Annali di Matematica Pura ed Applicata (1923 -) (2023) 202:1501–1529
https://doi.org/10.1007/s10231-022-01290-y

Nonlocal operators of small order

Pierre Aime Feulefack[1,2] · **Sven Jarohs**[1]

Received: 31 January 2022 / Accepted: 18 November 2022 / Published online: 15 December 2022

Abstract
In this work we study nonlocal operators and corresponding spaces with a focus on operators of order near zero. We investigate the interior regularity of eigenfunctions and of weak solutions to the associated Poisson problem depending on the regularity of the right-hand side. Our method exploits the variational structure of the problem and we prove that eigenfunctions are of class C^∞ if the kernel satisfies this property away from its singularity. Similarly in this case, if in the Poisson problem the right-hand is of class C^∞, then also any weak solution is of class C^∞.

Keywords Nonlocal operators · Regularity · Nonlocal function spaces

1 Introduction and main results

A crucial role in the investigation of differential operators is the study of eigenfunctions and corresponding eigenvalues, if they exist. In the classical case of the Laplacian $-\Delta$ in a bounded domain Ω in $\mathbb{R}^N$, it is well known that there exists a sequence of functions $u_n \in H_0^1(\Omega)$, $n \in \mathbb{N}$ and corresponding values $\lambda_n > 0$ such that

$$-\Delta u_n = \lambda_n u_n \quad \text{in } \Omega \text{ and } u_n = 0 \text{ in } \partial\Omega.$$

Here, $H_0^1(\Omega)$ is as usual the closure of $C_c^\infty(\Omega)$ with respect to the norm $u \mapsto \Big(\|u\|^2_{L^2(\Omega)} + \||\nabla u|\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}$.

With the well-known De Giorgi iteration in combination with the Sobolev embedding, it follows that u_n must be bounded and by a boot strapping argument using the regularity theory of the Laplacian it follows that u_n is smooth in Ω. In the model case of a nonlocal

✉ Sven Jarohs
jarohs@math.uni-frankfurt.de

Pierre Aime Feulefack
feulefac@math.uni-frankfurt.de; pierre.a.feulefack@aims-senegal.org

1 Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60629 Frankfurt, Germany

2 African Institute for Mathematical Sciences in Senegal (AIMS Senegal), KM2, Route de Joal, B.P. 14 18, Mbour, Senegal

problem, one usually studies the *fractional Laplacian* $(-\Delta)^s$ with $s \in (0,1)$. This operator can be defined via its Fourier symbol $|\cdot|^{2s}$ and it can be shown that for $\phi \in C_c^\infty(\mathbb{R}^N)$ we have

$$(-\Delta)^s\phi(x)=c_{N,s}\,p.v.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y=c_{N,s}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,\quad x\in\mathbb{R}^N$$

with a suitable normalization constant $c_{N,s} > 0$. As in the above classical case, it can be shown that there exists a sequence of functions $u_n \in \mathscr{H}_0^s(\Omega)$, $n \in \mathbb{N}$ and corresponding values $\lambda_{s,n} > 0$ such that

$$(-\Delta)^s u_n = \lambda_{s,n}u_n \text{ in } \Omega \text{ and } u_n = 0 \text{ in } \mathbb{R}^N\setminus\Omega.$$

Here, the space $\mathscr{H}_0^s(\Omega)$ is given by the closure of $C_c^\infty(\Omega)$—understood as functions on $\mathbb{R}^N$—with respect to the norm $u \mapsto \Big(\|u\|^2_{L^2(\Omega)} + \mathscr{E}_s(u,u)\Big)^{\frac{1}{2}}$, where for $u, v \in C_c^\infty(\mathbb{R}^N)$ we set

$$\mathscr{E}_s(u,v) := \frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y.$$

With similar methods as in the classical case, it follows that u_n is smooth in the interior of Ω.

In the following we investigate the above discussion to the case where the *kernel function* $z \mapsto c_{N,s}|z|^{-N-2s}$ is replaced by a measurable function $j : \mathbb{R}^N \to [0,\infty]$ such that for some $\sigma \in (0,2]$ we have

$$j(z) = j(-z) \text{ for all } z \in \mathbb{R}^N \textit{ and } \int_{\mathbb{R}^N}\min\{1,|z|^\sigma\}j(z)\,\mathrm{d}z < \infty. \tag{1.1}$$

With $\sigma = 2$, the above yields that $j\,\mathrm{d}z$ is a Lévy measure and the associated operator to this choice of kernel is of order below 2. In the following we focus on the case where the singularity of j is not too large, that is on the case $\sigma < 1$ so that the associated operator is of order strictly below one. We call the operator in this case also of *small order*.

Motivated by applications using nonlocal models, where a small order of the operator captures the optimal efficiency of the model [1, 24], nonlocal operators with possibly order near zero, i.e. if (1.1) is satisfied for all $\sigma > 0$, have been studied in linear and nonlinear integro-differential equations, see [5, 6, 11–13, 17, 18, 25] and the references in there. From a stochastic point of view, general classes of nonlocal operators appear as the generator of jump processes, where the jump behavior is modeled through types of *Lévy measures* and properties of *associated harmonic* functions have been studied, see [14, 16, 21, 23] and the references in there. In particular, operators of the form $\phi(-\Delta)$ for a certain class of functions ϕ are of interest from a stochastic and analytic point of view, see e.g. [2, 3] and the references in there.

In the following, we aim at investigating properties of bilinear forms and operators associated to a kernel j satisfying (1.1) for some $\sigma \in (0,2]$ from a variational point of view. For this, some further assumptions on j are needed in our method and we present certain explicit examples at the end of this introduction, where our results apply.

Let $\Omega \subset \mathbb{R}^N$ open, $u, v \in C_c^{0,1}(\Omega)$ understood as functions defined on $\mathbb{R}^N$, and consider the bilinear form

$$b_{j,\Omega}(u,v) := \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))j(x-y)\,\mathrm{d}x\mathrm{d}y, \tag{1.2}$$

where we also write $b_j(u, v) := b_{j,\mathbb{R}^N}(u, v)$ and $b_{j,\Omega}(u) := b_{j,\Omega}(u, u)$, $b_j(u) = b_j(u, u)$ resp. We denote

$$D^j(\Omega) := \{u \in L^2(\Omega) \ : \ b_{j,\Omega}(u) < \infty\}. \tag{1.3}$$

Associated to b_j there is a *nonlocal* self-adjoint operator I_j which for $u, v \in C_c^{0,1}(\Omega)$ satisfies

$$b_j(u, v) = \int_{\mathbb{R}^N} I_j u(x)v(x)\,\mathrm{d}x \ \text{ and is represented by}$$
$$I_j u(x) = p.v. \int_{\mathbb{R}^N} (u(x) - u(y))j(x - y)\,\mathrm{d}y, \quad x \in \Omega. \tag{1.4}$$

To investigate the eigenvalue problem for I_j, we further need the space

$$\mathscr{D}^j(\Omega) := \{u \in D^j(\mathbb{R}^N) \ : \ 1_{\mathbb{R}^N\setminus\Omega}u \equiv 0\}.$$

Note that clearly $\mathscr{D}^j(\mathbb{R}^N) = D^j(\mathbb{R}^N)$ and both $D^j(\Omega)$ and $\mathscr{D}^j(\Omega)$ are Hilbert spaces with scalar products

$$\langle\cdot,\cdot\rangle_{D^j(\Omega)} := \langle\cdot,\cdot\rangle_{L^2(\Omega)} + b_{j,\Omega}(\cdot,\cdot) \text{ and } \langle\cdot,\cdot\rangle_{\mathscr{D}^j(\Omega)} := \langle\cdot,\cdot\rangle_{L^2(\Omega)} + b_j(\cdot,\cdot) \text{ resp.}$$

Our first result concerns the eigenfunctions for the operator I_j.

Theorem 1.1 *Let* (1.1) *hold with* $\sigma = 2$ *and assume* j *satisfies additionally*

$$\int_{\mathbb{R}^N} j(z)\,\mathrm{d}z = \infty. \tag{1.5}$$

Let $\Omega \subset \mathbb{R}^N$ *be a bounded open set. Then there exists a sequence* $u_n \in \mathscr{D}^j(\Omega)$, $n \in \mathbb{N}$ *and values* λ_n *such that*

$$b_j(u_n, v) = \lambda_n \int_\Omega u_n(x)v(x)\,\mathrm{d}x \ \textit{ for all } v \in \mathscr{D}^j(\Omega), n \in \mathbb{N} \textit{ and}$$
$$0 < \lambda_1 < \lambda_2 \le \ldots \le \lambda_n \to \infty \ \textit{ for } n \to \infty.$$

Here, we have

$$\lambda_1 = \Lambda_1(\Omega) := \inf_{\substack{u\in\mathscr{D}^j(\Omega)\\ u\neq 0}} \frac{b_j(u)}{\|u\|^2_{L^2(\Omega)}} \tag{1.6}$$

and u_1 *is unique up to a multiplicative constant—that is,* λ_1 *is simple. Moreover,* u_1 *can be chosen to be positive in* Ω*. Furthermore, the following statements hold.*

(1) *If in addition* j *satisfies*

$$\int_{B_R(0)\setminus B_r(0)} j^2(z)\,\mathrm{d}z < \infty \textit{ for all } 0 < r < R, \tag{1.7}$$

then $u_n \in L^\infty(\Omega)$ *for every* $n \in \mathbb{N}$ *and there is* $C = C(\Omega, j, n) > 0$ *such that*

$$\|u_n\|_{L^\infty(\Omega)} \le C\|u_n\|_{L^2(\Omega)}.$$

(2) *Assume* (1.1) *holds with* $\sigma < \frac{1}{2}$, (1.7) *holds, and there is* $m \in \mathbb{N}\cup\{\infty\}$ *such that the following holds: We have* $j \in W^{l,1}(\mathbb{R}^N\setminus B_\epsilon(0)$ *for every* $l \in \mathbb{N}$ *with* $l \le 2m$*, and there is some constant* $C_j > 0$ *such that*

$$|\nabla j(z)| \le C_j|z|^{-1-\sigma-N} \textit{ for all } 0 < |z| \le 3.$$

Then $u_n \in H^m_{loc}(\Omega)$*. In particular,* $u_n \in C^\infty(\Omega)$ *if* $m = \infty$*.*

For the definition of the Sobolev spaces $W^{l,1}$ and H^m we refer to Sect. 2.1. The first part of Theorem 1.1 indeed follows immediately from the results of [19]. To show the boundedness, we emphasize that in our setting, there are no Sobolev embeddings available and thus it is not clear how to implement the approach via the De Giorgi iteration. We circumvent this, by generalizing the δ-decomposition introduced in [13]. The proof of the regularity statement is inspired by the approach used in [7], where the author studies regularity of solutions to equations involving nonlocal operators which are in some sense comparable to the fractional Laplacian and uses Nikol'skii spaces. We emphasize that some of our methods generalize to the situation where the operator is not translation invariant and maybe perturbed by a convolution type operator. We treat these in the present work, too, see e.g. Theorem 4.3 below. Using a probabilistic and potential theoretic approach, a local smoothness of bounded harmonic solutions solving in a certain very weak sense $I_j u = 0$ in Ω, have been obtained in [16, Theorem 1.7] for radial kernel functions using the same regularity as we impose in statement (2) of Theorem 1.1 (see also [14, 23]). See also [15] for related regularity properties of solutions.
To present our generalization of the above mentioned δ-decomposition, let us first note that the first equality in (1.4) can be extended, see Sect. 2. For this, let $\mathcal{V}^j(\Omega)$ denote the space of those functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $u|_\Omega \in D^j(\Omega)$ and

$$\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B_r(x)} |u(y)|j(x-y)\,\mathrm{d}y < \infty \text{ for all } r > 0. \tag{1.8}$$

Given $f \in L^2_{loc}(\Omega)$, we then call $u \in \mathcal{V}^j(\Omega)$ a *(weak) supersolution* of $I_j u = f$ in Ω, if

$$b_j(u, v) \geq \int_\Omega f(x)v(x)\,\mathrm{d}x \text{ for all } v \in C_c^\infty(\Omega). \tag{1.9}$$

In this situation, we also say that u satisfies in weak sense $I_j u \geq f$ in Ω. Similarly, we define *subsolutions* and *solutions*.
We emphasize that this definition of supersolution is larger than the one considered in [19]. In the case where $\sigma < 1$ in (1.1) this allows via a density result to extend the weak maximum principles presented in [19] as follows.

Proposition 1.2 (Weak maximum principle) *Assume* (1.1) *is satisfied with* $\sigma < 1$ *and assume that*

$$j \text{ does not vanish identically on } B_r(0) \text{ for any } r > 0. \tag{1.10}$$

Let $\Omega \subset \mathbb{R}^N$ *open and with Lipschitz boundary,* $c \in L^\infty_{loc}(\Omega)$*, and assume either*

(1) $c \leq 0$ *or*
(2) Ω *and* c *are such that* $\|c^+\|_{L^\infty(\Omega)} < \inf_{x\in\Omega} \int_{\mathbb{R}^N\setminus\Omega} j(x-y)\,\mathrm{d}y$.

If $u \in \mathcal{V}^j(\Omega)$ *satisfies in weak sense*

$$I_j u \geq c(x)u \ \text{ in } \Omega, u \geq 0 \ \text{ almost everywhere in } \mathbb{R}^N\setminus\Omega, \ \text{ and } \liminf_{|x|\to\infty} u(x) \geq 0,$$

then $u \geq 0$ *almost everywhere in* $\mathbb{R}^N$.

Remark 1.3 (1) The Lipschitz boundary assumption on Ω is a technical assumption for the approximation argument.
(2) Note that Assumption (1.10) readily implies the positivity $\Lambda_1(\Omega)$ defined in (1.6), whenever Ω is an open set in $\mathbb{R}^N$ which is bounded in one direction, that is Ω is contained (after a rotation) in a strip $(-a, a) \times \mathbb{R}^{N-1}$ for some $a > 0$ (see [11, 20]).

