



Bidisperse filtration problem with non-monotonic retention profiles

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Abstract

During deep bed filtration of suspensions and colloids in a porous medium, some particles are retained in the pores and form a fixed deposit. A one-dimensional mathematical model of filtration with particles of two types is considered. Exact solution is derived. The existence and the uniqueness of the solution are proved by the method of characteristics, and a solution in the form of a traveling wave is obtained. The profiles of total and partial retained concentrations, showing the dependence of the retained particles concentrations on the coordinate at a fixed time, are studied. It is shown by Taylor expansions that the retained profiles of large particles decrease monotonically, while the retained profiles of small particles are non-monotonic. At a short time, the profile of small particles decreases monotonically; with increasing time, a maximum point appears on it, moving from the inlet to the outlet of the porous medium. When the maximum point reaches the outlet, the profile becomes monotonically increasing. The condition for the non-monotonicity of the total retained profile is obtained.

Keywords Deep bed filtration · Quasilinear hyperbolic system · Exact solution · Unique solvability · Retention profiles

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1 Introduction

Various natural phenomena and technological processes are associated with the filtration of suspensions and colloids in porous media: oil production, soil strengthening, treatment of industrial and municipal wastewater, the spread of bacteria and viruses in groundwater, biological restoration of reservoirs, and much more [1–7].

In the course of filtration in a porous medium, the tiny particles are transported by the carrier fluid through the pores. During deep bed filtration, particles are retained not only at the inlet, but also in the entire depth of the porous medium and form a stationary deposit [8–12]. Depending on the physicochemical properties of the particles and of the porous medium, the particles are blocked by electrostatic, hydrodynamic and gravitational forces [13–16]. The retained particles either clog the pore throats singly (size-exclusion), or in groups (bridging), or enter dead-end pores, or settle on the walls of wide pores (attachment) [17–21].

If a suspension or colloid is filled with identical particles, it is called monodisperse; polydisperse suspensions contain particles of different sizes or different properties. A bidisperse suspension or colloid contains two types of particles.

The simplest macroscopic dimensionless model of filtration of a monodisperse suspension or colloid in a porous medium with a size-exclusion retention mechanism is used when the porous medium contains pores of both small and large sizes in comparison with the particle diameter. The model includes the equation for the balance of the suspended and retained particles concentrations $C(x, t)$, $S(x, t)$ and the kinetic equation for the deposit growth [22, 23]

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \frac{\partial S}{\partial t} = 0, \quad \frac{\partial S}{\partial t} = \lambda \left(1 - \frac{S}{S_m}\right) C. \quad (1)$$

Here S_m is the maximum concentration of retained particles.

When the number of particles in the carrier fluid is small, the growth of the retained particles concentration is proportional to the first degree of the suspended particles concentration. The proportionality coefficient depends on the retained concentration and is called the filtration function. Even when the filtration function is linear as in Eq. (1), the system of equations is quasilinear and the model generates nonlinear effects. When the retained particles concentration reaches the limit value, all the pores which are small in cross section are blocked by the retained particles and the suspended particles are being freely transported through the large pores without formation of deposit.

The initial and boundary conditions for the system (1)

$$x = 0 : C = 1, \quad t = 0 : C = 0, \quad S = 0 \quad (2)$$

correspond to the injection of a suspension or a colloid with the constant concentration 1 into a porous medium filled with a pure fluid without suspended and retained particles.

The exact solution to the system (1) with the conditions (2) has a discontinuity on the concentration front $t = x$

$$C(x, t) = \begin{cases} 0, & t < x, \\ \frac{e^{\lambda(t-x)/S_m}}{e^{\lambda(t-x)/S_m} + e^{\lambda x} - 1}, & t > x; \end{cases} \quad S(x, t) = \begin{cases} 0, & t < x, \\ \frac{S_m(e^{\lambda(t-x)/S_m} - 1)}{e^{\lambda(t-x)/S_m} + e^{\lambda x} - 1}, & t > x. \end{cases}$$

For the investigation of the filtration process, it is important to know the behavior of the retention profile, i.e. the dependence of the retained particles concentration on the x coordinate

at a fixed time t . Differentiating the second solution $S(x, t)$ with respect to the variable x , we see that $\partial S/\partial x < 0$, so the retention profile of identical particles decreases monotonically at any time t .

Exact solutions to one-dimensional filtration problems for monodisperse suspensions and colloids with any arbitrary filtration function are obtained in [24, 25]. However, the exact solutions were unknown for the problem of filtration of bidisperse suspensions and colloids. In a number of papers, the bidisperse filtration in a porous medium is studied experimentally and numerically [26–35]. It is shown that the total and partial retention profiles can be non-monotonic. However, there was no theoretical explanation for the profile non-monotonicity. This article provides an explanation for the non-monotonicity of the retention profiles. The exact solution of a 4×4 system for bidisperse suspension or colloid filtration in a porous medium is obtained, and the profiles of total and partial retained concentrations are studied. The conditions for the monotonicity/non-monotonicity of the total and partial retention profiles are obtained. Some of the results of the present paper were announced in a brief form and without proof in [36].

Section 2 presents a macroscopic deep bed filtration model for bidisperse suspensions and colloids in porous media; a solution in the form of a traveling wave is obtained. The main results are formulated in Sect. 3. Section 4 is devoted to proving the existence and uniqueness theorems for the solution. The theorems concerning the properties of retention profiles are proved in Sect. 5. Numerical calculations and profile plots are given in Sect. 6. Discussion and Conclusions in Sects. 7 and 8 end the article.

2 Mathematical model for the binary system

Let us consider deep bed filtration of a binary mixture in porous media. The particles, rock, and carrier water are incompressible. The additivity of volumes for both populations in water during retention is assumed (Amagat’s law [1]). It allows introducing volumetric concentration of retained and suspended particles. The volume of retained particles is significantly lower than the porous space, so the porosity is assumed to remain constant. For each population, the retention rate is proportional to its suspension concentration and to the vacant concentration on the rock surface (Langmuir’s blocking [4, 37]); the proportionality constants $\lambda_i, i = 1, 2$ are called the filtration coefficients, which are assumed to be constant. Filtration coefficient is the particle capture probability per unit length of its trajectory [38]. The term “vacant” is attributed to various particle capture mechanisms, like attachment, straining, size exclusion, and interception under electrostatic attraction [37]; to be specific, in this paper we consider particle attachment alone. We discuss the colloidal transport with a constant flow rate. This model describes suspension-colloidal-nano-transport of binary mixtures with retention and competition for free sites on the surface of a porous medium. Those assumptions are typical in the modelling of suspension-colloid-nano-transport in porous media and its numerous applications [4, 7, 12, 16, 21, 28].

In the quadrant $\Omega = \{x \geq 0, t \geq 0\}$ consider the system

$$\frac{\partial c_i}{\partial t} + \frac{\partial c_i}{\partial x} + \frac{\partial s_i}{\partial t} = 0, \tag{3}$$

$$\frac{\partial s_i}{\partial t} = (1 - b)\lambda_i c_i, \quad b = B_1 c_1^0 s_1 + B_2 c_2^0 s_2, \quad i = 1, 2, \tag{4}$$

where λ_i, B_i, c_i^0 are positive constants and $\lambda_1 > \lambda_2, c_1^0 + c_2^0 = 1$. Here $c_i, s_i, i = 1, 2$ are the suspended and retained particles concentrations, respectively, b is the concentration of occupied sites, B_i is the individual area that an attached particle occupies at the rock surface, and c_i^0 are the particle concentrations in the injected suspension.

