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Hypersurfaces of constant higher-order mean curvature in $M \times \mathbb{R}$

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Abstract

We consider hypersurfaces of products $M \times \mathbb{R}$ with constant rth mean curvature $H_r \geq 0$ (to be called H_r-hypersurfaces), where M is an arbitrary Riemannian n-manifold. We develop a general method for constructing them and employ it to produce many examples for a variety of manifolds M, including all simply connected space forms and the hyperbolic spaces $\mathbb{H}^m_{\mathbb{F}}$ (rank one symmetric spaces of noncompact type). We construct and classify complete rotational $H_r(\geq 0)$-hypersurfaces in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ and in $\mathbb{S}^n \times \mathbb{R}$ as well. They include spheres, Delaunay-type annuli and, in the case of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, entire graphs. We also construct and classify complete $H_r(\geq 0)$-hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which are invariant by either parabolic isometries or hyperbolic translations. We establish a Jellett–Liebmann-type theorem by showing that a compact, connected and strictly convex H_r-hypersurface of $\mathbb{H}^n \times \mathbb{R}$ or $\mathbb{S}^n \times \mathbb{R}$ $(n \geq 3)$ is a rotational embedded sphere. Other uniqueness results for complete H_r-hypersurfaces of these ambient spaces are obtained.

Keywords Higher-order mean curvature · r-minimal · Product space

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1 Introduction

In his pioneering work [29], H. Rosenberg initiated the study of minimal and constant mean curvature hypersurfaces of product spaces $M \times \mathbb{R}$, where M is an arbitrary Riemannian n-manifold. Since then, many results on this subject have been obtained by many authors, mostly in the particular case M is a simply connected space form.

Following this path, we approach here hypersurfaces of $M \times \mathbb{R}$ with constant rth mean curvature $H_r \geq 0$ (for some $r \in \{1, \ldots, n\}$), which we call *H_r-hypersurfaces*. Let us recall that the (nonnormalized) rth mean curvature H_r of a hypersurface is the rth elementary symmetric polynomial of its principal curvatures, so that it constitutes a natural extension of the mean curvature ($r = 1$) and the Gauss–Kronecker curvature ($r = n$).

We focus on constructing and classifying H_r-hypersurfaces of products $M \times \mathbb{R}$. With this purpose, we use a special type of graph built on families of parallel hypersurfaces of M. In fact, for any given constant $H_r \geq 0$, we obtain H_r-graphs in $M \times \mathbb{R}$ for those Riemannian manifolds M which admit a local family of parallel hypersurfaces, each of them having constant principal curvatures. Following [4], such hypersurfaces are called *isoparametric*. We point out that many Riemannian manifolds M admit isoparametric hypersurfaces, such as space forms, hyperbolic spaces, warped products and $\mathbb{E}(\kappa, \tau)$ spaces.

By suitably "gluing" pieces of H_r-graphs, we construct properly embedded H_r-hypersurfaces in $M \times \mathbb{R}$ when M is either the standard n-sphere $\mathbb{S}^n$ or one of the hyperbolic spaces $\mathbb{H}^m_{\mathbb{F}}$ (rank one symmetric spaces of noncompact type). In this setting, we show that there exists a rotational H_r-sphere in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ if and only if $H_r > C_{\mathbb{F}}(r)$, where the constant $C_{\mathbb{F}}(r)$ is defined as the limit of the rth mean curvature of a geodesic sphere of $\mathbb{H}^m_{\mathbb{F}}$ as its radius goes to infinity. (In particular, $C_{\mathbb{F}}(r)$ is positive for $1 \leq r < n = \dim \mathbb{H}^m_{\mathbb{F}}$ and vanishes for $r = n$.) On the other hand, as we also show, for any $r \in \{1, \ldots, n\}$ and any constant $H_r > 0$, there exists a rotational H_r-sphere in $\mathbb{S}^n \times \mathbb{R}$.

We remark that rotational hypersurfaces of a general product $M \times \mathbb{R}$ are defined here as those which are foliated by horizontal geodesic spheres centered at an axis $\{o\} \times \mathbb{R}, o \in M$.

We provide other examples of properly embedded rotational $H_r(> 0)$-hypersurfaces in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ and in $\mathbb{S}^n \times \mathbb{R}$ as well, including Delaunay-type annuli and, in the case of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, entire graphs over $\mathbb{H}^m_{\mathbb{F}}$. Then, we classify those complete connected rotational $H_r(> 0)$ -hypersurfaces of these product spaces whose height functions are Morse type (i.e., have isolated critical points), which include all the properly embedded rotational $H_r(> 0)$-hypersurfaces we obtain here.

We also construct and classify complete connected $H_r(> 0)$-hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ with no horizontal points (critical points of the height function) which are invariant by either parabolic or hyperbolic isometries of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$

(in the latter case, only the real hyperbolic space $\mathbb{H}^m_{\mathbb{R}} := \mathbb{H}^n$ is considered).

Our methods work equally well for H_r-hypersurfaces with $H_r = 0$, the so called *r-minimal* hypersurfaces. By applying them, we obtain a one-parameter family of rotational, properly embedded catenoid-type r-minimal n-annuli in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$. Similarly, we obtain a one-parameter family of rotational, properly embedded Delaunay-type r-minimal n-annuli in $\mathbb{S}^n \times \mathbb{R}$. Then, we show that these annuli are the only complete connected r-minimal rotational hypersurfaces of these product spaces (besides horizontal hyperplanes and, in the case $r = n$, cylinders over geodesic spheres).

Analogously to the case of $H_r(> 0)$-hypersurfaces, we construct and classify the complete connected r-minimal hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which are invariant by either parabolic or hyperbolic isometries of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$.

The study of H_r-hypersurfaces of a Riemannian manifold leads naturally to considerations on their uniqueness properties. On this matter, Montiel and Ros [23] (see also [21]) showed the following Alexandrov-type theorem:

The only compact, connected and embedded H_r-hypersurfaces in $\mathbb{R}^n$, $\mathbb{H}^n$, or an open hemisphere of $\mathbb{S}^n$ are geodesic spheres.

In [15], this result was extended to the context of H_r-hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$, where the geodesic spheres in the statement are replaced by rotational spheres.

Here, we establish uniqueness results for rotational H_r-spheres of $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$, $n \geq 3$. The case $n = 2$ was settled in [1] (for $r = 1$) and in [17] (for $r = 2$). More precisely, we show that, for $n \geq 3$, any compact connected strictly convex H_r-hypersurface Σ of either $\mathbb{H}^n \times \mathbb{R}$ or $\mathbb{S}^n \times \mathbb{R}$ is necessarily an embedded rotational sphere. Assuming Σ complete, instead of compact, the same conclusion holds if, in addition, the height function of Σ has a critical point and, in the case $\Sigma \subset \mathbb{H}^n \times \mathbb{R}$, the least principal curvature of Σ is bounded away from zero. Finally, we show that, for $n \geq 3$, any connected, properly immersed and strictly convex $H_r(> 0)$-hypersurface of $\mathbb{S}^n \times \mathbb{R}$ is necessarily an embedded rotational H_r -sphere.

It is worth mentioning that these uniqueness results constitute applications of the main theorems in [8], which concern convexity properties of hypersurfaces in $M \times \mathbb{R}$, M being either a Hadamard manifold or the sphere $\mathbb{S}^n$. Besides, the noncompact cases are based on height estimates we establish here for strictly convex vertical graphs in arbitrary products $M \times \mathbb{R}$.

The paper is organized as follows. In Sect. 2, we set notation and some basic concepts. In Sect. 3, we introduce graphs on parallel hypersurfaces and establish two key lemmas. In Sect. 4 (resp. Sect. 5), we construct and classify complete rotational $H_r(> 0)$-hypersurfaces (resp. r-minimal hypersurfaces) in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$, whereas in Sect. 6 (resp. Sect. 7) we do the same for complete translational ones, i.e, invariant by either parabolic or hyperbolic isometries. In Sect. 8, we prove the aforementioned uniqueness results for rotational H_r-spheres of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$.

2 Preliminaries

Let Σ be an oriented hypersurface of a Riemannian manifold $\overline{M}^{n+1}$, $n \geq 2$. Set $\overline{\nabla}$ for the Levi-Civita connection of $\overline{M}$, N for the unit normal field of Σ and A for its shape operator with respect to N, so that

$$AX = -\overline{\nabla}_X N, \;\; X \in T\Sigma,$$

where $T\Sigma$ stand for the tangent bundle of Σ. The principal curvatures of Σ, that is, the eigenvalues of the shape operator A, will be denoted by $k_1, \dots, k_n$.

Given an integer $r \geq 0$, we define the (nonnormalized) rth *mean curvature* H_r of the hypersurface $\Sigma \subset \overline{M}$ as:

$$H_r := \begin{cases} 1 & \text{if } r = 0. \\ \displaystyle\sum_{i_1 < \cdots < i_r} k_{i_1} \dots k_{i_r} & \text{if } 1 \leq r \leq n. \\ 0 & \text{if } r > n. \end{cases} \tag{1}$$

Notice that H_1 and H_n are the nonnormalized mean curvature and Gauss–Kronecker curvature functions of Σ, respectively, i.e.,

$$H_1 = \operatorname{trace} A \quad \text{and} \quad H_n = \det A.$$

Definition 1 With the above notation, given a constant $H_r \in \mathbb{R}$, we say that $\Sigma \subset \overline{M}$ is an *H_r-hypersurface* of $\overline{M}$ if its rth mean curvature is constant and equal to H_r. In the case $H_r = 0$, we say that Σ is an *r-minimal* hypersurface of $\overline{M}$.

Definition 2 A hypersurface $\Sigma \subset \overline{M}$ is said to be *convex* at $x \in \Sigma$ if, at this point, all the nonzero principal curvatures have the same sign. If, in addition, none of these principal curvatures is zero, then Σ is said to be *strictly convex* at x. We call Σ *convex* (resp. *strictly convex*) if it is convex (resp. strictly convex) at all of its points.

In some of our proofs, we shall consider the tangency principle obtained in [18]. Roughly speaking, this principle asserts that two H_r-hypersurfaces Σ_1 and Σ_2 which are tangent at a point—with one "above" the other in a neighborhood V of this point—must coincide on V, provided that one of them is strictly convex at one of its points (see [15, 18] for details). If, in addition, Σ_1 and Σ_2 are complete, then the continuation principle applies and gives that $\Sigma_1 = \Sigma_2$. Essentially, these tangency and continuation principles are due to the fact that the equation for prescribed rth mean curvature is elliptic (cf. [7, 27]).

2.1 Hypersurfaces of $M \times \mathbb{R}$

The ambient spaces we shall consider are the products $\overline{M}^{n+1} = M^n \times \mathbb{R}$—where M^n is some Riemannian manifold—endowed with the standard product metric:

$$\langle\, , \rangle = \langle\, , \rangle_M + dt^2.$$

In this setting, we denote by ∂_t the gradient of the projection $\pi_{\mathbb{R}}$ of $M \times \mathbb{R}$ on its second factor. Clearly, ∂_t is parallel on $M \times \mathbb{R}$, that is,

$$\overline{\nabla}_X \partial_t = 0 \ \ \forall X \in T(M \times \mathbb{R}). \tag{2}$$

Let Σ be a hypersurface of $M \times \mathbb{R}$. Its *height function* ξ and its *angle function* Θ are defined by the following identities:

$$\xi := \pi_{\mathbb{R}}|_{\Sigma} \quad \text{and} \quad \Theta(x) := \langle N(x), \partial_t \rangle, \ \ x \in \Sigma.$$

A critical point of ξ is called *horizontal*, whereas a point on which Θ vanishes is called *vertical*. Notice that $x \in \Sigma$ is horizontal if and only if $\Theta(x) = \pm 1$.

We shall denote the gradient field and the Hessian of a function ζ on Σ by $\nabla\zeta$ and $\operatorname{Hess}\zeta$, respectively, that is,

$$\operatorname{Hess}\zeta(X, Y) := \langle \overline{\nabla}_X \nabla\zeta, Y \rangle \ \ \forall X, Y \in T(\Sigma).$$

It is easily checked that

$$\nabla\xi = \partial_t - \Theta N \quad \text{and} \quad \nabla\Theta = -A\nabla\xi. \tag{3}$$

Also, from (2) and (3), one has $\overline{\nabla}_X \nabla\xi = -(\Theta\overline{\nabla}_X N + X(\Theta)N)$. Hence,

$$\operatorname{Hess} \xi(X, Y) = \Theta \langle AX, Y \rangle \;\; \forall X, Y \in T\Sigma. \tag{4}$$

Given $t \in \mathbb{R}$, the set $P_t := M \times \{t\}$ is called a *horizontal hyperplane* of $M \times \mathbb{R}$. Horizontal hyperplanes are all isometric to M and totally geodesic in $M \times \mathbb{R}$. In this context, we call a transversal intersection $\Sigma_t := \Sigma \pitchfork P_t$ a *horizontal section* of Σ. Any horizontal section Σ_t is a hypersurface of P_t. So, at any point $x \in \Sigma_t \subset \Sigma$, the tangent space $T_x\Sigma$ of Σ at x splits as the orthogonal sum

$$T_x\Sigma = T_x\Sigma_t \oplus \operatorname{Span}\{\nabla \xi\}. \tag{5}$$

We will denote by $\mathbb{Q}^n_\epsilon$ the simply connected space form of constant sectional curvature $\epsilon \in \{0, 1, -1\}$: the Euclidean space $\mathbb{R}^n$ ($\epsilon = 0$), the unit sphere $\mathbb{S}^n$ ($\epsilon = 1$), and the hyperbolic space $\mathbb{H}^n$ ($\epsilon = -1$).

2.2 The hyperbolic spaces $\mathbb{H}^m_{\mathbb{F}}$

Mostly, the first factor manifold M of the Riemannian products $M \times \mathbb{R}$ considered here is either the unit n-sphere $\mathbb{S}^n$ or one of the Riemannian manifolds known as *hyperbolic spaces* (which include the canonical n-dimensional hyperbolic space $\mathbb{H}^n$ of constant sectional curvature -1).

The hyperbolic spaces are precisely the rank one symmetric spaces of noncompact type, which can be described through the four normed division algebras: $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{K}$ (quaternions) and $\mathbb{O}$ (octonions). They are denoted by $\mathbb{H}^m_{\mathbb{R}}$, $\mathbb{H}^m_{\mathbb{C}}$, $\mathbb{H}^m_{\mathbb{K}}$ and $\mathbb{H}^2_{\mathbb{O}}$ and called *real hyperbolic space, complex hyperbolic space, quaternionic hyperbolic space* and *Cayley hyperbolic plane,* respectively.

We will adopt the unified notation $\mathbb{H}^m_{\mathbb{F}}$ for the hyperbolic spaces, where $m = 2$ for $\mathbb{F} = \mathbb{O}$. The real dimension of $\mathbb{H}^m_{\mathbb{F}}$ is $n = m \dim \mathbb{F}$. In particular, $\mathbb{H}^2_{\mathbb{O}}$ has dimension $n = 16$. We will keep the standard notation $\mathbb{H}^n$ for the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$.

We remark that, being symmetric, the hyperbolic spaces $\mathbb{H}^m_{\mathbb{F}}$ are homogeneous. In addition, they are included in a distinguished class of Lie groups known as *Damek–Ricci spaces* (see Example 3 in the next section). In this context, it can be shown that any hyperbolic space $\mathbb{H}^m_{\mathbb{F}}$ is a Hadamard–Einstein manifold with nonconstant (except for $\mathbb{H}^n$) sectional curvatures pinched between -1 and $-1/4$ (cf. [4, Sects. 4.1.9 and 4.2]).

3 H_r-graphs on parallel hypersurfaces

In this section, we give a detailed description of graphs in $M \times \mathbb{R}$ which are built on families of parallel hypersurfaces of M. As we mentioned before, they will constitute our main tool for constructing H_r-hypersurfaces in product spaces $M \times \mathbb{R}$.

With this purpose, consider an isometric immersion

$$f : M_0^{n-1} \to M^n$$

between two Riemannian manifolds M_0^{n-1} and M^n, and suppose that there exists a neighborhood $\mathscr{U}$ of M_0 in $TM_0^\perp$ without focal points of f, that is, the restriction of the normal exponential map $\exp^\perp_{M_0} : TM_0^\perp \to M$ to $\mathscr{U}$ is a diffeomorphism onto its image. In this case, denoting by η the unit normal field of f, there is an open interval $I \ni 0$ such that, for all $p \in M_0$, the curve

$$\gamma_p(s) = \exp_{_M}(f(p), s\eta(p)), \; s \in I, \tag{6}$$

is a well-defined geodesic of M without conjugate points. Thus, for all $s \in I$,

$$\begin{array}{rcl} f_s : M_0 & \to & M \\ p & \mapsto & \gamma_p(s) \end{array}$$

is an immersion of M_0 into M, which is said to be *parallel* to f. Observe that, given $p \in M_0$, the tangent space $f_{s_*}(T_pM_0)$ of f_s at p is the parallel transport of $f_*(T_pM_0)$ along γ_p from 0 to s. We also remark that, with the induced metric, the unit normal η_s of f_s at p is given by

$$\eta_s(p) = \gamma_p'(s).$$

Definition 3 Let $\phi : I \to \phi(I) \subset \mathbb{R}$ be an increasing diffeomorphism, i.e., $\phi' > 0$. With the above notation, we call the set

$$\Sigma := \{(f_s(p), \phi(s)) \in M \times \mathbb{R} \,; p \in M_0, \; s \in I\}, \tag{7}$$

the *graph* determined by $\{f_s \,; s \in I\}$ and ϕ, or (f_s, ϕ)-*graph*, for short.

We remark that, for a given (f_s, ϕ)-graph Σ, and for any $s \in I$, $f_s(M_0)$ is the projection on M of the horizontal section $\Sigma_{\phi(s)} \subset \Sigma$, that is, these sets are the level hypersurfaces of Σ.

For an arbitrary point $x = (f_s(p), \phi(s))$ of such a graph Σ, one has

$$T_x\Sigma = f_{s_*}(T_pM_0) \oplus \text{Span}\,\{\partial_s\}, \quad \partial_s = \eta_s + \phi'(s)\partial_t.$$

So, a unit normal to Σ is

$$N = \frac{-\phi'}{\sqrt{1+(\phi')^2}}\eta_s + \frac{1}{\sqrt{1+(\phi')^2}}\partial_t \,. \tag{8}$$

In particular, its angle function is

$$\Theta = \frac{1}{\sqrt{1+(\phi')^2}} \cdot \tag{9}$$

A key property of (f_s, ϕ)-graphs is that the trajectories of $\nabla\xi$ on them are lines of curvature, that is, $\nabla\xi$ is one of its principal directions. (Notice that, by (9), $0 < \Theta < 1$, so $\nabla\xi$ never vanishes on an (f_s, ϕ)-graph.) More precisely (cf. [9, 31]),

$$A\nabla\xi = \frac{\phi''}{(\sqrt{1+(\phi')^2})^3}\nabla\xi. \tag{10}$$

We point out that, besides being lines of curvature, the trajectories of $\nabla\xi$ on an (f_s, ϕ)-graph Σ, when properly reparametrized, are also geodesics. This follows from the fact that Θ, and consequently $\|\nabla\xi\|$, is constant along the horizontal sections of Σ (see [31, Lemma 5]). It should also be noticed that these trajectories project on the geodesics $\gamma_p = \gamma_p(s)$ given by (6) (Fig. 1).

