscosity solution to $\mathcal{P}_-(D^2u) \leq f$ in $B_{4\sqrt{n}}$, for some $f \in L^p(\Omega)$ satisfying $\|f\|_{L^p(B_{4\sqrt{n}})} \leq \delta_0$, with $p > p_0$, such that* $\inf_{B_{4\sqrt{n}}} u \geq -1$, $\inf_{Q_3} u \leq 0$ *and $u(x) \geq -|x|$ for all $x \in \Omega \setminus B_{2\sqrt{n}}$. Then*

$$|\underline{L}_m(u, \Omega) \cap Q_1| \leq \sigma,$$

where $m > 1$, $\delta_0 > 0$ and $\sigma \in (0, 1)$ are constants depending at most on n, λ, Λ and p.

Proof As in the proof of [6, Lemma 7.5], we consider an auxiliary function $w = u + 1 + \varphi$ on $B_{4\sqrt{n}}$, where $\varphi \in C^2(B_{4\sqrt{n}})$ is the barrier function found in [6, Lemma 4.1]. Due to the general maximum principle (which is available as $f \in L^p$ with $p > p_0$), one can argue as in the proof of [6, Lemma 4.5] (where the smallness of $\delta_0 \geq \|f\|_{L^p(B_{4\sqrt{n}})}$ is determined) and observe that

$$|\{w = \Gamma_w\} \cap Q_1| \geq 1 - \sigma,$$

for some constant $\sigma > 0$, depending only on n, λ, Λ and p, where Γ_w is the convex envelope of $-w^-$ in $B_{4\sqrt{n}}$. Our claim is, as again in the proof of [6, Lemma 7.5], that

$$\underline{L}_m(u, \Omega) \subset Q_1 \setminus \{w = \Gamma_w\},$$

for some large constant $m > 1$, depending only on n, λ and Λ.

The main observation here is that the gradient of the supporting hyperplanes for the convex envelope Γ_w at the contact set is universally bounded. This actually follows from a simple fact that by construction, $\Gamma_w = 0$ on $\partial B_{4\sqrt{n}}$, $w \geq 0$ in $B_{4\sqrt{n}} \backslash B_{2\sqrt{n}}$ and $-m \leq \inf_{Q_3} w \leq -1$, for some $m > 0$ depending only on n, λ and Λ. These inequalities follows from the specific choice of the barrier function φ in [6, Lemma 4.1], that $\varphi \geq 0$ on $\mathbb{R}^n \backslash B_{2\sqrt{n}}$ and $\inf_{B_{2\sqrt{n}}} \varphi \geq -m$.

In what follows, we shall let m denote a generic positive constant depending only on n, λ and Λ and allow it to vary at each occurrence.

Keeping in mind of these properties of φ, let $x_0 \in \{w = \Gamma_w\} \cap Q_1$ be any. As Γ_w is the convex envelope of w in $B_{4\sqrt{n}}$, we can find a linear polynomial ℓ (as one of the supporting hyperplanes of Γ_w at x_0) such that $w \geq \ell$ in $B_{4\sqrt{n}}$ and $w(x_0) = \ell(x_0)$. Since $\Gamma_w = 0$ on $\partial B_{4\sqrt{n}}$, $\ell \leq 0$ on $\partial B_{4\sqrt{n}}$. However, as $\ell(x_0) = w(x_0) \geq -m$ with $x_0 \in Q_1$, we deduce that $|D\ell| \leq m$, where $c_n > 0$ depends only on n. Thus,

$$w(x) \geq w(x_0) - m|x - x_0|, \tag{2.0.1}$$

for all $x \in \Omega$.

Next, we observe that we have freedom to choose φ in such a way that $\|D\varphi\|_{L^\infty(B_{4\sqrt{n}})} \leq M$. To see this, note that φ is constructed in such a way that $\varphi(x) = m_1 - m_2|x|^{-\alpha}$ for $x \in B_{4\sqrt{n}} \backslash B_{1/4}$, and it is extended smoothly inside $B_{1/4}$ such that $\mathcal{P}_-(D^2\varphi) \leq c_0\xi$ in $B_{4\sqrt{n}}$, where $\xi \in C(B_{4\sqrt{n}})$ is a continuous function with $0 \leq \xi \leq 1$ and $\operatorname{spt} \xi \subset \overline{Q_1}$; here all constants m_1, m_2, α and c_0 depend only on n, λ and Λ. Since $|D\varphi| \leq m$ in $B_{4\sqrt{n}} \setminus B_{1/4}$, and the extension leaves the gradient free, we can find an extension such that

$$\sup_{B_{4\sqrt{n}}} |D\varphi| \leq m, \tag{2.0.2}$$

by taking m larger if necessary.

Finally, by the definition of w, we deduce from (2.0.1), (2.0.2) and the assumption that $u(x) \geq -|x|^2$ for all $x \in \Omega \setminus B_{2\sqrt{n}}$ that

$$u(x) \geq u(x_0) - m|x - x_0|,$$

for all $x \in \Omega$. This proves that $x_0 \in Q_1 \backslash \underline{L}_m(u, \Omega)$, as desired. □

Now we may argue as in [6, Lemma 7.7] to deduce a universal decay of the measure of the set with large gradient "from below".

Lemma 4.3 *Let $u \in C(B_{4\sqrt{n}})$ be an L^p-viscosity solution to $\mathcal{P}_-(D^2u) \leq f$ in $B_{4\sqrt{n}}$, for some $f \in L^p(B_{4\sqrt{n}})$. Suppose that $\inf_{B_{4\sqrt{n}}} u \geq -1$, $\inf_{Q_3} u \leq 0$, and $\|f\|_{L^p(B_{4\sqrt{n}})} \leq \delta_0$. Let L_k and B_k denote $\underline{L}_{m^k}(u, B_{4\sqrt{n}}) \cap Q_1$ and respectively $\{M(|f|^p \chi_{B_{4\sqrt{n}}}) > \delta_0^p m^{kp}\}$. Then for each integer $k \geq 1$,*

$$|L_{k+1}| \leq \sigma |L_k \cup B_k|,$$

where $m > 1$, $\delta_0 > 0$ and $\sigma \in (0, 1)$ depend only on n, λ, Λ and p.

Proof The proof is almost the same with that of [6, Lemma 7.7]. The involvement of the Riesz potential, which replaces the maximal function in the statement of the latter lemma, is due to the linear rescaling of the solutions.

Fix an integer $k \geq 1$. Due to Lemma 4.2, $|L_1| \leq \sigma$. As $L_{k+1} \subset L_k \subset \cdots \subset L_1$, we have $|L_{k+1}| \leq \sigma$. Hence, due to the Calderón-Zygmund cube decomposition lemma, it suffices to prove that given any dyadic cube $Q \subset Q_1$, if $|L_k \cap Q| > \sigma|Q|$, then $\tilde{Q} \subset L_k \cup B_k$ where $\tilde{Q}$ is the predecessor of Q.

Suppose, by way of contradiction, that $\tilde{Q}\backslash(L_k \cup B_k) \neq \emptyset$. Let x_Q and s_Q be the center and respectively the side-length of Q, i.e., $Q = Q_{s_Q}(x_Q)$. Let us consider the rescaled version of u and respectively f,

$$u_Q(x) = \frac{u(x_Q + s_Q x) - u(x_Q)}{c s_Q m^k}, \quad f_Q(x) = \frac{s_Q}{cm^k} f(x_Q + s_Q x),$$

with $c > 1$ being a constant to be determined solely by n and p, and let $\Omega_Q = s_Q^{-1}(-x_Q + B_{4\sqrt{n}})$. Choose any $\tilde{x}_Q \in \tilde{Q} \setminus (L_k \cup B_k)$. Then since $\tilde{x}_Q \in Q_{3s_Q}(x_Q)$, one can easily verify from $\tilde{x}_Q \notin L_k$ that

$$\inf_{B_{4\sqrt{n}}} u_Q \geq -1, \quad \inf_{Q_3} u_Q \leq u_Q(0) = 0, \tag{2.0.3}$$

where the first inequality is ensured by choosing $c > 1$ large, depending only on n. Moreover, since $B_{4s_Q\sqrt{n}}(x_Q) \subset B_{6s_Q\sqrt{n}}(\tilde{x}_Q)$, $s_Q < 1$ and $\tilde{x}_Q \notin B_k$, we also obtain

$$\|f_Q\|_{L^p(B_{4\sqrt{n}})} \leq \frac{c_0 s_Q}{cm^k} (M(|f|^p \chi_{B_{4\sqrt{n}}})(\tilde{x}_Q))^{\frac{1}{p}} \leq \delta_0, \tag{2.0.4}$$

provided that we choose $c > c_0$. Furthermore,

$$\mathcal{P}_-(D^2 u_Q) \leq f_Q \quad \text{in } B_{4\sqrt{n}}, \tag{2.0.5}$$

in the L^p-viscosity sense.

Thanks to (2.0.3)–(2.0.5), u_Q and f_Q fall under the setting of Lemma 4.2, from which we deduce that

$$|\underline{L}_{c^{-1}m}(u_Q, \Omega_Q) \cap Q_1| \leq \sigma$$

by choosing $m > 1$ larger from the beginning so that $c^{-1}m$ becomes the constant appearing in the latter lemma. Rescaling back, we arrive at $|L_{k+1} \cap Q| \leq \sigma|Q|$, a contradiction to the choice of Q. Thus, the proof is finished. □

As a corollary, we obtain a universal decay estimate in the interior. We shall only present the statement and omit the proof, as it is essentially the same with that of [6, Lemma 7.8].

Lemma 4.4 *Under the setting of Lemma 4.3, for any* $t > 0$,

$$|\underline{L}_t(u, \Omega) \cap Q_1| \leq ct^{-\mu},$$

where $c > 1$ *and* $\mu > 0$ *depend at most on* n, λ, Λ *and* p.

From now on, we shall study the estimates near boundaries. As mentioned earlier, the idea to combine the interior estimate with the small measure of the thin slab that contains the boundary layer is originally from [30, Lemma 2.9]; here we simply extend the argument to the framework of Reifenberg flat domains.

Lemma 4.5 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* $0 \in \partial\Omega$ *such that* $\partial\Omega \cap B_2$ *is* $(\delta, 1)$-*Reifenberg flat, and* $u \in C(\Omega)$ *be an* L^p-*viscosity solution to* $\mathcal{P}_-(D^2u) \leq f$ *in* Ω *for some* $f \in L^p(\Omega)$, *for some* $p > p_0$. *Suppose that* $\|f\|_{L^p(\Omega\cap B_2)} \leq 1$, $\inf_{\Omega\cap B_2} u \geq -1$, $\inf_{\Omega\cap B_1} u \leq 0$ *and* $u(x) \geq -|x|$ *for all* $x \in \Omega \setminus B_1$. *Then*

$$|\underline{L}_t(u, \Omega) \cap B_1| \leq c(\delta^{-\mu} t^{-\mu} + \delta),$$

for any $t > 0$, *where* $c > 0$ *and* $\mu > 0$ *depend at most on* n, λ, Λ *and* p.

Proof Since $\partial\Omega \cap B_2$ is $(\delta, 1)$-Reifenberg flat and $0 \in \partial\Omega$, there exists a unit vector $\nu \in \mathbb{R}^n$ such that $\partial\Omega \cap B_1 \subset \{x \in B_1 : |x \cdot \nu| < \delta\}$. Due to (a properly rescaled form of) Lemma 4.4, we have, for any $t > 1$,

$$|\underline{L}_t(u, \Omega) \cap \{x \in B_1 : x \cdot \nu > 2\delta\}| \le c\delta^{-\mu} t^{-\mu}.$$

The conclusion follows easily from the observation that $|\{x \in B_1 : |x \cdot \nu| < 2\delta\}| \le c_n \delta$, and $\underline{L}_t(u, \Omega) \subset \Omega$ (hence $L_t(u, \Omega) \cap B_1 = L_t(u, \Omega) \cap \{x \in B_1 : x \cdot \nu > -\delta\}$). □

Next, we obtain universal decay estimates near boundary layers.

Lemma 4.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $0 \in \partial\Omega$ such that $\partial\Omega \cap B_2$ is $(\delta_0, 2)$-Reifenberg flat, and $u \in C(\Omega \cap B_2)$ be an L^p-viscosity solution to $\mathcal{P}_-(D^2 u) \le f$ in $\Omega \cap B_2$ for some $f \in L^p(\Omega \cap B_2)$, for some $p > p_0$. Denote by L_k and B_k the sets $\underline{L}_{m^k}(u, \Omega \cap B_2) \cap B_1$ and respectively $\{M(|f|^p \chi_{\Omega \cap B_2}) > m^{kp}\}$, where I_p is the Riesz potential of order p. Suppose that $\|u\|_{L^\infty(\Omega \cap B_2)} \le 1$ and $\|f\|_{L^p(\Omega \cap B_2)} \le 1$. Then for each integer $k \ge 1$,*

$$|L_{k+1}| \le \frac{1}{2} |L_k \cup B_k|,$$

where $m > 1$ and $\delta_0 \in (0, \frac{1}{2})$ are constants depending at most on n, λ, Λ and p.

Proof Let $m > 1$ be a constant to be determined later, and set $\eta = c_1 c_0 (\delta_0^{-\mu} m^{-\mu} + \delta_0)$, where $c_0 > 1$ and $\mu > 0$ are as in Lemma 4.5, and $c_1 > 1$ is a constant to be determined later, by n, λ, Λ and p only. Fix any integer $k \ge 1$. Then it follows from the latter lemma, as well as the relation $L_{k+1} \subset L_k \subset \cdots \subset L_1$, that $|L_{k+1}| \le \eta$.

Fix any ball $B \subset B_1$ with center in $\overline{\Omega} \cap B_1$ and rad $B \le 1$. Suppose that $|L_{k+1} \cap B| > \eta |B|$. We claim that

$$\Omega \cap B \subset L_k \cup B_k. \tag{2.0.6}$$

Assume for the moment that the claim is true. Then it follows from Lemma 2.7 (along with $\delta_0 < \frac{1}{2}$), that $|L_{k+1}| \le c_n \eta |L_k \cup B_k|$. Then we first choose δ_0 sufficiently small such that $4 c_n c_0 \delta_0 \le 1$. Selecting m accordingly large such that $4 c_n c_0 \delta^{-\mu} m^{-\mu} \le 1$, we obtain that $c_n \eta = c_n c_0 (\delta_0^{-\mu} m^{-\mu} + \delta_0) \le \frac{1}{2}$, which finishes the proof.

Henceforth, we shall prove the claim (2.0.6). Suppose by way of contradiction that $\Omega \cap B \backslash (L_k \cup B_k) \neq \emptyset$. Here it suffices to consider the case $2B \setminus (\Omega \cap B_2) \neq \emptyset$, since the other case can be handled as in the interior analysis (see the proof of Lemma 4.3).

Set $r_B = \text{rad } B$ and choose $x_B \in \partial\Omega \cap B_1$ in such a way that $2B \subset B_{4r_B}(x_B)$. Set $\Omega_B = -\frac{1}{2r_B}(-x_B + \Omega \cap B_2)$ and rescale u and f as follows,

$$u_B(x) = \frac{u(x_B + 2r_B x) - u(x_B)}{c m^k r_B}, \quad f_B(x) = \frac{r_B}{c m^k} f(x_B + 2 r_B x).$$

Arguing analogously as in the proof of Lemma 4.3, we may deduce from $\Omega \cap B \setminus (L_k \cup B_k) \neq \emptyset$ that $\inf_{\Omega \cap B_2} u_B \ge -1$, $\inf_{\Omega \cap B_1} u_B \le 0$, $u_B(x) \ge -|x|$ for all $x \in \Omega_B \backslash B_1$, and $\|f_B\|_{L^p(\Omega_B \cap B_2)} \le 1$, provided that $c > 1$ is a large constant, depending at most on n and p. Moreover,

$$\mathcal{P}_-(D^2 u_B) \le f_B \quad \text{in } \Omega_B \cap B_2,$$

in the L^p-viscosity solution. Thanks to the scaling invariance of the Reifenberg flatness, $\partial\Omega_B \cap B_2$ is $(\delta_0, 2)$-Reifenberg flat and contains the origin. Thus, we can employ Lemma 4.5 to deduce that

$$|\underline{L}_t(u_B, \Omega_B) \cap B_1| \le c_0 (\delta_0^{-\mu} t^{-\mu} + \delta_0),$$

for any $t > 0$. Rephrase the above inequality in terms of u, and deduce that $|\underline{L}_{cm^k t}(u, \Omega) \cap B_{2r_B}(x_B)| \leq c_0(\delta_0^{-\mu} t^{-\mu} + \delta_0)|2r_B|^n$. As $B \subset B_{2r_B}(x_B)$ and $r_B = \text{rad } B$, we derive that

$$|\underline{L}_{cm^k t}(u, \Omega) \cap B| \leq \frac{2^n c_0}{\omega_n}(\delta_0^{-\mu} t^{-\mu} + \delta_0)|B|,$$

where ω_n is the volume of the n-dimensional unit ball. Evaluating this inequality at $t = c^{-1}m$, we reach contradiction against $|L_{k+1} \cap B_r(x_0)| > \eta|B_r(x_0)|$ with $\eta = 2^n \omega_n^{-1} c^\mu c_0(\delta_0 m^{-\mu} + \delta_0)$ (i.e., $c_1 = 2^n \omega_n^{-1} c^\mu$ from the notation in the beginning of the proof). □

Now we have a boundary-analogue of Lemma 4.4. Let us skip the proof for the same reason as mentioned above the statement of the latter lemma.

Lemma 4.7 *Under the same hypothesis of Lemma 4.6, for any $t > 0$,*

$$|\underline{L}_t(u, \Omega) \cap B_1| \leq ct^{-\mu},$$

where $c > 1$ and $\mu > 0$ depend at most on n, λ, Λ and p.

Finally, we are ready to prove the global universal decay asserted in the beginning of this section.

Proof of Proposition 4.1 With Lemmas 4.4 and 4.7, the assertion of this proposition follows easily via a standard covering argument. The exponent μ can be taken as the minimum between those in both lemmas. We omit the detail. □

4.2 Set of large Hessian

Here we shall study universal decay estimates for the set of large Hessian. Note that an interior estimate is by now considered classical, and can be found in [6, Lemma 7.8], while an estimate near flat boundary is established rather recently in [30]. Here we extend the result to Reifenberg flat boundaries.

Here given any $u \in C(\Omega)$ and $t > 0$, $\underline{A}_t(u, \Omega)$ is defined, as in [6, Section 7], as a subset of Ω such that $x_0 \in \Omega \backslash \underline{A}_t(u, \Omega)$ if and only if there exists a linear polynomial ℓ for which $u(x) \geq \ell(x) - \frac{t}{2}|x - x_0|^2$ for all $x \in \Omega$.