Consider the solution of the system (3), (4) in the form of travelling wave [39]:

$$c_i = c_i(w), \quad s_i = s_i(w), \quad w = x - ut, \quad i = 1, 2, \tag{5}$$

where u is the yet unknown constant velocity of the traveling wave.

The substitution of the expressions (5) into the system (3), (4) yields a system of ordinary differential equations

$$(1 - u)c_i' - us_i' = 0, \quad -us_i' = (1 - b)\lambda_i c_i, \quad i = 1, 2. \tag{6}$$

For the uniqueness of the solution, the conditions are set at infinity:

$$w \rightarrow +\infty : c_i \rightarrow 0, \quad s_i \rightarrow 0, \quad i = 1, 2; \quad w \rightarrow -\infty : c_i \rightarrow c_i^0, \quad i = 1, 2. \tag{7}$$

Integrate the first Eq. (6) taking into account the conditions (7) at $w \rightarrow +\infty$

$$(1 - u)c_i = us_i, \quad i = 1, 2. \tag{8}$$

Substitute the second Eq. (6) into the first one

$$(1 - u)c_i' + (1 - b)\lambda_i c_i = 0, \quad i = 1, 2. \tag{9}$$

and express the term $1 - b$

$$1 - b = -\frac{(1 - u)c_i'}{\lambda_i c_i}, \quad i = 1, 2. \tag{10}$$

Equating the right-hand parts of Eqs. (10) at $i = 1$ and $i = 2$ and integrating them in w with conditions (7) at $w \rightarrow -\infty$, we obtain

$$c_2 = c_2^0 \left(\frac{c_1}{c_1^0} \right)^{\lambda_2/\lambda_1}. \tag{11}$$

Substitute the formulae (8), (11) into Eq. (9) for $i = 1$:

$$(1 - u)c_1' + \lambda_1 c_1 \left(1 - \frac{1 - u}{u} B_1 c_1^0 c_1 - \frac{1 - u}{u} B_2 (c_2^0)^2 \left(\frac{c_1}{c_1^0} \right)^{\lambda_2/\lambda_1} \right) = 0. \tag{12}$$

To single out a specific solution of Eq. (12), one should specify the solution at a given point, say, at $w = 0$:

$$c_1|_{w=0} = c_1^0 p, \quad 0 < p < 1. \tag{13}$$

The solution of Eq. (12) with the condition (13) is

$$\int_{c_{1p}^0}^{c_1} \frac{dc}{c \left(\frac{u}{1-u} - B_1 c_1^0 c - B_2 (c_2^0)^2 \left(\frac{c}{c_1^0} \right)^{\lambda_2/\lambda_1} \right)} = -\lambda_1 \frac{w}{u}. \tag{14}$$

From the formula (14) and the condition (7) at $w \rightarrow -\infty$, we obtain the travelling wave velocity

$$u = \frac{B_1 (c_1^0)^2 + B_2 (c_2^0)^2}{1 + B_1 (c_1^0)^2 + B_2 (c_2^0)^2}. \tag{15}$$

The solution (14) decreases monotonically from c_1^0 at $w \rightarrow -\infty$ to 0 at $w \rightarrow +\infty$.

The formulae (8), (11) and (14) determine the solution of the system (3), (4) in the form of travelling wave.

Consider the initial-boundary conditions for the system (3), (4)

$$x = 0 : c_i = c_i^0, \quad t = 0 : c_i = 0, \quad s_i = 0, \quad i = 1, 2. \tag{16}$$

The solutions $c_1(x, t)$, $c_2(x, t)$ have a discontinuity on the characteristic line $t = x$, because the initial and boundary conditions do not match at the origin. The line $t = x$ is the concentration front Γ of the suspended and retained particles which divides the interior Ω^0 of the quadrant Ω into two zones [40]. In the domain $\Omega_0 = \{x > 0, 0 < t < x\}$ the problem has a zero solution; in the domain $\Omega_1 = \{x > 0, t > x\}$ the solution is positive (see Theorem 2). It is shown below that in the domain Ω_1 the solution is given by formulae similar to (11), (14).

Using formula (11) the system (3), (16) can be reduced to a 3×3 system for the overall suspended and retained particles concentrations and the concentration of occupied sites. Denote

$$s = s_1 + s_2, \quad c = c_1 + c_2, \quad f(c) = \lambda_1 c_1 + \lambda_2 c_2, \quad d(c) = B_1 \lambda_1 c_1^0 c_1 + B_2 \lambda_2 c_2^0 c_2, \quad w(c) = \int_c^1 \frac{d(y)}{f(y)} dy.$$

$f(c)$ and $d(c)$ are called the suspension and occupation functions.

Adding in pairs at $i = 1$ and $i = 2$ Eqs. (3), (4) and (4) multiplied by $B_1 c_1^0$ and $B_2 c_2^0$ yields a system of equations for three unknowns b, s, c :

$$\frac{\partial b}{\partial t} = (1 - b)d(c), \quad \frac{\partial s}{\partial t} = (1 - b)f(c), \quad \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = -(1 - b)f(c).$$

The initial-boundary conditions for this system follow from the conditions (16):

$$x = 0 : c = 1, \quad t = 0 : c = 0, \quad b = 0, \quad s = 0.$$

Similar to the system (3), (16), in the domain Ω_0 the solution to the system 3×3 is zero; in the domain Ω_1 the solution is positive. On the concentration front $t = x$, the solution c is discontinuous, the solutions s and b are continuous, and $s = b = 0$. In the domain Ω_1 the solution is given by the formulae [41, Sect. 5.2]

$$\int_{c^-(x)}^{c(x,t)} \frac{dy}{f(y)w(y)} = t - x, \quad b(x, t) = 1 - \frac{w(c(x, t))}{w(c^-(x))}, \quad s(x, t) = \frac{c(x, t) - c^-(x)}{w(c^-(x))},$$

where c^- is the solution on the concentration front defined implicitly by the relation $\int_{c^-(x)}^1 \frac{dy}{f(y)} = x$.

3 Main results

Since the solutions are discontinuous in Ω on the line $t = x$, consider weak solutions of the system (3) [42]. A function φ is said to be piecewise continuous in a domain Ω if it is continuous in the closures $\overline{\Omega}_0$ and $\overline{\Omega}_1$ of the domains Ω_0 and Ω_1 and is discontinuous on the boundary Γ of these domains. A function φ is called piecewise smooth if its partial derivatives φ'_t and φ'_x are piecewise continuous in Ω .

Definition 1 A weak solution to the system (3), (4) with the conditions (16) is such a set of functions $c_i, s_i, i = 1, 2$ that

- a) the functions c_1, c_2 are piecewise differentiable in Ω ;
- b) the functions s_1, s_2 are piecewise differentiable and continuous in Ω ;
- c) Eqs. (3), (4) are satisfied in the weak sense, i.e. any function $\varphi(x, t) \in C_0^\infty(\Omega^0)$ satisfies the equalities

$$\iint_{\Omega} \left((c_i + s_i) \frac{\partial \varphi}{\partial t} + c_i \frac{\partial \varphi}{\partial x} \right) dxdt = 0, \quad \iint_{\Omega} \left(s_i \frac{\partial \varphi}{\partial t} + \lambda_i(1 - b)c_i \varphi \right) dxdt = 0, \quad i = 1, 2,$$

- d) the conditions (16) are satisfied in the strong sense (pointwise).