Let us compute now the principal curvatures of an (f_s, ϕ)-graph Σ. For that, let $\{X_1, \dots, X_n\}$ be the orthonormal frame of principal directions of Σ in which $X_n = \nabla\xi/\|\nabla\xi\|$. In this case,

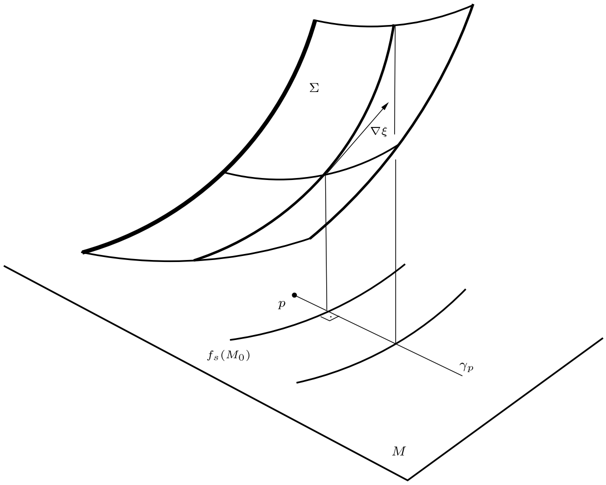


Fig. 1 Trajectory of $\nabla\xi$ on an (f_s, ϕ)-graph

for $1 \leq i \leq n-1$, the fields X_i are all horizontal, that is, tangent to M (cf. (5)). Therefore, setting

$$\varrho := \frac{\phi'}{\sqrt{1+(\phi')^2}} \tag{11}$$

and considering (8), we have, for all $i = 1, \dots, n-1$, that

$$k_i = \langle AX_i, X_i\rangle = -\langle \overline{\nabla}_{X_i} N, X_i\rangle = \varrho\langle \overline{\nabla}_{X_i}\eta_s, X_i\rangle = -\varrho k_i^s,$$

where k_i^s is the ith principal curvature of f_s. Also, it follows from (10) that $k_n = \varrho'$. Thus, the array of principal curvatures of the (f_s, ϕ)-graph Σ is

$$k_i = -\varrho k_i^s \ (1 \leq i \leq n-1) \quad \text{and} \quad k_n = \varrho'. \tag{12}$$

Now, considering the above identities and writing, for $1 \leq r \leq n$,

$$H_r = \sum_{i_1<\cdots<i_r \neq n} k_{i_1} \dots k_{i_r} + \sum_{i_1<\cdots<i_{r-1}} k_{i_1} \dots k_{i_{r-1}} k_n\,,$$

we have that the rth mean curvature of the (f_s, ϕ)-graph Σ is

$$H_r = (-1)^r H_r{}^s \varrho^r + (-1)^{r-1} H_{r-1}^s \varrho^{r-1}\varrho', \tag{13}$$

where $H_r{}^s$ denotes the rth mean curvature of f_s.

Due to equality (13), the function defined in (11)—to be called the *ϱ-function* of the (f_s, ϕ)-graph Σ—will play a major role in the sequel. We remark that, up to a constant, the ϱ-function of Σ determines its ϕ-function. Indeed, it follows from equality (11) that

$$\phi(s) = \int_{s_0}^{s} \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}}du + \phi(s_0), \quad s_0 \in I. \tag{14}$$

We introduce now a special type of family of parallel hypersurfaces which will be used for constructing H_r-hypersurfaces in $M \times \mathbb{R}$.

Definition 4 Following [4], we call a family of parallel hypersurfaces

$$\mathscr{F} := \{f_s : M_0 \to M\,; s \in I\}$$

isoparametric if, for each $s \in I$, any principal curvature k_i of f_s is constant (possibly depending on i and s). If so, each hypersurface f_s is also called *isoparametric*.

We should mention that there is some mismatch regarding the nomenclature for isoparametric hypersurfaces. In some contexts, isoparametric hypersurfaces are defined as those which, together with its parallel nearby hypersurfaces, have constant mean curvature. It is shown that some manifolds M admit hypersurfaces which are isoparametric in this sense and nonisoparametric as we defined.

Let Σ be an (f_s, ϕ)-graph such that the family $\mathscr{F} = \{f_s\,; s \in I\}$ is isoparametric. Then, for any $r = 1, \dots, n-1$, the rth mean curvature of f_s is a function of s alone, which we assume to be no vanishing. In this setting, writing $\tau := \varrho^r$, and considering (13) with H_r constant, we obtain the following result, which turns out to be our main lemma.

Lemma 1 *Let $\mathscr{F} = \{f_s : M_0 \to M\,; s \in I\}$ be an isoparametric family of hypersurfaces whose $r(< n)$-mean curvatures ${H_r}^s$ never vanish. Given $r \in \{1, \dots, n\}$ and $H_r \in \mathbb{R}$, let τ be a solution of the first-order differential equation*

$$y' = a(s)y + b(s), \quad s \in I, \tag{15}$$

where the coefficients a and b are the functions

$$a(s) := \frac{r{H_r}^s}{H^s_{r-1}} \quad \text{and} \quad b(s) := \frac{(-1)^{r-1} r H_r}{H^s_{r-1}} \cdot \tag{16}$$

Then, if $0 < \tau < 1$, the (f_s, ϕ)-graph Σ with ϱ-function $\tau^{1/r}$ is an H_r-hypersurface of the product $M \times \mathbb{R}$. Conversely, if an (f_s, ϕ)-graph Σ has constant rth mean curvature H_r, then $\tau := \varrho^r$ is a solution of (15).

Regarding Eq. (15), recall that its general solution is

$$\tau(s) = \frac{1}{\mu(s)}\left(\tau_0 + \int_{s_0}^{s} b(u)\mu(u)du\right), \; s_0, s \in I, \, \tau_0 \in \mathbb{R}, \tag{17}$$

where μ is the exponential function

$$\mu(s) = \exp\left(-\int_{s_0}^{s} a(u)du\right), \quad s \in I.$$

It follows from Lemma 1 that, as long as M admits isoparametric hypersurfaces with non-vanishing rth mean curvature, for any $H_r \in \mathbb{R}$, there exist H_r-graphs in $M \times \mathbb{R}$. (Notice that the interval I and the constant τ_0 in (17) can be chosen in such a way that the corresponding solution τ of (15) satisfies $0 < \tau < 1$.) This includes, as a trivial case, the Euclidean space $\mathbb{R}^n$. In the next examples, we shall consider other manifolds M to which Lemma 1 applies.

Example 1 (*sphere* $\mathbb{S}^n$) It is a well-known fact that isoparametric hypersurfaces in $\mathbb{S}^n$ are abundant and include all its geodesic spheres (see, e.g., [12]).

Example 2 (*warped products*) Let $M = I \times_\omega F^{n-1}$ be a warped product, where the basis $I \subset \mathbb{R}$ is an open interval and the fiber F is an arbitrary $(n-1)$-dimensional manifold. For each $s \in I$, define f_s as the standard immersion $F \hookrightarrow \{s\} \times_\omega F \subset M$. It is well known that, with the induced metric, $\mathscr{F} = \{f_s\,; s \in I\}$ is a family of parallel totally umbilical hypersurfaces of M with constant principal curvatures ω'/ω (see, e.g., [5]). In particular, $\mathscr{F}$ is isoparametric. Hence, if ω' never vanishes, Lemma 1 applies to M.

Example 3 (*Damek–Ricci spaces*) Let us consider the Riemannian manifolds known as *Damek–Ricci spaces*. These are Lie groups endowed with a left invariant metric with especial properties (see [4, 12]). For instance, all Damek–Ricci spaces are both Hadamard and Einstein manifolds. As we have mentioned, the hyperbolic spaces $\mathbb{H}^m_{\mathbb{F}}$ are Damek–Ricci spaces. In fact, they are the only ones which are symmetric. Their isoparametric hypersurfaces include their geodesic spheres, as well as their horospheres. Also, as shown in [10], there exist families of isoparametric hypersurfaces with nonvanishing rth curvatures in Damek–Ricci harmonic spaces.

Example 4 ($E(\kappa, \tau)$-*spaces*) In [13], it was proved that there exist isoparametric families of parallel surfaces with nonzero constant principal curvatures in $\mathbb{E}(k, \tau)$ spaces satisfying $k - 4\tau^2 \neq 0$. (Those include the products $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, the Heisenberg space Nil_3, the Berger spheres and the universal cover of the special linear group $\mathrm{SL}_2(\mathbb{R})$.)

In the next two sections, we construct properly embedded H_r-hypersurfaces in products $M \times \mathbb{R}$ by suitably "gluing" H_r-graphs. To this task, the following result will be considerably helpful.

Lemma 2 *Let* $\varrho : [a_1, a_2] \to \mathbb{R}$ *be a differentiable function which satisfies*:

$$0 < \varrho|_{(a_1,a_2)} < 1,$$

and consider the following conditions:

(i) $\varrho(a_2) = 1$ *and* $\varrho'(a_2) > 0$.
(ii) $\varrho(a_1) = 1$ *and* $\varrho'(a_1) < 0$.

Then, if (i) occurs, there exists $\delta > 0$ such that the improper integral

$$\int_{a_2-\delta}^{a_2} \frac{\varrho(s)ds}{\sqrt{1-\varrho^2(s)}} \tag{18}$$

is convergent. Analogously, the improper integral

$$\int_{a_1}^{a_1+\delta} \frac{\varrho(s)ds}{\sqrt{1-\varrho^2(s)}}$$

is convergent if (*ii*) *occurs*.

Proof Assume that (i) occurs. In this case, there exist positive constants, δ and C, such that $\varrho'(s) \geq C > 0\,\forall s \in (a_2-\delta, a_2)$. Therefore, since $0 < \varrho|_{(a_1,a_2)} < 1$,

$$\int_{a_2-\delta}^{a_2} \frac{\varrho(s)ds}{\sqrt{1-\varrho^2(s)}} \leq \int_{a_2-\delta}^{a_2} \frac{\varrho'(s)ds}{\varrho'(s)\sqrt{1-\varrho^2(s)}} \leq \frac{1}{C}\int_{\varrho(a_2-\delta)}^{1} \frac{d\varrho}{\sqrt{1-\varrho^2}}$$
$$=\frac{1}{C}\left(\frac{\pi}{2} - \arcsin(\varrho(a_2-\delta))\right) \leq \frac{\pi}{2C},$$

which proves the convergence of the integral (18). The case (ii) is analogous. □

4 Rotational $H_r(>0)$-hypersurfaces of $\mathbb{H}_{\mathbb{F}}^m \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$.

Rotational hypersurfaces in simply connected space forms $\mathbb{Q}_\epsilon^n$ or products $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ are among the most classical hypersurfaces of these spaces. In the case of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, they are obtained by rotating (with the aid of the group of isometries of $\mathbb{Q}_\epsilon^n$) a plane curve about an axis $\{o\} \times \mathbb{R}$, $o \in \mathbb{Q}_\epsilon^n$. Consequently, any connected component of any horizontal section Σ_t of a rotational hypersurface Σ in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ lies in a geodesic sphere of $\mathbb{Q}_\epsilon^n \times \{t\}$ with center at the axis. This fact suggests the following definition.

Definition 5 A hypersurface $\Sigma \subset M \times \mathbb{R}$ is called *rotational*, if there exists a fixed point $o \in M$ such that any connected component of any horizontal section Σ_t of Σ is contained in a geodesic sphere of $M \times \{t\}$ with center at $o \times \{t\}$. If so, the set $\{o\} \times \mathbb{R}$ is called the *axis* of Σ. In particular, any horizontal hyperplane $P_t := M \times \{t\}$ is a rotational hypersurface of $M \times \mathbb{R}$ with axis at any point $o \in M$.

In what follows, we construct and classify complete rotational $H_r(>0)$-hypersurfaces in $\mathbb{H}_{\mathbb{F}}^m \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$.

4.1 Rotational $H_r(>0)$-hypersurfaces of $\mathbb{H}_{\mathbb{F}}^m \times \mathbb{R}$

Let us consider a family

$$\mathscr{F} := \{f_s : \mathbb{S}^{n-1} \to \mathbb{H}_{\mathbb{F}}^m;\ s \in (0,+\infty)\} \tag{19}$$

of isoparametric concentric geodesic spheres of $\mathbb{H}_{\mathbb{F}}^m$ indexed by their radiuses, that is, for a fixed point $o \in \mathbb{H}_{\mathbb{F}}^m$, and for each $s \in (0,+\infty)$, $f_s(\mathbb{S}^{n-1})$ is the geodesic sphere $S_s(o)$ of $\mathbb{H}_{\mathbb{F}}^m$ with center at o and radius s.

We remark that any sphere $f_s \in \mathscr{F}$ is strictly convex. Also, in accordance to the notation of Sect. 3, for each $s \in (0,+\infty)$, we choose the outward orientation of f_s, so that *any principal curvature k_i^s of f_s is negative*. In this setting, the function $s \in (0,+\infty) \mapsto H_r^s$ is positive for r

even and negative for r odd. Hence, for any constant $H_r > 0$, the coefficients a and b in (16) are given by

$$a(s) = -\frac{r|H_r{}^s|}{|H^s_{r-1}|} \quad \text{and} \quad b(s) = \frac{rH_r}{|H^s_{r-1}|}\,. \tag{20}$$

The principal curvatures k_i^s of the geodesic spheres $f_s \in \mathscr{F}$ are ($n = \dim \mathbb{H}^m_{\mathbb{F}}$):

$$\begin{aligned} k_1^s &= -\frac{1}{2}\coth(s/2) \ \text{ with multiplicity } \ n-p-1 \\ k_2^s &= -\coth(s) \ \text{ with multiplicity } \ p \end{aligned}\,, \tag{21}$$

where $p = n-1$ for $\mathbb{H}^n$, $p = 1$ for $\mathbb{H}^m_{\mathbb{C}}$, $p = 3$ for $\mathbb{H}^m_{\mathbb{K}}$, and $p = 7$ for $\mathbb{H}^2_{\mathbb{O}}$ (see, e.g., [6, pgs. 353, 543] and [20]).

From equalities (21), we obtain the r-mean curvatures $H_r{}^s$ of the geodesic spheres f_s of $\mathbb{H}^m_{\mathbb{F}}$. For instance, in $\mathbb{H}^n$, $n \geq 2$, we have

$$H_r{}^s = (-1)^r \binom{n-1}{r} \coth^r(s) \quad (1 \leq r \leq n-1), \tag{22}$$

whereas for $\mathbb{H}^m_{\mathbb{C}}$, $m > 1$, one has

$$\begin{aligned} H_r{}^s =& \left(-\frac{1}{2}\right)^r \binom{n-2}{r} \coth^r(s/2) \\ &+ (-1)^r \left(\frac{1}{2}\right)^{r-1} \binom{n-2}{r-1} \coth^{r-1}(s/2)\coth(s) \end{aligned} \tag{23}$$

if $1 \leq r \leq n-2$, and

$$H^s_{n-1} = (-1)^{n-1}\coth(s)\coth^{n-2}(s/2)/2^{n-2}. \tag{24}$$

Analogously, one obtains the rth mean curvature functions $H_r{}^s$ for the other hyperbolic spaces. A direct computation from these data yields the following

Lemma 3 *The functions a e b defined in* (20) *have the following properties*:

(i) *a is negative and increasing for* $1 \leq r \leq n-1$, *and vanishes for* $r = n$.
(ii) *b is positive and increasing for* $1 < r \leq n$, *and* $b = H_1$ *for* $r = 1$.

In particular, we have the inequalities

$$a'(s) \geq 0, \quad b'(s) \geq 0, \quad \text{and} \quad a'(s) + b'(s) > 0 \ \ \forall s \in (0, +\infty). \tag{25}$$

We point out that, in the above setting, one has (cf. [22])

$$|H_r{}^s| = \binom{n-1}{r} s^{-r} + \mathcal{O}(s^{2-r}) \tag{26}$$

in a neighborhood of $s = 0$. In particular,

$$\lim_{s\to 0} |H_r{}^s| = +\infty. \tag{27}$$

In what follows, by means of the family $\mathscr{F}$, we will construct complete rotational H_r-hypersurfaces in $\mathbb{H}_{\mathbb{F}}^m \times \mathbb{R}$ which are made of pieces of (f_s, ϕ)-graphs. To this end, we will look for solutions $\tau(s)$ of the equation $y' = ay + b$ (with a and b as in (20)) satisfying suitable initial conditions. Let us recall that, in this context, the general solution of $y' = ay + b$ is

$$\begin{array}{l} \tau(s) = \dfrac{1}{\mu(s)}\left(\tau_0 + \displaystyle\int_{s_0}^{s} b(u)\mu(u)du\right) \\ \mu(s) = \exp\left(-\displaystyle\int_{s_0}^{s} a(u)du\right) \end{array}, \quad s_0\,, s \in (0,+\infty),\ \ \tau_0 \in \mathbb{R}. \tag{28}$$

Concerning the solutions $\tau(s)$, we will be also interested in those which can be extended to $s = 0$. Notice that, in principle, neither a nor b are defined at $s = 0$, which makes this point a singularity. However, the function b is easily extendable to $s = 0$. Indeed, we can set $b(0) = H_r$ if $r = 1$ and $b(0) = 0$ if $r > 1$ (from (27)). As for a, it follows from (26) that, for $1 \le r < n$,

$$\lim_{s\to 0} |a(s)| = +\infty \quad \text{and} \quad \lim_{s\to 0} s|a(s)| < +\infty, \tag{29}$$

which characterizes $s = 0$ as a regular singular point of $y' = ay + b$. This means that, despite the fact that a is not defined at $s = 0$, this equation has a nonnegative solution τ defined at $s = 0$ that satisfies $\tau(0) = 0$ (cf. [28, Theorem 3.1], [30, Lemma 4.4]). More precisely, this solution is

$$\tau(s) := \begin{cases} \frac{1}{\mu(s)} \int_0^s b(u)\mu(u)du & \text{if } s \in (0,+\infty) \\ 0 & \text{if } s = 0, \end{cases} \tag{30}$$

where μ is a solution of $y' + ay = 0$. Notice that the function τ defined in (30) is also the solution of $y' = ay + b$ in the case $r = n$, i.e., for $a = 0$. (Just set $\mu(s) = 1$.)

As we shall see, the geometry of the H_r-hypersurfaces we construct from (f_s, ϕ)-graphs is closely related to the growth of τ as $s \to +\infty$. Taking that into account, for a given family of parallel geodesic spheres $\mathscr{F}$ in $\mathbb{H}_{\mathbb{F}}^m$, we define

$$C_{\mathbb{F}}(r) := \lim_{s\to+\infty} |H_r{}^s|, \quad r = 1, \dots, n. \tag{31}$$

In particular, $C_{\mathbb{F}}(n) = 0$. Notice that, since $\mathbb{H}_{\mathbb{F}}^m$ is homogeneous, the constant $C_{\mathbb{F}}(r)$ is well defined, that is, it does not depend on the family $\mathscr{F}$ of geodesic spheres.

It follows from equalities (22)–(24) that

(i) $C_{\mathbb{R}}(r) = \binom{n-1}{r} \quad (1 \le r \le n-1)$.
(ii) $C_{\mathbb{C}}(r) = \left(\frac{1}{2}\right)^r \binom{n-2}{r} + \left(\frac{1}{2}\right)^{r-1} \binom{n-2}{r-1} \quad (1 \le r \le n-2)$.
(iii) $C_{\mathbb{C}}(n-1) = \frac{1}{2^{n-2}}\cdot$

Similarly, one can compute the other constants $C_{\mathbb{F}}(r)$ and easily conclude that

$$C_{\mathbb{F}}(r) > 0 \ \ \forall r \in \{1, \dots, n-1\}.$$

The next proposition shows the relation between the solution τ of $y' = ay + b$ and the constant $C_{\mathbb{F}}(r)$. Notice that, for $1 \leq r \leq n-1$, the identities (20) yield:

$$\lim_{s\to+\infty} \frac{-b(s)}{a(s)} = \frac{H_r}{C_{\mathbb{F}}(r)} \cdot \tag{32}$$

Proposition 1 *The following assertions hold*:

(i) *The solution* τ *defined in* (30) *is increasing*, *i.e.*, $\tau' > 0$ *in* $(0, +\infty)$.
(ii) *Both the solutions* τ *in* (28) *and* (30) *satisfy the following equality*:

$$\lim_{s\to+\infty} \tau(s) = \begin{cases} +\infty & \text{if } r = n. \\ H_r/C_{\mathbb{F}}(r) & \text{if } 1 \leq r \leq n-1. \end{cases} \tag{33}$$

Proof To prove (i), let us first observe that, since the solution τ in (30) is positive in $(0, +\infty)$ and $\tau(0) = 0$, we have that τ is increasing near 0. Assume that τ is not increasing in $(0, +\infty)$. In this case, τ has a first critical point s_0 in $(0, +\infty)$ which is necessarily a local maximum. However, considering (25) and the equality $\tau' = a\tau + b$, we have

$$\tau''(s_0) = a'(s_0)\tau(s_0) + a(s_0)\tau'(s_0) + b'(s_0) = a'(s_0)\tau(s_0) + b'(s_0) > 0,$$

which implies that s_0 is a local minimum—a contradiction. Therefore, τ is increasing in $(0, +\infty)$, which proves (i).