Proposition 4.8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded, (δ, R)-Reifenberg flat domain with $R \in (0, 1]$, $f \in L^p(\Omega)$ be given, with some $p > p_0$, and $u \in C(\Omega) \cap L^\infty(\Omega)$ be an L^p-viscosity solution to $\mathcal{P}_-(D^2 u) \leq f$ in Ω. Then for any $t > 0$,*

$$|\underline{A}_t(u, \Omega)| \leq \frac{C|\Omega|}{R^\mu t^\mu}(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)})^\mu,$$

where $C > 1$, $\delta > 0$ and $\mu > 0$ depend at most on n, λ, Λ and p.

Proof Since the proof repeats most of the argument presented in the section above, we shall pinpoint the difference and skip the detail. First, observe that we can replace $\underline{L}_t$ in Lemma 4.5 with $\underline{A}_t$, by simply applying [6, Lemma 7.5] (which holds equally well for L^p-viscosity solutions with L^p-integrable right-hand side, due to [8]) in place of Lemma 4.2 in the proof. Now the assertion of Lemma 4.6 holds true with L_k now denoting the set $\underline{A}_{m^k}(u, \Omega) \cap B_1$, since the proof only uses Lemma 4.2. Finally, iterating the modified version of Lemma 4.6 would yield Lemma 4.7, again with $\underline{L}_t(u, \Omega)$ replaced by $\underline{A}_t(u, \Omega)$. Thus, a standard covering argument along with the L^p-variant of [6, Lemma 7.5] and the modified version of Lemma 4.7 would yield the conclusion of this proposition. □

5 Uniform estimates in the interior

5.1 $W^{1,\frac{np}{n-p}}$-estimates

This section is devoted to uniform interior $W^{1,\frac{np}{n-p}}$-estimates in fully nonlinear homogenization problems, for any $p \in (p_0, n)$; note that $\frac{np}{n-p}$ is the (critical) Sobolev exponent of p. The estimate is optimal, and is even new in the context of standard fully nonlinear problems.

Theorem 5.1 *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.3), $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in C(\Omega) \cap L^p(\Omega)$, for some $p \in (p_0, n)$, and $u^\varepsilon \in C(\Omega)$. for some $\varepsilon > 0$, be a viscosity solution to*

$$F\left(D^2 u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f \quad in\ \Omega. \tag{2.0.1}$$

Then $G^\varepsilon_\Omega(u^\varepsilon) \in L^{\frac{np}{n-p}}_{loc}(\Omega)$, and for any subdomain $\Omega' \Subset \Omega$,

$$\|G^\varepsilon_\Omega(u^\varepsilon)\|_{L^{\frac{np}{n-p}}(\Omega')} \leq C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega)}}{\text{dist}\,(\Omega', \partial\Omega)^{2-\frac{n}{p}}} + \|f\|_{L^p(\Omega)}\right),$$

where $C > 0$ depends only on n, λ, Λ, q and p.

In what follows, we shall present our argument with $\Omega = B_{4\sqrt{n}}$ and $\Omega' = Q_1$, as our analysis will be of local character. Also, unless stated otherwise, we shall always assume that F is a continuous functional on $\mathcal{S}^n \times \mathbb{R}^n$ satisfying (2.0.1)–(2.0.3), u^ε is a viscosity solution to (2.0.1) with Ω replaced by $B_{4\sqrt{n}}$, for some $\varepsilon > 0$, and $f \in C(B_{4\sqrt{n}}) \cap L^p(B_{4\sqrt{n}})$ for some $p > p_0$.

Let us begin with an approximation lemma for the measure of the set with large “gradient".

Lemma 5.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $B_{4\sqrt{n}} \subset \Omega$, and suppose that $0 < \varepsilon < 1$, $\|u^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})} \leq 1$ and $|u^\varepsilon(x)| \leq |x|$ for all $x \in \Omega \setminus B_{2\sqrt{n}}$. Then for any $s > N$,*

$$|L^\varepsilon_s(u^\varepsilon, \Omega) \cap Q_1| \leq c s^{-\mu} \|f\|^\mu_{L^p(B_{4\sqrt{n}})},$$

where $N > 1$ and $\mu > 0$ depend at most on n, λ, Λ and p.

Proof Consider an auxiliary boundary value problem,

$$\begin{cases} F\left(D^2 h^\varepsilon, \frac{\cdot}{\varepsilon}\right) = 0 & \text{in } B_{4\sqrt{n}}, \\ h^\varepsilon = u^\varepsilon & \text{on } \partial B_{4\sqrt{n}}. \end{cases}$$

As $u^\varepsilon \in C(\partial B_{4\sqrt{n}})$ and $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ satisfying (2.0.1), there exists a unique viscosity solution $h^\varepsilon \in C(\overline{B}_{4\sqrt{n}})$ to the above problem. By the maximum principle, $\|h^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})} \leq \|u^\varepsilon\|_{L^\infty(\partial B_{4\sqrt{n}})} \leq 1$. Now it follows from Lemma 3.3, along with the Kyrlov theory [6, Corollary 5.7], that for any $x_0 \in B_{3\sqrt{n}}$, there exists a linear polynomial $\ell^\varepsilon_{x_0}$ such that $|D\ell^\varepsilon_{x_0}| \leq c$ and $|(h^\varepsilon - \ell^\varepsilon_{x_0})(x)| \leq c(|x - x_0|^{1+\alpha} + \varepsilon^{1+\alpha})$ for all $x \in B_{4\sqrt{n}}$, for some $c > 1$ and $\bar{\alpha} \in (0, 1)$ depending only on n, λ and Λ. Thus,

$$L^\varepsilon_c(h^\varepsilon, B_{4\sqrt{n}}) \cap Q_1 = \emptyset, \tag{2.0.2}$$

by taking $c > 1$ slightly larger if necessary.

On the other hand, due to Lemma 3.1, one can compute that $w^\varepsilon = \delta^{-1}(u^\varepsilon - h^\varepsilon)$ satisfies

$$\begin{cases} \mathcal{P}_-(D^2 w^\varepsilon) \leq \frac{f}{\delta} \leq \mathcal{P}_+(D^2 w^\varepsilon) & \text{in } B_{4\sqrt{n}}, \\ w^\varepsilon = 0 & \text{on } \partial B_{4\sqrt{n}}, \end{cases}$$

in the viscosity sense. Since we assume $\|f\|_{L^p(B_{4\sqrt{n}})} \leq \delta$, it follows from the general maximum principle that $\|w^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})} \leq c$. Therefore, one can deduce from Proposition 4.1 to both w^ε and $-w^\varepsilon$ that $|L_t(w^\varepsilon, B_{4\sqrt{n}})| \leq ct^{-\mu}$ for all $t > 0$. Rephrasing this inequality in terms of u^ε, we deduce that

$$|L_s(u^\varepsilon - h^\varepsilon, B_{4\sqrt{n}})| \leq c\delta^\mu s^{-\mu}, \tag{2.0.3}$$

for any $s > 0$. Thus, the conclusion follows from (2.0.2) and (2.0.3), as well as the assumption that $|u^\varepsilon(x)| \leq |x|$ for all $x \in \Omega \backslash B_{2\sqrt{n}}$. □

The following lemma is an analogue of [6, Lemma 7.12].

Lemma 5.3 *Suppose that $\|f\|_{L^p(B_{4\sqrt{n}})} \leq \delta$ for some $\delta \in (0, 1)$, and that $\|u^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})} \leq 1$. Let L_k^ε and B_k denote $L_{m^k}^\varepsilon(u^\varepsilon, \Omega) \cap Q_1$ and respectively $\{I_p(|f|^p \chi_{B_{4\sqrt{n}}}) > \delta^p m^{kp}\}$. Then for any integer $k \geq 1$,*

$$|L_{k+1}^\varepsilon| \leq \delta^\mu |L_k^\varepsilon \cup B_k|,$$

where $M > 1$ and $\mu > 0$ are constants depending at most on n, λ, Λ and p.

Proof Fix an integer $k \geq 1$. Let $M > N$ be a large constant such that $c_0 M^{-\mu} < 1$, with c_0, $N > 1$ and $\mu > 0$ as in Lemma 3.2; note that M depends only on n, λ, Λ and p. By Lemma 3.2 (along with $c_0 M^{-\mu} < 1$), $|L_1^\varepsilon| \leq \delta^\mu$. Since $L_{k+1}^\varepsilon \subset L_k^\varepsilon \subset \cdots \subset L_1^\varepsilon$, we have $|L_{k+1}^\varepsilon| \leq \delta^\mu < 1$.

The rest of our proof will resemble that of [6, Lemma 7.12]. Let $Q \subset Q_1$ be a dyadic cube such that $|L_{k+1}^\varepsilon \cap Q| > \delta^\mu |Q|$. We claim that $\tilde{Q} \subset L_k^\varepsilon \cup B_k$, where $\tilde{Q}$ is the predecessor of Q. Once this claim is justified, the conclusion is ensured by the Calderón-Zygmund cube decomposition lemma.

Suppose, by way of contradiction, that $\tilde{Q} \backslash (L_k^\varepsilon \cup B_k) \neq \emptyset$. Denote by x_Q and s_Q the center and respectively the side-length of Q; i.e., $Q = Q_{s_Q}(x_Q)$. Choose any point $\tilde{x}_Q \in \tilde{Q} \backslash (L_k^\varepsilon \cup B_k)$. Now since $|\tilde{x}_Q - x_Q| \leq \frac{3}{2} s_Q \sqrt{n}$, we have $B_{4s_Q\sqrt{n}}(x_Q) \subset B_{6s_Q\sqrt{n}}(\tilde{x}_Q)$.

Let us first remark that $s_Q > \varepsilon$, since $L_{k+1}^\varepsilon \cap Q \neq \emptyset$. The reason is as follows. Suppose that $s_Q \leq \varepsilon$. Then since $\tilde{x}_Q \in \tilde{Q} \backslash L_k^\varepsilon$, there exists some constant $a \in \mathbb{R}$ for which $|u^\varepsilon(x) - a| \leq m^k(|x - \tilde{x}_Q| + \varepsilon)$ for all $x \in B_{4\sqrt{n}}$. Therefore, as diam $(\tilde{Q}) = 2s_Q\sqrt{n} < 2\varepsilon\sqrt{n}$, it follows from the latter inequality that for any $x_0 \in \tilde{Q}$, $|u^\varepsilon(x) - a| \leq m^k(|x - x_0| + (2\sqrt{n} + 1)\varepsilon) \leq m^{k+1}(|x - x_0| + \varepsilon)$ for all $x \in B_{4\sqrt{n}}$, provided that $m > 2\sqrt{n} + 1$. This implies that $\tilde{Q} \cap L_{k+1}^\varepsilon = \emptyset$, a contradiction.

Since $\tilde{x}_Q \notin L_k$, we can choose a constant a for which $|u^\varepsilon(x) - a| \leq m^k(|x - \tilde{x}_Q| + \varepsilon)$ for all $x \in B_{4\sqrt{n}}$. This combined with $|\tilde{x}_Q - x_Q| \leq \frac{3}{2} s_Q \sqrt{n}$ and $\varepsilon < s_Q$ yields that

$$|u^\varepsilon(x) - a| \leq m^k \left(\frac{5}{2} s_Q \sqrt{n} + |x - x_Q| \right), \tag{2.0.4}$$

for all $x \in B_{4\sqrt{n}}$. In addition, due to the assumption $\tilde{x}_Q \notin B_k$, as well as $B_{4s_Q\sqrt{n}}(x_Q) \subset B_{6s_Q\sqrt{n}}(\tilde{x}_Q)$, one can compute that

$$\begin{aligned} \int_{B_{4s_Q\sqrt{n}}(x_Q)\cap B_{4\sqrt{n}}} |f(x)|^p\,dx &\le \int_{B_{6s_Q\sqrt{n}}(\tilde{x}_Q)\cap B_{4\sqrt{n}}} |f(x)|^p\,dx \\ &\le (6s_Q\sqrt{n})^{n-p} \int_{B_{4\sqrt{n}}} \frac{|f(x)|^p}{|x-\tilde{x}_Q|^{n-p}}\,dx \\ &\le c^p s_Q^{n-p}\delta^p, \end{aligned} \tag{2.0.5}$$

where $c > 1$ depends only on n and p.

In what follows, we shall use $c > 1$ to denote a positive constant depending at most on n and p, and allow it to vary at each occurrence.

Let us consider the following rescaled versions of u^ε and f,

$$\begin{aligned} u_Q^{\varepsilon_Q}(x) &= \frac{u^\varepsilon(x_Q + s_Q x) - a}{cm^k s_Q}, \quad \varepsilon_Q = \frac{\varepsilon}{s_Q}, \\ f_Q(x) &= \frac{s_Q}{cm^k} f(x_Q + s_Q x). \end{aligned}$$

Setting $\Omega_Q = s_Q^{-1}(-x_Q + B_{4\sqrt{n}})$, we have $B_{4\sqrt{n}} \subset \Omega_Q$, and thus in view of (2.0.1), $u_Q^{\varepsilon_Q}$ is a viscosity solution to

$$F_Q\left(D^2 u_Q^{\varepsilon_Q}, \frac{\cdot}{\varepsilon_Q}\right) = f_Q \quad \text{in } B_{4\sqrt{n}},$$

where $F_Q(P, y) = \frac{s_Q}{cm^k} F(\frac{cm^k}{s_Q} P, y + \frac{x_Q}{\varepsilon})$. Clearly, $F_Q \in C(\mathcal{S}^n \times \mathbb{R}^n)$ and it satisfies (2.0.1)–(2.0.3). On the other hand, it follows immediately from (2.0.4) and (2.0.5) that with $c > 1$ sufficiently large, $|u^{\varepsilon_Q}(x)| \le 1$ for all $x \in B_{4\sqrt{n}}$, $|u^{\varepsilon_Q}(x)| \le |x|$ for all $\Omega_Q \backslash B_{2\sqrt{n}}$, and $\|f_Q\|_{L^p(B_{4\sqrt{n}})} \le \delta$.

In all, ε_Q, F_Q, $u_Q^{\varepsilon_Q}$ and f_Q fall into the setting of Lemma 5.2, from which we obtain

$$|L_s^{\varepsilon_Q}(u_Q^{\varepsilon_Q}, \Omega_Q) \cap Q_1| \le c_0 \delta^\mu s^{-\mu},$$

for any $s > N$ with $N > 1$ as in the latter lemma. Thus, taking $M > N$ larger if necessary such that $c_0 c^{-\mu} M^{-\mu} < 1$, and then rescaling back to u^ε, we arrive at $|L_{k+1}^\varepsilon \cap Q| \le \delta^\mu |Q|$, a contradiction. □

We are ready to prove the uniform "large-scale" interior $W^{1,\frac{np}{n-p}}$-estimate.

Proof of Theorem 5.1 After some suitable rescaling argument, it suffices to consider the case where $\Omega = B_{4\sqrt{n}}$, $\Omega' = Q_1$, $\|u^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})} \le 1$ and $\|f\|_{L^p(B_{4\sqrt{n}})} \le \delta$, where $\delta \in (0, 1)$ is to be determined by n, λ, Λ and p only.

Set $p' = \frac{p+p_0}{2} \in (p_0, p)$, and apply Lemma 5.3, with p replaced by p'. Observe that $\|f\|_{L^{p'}(B_{4\sqrt{n}})} \le c\|f\|_{L^p(B_{4\sqrt{n}})} \le c\delta$ for some $c > 0$ depending only on n and p. Hence, with $\eta = (c\delta)^\mu < 1$ (by choosing δ smaller if necessary), $\alpha_k = |L_k^\varepsilon|$ and $\beta_k = |B_k| = |\{I_{p'}(|f|^{p'}\chi_{B_{4\sqrt{n}}}) > (c\delta m^k)^{p'}\} \cap B_{4\sqrt{n}}|$, we obtain that

$$\alpha_k \le \eta(\alpha_{k-1}^\varepsilon + \beta_{k-1}) \le \cdots \le \eta^k + \sum_{i=1}^k \eta^i \beta_{k-i}. \tag{2.0.6}$$

Now as $f \in L^p(B_{4\sqrt{n}})$, we have $|f|^{p'}\chi_{B_{4\sqrt{n}}} \in L^{\frac{p}{p'}}(\mathbb{R}^n)$ with $\frac{p}{p'} > 1$. Thus, according to the embedding theorem for the Riesz potential, $I_{p'}(|f|^{p'}\chi_{B_{4\sqrt{n}}}) \in L^{\frac{np}{p'(n-p)}}(\mathbb{R}^n)$, from which we can compute that

$$\begin{aligned}\sum_{k=1}^{\infty} M^{\frac{np}{n-p}k}\beta_k &\le c\int_0^\infty t^{\frac{np}{n-p}-1}|\{I_{p'}(|f|^{p'}\chi_{B_{4\sqrt{n}}}) > t^{p'}\}\cap B_{4\sqrt{n}}| \\ &= c\int_{B_{4\sqrt{n}}} (I_{p'}(|f|^{p'}\chi_{B_{4\sqrt{n}}}))^{\frac{np}{p'(n-p)}}\,dx \\ &\le c.\end{aligned} \tag{2.0.7}$$

To this end, we choose $\delta \in (0,1)$ as a sufficiently small constant such that $M^{\frac{np}{n-p}}\eta = M^{\frac{np}{n-p}}(c\delta)^\mu \le \frac{1}{2}$; clearly, it is the set of parameters n, λ, Λ and p that determines how small δ should be. Then it follows from (2.0.6) and (2.0.7) that

$$\sum_{k=1}^{\infty} M^{\frac{np}{n-p}k}\alpha_k \le \sum_{k=1}^{\infty} M^{\frac{np}{n-p}k}\eta^k + \sum_{k=1}^{\infty}\sum_{i=1}^{k}(M^{\frac{np}{n-p}i}\eta^i)(M^{\frac{np}{n-p}(k-i)}\beta_{k-i}) \le c.$$

Since $\{|D_\varepsilon u^\varepsilon| > t\} \subset L_t^\varepsilon(u^\varepsilon, B_{4\sqrt{n}})$, we have proved that $\|D_\varepsilon u^\varepsilon\|_{L^{\frac{np}{n-p}}(Q_1)} \le c$, as desired. □

With the additional assumption in the statement of the theorem, we can replace L_s^ε in Lemma 5.2 with L_s. For we can now invoke the uniform interior $C^{1,\alpha}$-estimate below ε-scale, [18, Theorem 4.1 (ii)], to deduce that the approximating solution h^ε belongs to $C^{1,\alpha}(B_{3\sqrt{n}})$ with $\|Dh^\varepsilon\|_{C^\alpha(B_{3\sqrt{n}})} \le c$, whence we can replace $L_N^\varepsilon(h^\varepsilon,\cdot)$ with $L_N(h^\varepsilon,\cdot)$ in (2.0.2).

