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Almost-formality and deformations of representations of the fundamental groups of Sasakian manifolds

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Abstract

For a $2n + 1$-dimensional compact Sasakian manifold, if $n \geq 2$, we prove that the analytic germ of the variety of representations of the fundamental group at every semi-simple representation is quadratic. To prove this result, we prove the almost-formality of de Rham complex of a Sasakian manifold with values in a semi-simple flat vector bundle. By the almost-formality, we also prove the vanishing theorem on the cup product of the cohomology of semi-simple flat vector bundles over a compact Sasakian manifold.

Keywords Deformations of representations · Harmonic bundle · Topology of Sasakian manifolds · Formality.

Mathematics Subject Classification 14F35 · 53C07 · 53C25 · 58A14

1 Introduction

In [6], Goldman and Millson establishes the correspondence between local structures of the varieties of representations of the fundamental groups of manifolds and the deformation theory of augmented differential graded Lie algebras. By using this correspondence, they prove that for a compact Kähler manifold M with a point x, the analytic germ of the variety $\mathcal{R}(\pi_1(M, x), GL_m(\mathbb{C}))$ of representations at ρ is quadratic for any representation $\rho : \pi_1(M, x) \to GL_m(\mathbb{C})$ which is the monodromy representation of a polarized variation of Hodge structure. In [14], Simpson generalizes this result for every semi-simple representation $\rho : \pi_1(M, x) \to GL_m(\mathbb{C})$ by using harmonic bundle structures of semi-simple flat bundles over compact Kähler manifolds.

The aim of this paper is to give an odd-dimensional analogue of this result. A Sasakian manifold is viewed as an important odd-dimensional analogue of a Kähler manifold [12]. Typical examples are odd-dimensional spheres and Heisenberg nilmanifolds. The main result of this paper is to prove the following statement.

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Theorem 1.1 *Let M be a $(2n+1)$-dimensional compact Sasakian manifold with a point x. If $n \geq 2$, then for any semi-simple representation $\rho : \pi_1(M, x) \to GL_m(\mathbb{C})$ of the fundamental group, the analytic germ of the variety $\mathcal{R}(\pi_1(M, x), GL_m(\mathbb{C}))$ of representations at ρ is quadratic.*

We notice that the statement does not hold for $n = 1$. 3-dimensional Heisenberg nilmanifolds are counter examples (see [6, Section 9.1]).

To prove the quadraticity, it is important to reduce the de Rham complex of a compact Kähler manifold with values in a flat bundle to its cohomology via quasi-isomorphisms so-called the formality as in [5]. In [14], by using harmonic bundle structures, Simpson proves the formality for every semi-simple flat bundle over a compact Kähler manifold. To prove Theorem 1.1, we prove the "almost-formality" of the de Rham complex of a Sasakian manifold with values in a semi-simple flat bundle (Corollary 4.3) by using (basic) harmonic bundle structures as in [1].

We have another application of such almost-formality. By the almost-formality, we also prove the following statement in the cup product.

Theorem 1.2 *Let M be a $(2n+1)$-dimensional compact Sasakian manifold and E and E' be semi-simple flat complex vector bundles. For $s, t < n$ with $s + t > n$, the cup product*

$$H^s(M,\ E) \otimes H^t(M,\ E') \to H^{s+t}(M,\ E \otimes E')$$

vanishes.

2 Harmonic bundles

Let M be a compact Riemannian manifold and E a flat complex vector bundle over M equipped with a flat connection D. For any Hermitian metric h on E, we have a unique decomposition

$$D = \nabla + \phi \tag{2.1}$$

such that ∇ is a unitary connection and ϕ is a 1-form on M with values in the self-adjoint endomorphisms of E with respect to h.

Theorem 2.1 *([4]) A flat complex vector bundle $(E,\ D)$ is semi-simple if and only if there exists a Hermitian metric (called the harmonic metric) h on E such that*

$$\nabla^* \phi = 0,$$

where ∇^* is the formal adjoint operator of ∇.