Suppose that τ is as in (28). Since b is increasing, if $r = n$ one has

$$\tau(s) = \tau_0 + \int_{s_0}^{s} b(u)du \geq \tau_0 + b(s_0)(s - s_0),$$

which implies that $\tau(s) \to +\infty$ as $s \to +\infty$.

Now, assume $1 \leq r \leq n-1$. We claim that, in this case, $\mu(s) \to +\infty$ as $s \to +\infty$. Indeed, for any fixed $s_0 > 0$, and $s > s_0$,

$$|a(s)| = \frac{r|H_r{}^s|}{|H^s_{r-1}|} > \frac{rC_{\mathbb{F}}(r)}{|H^{s_0}_{r-1}|} > 0.$$

Since $|a|$ is decreasing, this inequality gives that $\inf |a| > 0$ in $(0, +\infty)$. Hence,

$$\mu(s) = e^{-\int_{s_0}^{s} a(u)du} = e^{\int_{s_0}^{s} |a(u)|du} > e^{(\inf |a|)(s-s_0)} \ \forall s > s_0,$$

which clearly implies the claim on μ.

From the expression of τ, we have (apply l'Hôpital to the second summand):

$$\lim_{s\to+\infty} \tau(s) = \lim_{s\to+\infty} \frac{\tau_0}{\mu(s)} + \lim_{s\to+\infty} \frac{b(s)\mu(s)}{\mu'(s)} = \lim_{s\to+\infty} \frac{b(s)\mu(s)}{-a(s)\mu(s)} = \frac{H_r}{C_{\mathbb{F}}(r)},$$

where the last equality comes from (32). This finishes the proof of (33) when τ is defined as in (28). The proof for the solution τ in (30) is completely analogous. □

Now, it is clear from Proposition 1 that a solution τ as in (30) reaches the value 1 if and only if $H_r > C_{\mathbb{F}}(r)$. More specifically, we have the following

Corollary 1 *Let τ be as in (30). Then, there exists $s_0 \in (0, +\infty)$ satisfying*

$$0 < \tau|_{(0,s_0)} < 1 \quad \text{and} \quad \tau(s_0) = 1$$

if and only if $H_r > C_{\mathbb{F}}(r)$ (Fig. 2a). Consequently, for $1 \le r \le n-1$ and any constant $H_r \in (0, C_{\mathbb{F}}(r))$, the following inequality holds (Fig. 2b):

$$0 < \tau(s) < 1 \ \ \forall s \in (0, +\infty).$$

Now, we are in position to establish our first existence result.

Theorem 1 *Given $r \in \{1, \dots, n\}$ and a constant $H_r > 0$, the following hold:*

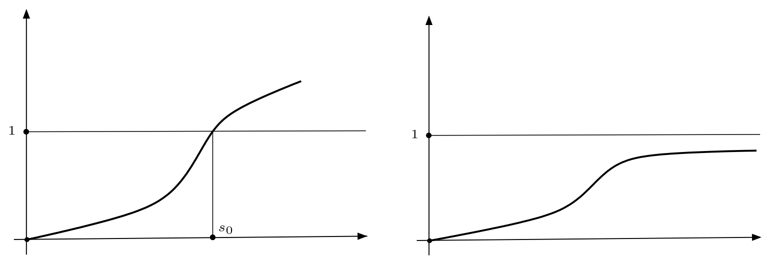
- (i) *If $H_r > C_{\mathbb{F}}(r)$, there exists an embedded strictly convex rotational H_r-sphere in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which is symmetric with respect to a horizontal hyperplane.*
- (ii) *If $0 < H_r \le C_{\mathbb{F}}(r)$, there exists an entire strictly convex rotational H_r-graph in $\mathbb{H}^m_{\mathbb{F}} \times [0, +\infty)$ which is tangent to $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$ at a single point, and whose height function is unbounded above. Consequently, there are no compact H_r-hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ for such values of H_r.*

Proof Let $\mathscr{F}$ be an arbitrary family of parallel geodesic spheres of $\mathbb{H}^m_{\mathbb{F}}$ as in (19). Consider the functions a and b defined in (20) and let τ be the solution (30) of the ODE $y' = ay + b$.

If $H_r > C_{\mathbb{F}}(r)$, we have from Corollary 1 that there exists $s_0 \in (0, +\infty)$ satisfying

$$\tau(0) = 0 < \tau|_{(0,s_0)} < 1 = \tau(s_0).$$

Hence, by Lemma 1, the (f_s, ϕ)-graph Σ' with ϱ-function $\varrho := \sqrt[r]{\tau|_{[0,s_0)}}$ is a rotational H_r-graph of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ over the open ball $B_{s_0}(o) \subseteq \mathbb{H}^m_{\mathbb{F}}$ such that



(a) $H_r > C_{\mathbb{F}}(r)$ **(b)** $H_r \le C_{\mathbb{F}}(r)$

Fig. 2 Graphs of τ (as in (30)) according to the sign of $H_r - C_{\mathbb{F}}(r)$

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}}du, \quad s \in [0, s_0). \tag{34}$$

From Proposition 1-(i), one has $\tau'(s_0) > 0$, which implies that $\varrho'(s_0) > 0$. Hence, by Lemma 2, ϕ extends to s_0 , i.e.,

$$\phi(s_0) := \lim_{s \to s_0} \phi(s)$$

is well defined. In particular, $\partial\Sigma' = S_{s_0} \times \{\phi(s_0)\}$.

Notice that $o \in \Sigma'$ is an isolated minimum of the height function ξ of Σ'. Thus, Σ' is strictly convex at o. In addition, by the identities (12), at any point of $\Sigma' - \{o\}$, all the principal curvatures are positive. Therefore, Σ' is strictly convex.

As we know, the angle function Θ of Σ' is given by

$$\Theta = \frac{1}{\sqrt{1+(\phi')^2}} \cdot \tag{35}$$

Since $\varrho(s_0) = 1$, we have from (34) that $\phi'(s) \to +\infty$ as $s \to s_0$. This, together with (35), implies that the tangent spaces of Σ' along $\partial\Sigma'$ are vertical. Hence, the trajectories of $\nabla\xi$ all emanate from o and meet $\partial\Sigma'$ orthogonally (Fig. 3). Recall that these trajectories are geodesics which foliate Σ'.

Now, set Σ'' for the reflection of Σ' with respect to $\mathbb{H}^m_{\mathbb{F}} \times \{\phi(s_0)\}$ and define

$$\Sigma := \text{closure}\,(\Sigma') \cup \text{closure}\,(\Sigma''),$$

that is, Σ is the "gluing" of Σ' and Σ'' along the $(n-1)$-sphere $S_{s_0}(o) \times \{\phi(s_0)\}$. Since the tangent spaces along $S_{s_0}(o) \times \{\phi(s_0)\}$ are all vertical, we have that Σ is a well-defined rotational strictly convex H_r-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which is homeomorphic to the n-sphere $\mathbb{S}^n$ and is symmetric with respect to $\mathbb{H}^m_{\mathbb{F}} \times \{\phi(s_0)\}$. This proves (i).

Under the hypotheses in (ii), it follows from Corollary 1 that τ satisfies:

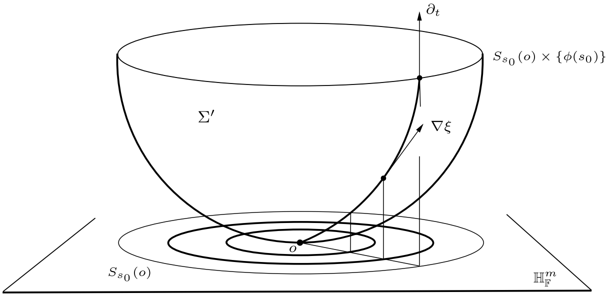


Fig. 3 Trajectories of $\nabla\xi$ on Σ' emanate from o and meet $\partial\Sigma'$ orthogonally

$$\tau(0) = 0 < \tau|_{(0,+\infty)} < 1,$$

so that the (f_s, ϕ)-graph Σ with ϱ-function $\varrho := \sqrt[r]{\tau|_{[0,+\infty)}}$ is an entire rotational H_r-graph of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ over $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$. Since $\phi(0) = 0$ and $\phi(s) > 0$ for any $s > 0$, Σ is contained in the closed half-space $\mathbb{H}^m_{\mathbb{F}} \times [0, +\infty)$ and is tangent to $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$ at o. Also, the height function of Σ is unbounded above. Indeed, from Proposition 1-(i), τ, and so ϱ, is increasing. Thus, for a fixed $\delta > 0$, and any $s > \delta$, one has

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du \geq \int_\delta^s \varrho(u) du \geq \varrho(\delta)(s - \delta),$$

which implies that ϕ is unbounded above. Also, arguing as for the graph Σ' in the preceding paragraphs, we conclude that Σ is strictly convex.

Observe that the mean curvature vector of Σ "points upwards," that is, its mean convex side Λ is the connected component of $(\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}) - \Sigma$ which contains the axis $\{o\} \times \mathbb{R}$. In particular, Λ is foliated by the balls $B_s(o) \times \{\phi(s)\},\ s \in (0, +\infty)$.

Let us suppose that there exists a compact H_r-hypersurface $\widetilde{\Sigma}$ such that $0 < H_r \leq C_{\mathbb{F}}(r)$. Considering the fact that Λ is "horizontally and vertically unbounded," it is easily seen that, after a suitable vertical translation, we can assume $\widetilde{\Sigma} \subset \Lambda$ (Fig. 4). Now, translate $\widetilde{\Sigma}$ downward until it has a first contact with Σ. Since Σ is strictly convex, the tangency principle applies (cf. Sect. 2) and gives that Σ coincides with $\widetilde{\Sigma}$, which is clearly impossible. This shows that such a $\widetilde{\Sigma}$ cannot exist and finishes the proof of (ii). □

Remark 1 Let $\Sigma \subset \mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ be an entire H_r-graph as in Theorem 1-(ii). Since $C_{\mathbb{F}}(n) = 0$, we must have $r < n$. Also, the associated function $\tau : [0, +\infty) \to \mathbb{R}$ is positive, bounded and increasing, so that

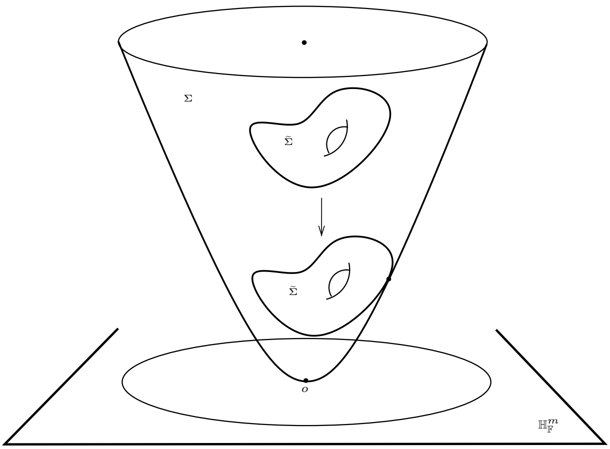


Fig. 4 After a downward translation of $\tilde{\Sigma}$, it has a contact with Σ

$$\lim_{s \to +\infty} \tau'(s) = 0,$$

which implies that $\varrho'(s) \to 0$ as $s \to +\infty$. This, together with (12), gives that the principal curvature $k_n = \varrho'(s)$ goes to zero as $s \to +\infty$. In particular, *the least principal curvature function of Σ is not bounded away from zero*.

Next, we apply the method of (f_s, ϕ)-graphs to produce one-parameter families of $H_r(>0)$-annuli in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, $1 \le r < n$. For that, fix $s_0 = \lambda > 0$ and consider the solution $\tau(s)$ of $y' = ay + b$ given in (28), which satisfies the initial condition $\tau_0 = \tau(\lambda) = 1$. Since we must have $0 < \tau < 1$, we look at those values of λ for which $\tau'(\lambda) < 0$. From (20), we have

$$\tau'(\lambda) = a(\lambda) + b(\lambda) = \frac{r(H_r - |{H_r}^\lambda|)}{|H^\lambda_{r-1}|}, \tag{36}$$

so that $\tau'(\lambda) < 0$ if and only if $|{H_r}^\lambda| > H_r$. This (and equality 27) suggest us to define, for $1 \le r < n$ and a given $H_r > 0$, the following constant:

$$\delta_{H_r} := \sup\{\lambda > 0\,;\, |{H_r}^\lambda| > H_r\}. \tag{37}$$

In this setting, we have the following result.

Theorem 2 *Given $r \in \{1, \dots, n-1\}$ and $H_r > 0$, let $\delta_{H_r} > 0$ be as in (37). Then, there exists a one-parameter family*

$$\mathscr{S} = \{\Sigma(\lambda)\,;\, \lambda \in (0, \delta_{H_r})\}$$

of properly embedded rotational H_r-hypersurfaces in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which are all homeomorphic to the n-annulus $\mathbb{S}^{n-1} \times \mathbb{R}$. In addition, the following assertions hold:

(i) *If $H_r > C_{\mathbb{F}}(r)$, each $\Sigma(\lambda) \in \mathscr{S}$ is Delaunay type, i.e., it is periodic in the vertical direction, and has unduloids as the trajectories of the gradient of its height function.*
(ii) *If $0 < H_r \le C_{\mathbb{F}}(r)$, each hypersurface $\Sigma(\lambda) \in \mathscr{S}$ is symmetric with respect to $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$ and has unbounded height function.*

Proof Given $\lambda \in (0, \delta_{H_r})$, let τ be the solution (28) such that $s_0 = \lambda$ and $\tau(\lambda) = 1$. By (36) and the definition of δ_{H_r}, one has $\tau'(\lambda) < 0$, so that τ is decreasing near λ.

It is clear from (28) that τ is positive in $(\lambda, +\infty)$. Thus, if $H_r > C_{\mathbb{F}}(r)$, it follows from (33) that there exists $\bar\lambda \in (\lambda, +\infty)$ such that (Fig. 5a)

$$0 < \tau|_{(\lambda,\bar\lambda)} < 1 = \tau(\bar\lambda) = \tau(\lambda).$$

Let us observe that a critical point s_1 of τ is necessarily a strict minimum, since $\tau''(s_1) = a'(s_1)\tau(s_1) + b'(s_1) > 0$. Therefore, τ must have a unique local minimum at a point between λ and $\bar\lambda$. In particular, $\tau'(\bar\lambda) > 0$.

Setting $\tau_\lambda := \tau|_{(\lambda,\bar\lambda)}$, it follows from the above considerations and Lemmas 1 and 2 that the (f_s, ϕ)-graph $\Sigma'(\lambda)$ with ϱ-function $\varrho(s) = \sqrt[r]{\tau_\lambda}$ is a bounded H_r-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$. Moreover, $\Sigma'(\lambda)$ is homeomorphic to $\mathbb{S}^{n-1} \times (\lambda, \bar\lambda)$ and has boundary (see Fig. 6):

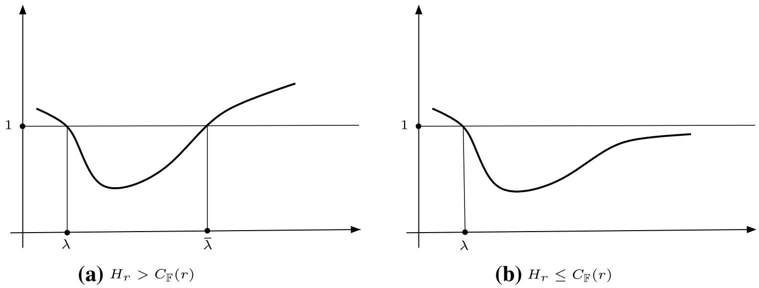


Fig. 5 Two types of solutions τ, as in (28), satisfying $\tau(\lambda) = 1$

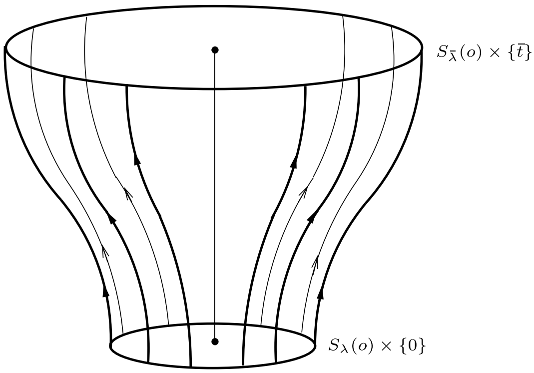


Fig. 6 "Block" of a Delaunay-type H_r-hypersurface in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$

$$\partial\Sigma'(\lambda) = (S_\lambda(o) \times \{0\}) \cup (S_{\bar{\lambda}}(o) \times \{\phi(\bar{\lambda})\}).$$

We also have that the tangent spaces of $\Sigma'(\lambda)$ are vertical along its boundary $\partial\Sigma'(\lambda)$, for $\varrho(\lambda) = \varrho(\bar{\lambda}) = 1$. Therefore, we obtain a properly embedded rotational H_r-hypersurface $\Sigma(\lambda)$ from $\Sigma'(\lambda)$ by continuously reflecting it with respect to the horizontal hyperplanes $\mathbb{H}^m_{\mathbb{F}} \times \{k\phi(\bar{\lambda})\}$, $k \in \mathbb{Z}$. This proves (i).

Now, let us assume $0 < H_r \le C_{\mathbb{F}}(r)$. In this case, (33) gives that (Fig. 5b)

$$0 < \tau|_{(\lambda,+\infty)} < 1,$$

so that the (f_s, ϕ)-graph $\Sigma'(\lambda)$ determined by $\varrho = \tau^{1/r}|_{(\lambda,+\infty)}$ is a rotational H_r-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ with boundary $\partial\Sigma'(\lambda) = S_\lambda(o) \times \{0\}$ (Fig. 7). By reflecting $\Sigma'(\lambda)$ with respect to $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$, as we did before, we obtain the embedded H_r-hypersurface $\Sigma(\lambda)$ as stated.

It remains to show that the height function of $\Sigma(\lambda)$ is unbounded. For that, we have just to observe that the infimum of τ in $[\lambda, +\infty)$ is positive, since τ itself is positive in this interval, and its limit as $s \to +\infty$ is $H_r/C_{\mathbb{F}}(r) > 0$. So, the same is true for $\varrho = \tau^{1/r}$. Therefore,

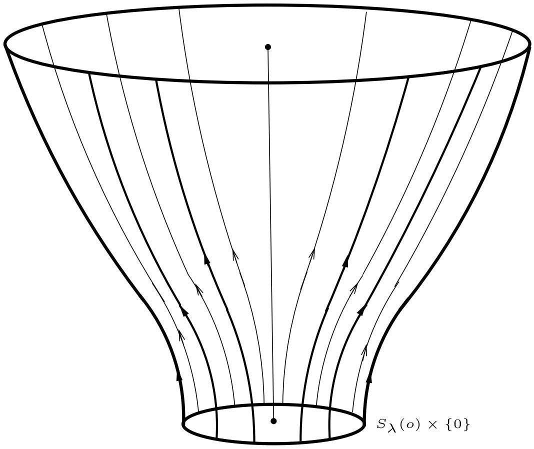


Fig. 7 (f_s, ϕ)-graph $\Sigma'(\lambda)$, on which all the trajectories of $\nabla\xi$ emanate from $\partial\Sigma'(\lambda)$ orthogonally

$$\phi(s) = \int_\lambda^s \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}}du > \int_\lambda^s \varrho(u)du > \inf \varrho|_{[\lambda,+\infty)}(s-\lambda),$$

from which we conclude that ϕ is unbounded. □

Remark 2 The case $\mathbb{F} = \mathbb{R}$ of Theorem 1 was previously established in [15], whereas the case $\mathbb{F} = \mathbb{R}$ and $r = 1$ of Theorem 2 was considered in [2]. Nevertheless, the methods employed in these works are different from ours and are not applicable to the products $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}, \mathbb{F} \neq \mathbb{R}$.

We proceed now to the classification of the complete rotational H_r-hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ whose height functions are Morse type, i.e., have isolated critical points, if any. As we shall see, besides cylinders over geodesic spheres, these hypersurfaces are precisely the ones we obtained in Theorems 1 and 2. In particular, any of them is embedded. We point out that, in [2], it was shown that, for any $H_1 > 0$, there exist complete rotational H_1-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ which are not embedded. In accordance with our results, the height function of none of these H_1-hypersurfaces is Morse type.