5.2 $W^{2,p}$-estimates

This section is devoted to interior $W^{2,p}$-estimates for viscosity solutions to a certain class of fully nonlinear homogenization problems.

Theorem 5.4 *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.4), $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in C(\Omega)\cap L^p(\Omega)$ for some $p \in (p_0,\infty)$, and $u^\varepsilon \in C(\Omega)$ be a viscosity solution to*

$$F\left(D^2u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f \quad in\ \Omega. \tag{2.0.1}$$

Then $H_\Omega^\varepsilon(u^\varepsilon) \in L^p_{loc}(\Omega)$ and for any subdomain $\Omega' \Subset \Omega$,

$$\|H_\Omega^\varepsilon(u^\varepsilon)\|_{L^p(\Omega')} \le C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega)}}{\operatorname{dist}(\Omega',\partial\Omega)^{2-\frac{n}{p}}} + \|f\|_{L^p(\Omega)}\right),$$

where $C > 0$ depends only on n, λ, Λ, ψ, κ and p.

Although Lemma 3.2 yields an error estimate between u^ε and $\bar{u}$ in L^∞ norm, we cannot expect $\bar{u}$ is close to u^ε in the viscosity sense (i.e., $\mathcal{P}_\pm(D^2(u^\varepsilon - \bar{u})) \ne o(1)$ with $\mathcal{P}_\pm$ being the Pucci extremal operators), since D^2u^ε is supposed to be rapidly oscillating around $D^2\bar{u}$ in the small scales.

In the next lemma, we obtain the closeness between D^2u^ε and $D^2\bar{u}$ in the viscosity sense by incorporating interior correctors. To do so, we shall assume VMO-condition, or more exactly

BMO_ψ-condition for some modulus of continuity ψ, for $D^2\bar{u}$. This condition replaces the small oscillation condition for standard problems, $F(D^2u, x) = f$, c.f. [6, Theorem 7.1]. Let us remark that BMO_ψ-regularity does neither imply nor follow from the boundedness of $D^2\bar{u}$, and that it also allows $D^2\bar{u}$ to be discontinuous.

With suitable rescaling argument, it suffices to take care of the case where $\Omega = B_{4\sqrt{n}}$ and $\Omega' = Q_1$. In what follows, we shall always assume that F is a continuous functional satisfying (2.0.1)–(2.0.3), $f \in C(B_{4\sqrt{n}}) \cap L^p(B_{4\sqrt{n}})$ with $p > p_0$, $u^\varepsilon \in C(B_{4\sqrt{n}})$ is a viscosity solution to (2.0.1), unless stated otherwise. Moreover, we shall let c denote a positive generic constant depending at most on a set of fixed quantities, shown in the statement of each lemmas below, and we allow it to vary at each occurrence.

Lemma 5.5 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $B_{4\sqrt{n}} \subset \Omega$, $\|u^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})} \le 1$ and $|u^\varepsilon(x)| \le |x|^2$ in $\Omega \setminus B_{2\sqrt{n}}$. Let $\eta > 0$, $s > 1$, $p \in (p_0, \infty)$ and $q \in (p, \infty)$ be given constants. Then there exists a constant $\varepsilon_{s,\eta} > 0$, depending only on n, λ, Λ, κ, ψ, q, p, η and s, such that if $0 < \varepsilon < \varepsilon_{s,\eta}$ and $\|f\|_{L^p(B_{4\sqrt{n}})} \le \varepsilon_{s,\eta}$, then*

$$|A_s^\varepsilon(u^\varepsilon, \Omega) \cap Q_1| \le c(\eta s^{-\mu} + s^{-q}), \tag{2.0.2}$$

where $\mu \in (0,1)$ depends only on n, λ, Λ, p, and $c > 1$ depends further on ψ, κ and q.

Proof Fix $s > 1$ and $\eta > 0$. Let $\delta > 0$ be a constant to be determined later. Since the hypothesis of Lemma 3.2 is met, we can find a constant $\varepsilon_\delta \in (0,1)$, corresponding to δ, and a function $\bar{u} \in C(\overline{B_{3\sqrt{n}}})$ such that $\bar{F}(D^2\bar{u}) = 0$ in $B_{3\sqrt{n}}$ in the viscosity sense, $\|\bar{u}\|_{L^\infty(B_{3\sqrt{n}})} \le 1$ and $\|u^\varepsilon - \bar{u}\|_{L^\infty(B_{3\sqrt{n}})} \le \delta$. By the assumption (2.0.4) on $\bar{F}$, $D^2\bar{u} \in BMO_\psi(B_{2\sqrt{n}})$ and

$$\int_{B_{2\sqrt{n}}} |D^2\bar{u}|\,dx + [D^2\bar{u}]_{BMO_\psi(B_{2\sqrt{n}})} \le c. \tag{2.0.3}$$

Due to the John–Nirenberg inequality, we have $\|D^2\bar{u}\|_{L^q(B_{2\sqrt{n}})} \le c$, so by the strong (q,q)-type inequality, $\|M(|D^2\bar{u}|\chi_{B_{2\sqrt{n}}})\|_{L^q(\mathbb{R}^n)} \le c$, which ensures that

$$|\{M(|D^2\bar{u}|\chi_{2\sqrt{n}}) > s\}| \le cs^{-q}. \tag{2.0.4}$$

In addition, by [21, Lemma 2.5], we also have $\|\Theta(\bar{u}, B_{2\sqrt{n}})\|_{L^q(B_{2\sqrt{n}})} \le c\|D^2\bar{u}\|_{L^q(B_{2\sqrt{n}})} \le c$ (see Definition 2.2 for the definition of Θ), whence it follows from the relation $\{\Theta(\bar{u}, B_{2\sqrt{n}}) > s\} = A_s(\bar{u}, B_{2\sqrt{n}})$ that

$$|A_s(\bar{u}, B_{2\sqrt{n}})| \le cs^{-q}. \tag{2.0.5}$$

Let $\rho_s \in (0, \frac{1}{4})$ be a constant to be determined later. Our idea is to subdivide Q_1 into two groups, say $\mathcal{F}$ and $\mathcal{G}$, of dyadic cubes with side-length in between ρ_s and $2\rho_s$ such that $Q \in \mathcal{F}$ if $|Q \cap \{M(|D^2\bar{u}|\chi_{B_{2\sqrt{n}}}) \le \frac{s}{c}\}| > 0$, and $Q \in \mathcal{G}$ if $Q \notin \mathcal{F}$; here $c > 1$ is a constant to be determined by n, λ, Λ, κ, ψ and q. Thanks to (2.0.4) and (2.0.5) (as well as an obvious fact that $A_s^\varepsilon(g,E) \subset A_s(g,E)$ for any $g \in C(E)$) yields that

$$\begin{aligned}
&|A_s^\varepsilon(u^\varepsilon, B_{2\sqrt{n}}) \cap Q_1| \\
&\le \sum_{Q\in\mathcal{F}} |A_s^\varepsilon(u^\varepsilon, B_{2\sqrt{n}}) \cap Q| + \sum_{Q\in\mathcal{G}} |Q| \\
&\le \sum_{Q\in\mathcal{F}} |A_s^\varepsilon(u^\varepsilon - \bar{u}, B_{2\sqrt{n}}) \cap Q| + |A_s(\bar{u}, B_{2\sqrt{n}}) \cap Q_1| + cs^{-q} \\
&\le \sum_{Q\in\mathcal{F}} |A_s^\varepsilon(u^\varepsilon - \bar{u}, B_{2\sqrt{n}}) \cap Q| + cs^{-q}.
\end{aligned} \tag{2.0.6}$$

Note that we can replace the domain $B_{2\sqrt{n}}$ in the leftmost side to Ω by utilizing the assumption that $|u^\varepsilon(x)| \le |x|^2$ for all $x \in \Omega\backslash B_{2\sqrt{n}}$. Thus, it will be enough to prove that with $\varepsilon_{s,\eta} > 0$ sufficiently small and $\varepsilon < \varepsilon_{s,\eta}$,

$$|A_s^\varepsilon(u^\varepsilon - \bar{u}, B_{2\sqrt{n}}) \cap Q| \le c\eta s^{-\mu}|Q|, \tag{2.0.7}$$

for all $Q \in \mathcal{F}$, where $\mu > 0$ is a constant depending only on n, λ and Λ.

To prove (2.0.7), let us fix a cube $Q \in \mathcal{F}$. By definition, there is some $x_0 \in Q \cap \{M(|D^2\bar{u}|\chi_{B_{2\sqrt{n}}}) \le \frac{s}{c}\} \ne \emptyset$ and $Q \subset B_{\rho_s\sqrt{n}}(x_0) \subset B_{2\sqrt{n}}$. To simplify the exposition, let us write by B the ball $B_{2\rho_s\sqrt{n}}(x_0)$, by r_B the radius of B, i.e., $r_B = 2\rho_s\sqrt{n}$, and by P the matrix $(D^2\bar{u})_B$, i.e., $P = \frac{1}{|B|}\int_B D^2\bar{u}\,dx$. Then by the choice of x_0, $|P| \le \frac{s}{c}$, and by (2.0.3) along with the John–Nirenberg inequality,

$$\frac{1}{|B|}\int_B |D^2\bar{u} - P|^n\,dx \le c_n\kappa^n\psi(r_B)^n, \tag{2.0.8}$$

where $c_n > 0$ is a constant depending only on n.

Consider an auxiliary function $w : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\begin{cases} F(D_y^2 w + P, y) = \bar{F}(P) & \text{in } \mathbb{R}^n, \\ w(y+k) = w(y) & \text{for all } y \in \mathbb{R}^n, k \in \mathbb{Z}^n, \\ w(0) = 0, \end{cases} \tag{2.0.9}$$

in the viscosity sense. According to [13, Lemma 3.1], such a periodic viscosity solution exists (in $C^{0,\alpha}(\mathbb{R}^n)$, with $\alpha \in (0,1)$ universal) and unique, and due to (2.0.1) as well as(2.0.3), it satisfies

$$\|w\|_{C^{0,\alpha}(\mathbb{R}^n)} \le c_0|P| \le \frac{c_0 s}{c}, \tag{2.0.10}$$

for some constant $c_0 > 0$ depending only on n, λ and Λ.

Consider auxiliary functions $\phi^\varepsilon, g : B \to \mathbb{R}$ defined by

$$\phi^\varepsilon = \frac{1}{\kappa\psi(r_B)}\left(u^\varepsilon - \bar{u} - \varepsilon^2 w\left(\frac{\cdot}{\varepsilon}\right)\right), \quad g = \frac{r_B^2}{\kappa\psi(r_B)}(f + c|D^2\bar{u} - P|). \tag{2.0.11}$$

Clearly, $\phi \in C(B)$ and $g \in L^n(B)$. By $\|u^\varepsilon - \bar{u}\|_{L^\infty(B_{3\sqrt{n}})} \le \delta$ and (2.0.10) (as well as $\varepsilon \in (0, \varepsilon_{s,\eta})$ and $B \subset B_{3\sqrt{n}}$),

$$\|\phi^\varepsilon\|_{L^\infty(B)} \le \frac{\delta + c_0 c^{-1} s\varepsilon_{s,\eta}^2}{\kappa\psi(r_B)}. \tag{2.0.12}$$

In addition, by $\|f\|_{L^p(B_{4\sqrt{n}})} \le \varepsilon_{s,\eta}$ and (2.0.8),

$$\left(\frac{1}{|B|}\int_B |g|^n\,dx\right)^{1/n} \le c_n r_B^2 \left(\frac{\varepsilon_{s,\eta}}{r_B \kappa \psi(r_B)} + 1\right). \tag{2.0.13}$$

We claim that ϕ is an L^p-viscosity solution to

$$\max\{\mathcal{P}_-(D^2\phi), -\mathcal{P}_+(D^2\phi)\} \le |g| \quad \text{in } B. \tag{2.0.14}$$

Suppose for the moment that the claim is true. With sufficiently small δ and $\varepsilon_{s,\eta}$, whose smallness condition depending only on r_B, κ, $\psi(r_B)$, c_0 and c, we can deduce from (2.0.12) and (2.0.13) that $\|\phi^\varepsilon\|_{L^\infty(B)} \le 1$ and respectively $\|g\|_{L^n(B)} \le c_n r_B$. Then we can apply (a rescaled form of) [6, Lemma 7.5][1] to ϕ^ε and deduce that for any $t > 0$,

$$|A_t(\phi^\varepsilon, B) \cap Q| \le c_0 t^{-\mu} |Q|, \tag{2.0.15}$$

where $\mu > 0$ depends only on n, λ and Λ; here we also used that dist $(Q, \partial B) \ge \frac{1}{2} r_B$, which is apparent from the choice of Q and B. Thus, setting $t = \frac{s}{2\kappa\psi(r_B)}$, we obtain $|A^\varepsilon_{\frac{s}{2}}(u^\varepsilon - \bar{u} - \varepsilon^2 w(\frac{\cdot}{\varepsilon})), B)) \cap Q| \le c_0(\kappa\psi(r_B))^\mu s^{-\mu}|Q|$. Utilizing (2.0.10) (as well as a simple observation that $A_{(2\ell+1)s}(f_1 - f_2, E) \subset A_s(f_1, E)$ for any $f_1, f_2 \in C(E)$ with $\|f_2\|_{L^\infty(E)} \le \ell\varepsilon^2$), we deduce that

$$\left|A^\varepsilon_{(\frac{c_0}{c}+\frac{1}{2})s}(u^\varepsilon - \bar{u}, B) \cap Q\right| \le c_0 \kappa^\mu \psi(r_B)^\mu s^{-\mu} |Q|, \tag{2.0.16}$$

for any $s > 0$. Since $\|u^\varepsilon - \bar{u}\|_{L^\infty(B_{3\sqrt{n}})} \le \delta$ with δ being small depending on r_B, one can also replace B above with $B_{2\sqrt{n}}$. At this point, we select $c \ge 2c_0$, and $\rho_s = \frac{r_B}{2\sqrt{n}} \in (0, \frac{1}{4})$ as a small constant such that $\kappa\psi(r_B) \le \eta^{1/\mu}$, so that we arrive at (2.0.7), as desired.

Thus, we are only left with proving that (2.0.14) holds in the L^p-viscosity sense. Let φ be a quadratic polynomial such that $D^2\varphi = P$. For the moment, let us denote by W^ε the function $\varepsilon^2(\varphi + w)(\frac{\cdot}{\varepsilon})$. Clearly, $F(D^2 W^\varepsilon, \frac{\cdot}{\varepsilon}) = \bar{F}(P)$, as well as $F(D^2 u^\varepsilon, \frac{\cdot}{\varepsilon}) = f$, in B in the viscosity sense, so it follows from Lemma 3.1 that

$$\mathcal{P}_-(D^2(u^\varepsilon - W^\varepsilon)) \le f - \bar{F}(P) \quad \text{in } B,$$

in the viscosity sense. Now since $u^\varepsilon - \bar{u} - \varepsilon^2 w(\frac{\cdot}{\varepsilon}) = u^\varepsilon - W^\varepsilon - (\bar{u} - \varepsilon^2\varphi(\frac{\cdot}{\varepsilon}))$, and $\bar{u} - \varepsilon^2\varphi(\frac{\cdot}{\varepsilon}) \in W^{2,p}(B)$, one can deduce, along with $\bar{F}(D^2\bar{u}) = 0$ in B in the L^p-strong sense, that

$$\begin{aligned}
\mathcal{P}_-(D^2\phi^\varepsilon) &\le \frac{1}{\kappa\psi(r_B)}(\mathcal{P}_-(D^2(u^\varepsilon - W^\varepsilon)) - \mathcal{P}_-(D^2\bar{u} - P)) \\
&\le \frac{1}{\kappa\psi(r_B)}(f - \bar{F}(P) - \mathcal{P}_-(D^2\bar{u} - P)) \\
&\le \frac{1}{\kappa\psi(r_B)}(f - \mathcal{P}_-(P - D^2\bar{u}) - \mathcal{P} - (D^2\bar{u} - P)) \\
&\le |g|
\end{aligned}$$

in B in the L^p-viscosity sense. Similarly, we obtain $\mathcal{P}_+(D^2\phi^\varepsilon) \ge -|g|$ in B in the L^p-viscosity sense. This finishes the proof. □

[1] Although this lemma is written for C-viscosity solutions and continuous right-hand side whose L^n-norm is under control, it works equally well for L^n-viscosity solutions and measurable right-hand side.

We are ready to use the by-now standard cube decomposition argument to obtain a geometric decay of $|A_s^\varepsilon(u^\varepsilon,\Omega)\cap Q_1|$. The idea is the same with [9] in the sense that we split the set $A_k = A_{m^k}^\varepsilon(u^\varepsilon,\Omega)\cap Q_1$ into two parts, say D_k and E_k, where D_k is the part of A_k intersected by its Calderón-Zygmund cube covering whose side-length is at least $\frac{\varepsilon}{\varepsilon_\eta}$ (these cubes are said to be of high frequency), whereas $E_k = A_k \setminus D_k$, i.e., the part of A_k intersected by the cubes of low frequency. As for D_k, we can deduce a geometric decay via Lemma 5.5, and this part is almost the same with the argument for standard problem, e.g., [6, Lemma 7.13]. As for E_k, the above lemma is no longer applicable, but at the same time we cannot argue as in [9] because we do not assume any structure condition on F so as to ensure sufficient regularity of u^ε in small scales. Here we control E_k directly from the fact that the set A_k (or more exactly $Q_1\setminus A_k$) allows error of order ε^2 for quadratic polynomials to touch u^ε.

Lemma 5.6 *Let $\eta>0$ and $p\in(p_0,\infty)$ be given. Assume that $0<\varepsilon<\varepsilon_\eta$, $\|f\|_{L^p(B_{4\sqrt{n}})}\leq \varepsilon_\eta$, and $\|u^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})}\leq 1$. Let A_k and B_k denote the sets $A_{m^k}^\varepsilon(u^\varepsilon,B_{4\sqrt{n}})\cap Q_1$ and respectively $\{M(|f|^p\chi_{B_{4\sqrt{n}}})>\varepsilon_\eta^p M^{pk}\}$. Then one has, for each $k\geq k_\eta$,*

$$|A_{k+1}|\leq(\eta+cm^{-q})|A_k\cup B_k|+|B_{k-k_\eta}|, \tag{2.0.17}$$

where $m>1$ depends at most on n, λ, Λ, ψ, κ, p and q, while $\varepsilon_\eta>0$ and $k_\eta>1$ may depend further on η.