3 Sasakian manifolds

Let M be a $(2n+1)$-dimensional real smooth manifold. A *CR-structure (of codimension one)* on M is an n-dimensional complex sub-bundle $T^{1,0}$ of the complexified tangent bundle $TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ such that $T^{1,0} \cap \overline{T^{1,0}} = \{0\}$ and $T^{1,0}$ is formally integrable, that is $[T^{1,0}, T^{1,0}] \subset T^{1,0}$. In this paper, we do not consider CR-structures of higher codimension. We shall denote $\overline{T^{1,0}}$ by $T^{0,1}$. For a CR-structure $T^{1,0}$ on M, there is a unique sub-bundle S of rank $2n$ of the real tangent bundle TM together with a vector bundle homomorphism $I : S \longrightarrow S$ satisfying the conditions that

(1) $I^2 = -\mathrm{Id}_S$, and
(2) $T^{1,0}$ is the $\sqrt{-1}$-eigenbundle of I.

A $(2n+1)$-dimensional manifold M equipped with a triple $(T^{1,0}, S, I)$ as above is called a *CR-manifold*. A *contact CR-manifold* is a CR-manifold M with a contact 1-form η on M such that $\ker \eta = S$. Let ξ denote the Reeb vector field for the contact form η. On a contact CR-manifold, the above homomorphism I extends to the entire TM by setting $I(\xi) = 0$.

Definition 3.1 A contact CR-manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$ is a *strongly pseudo-convex CR-manifold* if the Hermitian form L_η on S_x defined by $L_\eta(X, Y) = d\eta(X, IY)$, $X, Y \in S_x$, is positive definite for every point $x \in M$.

For a strongly pseudo-convex CR-manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, we have a canonical Riemannian metric g_η on M which is defined by

$$g_\eta(X, Y) := L_\eta(X, Y) + \eta(X)\eta(Y), \quad X, Y \in T_x M.$$

Definition 3.2 A Sasakian manifold is a strongly pseudo-convex CR-manifold

$$(M, (T^{1,0}, S, I), (\eta, \xi))$$

such that for any section ζ of $T^{1,0}$, $[\xi, \zeta]$ is also a section of $T^{1,0}$.

For a Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, the metric cone of (M, g_η) is a Kähler manifold. We can also define a Sasakian manifold as a contact metric manifold whose metric cone is Kähler (see [2]).

4 Almost-formality

Let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold. Then the Reeb vector field ξ defines a 1-dimensional foliation $\mathcal{F}_\xi$ on M. It is known that the map $I : TM \to TM$ associated with the CR-structure $T^{1,0}$ defines a transversely complex structure on the foliated manifold $(M, \mathcal{F}_\xi)$. Furthermore, the closed basic 2-form $d\eta$ is a transversely Kähler structure with respect to this transversely complex structure.

A differential form ω on M is called *basic* if the equations

$$i_\xi \omega = 0 = \mathcal{L}_\xi \omega \tag{4.1}$$

hold. We denote by $A^*_B(M)$ the subspace of basic forms in the de Rham complex $A^*(M)$. Then $A^*_B(M)$ is a sub-complex of the de Rham complex $A^*(M)$.

Corresponding to the decomposition $S_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$, we have the bigrading

$$A^r_B(M)_{\mathbb{C}} = \bigoplus_{p+q=r} A^{p,q}_B(M)$$

as well as the decomposition of the exterior differential

$$d_{|A^r_B(M)_{\mathbb{C}}} = \partial_\xi + \overline{\partial}_\xi$$

on $A^r_B(M)_{\mathbb{C}}$, so that

$$\partial_\xi : A^{p,q}_B(M) \to A^{p+1,q}_B(M) \text{ and } \overline{\partial}_\xi : A^{p,q}_B(M) \to A^{p,q+1}_B(M).$$

We have the transverse Hodge theory as in ([7, 9]). Consider the usual Hodge star operator

$$* : A^r(M) \to A^{2n+1-r}(M)$$

associated to the Sasakian metric g_η and the formal adjoint operator

$$\delta = - * d* : A^r(M) \to A^{r-1}(M).$$

We define the homomorphism

$$\star_\xi : A^r_B(M) \to A^{2n-r}_B(M)$$

to be $\star_\xi \omega = *(\eta \wedge \omega)$ for $\omega \in A^r_B(M)$. Also define the operators