Firstly, let us recall that the H_r-hypersurfaces in Theorems 1 and 2 were constructed from a single (f_s, ϕ)-graph whose associated τ-function is a solution of the ODE $y' = ay + b$, where a and b are as in (20). For such a τ, there is a maximal interval (s_0, s_1), $0 \leq s_0 < s_1 \leq +\infty$, such that $0 < \tau|_{(s_0,s_1)} < 1$.

Notice that each choice of H_r determines the function b and, so, the equation $y' = ay + b$. The corresponding graph, then, is determined by the ordering of the constants H_r and $C_{\mathbb{F}}(r)$, as well as by the values of s_0 and $\tau(s_0)$.

Below, we list all the occurrences of s_0 and $\tau(s_0)$ in Theorems 1 and 2 with respect to the ordering of H_r and $C_{\mathscr{F}}(r)$:

(C1) $s_0 = 0, \tau(s_0) = 0, H_r > C_{\mathbb{F}}(r)$.
(C2) $s_0 = 0, \tau(s_0) = 0, H_r \leq C_{\mathbb{F}}(r)$.
(C3) $s_0 > 0, \tau(s_0) = 1, H_r > C_{\mathbb{F}}(r)$.

(C4) $s_0 > 0, \tau(s_0) = 1, H_r \leq C_{\mathbb{F}}(r)$.

The cases C1 and C2 correspond to Theorem 1-(i) and Theorem 1-(ii), respectively, whereas C3 and C4 correspond to Theorem 2-(i) and Theorem 2-(ii). We also remark that $s_1 < +\infty$ in the cases C1 and C3, with $\tau(s_1) = 1$, and that $s_1 = +\infty$ in the cases C2 and C4.

Let M_0 be a hypersurface of a Riemannian manifold M. It is easily seen that $\Sigma := M_0 \times \mathbb{R}$ is a hypersurface of $M \times \mathbb{R}$ whose tangent spaces are all vertical, so that ∂_t is a principal direction of Σ with vanishing principal curvature. In particular, $H_n = 0$ on Σ. Also, for all $r \in \{1, \dots, n-1\}$, the rth mean curvatures of M_0 and Σ at $x \in M_0$ and $(x, t) \in \Sigma$ coincide. In particular, M_0 is an $H_{r(<n)}$-hypersurface of M if and only if Σ is an $H_{r(<n)}$-hypersurface of $M \times \mathbb{R}$. We call $\Sigma := M_0 \times \mathbb{R}$ the *cylinder* over M_0.

Theorem 3 *Let Σ be a connected complete rotational $H_r(>0)$-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ whose height function is Morse type. Then, Σ is either a cylinder over a geodesic sphere of $\mathbb{H}^m_{\mathbb{F}}$ or one of the embedded H_r-hypersurfaces of Theorems 1–2.*

Proof Suppose that Σ is not a cylinder. In this case, we have that the open set $\Sigma_0 \subset \Sigma$ on which $\Theta\nabla\xi$ never vanishes is nonempty. Since Σ_0 contains no vertical points, for a given $x_0 \in \Sigma_0$, there is an open neighborhood Σ' of x_0 in Σ_0 which is a graph over an open set Ω of $\mathbb{H}^m_{\mathbb{F}}$. Thus, since Σ is rotational and Σ_0 contains no horizontal points, after possibly a reflection with respect to a horizontal hyperplane, we can assume that Σ' is an (f_s, ϕ)-graph over Ω. (Recall that, in our setting, the ϕ-function of an (f_s, ϕ)-graph is required to be radially increasing.)

By Lemma 1, the function $\tau = \varrho^r$ associated with Σ' is a solution of $y' = ay + b$, with a and b as in (20). In addition, since Σ is complete, there exists a maximal interval (s_0, s_1), $0 \leq s_0 < s_1 \leq +\infty$, such that $0 < \tau|_{(s_0,s_1)} < 1$. In particular, we have the following two possibilities:

$$\tau(s_0) = 0 \quad \text{or} \quad \tau(s_0) = 1.$$

Suppose that $\tau(s_0) = 0$. After a vertical translation, we can assume that

$$\phi(s) = \int_{s_0}^{s} \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du,$$

which yields $\phi(s_0) = \phi'(s_0) = 0$. If $s_0 > 0$, these equalities imply that the sphere $S_{s_0}(o) \times \{0\}$ of $\mathbb{H}^m_{\mathbb{F}}$ is contained in $\partial\Sigma'$, and that $\nabla\xi$ vanishes at all of its points. This, however, contradicts that the height function of Σ is Morse type. Hence, $s_0 = 0$, so that the τ-function of Σ' satisfies the initial condition $\tau(0) = 0$.

If $H_r > C_{\mathbb{F}}(r)$, by the uniqueness of solutions of linear ODE's satisfying an initial condition, the function τ such that $\tau(0) = 0$ coincides with the one in the case C1 above. Thus, the corresponding ϕ-functions also coincide, which clearly implies that Σ' is an open set of the (strictly convex) H_r-sphere obtained in Theorem 1-(i). Therefore, by the tangency principle, Σ coincides with this H_r-sphere. If $H_r \leq C_{\mathbb{F}}(r)$, then τ coincides with the solution of case C2. Analogously, we conclude that Σ is an entire graph as in Theorem 1-(ii).

Let us suppose now that $\tau(s_0) = 1$. Since $0 < \tau < 1$ in (s_0, s_1), τ is decreasing near s_0, which implies that $r < n$. (Indeed, for $r = n$, we have $\tau' = b > 0$.) In this case, as we have discussed, $|a(s)| \to +\infty$ as $s \to 0$ (cf. (29)), and $b(0)$ is 0 or H_1. In particular, any solution τ of $y' = ay + b$ at $s = 0$ must satisfy $\tau(0) = 0$, so that $s_0 \neq 0$. Hence, $s_0 \in (0, \delta_{H_r})$, where δ_{H_r} is the positive constant defined in (37). Otherwise, τ would not be decreasing near s_0.

Setting $s_0 = \lambda$ and observing that any of the hypersurfaces obtained in Theorem 2 is strictly convex at some of its points, we can argue as in the second from the last paragraph and conclude that Σ is the H_r-hypersurface $\Sigma(\lambda)$ of either Theorem 2-(i) or Theorem 2-(ii), according to whether $H_r > C_{\mathbb{F}}(r)$ or $H_r \leq C_{\mathbb{F}}(r)$. This finishes the proof. $\square$

4.2 Rotational $H_r(> 0)$-hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$

In this section, we apply the method of (f_s, ϕ)-graphs to construct and classify rotational $H_r(> 0)$-hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$.

As we did before, let us fix a point $o \in \mathbb{S}^n$ and consider a family

$$\mathscr{F} := \{f_s : \mathbb{S}^{n-1} \to \mathbb{S}^n \,;\, s \in (0, \pi)\} \tag{38}$$

of parallel geodesic spheres f_s of $\mathbb{S}^n$ with radius s and center at o. As is well known, each f_s is totally umbilical, having principal curvatures all equal to $-\cot s$ with respect to the outward orientation. In particular, $\mathscr{F}$ is isoparametric.

From a direct computation, we get that the coefficients a and b of the ODE $y' = ay + b$ determined by $\mathscr{F}$ and any given $H_r > 0$ are

$$a(s) = -(n-r)\cot s \quad \text{and} \quad b(s) = b_r \tan^{r-1}(s)\,, \;\; b_r = rH_r \binom{n-1}{r-1}^{-1}, \tag{39}$$

and that the corresponding general solution is:

$$\tau(s) := \left(\frac{\sin s_0}{\sin s}\right)^{n-r} \left(\tau_0 + \frac{b_r}{\sin^{n-r}(s_0)} \int_{s_0}^{s} \frac{\sin^{n-1}(u)}{\cos^{r-1}(u)} du\right), \;\; s_0\,, s \in (0, \mathcal{R}), \tag{40}$$

where $\tau_0 = \tau(s_0) \in \mathbb{R}$ and

$$\mathcal{R} := \begin{cases} \pi/2 & \text{if } r > 1. \\ \pi & \text{if } r = 1. \end{cases}$$

Also, it is easily checked that

$$\tau(s) := \begin{cases} \frac{b_r}{\sin^{n-r}(s)} \int_0^s \frac{\sin^{n-1}(u)}{\cos^{r-1}(u)} du & \text{if } s \in (0, \mathcal{R}) \\ 0 & \text{if } s = 0 \end{cases} \tag{41}$$

is a well-defined solution of $y' = ay + b$ satisfying $y(0) = 0$.

Given an integer $n \geq 2$, it will be convenient to introduce the following constant:

$$S(n) = \int_0^{\pi/2} \sin^{n-1}(s) ds, \;\; n \geq 2. \tag{42}$$

Proposition 2 *Let* τ *be the solution* (41). *Then, the following hold:*

(i) $\tau' > 0$ *in* $(0, \mathcal{R})$.
(ii) $\lim_{s\to\mathcal{R}} \tau(s) = +\infty$.
(iii) $\lim_{s\to\frac{\pi}{2}} \tau(s) = H_1 S(n)$ *if* $r = 1$.

Proof Since the functions a and b in (39) are both increasing (when nonconstant), the proof of (i) is entirely analogous to the one given in Proposition 1-(i).

To prove (ii), let us first assume $r = 1$. In this case, since $\sin\pi = 0$ and the integral $\int_0^\pi \sin^{n-1}(u)du$ is positive, we have that τ satisfies (ii) for $\mathcal{R} = \pi$.

If $r > 1$, for a fixed $\delta \in (0, \pi/2)$ and any $s \in (\delta, \pi/2)$, one has

$$\int_0^s \frac{\sin^{n-1}(u)}{\cos^{r-1}(u)}du \geq \int_\delta^s \tan^{r-1}(u)\sin^{n-r}(u)du \geq \sin^{n-r}(\delta)\int_\delta^s \tan^{r-1}(u)du,$$

which implies that the first integral on the left goes to infinity as $s \to \pi/2$, since the same is true for the integral $\int_\delta^s \tan^{r-1}(u)du$. It follows from this fact that $\tau(s) \to +\infty$ as $s \to \pi/2$ if $r > 1$, which proves (ii).

The identity in (iii) follows directly from the definitions of τ (for $r = 1$) and $S(n)$ (as in (42)). □

From the above proposition, we get the following existence result for $H_r(>0)$-hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$.

Theorem 4 *Given* $r \in \{1, \dots, n\}$ *and a constant* $H_r > 0$, *there exists a rotational* H_r*-sphere* Σ *in* $\mathbb{S}^n \times \mathbb{R}$ *which is symmetric with respect to a horizontal hyperplane. Furthermore:*

(i) Σ *is strictly convex if either* $r = 1$ *and* $H_1 > 1/S(n)$ *or* $r > 1$.
(ii) Σ *is convex if* $r = 1$ *and* $H_1 = 1/S(n)$.
(iii) Σ *is nonconvex if* $r = 1$ *and* $0 < H_1 < 1/S(n)$.

Proof Let $\mathscr{F}$ be an arbitrary family of parallel geodesic spheres of $\mathbb{S}^n$ as given in (38). Consider the functions a and b defined in (39) and let τ be the solution (41) of the ODE $y' = ay + b$.

From Proposition 2-(ii), there exists a positive $s_0 < \mathcal{R}$ such that

$$0 = \tau(0) < \tau|_{(0,s_0)} < 1 = \tau(s_0),$$

so that $\tau|_{[0,s_0)}$ determines an (f_s, ϕ)-graph Σ' over $B_{s_0}(0) \subset \mathbb{S}^n$. Since $\tau(s_0) = 1$ and $\tau'(s_0) > 0$ (by Proposition 2-(i)), we can proceed just as in the proof of Theorem 1-(i) to obtain from Σ' the embedded H_r-sphere Σ of $\mathbb{S}^n \times \mathbb{R}$ which is symmetric with respect to $P_{\phi(s_0)} := \mathbb{S}^n \times \{\phi(s_0)\}$.

If either $r = 1$ and $H_1 > 1/S(n)$ or $r > 1$, we have from Proposition 2, items (ii) and (iii), that $0 < s_0 < \pi/2$. Hence, for $s \in (0, s_0)$, all spheres f_s have negative principal curvatures, which, together with equalities (12), gives that Σ is strictly convex. This proves (i).

If $r = 1$ and $H_1 = 1/S(n)$, Proposition 2-(iii) yields $s_0 = \pi/2$. However, $f_{\pi/2}$ is totally geodesic in $\mathbb{S}^n$, which implies that, except for $k_n = H_1 > 0$, the principal curvatures of Σ vanish at all points of the horizontal section $\Sigma_{\phi(\pi/2)} = \Sigma \cap P_{\phi(\pi/2)}$. Therefore, Σ is convex on $\Sigma_{\phi(\pi/2)}$ and strictly convex on $\Sigma - \Sigma_{\phi(\pi/2)}$.

Finally, assuming $r = 1$ and $0 < H_1 < 1/S(n)$, we have from Proposition 2-(iii) that $s_0 > \pi/2$. Observing that, for $\pi/2 < s < \pi$, f_s has positive principal curvatures, we conclude, as in the last paragraph, that Σ is strictly convex (resp. convex, nonconvex) on $\Sigma_{\phi(s)}$ if $s < \pi/2$ (resp. $s = \pi/2$, $s > \pi/2$). In particular, Σ is nonconvex. This shows (iii) and concludes our proof. □

Remark 3 Except for the assumptions on the convexity of Σ, the case $r = 1$ of Theorem 4 was proved in [25]. The case $r = n = 2$ was considered in [7]. It should also be mentioned that, for $n = 2$ and $r = 1$, the nonconvexity of Σ as stated in (iii) was pointed out in [1, Remark 2.8].

In our next theorem, we show the existence of one-parameter families of rotational Delaunay-type $H_r(> 0)$-annuli in $\mathbb{S}^n \times \mathbb{R}$. This result, then, generalizes the analogous one obtained in [26] for $r = 1$.

First, let us introduce the constant

$$C_r := \frac{n-r}{n}\binom{n}{r}$$

and observe that, for $1 \le r < n$, H_r, b_r (as in (39)), and C_r satisfy:

$$\frac{n-r}{b_r} = \frac{C_r}{H_r}\,. \tag{43}$$

In this setting, if we define

$$\delta_{H_r} := \arctan{(C_r/H_r)^{1/r}} \in (0, \pi/2), \tag{44}$$

then a solution τ of $y' = ay + b$ such that $\tau(s_0) = 1$, $s_0 \in (0, \pi/2)$, satisfies:

$$\tau'(s_0) < 0 \quad \Leftrightarrow \quad 0 < s_0 < \delta_{H_r}. \tag{45}$$

Theorem 5 *Given $n \ge 2$, $r \in \{1, \dots, n-1\}$, and $H_r > 0$, there exists a one-parameter family $\mathscr{S} = \{\Sigma(\lambda)\,;0 < \lambda < \delta_{H_r}\}$ of properly embedded Delaunay-type rotational H_r-hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$.*

Proof Given $\lambda \in (0, \delta_{H_r})$, consider the solution τ as in (40) such that $s_0 = \lambda$ and $\tau_0 = \tau(\lambda) = 1$. From (45), we have that τ is decreasing in a neighborhood of λ.

Observe that τ is positive in $(0, \mathcal{R})$. Also, setting

$$\mu(s) = \left(\frac{\sin s}{\sin \lambda}\right)^{n-r},$$

we have that $\mu > 1$ on $(\lambda, \pi/2)$. So, for $r > 1$ and $s > \lambda$,

$$\tau(s) > \frac{1}{\mu(s)}\left(1 + \int_\lambda^s b(u)du\right) = \frac{1}{\mu(s)}\left(1 + b_r \int_\lambda^s \tan^{r-1}(u)du\right),$$

which implies that $\tau(s) \to +\infty$ as $s \to \pi/2$.

If $r = 1$, since $\sin \pi = 0$ and $\int_\lambda^\pi \mu(s)ds$ is positive, we have that $\tau(s) \to +\infty$ as $s \to \pi$.

It follows from the above considerations that there exists $\bar{\lambda} \in (0, \mathcal{R})$ such that

$$\tau(\lambda) = \tau(\bar{\lambda}) = 1 \quad \text{and} \quad \tau'(\lambda) < 0 < \tau'(\bar{\lambda}). \tag{46}$$

From this point on, the proof is entirely analogous to that of Theorem 2-(i). □

A classification result for rotational H_r-hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ can be achieved as it was for their congeners in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$. To see this, assume that Σ is a complete connected rotational $H_r(> 0)$-hypersurface of $\mathbb{S}^n \times \mathbb{R}$ whose height function is Morse type. Assuming that Σ is noncylindrical, we have, as before, that there exists an open set $\Sigma' \subset \Sigma$ which is an (f_s, ϕ)-graph, $f_s \in \mathscr{F}$. The corresponding τ-function, restricted to a maximal interval (s_0, s_1), satisfies:

$$0 < \tau|_{(s_0,s_1)} < 1, \quad 0 \le s_0 < s_1 \le \mathcal{R},$$

which yields $\tau(s_0) = 0$ or $\tau(s_0) = 1$.

If $\tau(s_0) = 0$, then $s_0 = 0$. (Otherwise, the height function of Σ would not be Morse type.) In this case, τ coincides with the τ-function of the H_r-sphere of Theorem 4, and then, Σ itself coincides with this sphere. (Notice that any of the spheres obtained in Theorem 4 is strictly convex on an open set.)

If $\tau(s_0) = 1$, then τ is decreasing in a neighborhood of s_0 . Thus, $r < n$. In particular, $|a(s)| \to +\infty$ as $s \to 0$, so that $s_0 \in (0, \delta_{H_r})$. Analogously, this gives that Σ coincides with the H_r-annulus $\Sigma(\lambda)$ of Theorem 5, $\lambda = s_0$.

Summarizing, we have the following result.

Theorem 6 *Let Σ be a connected complete rotational $H_r(> 0)$-hypersurface of $\mathbb{S}^n \times \mathbb{R}$ whose height function is Morse type. Then, Σ is either a cylinder over a strictly convex geodesic sphere of $\mathbb{S}^n$ or one of the embedded H_r-hypersurfaces of Theorems 4–5.*

Remark 4 Regarding the hypothesis on the height function of Σ in Theorem 6, we point out that a rotational embedded $H_1(> 0)$-torus in $\mathbb{S}^n \times \mathbb{R}$ whose height function is non-Morse type was obtained in [25].

5 Rotational *r*-minimal hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$.

In this section, we shall see that the method of (f_s, ϕ)-graphs can be used for constructing and classifying rotational r-minimal hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$. A major distinction from the case of $H_r(>0)$-hypersurfaces is that the tangency principle is no longer available.

Theorem 7 *Given* $r \in \{1, \dots, n\}$, *there exists a one-parameter family*

$$\mathscr{S} = \{\Sigma(\lambda)\,;\, \lambda > 0\}$$

of complete rotational r-minimal n-annuli in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ *with the following properties*:

(i) *If* $r = n$, $\Sigma(\lambda)$ *is a cylinder over a geodesic sphere of* $\mathbb{H}^m_{\mathbb{F}}$ *of radius* $\lambda > 0$.
(ii) *If* $r < n$, $\Sigma(\lambda)$ *is catenoid type. More precisely, it is symmetric with respect to* $P_0 = \mathbb{H}^m_{\mathbb{F}} \times \{0\}$, *and* $P_0 \cap \Sigma(\lambda)$ *is the geodesic sphere of* $\mathbb{H}^m_{\mathbb{F}}$ *of radius* λ *centered at the point* $o \in \mathbb{H}^m_{\mathbb{F}}$ *of the axis. In addition, each of the parts of* $\Sigma(\lambda)$ *above and below* P_0 *is a rotational graph over* $\mathbb{H}^m_{\mathbb{F}} - B_\lambda(o)$ *(Fig. 8).*

Furthermore, up to ambient isometries, any complete connected rotational r-minimal hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ is either an element of $\mathscr{S}$ or a horizontal hyperplane.

Proof Given $\lambda > 0$, is immediate that a cylinder over a geodesic sphere of $\mathbb{H}^m_{\mathbb{F}}$ of radius λ is an n-minimal rotational annulus of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, which yields (i).