Proof As briefly mentioned in the discussion before the statement of this lemma, the set D_k is the part of A_k intersected by its Calderón-Zygmund covering whose side-length is no less than $\frac{\varepsilon}{\varepsilon_\eta}$. More exactly, we choose $M>1$ sufficiently large such that $|A_1|\leq\eta+cm^{-q}<1$ due to Lemma 5.5. We shall fix ε_η as the constant $\varepsilon_{c_0^{-1}m,\eta}$ from Lemma 5.5, with $c_0>1$ to be determined later. As $A_{k+1}\subset A_k\subset\cdots A_1$ for each integer $k\geq 1$, $|A_{k+1}|\leq\eta+cm^{-q}$, whence there exists a Calderón-Zygmund covering, denoted by $\mathcal{F}_{k+1}^\varepsilon$, of A_{k+1} corresponding to the level $\eta+cm^{-q}$. Define

$$D_{k+1}=\bigcup\left\{A_{k+1}\cap Q : Q\in\mathcal{F}_{k+1}^\varepsilon,\ \frac{\operatorname{diam}(Q)}{\sqrt{n}}>\frac{\varepsilon}{\varepsilon_\eta}\right\}.$$

Let us claim that $\tilde{Q}\subset A_k\cup B_k$ for any cube $Q\in\mathcal{F}_k^\varepsilon$ whose side-length is no less than $\frac{\varepsilon}{\varepsilon_\eta}$, where $\tilde{Q}$ is the predecessor of Q. Once this is proved, from the fact that Q belongs to the Calderón-Zygmund covering of A_k it follows immediately that

$$|D_{k+1}|\leq(\eta+cm^{-q})|A_k\cup B_k|. \tag{2.0.18}$$

The proof for the above claim mostly follows the argument for standard problems, e.g., [6, Lemma 7.12], except for the following two points: (i) we need to verify the hypothesis (2.0.4) for the effective functional at each iteration step, (ii) the set A_k involves error of order ε^2 and may vary along with different scalings in the domain.

Denote by x_Q and s_Q the center and respectively the side-length of Q. Since $Q\in\mathcal{F}_k^\varepsilon$, $|A_{k+1}\cap Q|>(\eta+cm^{-q})|Q|$. Assume, by way of contradiction, that $\tilde{Q}\backslash(A_k\cup B_k)\neq\emptyset$. Set $\Omega_Q=s_Q^{-1}(-x_Q+B_{4\sqrt{n}})$, the image of $B_{4\sqrt{n}}$ via the above rescaling. Then due to the assumption $\tilde{Q}\backslash(A_k\cup B_k)\neq\emptyset$, the functions $u_Q^{\varepsilon_Q}\in C(\Omega_Q)$, defined by

$$u_Q^{\varepsilon_Q}(x)=\frac{(u^\varepsilon-\ell)(x_Q+s_Qx)}{c_0m^ks_Q^2},\quad \varepsilon_Q=\frac{\varepsilon}{s_Q}, \tag{2.0.19}$$

for certain linear polynomial ℓ, satisfy that $\|u_Q^{\varepsilon_Q}\|_{L^\infty(\Omega_Q)} \leq 1$, $|u_Q^{\varepsilon_Q}(x)| \leq |x|^2$ for all $x \in \Omega \backslash B_{2\sqrt{n}}$, provided that we choose $c_0 > 1$ to be large (depending only on n). Now as $F(D^2 u^\varepsilon, \frac{\cdot}{\varepsilon}) = f$ in $B_{4\sqrt{n}}$ in the viscosity sense, we have

$$F_Q\left(D^2 u_Q^{\varepsilon_Q}, \frac{\cdot}{\varepsilon_Q}\right) = f_Q \quad \text{in } B_{4\sqrt{n}}, \tag{2.0.20}$$

in the viscosity sense, where $F_Q(P, y) = \frac{1}{c_0 m^k} F(c_0 m^k P, \frac{x_0}{\varepsilon} + y)$ and $f_Q(x) = \frac{1}{c_0 m^k} f(x_0 + s_Q x)$. Clearly, $F_Q \in C(\mathcal{S}^n \times \mathbb{R}^n)$ satisfies (2.0.1), (2.0.2) and (2.0.3). Noting that its effective functional $\bar{F}_Q : \mathcal{S}^n \to \mathbb{R}$ is given by $\bar{F}_Q(P) = \frac{1}{c_0 m^k} \bar{F}(c_0 m^k P)$, for any $P \in \mathcal{S}^n$, $\bar{H}$ satisfies (2.0.4) (with the same modulus of continuity ψ and constant κ as $\bar{F}$ does). Moreover, with $c_0 > 1$, it follows from the choice of $\tilde{Q} \backslash B_k \neq \emptyset$ that $\|f_Q\|_{L^\infty(B_{4\sqrt{n}})} \leq c_0^{-\frac{1}{p}} \varepsilon_\eta < \varepsilon_\eta$. Therefore, due to the choice of $\varepsilon_Q = s_Q^{-1} \varepsilon \leq \varepsilon_\eta = \varepsilon_{c_0^{-1} m, \eta}$, we can apply Lemma 5.5 to $u_Q^{\varepsilon_Q}$ and deduce that

$$\left| A^{\varepsilon_Q}_{\frac{m}{c_0}}(u_Q^{\varepsilon_Q}, \Omega_Q) \cap Q_1 \right| \leq \eta + c m^{-q}.$$

Rescaling back, we arrive at $|A_{k+1} \cap Q| \leq (\eta + c m^{-q})|Q|$, a contradiction to $Q \in \mathcal{F}_k^\varepsilon$. This finishes the proof for (2.0.18).

Next, we prove that

$$A_{k+1} \setminus D_{k+1} \subset B_{k-k_\eta}, \tag{2.0.21}$$

with $k_\eta = -\log_2 \varepsilon_\eta$. Let $Q \in \mathcal{F}_k^\varepsilon$ be a dyadic cube with side-length less than $\frac{\varepsilon}{\varepsilon_\eta}$. Then $|Q \cap A_{k+1}| > (\eta + c m^{-q})|Q|$, so $|Q \cap A^\varepsilon_{k-k_\eta}| > (\eta + c m^{-q})|Q|$. We claim that $\tilde{Q} \backslash A^\varepsilon_{k-k_\eta} \subset B_{k-k_\eta}$, from which (2.0.21) follows immediately. Suppose, by way of contradiction, that $\tilde{Q} \backslash (A^\varepsilon_{k-k_\eta} \cup B^\varepsilon_{k-k_\eta}) \neq \emptyset$. As we have chosen $k_\eta = -\log_2 \varepsilon_\eta$, that $\tilde{Q} \backslash A^\varepsilon_{k-k_\eta} \neq \emptyset$ implies $\tilde{Q} \cap A_{k+1} = \emptyset$. This is again a contradiction to the fact that $Q \subset \tilde{Q}$ and $|Q \cap A_{k+1}| > (\eta + c m^{-q})|Q|$. Therefore, $\tilde{Q} \setminus A_{k-k_\eta} \subset B_{k-k_\eta}$ as desired, and the inclusion in (2.0.21) follows. □

We are now ready to prove the large-scale interior $W^{2,p}$-estimates.

Proof of Theorem 5.4 Fix $p \in (p_0, \infty)$. With suitable rescaling of the problem, it is sufficient to prove the assertions for the case where $\Omega = B_{4\sqrt{n}}$, $\Omega' = Q_1$, $\|u^\varepsilon\|_{L^\infty(B_{4\sqrt{n}})} \leq 1$ and $\|f\|_{L^p(B_{4\sqrt{n}})} \leq \varepsilon_\eta$, with ε_η to be determined.

Choose $q = 2p$ and $p' = \frac{p+p_0}{2}$. Let $M > 1$ be as in Lemma 5.6 with p replaced by p'. We can assume that $cM^{p-q} \leq \frac{1}{4}$ by taking M larger, depending on the choice of p. Then we select $\eta > 0$ as a small constant satisfying $M^p \eta \leq \frac{1}{4}$. Then

$$M^p(\eta + c m^{-q}) \leq \frac{1}{2}.$$

Now let $\varepsilon_\eta > 0$ and $k_\eta > 1$ be as in Lemma 5.6 corresponding to the specific choice of η. Note that all the constants η, M, $\hat{c}$, ε_η and k_η involved in the statement of Lemma 5.6 now depend only on n, λ, Λ, ψ, κ and p (as q depending solely on p, and γ solely on n, λ, Λ and p).

With (2.0.17) at hand, we deduce that

$$\sum_{k=1}^{\infty} m^{kp}\alpha_k \le c + c\sum_{k=k_\eta}^{\infty} m^{kp}\beta_k,$$

with $\alpha_k = |A_k|$ and $\beta_k = |B_k|$, where A_k and B_k are as in the statement of Lemma 5.6. Since $\||f|^{p'}\|_{L^{p/p'}(Q_1)} \le \|f\|^{p'}_{L^p(B_{4\sqrt{n}})} \le \varepsilon_\eta^{p'}$, we have $\sum_{k=1}^{\infty} m^{kp}\beta_k \le c$, as well. The proof can now be finished, as in [6, Proposition 7.2], and we omit the detail. □

To prove the second assertion of Theorem 5.4, we shall modify some of the argument for the proof of Lemmas 5.5 and 5.6 in such a way that we estimate the measure of $A_s(u^\varepsilon, \cdot)$ instead of $A_s^\varepsilon(u^\varepsilon, \cdot)$. In what follows, we intend to highlight those changes, which are not so trivial, and then omit the detail for the argument that might repeat what is already written so far.

6 Estimates near boundary layers

6.1 $W^{1,\frac{np}{n-p}}$-estimates

Let us now turn to uniform $W^{1,\frac{np}{n-p}}$-estimates near boundary layers.

Theorem 6.1 *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.3), $\Omega \subset \mathbb{R}^n$ be a domain, $U \subset \mathbb{R}^n$ be an open neighborhood of a point of $\partial\Omega$ such that $\partial\Omega \cap U$ is (δ, R)-Reifenberg flat, for some $\delta > 0$ and some $R \in (0, 1]$, $f \in L^p(\Omega \cap U)$ and $g \in W^{1,\frac{np}{n-p}}(U)$, for some $p \in (p_0, n)$. Let $u^\varepsilon \in C(\overline{\Omega} \cap U)$ be a viscosity solution to*

$$\begin{cases} F\left(D^2u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f & \text{in } \Omega \cap U, \\ u^\varepsilon = g & \text{on } \partial\Omega \cap U. \end{cases} \tag{2.0.1}$$

Then $G^\varepsilon_{\Omega\cap U}(u^\varepsilon) \in L^{\frac{np}{n-p}}_{loc}(\overline{\Omega}_\varepsilon \cap U)$, and for any $U' \Subset U$,

$$\|G^\varepsilon_{\Omega\cap U}(u^\varepsilon)\|_{L^{\frac{np}{n-p}}(\Omega_\varepsilon\cap U')} \le C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega\cap U)}}{\operatorname{dist}(U', \partial U)^{2-\frac{n}{p}}} + \|f\|_{L^p(\Omega\cap U)} + \|Dg\|_{L^{\frac{np}{n-p}}(U)}\right),$$

where $\delta > 0$ depends only on n, λ, Λ and p, and $C > 1$ may depend further on R and diam (U).

The above estimate is optimal as the power of integrability reaches the critical Sobolev exponent. This estimate is even new in the setting of the standard problems. To the best of the author's knowledge, the boundary estimate is proved up to the subcritical Sobolev exponent, i.e., $W^{1,q}$ with $q < \frac{np}{n-p}$, in [30]. Following the spirit of [29], the proof in [30] relies heavily on pointwise $C^{1,\alpha}$-approximation, and hence the estimate could not reach the critical exponent.

We shall set the starting point of our analysis, however, at a sub-optimal estimate below. We shall provide some motivations and remarks after the statement.

Proposition 6.2 *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.3), $\Omega \subset \mathbb{R}^n$ be a domain, $U \subset \mathbb{R}^n$ be an open neighborhood of a point of $\partial\Omega$ such that $\partial\Omega \cap U$ is*

(δ, R)-Reifenberg flat from exterior,[2] *for some $\delta > 0$ and some $R \in (0,1]$ $f \in L^p(\Omega \cap U)$ and $g \in C^{0,2-\frac{n}{p}}(\Omega \cap U)$, for some $p \in (p_0, n)$. Let $u^\varepsilon \in C(\overline{\Omega} \cap U)$ be a viscosity solution to*

$$\begin{cases} F\left(D^2 u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f & \text{in } \Omega \cap U, \\ u^\varepsilon = g & \text{on } \partial\Omega \cap U. \end{cases}$$

Then $G^\varepsilon_{\Omega\cap U}(u^\varepsilon)u^\varepsilon \in L^q_{loc}(\overline{\Omega}_\varepsilon \cap U)$ for any $q \in [1, \frac{np}{n-p})$, and for any $U' \Subset U$,

$$\frac{\|G^\varepsilon_{\Omega\cap U}(u^\varepsilon)\|_{L^q(\Omega_\varepsilon\cap U')}}{\operatorname{dist}(U', \partial U)^{1-\frac{n}{p}+\frac{n}{q}}} \le C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega\cap U)}}{\operatorname{dist}(U', \partial U)^{2-\frac{n}{p}}} + \|f\|_{L^p(\Omega\cap U)} + [g]_{C^{0,2-\frac{n}{p}}(\partial\Omega\cap U)}\right),$$

where $\delta > 0$ depends only on n, λ, Λ and p, and $C > 1$ may depend further on q, R and $\operatorname{diam}(U)$.

We shall provide this estimate, mainly because of its independent interests. Of course, we will use this proposition in our subsequent analysis to ensure a better regularity for the approximating solutions. Still, this step could be conveniently replaced by the uniform boundary $C^{1,\alpha}$-estimates [18], as the approximating solutions solve a "clean" version of the homogenization problems (i.e., $f = 0$ and $g = a$, with a a constant).

In view of the compact embedding of $W^{1,\frac{np}{n-p}}$ to $C^{0,2-\frac{n}{p}}$, the above estimate seems to be the best one can expect with a $C^{0,2-\frac{n}{p}}$-regular boundary data. The estimate is quite interesting in the sense that a Hölder regular function may not be (weakly) differentiable a.e., that is, one cannot expect $C^{0,2-\frac{n}{p}}$-regular boundary data to be extended to a $W^{1,\frac{np}{n-p}}$-regular function in a neighborhood of the boundary layer. In particular, one cannot deduce the above sub-optimal estimate from the former optimal estimate (Theorem 6.1). Hence, the above proposition shows certain regularizing effect arising from the presence of boundary layers.

Let us also remark that the sub-optimal estimate above improves the one in [30], in terms of the regularity of the boundary data, as the latter estimate requires $C^{1,\alpha}$-regularity.

We shall first present a complete proof for Proposition 6.2 and then move onto that of Theorem 6.1. The proof for the former is based on a boundary $C^{0,2-\frac{n}{p}}$-estimate for any viscosity solution belonging to the Pucci class, up to an L^p-regular, with $p > p_0$, right hand side. In particular, the functional may not oscillate under certain pattern in small scales, whence it has nothing to do with homogenization.

Proof of Proposition 6.2 Let us prove the first part of the assertion, and then mention the changes in the argument for the second part, as the latter is almost the same with the former.

After some standard rescaling procedure, one may prove the assertion for the case where $0 \in \partial\Omega$, $U = B_1$, $U' = B_{1/2}$, $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_1)} \le 1$, $\|f\|_{L^p(\Omega\cap B_1)} \le 1$ and $[g]_{C^{0,2-\frac{n}{p}}(\partial\Omega\cap B_1)} \le 1$. Applying the boundary Hölder regularity [22, Theorem 1.1] to u^ε around each point $x_0 \in \partial\Omega \cap B_{1/2}$, we deduce that

$$|u^\varepsilon(x) - g(x_0)| \le c|x - x_0|^{2-\frac{n}{p}}, \tag{2.0.2}$$

for all $x \in \overline{\Omega} \cap B_1$; let us remark that although the statement of [22, Theorem 1.1] involves $\sup_{r>0} r^{\alpha-2}\|f\|_{L^n(\Omega\cap B_r(x_0))}$ on the right-hand side, one can easily replace this norm with $\|f\|_{L^p(\Omega\cap B_1)}$ by taking $\alpha = 2 - \frac{n}{p}$, as the modification in the proof there is straightforward.

[2] That is, $\partial\Omega \cap B_r(x_0) \subset \{x \in B_r(x_0) : (x - x_0)\cdot\nu > -\delta r\}$ for any $r \in (0, R)$ and any $x_0 \in \partial\Omega \cap U$.

Let $B \subset \Omega \cap B_{1/2}$ be a ball for which $\partial(2B) \cap \partial\Omega \cap B_{1/2} \neq \emptyset$. Let x_B and ρ_B denote the center and respectively the radius of B. Also let $x_{B,0}$ be a point of intersection between $\partial(2B)$ and $\partial\Omega \cap B_{1/2}$. Consider the rescaling

$$u_B^{\varepsilon_B}(x) = \frac{u^\varepsilon(x_B + \rho_B x) - g(x_{B,0})}{\rho_B^{2-\frac{n}{p}}},$$

of u^ε, where $\varepsilon = \rho_B^{-1}\varepsilon$. As $B_{2\rho_B}(x_B) \subset B_1$, one may observe that

$$F_B\left(D^2 u_B^{\varepsilon_B}, \frac{\cdot}{\varepsilon_B}\right) = f_B \quad \text{in } B_1,$$

in the viscosity sense, where we wrote $F_B(P, y) = \rho_B^{n/p} F(\rho_B^{-n/p} P, y)$ and $f_B(x) = \rho_B^{n/p} f(x_B + \rho_B x)$. Clearly, F_B is a continuous functional on $\mathcal{S}^n \times \mathbb{R}^n$ satisfying (2.0.1)–(2.0.3), and $f_B \in L^p(B_2) \cap C(B_2)$ with

$$\|f_B\|_{L^p(B_2)} \leq \|f\|_{L^p(\Omega \cap B_{\rho_B}(x_B))} \leq \delta.$$

Thanks to (2.0.2), we also have $\|u_B^{\varepsilon_B}\|_{L^\infty(B_2)} \leq c$. Therefore, we can apply Theorem 5.1 (i) to observe that $\|G_{\Omega_B}^{\varepsilon_B} u_B^{\varepsilon_B}\|_{L^q(B_1)} \leq c$, for each $q \in [1, \frac{np}{n-p})$, where $\Omega_B = \rho_B^{-1}(-x_B + \Omega \cap B_1)$. The latter estimate can be translated in term of u^ε as

$$\|G_{\Omega \cap B_1}^\varepsilon(u^\varepsilon)\|_{L^q(B_{\rho_B}(x_B))} \leq c\rho_B^{\frac{n}{q}+1-\frac{n}{p}}. \tag{2.0.3}$$

To this end, consider a covering $\mathcal{F}$ of $\Omega_\varepsilon \cap B_{1/2}$ by balls B for which $\partial(2B) \cap \partial\Omega \cap B_{1/2} \neq \emptyset$; i.e., $\mathcal{F} \subset \{B_{\operatorname{dist}(x, \partial\Omega \cap B_{1/2})}(x) : x \in \Omega_\varepsilon \cap B_{1/2}\}$. Thanks to the Besicovitch covering theorem, one can find a finite subcovering $\mathcal{G} \subset \mathcal{F}$ such that $\#\{\hat{B} \in \mathcal{G} : B \cap \hat{B} \neq \emptyset\} \leq c_n$ for all $B \in \mathcal{G}$, where $c_n > 0$ depends only on n. One can split $\mathcal{G}$ into a union of $\mathcal{G}_k$, with $k = 1, 2, \cdots, k_\varepsilon$, such that $B \in \mathcal{G}_k$ if $2^{-k-1} < \rho_B \leq 2^{-k}$, where k_ε is a large positive integer of order $-\log_2 \varepsilon$. Then

$$\begin{aligned}
\int_{\Omega_\varepsilon \cap B_{1/2}} |G_{\Omega \cap B_1}^\varepsilon(u^\varepsilon)|^q \, dx &\leq \sum_{k=1}^{k_\varepsilon} \sum_{B \in \mathcal{G}_k} \int_B |G_{\Omega \cap B_1}^\varepsilon(u^\varepsilon)|^q \, dx \\
&\leq c \sum_{k=1}^{k_\varepsilon} \sum_{B \in \mathcal{G}_k} \rho_B^{n+q-\frac{nq}{p}} \\
&\leq c \sum_{k=1}^{k_\varepsilon} 2^{-n-q+\frac{nq}{p}} \\
&\leq c,
\end{aligned} \tag{2.0.4}$$

where the last inequality is ensured from the fact that $q < \frac{np}{n-p}$. This finishes the proof, for the first part of the assertion of the theorem.