$$\begin{aligned}
\delta_\xi &= -\star_\xi d \star_\xi : A^r_B(M) \to A^{r-1}_B(M),\\
\partial^*_\xi &= -\star_\xi \overline{\partial}_\xi \star_\xi : A^{p,q}_B(M) \to A^{p-1,q}_B(M),\\
\overline{\partial}^*_\xi &= -\star_\xi \partial_\xi \star_\xi : A^{p,q}_B(M) \longrightarrow A^{p,q-1}_B(M)
\end{aligned}$$

and $\Lambda = -\star_\xi (d\eta\wedge) \star_\xi$. They are the formal adjoints of d, ∂_ξ, $\overline{\partial}_\xi$ and $(d\eta\wedge)$ respectively for the pairing

$$A^r_B(M) \times A^r_B(M) \ni (\alpha, \beta) \longmapsto \int_M \eta \wedge \alpha \wedge \star_\xi \beta.$$

Define the Laplacian operators

$$\Delta : A^r(M) \longrightarrow A^r(M) \quad \text{and} \quad \Delta_\xi : A^r_B(M) \longrightarrow A^r_B(M)$$

by

$$\Delta = d\delta + \delta d \quad \text{and} \quad \Delta_\xi = d\delta_\xi + \delta_\xi d$$

respectively. For $\omega \in A^r_B(M)$, since the relation $*\omega = (\star_\xi \omega) \wedge \eta$ holds, we have the relation

$$\delta\omega = \delta_\xi \omega + *(d\eta \wedge \star_\xi \omega).$$

Thus, for $\omega \in A^1_B(M)$, the equality $\delta_\xi \omega = \delta\omega$ holds, and hence for $f \in A^0_B(M)$, we have that $\Delta_\xi f = \Delta f$. The usual Kähler identities

$$[\Lambda, \partial_\xi] = -\sqrt{-1}\, \overline{\partial}^*_\xi \quad \text{and} \quad [\Lambda, \overline{\partial}_\xi] = \sqrt{-1}\, \partial^*_\xi$$

hold, and these imply that

$$\Delta_\xi = 2\Delta'_\xi = 2\Delta''_\xi,$$

where $\Delta'_\xi = \partial_\xi \partial^*_\xi + \partial^*_\xi \partial_\xi$ and $\Delta''_\xi = \overline{\partial}_\xi \overline{\partial}^*_\xi + \overline{\partial}^*_\xi \overline{\partial}_\xi$.

A *basic vector bundle* E over the foliated manifold $(M, \mathcal{F}_\xi)$ is a C^∞ vector bundle over M which has local trivializations with respect to an open covering $M = \bigcup_\alpha U_\alpha$ satisfying the condition that each transition function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C})$ is basic on $U_\alpha \cap U_\beta$ i.e. it is constant on the leaves of the foliation $\mathcal{F}_\xi$. For a basic vector bundle E, a differential form $\omega \in A^*(M, E)$ with values in E is called basic if ω is basic on every U_α, meaning $\omega_{|U_\alpha} \in A^*_B(U_\alpha) \otimes \mathbb{C}^r$ for every α. Let

$$A^*_B(M, E) \subset A^*(M, E)$$

denote the subspace of basic forms in the space $A^*(M, E)$ of differential forms with values in E. Corresponding to the decomposition $S_{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$, we have the bigrading

$$A^r_B(M, E) = \bigoplus_{p+q=r} A^{p,q}_B(M, E).$$

We shall consider any flat vector bundle (E, D) over M as a basic vector bundle by local flat frames. Then, $A^*_B(M, E)$ is a sub-complex of the de Rham complex $A^*(M, E)$ equipped with the differential D associated to the flat connection. We denote by $H^*(M, E)$ and $H^*_B(M, E)$ the cohomology of the de Rham complex $A^*(M, E)$ and the cohomology of the sub-complex $A^*_B(M, E)$ respectively. Suppose E is equipped with a Hermitian metric h. Let

$$D = \nabla + \phi$$

be the canonical decomposition of the connection D (see (2.1)). Then, using the pairing $A^r(M, E) \times A^{2n+1-r}(M, E) \longrightarrow A^{2n+1}(M)$ associated with h, define the Hodge star operator

$$*_h : A^r(M, E) \longrightarrow A^{2n+1-r}(M, E)$$

as well as the formal adjoint operator $D^* = - *_h D *_h$.