Assume that $1 \le r < n$ and let $\mathscr{F} = \{f_s\,;\, s \in (0, +\infty)\}$ be the parallel family of geodesic spheres of $\mathbb{H}^m_{\mathbb{F}}$ centered at the axis point $o \in \mathbb{H}^m_{\mathbb{F}}$. The ODE determined by $\mathscr{F}$ and $H_r = 0$ is

$$y' = ay, \quad a(s) = -\frac{r|H_r{}^s|}{|H_r{}^{s-1}|}, \quad s \in (0, +\infty). \tag{47}$$

Since $a < 0$, given $\lambda > 0$, the function

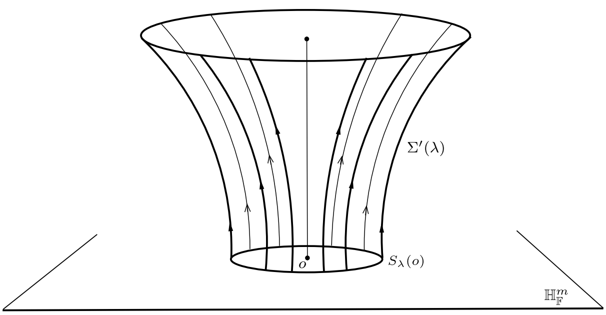


Fig. 8 Half r-minimal catenoid $\Sigma'(\lambda)$, on which all the trajectories of $\nabla\xi$ emanate from $\partial\Sigma'(\lambda)$ orthogonally

$$\tau_\lambda(s) = \exp\left(\int_\lambda^s a(u)du\right), \quad s \in [\lambda, +\infty),$$

is clearly a solution of (47) which satisfies

$$0 < \tau_\lambda(s) \le \tau_\lambda(\lambda) = 1 \ \ \forall s \in [\lambda, +\infty).$$

In addition, $\tau'_\lambda(\lambda) = a(\lambda) < 0$. So, setting $\varrho_\lambda = \tau_\lambda^{1/r}$, it follows from Lemma 2 that

$$\phi_\lambda(s) := \int_\lambda^s \frac{\varrho_\lambda(u)}{\sqrt{1 - \varrho_\lambda^2(u)}} du, \quad s \in [\lambda, +\infty),$$

is well defined. Therefore, by Lemma 1, the (f_s, ϕ_λ)-graph $\Sigma'(\lambda)$ is an r-minimal hypersurface of $\mathbb{H}^m_\mathbb{F} \times \mathbb{R}$. Notice that $\Sigma'(\lambda)$ is a graph over $\mathbb{H}^m_\mathbb{F} - \overline{B}_\lambda(o)$ with boundary $\partial\Sigma'(\lambda) = S_\lambda(o)$ (Fig. 8).

Also, since $\varrho_\lambda(\lambda) = 1$, the tangent spaces of $\Sigma'(\lambda)$ along $\partial\Sigma'(\lambda)$ are all vertical. Thus, considering the reflection $\Sigma''(\lambda)$ of $\Sigma'(\lambda)$ with respect to $\mathbb{H}^m_\mathbb{F} \times \{0\}$, as before, we have that $\Sigma(\lambda) := \text{closure}\,(\Sigma'(\lambda)) \cup \text{closure}\,(\Sigma''(\lambda))$ is the desired r-minimal hypersurface.

Suppose now that Σ is a complete connected rotational r-minimal hypersurface of $\mathbb{H}^m_\mathbb{F} \times \mathbb{R}$, $r \in \{1, \dots, n\}$, and set

$$\Sigma_0 := \{x \in \Sigma\, ; \Theta(x)\nabla\xi(x) \neq 0\}.$$

Notice that Σ is either a horizontal hyperplane or a cylinder if and only if $\Sigma_0 = \emptyset$. So, we can assume $\Sigma_0 \neq \emptyset$. We can also assume, without loss of generality, that Σ and all elements of $\mathscr{S}$ share the same axis $\{o\} \times \mathbb{R}$.

As we argued in previous proofs, under the above hypotheses, there exists an (f_s, ϕ) -graph $\Sigma' \subset \Sigma_0$ and a maximal open interval (s_0, s_1), $0 \le s_0 < s_1 \le +\infty$, such that the τ -function of Σ' satisfies $0 < \tau|_{(s_0,s_1)} < 1$.

For $r = n$, we have that τ, and so ϱ, is constant. Hence, up to a vertical translation, one has $\phi(s) = cs, \ s > 0$, for some constant $c > 0$. However, $\phi'(0) = c > 0$, which implies that the closure of Σ' in Σ meets the rotation axis nonorthogonally, i.e., Σ is not smooth at $\partial\Sigma'$—a contradiction. So, $\Sigma_0 = \emptyset$ if $r = n$.

For $r < n$, we have that τ is a solution of (47). In particular, τ is decreasing, which implies that $\tau(s_0) = 1$. As before, this yields $s_0 > 0$. Thus, setting $s_0 = \lambda$, we have $\tau = \tau_\lambda$, which implies that, up to a vertical translation, $\phi = \phi_\lambda$ and, then, Σ' coincides with the half-catenoid $\Sigma'(\lambda)$.

We conclude from the above that Σ_0 is the union of open half-catenoids $\Sigma'(\lambda)$, where $\Sigma(\lambda) \in \mathscr{S}$. In particular, the boundary $\partial\Sigma_0$ of Σ_0 in Σ has no horizontal points.

Now we prove that $\Sigma_1 := \Sigma - \Sigma_0$ has empty interior in Σ. Indeed, assuming otherwise, consider a nonempty maximum open set $U \subset \Sigma_1$ of Σ whose boundary intersects $\partial\Sigma_0$. Then, U is either horizontal or vertical. Since $\partial\Sigma_0$ has no horizontal points, U should be vertical and, so, part of a vertical rotational cylinder. However, rotational cylinders in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ are r-minimal if and only if $r = n$.

It follows from the above that Σ_0 is open and dense in Σ. Clearly, the intersection of two distinct elements of $\mathscr{S}$ is always transversal. This, together with the connectedness of Σ and the density of Σ_0 in Σ, gives that Σ coincides with some $\Sigma(\lambda) \in \mathscr{S}$. This finishes the proof. □

Theorem 8 *Given* $r \in \{1, \dots, n\}$, *there exist* $\mathcal{R}_0 > 0$ *and a one-parameter family*

$$\mathscr{S} = \{\Sigma(\lambda)\,;0 < \lambda < \mathcal{R}_0\}$$

of complete rotational r-minimal n-annuli in $\mathbb{S}^n \times \mathbb{R}$ *with the following properties*:

(i) *If* $r = n$, *then* $\mathcal{R}_0 = \pi$ *and* $\Sigma(\lambda)$ *is a cylinder over a geodesic sphere of* $\mathbb{S}^n$ *of radius* λ.
(ii) *If* $r < n$, *then* $\mathcal{R}_0 = \pi/2$ *and* $\Sigma(\lambda)$ *is Delaunay type*.

Furthermore, up to ambient isometries, any complete connected rotational r-minimal hypersurface of $\mathbb{S}^n \times \mathbb{R}$ is either an element of $\mathscr{S}$ or a horizontal hyperplane.

Proof Statement (i) is trivial. So, assume $r < n$ and let $\mathscr{F} = \{f_s\,; s \in (0, \pi)\}$ be the family of parallel geodesic spheres of $\mathbb{S}^n$ centered at some point $o \in \mathbb{S}^n$. In this setting, the ODE determined by $\mathscr{F}$ and $H_r = 0$ is

$$y' = ay, \quad a(s) = -(n-r)\cot s, \quad s \in (0, \pi). \tag{48}$$

Given $\lambda \in (0, \pi/2)$, the function

$$\tau_\lambda(s) = \left(\frac{\sin\lambda}{\sin s}\right)^{n-r}, \ s \in (0, \pi),$$

is easily seen to be the solution of (48) satisfying

$$0 < \tau_\lambda|_{(\lambda, \pi-\lambda)} < 1 = \tau(\lambda) = \tau(\pi - \lambda).$$

Henceforth, the reasoning in the proof of Theorem 5 applies and leads to the construction of the Delaunay-type r-minimal hypersurface $\Sigma(\lambda)$ as stated in (ii).

Now, suppose that Σ is a complete connected rotational r-minimal hypersurface of $\mathbb{S}^n \times \mathbb{R}$. Under this assumption, define

$$\Sigma_0 := \{x \in \Sigma\,; \Theta(x)\nabla\xi(x) \neq 0\}.$$

As in the preceding proof, $\Sigma_0 = \emptyset$ if $r = n$. Thus, in this case, Σ is either a horizontal hyperplane or a cylinder over a geodesic sphere of $\mathbb{S}^n$.

Suppose that $r < n$ and that the axis of Σ is $\{o\} \times \mathbb{R}$. If $\Sigma_0 \neq \emptyset$, then Σ is neither a horizontal hyperplane nor a cylinder. In addition, there exists an (f_s, ϕ)-graph $\Sigma' \subset \Sigma_0$ and a maximal interval (s_0, s_1), $0 \le s_0 < s_1 \le \pi$, such that the τ-function of Σ' satisfies $0 < \tau|_{(s_0, s_1)} < 1$. So, $\tau(s_0) = 0$ or $\tau(s_0) = 1$.

The formula of the general solution of the ODE (48) gives that τ is positive, not defined at $s = 0, \pi$, and bounded away from zero. In particular, $s_0 \neq 0, s_1 \neq \pi$ and $\tau(s_0) = \tau(s_1) = 1$, so that τ is given by

$$\tau(s) = \left(\frac{\sin s_0}{\sin s}\right)^{n-r}, \quad s_0\,, s \in (0, \pi).$$

It is clear from this last equality and the considerations preceding it that $s_0 < \pi/2 < s_1 < \pi$, which implies that τ coincides with τ_λ , $\lambda = s_0$. Therefore, Σ' coincides with the "block" $\Sigma'(\lambda)$ that generates $\Sigma(\lambda)$, so that Σ_0 is a union of open sets of elements of $\mathscr{S}$. In particular, the boundary $\partial\Sigma_0$ of Σ_0 in Σ has no horizontal points, and any of its vertical points lies on a geodesic sphere of a hyperplane of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ of radius different from $\pi/2$ (see Fig. 6).

It follows from the considerations in the last paragraph that $\Sigma_1 := \Sigma - \Sigma_0$ has empty interior in Σ. Indeed, assume by contradiction that $U \subset \Sigma_1$ is a nonempty maximal open set of Σ intersecting $\partial\Sigma_0$. Then, U must be vertical, since $\partial\Sigma_0$ has no horizontal points. This gives that U is part of the totally geodesic cylinder $S_{\pi/2} \times \mathbb{R}$. In this case, a boundary point of U in $\partial\Sigma_0$ is vertical and lies on a geodesic sphere of radius $\pi/2$ in a hyperplane of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, which is impossible, as we have shown.

We conclude from the above that Σ_0 is open and dense in Σ. Since Σ is connected and two distinct elements of $\mathscr{S}$ are never tangent, it follows that, for some $\lambda \in (0, \pi/2)$, Σ coincides with $\Sigma(\lambda) \in \mathscr{S}$. □

Remark 5 The case $\mathbb{F} = \mathbb{R}$ of Theorem 7 was considered in [16], whereas Theorem 8 was proved in [26] for $r = 1$. Again, the methods employed in these works is different from ours and cannot be applied to general products $M \times \mathbb{R}$, since they all rely on the Euclidean and Lorentzian geometries of the underlying spaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

6 Translational $H_r(> 0)$-hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$.

Given a Hadamard manifold M, recall that the *Busemann function* $\mathfrak{b}_\gamma$ of M corresponding to an arclength geodesic $\gamma : (-\infty, +\infty) \to M$ is defined as

$$\mathfrak{b}_\gamma(p) := \lim_{s\to+\infty} (\mathrm{dist}_M(p, \gamma(s)) - s), \quad p \in M.$$

The level sets $\mathscr{H}_s := \mathfrak{b}_\gamma^{-1}(s)$ of a Busemann function $\mathfrak{b}_\gamma$ are called *horospheres* of M. In this setting, as is well known, $\{\mathscr{H}_s\,; s \in (-\infty, +\infty)\}$ is a family of parallel hypersurfaces which foliates M. Furthermore, any horosphere $\mathscr{H}_s$ is homeomorphic to $\mathbb{R}^{n-1}$, and any geodesic of M which is asymptotic to γ—i.e., with the same point p_∞ on the asymptotic boundary $M(\infty)$ of M—is orthogonal to each horosphere $\mathscr{H}_s$. In this case, we say that the horospheres $\mathscr{H}_s$ are *centered* at p_∞ .

Therefore, in what concerns its horospheres, a Hadamard manifold can be pictured just as the Poincaré ball model of hyperbolic space $\mathbb{H}^n$, where the horospheres centered at a point $p_\infty \in \mathbb{H}^n(\infty)$ are the Euclidean $(n-1)$-spheres in $\mathbb{H}^n$ which are tangent to $\mathbb{H}^n(\infty)$ at p_∞ (Fig. 9).

In the real hyperbolic space $\mathbb{H}^n$, any horosphere is totally umbilical with constant principal curvatures equal to 1. As shown in [4, Proposition-(vi), pg. 88], any horosphere

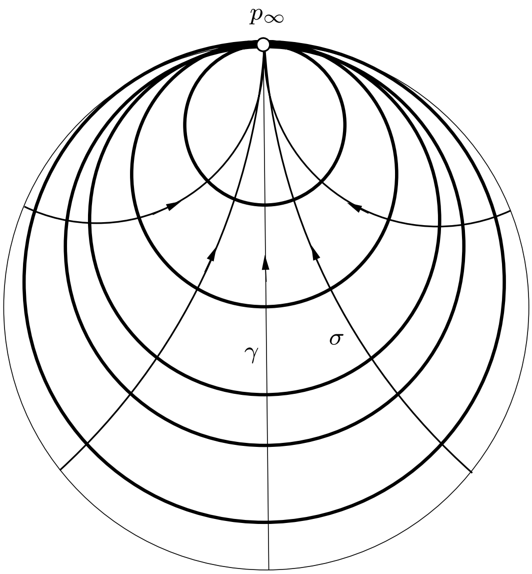


Fig. 9 A "ball model" for a Hadamard manifold

of $\mathbb{H}^m_{\mathbb{F}}$, $\mathbb{F} \neq \mathbb{R}$, has principal curvatures 1 and 1/2 with multiplicities 1 and $n-2$, respectively. Therefore, *any family $\mathscr{F}$ of parallel horospheres of $\mathbb{H}^m_{\mathbb{F}}$ is isoparametric and its elements are pairwise congruent. In addition, for any integer $r \in \{1, \dots, n-1\}$, all horospheres of $\mathbb{H}^m_{\mathbb{F}}$ have the same (positive) r-th mean curvature, which we denote by $H_r{}^0$.*

Theorem 9 *Let $\mathscr{F} := \{\mathscr{H}_s\,; s \in (-\infty, +\infty)\}$ be a family of parallel horospheres in hyperbolic space $\mathbb{H}^m_{\mathbb{F}}$. Then, for any even integer $r \in \{2, \dots, n-1\}$, and any constant $H_r \in (0, H_r{}^0)$, there exists two properly embedded, everywhere nonconvex H_r-hypersurfaces Σ_1 and Σ_2 in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, which are both homeomorphic to $\mathbb{R}^n$. In addition, the following hold*:

(i) *Σ_1 is a convex and nowhere strictly convex entire graph over $\mathbb{H}^m_{\mathbb{F}}$ with constant angle function and unbounded height function.*
(ii) *Σ_2 is foliated by horospheres, is symmetric with respect to the horizontal hyperplane $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$, and has unbounded height function.*

Furthermore, as $H_r \to H_r{}^0$, both Σ_1 and Σ_2 converge to a $H_r{}^0$-cylinder over a horosphere of the family $\mathscr{F}$.

Proof For each $s \in (-\infty, \infty)$, consider the isometric immersion $f_s : \mathbb{R}^{n-1} \to \mathbb{H}^m_{\mathbb{F}}$ such that $f_s(\mathbb{R}^{n-1}) = \mathscr{H}_s$. Since all the principal curvatures of f_s are constant and independent of s, the coefficients a e b of the ODE $y' = ay + b$ associated with this family are constants. Also, since r is even and $0 < H_r < H_r{}^0$, we have

$$b < 0 < a \quad \text{and} \quad 0 < -\frac{b}{a} = \frac{H_r}{{H_r}^0} < 1. \tag{49}$$

Therefore, the constant function

$$\tau(s) = -\frac{b}{a}, \ \ s \in (-\infty, +\infty),$$

is a trivial solution of $y' = ay + b$ satisfying $0 < \tau < 1$. This, together with Lemma 1 and equalities (9) and (14), gives that the (f_s, ϕ)-graph Σ_1 determined by $\varrho = \tau^{1/r}$ is entire, unbounded and constitutes an H_r-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ with constant angle function. Also, from identities (12), all principal curvatures of Σ_1 are nonpositive, with $k_n = \varrho' = 0$, so that Σ_1 is convex and nowhere strictly convex. Finally, it follows from (49) that $\tau = -b/a \to 1$ as $H_r \to {H_r}^0$, which implies that the angle function of Σ_1 goes to 0 as $H_r \to {H_r}^0$. Consequently, Σ_1 converges to a cylinder over a horosphere of $\mathscr{F}$ as $H_r \to {H_r}^0$.

Now, let us consider the following solution $\tau : (-\infty, 0] \to \mathbb{R}$ of $y' = ay + b$:

$$\tau(s) = e^{a(s-s_0)} - \frac{b}{a}, \ \ s_0 = \log(1 + b/a)^{-1/a}. \tag{50}$$

It is easily seen that

$$0 < -\frac{b}{a} < \tau(s) \le 1 = \tau(0) \ \ \forall s \in (-\infty, 0]. \tag{51}$$

So, by Lemma 1, the (f_s, ϕ)-graph Σ' with $\varrho = \sqrt[r]{\tau}$ is an H_r-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$. The function ϕ, in this case, is given by

$$\phi(s) := -\int_s^0 \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du, \ \ s \in (-\infty, 0).$$

Notice that, by (51), one has

$$\tau'(s) = a\tau(s) + b > a\frac{-b}{a} + b = 0,$$

so that $\tau'(0) > 0$. Thus, by Lemma 2, ϕ is well defined. Also, ϕ is negative on $(-\infty, 0)$ and is unbounded. Indeed, for all $s \in (-\infty, 0)$,

$$-\phi(s) = \int_s^0 \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du \ge \int_s^0 \varrho(u) du \ge -\inf \varrho|_{[s,0]} s = -s\varrho(s),$$

which implies that ϕ is unbounded, since $\varrho(s) \to (-b/a)^{1/r} > 0$ as $s \to -\infty$.

Denoting by B_0 the horoball of $\mathbb{H}^m_{\mathbb{F}}$ with boundary $\mathscr{H}_0$, it follows from the above considerations that Σ' is an H_r-graph over $\mathbb{H}^m_{\mathbb{F}} - B_0$ which is unbounded and has boundary $\partial\Sigma' = \mathscr{H}_0 \times \{0\}$ (Fig. 10). In particular, Σ' is homeomorphic to $\mathbb{R}^n$ and, from the identities (12), Σ' is everywhere nonconvex.

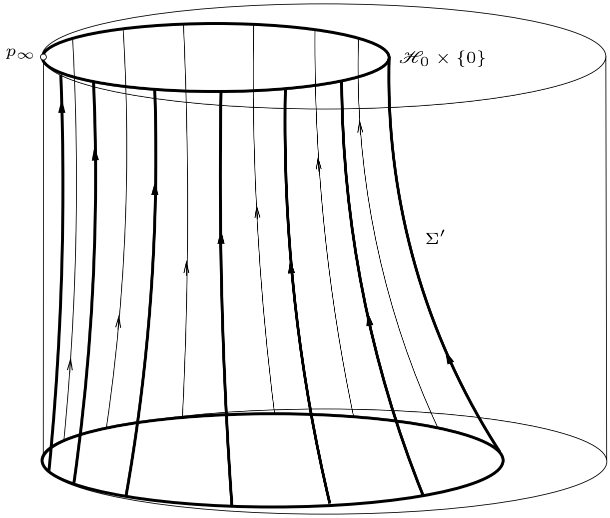


Fig. 10 A piece of the graph Σ', on which all the trajectories of $\nabla\xi$ meet $\mathscr{H}_0 \times \{0\}$ orthogonally

Since $\varrho(0) = 1$, as in the previous theorems, we have that any trajectory of $\nabla\xi$ on Σ' meets $\partial\Sigma'$ orthogonally. Consequently, setting Σ'' for the reflection of Σ' with respect to $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$ and defining

$$\Sigma_2 := \text{closure}\,(\Sigma') \cup \text{closure}\,(\Sigma''),$$

we have that Σ is a properly embedded H_r-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which is foliated by horospheres and is homeomorphic to $\mathbb{R}^n$ (Fig. 11).