As for the second part of the assertion, one may have already noticed that under the additional assumption, one can invoke Theorem 5.1 (ii) in place of (i) above, such that one can replace $D_\varepsilon u^\varepsilon$ with Du^ε first in (2.0.3), and then in (2.0.4), leaving everything else untouched. Thus, the conclusion follows. We leave out the detail to the reader. □

With Proposition 6.2 at hand, we may now proceed with the proof for Theorem 6.1. The idea is basically the same with the interior case (Lemma 5.2), but the presence of Reifenberg flat boundaries yields some additional technical difficulties.

As in the analysis in the interior case, we shall confine ourselves to the case $U = B_2$ and $U' = B_1$, as the analysis here is of local character around a boundary point. Unless specified otherwise, we shall always assume, from now on, that $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ verifies (2.0.1)–(2.0.3), $f \in C(\Omega \cap B_2) \cap L^p(\Omega \cap B_2)$, $g \in W^{1,\frac{np}{n-p}}(B_2)$, Ω is a bounded domain containing the origin, and u^ε is a viscosity solution to (2.0.1) with U replaced by B_2.

Lemma 6.3 *Assume that $\partial\Omega \cap B_2$ is $(\delta, 2)$-Reifenberg flat, $\|f\|_{L^p(\Omega\cap B_2)} \leq \eta$, $\|Dg\|_{L^{\frac{np}{n-p}}(B_2)} \leq \eta$ for some $\eta > 0$, and that $\|u^\varepsilon - a\|_{L^\infty(\Omega\cap B_2)} \leq 1$ and $|u^\varepsilon(x) - a| \leq |x|$ for all $x \in \Omega\backslash B_1$ for some constant $a \in \mathbb{R}$. Then for any $s > N$,*

$$|L_s^\varepsilon(u^\varepsilon, \Omega) \cap \Omega_\varepsilon \cap B_1| \leq c(\delta^{\frac{\alpha}{2}} + \eta)^\mu s^{-\mu},$$

where $\alpha \in (0, 1)$, $\mu > 0$ and $N > 1$ depend only on n, λ and Λ, while $\delta > 0$ and $c > 1$ $N > 1$ may depend further on p.

Proof Let $\hat{a}$ denote the integral average of g over B_2. By the Poincaré inequality, we have $\|g - \hat{a}\|_{W^{1,\frac{np}{n-p}}(B_2)} \leq c_0\|Dg\|_{L^{\frac{np}{n-p}}(B_2)} \leq c_0\eta$, where c_0 depends only on n and p. By the Sobolev embedding theorem, $g \in C^{0,2-\frac{n}{p}}(\overline{B_2})$ and

$$\|g - \hat{a}\|_{L^\infty(B_2)} + [g]_{C^{0,2-\frac{n}{p}}(\overline{B_2})} \leq c_0\eta, \tag{2.0.5}$$

by taking c_0 larger.

Denote by S the slab $S_{2\delta}(\nu)$ for some unit vector ν such that $\partial\Omega \cap B_2 \subset S$; such a unit vector exists owing to the Reifenberg flatness of $\partial\Omega \cap B_2$. Also let E be the half-space E, so that $\Omega \cap B_2 \subset E$.

Since $u^\varepsilon = g$ on $\partial\Omega \cap B_2$ and $|u^\varepsilon| \leq 1$ on $\overline{\Omega \cap B_2}$, it follows from (2.0.5) that $\|g\|_{L^\infty(B_2)} \leq 1 + c_0\eta < 2$. Extend u^ε by g outside Ω. It readily follows that $u^\varepsilon \in C(\overline{B_2})$ and $|u^\varepsilon - g| \leq 2$ in B_2. Consider a Dirichlet boundary value problem,

$$\begin{cases} F\left(D^2h^\varepsilon, \frac{\cdot}{\varepsilon}\right) = 0 & \text{in } E \cap B_{3/2}, \\ h^\varepsilon = u^\varepsilon - g & \text{on } \partial(E \cap B_{3/2}). \end{cases}$$

This problem admits a unique viscosity solution $h^\varepsilon \in C(\overline{E \cap B_{3/2}})$ because F and $u^\varepsilon - g$ are continuous, and $\partial(E \cap B_{3/2})$ satisfies a uniform exterior sphere condition. Since $|h^\varepsilon| \leq 2$ on $\partial(E \cap B_{3/2})$ and $h^\varepsilon = 0$ on $\partial E \cap B_2$, we can deduce from Lemma 3.4, the Krylov theory [6, Corollary 5.7], as well as the fact that $\partial E \cap B_{3/2}$ is flat, that for each $x_0 \in \partial E \cap B_1$, there exists a linear polynomial $\ell_{x_0}^\varepsilon$ for which $|D\ell_{x_0}^\varepsilon| \leq c_0$ and $|(h^\varepsilon - \ell_{x_0}^\varepsilon)(x)| \leq c_0(|x - x_0|^{1+\alpha} + \varepsilon^{1+\alpha})$ for all $x \in \overline{E \cap B_{3/2}}$, with both $c_0 > 1$ and $\alpha \in (0, 1)$ depending only on n, λ and Λ. In particular,

$$L_{c_0}^\varepsilon(h^\varepsilon, E_\varepsilon \cap B_{3/2}) \cap B_1 = \emptyset, \tag{2.0.6}$$

by taking c_0 larger if necessary, with $E_\varepsilon = \{x \in E : \operatorname{dist}(x, \partial E) > 0\}$.

In what follows, we shall denote by c_0 a constant which may depend at most on n, λ, Λ and p, and we shall allow it to vary at each occurrence.

As $\partial\Omega \cap B_2$ is $(\delta, 2)$-Reifenberg flat, one may invoke [21, Theorem 1.1] to deduce, along with $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \leq 1$, $\|f\|_{L^p(\Omega\cap B_2)} \leq \eta$ and $[g]_{C^{0,2-\frac{n}{p}}(B_2)} \leq c_0\eta$ that if $\delta \leq \delta_p$, with δ_p depending only on n, λ, Λ and p, then

$$|u^\varepsilon(x) - g(x_0)| \leq c_0|x - x_0|^{2-\frac{n}{p}},$$

for all $x \in \Omega \cap B_2$, for each $x_0 \in \partial\Omega \cap B_{3/2}$. Combining this inequality with [12, Lemma 2], we deduce that

$$[u^\varepsilon]_{C^{0,\alpha}(\overline{\Omega \cap B_{3/2}})} \le c_0. \tag{2.0.7}$$

With (2.0.7) at hand, we may now estimate the global Hölder norm of h^ε. As $h^\varepsilon = 0$ on $\partial E \cap B_{3/2}$, $h^\varepsilon = u^\varepsilon - g$ on $E \cap \partial B_{3/2}$ and $[u^\varepsilon - g]_{C^{0,\alpha}(E \cap \partial B_{3/2})} \le c_0$, we may apply a variant of [6, Proposition 4.13] to derive that

$$[h^\varepsilon]_{C^{0,\frac{\alpha}{2}}(\overline{E \cap B_{3/2}})} \le c_0. \tag{2.0.8}$$

Consider another auxiliary function $w^\varepsilon = \eta_0^{-1}(u^\varepsilon - \hat{a} - h^\varepsilon)$, with η_0 to be determined later. This function is well-defined on $\overline{\Omega \cap B_{3/2}}$, since $\Omega \cap B_{3/2} \subset E \cap B_{3/2}$. Due to Lemma 3.1, we may compute that

$$\begin{cases} \mathcal{P}_-(D^2 w^\varepsilon) \le \frac{f}{\eta_0} \le \mathcal{P}_+(D^2 w^\varepsilon) & \text{in } \Omega \cap B_{3/2}, \\ w^\varepsilon = \frac{g - \hat{a}}{\eta_0} & \text{on } \Omega \cap \partial B_{3/2}, \\ w^\varepsilon = \frac{g - \hat{a} - h^\varepsilon}{\eta_0} & \text{on } \partial\Omega \cap B_{3/2}, \end{cases}$$

Thanks to (2.0.8), $h^\varepsilon = 0$ on $E \cap B_{3/2}$ and $\partial\Omega \cap B_{3/2} \subset H_\delta(-\nu) \cap E$, we may deduce that $|h^\varepsilon| \le c_0 \delta^{\frac{\alpha}{2}}$ on $\partial\Omega \cap B_{3/2}$. This together with (2.0.5) yields that with $\eta_0 = c_0 \max\{\delta^{\frac{\alpha}{2}}, \eta\}$ and $c_0 > 1$,

$$\|w^\varepsilon\|_{L^\infty(\partial(\Omega \cap B_{3/2}))} \le \frac{c_0 \delta^{\frac{\alpha}{2}} + c_0 \eta}{\eta_0} \le 1.$$

With such a choice of η_0, we also have $\|f\|_{L^p(\Omega \cap B_{3/2})} \le \eta_0$, whence it follows from the general maximum principle that $\|w^\varepsilon\|_{L^\infty(B_{3/2})} \le c_0$.

Now we can apply Proposition 4.1 to obtain that for any $t > 0$,

$$|L_t(w^\varepsilon, \Omega \cap B_{3/2}) \cap B_1| \le c t^{-\mu}, \tag{2.0.9}$$

where $\mu > 0$ depends only on n, λ, Λ and p; this is another step that determines how small the Reifenberg flatness constant δ should be. Since the set $L_t(w^\varepsilon, \cdot)$ is invariant under vertical translation of given function w^ε, the conclusion follows immediately from (2.0.6), (2.0.9), the choice of η_0 and the assumption that $|u^\varepsilon(x)| \le |x|$ for all $x \in \Omega \setminus B_1$. $\square$

The following lemma is the boundary analogue of Lemma 5.3.

Lemma 6.4 *Suppose that $\partial\Omega \cap B_2$ is $(\delta, 2)$-Reifenberg flat, $\|f\|_{L^p(\Omega \cap B_2)} \le \eta$, $\|Dg\|_{L^{\frac{np}{n-p}}(B_{4\sqrt{n}})} \le \eta$, for some $p \in (p_0, n)$ and for some $\eta \in (0,1)$, and $\|u^\varepsilon\|_{L^\infty(\Omega \cap B_2)} \le 1$. Let L_k^ε, B_k and C_k denote the sets $L^\varepsilon_{m^k}(u^\varepsilon, \Omega \cap B_2) \cap \Omega_\varepsilon \cap B_1$, $\{I_p(|f|^p \chi_{\Omega \cap B_2}) > \eta^p m^{kp}\}$ and respectively $\{M(|Dg|\chi_{B_2}) > \eta^{\frac{np}{n-p}} M^{k\frac{np}{n-p}}\}$. Then*

$$|L_{k+1}^\varepsilon| \le c(\delta^{\frac{\alpha}{2}} + \eta)^\mu |L_k^\varepsilon \cup B_k \cup C_k|,$$

for any integer $k \ge 1$, where $\alpha \in (0,1)$, $\delta > 0$, $\mu > 0$, $c > 1$ and $m > 1$ depend at most on n, λ, Λ and p.

Proof The proof resembles with that of Lemma 4.6. A key difference here is that now we need to be careful of the changes made in boundary data, when rescaling the problem; note that we did not encounter this issue in the proof of Lemma 4.6, since we did not need to see the boundary value at all. Henceforth, we shall proceed with the proof focusing on this issue, and try to skip any argument that only requires a minor modification of what is already shown so far.

Fix an integer $k \geq 1$. Arguing as in the proof of Lemma 5.3, one may use Lemma 6.3, in place of Lemma 5.2, to deduce that $|L_{k+1}^{\varepsilon}| \leq \eta_0 |B_1|$, with $\eta_0 = c_0(\delta^{\frac{\alpha}{2}} + \eta)^{\mu}$, where $c_0 > 1$ and $m > 1$ are to be chosen later.

Let $B \subset B_1$ be any ball with center in $\overline{\Omega} \cap B_2$ and rad $(B) \leq 1$. Suppose that $|L_{k+1}^{\varepsilon} \cap B| > \eta_0 |B|$. As in the proof of Lemma 4.6, we assert that $\Omega \cap B \subset (L_k^{\varepsilon} \cup B_k \cup C_k)$. If the claim is true, then $|L_{k+1}^{\varepsilon}| \leq c_n \eta_0 |L_k^{\varepsilon} \cup B_k \cup C_k|$ as desired, by Lemma 2.7.

Assume by way of contradiction that $\Omega \cap B \backslash (L_k^{\varepsilon} \cup B_k \cup C_k) \neq \emptyset$. For the same reason as in the proof of Lemma 5.2, we have $2r_B > \varepsilon$, by taking $m > 1$ large, depending only on n. Choose a point $x_B \in \partial\Omega \cap B_1$ such that $2B \subset B_{4r_B}(x_B)$. Select any $\tilde{x}_B \in \Omega \cap B \backslash (L_k^{\varepsilon} \cup B_k \cup C_k)$. Since $\tilde{x}_B \notin \Omega \cap B_{2r_B}(x_B) \backslash L_k^{\varepsilon}$ and $2r_B > \varepsilon$, one can find some constant a for which

$$|u_B(x) - a| \leq m^k (4r_B + |x - x_B|), \tag{2.0.10}$$

for all $x \in \Omega \cap B_2$. Moreover, it follows from $\tilde{x}_B \in \Omega \cap B_{2r_B}(x_B) \backslash (B_k \cup C_k)$, one may deduce as in the proof of Lemma 5.3 that

$$\int_{\Omega \cap B_{2r_B}(x_B)} |f|^p \, dx \leq c r_B^{n-p} \eta^p, \quad \int_{B_{2r_B}(x_B)} |Dg|^{\frac{np}{n-p}} \, dx \leq c r_B^n \eta^{\frac{np}{n-p}}. \tag{2.0.11}$$

Consider the following rescaled versions of u^{ε}, f and g,

$$u_B^{\varepsilon_B}(x) = \frac{u^{\varepsilon}(x_B + 2r_B x) - a}{2cm^k r_B}, \quad \varepsilon_B = \frac{\varepsilon}{2r_B},$$
$$g_B(x) = \frac{g(x_B + 2r_B x) - a}{2cm^k r_B}, \quad f_B(x) = \frac{2r_B}{cm^k} f(x_B + 2r_B x),$$

where $c > 1$ is a constant to be determined later. In view of (2.0.1), we may compute that

$$\begin{cases} F_B\left(D^2 u_B^{\varepsilon_B}, \frac{\cdot}{\varepsilon_B}\right) = f_B & \text{in } \Omega_B \cap B_2, \\ u_B^{\varepsilon_B} = g_B & \text{on } \partial\Omega_B \cap B_2, \end{cases}$$

in the viscosity sense, where $\Omega_B = \frac{1}{2r_B}(-x_B + \Omega \cap B_2)$ and $F_B(P, y) = \frac{2r_B}{cm^k} F(\frac{cm^k}{2r_B} P, y + \frac{x_B}{\varepsilon})$. Note that $F_B \in C(\mathcal{S}^n \times \mathbb{R}^n)$ and it satisfies (2.0.1)–(2.0.3) for an obvious reason. Selecting a large $c > 1$, we observe from (2.0.10) and (2.0.11) that u_B, f_B and g_B verify the hypothesis of Lemma 6.3. Due to the scaling invariance of the Reifenberg flatness, $\partial\Omega_B \cap B_2$ is also $(\delta, 2)$-Reifenberg flat. Hence, all the hypotheses of Lemma 6.3 are verified, from which it follows that

$$|L_s^{\varepsilon_B}(u_B^{\varepsilon_B}, \Omega_B) \cap \Omega_{B,\varepsilon_B} \cap B_1| \leq c(\delta^{\frac{\alpha}{2}} + \eta)^{\mu} s^{-\mu},$$

for any $s > N$, for some $N > 1$ depending only on n, λ and Λ. To this end, we may follow the argument at the end of the proof of Lemma 5.3 to arrive at $|L_{k+1} \cap B| \leq \eta_0 |B|$, with suitable choice of η_0, a contradiction. This finishes the proof. □

The uniform boundary $W^{1,\frac{np}{n-p}}$-estimates can now be proved as follows.

Proof of Theorem 6.1 One may argue exactly as in the proof of Theorem 5.1, by substituting Lemma 5.3 with Lemma 6.4. The additional term, namely the measure of C_k in the notation of the latter lemma, is controlled by the $W^{1,\frac{np}{n-p}}$-regularity of the boundary data g, as well as the strong (q, q)-type inequality, with $q > 1$, for the maximal function. We omit the detail to avoid repeating arguments. □

6.2 $W^{2,p}$-estimates

Let us now move on to the uniform $W^{2,p}$-estimates around boundary points. As always, we use Ω_ε to denote the set $\{x \in \Omega : \text{dist}\,(x, \partial\Omega) > \varepsilon\}$.

Theorem 6.5 *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.4), $\Omega \subset \mathbb{R}^n$ be a domain, $U \subset \mathbb{R}^n$ be an open neighborhood of a point of $\partial\Omega$ such that $\partial\Omega \cap U$ is a C^1-graph, $f \in L^p(\Omega \cap U)$ and $g \in W^{2,p}(\Omega \cap U)$, for some $p \in (p_0, \infty)$. Suppose that there is a diffeomorphism $\Phi \in C^1(U; V)$ for which $\Phi(\Omega \cap U) = H(e_n) \cap V$ and $\Phi(\partial\Omega \cap U) = \partial H(e_n) \cap V$ such that*

(i) $\Phi \in W^{2,n}(U; V)$ *if* $p_0 < p < n$;
(ii) $\Phi \in W^{2,n+\sigma}(U; V)$ *for some* $\sigma > 0$ *if* $p = n$;
(iii) $\Phi \in W^{2,p}(U, V)$, *if* $p > n$.