Theorem 1 In the domain Ω there exists a weak unique solution of the system (3), (4) with the conditions (16).

Denote the distance between points $(t, x) \in \Omega_1$ and $(x, x) \in \Gamma$

$$\rho = t - x.$$

Theorem 2

1. The solution to the problem (3), (4), (16) is zero in the domain Ω_0 and positive in the domain Ω_1 .
2. In the domain $\overline{\Omega}_1$, the solution is given by the formulae

$$\int_{c_1^-}^{c_1} \frac{dc}{c \left(B_1 c_1^0 (c_1^0 - c) + B_2 c_2^0 \left(c_2^0 - c_2^0 \left(\frac{c}{c_1^0} \right)^{\lambda_2/\lambda_1} \right) \right)} = \lambda_1 \rho, \tag{17}$$

$$\int_{c_2^-}^{c_2} \frac{dc}{c \left(B_1 c_1^0 \left(c_1^0 - c_1^0 \left(\frac{c}{c_2^0} \right)^{\lambda_1/\lambda_2} \right) + B_2 c_2^0 (c_2^0 - c) \right)} = \lambda_2 \rho,$$

$$s_i = \frac{c_i - c_i^-}{B_1 c_1^0 (c_1^0 - c_1^-) + B_2 c_2^0 (c_2^0 - c_2^-)}, \quad i = 1, 2, \tag{18}$$

where

$$c_i^- = c_i(x, x) = c_i^0 e^{-\lambda_i x} \tag{19}$$

is the solution on the concentration front Γ .

In particular, at the inlet $x = 0$ the solution (18) takes the form

$$s_i^0(t) = \frac{\lambda_i c_i^0}{B} (1 - e^{-Bt}), \quad i = 1, 2, \quad B = \lambda_1 B_1 (c_1^0)^2 + \lambda_2 B_2 (c_2^0)^2. \tag{20}$$

Corollary 1 At $x > 0$, $\rho \rightarrow \infty$.

$$c_i \rightarrow c_i^0, \quad s_i \rightarrow \frac{c_2^0 - c_i^-}{B_1 c_1^0 (c_1^0 - c_1^-) + B_2 c_2^0 (c_2^0 - c_2^-)}, \quad i = 1, 2. \tag{21}$$

The following theorems are devoted to the retention profiles.

Theorem 3 In the domain $\overline{\Omega}_1$ the partial and total retention profiles $s_1(x, t)$, $s_2(x, t)$, $s(x, t)$ monotonously decrease

1. for a fixed $t > 0$ at $x \rightarrow t$,
2. for a fixed $x \geq 0$ at $t \rightarrow x$.

Theorem 4 Let $\lambda_1 > \lambda_2$. For a fixed $x > 0$ at $t \rightarrow \infty$.

1. the profile $s_1(x, t)$ monotonously decreases;
2. the profile $s_2(x, t)$ monotonously increases;
3. the profile $s(x, t)$ monotonously increases if $B_1 c_1^0 > B_2 c_2^0$, and monotonously decreases if $B_1 c_1^0 < B_2 c_2^0$.

Denote

$$t_0 = \frac{1}{B} \ln \frac{D + \lambda_2 B}{D + B^2 - B\sqrt{B^2 + 2D + (\lambda_2)^2}}, \quad T_0 = \frac{1}{B} \ln \frac{\lambda D + \mu B}{\lambda(D + B^2) - B\sqrt{\lambda^2(B^2 + 2D) + \mu^2}},$$

where

$$\lambda = \lambda_1 c_1^0 + \lambda_2 c_2^0, \quad D = (\lambda_1)^2 B_1 (c_1^0)^2 + (\lambda_2)^2 B_2 (c_2^0)^2, \quad \mu = (\lambda_1)^2 c_1^0 + (\lambda_2)^2 c_2^0.$$

Theorem 5 Let $\lambda_1 > \lambda_2$. In the domain $\bar{\Omega}_1$ at $x \rightarrow 0$.

1. The partial retention profile $s_1(x, t)$ decreases monotonically for all $t > x$;
2. The partial retention profile $s_2(x, t)$ decreases monotonically for $x < t < t_0$ and increases monotonically for $t > t_0$;
3. The total retention profile $s(x, t)$ decreases monotonically for all $t > x$ if $B_1 c_1^0 < B_2 c_2^0$, decreases monotonically at $x < t < T_0$ and increases monotonically at $t > T_0$ if $B_1 c_1^0 > B_2 c_2^0$.

It follows from Theorems 3–5 that any non-monotonic retention profile has a maximum point. Theorem 6 describes the positions of the maximum points of the profiles at large t and x .

Theorem 6 Let $\lambda_1 > \lambda_2$. Then

1. The integral

$$I = \int_0^{c_2^0} \left(\frac{1}{c \left(B_1 c_1^0 (c_1^0 - c_1^0 \left(\frac{c}{c_2^0} \right)^{\lambda_1/\lambda_2}) + B_2 c_2^0 (c_2^0 - c) \right)} - \frac{\lambda_2}{B(c_2^0 - c)} - \frac{1}{Ac} \right) dc$$

converges;

2. In the domain Ω_1 at $x \rightarrow \infty$, $\rho \rightarrow \infty$ the coordinates of the maximum points $t_m(x)$ and $T_m(x)$ of the profiles $s_2(x, t)$ and $s(x, t)$ allow estimates

$$t_m(x) = \left(\frac{\lambda_2}{B} + \frac{1}{A} + 1 \right) x + \frac{I}{\lambda_2} - \frac{1}{B} \ln \frac{\lambda_2 B_1 (c_1^0)^2}{B} + O(e^{-\lambda_2 x} + e^{-(\lambda_1 - \lambda_2)x}), \quad (22)$$

$$T_m(x) = \left(\frac{\lambda_2}{B} + \frac{1}{A} + 1 \right) x + \frac{I}{\lambda_2} - \frac{1}{B} \ln \frac{(\lambda_2)^2 c_1^0 c_2^0 (B_1 c_1^0 - B_2 c_2^0)}{B(\lambda_1 c_1^0 + \lambda_2 c_2^0)} + O(e^{-\lambda_2 x} + e^{-(\lambda_1 - \lambda_2)x}), \quad (23)$$

at $t \rightarrow \infty$ the maximum values of the profiles $s_2(x, t)$ and $s(x, t)$ tend to c_2^0/A and $1/A$, respectively. Here

$$A = B_1 (c_1^0)^2 + B_2 (c_2^0)^2.$$

Remark 1 After neglecting the residuals in formulae (22) and (23), we obtain linear expressions in x representing the asymptotes with respect to t_m and T_m .

Remark 2 At $x \rightarrow \infty$, $\rho \rightarrow \infty$ the limit speeds of the maximum points of the profiles $s_2(x, t)$ and $s(x, t)$ coincide and are equal to

$$v = 1 / \left(\frac{\lambda_2}{B} + \frac{1}{A} + 1 \right).$$

4 Proof of Theorems 1, 2

To prove Theorem 1, assume at first that a solution $c_i, s_i, i = 1, 2$ to the problem (3), (4), (16) exists and then derive the formulae of Theorem 2 for the solution. This will prove the uniqueness of the solution: if there exists a solution, then it is necessarily given by these formulae. To prove the existence of the solution, one should reverse the reasoning used to prove the uniqueness of the solution: the functions given by formulae (17)–(20) are a solution to the problem (3), (4), (16). (For brevity, these arguments are omitted.)