Up to our knowledge Proposition 1.2 is even new in the case of $j = |\cdot|^{-N-2s}$, that is, the case of the fractional Laplacian (up to a multiplicative constant). In this situation, it holds $\mathscr{V}^j(\Omega) = H^s(\Omega) \cap \mathscr{L}_s^1$, where

$$\mathscr{L}_s^1 = \left\{ u \in L^1_{loc}(\mathbb{R}^N) \ : \ \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N+2s}} \, \mathrm{d}y < \infty \right\},$$

and the proposition can be reformulated as follows.

Corollary 1.4 *Let $s \in (0, \frac{1}{2})$, $\Omega \subset \mathbb{R}^N$ open and with Lipschitz boundary, $c \in L^\infty_{loc}(\Omega)$, and assume either*

(1) *$c \leq 0$ or*
(2) *Ω and c are such that $\|c^+\|_{L^\infty(\Omega)} < \Lambda_1(\Omega)$.*

If $u \in H^s(\Omega) \cap \mathscr{L}_s^1$ satisfies in weak sense

$$(-\Delta)^s u \geq c(x)u \ \text{ in } \Omega, u \geq 0 \text{ almost everywhere in } \mathbb{R}^N \backslash \Omega, \ \text{ and } \liminf_{|x|\to\infty} u(x) \geq 0,$$

then $u \geq 0$ almost everywhere in $\mathbb{R}^N$.

Similarly to the extension of the weak maximum principle, we have also the following extension of the strong maximum principle presented in [19] in the case $\sigma < 1$.

Proposition 1.5 (Strong maximum principle) *Assume (1.1) is satisfied with $\sigma < 1$ and assume that j satisfies additionally (1.5). Let $\Omega \subset \mathbb{R}^N$ open and $c \in L^\infty_{loc}(\Omega)$ with $\|c^+\|_{L^\infty(\Omega)} < \infty$. Moreover, let $u \in \mathscr{V}^j(\Omega)$, $u \geq 0$ satisfy in weak sense $I_j u \geq c(x)u$ in Ω. Then the following holds.*

(1) *If Ω is connected, then either $u \equiv 0$ in Ω or $\operatorname{essinf}_K u > 0$ for any $K \subset\subset \Omega$.*
(2) *j given in (2.2) satisfies $\operatorname{essinf}_{B_r(0)} j > 0$ for any $r > 0$, then either $u \equiv 0$ in $\mathbb{R}^N$ or $\operatorname{essinf}_K u > 0$ for any $K \subset\subset \Omega$.*

Our last results concern the Poisson problem associated to the operator I_j.

Theorem 1.6 *Let (1.1) hold with $\sigma = 2$ and assume j satisfies additionally (1.10). Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Then for any $f \in L^2(\Omega)$ there is a unique solution $u \in \mathscr{D}^j(\Omega)$ of $I_j u = f$. Moreover, if j satisfies (1.7) and $f \in L^\infty(\Omega)$, then also $u \in L^\infty(\Omega)$ and there is $C = C(\Omega, j) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}.$$

Additionally, if (1.1) holds with $\sigma < \frac{1}{2}$, there is $m \in \mathbb{N} \cup \{\infty\}$ such that j satisfies the assumptions in Theorem 1.1(2), and it holds $f \in C^{2m}(\overline{\Omega})$, then $u \in H^m_{loc}(\Omega)$. More precisely in this case, for every $\beta \in \mathbb{N}_0^N$, $|\beta| \leq m$ and $\Omega' \subset\subset \Omega$ there is $C = C(\Omega, \Omega', j, \beta) > 0$ such that

$$\|\partial^\beta u\|_{L^2(\Omega')} \leq C \|f\|_{C^{2m}(\Omega)}.$$

In particular, if $m = \infty$ and $f \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Our approach to prove Theorems 1.1 and 1.6 is uniformly by considering an equation of the form

$$I_j u = h * u + \lambda u + f \text{ in } \Omega$$

for $f \in L^2(\Omega)$, $h \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and $\lambda \in \mathbb{R}$. Moreover, several of our results need I_j not to be translation invariant and this setup is discussed in Sect. 2.

1.1 Examples

We close this introduction with some classes of operators covered by our results.

(1) As introduced in [5, 13, 18] the *logarithmic Laplacian*

$$L_\Delta \phi(x)=c_N P.V. \int_{B_1(0)} \frac{\phi(x)-\phi(x+y)}{|y|^N}\, dy - c_N \int_{\mathbb{R}^N\setminus B_1(0)} \frac{\phi(x+y)}{|y|^N}\, dy + \rho_N\phi(x), \tag{1.11}$$

appears as the operator with Fourier-symbol $-2\ln(|\cdot|)$ and can be seen as the formal derivative in s of $(-\Delta)^s$ at $s=0$. Here

$$c_N = \frac{\Gamma(\frac{N}{2})}{\pi^{N/2}} = \frac{2}{|S^{N-1}|} \text{ and } \rho_N := 2\ln(2) + \psi\left(\frac{N}{2}\right) - \gamma \tag{1.12}$$

where $\psi := \frac{\Gamma'}{\Gamma}$ denotes the digamma function and $\gamma := -\psi(1) = -\Gamma'(1)$ is the Euler-Mascheroni constant. With $j(z) = c_N 1_{B_1(0)}(z)|z|^{-N}$ the operator L_Δ can be seen as a bounded perturbation of the operator class discussed in the introduction. The following sections cover in particular this operator.

(2) The logarithmic Schrödinger operator $(I-\Delta)^{\log}$ as in [12] is an integro-differential operator with Fourier-symbol $\log(1+|\cdot|^2)$ and also appears as the formal derivative in s of the relativistic Schrödinger operator $(I-\Delta)^s$ at $s=0$,

$$(I-\Delta)^{\log}u(x) = d_N P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(x+y)}{|y|^N}\omega(|y|)\, dy,$$

where $d_N = \pi^{-\frac{N}{2}}$, $\omega(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} K_{\frac{N}{2}}(r)$ and K_ν is the modified Bessel function of the second kind with index ν. More generally, operators with symbol $\log(1+|\cdot|^\beta)$ for some $\beta \in (0,2]$ are studied in [22].

(3) Finally, also nonradial kernels of the type considered in [19] satisfy in particular the assumptions (2.1) and (4.1). See also also [14, 22, 23] and references in there.

The paper is organized as follows. In Sect. 2 we collect some general results concerning the spaces used in this paper and resulting definitions of weak sub- and supersolutions. Section 3 is devoted to show several density results, which then are used to show the Propositions 1.2 and 1.5. In Sect. 4 we present a general approach to show boundedness of solutions and in Sect. 5 we give the proof of an interior H^1-regularity estimate for solutions from which we then deduce the interior regularity statement as claimed in Theorems 1.1(2) and 1.6.

Notation In the remainder of the paper, we use the following notation. Let $U, V \subset \mathbb{R}^N$ be nonempty measurable sets, $x \in \mathbb{R}^N$ and $r > 0$. We denote by $1_U : \mathbb{R}^N \to \mathbb{R}$ the characteristic function, $|U|$ the Lebesgue measure, and $\text{diam}(U)$ the diameter of U. The notation $V \subset\subset U$ means that $\overline{V}$ is compact and contained in the interior of U. The distance between V and U is given by $\text{dist}(V,U) := \inf\{|x-y| \,:\, x \in V,\, y \in U\}$. Note that this notation does *not* stand for the usual Hausdorff distance. If $V = \{x\}$ we simply write $\text{dist}(x,U)$. We let $B_r(U) := \{x \in \mathbb{R}^N \,:\, \text{dist}(x,U) < r\}$, so that $B_r(x) := B_r(\{x\})$ is the open ball centered at x with radius r. We also put $B := B_1(0)$ and $\omega_N := |B|$. Finally, given a function $u : U \to \mathbb{R}$, $U \subset \mathbb{R}^N$, we let $u^+ := \max\{u,0\}$ and $u^- := -\min\{u,0\}$ denote the positive and negative part of u, and we write $\text{supp}\, u$ for the support of u given as the closure in $\mathbb{R}^N$ of the set $\{x \in U \,:\, u(x) \neq 0\}$.

2 Preliminaries

In the following, we generalize the translation invariant setting of the introduction. For this and from now on, let $k : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty]$ be a measurable function satisfying for some $\sigma \in (0, 2]$

$$k(x, y) = k(y, x) \text{ for all } x, y \in \mathbb{R}^N \quad \text{and} \quad \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1, |x - y|^\sigma\}k(x, y)\, dy < \infty. \tag{2.1}$$

We let $j : \mathbb{R}^N \to [0, \infty]$ be the symmetric lower bound of k given by

$$j(z) := \operatorname{essinf}\{k(x, x \pm z) \ : \ x \in \mathbb{R}^N\} \text{ for } z \in \mathbb{R}^N. \tag{2.2}$$

For $\Omega \subset \mathbb{R}^N$ open let

$$\begin{aligned} b_{k,\Omega}(u, v) &:= \frac{1}{2}\int_\Omega \int_\Omega (u(x) - u(y))(v(x) - v(y))k(x, y)\, dx dy, \\ \kappa_{k,\Omega}(x) &:= \int_{\mathbb{R}^N\setminus\Omega} k(x, y)\, dy \in [0, \infty] \text{ for } x \in \mathbb{R}^N, \quad \text{and} \\ K_{k,\Omega}(u, v) &:= \int_\Omega u(x)v(x)\kappa_{k,\Omega}(x)\, dx, \end{aligned}$$

where, if $u = v$ we put

$$b_{k,\Omega}(u) := b_{k,\Omega}(u, u) \text{ and } K_{k,\Omega}(u) := K_{k,\Omega}(u, u)$$

and we drop the index Ω, if $\Omega = \mathbb{R}^N$. Note that we have for any fixed $x \in \Omega$ that $\kappa_{k,\Omega}(x) < \infty$ by (2.1). We consider the function spaces

$$\begin{aligned} D^k(\Omega) &:= \Big\{u \in L^2(\Omega) \ : \ b_{k,\Omega}(u) < \infty\Big\}, \\ \mathscr{D}^k(\Omega) &:= \Big\{u \in D^k(\mathbb{R}^N) \ : \ u = 0 \text{ on } \mathbb{R}^N\setminus\Omega\Big\}, \\ \mathscr{V}^k(\Omega) &:= \Big\{u : \mathbb{R}^N \to \mathbb{R} \ : \ u|_\Omega \in D^k(\Omega) \ \text{ and, for all } r > 0, \\ &\qquad \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B_r(x)} |u(y)|k(x, y)\, dy < \infty\Big\} \text{ and} \\ \mathscr{V}^k_{loc}(\Omega) &:= \Big\{u : \mathbb{R}^N \to \mathbb{R} \ : \ u|_{\Omega'} \in \mathscr{V}^k(\Omega') \ \text{ for all } \Omega' \subset\subset \Omega\Big\}. \end{aligned}$$

Lemma 2.1 *Let $U \subset \Omega \subset \mathbb{R}^N$ open and $u : \mathbb{R}^N \to \mathbb{R}$. Then the following hold:*

(1) $u \in \mathscr{D}^k(\Omega) \ \Rightarrow \ u|_\Omega \in D^k(\Omega)$.
(2) $\mathscr{D}^k(U) \subset \mathscr{D}^k(\Omega) \subset \mathscr{V}^k(\Omega) \subset \mathscr{V}^k(U) \subset \mathscr{V}^k_{loc}(U)$.

Proof This follows immediately from the definitions (see also [19, Section 3]). □

Lemma 2.2 (See Proposition 3.3 in [19] or Proposition 1.7 in [20]) *For $\Omega \subset \mathbb{R}^N$ open, let $\Lambda_1(\Omega)$ be given by (c.f. (1.6))*

$$\Lambda_1(\Omega) := \inf_{\substack{u\in\mathscr{D}^j(\Omega)\\ u\neq 0}} \frac{b_j(u)}{\|u\|^2_{L^2(\Omega)}}$$

and let

$$\lambda(r) = \inf\{\Lambda_1(\Omega) \;:\; \Omega \subset \mathbb{R}^N \text{ open with } |\Omega| = r\}.$$

Then $\lim_{r\to\infty} \lambda(r) \geq \int_{\mathbb{R}^N} j(z)\,\mathrm{d}z$ *with* $j(z) := \operatorname{essinf}\{k(x, x \pm z) \;:\; z \in \mathbb{R}^N\}$ *for* $z \in \mathbb{R}^N$ *as in* (2.2).

Lemma 2.3 *Let* $\Omega \subset \mathbb{R}^N$ *open and let* X *be any of the above function spaces. Then the following hold:*

(1) $b_{k,\Omega}$ *is a bilinear form and in particular we have* $b_{k,\Omega}(u, v) \leq b_{k,\Omega}^{1/2}(u) b_{k,\Omega}^{1/2}(v)$. *Moreover,* $D^k(\Omega)$ *and* $\mathscr{D}^k(\Omega)$ *are Hilbert spaces with scalar products*

$$\langle u, v\rangle_{D^k(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + b_{k,\Omega}(u, v),$$
$$\langle u, v\rangle_{\mathscr{D}^k(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + b_{k,\mathbb{R}^N}(u, v).$$

(2) *If* $u \in X$, *then* $u^\pm, |u| \in X$ *and we have* $b_{k,\Omega'}(u^+, u^-) \leq 0$ *for all* $\Omega' \subset \Omega$ *with* $b_{k,\Omega'}(u) < \infty$.
(3) *If* $g \in C^{0,1}(\mathbb{R}^N)$, $u \in X$, *then* $g \circ u \in X$.
(4) $C_c^{0,1}(\Omega) \subset X$.
(5) $\phi \in C_c^{0,1}(\Omega)$, $u \in X$, *then* $\phi u \in \mathscr{D}^k(\Omega)$, *where if necessary we extend* u *trivially to a function on* $\mathbb{R}^N$. *Moreover, there is* $C = C(N, k, \|\phi\|_{C^{0,1}(\Omega)}) > 0$ *such that*

$$b_{k,\mathbb{R}^N}(\phi u) \leq C\Big(\|u\|^2_{L^2(\Omega')} + b_{k,\Omega'}(u)\Big)$$

for any $\Omega' \subset \Omega$ *with* $\operatorname{supp}\phi \subset\subset \Omega'$.