Assume the Hermitian structure h to be basic (equivalently, $\phi(\xi) = 0$ [1, Proposition 4.1]). By $D_\xi = \nabla_\xi$, the unitary connection ∇ restricts to an operator

$$\nabla : A^r_B(M, E) \longrightarrow A^{r+1}_B(M, E).$$

Now, on $A^{p,q}_B(M, E)$, decompose

$$\nabla = \partial_{h,\xi} + \overline{\partial}_{h,\xi}$$

such that $\partial_{h,\xi} : A^{p,q}_B(M, E) \longrightarrow A^{p+1,q}_B(M, E)$ and $\overline{\partial}_{h,\xi} : A^{p,q}_B(M, E) \longrightarrow A^{p,q+1}_B(M, E)$. Since ∇ is a unitary connection, we have that

$$\overline{\partial}_\xi h(s_1, s_2) = h(\overline{\partial}_{h,\xi} s_1, s_2) + h(s_1, \partial_{h,\xi} s_2)$$

for $s_1, s_2 \in A^{0,0}_B(M, E)$. We define the operator

$$\star_{h,\xi} : A^r_B(M, E) \longrightarrow A^{2n-r}_B(M, E)$$

and the formal adjoint operators

$$\begin{aligned}
(\nabla)^*_\xi &= -\star_{h,\xi} \nabla \star_{h,\xi} : A^r_B(M, E) \longrightarrow A^{r-1}_B(M, E),\\
\partial^*_{h,\xi} &= -\star_{h,\xi} \overline{\partial}_{h,\xi} \star_{h,\xi} : A^{p,q}_B(M, E) \longrightarrow A^{p-1,q}_B(M, E),\\
\overline{\partial}^*_{h,\xi} &= -\star_{h,\xi} \partial_{h,\xi} \star_{h,\xi} : A^{p,q}_B(M, E) \longrightarrow A^{p,q-1}_B(M, E),
\end{aligned}$$

and $\Lambda_h := -\star_{h,\xi}(d\eta\wedge)\star_{h,\xi}$ in the same way as above. We now have the Kähler identities

$$[\Lambda, \partial_{h,\xi}] = -\sqrt{-1}\,\overline{\partial}^*_{h,\xi} \quad \text{and} \quad [\Lambda, \overline{\partial}_{h,\xi}] = \sqrt{-1}\,\partial^*_{h,\xi}. \tag{4.2}$$

Theorem 4.1 *([1, Theorem 4.2]) Let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold and (E, D) a flat complex vector bundle over M with a Hermitian metric h. Then the following two conditions are equivalent:*

- *The Hermitian structure h is harmonic, i.e., $(\nabla)^*\phi = 0$.*

- *The Hermitian structure h is basic and for the decomposition*

$$\phi = \theta + \bar{\theta}$$

with $\theta \in A_B^{1,0}(M, \mathrm{End}(E))$ *and* $\bar{\theta} \in A_B^{0,1}(M, \mathrm{End}(E))$, *the equalities*

$$\bar{\partial}_{h,\xi}\bar{\partial}_{h,\xi} = 0, \quad [\theta, \theta] = 0 \quad \text{and} \quad \bar{\partial}_{h,\xi}\theta = 0$$

hold.

For a semi-simple flat complex vector bundle (E, D) over a compact Sasakian manifold M, we have a harmonic metric h by Corlette's Theorem. On $A_B^*(M, E)$, we define the operators $D' = \partial_{h,\xi} + \bar{\theta} : A_B^*(M, E) \to A_B^{*+1}(M, E)$ and $D'' = \bar{\partial}_{h,\xi} + \theta : A_B^*(M, E) \to A_B^{*+1}(M, E)$. By Theorem 4.1, we have $D''D'' = 0$. Since D is flat, $\nabla = \partial_{h,\xi} + \bar{\partial}_{h,\xi}$ is unitary and $\phi = \theta + \bar{\theta}$ is self-adjoint, we have the equalities $D'D' = 0$ and $D'D'' + D''D' = 0$. Define $D^c = \sqrt{-1}(D'' - D')$. By using Kähler identities (4.2) and the similar equations for θ and $\bar{\theta}$, we have the Kähler identities for the operators D' and D''

$$[\Lambda, D'] = -\sqrt{-1}(D'')^* \quad \text{and} \quad [\Lambda, D''] = \sqrt{-1}(D')^* \tag{4.3}$$

as [13, Lemma 3.1]. By the similar arguments in [5, Lemma 5.11], we have the DD^c-Lemma:

$$\ker D \cap \ker D^c \cap \mathrm{im} D = \ker D \cap \ker D^c \cap \mathrm{im} D^c = \mathrm{im} DD^c.$$