4.1 Proof of item 1 of Theorem 2

Substitute Eqs. (4) into (3)

$$\frac{\partial c_i}{\partial t} + \frac{\partial c_i}{\partial x} + (1 - b)\lambda_i c_i = 0, \quad i = 1, 2, \tag{24}$$

and then apply the standard method of characteristics [43] to Eqs. (24). The characteristic equations corresponding to the system (24) have the form

$$\dot{t} = 1, \quad \dot{x} = 1, \quad \dot{c}_i + (1 - b)\lambda_i c_i = 0, \quad i = 1, 2, \tag{25}$$

where the dot means the derivative by τ (the intrinsic time along the characteristics). The conditions for the system (25) in the domains $\bar{\Omega}_0$ and $\bar{\Omega}_1$ are determined by the initial and boundary conditions (16). In the domain $\bar{\Omega}_0$

$$t|_{\tau=0} = 0, \quad x|_{\tau=0} = x_0, \quad c_i|_{\tau=0} = 0, \quad i = 1, 2; \tag{26}$$

in the domain $\bar{\Omega}_1$

$$t|_{\tau=0} = t_0, \quad x|_{\tau=0} = 0, \quad c_i|_{\tau=0} = c_i^o, \quad i = 1, 2. \tag{27}$$

The characteristics, being the solutions of the system (25) with the conditions (26) and (27), are straight lines $x = t + x_0, x = t - t_0$, everywhere densely filling the domains $\bar{\Omega}_0$ and $\bar{\Omega}_1$. In the domain $\bar{\Omega}_0$ the solutions $c_i = 0, i = 1, 2$; according to the initial conditions (16), the solutions to Eqs. (4) are also zero. In the domain $\bar{\Omega}_1$, the solution $c_i = c_i^o \exp\left(-\lambda_i \int_0^\tau (1 - b)d\tau\right), i = 1, 2$ to Eqs. (25) with the conditions (27) is positive.

Multiply Eqs. (4) by $B_i c_i^0$ and add them together. The resulting equation has the form

$$\frac{\partial b}{\partial t} = (B_1 c_1^0 \lambda_1 c_1 + B_2 c_2^0 \lambda_2 c_2)(1 - b). \tag{28}$$

The condition for Eq. (28) follows from the initial conditions (16):

$$t = 0 : b = 0. \tag{29}$$

The solution of Eq. (28) with the condition (29) can be written in the form $b = 1 - \exp\left(-\int_0^t (B_1 c_1^0 \lambda_1 c_1 + B_2 c_2^0 \lambda_2 c_2) dt\right)$. Consequently, $1 - b > 0$ and the solution $s_i = \int_0^t (1 - b) dt$, $i = 1, 2$ of Eqs. (4) with the conditions (16) is positive in Ω_1 .

Item 1 of Theorem 2 is proved.

4.2 Proof of item 2 of Theorem 2

Let us prove the validity of the formulae (19), (20). At the inlet $x = 0$ Eqs. (4) take the form

$$\frac{\partial s_i}{\partial t} = (1 - b) \lambda_i c_i^0, \quad i = 1, 2. \tag{30}$$

Multiply Eqs. (30) by $B_i c_i^0$ and add them together. According to the formula (4), the resulting equation has the form

$$\frac{\partial b}{\partial t} = (1 - b)B. \tag{31}$$

The solution of Eq. (31) with the initial condition (29) is

$$b(0, t) = 1 - e^{-Bt}. \tag{32}$$

The substitution of (32) into Eq. (30) and the integration in t taking into account the initial conditions (16) yield the formulae (20).

Since the solutions s_i , $i = 1, 2$ are continuous in the domain Ω , then in $\overline{\Omega}_1$ on the concentration front Γ the solution b is zero and Eqs. (24) take the form

$$\frac{\partial c_i}{\partial t} + \frac{\partial c_i}{\partial x} + \lambda_i c_i = 0, \quad i = 1, 2. \tag{33}$$

The solution of Eqs. (33) with the boundary conditions (16) is given by the formulae (19).

Proposition 1 *In the domain $\overline{\Omega}_1$ the solutions $c_1(x, t)$, $c_2(x, t)$ satisfy the relations*

$$c_1 = c_1^0 \left(\frac{c_2}{c_2^0} \right)^{\lambda_1/\lambda_2}, \quad c_2 = c_2^0 \left(\frac{c_1}{c_1^0} \right)^{\lambda_2/\lambda_1}. \tag{34}$$

Proof Express $(1 - b)$ from Eq. (24)

$$1 - b = -\frac{1}{\lambda_i c_i} \left(\frac{\partial c_i}{\partial t} + \frac{\partial c_i}{\partial x} \right), \quad i = 1, 2. \tag{35}$$

Equating the right-hand sides of Eqs. (35) at $i = 1$ and $i = 2$, integrating the resulting equation in x and taking into account the boundary conditions (16), we obtain

$$\frac{1}{\lambda_1} \ln \frac{c_1}{c_1^0} = \frac{1}{\lambda_2} \ln \frac{c_2}{c_2^0}. \tag{36}$$

The formulae (34) follow from the relation (36). Proposition 1 is proved.

To prove the formulae (17), substitute (4) into relation (34) for $i = 1$ and express s_2 :

$$s_2 = \frac{1 - B_1 c_1^0 s_1}{B_2 c_2^0} + \frac{1}{\lambda_1 c_1 B_2 c_2^0} \left(\frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial x} \right). \tag{37}$$

The substitution of (35) for $i = 1$ and of (37) into Eq. (4) for $i = 2$ yields

$$\frac{\partial}{\partial t} \left(\frac{1 - B_1 c_1^0 s_1}{B_2 c_2^0} + \frac{1}{\lambda_1 c_1 B_2 c_2^0} \left(\frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial x} \right) \right) = -\frac{\lambda_2 c_2}{\lambda_1 c_1} \left(\frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial x} \right). \tag{38}$$

Since the function $\ln c_1(x, t)$ is continuous and piecewise smooth, its derivatives in the sense of distributions coincide with the classical derivatives. This yields the chain of relations

$$\frac{\partial}{\partial t} \left(\frac{1}{c_1} \frac{\partial c_1}{\partial x} \right) = \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x} \ln c_1 \right) = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} \ln c_1 \right) = \frac{\partial}{\partial x} \left(\frac{1}{c_1} \frac{\partial c_1}{\partial t} \right).$$

Equation (3) at $i = 1$ can be written in a form

$$\frac{\partial s_1}{\partial t} = - \left(\frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial x} \right).$$

Using formula (34) with $i = 2$, transform the right-hand side of Eq. (38)

$$-\frac{\lambda_2 c_2}{\lambda_1 c_1} \left(\frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial x} \right) = - \left(\frac{\partial c_2}{\partial t} + \frac{\partial c_2}{\partial x} \right).$$

Now (38) becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left(\frac{B_1 c_1^0 c_1 + \frac{\partial c_1 / \partial t}{\lambda_1 c_1}}{B_2 c_2^0} + c_2 \right) = 0. \tag{39}$$

Lower the order of Eq. (39)

$$\frac{B_1 c_1^0 c_1 + \frac{\partial c_1 / \partial t}{\lambda_1 c_1}}{B_2 c_2^0} + c_2 = K(t - x). \tag{40}$$

The integration constant $K(t - x)$ is determined from the boundary conditions (16):

$$K = \frac{B_1 (c_1^0)^2 + B_2 (c_2^0)^2}{B_2 c_2^0}.$$

Equation (40) takes the form

$$\frac{\partial c_1 / \partial t}{\lambda_1 c_1} = B_1 c_1^0 (c_1^0 - c_1) + B_2 c_2^0 (c_2^0 - c_2). \tag{41}$$

The first formula (17) is the solution of Eq. (41) with the initial condition (19) on the concentration front $t = x$. The second formula (17) is obtained similarly.