Proof Theses statements follow directly from the definition (c.f. [19, Section 3]). To be precise in the last part, let $\phi \in C_c^{0,1}(\Omega)$ and fix $L := \|\phi\|_{C^{0,1}(\Omega)}$. That is, we have

$$|\phi(x)| \leq L \text{ and } |\phi(x) - \phi(y)| \leq L|x - y|.$$

Then using the inequality for $x, y \in \mathbb{R}^N$

$$|\phi(x)u(x) - \phi(y)u(y)|^2 \leq 2|\phi(x) - \phi(y)|^2|u(x)|^2 + 2|\phi(y)|^2|u(x) - u(y)|^2$$

we find by the assumptions (2.1)

$$\begin{aligned}
b_{k,\mathbb{R}^N}(\phi u) &\leq b_{k,\Omega'}(\phi u) + L^2 \int_{\operatorname{supp}\phi} |u(x)|^2 \kappa_{k,\Omega'}(x)\,\mathrm{d}x \\
&\leq 2L^2 \int_{\Omega'}\int_{\Omega'} |u(x)|^2 |x - y|^2 k(x, y)\,\mathrm{d}y\,\mathrm{d}x + 2L^2 b_{k,\Omega'}(u) \\
&\quad + L^2 \sup_{x\in\operatorname{supp}\phi} \kappa_{k,\Omega'}(x)\|u\|^2_{L^2(\Omega')} \\
&\leq 2L^2\Big(\sup_{x\in\Omega'} \int_{\Omega'} |x - y|^2 k(x, y)\,\mathrm{d}y + \sup_{x\in\operatorname{supp}\phi} \kappa_{k,\Omega'}(x)\Big)\|u\|^2_{L^2(\Omega')} \\
&\quad + 2L^2 b_{k,\Omega'}(u) < \infty.
\end{aligned}$$

□

Remark 2.4 (1) Note that for $u, v \in \mathscr{D}^k(\Omega)$ we have

$$b_k(u, v) = b_{k,\mathbb{R}^N}(u, v) = b_{k,\Omega}(u, v) + K_{k,\Omega}(u, v).$$

(2) It follows in particular that there is a nonnegative self-adjoint operator I_k associated to $b_{k,\mathbb{R}^N} = b_k$ as mentioned in the introduction.

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^N$ open and $u \in \mathscr{V}^k_{loc}(\Omega)$. Then $b_k(u,\phi)$ is well-defined for any $\phi \in C_c^\infty(\Omega)$.*

Proof Let $\phi \in C_c^\infty(\Omega)$ and fix $U \subset\subset \Omega$ such that $\operatorname{supp}\phi \subset\subset U$. Then with the symmetry of k

$$|b_k(u,\phi)| \le |b_{k,U}(u,\phi)| + \int_U |\phi(x)| \int_{\mathbb{R}^N\setminus U} |u(x)-u(y)|k(x,y)\,\mathrm{d}y\mathrm{d}x$$
$$\le b_{k,U}^{1/2}(u)b_{k,U}^{1/2}(\phi) + \int_{\operatorname{supp}\phi} |\phi(x)|\,\mathrm{d}x \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B_\epsilon(x)} |u(y)|k(x,y)\,\mathrm{d}y$$
$$+ \int_{\operatorname{supp}\phi} |\phi(x)u(x)|\,\mathrm{d}x \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B_\epsilon(x)} k(x,y)\,\mathrm{d}y < \infty,$$

where $\epsilon = \operatorname{dist}(\operatorname{supp}\phi, \mathbb{R}^N\setminus U) > 0$. □

Definition 2.6 Let $\Omega \subset \mathbb{R}^N$ open and $f \in L^1_{loc}(\Omega)$. Then $u \in \mathscr{V}^k_{loc}(\Omega)$ is called a *weak supersolution* of $I_k u = f$ in Ω, if

$$b_k(u,\phi) \ge \int_\Omega f(x)\phi(x)\,\mathrm{d}x \text{ for all nonnegative } \phi \in C_c^\infty(\Omega).$$

We also say that *u satisfies $I_k u \ge f$ weakly in Ω.*
Similarly, we define *weak subsolutions* and *solutions*.

Remark 2.7 (1) We note that by Assumption 2.1, it follows that for any function $u \in \mathscr{V}^k_{loc}(\Omega)$ with $u|_\Omega \in C^{0,1}(\Omega)$ for $\Omega \subset \mathbb{R}^N$ open, we have $I_k u|_U \in L^\infty(U)$ for any $U \subset\subset \Omega$ and

$$I_k u(x) = \int_{\mathbb{R}^N} (u(x)-u(y))k(x,y)\,\mathrm{d}y \text{ for } x \in \Omega.$$

This follows similarly to the proof of the statements in Lemma 2.3.

(2) If $u \in \mathscr{V}^k(\Omega)$, then indeed also $b_k(u,\phi)$ is well-defined for all $\phi \in \mathscr{D}^k(\Omega)$. Hence also b_k is well defined on $\mathscr{V}^k_{loc}(\Omega) \times \mathscr{D}^k(U)$ for all $U \subset\subset \Omega$. In some of our results the statements need a Lipschitz-boundary of Ω, which comes into play due to approximation with $C_c^\infty(\Omega)$-functions (see Section 3 below). However, this can be weakened, if $u \in \mathscr{V}^k(\Omega)$ and the space of test-functions is adjusted.

Lemma 2.8 *Let $\Omega \subset \mathbb{R}^N$ open. Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Let $\eta \in C_c^{0,1}(\mathbb{R}^N)$ such that $0 \le \eta \le 1$ in $\mathbb{R}^N$ and we have*

$$\eta = 1 \text{ in } \Omega_2 \text{ and } \eta = 0 \text{ in } \mathbb{R}^N\setminus\Omega_3.$$

Let $f \in L^1_{loc}(\Omega)$ and let $u \in \mathscr{V}^k_{loc}(\Omega)$ satisfy in weak sense $Iu \ge f$ in Ω. Then the function $v = \eta u \in \mathscr{D}^k(\Omega_3)$ satisfies in weak sense $Iv \ge f + g_{\eta,u}(x)$ in Ω_1, where

$$g_{\eta,u}(x) = \int_{\mathbb{R}^N\setminus\Omega_2} (1-\eta(y))u(y)k(x,y)\,\mathrm{d}y \text{ for } x \in \Omega_1.$$

Proof The fact, that $v \in \mathscr{D}^k(\Omega_3)$ follows from Lemma 2.3. Let $\phi \in C_c^\infty(\Omega_1)$, then

$$\int_{\mathbb{R}^N} v(x)I_k\phi(x)\,\mathrm{d}x \ge \int_{\mathbb{R}^N} f(x)\phi(x)\,\mathrm{d}x - \int_{\mathbb{R}^N} (1-\eta(x))u(x)I_k\phi(x)\,\mathrm{d}x.$$

Here, since $(1-\eta)u \equiv 0$ on Ω_2, we have

$$\int_{\mathbb{R}^N} (1-\eta(x))u(x)I_k\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \phi(x)[I_k(1-\eta)u](x)\,\mathrm{d}x$$
$$= -\int_{\Omega_1} \phi(x) \int_{\mathbb{R}^N\setminus\Omega_2} (1-\eta(y))u(y)k(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Thus the claim follows. □

Remark 2.9 The same result as in Lemma 2.8 also holds if "≥" in the solution type is replaced by "≤" or "=".

In the following, it is useful to understand functions $u \in D^k(\Omega)$ satisfying $b_{k,\Omega}(u) = 0$.

Proposition 2.10 *Assume that the symmetric lower bound j of k defined in (2.2) satisfies $\int_{\mathbb{R}^N} j(z)\,\mathrm{d}z = \infty$. Let $\Omega \subset \mathbb{R}^N$ open and bounded and let $u \in D^k(\Omega)$ such that $b_{k,\Omega}(u) = 0$. Then u is constant.*

Proof Let $x_0 \in \Omega$ and fix $r > 0$ such that $B_{2r}(x_0) \subset \Omega$. Denote $q(z) := \min\{c, j(z)\}1_{B_r(0)}(z)$, where we may fix $c > 0$ such that $|\{q > 0\}| > 0$ due to the assumption on j. Then by Lemma A.1 we have

$$0 = 2b_{k,\Omega}(u) \geq 2b_{q,\Omega}(u) \geq \frac{1}{2\|q\|_{L^1(\mathbb{R}^N)}} b_{q*q,B_r(x_0)}(u),$$

where $a * b = \int_{\mathbb{R}^N} a(\cdot - y)b(y)\,\mathrm{d}y$ denotes as usual the convolution. Note that since $q \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $q = 0$ on $\mathbb{R}^N \setminus B_r(0)$, it follows that $q * q \in C(\mathbb{R}^N)$ with support in $B_{2r}(0)$ and we have

$$q * q(0) = \int_{\mathbb{R}^N} q(z)^2\,\mathrm{d}z > 0$$

by the assumption on j. Hence there is $R > 0$ with $q * q \geq \epsilon$ for some $\epsilon > 0$ and thus we have

$$0 = b_{q*q,B_r(x_0)}(u) \geq b_{q*q,B_\rho(x_0)}(u) \geq \frac{\epsilon}{2} \int_{B_\rho(x_0)} \int_{B_\rho(x_0)} (u(x) - u(y))^2\,\mathrm{d}x\mathrm{d}y.$$

for any $\rho \in (0, \frac{R}{2}]$. But then $u(x) = u(y)$ for almost every $x, y \in B_{R/2}(x_0)$ so that u is constant a.e. in $B_\rho(x_0)$. Since $b_{k,\Omega}(u) = b_{k,\Omega}(u - m)$ for any $m \in \mathbb{R}$, we may next assume that $u = 0$ in $B_{R/2}(x_0)$ and show that indeed we have $u = 0$ a.e. in Ω. Denote by W the set of points $x \in \Omega$ such that there is $r > 0$ with $u = 0$ a.e. in $B_r(x)$. By definition W is open and the above shows that W is nonempty. Next, let $(x_n)_n \subset W$ be a sequence with $x_n \to x \in \Omega$ for $n \to \infty$. Then there is $r_x > 0$ such that $B_{4r_x}(x) \subset \Omega$ and we can find $n_0 \in \mathbb{N}$ such that $x \in B_{r_x}(x_n) \subset B_{2r_x}(x_n) \subset \Omega$ for $n \geq n_0$. Repeating the above argument, it follows that u must be zero in $B_{r_x}(x_n)$ and thus $x \in W$. Hence, W is relatively open and closed in Ω and since W is nonempty, we have $W = \Omega$. That is $u = 0$ in Ω. □

2.1 On Sobolev and Nikol'skii spaces

We recall here the notations and properties of Sobolev and Nikol'skii spaces as introduced in [7, 26]. In the following, let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^N$ open.

2.1.1 Sobolev spaces

If $k \in \mathbb{N}_0$, we set as usual

$$W^{k,p}(\Omega) := \Big\{u \in L^p(\Omega) \ : \ \partial^\alpha u \ \text{ exists for all } \alpha \in \mathbb{N}_0^n, |\alpha| \le k \ \text{ and belongs to } L^p(\Omega)\Big\}$$

for the Banach space of k-times (weakly) differentialable functions in $L^p(\Omega)$. Moreover, as usual, for $\sigma \in (0,1)$, $p \in [1,\infty)$ we set

$$W^{\sigma,p}(\Omega) := \Big\{u \in L^p(\Omega) \ : \ \frac{u(x)-u(y)}{|x-y|^{\frac{n}{p}+\sigma}} \in L^p(\Omega \times \Omega)\Big\}.$$

With the norm

$$\|u\|_{W^{\sigma,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + [u]_{W^{\sigma,p}(\Omega)}, \quad \text{where}$$
$$[u]_{W^{\sigma,p}(\Omega)} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\sigma p}}\, dxdy\right)^{1/p}$$

the space $W^{\sigma,p}(\Omega)$ is a Banach space. For general $s = k+\sigma$, $k \in \mathbb{N}_0$, $\sigma \in [0,1)$ the Sobolev space is defined as

$$W^{s,p}(\Omega) := \Big\{u \in W^{k,p}(\Omega) \ : \ \partial^\alpha u \in W^{\sigma,p}(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = k\Big\}.$$

Finally, in the particular case $p = 2$ the space $H^s(\Omega) := W^{s,2}(\Omega)$ is a Hilbert space.

2.1.2 Nikol'skii spaces

For $u : \Omega \to \mathbb{R}$ and $h \in \mathbb{R}$, let $\Omega_h := \{x \in \Omega \ : \ \mathrm{dist}(x,\partial\Omega) > h\}$ and, with $e \in \partial B_1(0)$, we let

$$\delta_h u(x) = \delta_{h,e} u(x) := u(x+he) - u(x).$$

Moreover, for $l \in \mathbb{N}$, $l > 1$ let

$$\delta_h^l u(x) = \delta_h(\delta_h^{l-1} u)(x).$$

For $s = k + \sigma > 0$ with $k \in \mathbb{N}_0$ and $\sigma \in (0,1]$ define

$$N^{s,p}(\Omega) := \Big\{u \in W^{k,p}(\Omega) \ : \ [\partial^\alpha u]_{N^{\sigma,p}(\Omega)} < \infty \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = k\Big\},$$

where

$$[u]_{N^{\sigma,p}(\Omega)} = \sup_{\substack{e\in\partial B_1(0)\\ h>0}} h^{-\sigma} \|\delta^2_{h,e} u\|_{L^p(\Omega_{2h})}.$$

It follows that $N^{s,p}(\Omega)$ is a Banach space with norm $\|u\|_{N^{s,p}(\Omega)} := \|u\|_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} [\partial^\alpha u]_{N^{\sigma,p}(\Omega)}$. It can be shown that this norm is equivalent to

$$\|u\|_{L^p(\Omega)} + \sum_{|\alpha|=k} \sup_{\substack{e\in\partial B_1(0)\\ h>0}} h^{m-\sigma} \|\delta^l_{h,e} u\|_{L^p(\Omega_{lh})}$$

for any fixed $m, l \in \mathbb{N}_0$ with $m < \sigma$ and $l > \sigma - m$ (see [26, Theorem 4.4.2.1]).