Consider the sub-complex $\ker D^c \subset A_B^*(M, E)$ and the cohomology $H_{D^c}^*(A_B^*(M, E))$ of $A_B^*(M, E)$ for the differential D^c. By the similar arguments in [5, Section 6], the inclusion $\ker D^c \subset A_B^*(M, E)$ is a quasi-isomorphism, the differential on $H_{D^c}^*(A_B^*(M, E))$ induced by D is trivial and the quotient $q : \ker D^c \to H_{D^c}^*(A_B^*(M, E))$ is a quasi-isomorphism. Hence, we have the sequence

$$A_B^*(M, E) \leftarrow \ker D^c \to H_B^*(M, E) \tag{4.4}$$

of quasi-isomorphisms. By $d\eta \in A_B^{1,1}(M)$ and $\partial_\xi d\eta = \bar{\partial}_\xi d\eta = 0$, we have the sub-complexes

$$\ker D^c \oplus \ker D^c \wedge \eta \subset A_B^*(M, E) \oplus A_B^*(M, E) \wedge \eta \subset A^*(M, E)$$

and the cochain complex $H_B^*(M, E) \oplus H_B^*(M, E) \otimes \langle\eta\rangle$. Define the decreasing filtration F^* on $A_B^*(M, E) \oplus A_B^*(M, E) \wedge \eta$ (resp. $H_B^*(M, E) \oplus H_B^*(M, E) \otimes \langle\eta\rangle$) such that $F^{-1} = A_B^*(M, E) \oplus A_B^*(M, E) \wedge \eta$, $F^0 = A_B^*(M, E)$ and $F^1 = 0$ and define the one on $H_B^*(M, E) \oplus H_B^*(M, E) \otimes \langle\eta\rangle$ in the same manner. By the standard argument on the spectral sequences of these filtrations, we have the sequence

$$A_B^*(M, E) \oplus A_B^*(M, E) \wedge \eta \leftarrow \ker D^c \oplus \ker D^c \wedge \eta \to H_B^*(M, E) \oplus H_B^*(M, E) \otimes \langle\eta\rangle \tag{4.5}$$

of quasi-isomorphisms.

Proposition 4.2 *The inclusion* $A_B^*(M, E) \oplus A_B^*(M, E) \wedge \eta \subset A^*(M, E)$ *is a quasi-isomorphism.*

Proof Consider the covariant derivative D_ξ on $A^*(M, E)$ at the Reeb vector field ξ. Then, a form $\omega \in A^*(M, E)$ is basic if and only if $D_\xi\omega = i_\xi\omega = 0$. For any $\omega \in \ker D_\xi$, we have the decomposition

$$\omega = (\omega - i_\xi\omega \wedge \eta) + i_\xi\omega \wedge \eta \in A_B^*(M, E) \oplus A_B^*(M, E) \wedge \eta$$

and hence we can say $\ker D_\xi = A^*_B(M, E) \oplus A^*_B(M, E) \wedge \eta$. Since we have $D_\xi = \nabla_\xi$ for the unitary connection ∇ and ξ is Killing, we have $(D_\xi)^* = -D_\xi$ and hence D_ξ and the Laplacian operator $\Delta_D = DD^* + D^*D$ commute. Hence, if $\omega \in A^*(M, E)$ is harmonic, then $D_\xi\omega$ is also harmonic. Since D_ξ induces a trivial map on the cohomology $H^*(M, E)$ of $A^*(M, E)$, we have $D_\xi\omega = 0$. Thus, the inclusion $\ker D_\xi = A^*_B(M, E) \oplus A^*_B(M, E) \wedge \eta \subset A^*(M, E)$ induces a surjection on cohomology.