To prove the formulae (18), differentiate equalities (17) with respect to t and x

$$\begin{aligned} \frac{1}{c_i (B_1 c_1^0 (c_1^0 - c_1) + B_2 c_2^0 (c_2^0 - c_2))} \frac{\partial c_i}{\partial t} &= \lambda_i, \\ \frac{1}{c_i (B_1 c_1^0 (c_1^0 - c_1) + B_2 c_2^0 (c_2^0 - c_2))} \frac{\partial c_i}{\partial x} - \frac{1}{c_i^- (B_1 c_1^0 (c_1^0 - c_1^-) + B_2 c_2^0 (c_2^0 - c_2^-))} \frac{\partial c_i^-}{\partial x} &= -\lambda_i, \quad i = 1, 2. \end{aligned} \tag{42}$$

Add Eqs. (42) together for $i = 1$:

$$\frac{1}{c_1 (B_1 c_1^0 (c_1^0 - c_1) + B_2 c_2^0 (c_2^0 - c_2))} \left(\frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial x} \right) - \frac{1}{c_1^- (B_1 c_1^0 (c_1^0 - c_1^-) + B_2 c_2^0 (c_2^0 - c_2^-))} \frac{\partial c_1^-}{\partial x} = 0$$

Transform this relation using Eqs. (3), (4) at $i = 1$ and the formula (19):

$$\frac{-\lambda_1 (1 - b)}{B_1 c_1^0 (c_1^0 - c_1) + B_2 c_2^0 (c_2^0 - c_2)} + \frac{\lambda_1}{B_1 c_1^0 (c_1^0 - c_1^-) + B_2 c_2^0 (c_2^0 - c_2^-)} = 0$$

and then express the function b :

$$b = 1 - \frac{B_1 c_1^0 (c_1^0 - c_1) + B_2 c_2^0 (c_2^0 - c_2)}{B_1 c_1^0 (c_1^0 - c_1^-) + B_2 c_2^0 (c_2^0 - c_2^-)}. \tag{43}$$

The substitution of the first relation (42) and the formula (43) into Eq. (4) at $i = 1$ gives

$$\frac{\partial s_1}{\partial t} = \frac{1}{B_1 c_1^0 (c_1^0 - c_1^-) + B_2 c_2^0 (c_2^0 - c_2^-)} \frac{\partial c_1}{\partial t}. \tag{44}$$

The continuity of the solution $s_i(x, t)$, $i = 1, 2$ in Ω and item 1 of Theorem 2 imply the following condition on the concentration front:

$$t = x : \quad s_i = 0, \quad i = 1, 2. \tag{45}$$

Integrating Eq. (44) in the variable t from x to t , we obtain the formula (18) at $i = 1$. For $i = 2$ the proof is similar.

Theorem 2 is proved.

From Theorem 2 it follows that the solution of the problem (3), (4), (16) must be given by formulae (17)–(20). Consequently, the problem has at most one solution. Inverting the proof of Theorem 2, we obtain the existence of a solution to the problem (3), (4), (16). Theorem 1 is proved.

5 Proof of Theorems 3–6

To calculate the derivatives of the solutions $s_1(x, t)$, $s_2(x, t)$ and of $s = s_1 + s_2$ with respect to the variable x differentiate formula (18):

$$\frac{\partial s_i}{\partial x} = \frac{\left(\frac{\partial c_i}{\partial x} - \frac{\partial c_i^-}{\partial x}\right)(B_1 c_1^0(c_1^0 - c_1^-) + B_2 c_2^0(c_2^0 - c_2^-)) + (c_i - c_i^-)\left(B_1 c_1^0 \frac{\partial c_1^-}{\partial x} + B_2 c_2^0 \frac{\partial c_2^-}{\partial x}\right)}{(B_1 c_1^0(c_1^0 - c_1^-) + B_2 c_2^0(c_2^0 - c_2^-))^2}, \quad i = 1, 2. \tag{46}$$

Express the derivatives $\partial c_i / \partial x$, $\partial c_i^- / \partial x$ using solution (19) and the second formula (42)

$$\frac{\partial c_i^-}{\partial x} = -\lambda_i c_i^-, \quad \frac{\partial c_i}{\partial x} = -\lambda_i c_i (B_1 c_1^0(c_1^0 - c_1) + B_2 c_2^0(c_2^0 - c_2)) \left(1 + \frac{1}{B_1 c_1^0(c_1^0 - c_1^-) + B_2 c_2^0(c_2^0 - c_2^-)}\right).$$

Formulae (46) take the form

$$\frac{\partial s_i}{\partial x} = \frac{-\lambda_i c_i (B_1 c_1^0(c_1^0 - c_1) + B_2 c_2^0(c_2^0 - c_2))(1 + G(x)) + \lambda_i c_i^- G(x) - (c_i - c_i^-)(\lambda_1 B_1 c_1^0 c_1^- + \lambda_2 B_2 c_2^0 c_2^-)}{G^2(x)}, \tag{47}$$

where

$$G(x) = B_1 c_1^0(c_1^0 - c_1^-) + B_2 c_2^0(c_2^0 - c_2^-).$$

Adding formulae (47) at $i = 1$ and $i = 2$ gives the derivative of $s = s_1 + s_2$:

$$\begin{aligned} \frac{\partial s}{\partial x} = & \frac{-(\lambda_1 c_1 + \lambda_2 c_2)(B_1 c_1^0(c_1^0 - c_1) + B_2 c_2^0(c_2^0 - c_2))(1 + G(x))}{G^2(x)} \\ & + \frac{(\lambda_1 c_1^- + \lambda_2 c_2^-)G(x) - (c_1 + c_2 - c_1^- - c_2^-)(\lambda_1 B_1 c_1^0 c_1^- + \lambda_2 B_2 c_2^0 c_2^-)}{G^2(x)}. \end{aligned} \tag{48}$$

In the domain Ω_1 the monotonicity of the partial and total retained concentration profiles $s_1(x, t)$, $s_2(x, t)$, $s(x, t)$ for fixed $t \geq x$ depends on the signs of derivatives (47), (48).

Proof of Theorem 3: Both cases of Theorem 3 ($x \rightarrow t$ and $t \rightarrow x$) mean that $\rho = t - x \rightarrow 0$, so the solutions $c_1 \rightarrow c_1^-$, $c_2 \rightarrow c_2^-$. Application of the mean value theorem to the integrals on the left-hand side of (17) gives the representation.

$$c_i = c_i^- + O(\rho), \quad i = 1, 2. \tag{49}$$

Substitute the expansions (49) into derivatives (47), (48)

$$\frac{\partial s_i}{\partial x} = -\lambda_i c_i + O(\rho), \quad i = 1, 2, \tag{50}$$

$$\frac{\partial s}{\partial x} = -(\lambda_1 c_1 + \lambda_2 c_2) + O(\rho). \tag{51}$$

For small ρ the derivatives (50), (51) are negative and the partial and total retained concentration profiles are monotonously decreasing in x at $\rho \rightarrow 0$.