Proposition 2.11 (See e.g. Propositions 3 and 4 in [7]) *Let $\Omega \subset \mathbb{R}^N$ open and with C^∞ boundary. Moreover, let $t > s > 0$ and $1 \leq p < \infty$. Then*

$$N^{t,p}(\Omega) \subset W^{s,p}(\Omega) \subset N^{s,p}(\Omega).$$

3 Density results and maximum principles

The main goal of this section is to show the following.

Theorem 3.1 *Let either $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ open and bounded with Lipschitz boundary. In the following, let $X(\Omega) := \mathscr{D}^k(\Omega)$ or $D^k(\Omega)$. Then $C_c^\infty(\Omega)$ is dense in $X(\Omega)$. Moreover, if $u \in X(\Omega)$ is nonnegative, then we have*

(1) *There exists a sequence $(u_n)_n \subset X(\Omega) \cap L^\infty(\Omega)$ with $\lim_{n\to\infty} u_n = u$ in $X(\Omega)$ satisfying that for every $n \in \mathbb{N}$ there is $\Omega_n' \subset\subset \Omega$ with $u_n = 0$ on $\Omega \setminus \Omega_n'$ and $0 \leq u_n \leq u_{n+1} \leq u$.*
(2) *There exists a sequence $(u_n)_n \subset C_c^\infty(\Omega)$ with $u_n \geq 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_n = u$ in $X(\Omega)$.*

Remark 3.2 To put Theorem 3.1 into perspective, we consider the following examples.

(1) In the case $k(x, y) = |x - y|^{-2s-N}$ for some $s \in (0, \frac{1}{2})$, the above Theorem is well-known and leads to the interesting property that for any open, bounded Lipschitz set $\Omega \subset \mathbb{R}^N$ we have

$$D^k(\Omega) = H^s(\Omega) = H_0^s(\Omega).$$

We emphasize that the above equality also holds for $s = \frac{1}{2}$. Moreover, if $s < \frac{1}{2}$, it also holds $H^s(\Omega) = \{u \in H^s(\mathbb{R}^N) \ : \ 1_{\mathbb{R}^N \setminus \Omega} u \equiv 0\}$.
(2) If $k(x, y) = 1_{B_1(0)}(x - y)|x - y|^{-N}$, $D^k(\Omega)$ is associated to the function space of the *localized* logarithmic Laplacian (see [5]).

The proof is split into several smaller steps. Recall that $\mathscr{D}^k(\mathbb{R}^N) = D^k(\mathbb{R}^N)$ by definition.

Lemma 3.3 *Let $u \in D^k(\mathbb{R}^N)$. Then there is a sequence $(u_n)_n \subset D^k(\mathbb{R}^N)$ with $\lim_{n\to\infty} u_n = u$ in $D^k(\mathbb{R}^N)$ satisfying that for every $n \in \mathbb{N}$ there is $\Omega_n \subset\subset \mathbb{R}^N$ with $u_n = 0$ on $\mathbb{R}^N \setminus \Omega_n$. Moreover, if $u \geq 0$, then $(u_n)_n$ can be chosen to satisfy in addition $0 \leq u_n \leq u_{n+1} \leq u$.*

Proof For $n \in \mathbb{N}$ let $\phi_n \in C_c^{0,1}(\mathbb{R}^N)$ be radially symmetric and such that $\phi_n \equiv 1$ on $B_n(0)$, $\phi_n \equiv 0$ on $B_{n+1}(0)^c$. Clearly, we may assume that $[\phi_n]_{C^{0,1}(\mathbb{R}^N)} = 1$. By Lemma 2.3 there is hence some $C = C(N, k) > 0$ with $b_{k,\mathbb{R}^N}(\phi_n u) \leq C\|u\|_{D^k(\mathbb{R}^N)}$ for all $n \in \mathbb{N}$. In the following, let $u_n := \phi_n u$ and without loss of generality we may assume $u \geq 0$. Since then $0 \leq u - u_n \leq u$ on $\mathbb{R}^N$ and $u - u_n = 0$ on B_n, by dominated convergence we have $\lim_{n\to\infty} \|u - u_n\|_2 = 0$. Moreover, by choice of ϕ_n we have for $x, y \in \mathbb{R}^N$

$$\begin{aligned}|u(x)(1-\phi_n(x))-u(y)(1-\phi_n(y))| &\leq |u(x) - u(y)|(1 - \phi_n(x)) + |u(y)||\phi_n(x) - \phi_n(y)| \\ &\leq |u(x) - u(y)| + |u(y)| \min\{1, |x - y|\} =: U(x, y).\end{aligned}$$

Here, $U(x, y) \in L^2(\mathbb{R}^N \times \mathbb{R}^N, k(x, y)\, d(x, y))$, since

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} U(x,y)k(x,y)\,\mathrm{d}x\mathrm{d}y = b_{k,\mathbb{R}^N}(u) + \int_{\mathbb{R}^N}|u(y)|^2\int_{\mathbb{R}^N}\min\{1,|x-y|^2\}k(x,y)\,\mathrm{d}x\mathrm{d}y$$
$$\le b_{k,\mathbb{R}^N}(u) + \int_{\mathbb{R}^N}|u(y)|^2\,\mathrm{d}y$$
$$\mathrm{supp}_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N}\min\{1,|x-y|^2\}k(x,y)\mathrm{d}y < \infty.$$

Thus $\lim\limits_{n\to\infty} b_{k,\mathbb{R}^N}(u-u_n) = 0$ by the dominated convergence Theorem. □

Proposition 3.4 *We have that $C_c^\infty(\mathbb{R}^N)$ is dense in $D^k(\mathbb{R}^N)$. Moreover, if $u \in D^k(\mathbb{R}^N)$ is nonnegative, then there exists $(\phi_n)_n \subset C_c^\infty(\mathbb{R}^N)$ with $\phi_n \ge 0$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty}\phi_n = u$ in $D^k(\mathbb{R}^N)$.*

Proof Let $u \in D^k(\mathbb{R}^N)$. Moreover, let $\phi_n \in C_c^{0,1}(\mathbb{R}^N)$ for $n\in\mathbb{N}$ be given by Lemma 3.3 such that $\|u-\phi_n u\|_{s,p} < \frac{1}{n}$. Then $v_n := \phi_n u \in D^k(\mathbb{R}^N)$ and there is $R_n > 0$ with $v_n \equiv 0$ on $\mathbb{R}^N\setminus B_{R_n}(0)$. Next, let $(\rho_\epsilon)_{\epsilon\in(0,1]}$ by a Dirac sequence and denote $v_{n,\epsilon} := \rho_\epsilon * v_n$. Then $v_{n\epsilon} \in C_c^\infty(\mathbb{R}^N)$ for all $n\in\mathbb{N}$, $\epsilon\in(0,1]$ and

$$b_{k,\mathbb{R}^N}(u-v_{n,\epsilon}) \le b_{k,\mathbb{R}^N}(u-v_n) + b_{k,\mathbb{R}^N}(v_n-v_{n,\epsilon}) \le \frac{1}{n} + b_{k,\mathbb{R}^N}(v_n-v_{n,\epsilon}).$$

It is hence enough to show that $v_{n,\epsilon}\to v_n$ in $D^k(\mathbb{R}^N)$ for $\epsilon\to 0$. In the following, we write v in place of v_n and $v_\epsilon = \rho_\epsilon * v$ in place of $v_{n,\epsilon}$ for $\epsilon\in(0,1]$. Moreover, let $R = R_n > 0$ with $v = v_n = 0$ on $\mathbb{R}^N\setminus B_R(0)$. Clearly, $v_\epsilon \to v$ in $L^2(\mathbb{R}^N)$ for $\epsilon\to 0$ and this convergence is also pointwise almost everywhere. Hence it is enough to analyze the convergence of $b_{k,\mathbb{R}^N}(v-v_\epsilon)$ as $\epsilon\to 0$. From here, the proof follows along the lines of [19, Proposition 4.1] noting that there it is not used that k only depends on the difference of x and y. Note here, that if u is nonnegative then the above constructed sequence is also nonnegative. □

Lemma 3.5 *Let $\Omega\subset\mathbb{R}^N$ open and such that $\partial\Omega$ is bounded. Denote $\delta(x) := \mathrm{dist}(x,\mathbb{R}^N\setminus\Omega)$. Then the following is true.*

(1) *There is $C = C(N,\Omega,k) > 0$ such that $\kappa_{k,\Omega}(x) \le C\delta^{-\sigma}(x)$ for $x\in\Omega$.*
(2) *If Ω is bounded, then $1_\Omega \in D^k(\mathbb{R}^N)$.*

Proof Let $C = C(N,\Omega,k) > 0$ be constants varying from line to line and denote $U := \{x\in\mathbb{R}^N : \mathrm{dist}(x,\Omega)\le 1\}$. To see item 1., let $x\in\Omega$ and fix $p\in\partial\Omega$ such that $\delta(x) = |x-p|$. Then

$$\kappa_{k,\Omega}(x) \le C + \int_{U\setminus\Omega}\frac{|x-p|^\sigma}{|x-p|^\sigma}k(x,y)\,\mathrm{d}y \le C + \delta(x)^{-\sigma}\int_{U\setminus\Omega}|x-y|^\sigma k(x,y)\,\mathrm{d}y \le C\delta^{-\sigma}(x),$$

where we have used that $|x-p|\le|x-y|$ for $y\in\mathbb{R}^N\setminus\Omega$. Now 2. follows immediately from 1., since we have

$$b_{k,\mathbb{R}^N}(1_\Omega) = \int_\Omega\int_{\mathbb{R}^N\setminus\Omega}k(x,y)\,\mathrm{d}y\mathrm{d}x \le C\int_\Omega\delta^{-\sigma}(x)\,\mathrm{d}x < \infty.$$

□

Theorem 3.6 (See Theorem 3.1) *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Then $C_c^\infty(\Omega)$ is dense in $\mathscr{D}^k(\Omega)$. Moreover, if $u\in\mathscr{D}^k(\Omega)$ is nonnegative, then we have*

(1) *There exists a sequence* $(u_n)_n \subset \mathscr{D}^k(\Omega)$ *with* $\lim_{n\to\infty} u_n = u$ *in* $\mathscr{D}^k(\Omega)$ *satisfying that for every* $n \in \mathbb{N}$ *there is* $\Omega'_n \subset\subset \Omega$ *with* $u_n = 0$ *on* $\mathbb{R}^N\setminus\Omega'_n$ *and* $0 \le u_n \le u_{n+1} \le u$.
(2) *There exists a sequence* $(u_n)_n \subset C_c^\infty(\Omega)$ *with* $u_n \ge 0$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} u_n = u$ *in* $\mathscr{D}^k(\Omega)$.

Proof Note that the second claim follows immediately from the first one using [19, Proposition 4.1] as in the proof of Proposition 3.4. Then also the main claim follows by considering $u^\pm$ separately. Hence it is enough to show 1. We proceed similar to [5, Theorem 3.1]. Denote $\delta(x) := \operatorname{dist}(x, \mathbb{R}^N\setminus\Omega)$. For $r > 0$, define the Lipschitz map

$$\phi_r : \mathbb{R}^N \to \mathbb{R}, \quad \phi_r(x) = \begin{cases} 0 & \delta(x) \ge 2r, \\ 2 - \dfrac{\delta(x)}{r} & r \le \delta(x) \le 2r, \\ 1 & \delta(x) \le r. \end{cases}$$

Note that we have $\phi_s \le \phi_r$ for $0 < s \le r$. We show

$$u\phi_r \in \mathscr{D}^k(\Omega) \ \text{ for } r > 0 \text{ sufficiently small and } \ b_{k,\mathbb{R}^N}(u\phi_r) \to 0 \ \text{ for } r \to 0. \tag{3.1}$$

Note that once this is shown, we have $u(1-\phi_r) \in \mathscr{D}^k(\Omega)$ for $r > 0$ sufficiently small and $u(1-\phi_r) \to u$ for $r \to 0$. Since also $0 \le u(1-\phi_r) \le u(1-\phi_s)$ for $0 < s \le r$ and $u(1-\phi_r) = 0$ for $x \in \mathbb{R}^N$ with $\delta(x) \le r$, it follows that (3.1) implies 1.