Consider the operator H_D on $A^*(M, E)$ defined as the projection to harmonic forms. By the above argument, we have $D_\xi H_D = 0$. Since H_D is self-adjoint, we have $H_D D_\xi = 0$. Consider the Green operator G_D for Δ_D. For $\omega \in \ker D_\xi$, we have $\Delta_D D_\xi G\omega = D_\xi \Delta_D G\omega = D_\xi(\omega - H_D\omega) = 0$ and hence $D_\xi G\omega = H_D D_\xi G\omega = 0$. If $D\omega = 0$, then we have $\omega = H_D(\omega) + DD^*G\omega$ with $G\omega \in \ker D_\xi$. We notice that D^* preserves $\ker D_\xi$ since D_ξ and D^* commute. If $[\omega] = 0$ in $H^*(M, E)$, then $H_D(\omega) = 0$ and so we have $\omega = DD^*G\omega$ for $D^*G\omega \in \ker D_\xi$. Thus, the inclusion $\ker D_\xi = A^*_B(M, E) \oplus A^*_B(M, E) \wedge \eta \subset A^*(M, E)$ induces an injection on cohomology. Hence the proposition follows. □

Corollary 4.3 *We have the sequence*

$$A^*(M, E) \leftarrow \ker D^c \oplus \ker D^c \wedge \eta \rightarrow H^*_B(M, E) \oplus H^*_B(M, E) \otimes \langle\eta\rangle \tag{4.6}$$

of quasi-isomorphisms.

Remark 4.4 If E is trivial, then this statement is proved in [15].

5 Deformations of representations of the fundamental groups of Sasakian manifolds

We review the work of Goldman-Millson [6]. Let M be a compact manifold with a point x. We consider the variety $\mathcal{R}(\pi_1(M, x), GL_m(\mathbb{C}))$ of representations of the fundamental group $\pi_1(M, x)$. This is the set $\mathrm{Hom}(\pi_1(M, x), GL_m(\mathbb{C}))$ with the natural structure of an algebraic variety induced by the group structure of $\pi_1(M, x)$ and the algebraic group $GL_m(\mathbb{C})$. For $\rho \in \mathcal{R}(\pi_1(M, x), GL_m(\mathbb{C}))$, we define the flat vector bundle $(E = \pi_1(M, x)\backslash(\tilde{M} \times \mathbb{C}^m), D)$ where $\tilde{M}$ is the universal covering associated with the base point x. Consider the differential graded Lie algebra $A^*(M, \mathrm{End}(E))$ and the augmentation map $\epsilon_x : A^0(M, \mathrm{End}(E)) \rightarrow \mathrm{End}(E_x)$. Then the analytic germ of $\mathcal{R}(\pi_1(M, x), GL_m(\mathbb{C}))$ at ρ pro-represents the "deformation functor" of the differential graded Lie algebra $A^*(M, \mathrm{End}(E))_x = \ker\epsilon_x$ (Theorem [6, Theorem 6.8]). If a finite dimensional $\mathrm{End}(E_x)$-augmented differential graded Lie algebra L^* with an injective augmentation $\epsilon : L^0 \rightarrow \mathrm{End}(E_x)$ is quasi-isomorphic to the $\mathrm{End}(E_x)$-augmented differential graded Lie algebra $A^*(M, \mathrm{End}(E))$, then the deformation functor of $A^*(M, \mathrm{End}(E))_x$ is pro-represented by the analytic germ of

$$\left\{\omega \in L^1 \,\middle|\, d\omega + \frac{1}{2}[\omega, \omega] = 0\right\} \times \mathrm{End}(E_x)/\epsilon(L^0)$$

at the origin (see [6, Section 3]).

Theorem 5.1 *Let M be a $(2n+1)$-dimensional compact Sasakian manifold with a point x. If $n \geq 2$, then for any semi-simple representation $\rho : \pi_1(M, x) \rightarrow GL_m(\mathbb{C})$, the analytic germ of $\mathcal{R}(\pi_1(M, x), GL_m(\mathbb{C}))$ at ρ is quadratic.*