Theorem 3 is proved.

Proof of Theorem 4: The signs of derivatives (47), (48) coincide with the signs of its numerators. If $t \rightarrow \infty$, then the solution $c_1 \rightarrow c_1^0$, $c_2 \rightarrow c_2^0$. The numerators of the derivatives (47), (48) can be expanded in a form.

$$\begin{aligned}
 N_1 &= B_2 c_2^0 F(\lambda_1, \lambda_2, x) + O((c_2^0 - c_2) + (c_1^0 - c_1)), \\
 N_2 &= -B_1 c_1^0 F(\lambda_1, \lambda_2, x) + O((c_2^0 - c_2) + (c_1^0 - c_1)), \\
 N &= (B_2 c_2^0 - B_1 c_1^0) F(\lambda_1, \lambda_2, x) + O((c_2^0 - c_2) + (c_1^0 - c_1)),
 \end{aligned}
 \tag{52}$$

where

$$F(\lambda_1, \lambda_2, x) = \lambda_1 c_1^-(c_2^0 - c_2^-) - \lambda_2 c_2^-(c_1^0 - c_1^-) = c_1^0 c_2^0 (\lambda_1 e^{-\lambda_1 x} (1 - e^{-\lambda_2 x}) - \lambda_2 e^{-\lambda_2 x} (1 - e^{-\lambda_1 x})).
 \tag{53}$$

At the inlet $x = 0$ the function (53) is zero. The sign of (53) at $x > 0$ is determined by Proposition 2.

Proposition 2 *Let $\lambda_1 > \lambda_2$. Then the function (53) is negative for all $x > 0$.*

Proof Transform the function (53) as follows:

$$F(\lambda_1, \lambda_2, x) = c_1^0 c_2^0 \frac{\lambda_1 (e^{\lambda_2 x} - 1) - \lambda_2 (e^{\lambda_1 x} - 1)}{e^{\lambda_1 x} e^{\lambda_2 x}}.
 \tag{54}$$

Consider the numerator

$$F_1(\lambda_1, \lambda_2, x) = \lambda_1 (e^{\lambda_2 x} - 1) - \lambda_2 (e^{\lambda_1 x} - 1).$$

At $x = 0$ the numerator $F_1(\lambda_1, \lambda_1, 0) = 0$. If $x > 0$, then

$$F_1'(\lambda_1, \lambda_2, x) = \lambda_1 \lambda_2 (e^{\lambda_2 x} - e^{\lambda_1 x}) < 0,$$

and the function $F_1(\lambda_1, \lambda_2, x)$ is decreasing at $x > 0$. Therefore, the numerator $F_1(\lambda_1, \lambda_2, x)$ is negative.

The denominator of the ratio (54) is positive, so the function $F(\lambda_1, \lambda_2, x)$ is negative at $x > 0$.

Proposition 2 is proved.

Theorem 4 follows from the formulae (52) and Proposition 2.

Proof of Theorem 5 Represent the solution of the problem (3), (4), (16) in the form of a series in the powers of x with coefficients depending on $\rho = t - x$:

$$c_i = c_i^0 + c_i^1(\rho)x + O(x^2), \quad s_i = s_i^0(\rho) + s_i^1(\rho)x + O(x^2), \quad i = 1, 2,
 \tag{55}$$

where the functions s_i^0 are given by the formulae (20). Then the functions $s(x, t)$, $b(x, t)$ are expanded in series as follows:

$$\begin{aligned}
 s &= s^0(\rho) + s^1(\rho)x + O(x^2), \quad s^0(\rho) = s_1^0(\rho) + s_2^0(\rho), \quad s^1(\rho) = s_1^1(\rho) + s_2^1(\rho), \\
 b &= b_0(\rho) + b_1(\rho)x + O(x^2), \quad b_0 = B_1 c_1^0 s_1^0 + B_2 c_2^0 s_2^0, \quad b_1 = B_1 c_1^0 s_1^1 + B_2 c_2^0 s_2^1.
 \end{aligned}$$

Substituting the expansions (55) into Eqs. (3), (4) and equating the coefficients at equal powers of x , we obtain:

$$c_i^1 = -(1 - b_0)\lambda_i c_i^0, \quad i = 1, 2; \quad (s_i^1)' = -b_1 \lambda_i c_i^0 + (1 - b_0)\lambda_i c_i^1, \quad i = 1, 2.
 \tag{56}$$

Substitute the first Eq. (56) into the second one

$$(s_i^1)' = -b_1 \lambda_i c_i^0 - (1 - b_0)^2 (\lambda_i)^2 c_i^0, \quad i = 1, 2. \tag{57}$$

Multiply Eqs. (57) by $B_i c_i^0$ and add them together:

$$b_1' = -Bb_1 - D(1 - b_0)^2, \tag{58}$$

where

$$B = \lambda_1 B_1 (c_1^0)^2 + \lambda_2 B_2 (c_2^0)^2, \quad D = (\lambda_1)^2 B_1 (c_1^0)^2 + (\lambda_2)^2 B_2 (c_2^0)^2$$

The initial condition for Eq. (58) follows from the conditions (45) on the concentration front

$$\rho = 0 : \quad b_1 = 0. \tag{59}$$

Using the formula (32), the solution to Eq. (58) with the condition (59) is obtained:

$$b_1 = \frac{D}{B} e^{-B\rho} (e^{-B\rho} - 1). \tag{60}$$

Substitute the formulae (32), (60) in Eq. (57):

$$(s_i^1)' = \lambda_i c_i^0 \frac{D}{B} e^{-B\rho} (1 - e^{-B\rho}) - (\lambda_i)^2 c_i^0 e^{-2B\rho}, \quad i = 1, 2. \tag{61}$$

The integration of Eqs. (61) with the conditions (45) gives

$$s_i^1 = \frac{\lambda_i c_i^0}{2B^2} (1 - e^{-B\rho}) ((D - \lambda_i B) - (D + \lambda_i B) e^{-B\rho}), \quad i = 1, 2. \tag{62}$$

Represent the functions (20), (62) in the form

$$s_i^0(\rho) = s_i^0(t) - \lambda_i c_i^0 e^{-Bt} x + O(x^2), \quad s_i^1(\rho) = s_i^1(t) + O(x), \quad i = 1, 2 \tag{63}$$

and substitute the expansions (63) into the second formula (55):

$$s_i(x, t) = \frac{\lambda_i c_i^0}{B} (1 - e^{-Bt}) + \frac{\lambda_i c_i^0}{2B^2} ((1 - e^{-Bt}) ((D - \lambda_i B) - (D + \lambda_i B) e^{-Bt}) - 2B^2 e^{-Bt}) x + O(x^2), \quad i = 1, 2. \tag{64}$$

At small x the monotonicity of the profiles $s_i(x, t)$ is determined by the sign of the coefficient at x in the expansion (64). The monotonicity of the total retention profile $s(x, t)$ is determined by the sign of the coefficient at x in the sum of the expansions (64) at $i = 1$ and $i = 2$.