To finish the proof, let us just observe that, from (49) and (50), one has

$$\lim_{H_r \to {H_r}^0} \tau(s) = 1 \ \ \forall s \in (-\infty, 0].$$

Therefore, just as the graph Σ_1, Σ_2 converges to a cylinder over a horosphere of $\mathscr{F}$ as $H_r \to {H_r}^0$. □

Our next result establishes that the conditions on the parity of r and on the sign of $H_r - {H_r}^0$ in Theorem 9 are, in fact, necessary.

Theorem 10 *Let $\mathscr{F}$ be a family of parallel horospheres in $\mathbb{H}^m_{\mathbb{F}}$. Assume that, for some $r \in \{1, \dots, n\}$, Σ is a complete connected $H_r(>0)$-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ with no horizontal points and that each connected component of any horizontal section $\Sigma_t \subset \Sigma$ is a (vertically translated) horosphere of $\mathscr{F}$. Under these conditions, one has $r < n$. Assume, in addition, that either of the following assertions holds:*

(i) *r is even and $H_r \geq {H_r}^0$.*
(ii) *r is odd.*

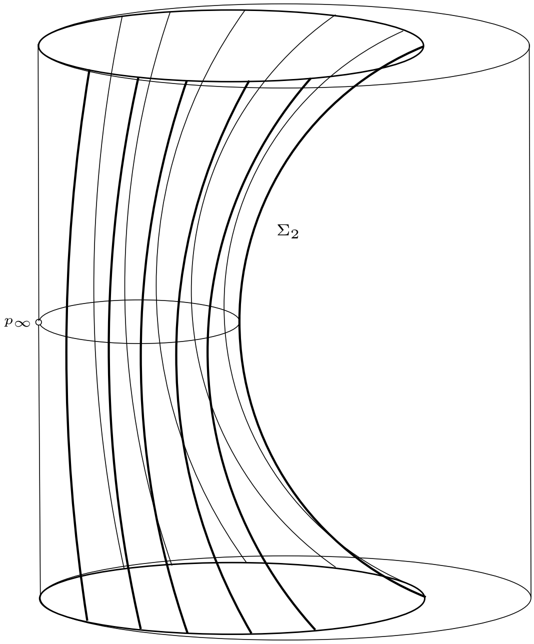


Fig. 11 A piece of a properly embedded everywhere nonconvex $H_r(>0)$-hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which is foliated by horospheres

Then, $\Sigma = \mathscr{H}_s \times \mathbb{R}$ for some $\mathscr{H}_s \in \mathscr{F}$. In particular, $H_r = {H_r}^0$.

Proof Let $\Sigma_0 \subset \Sigma$ be the open set of points $x \in \Sigma$ satisfying $\Theta(x) \neq 0$. Our aim is to prove that Σ_0 is empty. Assuming otherwise, choose $x_0 \in \Sigma_0$. Since Σ has no horizontal points, we can suppose (after possibly a reflection about a horizontal hyperplane) that there is an open neighborhood $\Sigma' \subset \Sigma_0$ of x_0 which is an (f_s, ϕ)-graph, $f_s \in \mathscr{F}$.

The τ-function associated with Σ' satisfies $\tau' = a\tau + b$, where a and $b \neq 0$ are the (constant) functions (16) determined by $\mathscr{F}$ and H_r. Also, there is a maximal interval $I = (s_1, s_2) \subset \mathbb{R}, -\infty \leq s_1 < s_2 \leq +\infty$, such that $\tau(I) \subset (0, 1)$.

Let us suppose that $r = n$. In this case, we have $a = 0$, which gives $\tau'(s) = b \neq 0$, that is, $\tau(s) = bs + c$, $c \in \mathbb{R}$. In particular, $s_1 > -\infty$ and $s_2 < +\infty$, and τ is increasing (if $b > 0$), or decreasing (if $b < 0$) in (s_1, s_2). So, τ vanishes at s_1 or at s_2. Assuming the former, we have that ϕ is defined at s_1 and $\phi'(s_1) = 0$. Thus, for any $p \in \mathbb{R}^{n-1}$, the point $x = (f_{s_1}(p), \phi(s_1)) \in \Sigma'$ is horizontal, contrary to our assumption. Therefore, if $r = n$, then $\Sigma_0 = \emptyset$, which implies that $\Sigma = \mathscr{H}_s \times \mathbb{R}$ for some $s \in \mathbb{R}$. But this contradicts the assumed positiveness of H_n. Hence, we must have $r < n$.

Let us assume now that (i) holds. Then, we have $b < 0 < a$. Also, on (s_1, s_2),

$$\tau' = a\tau + b < a + b = \frac{r({H_r}^0 - H_r)}{H^0_{r-1}} \leq 0 \quad \text{and} \quad \tau'' = a\tau' + b < 0,$$

that is, τ is decreasing and concave in (s_1, s_2), which clearly implies that $s_2 < +\infty$ and $\tau(s_2) = 0$. As in the preceding paragraph, this leads to the existence of a horizontal point of Σ. Therefore, $\Sigma_0 = \emptyset$ if (i) holds, which implies that $\Sigma = \mathscr{H}_s \times \mathbb{R}$ for some $s \in \mathbb{R}$.

Finally, let us assume that (ii) holds. In this case, one has $a, b > 0$, which gives that τ is increasing and convex. From this point, we get easily to the conclusion by reasoning just as in the last paragraph. □

An isometry φ of $\mathbb{H}^m_{\mathbb{F}}$ which fixes only one point $p_\infty \in \mathbb{H}^m_{\mathbb{F}}(\infty)$ is called *parabolic*. Such isometries have the following fundamental property: *The horospheres of* $\mathbb{H}^m_{\mathbb{F}}$ *centered at* $p_\infty \in \mathbb{H}^m_{\mathbb{F}}(\infty)$ *are invariant by parabolic isometries of* $\mathbb{H}^m_{\mathbb{F}}$ that fix p_∞ (cf. [14, Proposition 7.8]).

We point out that any isometry φ of $\mathbb{H}^m_{\mathbb{F}}$ has a natural extension to an isometry Φ of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$. Namely,

$$\Phi(p, t) = (\varphi(p), t), \ \ (p, t) \in \mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}.$$

We call Φ *parabolic* if φ is parabolic. More specifically, if

$$\mathscr{F} = \{f_s : \mathbb{R}^{n-1} \to \mathbb{H}^m_{\mathbb{F}}\,; s \in (-\infty, +\infty)\}, \ \ f_s(\mathbb{R}^{n-1}) = \mathscr{H}_s,$$

is the family of parallel horospheres which are invariant by φ, we say that φ and Φ are $\mathscr{F}$*-parabolic* isometries.

In the upper half-space model of $\mathbb{H}^n = \mathbb{H}^n_{\mathbb{R}}$, Euclidean horizontal translations in a fixed direction are parabolic. As for the other hyperbolic spaces, the parabolic isometries are more involved (see, e.g., [19]). Nevertheless, inspired by the real case, we say that parabolic isometries are *translational*.

Finally, let us remark that, given a family $\mathscr{F}$ of parallel horospheres in $\mathbb{H}^m_{\mathbb{F}}$, if a hypersurface Σ of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ is invariant by $\mathscr{F}$-parabolic isometries of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, then any connected component of any horizontal section $\Sigma_t \subset \Sigma$ is contained in a (vertically translated) horosphere of $\mathscr{F}$.

Now, we are in position to classify all complete connected $H_r(>0)$-hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ with no horizontal points which are invariant by parabolic isometries.

Theorem 11 *Let* $\mathscr{F}$ *be a family of parallel horospheres of* $\mathbb{H}^m_{\mathbb{F}}$. *Assume that* Σ *is a complete connected* $H_r(>0)$*-hypersurface of* $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$, $r \in \{1, \dots, n\}$, *with no horizontal points, which is invariant by* $\mathscr{F}$*-parabolic isometries. Then, up to ambient isometries,* Σ *is either a cylinder over a horosphere of* $\mathbb{H}^m_{\mathbb{F}}$ *or one of the embedded* H_r*-hypersurfaces obtained in Theorem* 9.

Proof Assume that Σ is not a cylinder over a horosphere of $\mathbb{H}^m_{\mathbb{F}}$. By Theorem 10, we have that $r(< n)$ is even and $0 < H_r < {H_r}^0$.

In this case, the open set

$$\Sigma_0 := \{x \in \Sigma\,; \Theta(x) \neq 0\}$$

is dense in Σ. Otherwise, there would be a vertical open set $U \subset \Sigma$ which would be necessarily contained in a cylinder over a horosphere of $\mathscr{F}$. But such a cylinder has constant $r(< n)$th mean curvature ${H_r}^0 > H_r$. Hence, the open set U cannot exist.

Given $x_0 \in \Sigma_0$, there exists an (f_s, ϕ)-graph $\Sigma' \ni x_0$ in Σ_0 with $f_s \in \mathscr{F}$. As we know, its associated τ-function is a solution of the ODE $y' = ay + b$ determined by $\mathscr{F}$ and H_r, which satisfies $0 < \tau < 1$ when restricted to a maximal interval $(s_0, s_1), -\infty \le s_0 < s_1 \le +\infty$.

It is easily seen that Σ_0, and so Σ, coincides with the entire graph Σ_1 of Theorem 9 if τ is the constant solution. Hence, we can assume that τ is nonconstant. Then, the conditions on the parity of r and the sign of $H_r - {H_r}^0$, as in the proof of Theorem 9, give that τ is increasing and convex, which implies that $s_1 < +\infty$ and that $\tau(s_1) = 1$.

Consider now the solution (50) of $y' = ay + b$ and denote it by $\tilde{\tau}$. Since $\tilde{\tau}(0) = \tau(s_1) = 1$, and the coefficients a and b are constants, by the uniqueness of solutions satisfying initial conditions, we have that $\tau(s) = \tilde{\tau}(s - s_1)$. This, together with the homogeneity of the horospheres of $\mathbb{H}^m_{\mathbb{F}}$, implies that Σ' coincides with the (f_s, ϕ)-graph determined by $\tilde{\tau}$. From this fact and the density of Σ_0 in Σ, we conclude that Σ coincides with the embedded H_r-hypersurface Σ_2 obtained in Theorem 9, as we wished to prove. □

Given a totally geodesic hyperplane $\mathscr{E}_0$ of $\mathbb{H}^n$, let us recall that there exists a family $\mathscr{F} := \{\mathscr{E}_s\,; s \in (-\infty, +\infty)\}$ of parallel hypersurfaces of $\mathbb{H}^n$ such that the distance of any point of $\mathscr{E}_s$ to $\mathscr{E}_0$ is $|s|$. The family $\mathscr{F}$ foliates $\mathbb{H}^n$, and each element $\mathscr{E}_s$ of $\mathscr{F}$, which is called an *equidistant hypersurface*, is properly embedded and homeomorphic to $\mathbb{R}^{n-1}$ (Fig. 12).

We shall also write $\mathscr{F}$ as a family of immersions:

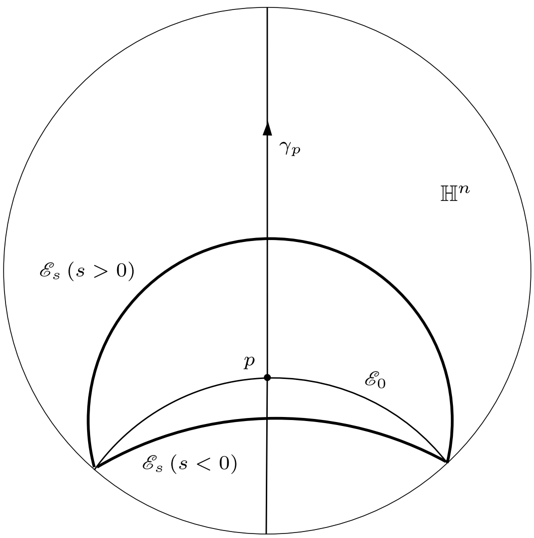


Fig. 12 Equidistant hypersurfaces in the Poincaré ball model of $\mathbb{H}^n$

$$\mathscr{F} = \{f_s : \mathbb{R}^{n-1} \to \mathbb{H}^n ; s \in (-\infty, +\infty)\}, \tag{52}$$

that is, for each $s \in (-\infty, +\infty)$, $f_s(\mathbb{R}^{n-1})$ is the equidistant $\mathscr{E}_s$ to $\mathscr{E}_0 = f_0(\mathbb{R}^{n-1})$.

Given a geodesic γ_p orthogonal to the elements of $\mathscr{F}$, $p \in \mathscr{E}_0$, any equidistant hypersurface $\mathscr{E}_s$ is totally umbilical with constant principal curvatures all equal to

$$k^s = -\tanh(s)$$

with respect to the unit normal $\eta_s = \gamma_p'$ (see Sect. 3). In particular, $\mathscr{F}$ is isoparametric. Also, given a constant H_r , the coefficients a and b of the differential equation $y' = ay + b$ associated with $\mathscr{F}$ and H_r are:

$$a(s) = -(n-r)\tanh(s) \quad \text{and} \quad b(s) = b_r \tanh^{1-r}(s), \;\; b_r = rH_r \binom{n-1}{r-1}^{-1}. \tag{53}$$

It will be convenient to reconsider the constant

$$C_r := \frac{n-r}{n}\binom{n}{r}$$

and recall that, for $1 \le r < n$, the following identity holds:

$$\frac{b_r}{n-r} = \frac{H_r}{C_r} \cdot \tag{54}$$

Our next result establishes that, for $H_r \in (0, C_r)$, $1 \le r < n$, there exists a one-parameter family of properly embedded H_r-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ which are foliated by (vertical translations of) parallel equidistant hypersurfaces of $\mathbb{H}^n$. In this setting, we have $0 < H_r/C_r < 1$, so we can define:

$$s_r := \operatorname{arctanh}(H_r/C_r)^{1/r}. \tag{55}$$

Notice that, for $1 \le r < n$, one has $C_r = \binom{n-1}{r}$. This, together with (55), yields

$$|H_r^{s_r}| = H_r \tag{56}$$

Theorem 12 *For any given integer $r \in \{1, \dots, n-1\}$ and any constant $H_r \in (0, C_r)$, there exists a one-parameter family*

$$\mathscr{S} := \{\Sigma(\lambda) ; \lambda \in (s_r, +\infty)\}$$

of properly embedded and everywhere nonconvex H_r-hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$. Each element $\Sigma(\lambda)$ of $\mathscr{S}$ is homeomorphic to $\mathbb{R}^n$ and is foliated by equidistant hypersurfaces. Moreover, $\Sigma(\lambda)$ is symmetric with respect to $\mathbb{H}^n \times \{0\}$, and its height function is unbounded.

Proof Let $\mathscr{F}$ be the family of parallel equidistant hypersurfaces of $\mathbb{H}^n$ as in (52). Given $\lambda \in (s_r, +\infty)$, let τ_λ be the solution of $y' = ay + b$ satisfying $y(\lambda) = 1$, where a and b are the functions in (53). From (54) and the definition of s_r, one has

$$\tanh^r(\lambda) > \frac{b_r}{n-r}, \tag{57}$$

which implies that

$$\tau'_\lambda(\lambda) = -(n-r)\tanh(\lambda) + b_r \tanh^{1-r}(\lambda) < 0,$$

i.e., τ_λ is decreasing near λ.

We claim that τ_λ is decreasing on the whole interval $[\lambda, +\infty)$. To show that, it suffices to prove that τ_λ has no critical points in $(\lambda, +\infty)$. Assuming otherwise, consider $s_1 > \lambda$ satisfying $\tau'_\lambda(s_1) = 0$. Since $\tau'_\lambda(s_1) = a(s_1)\tau_\lambda(s_1) + b(s_1)$, we have that $\tau_\lambda(s_1) = -b(s_1)/a(s_1) > 0$. We also have $a' < 0$ and $b' \leq 0$. Thus,

$$\tau''_\lambda(s_1) = a'(s_1)\tau_\lambda(s_1) + b'(s_1) < 0,$$

which implies that s_1 is necessarily a local maximum for τ_λ. This proves the claim, for τ_λ is decreasing near λ, so that a local maximum $s_1 > \lambda$ for τ_λ should be preceded by a local minimum.

We also have that τ_λ is positive in $[\lambda, +\infty)$. Indeed, if we had $\tau_\lambda(s) \leq 0$ for some $s > \lambda$, it would give $\tau'_\lambda(s) = a(s)\tau(s) + b(s) > 0$, and then τ_λ would be increasing near s.

It follows from the above considerations that

$$0 < \tau_\lambda(s) \leq 1 = \tau_\lambda(\lambda) \quad \forall s \in [\lambda, +\infty).$$

Furthermore, since τ_λ is decreasing and positive, one has $\tau'_\lambda(s) \to 0$ as $s \to +\infty$. This, together with the equalities $\tau'_\lambda = a\tau_\lambda + b$ and $\varrho^r_\lambda = \tau_\lambda$, gives

$$\lim_{s\to+\infty} \varrho_\lambda(s) = \left(H_r/C_r\right)^{1/r} > 0. \tag{58}$$

Therefore, the (f_s, ϕ) graph $\Sigma'(\lambda)$ associated with ϱ_λ (see Fig. 13) is an H_r-hypersurface of $\mathbb{H}^n \times \mathbb{R}$ whose ϕ-function is

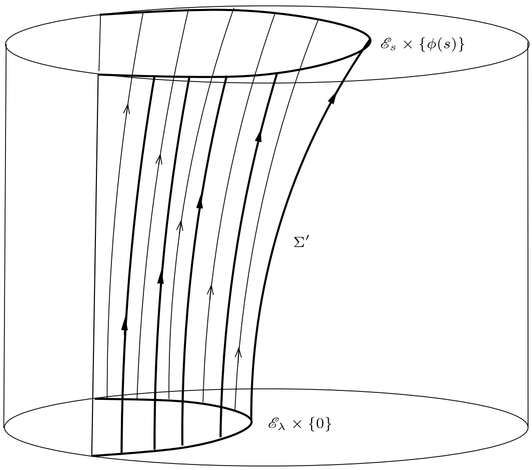


Fig. 13 A piece of the graph Σ', on which all trajectories of $\nabla\xi$ emanate from $\mathscr{E}_\lambda \times \{0\}$ orthogonally

$$\phi_\lambda(s) = \int_\lambda^s \frac{\varrho_\lambda(u)}{\sqrt{1 - \varrho_\lambda^2(u)}} du, \quad s \in [\lambda, +\infty).$$

As in the preceding proofs, we obtain a properly embedded H_r-hypersurface $\Sigma(\lambda)$ of $\mathbb{H}^n \times \mathbb{R}$ by reflecting $\Sigma'(\lambda)$ with respect to $\mathbb{H}^n \times \{0\}$, since $\varrho_\lambda(\lambda) = 1$ and $\varrho'_\lambda(\lambda) < 0$. It is also clear from equalities (12) that, except for $k_n = \varrho'$, its principal curvatures k_i are all positive, so that $\Sigma(\lambda)$ is nowhere convex.

Finally, considering (58) and the fact that ϱ_λ is decreasing, we have

$$\phi_\lambda(s) = \int_\lambda^s \frac{\varrho_\lambda(u)}{\sqrt{1 - \varrho_\lambda^2(u)}} du \geq \int_\lambda^s \varrho_\lambda(u) du \geq \left(H_r/C_r\right)^{1/r}(s - \lambda),$$

which clearly implies that the height function of $\Sigma(\lambda)$ is unbounded. □

Let us see now that, for any $H_r \in (0, C_r)$, there exists an H_r-hypersurface Σ in $\mathbb{H}^n \times \mathbb{R}$ which, as the ones in the above theorem, is foliated by equidistant hypersurfaces. However, Σ is not symmetric with respect to any horizontal hyperplane. Instead, it is asymptotic to a half-cylinder over an equidistant hypersurface of $\mathbb{H}^n$. The precise statement is as follows.