Let $u^\varepsilon \in C(\overline{\Omega} \cap U)$ be a viscosity solution to

$$\begin{cases} F\left(D^2u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f & \text{in } \Omega \cap U, \\ u^\varepsilon = g & \text{on } \partial\Omega \cap U. \end{cases} \tag{2.0.1}$$

Then $H^\varepsilon_{\Omega\cap U}(u^\varepsilon) \in L^p_{loc}(\overline{\Omega}_\varepsilon \cap U)$, provided that and for any $U' \Subset U$,

$$\|H^\varepsilon_{\Omega\cap U}(u^\varepsilon)\|_{L^p(\Omega_\varepsilon\cap U')} \leq C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega\cap U)}}{\text{dist}\,(U', \partial U)^{2-\frac{n}{p}}} + \||f| + |D^2g|\|_{L^p(\Omega\cap U)}\right),$$

where $C > 0$ depends at most on n, λ, Λ, ψ, κ, p, σ and diam (U).

Let us begin with a sub-optimal estimate, namely a uniform boundary $W^{2,q}$-estimates, with $q < p$, provided that $f \in L^p(\Omega \cap U)$ and $g \in C^{1,1-\frac{n}{p}}(\partial\Omega \cap U)$. Note that $g \in C^{1,1-\frac{n}{p}}(\partial\Omega \cap U)$ is a more relaxed assumption than $g \in W^{2,p}(\Omega \cap U)$. The proposition below will be used later in our approximation lemma (Lemma 6.7 for the boundary estimate.

Proposition 6.6 *Let $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ be a functional satisfying (2.0.1)–(2.0.4), $\Omega \subset \mathbb{R}^n$ be a domain, $U \subset \mathbb{R}^n$ be an open neighborhood of a point of $\partial\Omega$ such that $\partial\Omega \cap U$ is (δ, R)-Reifenberg flat, $g \in C^{0,\alpha}(\partial\Omega \cap U)$ for some $\alpha \in (0, 1]$, $f \in C(\Omega \cap U) \cap L^p(\Omega \cap U)$ for some $p > p_0$, and $u^\varepsilon \in C(\overline{\Omega} \cap U)$ be a viscosity solution to*

$$\begin{cases} F\left(D^2u^\varepsilon, \frac{\cdot}{\varepsilon}\right) = f & \text{in } \Omega \cap U, \\ u^\varepsilon = g & \text{on } \partial\Omega \cap U. \end{cases} \tag{2.0.2}$$

(i) *If $p_0 < p \leq n$, $H^\varepsilon_{\Omega\cap U}(u^\varepsilon) \in L^q_{loc}(\overline{\Omega}_\varepsilon \cap U)$ for all $q \in (p_0, \min\{p, \frac{n}{2-\alpha}\})$, and for any subdomain $U' \Subset U$,*

$$\|H^\varepsilon_{\Omega\cap U}(u^\varepsilon)\|_{L^q(\Omega_\varepsilon\cap U')} \leq C\left(\frac{\|u^\varepsilon\|_{L^\infty(\Omega\cap U)}}{\text{dist}\,(U', \partial U)^{2-\frac{n}{q}}} + \|f\|_{L^p(\Omega\cap U)} + \|g\|_{C^{0,\alpha}(\partial\Omega\cap U)}\right),$$

where $C > 0$ depends only on n, λ, Λ, ψ, κ, q, α and diam (U).

(ii) *If $p > n$, assume that $\partial\Omega \cap U$ is a $C^{1,\alpha}$-hypersurface, and $g \in C^{1,\alpha}(\partial\Omega \cap U)$. Then $H^{\varepsilon}_{\Omega\cap U}(u^{\varepsilon}) \in L^q_{loc}(\overline{\Omega}_{\varepsilon} \cap U)$ for all $q \in (p_0, \min\{p, \frac{n}{1-\alpha}\})$, and for any subdomain $U' \Subset U$,*

$$\|H^{\varepsilon}_{\Omega\cap U}(u^{\varepsilon})\|_{L^q(\Omega_{\varepsilon}\cap U')} \leq C \left(\frac{\|u^{\varepsilon}\|_{L^{\infty}(\Omega\cap U)}}{\text{dist}\,(U', \partial U)^{2-\frac{n}{q}}} + \|f\|_{L^p(\Omega\cap U)} + \|g\|_{C^{1,\alpha}(\partial\Omega\cap U)} \right).$$

Proof The proof is essentially the same with that of Proposition 6.2. After a suitable rescaling argument, it may suffice to prove the case where $U = B_1$, $U' = B_{1/2}$, $\|u^{\varepsilon}\|_{L^{\infty}(\Omega\cap B_1)} \leq 1$, $\|f\|_{L^p(\Omega\cap B_1)} \leq 1$ and $\|g\|_{C^{0,\alpha}(\partial\Omega\cap B_1)} \leq 1$ if $p \leq n$ and $\|g\|_{C^{1,\alpha}(\partial\Omega\cap B_1)} \leq 1$ if $p > n$. Let $p_n = \min\{p, \frac{n}{2-\alpha}\}$ if $p < n$, $p_n = \min\{p, \frac{n}{1-\alpha}\}$ if $p > n$, and $p_n = \gamma$ for some $\gamma \in (\frac{n}{2}, \min\{n, \frac{n}{2-\alpha}\})$. By [22, Theorem 1] if $p \leq n$ or Lemma 3.4 if $p > n$, one can find, for each $x_0 \in \partial\Omega \cap B_{1/2}$, a linear polynomial ℓ_{x_0} (in case $p \leq n$ the linear polynomial ℓ_{x_0} is taken by the constant $u^{\varepsilon}(x_0)$) such that

$$|(u^{\varepsilon} - \ell_{x_0})(x)| \leq c(|x - x_0|^{2-\frac{n}{p_n}} + \varepsilon^{2-\frac{n}{p_n}}), \tag{2.0.3}$$

for all $x \in \overline{\Omega} \cap B_1$, where $c > 0$ depends only on n, λ, Λ, κ and p_n.

Now for each ball $B \subset \Omega_{\varepsilon} \cap B_{1/2}$ with $\partial(2B) \cap \partial\Omega \cap B_{1/2} \neq \emptyset$, we can make the following rescaling,

$$u^{\varepsilon_B}_B(x) = \frac{(u^{\varepsilon} - \ell_{x_{B,0}})(x_B + \rho_B x)}{\rho_B^{2-\frac{n}{p_n}}},$$

of u^{ε}, where x_B is the center of B, ρ_B its radius and $x_{B,0}$ the point of intersection between $\partial(2B)$ and $\partial\Omega \cap B_{1/2}$. Then we may repeat the proof of Proposition 6.2, utilizing Theorem 5.4 in place of Theorem 5.1, to deduce that $\|H^{\varepsilon_B}_{\Omega_B}(u^{\varepsilon_B}_B)\|_{L^q(B_1)} \leq c$ for any $q \leq p$, with $\Omega_B = \rho_B^{-1}(-x_B + \Omega \cap B_1)$, whence

$$\|H^{\varepsilon}_{\Omega\cap B_1}(u^{\varepsilon})\|_{L^q(B)} \leq c\rho_B^{\frac{n}{q}-\frac{n}{p_n}}.$$

Fix any $q < p_n$. Then we can consider the same Besicovitch covering $\mathcal{G}$, as in the proof of Proposition 6.2, of $\Omega_{\varepsilon} \cap B_{1/2}$ by balls B, such that the summation of the right-hand side of over all $B \in \mathcal{G}$ is bounded by a constant c. This finishes the proof. □

As for the proof of Theorem 6.5, it suffices to consider the case where $f \in L^p$ and $g = 0$, since one may always substitute u^{ε} with $u^{\varepsilon} - g$ and f with $f + c|D^2 g|$.

Since our analysis will be of local nature around a boundary point, and will be invariant under translation, we shall work from now on with domains Ω with $0 \in \partial\Omega$, $U = B_2$ and $U' = B_1$. Unless specified otherwise, from now on, $F \in C(\mathcal{S}^n \times \mathbb{R}^n)$ satisfies (2.0.1)–(2.0.4), $\partial\Omega \cap B_2$ is a C^1-hypersurface containing the origin, and that there is a diffeomorphism $\Phi \in C^1(B_2; V)$, with $V \subset \mathbb{R}^n$ a neighborhood of the origin, such that $\Phi(0) = 0$, $\Phi(\Omega \cap B_2) = H_0(e_n) \cap V$ and $\Phi(\partial\Omega \cap B_2) = \partial H_0(e_n) \cap V$. We shall call Φ boundary flattening map (of $\partial\Omega$) around the origin. Moreover, $f \in C(\Omega \cap B_2) \cap L^p(\Omega \cap B_2)$ for some $p > p_0$, and $u^{\varepsilon} \in C(\overline{\Omega} \cap B_2)$ is a viscosity solution to (2.0.1) with $U = B_2$ and $g = -\ell$, with ℓ a linear polynomial; we shall discuss later in detail the reason for the involvement of a linear polynomial in the boundary condition.

The difficulty of our analysis arises from the fact that homogenization problems are unfavorable towards boundary flattening argument, as one loses the oscillating pattern by the transformation. In one way or another, one will resort to the fact that the original problem in small scales is homogenized to a "nice" effective problem to improve the regularity, whence the level of difficulty remains the same.

For this reason, we shall study our problem (2.0.1) without flattening the boundary. This readily implies some notable changes in the approximation lemma below for the measure of the set of large "Hessian", compared to the interior case (Lemma 5.5) as well as those for standard problems in the setting of flat boundaries (e.g., [30, Lemma 2.14]).

Lemma 6.7 *Let ε, δ, α, ρ, p and q be constants with $0 < \varepsilon < 1$, $0 < \delta \le \delta_0$, $0 < \alpha < 1$, $\rho > 0$ and $p_0 < p < q < \infty$ be given. Suppose that $|\xi^t D\Phi(0)\xi - 1| \le \delta$ for any $\xi \in \partial B_1$, $\operatorname{osc}_{B_2} D\Phi \le \delta$, $\|f\|_{L^p(\Omega\cap B_2)} \le \delta$, $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \le 1$, $|u^\varepsilon(x)| \le |x|^2$ for all $x \in \Omega\backslash B_1$, and $u^\varepsilon = -\ell$ on $\partial\Omega \cap B_2$, for some linear polynomial ℓ. Assume either of the following:*

(i) $|D\ell| \le \dfrac{1}{\rho}$, $\|D^2\Phi\|_{L^q(B_2)} \le \delta\rho$ *and* $q < n$;

(ii) $|D\ell| \le \dfrac{1}{\rho}$, $\|D^2\Phi\|_{L^q(B_2)} \le \delta\rho$, *and* $n < q < \frac{n}{1-\alpha}$;

(iii) $|D\ell| \le \dfrac{1}{\rho^{1-\alpha}}$, $\|D^2\Phi\|_{L^n(B_2)} \le \delta\rho$ *and* $q > n$.

Then for any $s > 0$,

$$|A_s^\varepsilon(u^\varepsilon, \Omega) \cap \Omega_\varepsilon \cap B_1| \le c(\delta^{\gamma\mu} s^{-\mu} + s^{-q}),$$

where $\mu > 0$ depends only on n, λ and Λ, $\delta_0 > 0$ and $0 < \gamma \le 1$ depend in addition to q and respectively α, and $c > 1$ may depend further on κ and ψ. Nevertheless, none of μ, δ_0, γ and c depends on ρ or ε.

Proof Set $T = \frac{1}{|B_2|}\int_{B_2} D\Phi\, dx$, and let $L_T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation induced by T; i.e., $L_T(0) = 0$ and $DL_T = T$. In what follows, we shall denote by c_0 a constant depending at most on n and q, and we shall allow it to vary from one line to another.

Case 1 $|D\ell| \le \rho^{-1}$, $\|D^2\Phi\|_{L^q(B_2)} \le \delta\rho$ and $q < n$.

By the Poincaré inequality, together with $\|D^2\Phi\|_{L^q(B_2)} \le \delta\rho$,

$$\int_{B_2} |D\Phi - T|^q\, dx \le c_0\delta^n\rho^n. \tag{2.0.4}$$

Let $L_T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation such that $DL_T = T$. Then by the Sobolev embedding theorem, one can infer that

$$[\Phi - L_T]_{C^{0,2-\frac{n}{q}}(B_2)} \le c_0\delta\rho. \tag{2.0.5}$$

Now by the assumption that $\Phi(\partial\Omega \cap B_2) \subset \partial H_0(e_n)$, i.e., $e_n \cdot \Phi = 0$ on $\partial\Omega \cap B_2$, (2.0.5) yields that $|e_n \cdot T(x-y)| \le c_0\delta\rho|x-y|^{2-\frac{n}{q}}$ for any $x, y \in \partial\Omega \cap B_2$. Recall that $T = \frac{1}{|B_2|}\int_{B_2} D\Phi\, dx$. By the assumption on Φ, $|\xi^t T\xi - 1| \le 2\delta$ for all $\xi \in \partial B_1$, and hence $|Te_n| \ge e_n^t Te_n \ge 1 - 2\delta > \frac{1}{2}$, provided that $\delta \le \frac{1}{4}$. Set $\nu = \frac{Te_n}{|Te_n|}$. Then the latter observation implies that

$$|\nu \cdot (x-y)| \le 2c_0\delta\rho|x-y|^{2-\frac{n}{q}}, \tag{2.0.6}$$

for any $x, y \in \partial\Omega \cap B_2$.

Let us now turn to the regularity of the linear polynomial ℓ, for which $u^\varepsilon = -\ell$ on $\partial\Omega\cap B_2$. Since $|u^\varepsilon| \le 1$ in $\Omega \cap B_2$, $|\ell| \le 1$ on $\partial\Omega \cap B_2$. Now we may deduce from $|D\ell| \le \rho^{-1}$, (2.0.6) and $0 \in \partial\Omega$ that with $S = S_{2c_0\delta\rho}(\nu)$,

$$\sup_{S\cap B_2} |\ell| < 1 + \frac{4\delta\rho}{\rho} \le 2. \tag{2.0.7}$$

Therefore, denoting by $P_\nu(e)$ the orthogonal projection of a vector e in direction ν, one can compute that

$$|P_\nu(D\ell)| \le \operatorname{osc}_{S\cap B_2}\ell \le 4. \tag{2.0.8}$$

Putting (2.0.8) together with (2.0.6), we also derive that for each $\alpha \in (0,1)$, we can take δ small, depending at most on n and β, such that

$$[\ell]_{C^{0,2-\frac{n}{q}}(\partial\Omega\cap B_2)} \le 2^{\frac{n}{q}-1}|P_\nu(D\ell)| + c_0\delta\rho|D_\nu\ell| \le 9, \tag{2.0.9}$$

where the last inequality holds for any small δ, whose smallness condition depends only on n and q.

Let $h^\varepsilon \in C(\overline{\Omega\cap B_2})$ be a viscosity solution to

$$\begin{cases} F\left(D^2h^\varepsilon, \frac{\cdot}{\varepsilon}\right) = 0 & \text{in } \Omega\cap B_2,\\ h^\varepsilon = u^\varepsilon & \text{on } \partial(\Omega\cap B_2). \end{cases} \tag{2.0.10}$$

The existence of such a viscosity solution is ensured by [27, Theorem 1], since $\partial\Omega\cap B_2$ is a C^1-hypersurface, and that $F \in C(S^n \times \mathbb{R}^n)$ and $u^\varepsilon \in C(\partial(\Omega\cap B_2))$. By the maximum principle, we have

$$\|h^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \le 1. \tag{2.0.11}$$

By the assumption on Φ, $\partial\Omega\cap B_2$ is a C^1-hypersurface whose Lipschitz norm is less than $c_0\delta$. Thus, by taking δ smaller if necessary, depending now on n, λ, Λ and q, we can deduce from Proposition 6.6 (with $\alpha = 2 - \frac{n}{q}$), (2.0.9) and (2.0.11) that $H^\varepsilon_{\Omega\cap B_2}(h^\varepsilon) \in L^q(\Omega_\varepsilon\cap B_1)$ and

$$\|H^\varepsilon_\Omega(h^\varepsilon, \Omega\cap B_2)\|_{L^q(\Omega_\varepsilon\cap B_1)} \le c, \tag{2.0.12}$$

for some constant $c > 0$ depending only on n, λ, Λ, κ, ψ and q. In particular, by the definition $A^\varepsilon_s(h^\varepsilon, E) = \{H^\varepsilon_E(h^\varepsilon) > s\}$, we obtain

$$|A^\varepsilon_s(h^\varepsilon, \Omega\cap B_2)\cap\Omega_\varepsilon\cap B_1| \le cs^{-q}, \tag{2.0.13}$$

for any $s > 0$.

In comparison of (2.0.10) with (2.0.1) (with $g = -\ell$), Lemma 3.1 ensures that $w^\varepsilon = \delta^{-1}(u^\varepsilon - h^\varepsilon)$ is a viscosity solution to

$$\begin{cases} \mathcal{P}_-(D^2w^\varepsilon) \le \frac{f}{\delta} \le \mathcal{P}_+(D^2w^\varepsilon) & \text{in } \Omega\cap B_2,\\ w^\varepsilon = 0 & \text{on } \partial(\Omega\cap B_2), \end{cases} \tag{2.0.14}$$

Owing to $\|f\|_{L^p(\Omega\cap B_2)} \le \delta$, the generalized maximum principle ensures that $\|w^\varepsilon\|_{L^\infty(\Omega\cap B_{3/2})} \le c$. Then by Proposition 4.8, we obtain

$$|A_t(w^\varepsilon, \Omega\cap B_{3/2})| \le c_0t^{-\mu}, \tag{2.0.15}$$

for any $t > 1$, where $\mu \in (0, 1)$ depends only on n, λ and Λ. Combining (2.0.15) with (2.0.13), we arrive at

$$|A_s(u^\varepsilon, \Omega \cap B_2) \cap \Omega_\varepsilon \cap B_1| \le c(\delta^\mu s^{-\mu} + s^{-q}).$$

We can then replace $\Omega \cap B_2$ above with Ω by invoking the inequality $|u^\varepsilon(x)| \le |x|^2$ for all $x \in \Omega \setminus B_1$.

Case 2 $|D\ell| \le \rho^{-1}$, $\|D^2\Phi\|_{L^q(B_2)} \le \delta\rho$ and $n \le q < \frac{n}{1-\alpha}$.