Denote

$$z = e^{-Bt}, \quad 0 < z \leq 1,$$

and consider the coefficients at x in (64):

$$w_i(z) = (1 - z) ((D - \lambda_i B) - (D + \lambda_i B) z) - 2B^2 z, \quad i = 1, 2.$$

Proposition 3 *Let $\lambda_1 > \lambda_2$. Then the function*

1. $w_1(z)$ is negative for all $z \in [0, 1][0, 1]$;
2. $w_2(z)$ is negative for $z_0 < z \leq 1$ and positive for $0 \leq z < z_0$, where $z_0 = \frac{D+B^2-B\sqrt{B^2+2D+(\lambda_2)^2}}{D+\lambda_2 B} \in (0, 1)$;
3. $W(z) = w_1(z) + w_2(z)$ is negative for all $z \in [0, 1]$ if $B_1 c_1^0 \leq B_2 c_2^0$; $W(z)$ is negative for $Z_0 < z \leq 1$ and positive for $0 \leq z < Z_0$, where $Z_0 = \frac{\lambda(D+B^2)-B\sqrt{\lambda^2(B^2+2D)+\mu^2}}{\lambda D+\mu B} \in (0, 1)$, $\lambda = \lambda_1 c_1^0 + \lambda_2 c_2^0$, $\mu = (\lambda_1)^2 c_1^0 + (\lambda_2)^2 c_2^0$, if $B_1 c_1^0 > B_2 c_2^0$.

Proposition 3 follows from the properties of the square trinomials $w_1(z)$ and $w_2(z)$.

Theorem 5 follows from the formulae (64) and Proposition 3.

Proof of Theorem 6 1. If $\rho \rightarrow \infty$, $x \rightarrow \infty$, then solutions $c_2 \rightarrow c_2^0$, $c_2^- \rightarrow 0$. The denominator $R(c) = c(B_1 c_1^0 (c_1^0 - c_1^0 (c/c_2^0)^{\lambda_1/\lambda_2}) + B_2 c_2^0 (c_2^0 - c))$ of the first term in the integral I has singularities at the points $c = c_2^0$, $c = 0$ and admits the following estimates in vicinities of the singular points:

$$R(c) = B(c_2^0 - c)/\lambda_2 + O(c_2^0 - c)^2, \quad R(c) = Ac(B_1(c_1^0)^2 + B_2(c_2^0)^2) + O(c^2). \tag{65}$$

Consequently, the integrand of the integral I is bounded at $c \rightarrow c_2^0$ and $c \rightarrow 0$, and the integral I converges.

2. The integral on the left-hand side of the second formula (17) can be represented as

$$\int_{c_2^-}^{c_2^0} \frac{dc}{c \left(B_1 c_1^0 (c_1^0 - c_1^0 \left(\frac{c}{c_2^0} \right)^{\lambda_1/\lambda_2}) + B_2 c_2^0 (c_2^0 - c) \right)} = I - \int_{c_2^-}^{c_2^0} g(c)dc - \int_0^{c_2^-} g(c)dc + \int_{c_2^-}^{c_2^0} \left(\frac{\lambda_2}{B(c_2^0 - c)} + \frac{1}{Ac} \right) dc \tag{66}$$

$$= \frac{1}{A} \ln \frac{c_2^-}{c_2^0} - \frac{\lambda_2}{B} \ln \frac{c_2^0 - c_2^-}{c_2^0 - c_2^-} + I + O(c_2^0 - c_2^-) + O(c_2^-).$$

Here $g(c)$ is the integrand of the integral I .

Substitute the expansion (66) into the second formula (17) and express the difference $c_2^0 - c_2^-$:

$$c_2^0 - c_2^- = c_2^0 e^{-B(t-x-x/A-1/\lambda_2)} \left(1 + O(e^{-B(t-x-x/A)} + e^{-\lambda_2 x}) \right).$$

The first formula in (34) implies the estimate

$$c_1^0 - c_1^- = \frac{c_1^0}{c_2^0} \frac{\lambda_1}{\lambda_2} (c_2^0 - c_2^-) + O(c_2^0 - c_2^-)^2.$$

The asymptotics of the maximum point $t_m(x)$ can be determined from the equation $\partial s_2/\partial x = 0$. Using the estimates of the differences $c_i^0 - c_i$, we represent the numerator in the formula (47) in the form

$$N_2(x, t) = -Bc_2^0 e^{-B(t-x-x/A-I/\lambda_2)} + (c_1^0)^2 c_2^0 B_1 (\lambda_2 e^{-\lambda_2 x} - \lambda_1 e^{-\lambda_1 x}) + O(e^{-B(t-x-x/A)} + e^{-\lambda_2 x})^2.$$

The formula (22) is obtained by solving the equation $N_2(x, t) = 0$.

Arguing similarly to the investigation of the profile $s_2(x, t)$, we represent the numerator of the derivative (48) of the total retention profile $s(x, t)$ in the form

$$N(x, t) = (c_2^0 - c_2) \left(-\lambda_1 B_1 (c_1^0)^2 \frac{c_1^0 \lambda_1}{c_2^0 \lambda_2} - \lambda_1 B_2 c_2^0 c_1^- - \lambda_2 B_1 c_1^0 c_2^0 \frac{c_1^0 \lambda_1}{c_2^0 \lambda_2} - \lambda_2 B_2 (c_2^0)^2 \right) - \lambda_1 B_1 c_1^0 c_2^0 c_1^- + \lambda_1 B_2 (c_2^0)^2 c_1^- + \lambda_2 B_1 (c_1^0)^2 c_2^- - \lambda_2 B_2 c_2^0 c_1^0 c_2^- + O((c_2 - c_2^0)^2 + (c_2 - c_2^0) e^{-\lambda_2 x} + e^{-(\lambda_1 + \lambda_2)x}).$$

Solving the equation $N(x, t) = 0$, we obtain the asymptotics (23) of the maximum point $T_m(x)$ of the total retention profile $s(x, t)$.

The limit maximum values of the profiles $s_2(x, t)$ and $s(x, t)$ follow from formulae (21).

Theorem 6 is proved.

Examples When calculating filtration problems, explicit finite difference schemes are usually used. The ratio of steps in time and coordinate must satisfy the Courant condition to ensure the convergence of the difference solution to the exact one [44]. Since the solution is zero ahead of the concentration front, the construction of a discontinuous solution on the front can be bypassed by considering the problem only behind the front, where the solution is continuous and positive. In a triangular domain $\{x \geq 0, t \geq x\}$, the Goursat problem is solved with conditions at the inlet $x = 0$ and on the front $t = x$ [45].

The authors use a different approach based on the known exact solution of the problem. The calculation of the solution is carried out according to the formulas (17), (18). Numerical calculation of examples is given below. The Python program code for calculating integrals can be seen in the Supplement.

The examples illustrate the change in the monotonicity of the retention profiles and the movement of the maximum points. Numerical calculations were performed at $\lambda_1 = 24, \lambda_2 = 6, c_1^0 = 0.6, c_2^0 = 0.4$.