Theorem 13 *Let $\mathscr{F}$ be the family of parallel equidistant hypersurfaces of $\mathbb{H}^n$ as in (52). Given $r \in \{1, \ldots, n-1\}$ and $H_r \in (0, C_r)$, let $s_r > 0$ be the constant defined in (55). Then, there exists a complete everywhere nonconvex H_r-hypersurface Σ in $\mathbb{H}^n \times \mathbb{R}$ which is an (f_s, ϕ)-graph, $s \in (s_r, +\infty)$. Furthermore, the height function of Σ is unbounded above and below, and Σ is asymptotic to $\mathscr{E}_{s_r} \times (-\infty, 0)$.*

Proof Let τ be the solution of the differential equation $y' = ay + b$ associated with H_r and $\mathscr{F}$ (i.e., with a and b as in (53)) which satisfies the initial condition $\tau(s_r) = 1$.

From its definition, we have that s_r satisfies $\tau'(s_r) = 0$. In addition,

$$\tau''(s_r) = a'(s_r) + b'(s_r) < 0,$$

so that s_r is a local maximum of τ. Reasoning as in the preceding proof, we get that τ, and so $\varrho = \tau^{1/r}$, is positive and decreasing in $(s_r, +\infty)$. From this, we conclude analogously that $\varrho(s) \to (H_r/C_r)^{1/r}$ as $s \to +\infty$.

Now, for a fixed $s_0 > s_r$, define

$$\phi(s) := \int_{s_0}^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du, \quad s \in (s_r, +\infty),$$

and let Σ be the corresponding (f_s, ϕ)-graph. As before, we have that Σ is nowhere convex. Denoting by Ω_{s_r} the convex connected component of $\mathbb{H}^n - \mathscr{E}_{s_r}$, we also have that Σ is a graph over $\mathbb{H}^n - \Omega_{s_r}$ (Fig. 14).

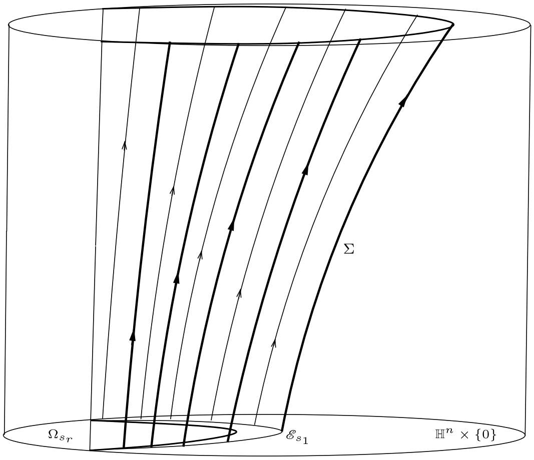


Fig. 14 A piece of the (f_s, ϕ)-graph Σ which is above $\mathbb{H}^n \times \{0\}$. As $s \to -\infty$, the trajectories of $\nabla\xi$ converge asymptotically to $\mathscr{E}_{s_r} \times (0, -\infty)$

For $s > s_0$, we have

$$\phi(s) = \int_{s_0}^{s} \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}}du \geq \int_{s_0}^{s} \varrho(u)du \geq (H_r/C_r)^{1/r}(s - s_0),$$

which gives that ϕ, and so the height function of Σ, is unbounded above.

Finally, given a constant $C > 0$, there exists $\bar{s} \in (s_r, s_0)$ such that

$$\frac{1}{\varrho'(s)} < -C \ \ \forall s \in (s_r, \bar{s}),$$

for $\varrho'(s_r) = 0$. Also, since $\varrho(s_r) = 1$, we can choose $\bar{s}$ sufficiently close to s_r in such a way that, for a fixed small $\delta > 0$, the inequalities

$$\arcsin\varrho(\bar{s}) - \arcsin\varrho(s_0) > (\pi/2 - \arcsin\varrho(s_0)) - \delta > 0 \quad \text{and} \quad \varrho(\bar{s}) > 1 - \delta > 0$$

hold. In this manner, for all $s \in (s_r, \bar{s})$, one has

$$\begin{aligned}
\phi(s) &= \int_{s_0}^{s} \frac{\varrho'(u)\varrho(u)}{\varrho'(u)\sqrt{1-\varrho^2(u)}}du \leq -C\varrho(\bar{s}) \int_{s_0}^{s} \frac{\varrho'(u)}{\sqrt{1-\varrho^2(u)}}du \\
&= - C\varrho(\bar{s}) \int_{\varrho(s_0)}^{\varrho(s)} \frac{d\varrho}{\sqrt{1-\varrho^2}} = -C\varrho(\bar{s})(\arcsin\varrho(s) - \arcsin\varrho(s_0)) \\
&\leq - C\varrho(\bar{s})(\arcsin\varrho(\bar{s}) - \arcsin\varrho(s_0)) \\
&\leq - C(1-\delta)((\pi/2 - \arcsin\varrho(s_0)) - \delta),
\end{aligned}$$

which implies that $\phi(s) \to -\infty$ as $s \to s_r$, since δ is fixed and the positive constant C is arbitrary. Therefore, the height function of Σ is unbounded below, and Σ is asymptotic to $\mathscr{E}_{s_r} \times (-\infty, 0)$ in $\mathbb{H}^n \times (-\infty, 0)$, as we wished to prove. □

A family $\mathscr{F} = \{\mathscr{E}_s\,; s \in (-\infty, +\infty)$ of equidistant hypersurfaces in $\mathbb{H}^n$ determines a group of translational isometries which we shall call $\mathscr{F}$*-hyperbolic*. In the upper half-space model of $\mathbb{H}^n$, taking $\mathscr{E}_0$ as a Euclidean half vertical hyperplane orthogonal to $\partial_\infty \mathbb{H}^n$ through the "origin" o, we have that the $\mathscr{F}$-hyperbolic isometries are the Euclidean homotheties from o. It should be noticed that the equidistant hypersurfaces of $\mathscr{F}$ are all invariant by $\mathscr{F}$-hyperbolic isometries.

The natural extension of an $\mathscr{F}$-hyperbolic isometry of $\mathbb{H}^n$ to $\mathbb{H}^n \times \mathbb{R}$ will also be called $\mathscr{F}$*-hyperbolic*. If Σ is a hypersurface of $\mathbb{H}^n \times \mathbb{R}$ which is invariant by $\mathscr{F}$-hyperbolic isometries, it is clear that any connected component of any horizontal section Σ_t of Σ is contained in $\mathscr{E}_s \times \{t\}$ for some $s \in (-\infty, +\infty)$.

Next, we classify $H_r(>0)$-hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ (without horizontal points or totally geodesic horizontal sections) which are invariant by hyperbolic translations.

Theorem 14 *Let $\mathscr{F}$ be the family of parallel equidistant hypersurfaces of $\mathbb{H}^n$ as in* (52). *Assume that, for some $r \in \{1, \dots, n\}$, Σ is a complete connected $H_r(>0)$-hypersurface of $\mathbb{H}^n \times \mathbb{R}$ which is invariant by $\mathscr{F}$-hyperbolic translations. Assume further that Σ has no horizontal points and that no horizontal section Σ_t of Σ is totally geodesic in $\mathbb{H}^n \times \{t\}$ (i.e., $\Sigma_t \not\subset \mathscr{E}_0 \times \{t\}$). Under these conditions, the following assertions hold:*

(i) $r < n$.
(ii) $0 < H_r < C_r$.
(iii) *Σ is either the cylinder over the equidistant $\mathscr{E}_{s_r}$ or, up to an ambient isometry, one of the embedded hypersurfaces obtained in Theorems 12–13.*

Proof Set $\Sigma_0 := \{x \in \Sigma\,; \Theta(x) \neq 0\}$ and assume $\Sigma_0 \neq \emptyset$. Given $x_0 \in \Sigma_0$, as in previous proofs, we can assume there is an open set $\Sigma' \subset \Sigma_0$ which is an (f_s, ϕ)-graph containing x_0. Its τ function satisfies $\tau' = a\tau + b$, where a and b are the functions given in (53). Also, τ is defined in a maximal interval $(s_0, s_1) \subset \mathbb{R}$ such that $0 < \tau|_{(s_0,s_1)} < 1$. Since no horizontal section of Σ is totally geodesic, we can assume $0 < s_0 < s_1 \leq +\infty$.

The maximality of (s_0, s_1) gives that $\tau(s_0) = 0$ or $\tau(s_0) = 1$. In the former case, we have $\tau'(s_0) = b(s_0) > 0$. Then, $\phi(s_0)$ is well defined (by Lemma 2) and $\phi'(s_0) = 0$, so that $x = (f_{s_0}(p), \phi(s_0))$, $p \in \mathbb{R}^{n-1}$, is a horizontal point of Σ, contrary to our hypothesis. Then, we must have $\tau(s_0) = 1$. In particular, near s_0, τ is decreasing in (s_0, s_1), which implies that $r < n$. Indeed, for $r = n$, $\tau' = b > 0$.

Assume now that $H_r \geq C_r$, $r < n$. Then, we have

$$
\begin{aligned}
\tau'(s_0) =& a(s_0) + b(s_0) = -(n-r)\tanh(s_0) + b_r \tanh^{1-r}(s_0) \\
=& (n-r)((H_r/C_r)\tanh^{1-r}(s_0) - \tanh(s_0)) \\
\geq& (n-r)(\tanh^{1-r}(s_0) - \tanh(s_0)) > 0,
\end{aligned}
$$

which contradicts that τ is decreasing near s_0.

It follows from the above considerations that, if $\Sigma_0 \neq \emptyset$, then $r < n$ and $H_r < C_r$. Furthermore, a direct computation gives that $\tau'(s_0) \leq 0$ if and only if $s_0 \geq s_r$. If $s_0 = \lambda > s_r$, then τ coincides with the function τ_λ of the (f_s, ϕ)-graph associated with the hypersurface $\Sigma(\lambda)$ of Theorem 12. From this, arguing as in preceding proofs, we conclude that

$\Sigma = \Sigma(\lambda)$. By the same token, if $s_0 = s_r$, then $\Sigma = \Sigma'$ is the complete graph obtained in Theorem 13.

Let us suppose now that $\Sigma_0 = \emptyset$. In this case, we must have $\Sigma = \mathscr{E}_s \times \mathbb{R}$, where $\mathscr{E}_s = f_s(\mathbb{R}^{n-1})$ is an equidistant hypersurface with rth mean curvature $H_r{}^s = H_r$, so that $s = s_r$ (see (56)). Therefore, we have $r < n$ (since we are assuming $H_r > 0$) and

$$H_r = |H_r{}^{s_r}| = \binom{n-1}{r} \tanh^r(s_r) = C_r \tanh^r(s_r) < C_r \,,$$

which concludes our proof. □

7 Translational *r*-minimal Hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$.

In this section, we construct and classify r-minimal hypersurfaces in $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which are invariant by translational isometries. It will be convenient to consider first the case of hyperbolic isometries of $\mathbb{H}^n$.

Theorem 15 *Let $\mathscr{F} = \{f_s \,; s \in (-\infty, +\infty)\}$ be a family of parallel equidistant hypersurfaces to a totally geodesic hyperplane $\mathscr{E}_0 = f_0(\mathbb{R}^{n-1})$ of $\mathbb{H}^n$. Then, for each $r \in \{1, \dots, n\}$, there exists a one-parameter family $\mathscr{S} = \{\Sigma(\lambda) \,; \lambda > 0\}$ of properly embedded r-minimal hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ which are all homeomorphic to $\mathbb{R}^n$ and invariant by $\mathscr{F}$-hyperbolic translations. Each element $\Sigma(\lambda) \in \mathscr{S}$ has the following additional properties*:

(i) *For $r = n$, $\Sigma(\lambda)$ is a constant angle entire r-minimal graph over $\mathbb{H}^n$ whose height function is unbounded above and below.*

For $r < n$, we distinguish the following cases:

(ii) $\lambda > 1$: *$\Sigma(\lambda)$ is symmetric with respect to the horizontal hyperplane $P_0 = \mathbb{H}^n \times \{0\}$ and is contained in a slab $\mathbb{H}^n \times (-\alpha, \alpha), \alpha > 0$.*
(iii) $\lambda = 1$: *$\Sigma(\lambda)$ is an (f_s, ϕ)-graph* $(s > 0)$ *which is bounded above*, *unbounded below*, *and asymptotic to $\mathscr{E}_0 \times (0, -\infty)$.*
(iv) $\lambda < 1$: *$\Sigma(\lambda)$ is an entire graph over $\mathbb{H}^n$ which is symmetric with respect to $\mathscr{E}_0$, and is contained is a slab $\mathbb{H}^n \times (-\alpha, \alpha), \alpha > 0$.*

Furthermore, except for the cylinders $\mathscr{E}_s \times \mathbb{R}$, $s \neq 0$ (in the case $r = n$), and up to ambient isometries, the elements of $\mathscr{S}$ are the only complete nontotally geodesic r-minimal hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ which are invariant by hyperbolic translations.

Proof The Eq. (15) determined by $\mathscr{F}$ and $H_r = 0$ is:

$$y' = a(s)y, \quad a(s) = -(n-r)\tanh(s). \tag{59}$$

For $r = n$, its solution τ is constant. So, given $\lambda > 0$, defining

$$\phi(s) = \lambda s, \ s \in (-\infty, +\infty),$$

we have that the corresponding (f_s, ϕ)-graph $\Sigma(\lambda)$ is an entire n-minimal graph whose level hypersurfaces are the leaves of $\mathscr{F}$. Clearly, the height function of $\Sigma(\lambda)$ is unbounded above and below. Moreover, it follows from (9) that $\Sigma(\lambda)$ is a constant angle hypersurface. This proves (i).

Let us suppose now that $1 \le r < n$. Given $\lambda > 0$, set

$$\tau_\lambda(s) := \lambda \cosh^{r-n}(s), \quad s \in (-\infty, +\infty).$$

It is easily checked that τ_λ is the solution of (59) satisfying $\tau_\lambda(0) = \lambda$.

Assume that $\lambda > 1$. Then, defining $s_\lambda := \operatorname{arccosh}(\lambda^{1/(n-r)})$, one has

$$0 < \tau_\lambda(s) \le 1 = \tau_\lambda(s_\lambda) \ \ \forall s \in [s_\lambda, +\infty).$$

Hence, setting

$$\phi_\lambda(s) := \int_{s_\lambda}^{s} \frac{\varrho_\lambda(u)}{\sqrt{1 - \varrho_\lambda^2(u)}} du, \quad \varrho_\lambda = \tau_\lambda^{1/r}, \quad s \in (s_\lambda, +\infty), \tag{60}$$

we have that the (f_s, ϕ_λ)-graph $\Sigma'(\lambda)$ is a well-defined r-minimal hypersurface, for $\tau'(s_\lambda) < 0$. Also, since $\tau_\lambda(s_\lambda) = 1$, the closure of $\Sigma'(\lambda)$ intersects P_0 orthogonally. Thus, we obtain an r-minimal hypersurface $\Sigma(\lambda)$ by reflecting $\Sigma'(\lambda)$ about P_0.

As for the boundedness of ϕ_λ, we first observe that, from the equalities $\tau_\lambda = \varrho_\lambda^r$ and $\tau_\lambda' = a\tau_\lambda$, we have $\varrho_\lambda = (r/a)\varrho_\lambda'$. In addition, the function $1/a$ is bounded above by $-1/(n-r)$ in $(0, +\infty)$. Hence,

$$\begin{aligned}\phi_\lambda(s) &= \int_{s_\lambda}^{s} \frac{r\varrho_\lambda'(u)}{a(u)\sqrt{1 - \varrho_\lambda^2(u)}} du \le -\frac{r}{n-r} \int_{\varrho_\lambda(s_\lambda)}^{\varrho_\lambda(s)} \frac{d\varrho_\lambda}{\sqrt{1 - \varrho_\lambda^2}} \\ &= \frac{r}{n-r}(\arcsin \varrho_\lambda(s_\lambda) - \arcsin \varrho_\lambda(s)) \le \frac{\pi r}{2(n-r)},\end{aligned} \tag{61}$$

which finishes the proof of (ii).

Assuming now $\lambda = 1$, let us fix $s_0 > 0$ and define

$$\phi(s) = \int_{s_0}^{s} \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du, \quad \varrho = \tau_1^{1/r}, \quad s \in (0, +\infty).$$

Since $\tau'(0) = 0$, we can mimic the final part of the proof of Theorem 13 and conclude that ϕ is unbounded below and that the corresponding (f_s, ϕ)-graph Σ is asymptotic to $\mathscr{E}_0 \times (-\infty, 0)$. Also, proceeding as in (61), we can show that ϕ is bounded above. This proves (iii).

Given $0 < \lambda < 1$, we have that $0 < \tau_\lambda < 1$ in $(0, +\infty)$. So, we can define ϕ_λ as in (60), replacing s_λ by 0. Analogously, we have that ϕ_λ is bounded above and that the boundary of the (f_s, ϕ_λ)-graph $\Sigma'(\lambda)$ is $\mathscr{E}_0 \times \{0\} \subset P_0$.

Notice that $\phi'_\lambda(0) = \varrho_\lambda(0)/\sqrt{1-\varrho^2_\lambda(0)}$ is well defined and positive, since $\varrho_\lambda(0)$ is neither 0 nor 1. Thus, we obtain a complete properly embedded r-minimal hypersurface $\Sigma(\lambda)$ from $\Sigma'(\lambda)$ by reflecting it with respect to P_0 , and then with respect to the totally geodesic vertical hyperplane $\mathscr{E}_0 \times \mathbb{R}$ (Fig. 15). This shows (iv).

Assume now that Σ is a complete nontotally geodesic r-minimal hypersurface of $\mathbb{H}^n \times \mathbb{R}$ which is invariant by $\mathscr{F}$-hyperbolic translations. Set

$$\Sigma_0 := \{x \in \Sigma\,;\, \Theta(x)\nabla\xi(x) \neq 0\}$$

and suppose that $1 \le r < n$. Then, $\Sigma_0 \neq \emptyset$. Otherwise, Σ would be either a horizontal hyperplane or a cylinder over the hyperplane $\mathscr{E}_0$ of $\mathbb{H}^n$. In both cases, Σ would be totally geodesic, which is contrary to our assumption.

Therefore, if $1 \le r < n$, for each $x_0 \in \Sigma_0$, there is an (f_s, ϕ)-graph $\Sigma' \subset \Sigma_0$ which contains x_0 , and whose τ-function is a solution of (59). More precisely, for some $c > 0$, this solution is given by

$$\tau(s) = c\left(\frac{\cosh s_0}{\cosh s}\right)^{n-r}, \quad s_0\,, s \in (-\infty, +\infty).$$

Now, recall that in the cases (ii)–(iv) above, the corresponding function τ_λ satisfies $\tau_\lambda(0) = \lambda$. Thus, for $\lambda := \tau(0) = c\cosh^{n-r}(s_0)$, the function τ coincides with τ_λ, which implies that $\Sigma' \subset \Sigma(\lambda)$. In addition, no $\Sigma(\lambda) \in \mathscr{S}$ has horizontal or totally geodesic points, which gives that Σ_0 is open and dense in Σ. Therefore, Σ coincides with $\Sigma(\lambda)$.