As for this case, by the Sobolev embedding theorem,

$$[D\Phi]_{C^{0,1-\frac{n}{q}}(B_2)} \le c_0\delta\rho. \tag{2.0.16}$$

Especially, since $|e_n \cdot D\Phi| \ge e_n^t D\Phi e_n \ge 1 - 2\delta > \frac{1}{2}$ on B_2, denoting by ν_x the unit vector $\frac{e_n \cdot \Phi(x)}{|e_n \cdot D\Phi(x)|}$, we obtain that

$$|\nu_x - \nu_y| \le 2c_0\delta\rho|x - y|^{1-\frac{n}{q}}, \tag{2.0.17}$$

for any $x, y \in \partial\Omega \cap B_2$. Moreover, one can also deduce from (2.0.16) that $|\Phi(y) - \Phi(x) - D\Phi(x) \cdot (y - x)| \le c_0\delta\rho|y - x|^{2-\frac{n}{q}}$, for any $x, y \in B_2$. Utilizing $e_n \cdot \Phi = 0$ on $\partial\Omega \cap B_2$, we also obtain that

$$|\nu_x \cdot (y - x)| \le 2c_0\delta\rho|y - x|^{2-\frac{n}{q}}, \tag{2.0.18}$$

and In other words, $\partial\Omega \cap B_2$ is a $C^{1,1-\frac{n}{q}}$-hypersurface whose $C^{1,\alpha}$-norm is bounded by $2c_0\delta\rho$.

For the rest of the proof, we shall denote by c_α a positive constant depending at most on n and α, and it may vary at each occurrence. With (2.0.18) at hand, we claim that $\ell \in C^{1,1-\frac{n}{q}}(\partial\Omega \cap B_2)$ and

$$\|\ell\|_{C^{1,1-\frac{n}{q}}(\partial\Omega\cap B_2)} \le c_0. \tag{2.0.19}$$

Note that the hypothesis of Case 2 is stronger than that of Case 1, whence (2.0.7) and (2.0.8) continue to hold, with the bounds possibly replaced by c_0. Thus, (2.0.8) together with (2.0.18) (with $x_0 = 0$) and $|D\ell| \le \rho^{-1}$ implies that

$$[\ell]_{C^{0,1}(\partial\Omega\cap B_2)} \le |P_\nu(D\ell)| + c_0\delta\rho|D_\nu\ell| \le c_0.$$

Moreover, we may also compute, via (2.0.17) and $|D\ell| \le \rho^{-1}$, that

$$[D_\tau\ell]_{C^{0,1-\frac{n}{q}}(\partial\Omega\cap B_2)} \le |D\ell| \sup_{x,y\in\partial\Omega\cap B_2} \frac{|\nu_x - \nu_y|}{|x - y|^{1-\frac{n}{q}}} \le c_0,$$

where D_τ is the tangential gradient to $\partial\Omega \cap B_2$. Combining the last two displays altogether with (2.0.7), we verify the claim (2.0.19).

Now let h^ε be the viscosity solution to (2.0.10), as in Case 1. The inequality in (2.0.11) continues to hold here. However, now with (2.0.18) and (2.0.19) at hand, Proposition 6.6 ensures that (2.0.12), hence (2.0.13) as well, holds for $q \le \frac{n}{1-\alpha}$. The rest of the proof repeats that of Case 1 verbatim, so it is omitted.

Case 3 $|D\ell| \le \rho^{\alpha-1}$, $\|D^2\Phi\|_{L^n(B_2)} \le \delta\rho$ and $q > n$.

In the above cases, the auxiliary Dirichlet problem for the approximating function h^ε was imposed on the same domain $\Omega \cap B_2$. Thus, the integrability of $H^\varepsilon_{\Omega\cap B_2}(h^\varepsilon)$ cannot exceed the exponent determined by the regularity of the boundary layer. On contrast, this last case asks for q to go over the threshold. To achieve this goal, we shall consider another Dirichlet problem, whose boundary layer is much smoother (in fact, a hyperplane). However, the choice of such an auxiliary problem cannot be arbitrary, since the newly obtained approximating function should also be sufficiently close to the original solution on the boundary layer, so that their difference satisfies (2.0.15). To meet the latter requirement, we need ℓ to be $C^{0,\alpha}$-regular for some $\alpha > 0$, not only on $\partial\Omega \cap B_2$ (as in (2.0.9)), but also over a slab $S \cap B_2$, with $S = S_{c_0\delta\rho}(\nu)$, that contains $\partial\Omega \cap B_2$. This is where a stronger assumption, $|D\ell| \leq \rho^{\alpha-1}$, is used.

Let us explain this in more detail. In what follows, let E denote the half-space $H_{-c_0\delta\rho}(\nu)$; note that $\Omega \cap B_2 \subset E$. We shall keep writing by S the slab $S_{c_0\delta\rho}(\nu)$; recall from (2.0.6) that $\partial\Omega \cap B_2 \subset S$.

With α being the exponent for which $|D\ell| \leq \rho^{\alpha-1}$, we may compute, by using (2.0.8), that

$$\begin{aligned} [\ell]_{C^{0,\alpha}(S\cap B_2)} &\leq 2^{1-\alpha}|P_\nu(D\ell)| + |D_\nu \ell| \sup_{x,y\in S\cap B_2} \frac{|\nu\cdot(x-y)|}{|x-y|^\alpha} \\ &\leq 8 + \rho^{\alpha-1} \sup_{x,y\in S\cap B_2} |\nu\cdot(x-y)|^{1-\alpha} \\ &\leq 9, \end{aligned} \tag{2.0.20}$$

where the last inequality again follows by choosing δ small, depending only on n and α.

Let us remark that $u^\varepsilon \in C(\overline{\Omega} \cap B_2)$, $\mathcal{P}_-(D^2u^\varepsilon) \leq f \leq \mathcal{P}_+(D^2u^\varepsilon)$ in $\Omega \cap B_2$ in the viscosity sense, $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \leq 1$, $\|f\|_{L^p(\Omega\cap B_2)} \leq \delta$ and $u^\varepsilon = -\ell$ on $\partial\Omega \cap B_2$. Moreover, $\Omega \cap B_2$ satisfies a uniform exterior cone condition, where the size of the cone is bounded by an absolute constant, because of the assumption on the boundary flattening map Φ. Therefore, one may employ [27, Theorem 2], along with (2.0.20) (in fact, (2.0.9) works equally well here) to deduce that $u^\varepsilon \in C^{0,2\gamma}(\overline{\Omega \cap B_{3/2}})$, and

$$[u^\varepsilon]_{C^{0,2\gamma}(\overline{\Omega\cap B_{3/2}})} \leq c, \tag{2.0.21}$$

for some $\gamma \in (0, \frac{\alpha}{4}]$ and $c > 0$ depending at most on n, λ, Λ, p and α.

Now let $\phi^\varepsilon \in C(\overline{E \cap B_{3/2}})$ be a viscosity solution to

$$\begin{cases} F\left(D^2\phi^\varepsilon, \frac{\cdot}{\varepsilon}\right) = 0 & \text{in } E \cap B_{3/2}, \\ \phi^\varepsilon = u^\varepsilon & \text{on } \Omega \cap \partial B_{3/2}, \\ \phi^\varepsilon = -\ell & \text{on } \partial(E \cap B_{3/2}) \setminus \Omega. \end{cases} \tag{2.0.22}$$

Note that $\phi^\varepsilon \in C(\partial(E \cap B_2))$, since $u^\varepsilon = -\ell$ on $\partial\Omega \cap B_2$, and that $E \cap B_2$ satisfies a uniform exterior sphere condition with radius 1. Hence, the existence of a viscosity solution to the above problem is obvious. Also by (2.0.20) and (2.0.21), $\phi^\varepsilon \in C^{0,2\gamma}(\partial(E \cap B_{3/2}))$, and thus, it follows from e.g., [6, Proposition 4.13, 4.14], that

$$[\phi^\varepsilon]_{C^{0,\gamma}(\overline{E\cap B_{3/2}})} \leq c. \tag{2.0.23}$$

On the other hand, by the maximum principle, $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \leq 1$ and (2.0.7), we have $\|\phi^\varepsilon\|_{L^\infty(E\cap B_{3/2})} \leq 2$. Moreover, as ∂E is an hyperplane orthogonal to ν, $|D_\tau \ell| = |P_\nu(D\ell)|$ and $|D^2_\tau \ell| = 0$ on ∂E, with D_τ being the tangential gradient on ∂E. Therefore, it follows from (2.0.7) and (2.0.8) that

$$\|\ell\|_{C^{1,1}(\partial E\cap B_{3/2})} \leq 6. \tag{2.0.24}$$

By (2.0.24) (as well as $\|\phi^\varepsilon\|_{L^\infty(E\cap B_2)} \leq 2$, Proposition 6.6 (now with $f=0$, $g=-\ell$, $p=q$ and $\alpha=1$ there) yields that $H^\varepsilon_{E\cap B_{3/2}}(h^\varepsilon) \in L^q(E_\varepsilon \cap B_1)$, with the chosen $q\in(n,\infty)$ and $E_\varepsilon = \{x\in E : \operatorname{dist}(x,\partial E) > \varepsilon\}$, and hence arguing as in the derivation of (2.0.12), we similarly obtain that

$$|A^\varepsilon_s(\phi^\varepsilon, E\cap B_{3/2}) \cap E_\varepsilon \cap B_1| \leq cs^{-q}, \tag{2.0.25}$$

where c may now depend at most on n, λ, Λ, ψ, κ and q.

Set $v^\varepsilon = \frac{u^\varepsilon - \phi^\varepsilon}{c\delta^\gamma}$, with a possibly different $c>1$ to that in the last display, yet depending on the same quantities. Then again by Lemma 3.1, we may compare (2.0.1) (with $g=-\ell$) with (2.0.22) and compute that

$$\begin{cases} \mathcal{P}_-(D^2 v^\varepsilon) \leq \frac{f}{c\delta^\gamma} \leq \mathcal{P}_+(D^2 v^\varepsilon) & \text{in } \Omega\cap B_{3/2},\\ v^\varepsilon = 0 & \text{on } \Omega\cap\partial B_{3/2},\\ v^\varepsilon = \frac{-\ell-\phi^\varepsilon}{c\delta^\gamma} & \text{on } \partial\Omega\cap B_{3/2}. \end{cases} \tag{2.0.26}$$

As $c>1$ and $0<\delta,\gamma<1$, it follows from the assumption that $\|f\|_{L^p(\Omega\cap B_2)} \leq c\delta^\gamma$. This implies, by the general maximum principle, that

$$\|v^\varepsilon\|_{L^\infty(\Omega\cap B_{3/2})} \leq c_0 + \sup_{\partial\Omega\cap B_{3/2}} \frac{|\phi^\varepsilon + \ell|}{c\delta^\gamma}.$$

Therefore, once the rightmost term is proved to be bounded by an absolute constant (by choosing $c>1$ large), we may repeat the final part of the proof for Case 1, and derive the desired decay estimate. Since the latter implication is already shown above, let us finish the proof by justifying that $|\phi^\varepsilon + \ell| \leq c\delta^\gamma$ on $\partial\Omega\cap B_{3/2}$.

To this end, let $x\in\partial\Omega\cap B_{3/2}$ be any, and find $x_0\in\partial E\cap B_{3/2}$ such that $|x-x_0|\leq c_0\delta\rho$. Such a point x_0 always exists because $\partial\Omega\cap B_2 \subset S$ with S being a slab with width $c_0\delta\rho$. Then by (2.0.20) and (2.0.23),

$$\begin{aligned} |\phi^\varepsilon(x) + \ell(x)| &\leq |\phi^\varepsilon(x) + \ell(x_0)| + |\ell(x) - \ell(x_0)| \\ &\leq c((\delta\rho)^\gamma + (\delta\rho)^\alpha) \\ &\leq c\delta^\gamma, \end{aligned}$$

where the last inequality is ensured by $\gamma \leq \frac{\alpha}{4}$, $\delta<1$ and $\rho\leq 1$. This completes the proof. □

Our next step is to design a suitable iteration argument for the boundary estimate.

Lemma 6.8 *Let* ε, δ, α, ρ, p *and* q *be constants with* $0<\varepsilon<1$, $0<\delta\leq\delta_0$, $0<\alpha<1$, $\rho>0$ *and* $p_0<p<q<\infty$ *be given. Suppose that* $|\xi^t D\Phi(0)\xi - 1| \leq \delta$ *for any* $\xi\in\partial B_1$, $\operatorname{osc}_{B_2} D\Phi \leq \delta$, $\|D^2\Phi\|_{L^n(B_2)} \leq \delta$, $\|f\|_{L^p(\Omega\cap B_2)} \leq \delta$, $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \leq 1$, *and* $u^\varepsilon = 0$ *on* $\partial\Omega\cap B_2$. *Set* $A_k = A^\varepsilon_{m^k}(u^\varepsilon, \Omega\cap B_2)\cap\Omega_\varepsilon\cap B_1$ *and* $B_k = \{M(|f|^p\chi_{\Omega\cap B_2})) > \delta^p m^{kp}\}$. *Assume either of the following:*

(i) $C_k = L^\varepsilon_{m^{k(1-p/n)}}(u^\varepsilon, \Omega\cap B_2)\cap\Omega_\varepsilon\cap B_{3/2}$, $D_k = \{M(|D^2\Phi|^q\chi_{B_2}) > \delta^q m^{\frac{kpq}{n}}\}$, *and* $p<q<n$;

(ii) $C_k = L^\varepsilon_{m^{k\alpha}}(u^\varepsilon, \Omega\cap B_2)\cap\Omega_\varepsilon\cap B_{3/2}$, $D_k = \{M(|D^2\Phi|^q\chi_{B_2}) > \delta^q m^{kn(1-\alpha)q}\}$ *and* $(1-\alpha)n = p < n < q < \frac{n}{1-\alpha}$;

(iii) $\|D^2\Phi\|_{L^p(B_2)} \leq \delta$, $C_k = \emptyset$, $D_k = \{M(|D^2\Phi|^p\chi_{B_2}) > \delta m^{kp}\}$ *and* $n<p<q$.

Then for each integer

$$1 \le k \le -c(n,p)\frac{\log \varepsilon}{\log m}, \quad \text{with } c(n,p) = \begin{cases} \frac{n}{p}, & \text{for case (i)}, \\ \frac{1}{1-\alpha}, & \text{for case (ii)}, \\ \frac{p}{n}, & \text{for case (iii)}, \end{cases}$$

one has

$$|A_{k+1}| \le c(\delta^{\gamma\mu} m^{-\mu} + m^{-q})|A_k \cup B_k \cup C_k \cup D_k|,$$

where $\mu > 0$ *depends only on* n, λ *and* Λ, $\delta_0 \in (0, \frac{1}{4})$ *and* $\gamma \in (0,1]$ *depends at most on* q *and* α, *while* $c > 0$ *and* $m > 1$ *may depend further on* ψ *and* κ.

Proof Since $u^\varepsilon = 0$ on $\partial\Omega \cap B_2$, we may apply Lemma 6.7 (with $\ell = 0$ and $\rho = 1$) to deduce that $|A_1| \le c(\delta^{\gamma\mu} m^{-\mu} + m^{-q}) \le \eta_0 |B_1|$. Let us remark that this initial step applies to all three cases considered in the statement. Fix an integer $k \ge 1$. Since $A_{k+1} \subset A_k \subset \cdots \subset A_1$, we readily obtain $|A_{k+1}| \le \eta_0 |B_1|$.

Next, let $B \subset B_1$ be a ball whose center lies in $\Omega_\varepsilon \cap B_1$ and rad $(B) \le 1$. Assume that $|A_{k+1} \cap B| > \eta_0 |B|$. Our goal is to show that $\Omega_\varepsilon \cap B \subset (A_k \cup B_k \cup C_k \cup D_k)$. Thus, by Lemma 2.7 (which applies to $\Omega_\varepsilon \cap B_1$, since $\partial\Omega_\varepsilon \cap B_1$ is $(c_0\delta, 2)$-Reifenberg flat, which can be easily inferred from the C^1-character of $\partial\Omega \cap B_1$), $|A_{k+1}| \le c\eta_0 |A_k \cup B_k \cup C_k \cup D_k|$.

To the rest of the proof, we shall assume $\Omega_\varepsilon \cap B \backslash (A_k \cup B_k \cup C_k \cup D_k) \ne \emptyset$, and attempt to derive a contradiction against $|A_{k+1} \cap B| > \eta_0 |B|$. Arguing as in Lemma 5.3, we observe that

$$\varepsilon < 2r_B, \tag{2.0.27}$$

by selecting $m > 1$ large, yet depending only on n.

Choose any $\tilde{x}_B \in \Omega_\varepsilon \cap B \backslash (A_k \cup B_k \cup C_k \cup D_k)$. We shall only consider the case $B_{2r_B}(\tilde{x}_B) \backslash \Omega \ne \emptyset$, as the other case can be treated as the interior case. Under this setting, we can find $x_B \in \partial\Omega \cap B_1$ such that $\varepsilon < |x_B - \tilde{x}_B| = \text{dist}\,(\tilde{x}_B, \partial\Omega) < 2r_B$.

For the rest of the proof, we shall denote by $c > 1$ a constant depending at most on n, λ, Λ, κ, ψ, p, q and σ, and we allow it to vary at each occurrence.

Case 1. $C_k = L^\varepsilon_{m^{k(1-\frac{p}{n})}}(u^\varepsilon, \Omega \cap B_2) \cap \Omega_\varepsilon \cap B_{3/2}$, $D_k = \{M(|D^2\Phi|^n \chi_{B_2}) > \delta m^{kp}\}$, and $p < q < n$.