Proof By the above arguments and Corollary 4.3, it is sufficient to prove that the analytic germ of

$$\left\{\omega \in H^1_B(M,\ \mathrm{End}(E)) \oplus H^0_B(M,\ \mathrm{End}(E)) \otimes \langle\eta\rangle \,\middle|\, d\omega + \frac{1}{2}[\omega, \omega] = 0\right\}$$

at the origin is quadratic. For $\omega = \alpha + \beta \otimes \eta \in H^1_B(M,\ \mathrm{End}(E)) \oplus H^0_B(M,\ \mathrm{End}(E)) \otimes \langle\eta\rangle$, the equality $d\omega + \frac{1}{2}[\omega, \omega] = 0$ holds if and only if $\beta \wedge d\eta + \frac{1}{2}[\alpha, \alpha] = 0$ and $[\alpha, \beta] \otimes \eta = 0$. On the other hand, $\beta \wedge d\eta + \frac{1}{2}[\alpha, \alpha] = 0$ implies

$$[\alpha, \beta] \wedge d\eta = [\alpha, \beta \wedge d\eta] = -\frac{1}{2}[\alpha, [\alpha, \alpha]] = 0$$

by the graded Jacobi identity. By the Kähler identities (4.3), if $n \geq 2$, we can say that the map $H^1_B(M,\ \mathrm{End}(E)) \ni a \mapsto a \wedge d\eta \in H^3_B(M,\ \mathrm{End}(E))$ is injective as the usual Lefschetz decomposition. Hence, if $n \geq 2$, $\beta \wedge d\eta + \frac{1}{2}[\alpha, \alpha] = 0$ implies $[\alpha, \beta] = 0$ and this means that the equality $d\omega + \frac{1}{2}[\omega, \omega] = 0$ is equivalent to the quadratic equation $\beta \wedge d\eta + \frac{1}{2}[\alpha, \alpha] = 0$. Hence the theorem follows. □

Remark 5.2 If ρ is trivial, the statement follows from [10, Corollary 6.10] (see also [8])

6 Vanishing cup products

Theorem 6.1 *Let M be a $(2n + 1)$-dimensional compact Sasakian manifold and E and E' be semi-simple flat complex vector bundles. For $s, t < n$ with $s + t > n$, the cup product*

$$H^s(M,\ E) \otimes H^t(M,\ E') \to H^{s+t}(M,\ E \otimes E')$$

vanishes.

Proof Considering the quasi-isomorphisms as in Corollary 4.3 for E, E' and $E \otimes E'$, we can say that the cup product $H^*(M,\ E) \otimes H^*(M,\ E') \to H^*(M,\ E \otimes E')$ is induced by the product

$$\begin{aligned}&\left(H^*_B(M,\ E) \oplus H^*_B(M,\ E) \otimes \langle\eta\rangle\right) \bigotimes \left(H^*_B(M,\ E') \oplus H^*_B(M,\ E') \otimes \langle\eta\rangle\right)\\ &\quad \to H^*_B(M,\ E \otimes E') \oplus H^*_B(M,\ E \otimes E') \otimes \langle\eta\rangle.\end{aligned}$$

By the Kähler identities (4.3), we can say that the map $H^r_B(M,\ E) \ni a \mapsto a \wedge d\eta \in H^{r+2}_B(M,\ E)$ is injective for $r \leq n - 1$ and surjective for $r \geq n - 1$ as the usual Lefschetz decomposition. Thus, any r-th cohomology class of the complex $H^*_B(M,\ E) \oplus H^*_B(M,\ E) \otimes \langle\eta\rangle$ has a representative in $H^r_B(M,\ E)$ for $r \leq n$. By the injectivity, for $s, t < n$, any cup product in

$$\begin{aligned}&H^s\left(H^*_B(M,\ E) \oplus H^*_B(M,\ E) \otimes \langle\eta\rangle\right) \bigotimes H^t\left(H^*_B(M,\ E') \oplus H^*_B(M,\ E') \otimes \langle\eta\rangle\right)\\ &\quad \to H^{s+t}\left(H^*_B(M,\ E \otimes E') \oplus H^*_B(M,\ E \otimes E') \otimes \langle\eta\rangle\right)\end{aligned}$$

has a representative in $H^{s+t}_B(M,\ E \otimes E')$. By the surjectivity, if $s + t > n$, any element in $H^{s+t}_B(M,\ E \otimes E')$ is exact. Hence the theorem follows. □

Remark 6.2 If E and E' are trivial, the statement follows from [3].

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