The remainder of the proof is to show (3.1). For this, let $C = C(N, \Omega, k) > 0$ be a constant which may vary from line to line. Let $A_t := \{x \in \Omega \ : \ \delta(x) \le t\}$. Note that $u\phi_r$ vanishes on $\mathbb{R}^N\setminus A_{2r}$, we have $0 \le \phi_r \le 1$ and, moreover,

$$|\phi_r(x) - \phi_r(y)| \le \min\left\{C\frac{|x-y|}{r}, 1\right\} \text{ for } x, y \in \mathbb{R}^N.$$

Then proceeding similarly to the proof of Lemma 2.3.(5) we find for r small enough

$$\begin{aligned}
b_{k,\mathbb{R}^N}(u\phi_r) &= \frac{1}{2}\int_{A_{4r}}\int_{A_{4r}} (u(x)\phi_r(x) - u(y)\phi_r(y))^2 k(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \int_{A_{2r}} u^2(x)\phi_r^2(x)\kappa_{k,A_{4r}}(x)\,\mathrm{d}x \\
&\le \int_{A_{4r}}\int_{A_{4r}} \Big(u(x)^2(\phi_r(x)-\phi_r(y))^2 + (u(x)-u(y))^2\phi_r^2(y)^2\Big)k(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,\mathrm{d}x \\
&\le \frac{C}{r^2}\int_{A_{4r}} u(x)^2 \int_{B_r(x)} |x-y|^2 k(x,y)\,\mathrm{d}y\mathrm{d}x + \int_{A_{4r}} u(x)^2 \int_{\mathbb{R}^N\setminus B_r(x)} k(x,y)\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_{A_{4r}}\int_{A_{4r}} (u(x)-u(y))^2 k(x,y)\,\mathrm{d}x\mathrm{d}y + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,\mathrm{d}x \\
&\le \frac{C}{r^\epsilon}\int_{A_{4r}} u(x)^2 \int_{B_1(x)} |x-y|^\epsilon k(x,y)\,\mathrm{d}y\mathrm{d}x \\
&\quad + C\int_{A_{4r}} u(x)^2\mathrm{d}x + b_{k,A_{4r}}(u) + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,\mathrm{d}x \\
&\le \frac{C}{r^\epsilon}\int_{A_{4r}} u(x)^2\,\mathrm{d}x + C\int_{A_{4r}} u(x)^2\mathrm{d}x + b_{k,A_{4r}}(u) + \int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\,\mathrm{d}x.
\end{aligned}$$

Note here, since $u \in D^k(\mathbb{R}^N)$, we have $\int_{A_{4r}} u(x)^2 \mathrm{d}x + b_{k,A_{4r}}(u) \to 0$ for $r \to 0$. Moreover, we have by Lebesgue's differentiation theorem

$$\begin{aligned}\frac{C}{r^\epsilon}\int_{A_{4r}} u(x)^2 \mathrm{d}x &\leq \frac{C|B_{4r}|}{r^\epsilon}\int_{\partial\Omega}\frac{1}{|B_{4r}|}\int_{B_{4r}(\theta)} u(x)^2\, \mathrm{d}x\sigma(d\theta)\\ &\leq Cr^{1-\epsilon}\int_{\partial\Omega}\frac{1}{|B_{4r}|}\int_{B_{4r}(\theta)} u(x)^2\, \mathrm{d}x\sigma(d\theta) \to 0 \text{ for } r \to 0^+.\end{aligned}$$

Finally, since

$$K_{k,\Omega}(u) = \int_\Omega u^2(x)\kappa_{k,\Omega}(x)\, \mathrm{d}x < \infty \tag{3.2}$$

and, by Lemma 3.5, we have

$$\kappa_{k,A_{4r}}(x) \leq \int_{\mathbb{R}^N\setminus\Omega} k(x,y)\, \mathrm{d}y + \int_{\Omega\setminus A_{4r}} k(x,y)\, \mathrm{d}y \leq C\kappa_{k,\Omega}(x) + Cr^{-\epsilon}$$

for $x \in A_{2r}$, so that also $\int_{A_{2r}} u^2(x)\kappa_{k,A_{4r}}(x)\, \mathrm{d}x \to 0$ for $r \to 0$ with a similar argument. □

Proof of Theorem 3.1 for $X(\Omega) = \mathscr{D}^k(\Omega)$ This statement now follows from Theorem 3.6, Lemma 3.3, and Proposition 3.4. □

Theorem 3.7 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Then $C_c^\infty(\Omega)$ is dense in $D^k(\Omega)$. Moreover, if $u \in D^k(\Omega)$ is nonnegative, then we have*

(1) *There exists a sequence $(u_n)_n \subset D^k(\Omega)\cap L^\infty(\Omega)$ with $\lim_{n\to\infty} u_n = u$ in $D^k(\Omega)$ satisfying that for every $n \in \mathbb{N}$ there is $\Omega_n' \subset\subset \Omega$ with $u_n = 0$ on $\Omega\setminus\Omega_n'$ and $0 \leq u_n \leq u_{n+1} \leq u$.*
(2) *There exists a sequence $(u_n)_n \subset C_c^\infty(\Omega)$ with $u_n \geq 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} u_n = u$ in $D^k(\Omega)$.*

Proof Consider the Lipschitz map

$$g_n : \mathbb{R} \to \mathbb{R}, \qquad g_n(t) = \begin{cases} 0 & t \leq 0\\ t & 0 < t < n\\ n & t \geq n.\end{cases}$$

Then $v_n := g_n(u) \in D^k(\Omega)\cap L^\infty(\Omega)$ and we have with ϕ_r as in the proof of Proposition 3.6

$$b_{k,\Omega}(u - (1-\phi_r)v_n) \leq b_{k,\Omega}(u - v_n) + b_{k,\Omega}(\phi_r v_n).$$

Clearly, $b_{k,\Omega}(u - v_n) \to 0$ for $n \to \infty$ by dominated convergence and $b_{k,\Omega}(\phi_r v_n) \to 0$ for $r \to 0$ analogously to the proof of Proposition 3.6, noting that the term in (3.2) reads in this case

$$K_{k,\Omega}(v_n) \leq n^2 \int_\Omega \kappa_{k,\Omega}(x)\, \mathrm{d}x < \infty \text{ for every } n \in \mathbb{N}.$$

In particular, statement (1) follows. Now statement (2) and the density statement follow analogously, again, to the proof of Proposition 3.6. □

Proof of Theorem 3.1 for $X(\Omega) = D^k(\Omega)$ This statement now follows from Theorem 3.7, Lemma 3.3, and Proposition 3.4. □

Remark 3.8 It is tempting to conjecture the following type of Hardy inequality: There is $C > 0$ such that

$$K_{k,\Omega}(\phi) \le C\Big(\|\phi\|^2_{L^2(\Omega)} + b_{k,\Omega}(\phi)\Big) \text{ for all } \phi \in C_c^\infty(\Omega)$$

if Ω is a bounded Lipschitz set and k is such that its symmetric lower bound j is not in $L^1(\mathbb{R}^N)$. Let us mention that for $k(x, y) = |x-y|^{-2s-N}$ this holds for $s \in (0, 1)$, $s \neq \frac{1}{2}$, see [4, 8]. Moreover, for $k(x, y) = 1_{B_1(0)}(x-y)|x-y|^{-N}$, this has been shown in [5]. In the general framework presented here, however, it is not clear if this is true.

Remark 3.9 With the above density results, we can now note that our definition of weak supersolutions (and similarly of weak subsolutions and solutions), see Definition 2.6, can be extended slightly:
Let $u \in \mathscr{V}^k_{loc}(\Omega)$ satisfy weakly $I_k u \ge f$ in Ω for some $f \in L^1_{loc}(\Omega)$ and $\Omega \subset \mathbb{R}^N$ open and bounded with Lipschitz boundary.

(1) If $f \in L^2_{loc}(\Omega)$, then by density it also holds

$$b_k(u, v) \ge \int_U f(x)v(x)\,\mathrm{d}x \text{ for all nonnegative } v \in \mathscr{D}^k(U), U \subset\subset \Omega. \tag{3.3}$$

(2) If $u \in \mathscr{V}^k(\Omega) \cap L^\infty(\mathbb{R}^N)$ and $f \in L^2(\Omega)$, then by density it also holds

$$b_k(u, v) \ge \int_\Omega f(x)v(x)\,\mathrm{d}x \text{ for all nonnegative } v \in \mathscr{D}^k(\Omega). \tag{3.4}$$

Finally note that if $u : \mathbb{R}^N \to \mathbb{R}$ satisfies $u1_U \in D^k(U)$ for some $U \subset\subset \mathbb{R}^N$ and $u \in L^\infty(\mathbb{R}^N \setminus U)$, then $u \in \mathscr{V}^k_{loc}(U)$.

Proposition 3.10 (Weak maximum principle) *Assume that the symmetric lower bound j of k defined in (2.2) satisfies*

$$j \text{ does not vanish identically on } B_r(0) \text{ for any } r > 0. \tag{3.5}$$

Let $\Omega \subset \mathbb{R}^N$ open and with Lipschitz boundary, $c \in L^\infty_{loc}(\Omega)$, and assume either

(1) *$c \le 0$ or*
(2) *Ω and c are such that $\|c^+\|_{L^\infty(\Omega)} < \inf_{x\in\Omega} \int_{\mathbb{R}^N\setminus\Omega} k(x, y)\,\mathrm{d}y$.*

If $u \in \mathscr{V}^k(\Omega)$ satisfies in weak sense

$$I_k u \ge c(x)u \;\text{ in } \Omega, u \ge 0 \text{ almost everywhere in } \mathbb{R}^N\setminus\Omega, \;\text{ and } \liminf_{|x|\to\infty} u(x) \ge 0,$$

then $u \ge 0$ almost everywhere in $\mathbb{R}^N$.

Proof Note that also $u^- \in \mathscr{V}^k(\Omega)$ and in particular $u^- \in D^k(\Omega)$. Hence, we can find $(v_n)_n \subset C_c^\infty(\Omega)$ with $v_n \to u^-$ in $D^k(\Omega)$ for $n \to \infty$ with $0 \le v_n \le v_{n+1} \le u^-$ by Proposition 3.7. Then

$$b_{k,\mathbb{R}^N}(u, v_n) \ge \int_\Omega c(x)u(x)v_n(x)\,\mathrm{d}x \ge -\|c^+\|_{L^\infty(\Omega)} \int_\Omega u^-(x)v_n(x)\,\mathrm{d}x.$$

On the other hand, since $u^+v_n = 0$ for all $n \in \mathbb{N}$ and $u \ge 0$ almost everywhere in $\mathbb{R}^N\setminus\Omega$, we find

$$
\begin{aligned}
b_{k,\mathbb{R}^N}(u,v_n) &= b_{k,\Omega}(u,v_n) + \int_\Omega v_n(x)\int_{\mathbb{R}^N\setminus\Omega}(u(x)-u(y))k(x,y)\,\mathrm{d}y\mathrm{d}x \\
&\le b_{k,\Omega}(u^+,v_n) - b_{k,\Omega}(u^-,v_n) + \int_\Omega v_n(x)u(x)\int_{\mathbb{R}^N\setminus\Omega} k(x,y)\,\mathrm{d}y\mathrm{d}x \\
&\le -b_{k,\Omega}(u^-,v_n) - K_{k,\Omega}(u^-,v_n).
\end{aligned}
$$

Hence

$$
0 \le \int_\Omega u^-(x)v_n(x)\Big(\|c^+\|_{L^\infty(\Omega)} - \kappa_{k,\Omega}(x)\Big)\,\mathrm{d}x - b_{k,\Omega}(u^-,v_n) \le -b_{k,\Omega}(u^-,v_n).
$$

Since $v_n \to u^-$ in $D^k(\Omega)$, it follows that $b_{k,\Omega}(u^-,u^-) = 0$, but then u^- is constant by Proposition 2.10 in Ω. Assume by contradiction that $u^- = m > 0$. Then the above calculation gives

$$
0 \le m\int_\Omega v_n(x)\Big(\|c^+\|_{L^\infty(\Omega)} - \kappa_{k,\Omega}(x)\Big)\,\mathrm{d}x, \tag{3.6}
$$

which is in both cases a contradiction: If in case 1. $c \le 0$, then since $\kappa_{k,\Omega}(x) \not\equiv 0$ and since $v_n \to m$ in $D^k(\Omega)$ the right-hand side of (3.6) is negative.
In case 2. this contradiction is immediate in a similar way. □

Proof of Proposition 1.2 The statement follows immediately from Proposition 3.10. □

Remark 3.11 Usually, the weak maximum principle is stated with an assumption on the first eigenvalue $\Lambda_1(\Omega)$ in place of $\inf_{x\in\Omega}\kappa_{k,\Omega}(x)$. This can be done once the Hardy inequality in Remark 3.8 is shown. Indeed, following the proof of Proposition 3.10 gives

$$
\begin{aligned}
-\|c^+\|_{L^\infty(\Omega)}\int_\Omega u^-(x)v_n(x)\,\mathrm{d}x &\le b_{k,\mathbb{R}^N}(u,v_n) \le -b_{k,\Omega}(u^-,v_n) \\
&-K_{k,\Omega}(u^-,v_n) = -b_{k,\mathbb{R}^N}(u^-,v_n).
\end{aligned}
$$

With $n\to\infty$ and using that by the Hardy inequality it holds $\mathscr{D}^k(\Omega) = D^k(\Omega)$, it follows that

$$
-\Lambda_1(\Omega)\|u^-\|^2_{L^2(\Omega)} \ge -b_{k,\mathbb{R}^N}(u^-,u^-) \ge -\|c^+\|_{L^\infty(\Omega)}\|u^-\|^2_{L^2(\Omega)}
$$

and the conclusion follows similarly.

Proof of Corollary 1.4 This statement now follows from Remark 3.11 using Remark 3.8. □

Proposition 3.12 (Strong maximum principle) *Assume k satisfies additionally* (4.1). *Let $\Omega \subset \mathbb{R}^N$ open and $c \in L^\infty_{loc}(\Omega)$ with $\|c^+\|_{L^\infty(\Omega)} < \infty$. Moreover, let $u \in \mathscr{V}^k(\Omega)$, $u \ge 0$ satisfy in weak sense $I_k u \ge c(x)u$ in Ω.*

(1) *If Ω is connected, then either $u \equiv 0$ in Ω or* $\operatorname{essinf}_K u > 0$ *for any $K \subset\subset \Omega$.*
(2) *If j given in* (2.2) *satisfies* $\operatorname{essinf}_{B_r(0)} j > 0$ *for any $r > 0$, then either $u \equiv 0$ in $\mathbb{R}^N$ or* $\operatorname{essinf}_K u > 0$ *for any $K \subset\subset \Omega$.*

Proof This statement follows by approximation from [19, Theorem 2.5 and 2.6]. Here, the statement $j \notin L^1(\mathbb{R}^N)$ comes into play since we need

$$
\inf_{x\in B_r(x_0)}\kappa_{k,B_r(x_0)}(x) \to \infty \text{ for } r\to 0
$$

to conclude the statement for arbitrary c as stated. □

Proof of Proposition 1.5 The statement follows immediately from Proposition 3.12. □

4 On boundedness

In the following, let $h * u(x) = \int_{\mathbb{R}^N} h(x-y)u(y)\,\mathrm{d}y$ as usual denote the convolution of two functions.