Finally, let us suppose that $r = n$. If $\Sigma_0 = \emptyset$, then $\Sigma = \mathscr{E}_s$ for some $s \neq 0$. If $\Sigma_0 \neq \emptyset$, then there exists an (f_s, ϕ)-graph $\Sigma' \subset \Sigma_0$ whose τ-function is constant. In particular, up to a

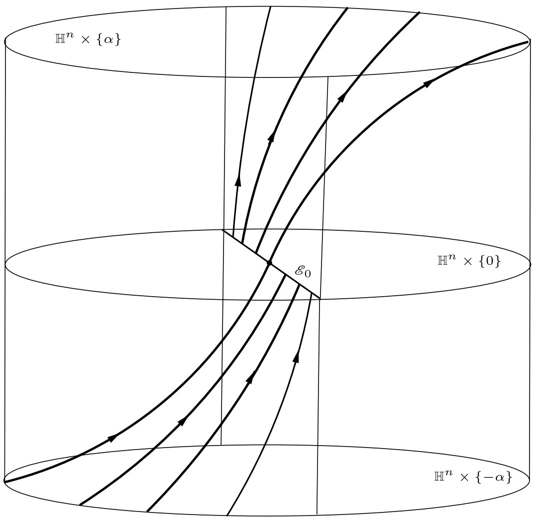


Fig. 15 Half of the (f_s, ϕ)-graph $\Sigma'(\lambda)$ (above $\mathbb{H}^n \times \{0\}$) and half of its reflection with respect to $\mathscr{E}_0$ (below $\mathbb{H}^n \times \{0\}$)

vertical translation, we have $\phi(s) = \lambda s$ for some $\lambda > 0$, so that $\Sigma' = \Sigma$ is the entire graph $\Sigma(\lambda)$ given in (i). □

Remark 6 It should be mentioned that, with an approach different from ours, the particular case $r = 1$ of Theorem 15 was considered in [3].

Next, we obtain all complete nontotally geodesic r-minimal hypersurfaces of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which are invariant by parabolic isometries.

Theorem 16 *Let $\mathscr{F} = \{f_s\,; s \in (-\infty, +\infty)\}$ be a family of parallel horospheres in $\mathbb{H}^m_{\mathbb{F}}$. Then, for any $r \in \{1, \dots, n\}$, there exists a properly embedded r-minimal hypersurface Σ of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ which is invariant by $\mathscr{F}$-parabolic isometries. In addition, Σ is homeomorphic to $\mathbb{R}^n$ and has the following properties*:

- (i) *For $r = n$, Σ is a constant angle entire graph over $\mathbb{H}^m_{\mathbb{F}}$ whose height function is unbounded above and below.*
- (ii) *For $r < n$, Σ is symmetric with respect to $\mathbb{H}^m_{\mathbb{F}} \times \{0\}$ and is contained in a slab $\mathbb{H}^m_{\mathbb{F}} \times (-\alpha, \alpha)$.*

Furthermore, except for the cylinders $\mathscr{H}_s \times \mathbb{R}$ (in the case $r = n$), and up to ambient isometries, Σ is the only complete nontotally geodesic r-minimal hypersurface of $\mathbb{H}^n \times \mathbb{R}$ which is invariant by parabolic isometries.

Proof The proof of the existence of Σ as in (i) is analogous to the one given in the preceding theorem. So, let us assume $r < n$. In this case, the Eq. (15) determined by $\mathscr{F}$ and $H_r = 0$ takes the form

$$y' = ay, \quad a = \frac{rH_r{}^0}{H^0_{r-1}} > 0\,, \tag{62}$$

and its positive solutions are

$$\tau_\lambda(s) = \lambda e^{as}, \quad \lambda > 0, \;\; s \in (-\infty, +\infty). \tag{63}$$

It is easily checked that, since the horospheres of $\mathbb{H}^m_{\mathbb{F}}$ are pairwise congruent, an (f_s, ϕ) graph with τ-function τ_λ does not depend on λ. More precisely, two such graphs obtained from functions τ_{λ_1} and τ_{λ_2}, $\lambda_1 \neq \lambda_2$, are isometric. Therefore, we can assume $\lambda = 1$ and set $\tau := \tau_1$. Then, we have

$$0 < \tau(s) < 1 = \tau(0) \;\; \forall s \in (-\infty, 0).$$

Since $\tau'(0) = a\tau(0) = a > 0$, writing $\varrho^r = \tau|_{(-\infty,0)}$, we have

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du, \quad s \in (-\infty, 0),$$

is well defined, and so is the corresponding (f_s, ϕ)-graph Σ'. Also, $\tau(0) = 1$, so that the tangent spaces of Σ' along its boundary are all vertical. Therefore, we obtain the stated r-minimal hypersurface Σ by reflecting Σ' about $P_0 = \mathbb{H}^m_{\mathbb{F}} \times \{0\}$.

Observe that, for all $s \in (-\infty, 0)$, one has

$$-\phi(s) = \int_s^0 \frac{e^{au/r}}{\sqrt{1 - e^{2au/r}}} du = \frac{r}{a}(\pi/2 - \arcsin(e^{as/r})).$$

Hence, setting $\alpha := \pi r/2a > 0$, we have that $\phi(s) \to -\alpha$ as $s \to -\infty$, which proves that Σ is contained in the slab $\mathbb{H}^m_{\mathbb{F}} \times (-\alpha, \alpha)$.

As for the uniqueness of Σ, notice that the following hold:

- The τ-function of any r-minimal (f_s, ϕ)-graph, $f_s \in \mathscr{F}$, is a positive solution of (62) (if $r < n$) or is a positive constant (if $r = n$).
- Σ has no horizontal points.
- A vertical $\mathscr{F}$-invariant hypersurface of $\mathbb{H}^m_{\mathbb{F}} \times \mathbb{R}$ is r-minimal if and only if $r = n$.
- The graph in (i) has no vertical points.

These facts allow us to argue just as in preceding proofs, and then show the uniqueness of Σ as asserted. □

8 Uniqueness of Rotational H_r-spheres of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$

In this concluding section, we concern the uniqueness of the rotational H_r-spheres we constructed in Sect. 4. We restrict ourselves to $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, with $\epsilon \in \{-1, 1\}$ and $n \geq 3$. As we mentioned before, the case $n = 2$ was considered in [1, 17].

We obtain a Jellett–Liebmann-type theorem by showing that a compact, connected and strictly convex H_r-hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ is a rotational embedded sphere (cf. Theorem 17). We also show the uniqueness of these spheres under completeness or properness assumptions, instead of compactness (cf. Theorem 18 and Corollary 2).

For the proof of Theorem 18, we make use of a height estimate for convex graphs in $M \times \mathbb{R}$ which we establish in the next proposition. First, we compute the Laplacian of both the height function ξ and the angle function Θ of an arbitrary hypersurface Σ of a general product $M \times \mathbb{R}$.

Given a smooth function ζ on Σ, let us denote its Laplacian by $\Delta\zeta$, i.e.,

$$\Delta\zeta := \mathrm{trace}(\mathrm{Hess}\,\zeta).$$

In particular, from Eq. (4), the Laplacian of ξ is given by

$$\Delta\xi = \Theta H, \quad H = H_1\,. \tag{64}$$

Recall that, for $X, Y \in T\Sigma$, the Codazzi Eq. reads as

$$(\overline{R}(X, Y)N)^\top = \left(\nabla_Y A\right)X - \left(\nabla_X A\right)Y,$$

where $\overline{R}$ is the curvature tensor of $M \times \mathbb{R}$, $\top$ denotes the tangent component of the tangent bundle $T\Sigma$ of Σ, and, by definition,

$$\left(\nabla_Y A\right)X := \nabla_Y AX - A\nabla_Y X.$$

Observing that

$$\nabla_X \nabla \xi = \left(\overline{\nabla}_X \nabla \xi\right)^{\top} = -\left(\overline{\nabla}_X \Theta N\right)^{\top} = \Theta A X,$$

we have from (3) that

$$-\nabla_X \nabla \Theta = \nabla_X A \nabla \xi = \left(\nabla_X A\right)\nabla \xi + A \nabla_X \nabla \xi = \left(\nabla_X A\right)\nabla \xi + \Theta A^2 X,$$

which yields

$$\nabla_X \nabla \Theta = -(\overline{R}(\nabla \xi, X)N)^{\top} - \left(\nabla_{\nabla \xi} A\right)X - \Theta A^2 X. \tag{65}$$

Now, let us fix $x \in \Sigma$ and an orthonormal frame $\{X_1, \dots, X_n\}$ in a neighborhood of x in Σ, which is geodesic at x, that is

$$\nabla_{X_i} X_j(x) = 0 \ \ \forall i,j = 1, \dots, n.$$

Writing $\xi_j = X_j(\xi)$, we have $\nabla \xi = \sum_j \xi_j X_j$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} \langle \left(\nabla_{\nabla \xi} A\right) X_i, X_i \rangle &= \sum_{i=1}^{n} \left(\langle \nabla_{\nabla \xi} A X_i, X_i \rangle - \langle A \nabla_{\nabla \xi} X_i, X_i \rangle\right) \\
&= \sum_{i,j=1}^{n} \xi_j (\langle \nabla_{X_j} A X_i, X_i \rangle - \langle A \nabla_{X_j} X_i, X_i \rangle) \\
&= \sum_{i,j=1}^{n} \xi_j (X_j \langle A X_i, X_i \rangle - \langle A X_i, \nabla_{X_j} X_i \rangle - \langle A \nabla_{X_j} X_i, X_i \rangle) \\
&= \langle \nabla \xi, \nabla H \rangle - \sum_{i,j=1}^{n} \xi_j (\langle A X_i, \nabla_{X_j} X_i \rangle - \langle A \nabla_{X_j} X_i, X_i \rangle),
\end{aligned}$$

which implies that, at the chosen point $x \in \Sigma$,

$$\sum_{i=1}^{n} \langle \left(\nabla_{\nabla \xi} A\right) X_i, X_i \rangle = \langle \nabla \xi, \nabla H \rangle.$$

Since x is arbitrary, we get from this last equality and (65) that, on Σ,

$$\Delta \Theta = \overline{\mathrm{Ric}}(\nabla \xi, N) - \langle \nabla \xi, \nabla H \rangle - \Theta \|A\|^2, \tag{66}$$

where $\overline{\mathrm{Ric}}$ denotes the Ricci curvature tensor of $M \times \mathbb{R}$ and $\|A\|^2 := \mathrm{trace}\, A^2$.

Remark 7 For the next results, except for Theorem 19, we order the principal curvatures of a hypersurface Σ of $M \times \mathbb{R}$ as

$$k_1 \le k_2 \le \cdots \le k_{n-1} \le k_n .$$

Proposition 3 *Consider an arbitrary Riemannian manifold M, and let $\Sigma \subset M \times \mathbb{R}$ be a compact vertical graph of a nonnegative function defined on a domain $\Omega \subset M \times \{0\}$. Assume Σ strictly convex up to $\partial \Sigma \subset M \times \{0\}$. Under these conditions, the following height estimate holds*:

$$\xi(x) \leq \frac{1}{\inf_\Sigma k_1} \quad \forall x \in \Sigma. \tag{67}$$

Proof Consider in Σ the "inward" orientation, so that its angle function Θ is nonpositive. Choose $\delta > 0$ satisfying $1/\delta < \inf_\Sigma k_1$ and define on Σ the function

$$\varphi = \xi + \delta\Theta.$$

We claim that φ has no interior maximum. Indeed, assuming otherwise, let $x \in \Sigma - \partial\Sigma$ be a maximum point of φ. In this case, from (3), we have

$$0 = \nabla\varphi(x) = \nabla\xi(x) + \delta\nabla\Theta(x) = \nabla\xi(x) - \delta A\nabla\xi(x).$$

Hence, if we had $\nabla\xi(x) \neq 0$, then $1/\delta$ would be an eigenvalue of A at x, which is impossible, by our choice of δ. Thus, x is a critical point of ξ. Since Σ is strictly convex, x is necessarily its highest point. In particular, $\Theta(x) = -1$. This, together with identities (64) and (66), gives that, at x,

$$0 \geq \Delta\varphi = -H + \delta\|A\|^2. \tag{68}$$

However, from our choice of δ and the strict convexity of Σ, we have

$$\frac{H}{\delta} < k_1 H = k_1(k_1 + \cdots + k_n) \leq k_1^2 + \cdots + k_n^2 = \|A\|^2,$$

which contradicts (68). Therefore, φ attains its maximum on $\partial\Sigma$, which implies that $\varphi \leq 0$ on Σ, for $\varphi|_{\partial\Sigma} = \delta\Theta \leq 0$. Hence,

$$\xi(x) \leq -\delta\Theta(x) \leq \delta \quad \forall x \in \Sigma.$$

The result, then, follows from this last inequality, since it holds for any positive $\delta > 1/\inf_\Sigma k_1$. □

Remark 8 Proposition 3 has its own importance, since it establishes height estimates for vertical graphs in $M \times \mathbb{R}$ making no assumptions on M. In addition, no curvature of such a graph is assumed to be constant.

In the next two theorems, we apply the Alexandrov reflection technique. Since the arguments are standard, the proofs will be somewhat sketchy on this matter (see, e.g., [7, Theorems 4.2 and 5.1] and [24, Theorem 1.1]). We add that the proof of Theorem 17 is, essentially, the one for [8, Corollary 1], in which the case $r = 1$ was considered.

Theorem 17 (Jellett–Liebmann-type theorem) *Let Σ be a compact connected strictly convex $H_r(> 0)$-hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ $(n \geq 3)$. Then, Σ is an embedded rotational H_r-sphere.*

Proof Since Σ is compact, its height function ξ has a maximal point x. This, together with the strict convexity of Σ, allows us to apply [8, Theorems 1 and 2] and conclude that Σ is embedded and homeomorphic to $\mathbb{S}^n$. Thus, for $\epsilon = -1$, the result follows from [15, Theorem 7.6], the Alexandrov-type theorem we mentioned in the introduction.

For $\epsilon = 1$, we can perform Alexandrov reflections on Σ with respect to horizontal hyperplanes $P_t := \mathbb{S}^n \times \{t\}$ coming down from above Σ. For some $t_0 < \xi(x)$, the reflection of the part of Σ above P_{t_0} will have a first contact with Σ. Then, by the tangency principle Σ is symmetric with respect to P_{t_0}. Therefore, assuming $t_0 = 0$ and identifying $\mathbb{S}^n \times \{0\}$ with $\mathbb{S}^n$, we conclude that Σ is a bigraph over its projection $\pi(\Sigma)$ to $\mathbb{S}^n$. As a consequence, $\Sigma_0 := \Sigma \cap \mathbb{S}^n$ is the boundary of $\pi(\Sigma)$ in $\mathbb{S}^n$.

By [8, Lemma 1], the second fundamental form of Σ_0, as a hypersurface of $\mathbb{S}^n$, is positive definite. In particular, Σ_0 is nontotally geodesic in $\mathbb{S}^n$. Thus, by [11, Theorem 1], Σ_0 is contained in an open hemisphere $\mathbb{S}^n_+$ of $\mathbb{S}^n$, which implies that the same is true for $\pi(\Sigma)$, that is, $\Sigma \subset \mathbb{S}^n_+ \times \mathbb{R}$. In this setting, we can apply Alexandrov reflections on "vertical hyperplanes" $(\mathbb{S}^{n-1} \cap \mathbb{S}^n_+) \times \mathbb{R}$, where $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ is a totally geodesic $(n-1)$-sphere of $\mathbb{S}^n$, and conclude that Σ is rotational. □

Let us show now that, regarding Theorem 17, the compactness hypothesis can be replaced by completeness if we add a one point condition on the height function of Σ. In the case $\epsilon = -1$, we also have to impose a condition on the second fundamental form of Σ, which turns out to be a necessary hypothesis (see Remark 10, below).

Theorem 18 *Let Σ be a complete connected strictly convex $H_r(>0)$-hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ $(n \geq 3)$ whose height function ξ has a local extreme point. For $\epsilon = -1$, assume further that the least principal curvature k_1 of Σ is bounded away from zero. Then, Σ is an embedded rotational sphere.*

Proof As in the previous theorem, Σ fulfills the hypotheses of [8, Theorems 1 and 2], which implies that Σ is properly embedded and homeomorphic to either $\mathbb{S}^n$ or $\mathbb{R}^n$. Furthermore, in the latter case, the height function of Σ is unbounded and has a single extreme point x, which we assume to be a maximum.

For $\epsilon = 1$, the height estimates obtained in [7, Theorem 4.1-(i)] forbid ξ to be unbounded. Thus, in this case, Σ is homeomorphic to $\mathbb{S}^n$ and the result follows from Theorem 17.

Let us consider now the case $\epsilon = -1$. Assume, by contradiction, that Σ is homeomorphic to $\mathbb{R}^n$, so that ξ is unbounded below. Hence, given a horizontal hyperplane $P_t = M \times \{t\}$ with $t < \xi(x)$, the part Σ_t^+ of Σ which lies above P_t must be a vertical graph with boundary in P_t. If not, for some t' between t and $\xi(x)$, $P_{t'}$ would be orthogonal to Σ at one of its points. Then, the Alexandrov reflection method would give that Σ is symmetric with respect to $P_{t'}$, which is impossible, since we are assuming ξ unbounded, and the closure of $\Sigma_{t'}^+$ in Σ is compact.

It follows from the above that, for $|t|$ sufficiently large, one has

$$\xi(x) - t > \frac{1}{\inf_\Sigma k_1} \geq \frac{1}{\inf_{\Sigma_t^+} k_1},$$

which clearly contradicts Proposition 3. Therefore, Σ is homeomorphic to $\mathbb{S}^n$ and, again, the result follows from Theorem 17. □

Remark 9 In Theorems 17 and 18, the hypothesis of strict convexity of Σ is automatically satisfied for $r = n$, so it can be dropped in this case. Indeed, in both theorems, the height function ξ has a critical point $x \in \Sigma$, which can be assumed to be a maximum. Then, taking the inward orientation on Σ, we have that $\Theta(x) = -1$, which, together with equality (4), yields

$$\langle AX, X\rangle = -\operatorname{Hess}\xi(X,X) \geq 0 \ \ \forall X \in T_x\Sigma.$$

However, $H_n = \det A > 0$ on Σ. Thus, at x, and then on all of Σ, the second fundamental form is positive definite, that is, Σ is strictly convex.

Remark 10 It follows from the considerations of Remark 1 in Sect. 4 that, for $r < n$, the hypothesis on the least principal curvature of Σ in Theorem 18 is necessary. As shown by Theorem 5, the same is true for the hypothesis on the height function ξ in the case $\epsilon = 1$ and $r < n$.

Now, we consider the dual case of Theorem 18, assuming that the height function of the hypersurface Σ has no critical points. First, we recall that if M is an arbitrary Riemannian manifold, a hypersurface $\Sigma \subset M \times \mathbb{R}$ is said to be *cylindrically bounded* if there exists a closed geodesic ball $B \subset M$ such that $\Sigma \subset B \times \mathbb{R}$. In particular, *if M is compact, any hypersurface $\Sigma \subset M \times \mathbb{R}$ is cylindrically bounded.*

Theorem 19 *Assume $n \geq 3$, and let Σ be a proper, convex, connected $H_r(>0)$-hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with no horizontal points. Then, if Σ is cylindrically bounded, it is a cylinder over a geodesic sphere of $\mathbb{Q}^n_\epsilon$. In particular, $r < n$.*

Proof From the hypothesis and [8, Theorem 3], $\Sigma = \Sigma_0 \times \mathbb{R}$, where Σ_0 is an embedded convex topological sphere of $\mathbb{Q}^n_\epsilon$. Moreover, in the case $\epsilon = 1$, Σ_0 is contained in an open hemisphere of $\mathbb{S}^n$.

At a given point $x \in \Sigma$, the principal curvatures are $k_1, \dots, k_{n-1}, 0$, where $k_1, \dots, k_{n-1}$ are the principal curvatures of $\Sigma_0 \subset \mathbb{Q}^n_\epsilon$ at $\pi_{\mathbb{Q}^n_\epsilon}(x) \in \Sigma_0$. In particular, Σ_0 has constant rth mean curvature H_r if $r < n$, which implies that, in this case, Σ_0 is a geodesic sphere of $\mathbb{Q}^n_\epsilon$ (see [21, 23]). Also, $H_n = 0$ on Σ, so we must have $r < n$, since we are assuming $H_r > 0$. □

Since a cylinder $\Sigma_0 \times \mathbb{R} \subset \mathbb{Q}^n_\epsilon \times \mathbb{R}$ is nowhere strictly convex, it follows from the above theorem that, for $n \geq 3$, a connected, proper, cylindrically bounded and strictly convex H_r-hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ must have a horizontal point. This fact, together with Theorem 18, gives our last result:

Corollary 2 *For $n \geq 3$, any connected, properly immersed and strictly convex $H_r(>0)$-hypersurface $\Sigma \subset \mathbb{S}^n \times \mathbb{R}$ is necessarily an embedded rotational H_r-sphere. The same is true for $\Sigma \subset \mathbb{H}^n \times \mathbb{R}$ if we assume further that Σ is cylindrically bounded and has least principal curvature bounded away from zero.*

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