By $\tilde{x}_B \in B \backslash D_k$ and $B_{2r_B}(x_B) \subset B_{4r_B}(\tilde{x}_B)$,

$$\int_{B_{2r_B}(x_B)} |D^2\Phi|^q \, dx \le (4r_B)^n (\delta m^{\frac{kp}{n}})^q. \tag{2.0.28}$$

Moreover, thanks to $\tilde{x}_B \notin A_k$, there exists a linear polynomial ℓ such that for all $x \in \Omega \cap B_2$,

$$|(u^\varepsilon - \ell)(x)| \le \frac{m^k}{2}(|x - \tilde{x}_B|^2 + \varepsilon^2), \tag{2.0.29}$$

for all $x \in \Omega \cap B_2$. Also observe from $\tilde{x}_B \notin C_k$ that $|u^\varepsilon(x) - a| \le m^{k(1-\frac{p}{n})}(|x - \tilde{x}_B| + \varepsilon)$ for all $x \in \Omega \cap B_2$ for some constant $a \in \mathbb{R}$. Putting this observation together with (2.0.29), $m^{\frac{kp}{n}}\varepsilon \le 1$ and $B_\varepsilon(\tilde{x}_B) \subset \Omega$, we obtain $|\ell(x) - a| \le m^k \varepsilon^2 + 2\,m^{k(1-\frac{p}{n})}\varepsilon \le 3\,m^{k(1-\frac{p}{n})}\varepsilon$ for all $x \in B_\varepsilon(\tilde{x}_B)$. In particular, we arrive at

$$|D\ell| \le 6m^{k(1-\frac{p}{n})}. \tag{2.0.30}$$

In addition, from $\tilde{x}_B \notin B_k$ and $B_{2r_B}(x_B) \subset B_{4r_B}(\tilde{x}_B)$, we also have

$$\int_{\Omega\cap B_{2r_B}(x_B)} |f|^p \, dx \le (4r_B)^n (\delta m^k)^p. \tag{2.0.31}$$

Denote by Ω_B the rescaled domain $\frac{1}{2r_B}(-x_B + \Omega \cap B_2)$, and define Φ_B by

$$\Phi_B(x) = \frac{1}{2r_B}(\Phi(x_B + 2r_B x) - \Phi(x_B)). \tag{2.0.32}$$

Since $\operatorname{osc}_{B_r(x)} D\Phi \le \delta r$ for any $x \in \partial\Omega \cap B_2$ and any $r \in (0, 1)$, we have $\operatorname{osc}_{B_2} D\Phi_B \le \delta$. Moreover, since $|\xi^t D\Phi(0)\xi - 1| \le \delta$ for all $\xi \in \partial B_1$ and $\operatorname{osc}_{B_1} D\Phi \le \delta$, it follows that $|\xi^t D\Phi_B(0)\xi - 1| \le 2\delta$ for all $\xi \in \partial B_1$. Furthermore, according to (2.0.28), $\|D^2\Phi_B\|_{L^q(B_2)} \le 4\delta m^{\frac{kp}{n}} r_B$.

Consider the following rescaled versions of u^ε, ℓ and f,

$$\begin{aligned} u_B^{\varepsilon_B}(x) &= \frac{(u^\varepsilon - \ell)(x_B + 2r_B x)}{cm^k r_B^2}, \quad \varepsilon_B = \frac{\varepsilon}{2r_B} \\ \ell_B(x) &= \frac{\ell(x_B + 2r_B x)}{cm^k r_B^2}, \quad f_B(x) = \frac{f(x_B + 2r_B x)}{cm^k}, \end{aligned} \tag{2.0.33}$$

with $c > 1$ to be determined later. By (2.0.29), we have $|u_B^{\varepsilon_B}(x)| \le 1$ for all $x \in \Omega_B \cap B_2$ and $|u_B^{\varepsilon_B}(x)| \le |x|^2$ for all $x \in \Omega \backslash B_1$, and by (2.0.31), $\|f_B\|_{L^p(\Omega_B \cap B_2)} \le \delta$, while (2.0.30) ensures $|D\ell_B| \le (m^{\frac{pk}{n}} r_B)^{-1}$, provided that we take $c > 1$ larger if necessary. In view of (2.0.1) and $u^\varepsilon = 0$ on $\partial\Omega \cap B_2$, one may also compute that

$$\begin{cases} F_B\left(D^2 u_B^{\varepsilon_B}, \frac{\cdot}{\varepsilon_B}\right) = f_B & \text{in } \Omega_B \cap B_2, \\ u_B^{\varepsilon_B} = -\ell_B & \text{on } \partial\Omega_B \cap B_2, \end{cases} \tag{2.0.34}$$

in the viscosity sense, where $F_B(P, y) = \frac{1}{cm^k} F(cm^k P, y + \frac{x_B}{\varepsilon})$. Obviously, $F_B \in C(\mathcal{S}^n \times \mathbb{R}^n)$ and it verifies (2.0.1)–(2.0.3). One may also check (2.0.4) for $\bar{F}_B$, for the same reason shown in the proof of Lemma 5.6. Besides, $\varepsilon_B < 1$ because of (2.0.27).

Summing up all the observations above, ε_B, F_B, $u_B^{\varepsilon_B}$, ℓ_B, $\partial\Omega_B \cap B_2$, Φ_B and f_B verify all the hypotheses for the first case of Lemma 6.7 (with $\rho = m^{\frac{kp}{n}} r_B$ and $c\delta$ in place of δ there). Therefore, we obtain that for any $s > 0$,

$$|A_s^\varepsilon(u_B^{\varepsilon_B}, \Omega_B) \cap \Omega_{B,\varepsilon_B} \cap B_1| \le c(\delta^\mu s^{-\mu} + s^{-q}),$$

where Ω_{B,ε_B} denotes the set $\{x \in \Omega_B : \operatorname{dist}(x, \partial\Omega_B) > \varepsilon_B\}$. Rephrasing the inequality in terms of u^ε, we obtain $|A_{cm^k s}(u^\varepsilon, \Omega \cap B_2) \cap \Omega_\varepsilon \cap B_{2r_B}(x_B)| \le c(\delta^\mu s^{-\mu} + s^{-q}) r_B^n \le \eta_0 |B|$. Evaluating this inequality at $s = c^{-1} m$ and using $B \subset B_{2r_B}(x_B)$, we arrive at $|A_{k+1} \cap B| \le \eta_0 |B|$, a contradiction. This finishes the proof for Case 1.

Case 2. $C_k = L^\varepsilon_{cm^{k\alpha}}(u^\varepsilon, \Omega \cap B_2) \cap \Omega_\varepsilon \cap B_{3/2}$, $D_k = \{M(|D^2\Phi|^q \chi_{B_2}) > \delta^q m^{kn(1-\alpha)q}\}$ and $(1-\alpha)n = p < n < q < \frac{n}{1-\alpha}$.

Let us remark that C_k in Case 2 is the same with Case 1 by taking $p = (1-\alpha)n$, while D_k in Case 2 replaces n and p in that of Case 1 with $\frac{n}{1-\alpha}$ and respectively n. Keeping this in mind, we follow the lines of the proof for Case 1. Then one may observe that under the new hypothesis in Case 2, (2.0.28) is replaced by $\int_{B_{2r_B}(x_B)} |D^2\Phi|^q \, dx \le (4r_B)^n \delta^q m^{kn(1-\alpha)q}$, (2.0.29) remains the same, (2.0.30) is replaced by $|D\ell| \le 6m^{k\alpha}$ (here we also need $k \le \frac{1}{1-\alpha} \log_m \frac{1}{\varepsilon}$, which is ensured from the statement of this lemma) and (2.0.31) is replaced by $\int_{\Omega\cap B_{2r_B}(x_B)} |f|^p\,dx \le (4r_B)^n(\delta m^k)^{n(1-\alpha)}$. Thus, with Φ_B, $u_B^{\varepsilon_B}$, ε_B, ℓ_B, f_B, F_B as in (2.0.32), (2.0.33) and (2.0.34), one can verify that hypotheses for the second case of Lemma 6.7 is satisfied (with $\rho = m^{k(1-\alpha)}r_B$ and $c\delta$ in place of δ there). The rest of the proof is the same. Let us skip the detail in order to avoid redundant argument.

Case 3. $\|D^2\Phi\|_{L^p(B_2)} \le \delta$, $C_k = \emptyset$, $D_k\{M(|D^2\Phi|^n\chi_{B_2}) > \delta^n m^{kn}\}$ and $n < p < q$.

Unlike the first two case, we need to treat the last case differently. By the additional assumption $\|D^2\Phi\|_{L^p(B_2)} \le \delta$ and $p > n$, the Sobolev embedding theorem implies that $[D\Phi]_{C^{0,1-n/p}(B_2)} \le c\delta$. As Φ is the boundary flattening map of $\partial\Omega\cap B_2$, it follows that $\partial\Omega\cap B_2$ is a $C^{1,1-n/p}$-hypersurface, whose norm is bounded by $c\delta$; since this implication is already rigorously justified in the proof of Lemma 6.7, we shall not repeat it here. Since $\mathcal{P}_-(D^2u^\varepsilon) \le f \le \mathcal{P}_+(D^2u^\varepsilon)$ in $\Omega\cap B_2$ in the viscosity sense, and $u^\varepsilon = 0$ on $\partial\Omega\cap B_2$, one can find a linear polynomial ℓ_{x_B}, according to [23, Theorem 1.6] together with $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \le 1$ and $\|f\|_{L^p(\Omega\cap B_2)} \le \delta$, that

$$|(u^\varepsilon - \ell_{x_B})(x)| \le c|x - x_B|^{1+\alpha}, \tag{2.0.35}$$

for all $x \in \Omega\cap B_2$, where $c > 1$ and $\alpha \in (0, 1 - \frac{n}{p})$ depend at most on n, λ, Λ and p.

Next, since $F(D^2(u^\varepsilon - \ell_{x_B}), \frac{\cdot}{\varepsilon}) = f$ in $B_{d_B}(\tilde{x}_B)$, with $d_B = \operatorname{dist}(\tilde{x}_B, \partial\Omega) = |x_B - \tilde{x}_B|$, in the viscosity sense, we may apply Lemma 3.3 to u^ε (with $\Omega = B_{d_B}(\tilde{x}_B)$ and $x_0 = \tilde{x}_B$ there), and deduce from (2.0.35), as well as an obvious inequality $(I_{(1-\bar{\alpha})p}(|f|^p\chi_{\Omega\cap B_2})(\tilde{x}_B))^{\frac{1}{p}} \le c\|f\|_{L^p(B_{d_B}(\tilde{x}_B))} \le c\delta$, that

$$|(u^\varepsilon - \ell_{\tilde{x}_B})(x)| \le c(|x - \tilde{x}_B|^{1+\alpha} + \varepsilon^{1+\alpha}), \tag{2.0.36}$$

for any $x \in B_{d_B}(\tilde{x}_B)$, for some other linear polynomial $\ell_{\tilde{x}_B}$. Since $d_B = |x_B - \tilde{x}_B| > \varepsilon$, we may compare this with (2.0.29) in $B_\varepsilon(\tilde{x}_B) \subset B_{d_B}(\tilde{x}_B)$ and utilize $m^k\varepsilon^2 \le \varepsilon^{1+\alpha}$ (which is ensured by the choice $k \le \frac{p}{n}\log_m \varepsilon$ and $\alpha \le 1 - \frac{n}{p}$) to deduce that $|(\ell - \ell_{\tilde{x}_B})(x)| \le c\varepsilon^{1+\alpha}$ for all $x \in B_\varepsilon(\tilde{x}_B)$. Especially,

$$|D\ell - D\ell_{\tilde{x}_B}| \le cm^{k\alpha}\varepsilon^\alpha. \tag{2.0.37}$$

On the other hand, we may also derive from $\tilde{x}_B \notin D_k$ and $|x_B - \tilde{x}_B| < 2r_B$ that

$$\int_{B_{2r_B}(x_B)} |D^2\Phi|\,dx \le 4^n\delta^p r_B^n m^{kp}, \tag{2.0.38}$$

Now we define Φ_B, $u_B^{\varepsilon_B}$, ε_B, f_B and F_B as in (2.0.32), (2.0.33) and (2.0.34), but redefine ℓ_B by

$$\ell_B(x) = \frac{(\ell - \ell_{\tilde{x}_B})(x_B + 2r_B x)}{cm^k r_B^2}.$$

By the obvious identity, $u^\varepsilon - \ell = u^\varepsilon - \ell_{\tilde{x}_B} - (\ell - \ell_{\tilde{x}_B})$, we have $u_B^{\varepsilon_B} = -\ell_B$ on $\partial\Omega_B \cap B_2$. Moreover, due to (2.0.37) and (2.0.27), $|D\ell_B| \le cm^{k(\alpha-1)}r_B^{\alpha-1}$. In addition, by (2.0.38), we have $\|D^2\Phi_B\|_{L^p(B_2)} \le 4\delta m^k r_B$. The other properties concerning $u_B^{\varepsilon_B}$, f_B, F_B and Φ_B remain the same as in the proof of Case 1. Thus, ε_B, $u_B^{\varepsilon_B}$, f_B, ℓ_B, Φ_B and F_B altogether now verify the hypotheses of the last case of Lemma 6.7 (with $\rho = m^k r_B$ and α as above). The rest of the argument is the same with that of Case 1, whence it is left out to the reader. □

We are ready to prove the uniform $W^{2,p}$-estimates around boundary points.

Proof of Theorem 6.5 Fix any (finite) exponent $p > p_0$. Consider the case $0 \in \partial\Omega$, $U = B_2$, $U' = B_1$, $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \le 1$, $\|f\|_{L^p(\Omega\cap B_2)} \le \delta$, $g = 0$ on B_2, $\Phi(0) = 0$, $\operatorname{osc}_{B_2} D\Phi \le \delta$ and $|\xi^t D\Phi(0)\xi - 1| \le \delta$ for all $\xi \in \partial B_1$. Also assume that $\|D^2\Phi\|_{L^n(B_2)} \le \delta$ if $p < n$, $\|D^2\Phi\|_{L^{n+\sigma}(B_2)} \le \delta$ if $p = n$, and $\|D^2\Phi\|_{L^p(B_2)} \le \delta$ if $p > n$.

Throughout this proof, c will denote a positive constant depending at most on n, λ, Λ, κ, ψ, p and σ.

Choose $p' < p$ by $p' = \frac{p_0+p}{2}$ if $p < n$, $p' = \frac{n}{n+\sigma}$ if $p = n$, with σ as in the statement of the theorem, and $p' = \frac{n+p}{2}$ if $p > n$. Let α_k, β_k, γ_k and δ_k be the measure of A_k, B_k, C_k and respectively D_k as in Lemma 6.8 with p' in place of p, and $\alpha = \frac{\sigma}{n+\sigma}$ there. Since u^ε, f, F and Φ verify the hypotheses of Lemma 6.8, depending on the value of p', we obtain, after iteration, that

$$\alpha_k \le \eta^k + \sum_{i=1}^{k} \eta^i(\beta_{k-i} + \gamma_{k-i} + \delta_{k-i}),$$

with $\eta = c(\delta^{\gamma\mu} m^\mu + m^{-q})$.

Fix any $q > p$ such that $q < n$ if $p < n$, $q \le n + \sigma$ if $p = n$ and $q = 2p$ if $p > n$. We may choose m larger in Lemma 6.8, but still depending on the quantities specified in the statement of the lemma, such that $cm^{p-q} \le \frac{1}{4}$. Then we take δ sufficiently small such that $c\delta^{\gamma\mu} m^\mu \le \frac{1}{4}$, which ensures that $m^p\eta \le \frac{1}{2}$.

With such a choice of m and δ, one can derive, by following computations in the proof of Theorem 5.1, that

$$\sum_{k=1}^{-c(p',n)\log_m \varepsilon} m^{kp}\alpha_k \le c + c\sum_{k=1}^{\infty} m^{kp}(\beta_k + \gamma_k + \delta_k). \tag{2.0.39}$$

for some $c > 0$, depending only on n, λ, Λ, κ, ψ, p and σ (only for the case $p = n$). Hence, it suffices to prove a uniform bound of the rightmost term in the above display.

By the strong $(\frac{p}{p'}, \frac{p}{p'})$-type inequality for the maximal function and the assumption that $\|f\|_{L^p(\Omega\cap B_2)} \le \delta$, one can immediately prove that

$$\sum_{k=1}^{\infty} m^{kp}\beta_k \le c.$$

As for the summability of $m^{kp}\gamma_k$, we only need to take care of the case $p \le n$, since for the other case, $p > n$, Lemma 6.8 (iii) assumes $C_k = \emptyset$, i.e., $\gamma_k = 0$. For the case $p \le n$, it follows from Theorem 6.1 (along with $\|u^\varepsilon\|_{L^\infty(\Omega\cap B_2)} \le 1$, $\|f\|_{L^p(\Omega\cap B_2)} \le \delta$, $g = 0$ on $\partial\Omega \cap B_2$ and $\partial\Omega \cap B_2$ is a C^1-hypersurface whose Lipschitz norm is less than $c\delta$) that $G^\varepsilon_{\Omega\cap B_2}(u^\varepsilon) \in L^{np/(n-p')}(\Omega_\varepsilon \cap B_{3/2})$ (note $p' < n$ when $p \le n$) and

$$\int_{\Omega_\varepsilon\cap B_2} (G^\varepsilon_{\Omega\cap B_2}(u^\varepsilon))^{\frac{np}{n-p'}}\, dx \le c.$$

Thus, writing by φ the function $G^\varepsilon_{\Omega\cap B_2}(u^\varepsilon)^{\frac{n}{n-p'}}$, the above display implies that $\int_{\Omega_\varepsilon\cap B_2} \varphi^p\, dx \le c$. By the relation between the function $G^\varepsilon_E(v)$ and the set $L^\varepsilon(v, E)$ (see Definition 2.2), $\{\varphi > m^k\} = \{G^\varepsilon_{\Omega\cap B_2}(u^\varepsilon) > m^{k(1-\frac{p'}{n})}\} = L^\varepsilon_{m^{k(1-p'/n)}}(u^\varepsilon, \Omega \cap B_2)$, so

$$\sum_{k=1}^{\infty} m^{kp}\gamma_k = \sum_{k=1}^{\infty} m^{kp}|\{\varphi > m^k\}| \le c.$$

Finally, let us verify the summability of $m^{kp}\delta_k$. As for the case $p > n$ (hence $p' > n$), it follows from the strong $(\frac{p}{p'}, \frac{p}{p'})$-type inequality for the maximal function and the assumption that $\|D^2\Phi\|_{L^p(B_2)} \le \delta$. As for the case $p \le n$, hence either $p < q < n$ or $p = n < q < n + \sigma$, we invoke strong $(\frac{N}{q}, \frac{N}{q})$-type inequality, with $N = n$ if $p < n$ or $N = n + \sigma$ if $p = n$, for the maximal function. Then from the assumption $\|D^2\Phi\|_{L^N(B_2)} \le \delta$, we have $\int_{\mathbb{R}^n} M(|D^2\Phi|^q \chi_{B_2})^{\frac{N}{q}}\,dx \le c\delta$ and thus,

$$\sum_{k=1}^{\infty} m^{kp}\delta_k = \sum_{k=1} (m^{\frac{pq}{N}})^{\frac{kN}{q}} |\{M(|D^2\Phi|^q \chi_{B_2}) > \delta^q (m^{\frac{pq}{N}})^k\}| \le c.$$

In all, we have proved that the rightmost term of (2.0.39) is bounded by c, uniformly in ε. This finishes the proof, for the special case. By a standard rescaling argument, one may recover the case for general $\partial\Omega$, U, U', u^ε, f yet $g = 0$. Now for non-trivial $g \in W^{2,p}(U)$, we observe that $w^\varepsilon = u^\varepsilon - g$ satisfies the special case, with f replaced by $f + c_0|D^2 g|$, with c_0 depending only on n, λ and Λ; for more detail, see the proof of [30, Theorem 4.5]. □

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