Theorem 4.1 *Assume k is such that*

$$\text{the symmetric lower bound } j \text{ defined in (2.2) satisfies } \int_{\mathbb{R}^N} j(z)\,\mathrm{d}z = \infty \tag{4.1}$$

and it holds

$$\sup_{x\in\mathbb{R}^N} \int_{K\setminus B_\epsilon(x)} k(x,y)^2\,\mathrm{d}y < \infty \text{ for all } K \subset\subset \mathbb{R}^N \text{ and } \epsilon > 0. \tag{4.2}$$

Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $f \in L^\infty(\Omega)$, $h \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and let $u \in \mathscr{V}^k_{loc}(\Omega)$ satisfy in weak sense

$$I_k u \le \lambda u + h * u + f \text{ in } \Omega \text{ for some } \lambda > 0.$$

If $u^+ \in L^\infty(\mathbb{R}^N\setminus\Omega')$ for some $\Omega' \subset\subset \Omega$, then $u^+ \in L^\infty(\mathbb{R}^N)$ and there is $C = C(\Omega,\Omega',k,h,\lambda) > 0$ such that

$$\|u^+\|_{L^\infty(\Omega')} \le C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).$$

Proof Let $\Omega_1, \Omega_2, \Omega_3 \subset \mathbb{R}^N$ be with Lipschitz boundary and such that

$$\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega.$$

Let $\eta \in C^{0,1}_c(\Omega_3)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on Ω_2. Put $v = \eta u$ and, for $\delta > 0$, denote $J_\delta(x,y) := 1_{B_\delta(0)}(x-y)k(x,y)$ and $k_\delta(x,y) = k(x,y) - J_\delta(x,y)$. Note that by Assumption (2.1) it follows that $y \mapsto k_\delta(x,y) \in L^1(\mathbb{R}^N)$ for all $x \in \mathbb{R}^N$. Moreover, by Assumption (4.1)

$$c_\delta := \inf_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} k_\delta(x,y)\,\mathrm{d}y \ge \int_{\mathbb{R}^N\setminus B_\delta(0)} j(z)\,\mathrm{d}z \to \infty \text{ for } \delta \to 0.$$

Hence, we may fix $\delta > 0$ such that

$$c_\delta > \lambda.$$

In the following, $C_i > 0$, $i = 1, \ldots$ denote constants depending on $\Omega', \Omega_i, \lambda, \delta, \Omega, \eta, k$, and h but may vary from line to line—clearly, by the choices of these dependencies are actually only through $\lambda, \Omega, \Omega', \eta, k$, and h. First note that by Lemma 2.8 we have in weak sense

$$I_k v \le \lambda u + h * u + \tilde{f} \text{ in } \Omega_1 \text{ with } \tilde{f}(x) = f(x) + \int_{\mathbb{R}^N\setminus\Omega_2} (1-\eta(y))u(y)k(x,y)\,\mathrm{d}y.$$

In the following, put

$$A := \|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}.$$

Then note that for $x \in \mathbb{R}^N$ we have

$$|h * u(x)| \le \|h\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\|h\|_{L^1(\mathbb{R}^N)} \le C_1 A$$

and, since $\sup_{x\in\Omega_1} \int_{\mathbb{R}^N\setminus\Omega_2}(1-\eta(y))k(x,y)\,\mathrm{d}y \le C_2 \sup_{x\in\Omega_1}\int_{\mathbb{R}^N\setminus\Omega_2}\min\{1,|x-y|^\sigma\}k(x,y)\,\mathrm{d}y < \infty$, it also holds that

$$\|\tilde{f}\|_{L^\infty(\Omega_1)} \le \|f\|_{L^\infty(\Omega)} + \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega_1)}C_2 \le C_3A.$$

Whence, since $u = v$ in Ω_1, we have in weak sense

$$I_k v \le \lambda v + C_4 A \text{ in } \Omega_1.$$

Next, let $\mu \in C_c^\infty(\Omega'')$ for some $\Omega' \subset\subset \Omega'' \subset\subset \Omega_1$ such that $0 \le \mu \le 1$, $\mu = 1$ on Ω', and $\mu = 0$ on $\mathbb{R}^N\setminus\Omega''$. Let $\phi_t = \mu^2(v-t)^+ \in \mathscr{D}^k(\Omega'')$ for $t > 0$ and note that

$$b_k(v,\phi_t) \le \int_{\Omega''}(\lambda v(x) + C_4A)\phi_t(x)\,\mathrm{d}x. \tag{4.3}$$

Fix $t > 0$ such that

$t \ge \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}$ and $C_6A + (\lambda - c_\delta)t \le 0$, where

$$C_6 = C_4 + C_5 \text{ with } C_5 = \sup_{x\in\Omega'}\int_{\Omega'}k_\delta^2(x,y)\,\mathrm{d}y + \sup_{x\in\Omega'}\int_{\mathbb{R}^N\setminus\Omega'}k_\delta(x,y)\,\mathrm{d}y.$$

That is, we fix

$$t = A\Big(1 + \frac{C_6}{c_\delta - \lambda}\Big).$$

Then with (4.3)

$$\begin{aligned} b_{J_\delta}(v,\phi_t) &= b_k(v,\phi_t) - b_{k_\delta}(v,\phi_t) \\ &\le \int_{\Omega''}\lambda v(x)\phi_t(x) + C_4A\phi_t(x)\,\mathrm{d}x - \int_{\mathbb{R}^N}v(x)\phi_t(x)\int_{\mathbb{R}^N}k_\delta(x,y)\,\mathrm{d}y\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N}\phi_t(x)\int_{\mathbb{R}^N}v(y)k_\delta(x,y)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Note here, that for $x \in \mathbb{R}^N$ we have by the integrability assumptions on k_δ and k

$$\int_{\mathbb{R}^N}v(y)k_\delta(x,y)\,\mathrm{d}y \le \int_{\mathbb{R}^N}u(y)k_\delta(x,y)\,\mathrm{d}y \le C_5(\|u\|_{L^2(\Omega')} + \|u^+\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}) \le C_5A$$

so that using that $v \ge t$ in $\operatorname{supp}\phi_t$ we have

$$b_{J_\delta}(v,\phi_t) \le \int_{\Omega''}(C_6A + (\lambda - c_\delta)v(x))\phi_t(x)\,\mathrm{d}x \le (C_6A + (\lambda - c_\delta)t)\int_{\Omega''}\phi_t(x)\,\mathrm{d}x. \tag{4.4}$$

On the other hand, with $v_t(x) = v(x) - t$, we have

$$\begin{aligned} &(v(x) - v(y))(\phi_t(x) - \phi_t(y)) - (\mu(x)v_t^+(x) - \mu(y)v_t^+(y))^2 \\ &\quad = 2\mu(x)\mu(y)v_t^+(x)v_t^+(y) - v_t(y)\mu^2(x)v_t^+(x) - \mu^2(y)v_t^+(y)v_t(x) \\ &\quad = -v_t^+(x)v_t^+(y)(\mu(x) - \mu(y))^2 + v_t^-(y)\mu^2(x)v_t^+(x) + \mu^2(y)v_t^+(y)v_t^-(x) \\ &\quad \ge -v_t^+(x)v_t^+(y)(\mu(x) - \mu(y))^2. \end{aligned}$$

Whence with Poincaré's inequality, using that by Assumption 4.1 there is for any $K \subset \mathbb{R}^N$ open and bounded, some $C > 0$ such that $b_{J_\delta}(u) \geq C\|u\|^2_{L^2(\mathbb{R}^N)}$ for $u \in \mathscr{D}^{J_\delta}(K)$, we find for some constant C_7

$$\begin{aligned} b_{J_\delta}(v, \phi_t) &\geq b_{J_\delta}(\mu v_t^+) - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2 J_\delta(x,y)\,\mathrm{d}x\mathrm{d}y \\ &\geq C_7\int_{\mathbb{R}^N}\mu^2(x)(v_t^+(x))^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2 J_\delta(x,y)\,\mathrm{d}x\mathrm{d}y \end{aligned} \tag{4.5}$$

$$= C_7\int_{\mathbb{R}^N}\mu^2(x)(v_t^+(x))^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega'}\int_{\Omega'} v_t^+(x)v_t^+(y)(\mu(x)-\mu(y))^2 J_\delta(x,y)\,\mathrm{d}x\mathrm{d}y \tag{4.6}$$

$$= C_7\int_{\mathbb{R}^N}\mu^2(x)(v_t^+(x))^2\,\mathrm{d}x \geq C_7\int_{\Omega'}(v_t^+(x))^2\,\mathrm{d}x. \tag{4.7}$$

Combining (4.7) and (4.4) we have

$$C_7\int_{\Omega'}(v_t^+(x))^2\,\mathrm{d}x \leq \Big(C_6 A + (\lambda - c_\delta)t\Big)\int_{\Omega''}\phi_t(x)\,\mathrm{d}x \leq 0.$$

Whence $v_t^+ = 0$ in Ω' and thus $u = v \leq t = A \cdot C_8$ in Ω' as claimed. □

Corollary 4.2 *If in the situation of Theorem 4.1 we have in weak sense $I_k u = \lambda u + h * u + f$ in Ω, then we have $u \in L^\infty(\Omega')$ and there is $C = C(\Omega, \Omega', k, \lambda, h) > 0$ such that*

$$\|u\|_{L^\infty(\Omega')} \leq C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).$$

Proof This follows by replacing u with $-u$ (and f with $-f$) in the statement of Theorem 4.1. □

Theorem 4.3 *If in the situation of Theorem 4.1 we have in weak sense $I_k u = \lambda u + h * u + f$ in Ω and $u \in \mathscr{D}^k(\Omega)$, then we have $u \in L^\infty(\Omega)$ and there is $C = C(\Omega, k, \lambda, h) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\Big).$$

Proof Using in the proof of Theorem 4.1 the test-function u_t^+ instead of ϕ_t (and similarly for Corollary 4.2), we find

$$\|u\|_{L^\infty(\Omega)} \leq C\Big(\|f\|_{L^\infty(\Omega)} + \|u\|_{L^2(\Omega)}\Big).$$

as claimed. □

5 On differentiability of solutions

In the following, $\Omega \subset \mathbb{R}^N$ is an open bounded set and k satisfies through out the assumptions (2.1) with some $\sigma < \frac{1}{2}$, (4.1), and (4.2). Moreover, we assume that there is $j : \mathbb{R}^N \to [0,\infty]$ such that $k(x,y) = j(x-y)$ for $x, y \in \mathbb{R}^N$ and that for some $m \in \mathbb{N} \cup \{\infty\}$ the following holds: We have $j \in W^{l,1}(\mathbb{R}^N \setminus B_\epsilon(0)$ for every $l \in \mathbb{N}$ with $l \leq 2m$, and there is some constant $C_j > 0$ such that

$$|\nabla j(z)| \leq C_j |z|^{-1-\sigma-N} \text{ for all } 0 < |z| \leq 3.$$

For simplicity, we write j in place of k and fix

$$\alpha := 1 - \sigma \in \left(\frac{1}{2}, 1\right).$$

Theorem 5.1 *Let $f \in H^1(\Omega)$, $\lambda \in \mathbb{R}$ and $u \in \mathscr{V}^j_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_j u = f + \lambda u$ in Ω. Then for any $\Omega' \subset\subset \Omega$ there is $C = C(N, \Omega, \Omega', j, \lambda) > 0$ such that*

$$\|\delta_{h,e} u\|_{L^2(\Omega')} \leq h^\alpha C\Big(\|f\|^2_{H^1(\Omega)} + \|u\|^2_{L^\infty(\mathbb{R}^N)}\Big)^{\frac{1}{2}} \text{ for all } h > 0, e \in \partial B_1(0). \tag{5.1}$$

Proof Let $\Omega' \subset\subset \Omega$ and fix $r \in (0, \frac{1}{8})$ small such that $8r \leq \text{dist}(\Omega', \mathbb{R}^N \backslash \Omega)$. Moreover, fix $x_0 \in \Omega'$ and denote $B_n := B_{nr}(x_0)$. Note that by using assumption (4.1) with Lemma 2.2 we achieve, by making $r > 0$ small enough,

$$\lambda < \lambda_1 = \min_{\substack{w \in \mathscr{D}^j(B_4) \\ w \neq 0}} \frac{b_j(w)}{\|w\|_{L^2(\mathbb{R}^N)^2}}.$$

Let $\eta \in C^{0,1}_c(B_4)$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_2. Note that it holds

$$|\eta(x) - \eta(y)| \leq 2\|\eta\|_{C^{0,1}(\mathbb{R}^N)} \min\{1, |x - y|\},$$

where we put as usual

$$\|\eta\|_{C^{0,1}(\mathbb{R}^N)} := \sup_{x \in \mathbb{R}^N} |\eta(x)| + \sup_{\substack{x, y \in \mathbb{R}^N \\ x \neq y}} \frac{|\eta(x) - \eta(y)|}{|x - y|}.$$

Note that by choice we have $\|\eta\|_{C^{0,1}(\mathbb{R}^N)} \leq 1 + \frac{1}{r} \leq \frac{2}{r}$, so that for all $x, y \in \mathbb{R}^N$

$$|\eta(x) - \eta(y)| \leq \frac{4}{r} \min\{1, |x - y|\}. \tag{5.2}$$

Fix $e \in \partial B_1(0)$ and $h \in (0, r)$. Let

$$A := \|u\|_{L^\infty(\mathbb{R}^N)}.$$

Let $\psi = \eta^2 \delta_h u \in \mathscr{D}^j(B_4)$, where in the following $\delta_h u := \delta_{h,e} u$. Note that

$$\begin{aligned}(\delta_h u(x) - \delta_h u(y))(\psi(x) - \psi(y)) &= (\eta(x)\delta_h u(x) - \eta(y)\delta_h u(y))^2 \\ &\quad - \delta_h u(x)\partial_h u(y)(\eta(x) - \eta(y))^2.\end{aligned}$$