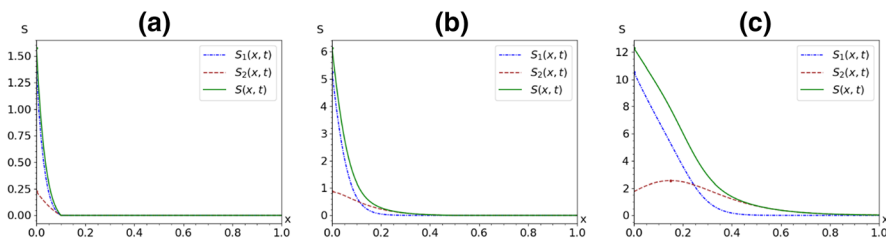


Fig. 1 Total and partial retention profiles at $B_1 = 0.1, B_2 = 0.5$. **a** $t = 0.1$, **b** $t = 0.5$, **c** $t = 3$

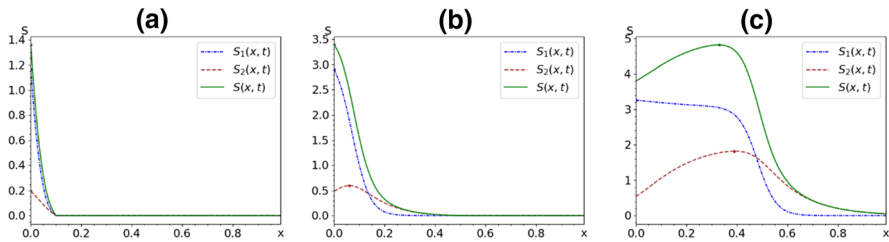


Fig. 2 Total and partial retention profiles at $B_1 = 0.5$, $B_2 = 0.1$. **a** $t = 0.1$, **b** $t = 0.5$, **c** $t = 3$

Fig. 3 Dynamics of maximum points of retention profiles at $B_1 = 0.5$, $B_2 = 0.1$

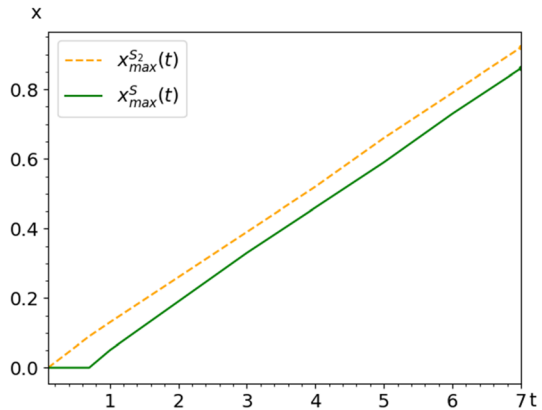


Figure 1 shows that when $B_1c_1^0 < B_2c_2^0$, the retention profiles $s_1(x, t)$ and $s(x, t)$ always decrease monotonically in x , the profile $s_2(x, t)$ being not monotonic in x at $t > 0.66$.

When $B_1c_1^0 > B_2c_2^0$, the retention profiles $s_2(x, t)$ and $s(x, t)$ are not monotonic in x (see Fig. 2). The peak appears on the partial profile $s_2(x, t)$ at $t_0 = 0.21$ and on the total retention profile $s(x, t)$ at $T_0 = 0.73$.

In Fig. 3, the graphs of the maximum points $x_{\max}(t)$ of the profiles $s_2(x, t)$ and $s(x, t)$ are shown. According to Remark 1 of Theorem 6, two graphs have parallel asymptotes.

Since the limit speed of the maximum points of the non-monotonic retention profiles is constant and less than the speed $v = 1$ of the concentration front Γ , it is useful to consider the profiles in the normalized coordinates (t, X) , $X = x/t$, $X \in [0; 1]$: $S_2(X, t) = s_2(Xt, t)$ and $S(X, t) = s(Xt, t)$. According to Theorem 6, the position of both peaks approaches the point $X = 0.13$, the limit maximum values of the profiles $S_2(X, t)$ and $S(X, t)$ are 2.04 and 5.1 (see Fig. 4).

6 Discussion

The size-exclusion retention mechanism assumes that particles freely pass through large pores and get stuck at the throats of pores that are smaller than the particles diameter. This retention mechanism occurs if the particle and pore size distributions overlap. In this case,

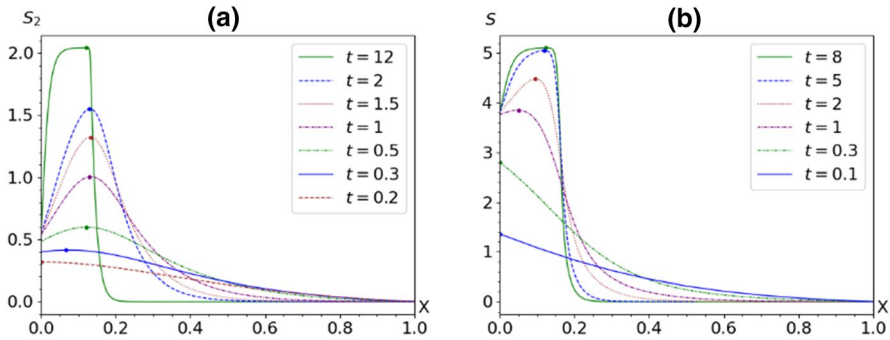


Fig. 4 Partial and total retention profiles at $B_1 = 0.5$, $B_2 = 0.1$ in coordinates (t, X) . **a** $S_2(X, t)$ **b** $S(X, t)$

large particles get stuck in the pores more often than small ones. According to Eqs. (4), the deposit growth rate is proportional to the coefficient λ_i . Therefore, the condition $\lambda_1 > \lambda_2$ means that particles of type 1 are larger than particles of type 2. Note that such a description of the types of particles is rather arbitrary. The particles can have the same size and differ in form, density or other physicochemical properties.

The profile maximum point appears at the inlet $x = 0$ at the moment $t_0 > 0$ and moves deep into the porous medium. For a porous sample of finite length l , at some instant $t_1 > t_0$, the maximum point reaches the outlet $x = l$ and the profile $s_2(x, t)$ becomes monotonically increasing in x . This means that the retained concentration of small particles is maximum at the outlet. Since the retention profile $s_1(x, t)$ always decreases monotonically, the retained concentration of large particles is maximum at the porous medium inlet. Thus, the long-term deep bed filtration of a bidisperse suspension or colloid in a porous medium makes it possible to separate large and small retained particles from each other.

For a water injection problem in a well, water is pumped down a wellbore and then dispersed radially into the formation. Assuming radial symmetry, this process is actually a radial flow scheme with an explicit radius dependency. For a monodisperse suspension, we obtain a one-dimensional filtration problem with a filtration function that depends on the coordinate. It corresponds to the filtration of suspension or colloid in a porous medium with variable porosity. The problem is reduced to a single first-order equation, which in the general case does not have an analytical solution [46]. The filtration problem in polar coordinates for a bidisperse suspension requires a separate study.

If the number of particle types $n > 2$, then the properties of the profiles of the polydisperse suspensions or colloids are not known. This problem will be considered later.

To calculate the pressure distribution in the porous medium during particle transport and retention, it is necessary to set the empirical function of permeability $k(s_1, s_2)$ and viscosity of the carrier water $\mu(c_1, c_2)$. Then pressure is determined using the Darcy’s law with varying permeability and viscosity [1].

7 Conclusions

The study of a one-dimensional deep bed filtration model of bidisperse suspensions and colloids in a porous medium leads to the following conclusions:

- The existence and the uniqueness of the solution to the bidisperse filtration problem are proved.
- Exact analytical formulae for the solution have been obtained.
- Retention profiles of large particles always decrease monotonically; the profiles of small particles are non-monotonic.
- The condition for the monotonicity/non-monotonicity of the total retention profile has been obtained.
- Starting from a certain moment of time, each non-monotonic profile has a maximum point.

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Availability of data and material No datasets were generated or analyzed for the results of this paper.

Code availability Python program code for the numerical solution of the filtration problem.

Declarations

Ethics We affirm that the submission represents original work that has not been published previously and is not currently being considered or submitted to another journal, until a decision has been made. Also, we confirm that each author has seen and approved the contents of the submitted manuscript.

Conflict of interest The authors declare that there are no conflicts of interest and competing interests.

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