Hence, we have

$$b_j(\delta_h u, \psi) = b_j(\eta\delta_h u) - \frac{1}{2}\iint_{\mathbb{R}^N \times \mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x) - \eta(y))^2 j(x - y)\, \mathrm{d}x\mathrm{d}y.$$

and using the translation invariance, we also have

$$b_j(\delta_h u, \psi) = \int_\Omega [\delta_h f(x) + \lambda\delta_h u]\psi(x)\, \mathrm{d}x.$$

In the following, for simplicity, we put $v(x) = \eta(x)\delta_h u(x)$, $x \in \mathbb{R}^N$. Note that by Definition, $v \in \mathscr{D}^j(B_4)$. Then with the help of Young's inequality for some $\mu \in (0, 1)$ such that

$$2\mu < \lambda_1 - \lambda \tag{5.3}$$

we find

$$\begin{aligned}\lambda_1\|v\|^2_{L^2(\Omega'')} &\leq b_j(v) = b_j(\delta_h u, \psi) + \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)\\ &\quad - \eta(y))^2 j(x-y)\,\mathrm{d}x\mathrm{d}y\\ &= \int_{\Omega''}[\delta_h f(x) + \lambda\delta_h u]\eta^2(x)\delta_h u(x)\,\mathrm{d}x + \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)\\ &\quad - \eta(y))^2 j(x-y)\,\mathrm{d}x\mathrm{d}y\\ &\leq (\mu+\lambda)\|v\|^2_{L^2(\Omega'')} + \mu^{-1}h^2\|\frac{\delta_h f}{h}\|^2_{L^2(\Omega'')} + \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)\\ &\quad - \eta(y))^2 j(x-y)\,\mathrm{d}x\mathrm{d}y. \qquad (5.4)\end{aligned}$$

By a rearrangement of the double integral with Young's inequality for the same $\mu \in (0, 1)$ as above we have

$$\begin{aligned}&\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^N} \delta_h u(x)\delta_h u(y)(\eta(x)-\eta(y))^2 j(x-y)\,\mathrm{d}x\mathrm{d}y\\ &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \delta_h u(x)\delta_h u(y)\eta(x)(\eta(x)-\eta(y)) j(x-y)\,\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{R}^N} \eta(x)\delta_h u(x)\int_{\mathbb{R}^N} u(y)\delta_{-h,y}\Big((\eta(x)-\eta(y))j(x-y)\Big)\,\mathrm{d}y\mathrm{d}x\\ &\leq \mu\|v\|^2_{L^2(B_4)} + \mu^{-1}\int_{B_4}\Big(\int_{\mathbb{R}^N}|u(y)|\Big|\delta_{-h,y}\Big((\eta(x)-\eta(y))j(x-y)\Big)\Big|\,\mathrm{d}y\Big)^2\mathrm{d}x\\ &\leq \mu\|v\|^2_{L^2(B_4)} + \mu^{-1}A^2\int_{B_4}\Big(\int_{\mathbb{R}^N}\Big|\delta_{-h,y}\Big((\eta(x)-\eta(y))j(y-x)\Big)\Big|\,\mathrm{d}y\Big)^2\mathrm{d}x\\ &\leq \mu\|v\|^2_{L^2(B_4)} + \mu^{-1}A^2\int_{B_4}\Big(\int_{\mathbb{R}^N}\Big|\delta_{-h,z}\Big((\eta(x)-\eta(z+x))j(z)\Big)\Big|\,\mathrm{d}z\Big)^2\mathrm{d}x. \qquad (5.5)\end{aligned}$$

Here, we indicate with $\delta_{-h,y}$ (resp. $\delta_{-h,z}$) that δ_{-h} acts on the y (resp. z) variable. Note that

$$\begin{aligned}&\delta_{-h,z}\Big((\eta(x)-\eta(z+x))j(z)\Big)\\ &= \delta_{-h,z}(\eta(x)-\eta(z+x))j(z) + (\eta(x)-\eta(z+x-he))\delta_{-h}j(z)\\ &= \Big(\eta(z+x)-\eta(z+x-he)\Big)j(z) + (\eta(x)-\eta(z+x-he))\Big(j(z-he)-j(z)\Big) \qquad (5.6)\\ &= \Big(\eta(z+x)-\eta(z)\Big)j(z) + \Big(\eta(x)-\eta(z+x-he)\Big)j(z-he). \qquad (5.7)\end{aligned}$$

Note here, that (5.6) satisfies

$$\begin{aligned}&\Big|\Big(\eta(z+x)-\eta(z+x-he)\Big)j(z) + (\eta(x)-\eta(z+x-he))\Big(j(z-he)-j(z)\Big)\Big|\\ &\qquad \leq \frac{4h}{r}J(z) + \frac{4h}{r}\min\{1,|z-he|\}\int_0^1 |\nabla j(z-\tau he)|\,d\tau \qquad (5.8)\end{aligned}$$

and (5.7) can be written as

$$\begin{aligned} &\left|\Big(\eta(z+x)-\eta(z)\Big)j(z)+\Big(\eta(x)-\eta(z+x-he)\Big)j(z-he)\right| \\ &\qquad\le \frac{4}{r}\min\{1,|z|\}j(z)+\frac{4}{r}\min\{1,|z-he|\}j(z-he). \end{aligned} \tag{5.9}$$

For $h\in(0,r)$, $z\in\mathbb{R}^N\setminus\{0\}$ put

$$\begin{aligned} k_h(z)=\min\Big\{&h\Big(j(z)+\min\{1,|z-he|\}\int_0^1|\nabla j(z-\tau he)|\,d\tau\Big), \\ &\min\{1,|z|\}j(z)+\min\{1,|z-he|\}j(z-he)\Big\}. \end{aligned}$$

Then, by combining (5.4) and (5.5), we find

$$\begin{aligned} \|\delta_h u\|^2_{L^2(B_2)} &\le \|v\|^2_{L^2(B_4)} \\ &\le \frac{\mu^{-1}|B_4|}{\lambda_1-\lambda-2\mu}\left(h^2\Big\|\frac{\delta_h f}{h}\Big\|^2_{L^2(B_4)}+\frac{16}{r^2}\|u\|^2_{L^\infty(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N}k_h(z)\,\mathrm{d}z\Big)^2\right). \end{aligned} \tag{5.10}$$

Next we show that we have $\int_{\mathbb{R}^N}k_h(z)\,\mathrm{d}z\le Ch^\alpha$ for some $C>0$. Clearly, we can bound

$$\int_{\mathbb{R}^N\setminus B_2(0)}k_h(z)\,\mathrm{d}z\le C_1h \tag{5.11}$$

for some $C_1=C_1(N,j)>0$, using that $B_1(0)\cup B_1(he)\subset B_2(0)$ and the properties of j. In the following, by making C_j larger if necessary, we may also assume that assumption (2.1) reads

$$\sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N}\min\{1,|x-y|^\sigma\}j(x-y)\,\mathrm{d}y=\int_{\mathbb{R}^N}\min\{1,|z|^\sigma\}j(z)\,\mathrm{d}z\le C_j.$$

Then note that $B_{2h}(he)\subset B_{3h}(0)$ and we have

$$\begin{aligned} &\int_{B_{2h}(0)}\min\{1,|z|\}j(z)+\min\{1,|z-he|\}j(z-he)\,\mathrm{d}z \\ &\quad\le C_j\int_{B_{3h}(0)}|z|^{1-\sigma-n}\,\mathrm{d}z+C_j\int_{B_{3h}(he)}|z-he|^{1-\sigma-n}\,\mathrm{d}z \\ &\quad=\frac{2|B_1(0)|C_j}{n}\int_0^{3h}\rho^{-\sigma}\,d\rho=\frac{2|B_1(0)|C_j}{n(1-\sigma)}(3h)^{1-\sigma}. \end{aligned} \tag{5.12}$$

While with $b_\sigma(t)=\frac{1}{\sigma}t^{-\sigma}$ we have

$$\begin{aligned} &h\int_{B_2(0)\setminus B_{2h}(0)}j(z)+\min\{1,|z-he|\}\int_0^1|\nabla j(z-\tau he)|\,d\tau\,\mathrm{d}z \\ &\le\frac{h|B_1(0)|C_j}{n}\int_{2h}^2\rho^{-\sigma-1}\,d\rho+hC_j\int_0^1\int_{B_3(\tau he)\setminus B_h(\tau he)}|z||z-\tau he|^{-1-\sigma-n}\,\mathrm{d}z\,d\tau \\ &\le\frac{h|B_1(0)|C_j}{n}b_\sigma(2h)+hC_j\int_0^1\int_{B_3(0)\setminus B_h(0)}|z+\tau he||z|^{-1-\sigma-n}\,\mathrm{d}z\,d\tau \end{aligned}$$

$$\leq \frac{h|B_1(0)|C_j}{n} b_\sigma(2h) + hC_J \int_{B_3(0)\setminus B_h(0)} |z|^{-\sigma-n}\, \mathrm{d}z + h^2 C_j \int_{B_3(0)\setminus B_h(0)} |z|^{-1-\sigma-n}\, \mathrm{d}z$$
$$\leq \frac{2h|B_1(0)|C_j}{n} b_\sigma(h) + \frac{h^2|B_1(0)|C_j}{n} \int_h^3 \rho^{-2-\sigma}\, d\rho$$
$$\leq \frac{2|B_1(0)|C_j}{n} h b_\sigma(h) + \frac{|B_1(0)|C_j}{n(1+\sigma)} h^{1-\sigma}. \tag{5.13}$$

Combining (5.11) with (5.12) and (5.13) and the choice $\alpha = 1-\sigma \in (0,1)$ we find $C_2 = C_2(N, j, \alpha) > 0$ such that

$$\int_{\mathbb{R}^N} k_h(z)\, \mathrm{d}z \leq C_2 h^\alpha. \tag{5.14}$$

Whence, from (5.10) with (5.14) we have

$$\|\delta_h u\|^2_{L^2(B_2)} \leq \|v\|^2_{L^2(B_4)} \leq h^{2\alpha} C_4\Big(\|\frac{\delta_h f}{h}\|^2_{L^2(B_4)} + \|u\|^2_{L^\infty(\mathbb{R}^N)}\Big), \tag{5.15}$$

for a constant $C_4 = C_4(N, j, r, \alpha, \lambda) > 0$. By a standard covering argument, we then also find with a constant $C_5 = C_5(N, j, \Omega, \Omega', \alpha, \lambda) > 0$ and $\Omega'' = \{x \in \Omega \ :\ \mathrm{dist}(x, \mathbb{R}^N\setminus\Omega) > 4r\}$

$$\|\delta_h u\|^2_{L^2(\Omega')} \leq h^{2\alpha} C_4\Big(\|\frac{\delta_h f}{h}\|^2_{L^2(\Omega'')} + \|u\|^2_{L^\infty(\mathbb{R}^N)}\Big), \tag{5.16}$$

The claim (5.1) then follows since $f \in H^1(\Omega)$. □

Remark 5.2 If additionally $f \in L^\infty(\Omega)$, combining Theorem 5.1 with Corollary 4.2 it follows that we have in the situation of Theorem 5.1 for every $\Omega' \subset\subset \Omega$

$$\|\delta_{h,e} u\|_{L^2(\Omega')} \leq h^\alpha C\Big(\|f\|^2_{C^1(\Omega)} + \|u\|^2_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}} \text{ for all } h>0,\, e \in \partial B_1(0). \tag{5.17}$$

Corollary 5.3 *Assume* $m = 1$. *Let* $f \in C^2(\overline{\Omega})$, $\lambda \in \mathbb{R}$, *and let* $u \in \mathcal{V}^j_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ *satisfy in weak sense* $I_j u = \lambda u + f$ *in* Ω. *Then* $u \in H^1(\Omega')$ *and* $\partial_i u \in D^j(\Omega')$ *for any* $\Omega' \subset\subset \Omega$. *More precisely, with* α *as above there is for any* $\Omega' \subset\subset \Omega$ *a constant* $C = C(N, \Omega, \Omega', j, \lambda) > 0$ *such that*

$$\sup_{\substack{e\in\partial B_1(0)\\ h>0}} h^{-2\alpha}\|\delta^2_{h,e} u\|_{L^2(\Omega')} \leq C\Big(\|f\|^2_{C^2(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}}, \tag{5.18}$$

so that $u \in N^{2\alpha,2}(\Omega') \subset H^1(\Omega')$, *that is, there is also* $C' = C'(N, j, \Omega, \Omega', \alpha, \lambda) > 0$ *such that*

$$\|\nabla u\|_{L^2(\Omega')} \leq C'\Big(\|f\|^2_{C^2(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}} \tag{5.19}$$

and, moreover,

$$b_{j,\Omega'}(\partial_i u) \leq C' \text{ for } i = 1, \ldots, N.$$

Proof Let $\Omega_i \subset\subset \Omega$, $i = 1, \ldots, 7$ such that

$$\Omega' \subset\subset \Omega_i \subset\subset \Omega_j \text{ for } 1 \leq i < k \leq 7.$$

Let $\eta \in C_c^\infty(\Omega_7)$ with $\eta = 1$ on Ω_6 and $0 \le \eta \le 1$. Fix $e \in \partial B_1(0)$ and $h \in (0, \frac{1}{2}r)$, where $r = \min\{\operatorname{dist}(\Omega_i, \Omega\setminus\Omega_{i+1}) \ : \ i = 1, \dots, 6\}$. Then by Lemma 2.8 the function $v = \eta\delta_h u$, where we write δ_h instead of $\delta_{h,e}$, satisfies $I_j v = \lambda v + \tilde{f}$ in Ω_5, where $\tilde{f} = \delta_h f + g_{\eta,\delta_h u}$. Following the proof of Theorem 5.1 to (5.16) it follows with Theorem 4.1 that there is $C = C(N, j, r, \alpha, \lambda) > 0$ (changing from line to line) such that

$$\begin{aligned}
\|\delta_h^2 u\|_{L^2(\Omega')}^2 = \|\delta_h v\|_{L^2(\Omega')}^2 &\le h^{2\alpha} C\Big(\|\frac{\delta_h \tilde{f}}{h}\|_{L^2(\Omega_1)}^2 + \|v\|_{L^\infty(\mathbb{R}^N)}^2\Big)\\
&\le h^{2\alpha} C\Big(\|\frac{\delta_h \tilde{f}}{h}\|_{L^2(\Omega_1)}^2 + \|\tilde{f}\|_{L^\infty(\Omega_4)}^2 + \|v\|_{L^2(\Omega_3)}^2 + \|v\|_{L^\infty(\mathbb{R}^N\setminus\Omega_3)}^2\Big)\\
&\le h^{2\alpha} C\Big(\|\frac{\delta_h \tilde{f}}{h}\|_{L^2(\Omega_1)}^2 + \|\tilde{f}\|_{L^\infty(\Omega_4)}^2 + \|\delta_h u\|_{L^2(\Omega_3)}^2\Big)\\
&\le h^{2\alpha} C\Big(\|\frac{\delta_h \tilde{f}}{h}\|_{L^2(\Omega_1)}^2 + \|\tilde{f}\|_{L^\infty(\Omega_4)}^2 + h^{2\alpha}\Big(\|f\|_{C^1(\Omega)}^2 + \|u\|_{L^\infty(\mathbb{R}^N)}^2\Big)\Big),
\end{aligned}$$

where we applied once more Theorem 5.1. Here, for $x \in \Omega_4$ using the assumptions on the differentiability of j it follows that there is $C = C(j) > 0$ such that

$$\begin{aligned}
|\tilde{f}(x)| &\le |\delta_h f(x)| + \left|\int_{\mathbb{R}^N\setminus\Omega_6} (1-\eta(y))\delta_h u(y) j(x-y)\,\mathrm{d}y\right|\\
&= |\delta_h f(x)| + \left|\int_{\mathbb{R}^N\setminus\Omega_5} |u(y)|\delta_h[(1-\eta(y))j(x-y)]\,\mathrm{d}y\right.\\
&\le hC\Big(\|\nabla f\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).
\end{aligned}$$

Moreover, for $x \in \Omega_1$ in a similar way, there is $C = C(j) > 0$ such that

$$\begin{aligned}
|\delta_h \tilde{f}(x)| &\le |\delta_h^2 f(x)| + \left|\int_{\mathbb{R}^N\setminus\Omega_6} (1-\eta(y))\delta_h u(y)\delta_h j(x-y)\,\mathrm{d}y\right|\\
&\le h^2\|f\|_{C^2(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\left|\int_{\mathbb{R}^N\setminus\Omega_5} \delta_h[(1-\eta(y))\delta_h j(x-y)]\,\mathrm{d}y\right|\\
&\le h^2 C\Big(\|f\|_{C^2(\Omega)} + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big).
\end{aligned}$$

Thus we have

$$\|\delta_h^2 u\|_{L^2(\Omega')}^2 \le Ch^{4\alpha}\Big(\|f\|_{C^2(\Omega)}^2 + \|u\|_{L^2(\Omega')}^2 + \|u\|_{L^\infty(\mathbb{R}^N\setminus\Omega')}^2\Big).$$

The proof of the first part then is finished with Proposition 2.11 since $2\alpha > 1$. Next, write $D_h p(x) = \frac{p(x+he)-p(x)}{h}$ for any function $p : \mathbb{R}^N \to \mathbb{R}$, with $e \in \partial B_1(0)$ fixed and $h \in \mathbb{R}\setminus\{0\}$. Then with Lemma 2.8 for some $\eta \in C_c^\infty(\Omega)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $\Omega_2 \subset\subset \Omega$ with $\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2$ we have with $v = \eta u$,

$$I_j v = f + \lambda v + g_{\eta,u} \text{ in } \Omega_1, \quad \text{where } g_{\eta,u} = \int_{\mathbb{R}^N\setminus\Omega_2} (1-\eta(y))u(y)j(x-y)\,\mathrm{d}y.$$

Next, let $\mu \in C_c^\infty(\Omega_1)$ with $0 \le \mu \le 1$ and $\mu \equiv 1$ on Ω'. Then with $\phi = D_{-h}[\mu^2 D_h v] \in \mathscr{D}^j(\Omega_1)$ for h small enough we have for some $C > 0$ (which may change from line to line independently of h)

$$|b_j(v,\phi)| = \left| \int_{\Omega_1} D_h f \mu^2 D_h v + \lambda(\mu D_h v)^2 + D_h g_{\eta,u} \mu^2 D_h v \,\mathrm{d}x \right| \le C, \tag{5.20}$$

since

$$\int_{\Omega_1} |D_h f \mu^2 D_h v| \,\mathrm{d}x \le C\|f\|_{C^1(\Omega)} \|\nabla u\|_{L^2(\Omega_2)} < \infty,$$
$$\int_{\Omega_1} |\lambda(\mu D_h v)^2| \,\mathrm{d}x \le 2|\lambda| \|\nabla u\|^2_{L^2(\Omega_2)} < \infty,$$

and

$$\int_{\Omega_1} |D_h g_{\eta,u} \mu^2 D_h v| \,\mathrm{d}x \le C\Big(\int_{\Omega_1} \int_{\mathbb{R}^N \setminus \Omega_2} |(1-\eta(y))u(y)| [D_h j](x-y)| \,\mathrm{d}y \,\mathrm{d}x \Big)^{1/2} \|\nabla u\|_{L^2(\Omega_2)} < \infty$$

due to assumptions on the differentiability of j. Moreover, with a similar calculation as in the proof of Theorem 5.1 we have

$$b_j(v,\phi) = b_j(\mu D_h v, \mu D_h v) - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} D_h v(x) D_h v(y) (\mu(x) - \mu(y))^2 j(x-y) \,\mathrm{d}x\mathrm{d}y,$$

where for some $\Omega_2 \subset\subset \Omega_3 \subset\subset \Omega_4 \subset\subset \Omega$ with h small enough

$$\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |D_h v(x) D_h v(y)(\mu(x)-\mu(y))^2 j(x-y)| \,\mathrm{d}x\mathrm{d}y \\
&\le C \int_{\Omega_3} \int_{\Omega_3} |D_h(\eta u)(x) D_h(\eta u)(y)| |x-y|^2 j(x-y) \,\mathrm{d}x\mathrm{d}y \\
&\le C \int_{\Omega_3} |D_h(\eta u)(x)|^2 \int_{\Omega_3} |x-y|^2 j(x-y) \,\mathrm{d}y\mathrm{d}x \\
&\le C \|\nabla u\|_{L^2(\Omega_4)} \int_{\mathbb{R}^N} \min\{1, |z|^2\} j(z) \,\mathrm{d}z < \infty.
\end{aligned}$$

Combining this with (5.20) we find

$$b_j(\mu D_h v, \mu D_h v) \le C \text{ for all } h > 0 \text{ small enough.}$$

Since also $\mu D_h v \in \mathscr{D}^j(\Omega_2)$ for all $h > 0$ small enough (see Lemma 2.3) and since $D^j(\Omega_2)$ is a Hilbert space, we conclude that $\mu \partial_e v \in \mathscr{D}^j(\Omega_2)$ with

$$b_j(\mu \partial_e v) \le C$$

for $h \to 0$. This finishes the proof. □

Corollary 5.4 *Let $f \in C^{2m}(\overline{\Omega})$, $\lambda \in \mathbb{R}$, and let $u \in \mathscr{V}^j_{loc}(\Omega) \cap L^\infty(\mathbb{R}^N)$ satisfy in weak sense $I_j u = \lambda u + f$ in Ω. Then $u \in H^m(\Omega')$ for any $\Omega' \subset\subset \Omega$ and there is $C = C(N, j, \Omega, \Omega', m) > 0$ such that*

$$\|u\|_{H^m(\Omega')} \le C \Big(\|f\|^2_{C^{2m}(\Omega)} + \|u\|^2_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N \setminus \Omega')} \Big)^{\frac{1}{2}}. \tag{5.21}$$

In particular, if $m = \infty$, then $u \in C^\infty(\Omega)$.

Proof By Corollary 5.3 the claim holds for $m = 1$ in particular with $u|_{\Omega'} \in D^j(\Omega')$ for all $\Omega' \subset\subset \Omega$. Assume next, the claim holds for $m-1$ with $m \in \mathbb{N}$, $m \geq 2$ in the following way: We have $u \in H^{m-1}(\Omega')$ and $\partial^\beta u|_{\Omega'} \in D^j(\Omega')$ for any $\Omega' \subset\subset \Omega$ and $\beta \in \mathbb{N}_0^N$ with $|\beta| \leq m-1$, and there is $C = C(N, j, \Omega, \Omega', m) > 0$ such that

$$\|u\|_{H^{m-1}(\Omega')} \leq C\Big(\|f\|^2_{C^{2m-2}(\Omega)} + \|u\|_{L^2(\Omega')} + \|u\|^2_{L^\infty(\mathbb{R}^N\setminus\Omega')}\Big)^{\frac{1}{2}}. \tag{5.22}$$

Fix $\Omega' \subset\subset \Omega$ and let $\Omega_i \subset\subset \Omega$, $i = 1, \ldots, 7$ and $\eta \in C_c^\infty(\Omega_7)$ as in the proof of Corollary 5.3. Put $v = \partial^\beta(\eta u)$ for some $\beta \in \mathbb{N}_0^N$, $|\beta| = m-1$. Then $Iv = \partial^\beta f + \lambda v + \partial^\beta g_{\eta,u}$ in Ω_5 by Lemma 2.8 and direct computation using the assumptions on J. From here, proceeding as in the proof of Corollary 5.3 by applying Theorem 5.1 the claim follows. □

Proof of Theorem 1.1 By assumption, it follows from [20] that $\mathscr{D}^j(\Omega)$ is compactly embedded into $L^2(\Omega)$. This gives the existence of the sequence of eigenfunctions and corresponding eigenvalues. The fact that the first eigenfunction can be chosen to be positive follows from the fact that $b_j(|u|) \leq b_j(u)$, Proposition 1.2 and Proposition 1.5 (see also [19]). Now statement (1) follows from Theorem 4.3 (with $h = f = 0$) and statement (2) follows directly from Corollary 5.4. □

Proof of Theorem 1.6 The first part follows from the Poincaré inequality, i.e. under the assumptions it holds $\Lambda_1(\Omega) > 0$, and Theorem 4.3 with $h = 0 = \lambda$. The last assertion follows from Corollary 5.4. □

Acknowledgements This work is supported by DAAD and BMBF (Germany) within the project 57385104. The authors thank Mouhamed Moustapha Fall and Tobias Weth for helpful discussions. We also thank the anonymous referee for valuable suggestions.

Funding Open Access funding enabled and organized by Projekt DEAL.

Appendix A: An inequality

The following is a variant of [9, Lemma 10] (see also [20, Lemma 5.1]).

Lemma A.1 *Let $q \in L^1(\mathbb{R}^N)$ be a nonnegative even function with $q = 0$ on $\mathbb{R}^N \setminus B_r(0)$ for some $r > 0$. Let $\Omega \subset \mathbb{R}^N$ open and $x_0 \in \Omega$ such that $B_{2r}(x_0) \subset \Omega$. Then for all measurable functions $u : \Omega \to \mathbb{R}$ we have*

$$b_{q*q,B_r(x_0)}(u) \leq 4\|q\|_{L^1(\mathbb{R}^N)} b_{q,\Omega}(u).$$

Proof In the following, we identify u with its trivial extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}$, $\tilde{u}(x) = u(x)$ for $x \in \Omega$ and $\tilde{u}(x) = 0$ otherwise. Denote $g(x, y) = (u(x) - u(y))^2$ for $x, y \in \mathbb{R}^N$. Note that we have

$$0 \leq g(x, y) = g(y, x) \leq 2g(x, z) + 2g(y, z) \text{ for all } x, y, z \in \mathbb{R}^N.$$

By Fubini's theorem we have

$$\begin{aligned}
&\int_{B_r(x_0)}\int_{B_r(x_0)} g(x,y)(q*q)(x-y)\,\mathrm{d}x\mathrm{d}y = \\
&\int_{B_r(x_0)}\int_{B_r(x_0)}\int_{\mathbb{R}^N} g(x,y)q(x-z)q(y-z)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y \\
&\le 2\int_{B_r(x_0)}\int_{B_r(x_0)}\int_{\mathbb{R}^N} [g(x,z)+g(y,z)]q(x-z)q(y-z)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y \\
&\le 4\int_{B_r(x_0)}\int_{\mathbb{R}^N} g(x,z)q(x-z) \\
&\int_{\mathbb{R}^N} q(y-z)\,\mathrm{d}y\mathrm{d}z\mathrm{d}x = 4\|q\|_{L^1(\mathbb{R}^N)}\int_{B_r(x_0)}\int_{\mathbb{R}^N} g(x,z)q(x-z)\,\mathrm{d}z\mathrm{d}x.
\end{aligned}$$

Note that since $q=0$ on $\mathbb{R}^N\setminus B_r(0)$, q is even, and $B_r(x)\subset B_{2r}(x_0)\subset\Omega$ for any $x\in B_r(x_0)$, we have

$$\int_{B_r(x_0)}\int_{\mathbb{R}^N} g(x,z)q(x-z)\,\mathrm{d}z\mathrm{d}x = \int_{B_r(x_0)}\int_{B_r(x)} (u(x)-u(z))^2 q(x-z)\,\mathrm{d}z\mathrm{d}x \le 2b_{q,\Omega}(u).$$

□

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