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A new approach to projectivity in the categories of complexes

Driss Bennis[1] · **J. R. García Rozas**[2] · **Hanane Ouberka**[1] · **Luis Oyonarte**[2]

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Abstract

Recently, several authors have adopted new alternative approaches in the study of some classical notions of modules. Among them, we find the notion of subprojectivity which was introduced to measure in a way the degree of projectivity of modules. The study of subprojectivity has recently been extended to the context of abelian categories, which has brought to light some interesting new aspects. For instance, in the category of complexes, it gives a new way to measure, among other things, the exactness of complexes. In this paper, we prove that the subprojectivity notion provides a new sight of null-homotopic morphisms in the category of complexes. This will be proven through two main results. Moreover, various results which emphasize the importance of subprojectivity in the category of complexes are also given. Namely, we give some applications by characterizing some classical rings and establish various examples that allow us to reflect the scope and limits of our results.

Keywords Subprojectivity domain · Projective complex · Null-homotopic morphism · Contractible complex

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✉ J. R. García Rozas
jrgrozas@ual.es

Driss Bennis
driss.bennis@fsr.um5.ac.ma; driss_bennis@hotmail.com

Hanane Ouberka
hanane_ouberka@um5.ac.ma; ho514@inlumine.ual.es

Luis Oyonarte
oyonarte@ual.es

1 Faculty of Sciences, Mohammed V University in Rabat, Rabat, Morocco

2 Departamento de Matemáticas, Universidad de Almería, 04071 Almería, Spain

1 Introduction

In this paper, we will work mainly on an abelian category with enough projectives, although we will find the biggest applications in Sect. 4 on the category of modules over an associative ring with unit. The connection between abelian categories and module categories is well known from Gabriel's theorem: an abelian category is equivalent to a module category if and only if it is cocomplete and has a finite projective generator (see, for instance, [1, page 211]).

Throughout the paper, $\mathscr{A}$ will denote an abelian category with enough projectives and R will denote an associative (non-necessarily commutative) ring with a unit element $1_R \in R$. The category of left R-modules will be denoted by $R-$Mod. Modules are, unless otherwise explicitly stated, left R-modules.

The notion of subprojectivity was introduced in [2] as a new treatment in the analysis of the projectivity of a module. However, the study of the subprojectivity goes beyond that goal and, indeed, provides, among other things, a new and interesting perspective on some other known notions. Flatness has also been studied recently with a similar approach in [3], by defining general (sub)domains and then studying their particularities related to flatness. However, the results presented in [3] are quite different from those we give in this paper, mainly due to significant differences between the classes of flat and projective modules.

An alternative perspective on the projectivity of an object of an abelian category $\mathscr{A}$ with enough projectives was investigated in [4], where, in addition, it was shown that subprojectivity can be used to measure characteristics different from the projectivity and that subprojectivity domains may not be restricted to a single object. On the contrary, the subprojectivity domains of a whole class of objects can be computed, giving rise to very interesting characterizations. For instance, the subprojectivity domain of the whole class of DG-projective complexes is very useful to measure the exactness of complexes (see [4, Proposition 2.5]).

Recall that, given two objects M and N of $\mathscr{A}$, M is said to be N-subprojective if for every epimorphism $g : B \to N$ and every morphism $f : M \to N$, there exists a morphism $h : M \to B$ such that $gh = f$, or equivalently, if every morphism $M \to N$ factors through a projective object (see [4, Proposition 2.7]). The subprojectivity domain of any object M, denoted $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$, is defined as the class of all objects N such that M is N-subprojective, and the subprojectivity domain of a whole class $\mathfrak{C}$ of $\mathscr{A}$, $\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(\mathfrak{C})$, is defined as the class of objects N such that every C of $\mathfrak{C}$ is N-subprojective.

In this paper, we go deeper in the investigation of subprojectivity in the category of complexes of $\mathscr{A}$ which has enough projectives since $\mathscr{A}$ is supposed to have enough projectives. In this sense, when studying subprojectivity of complexes, it is observed that the concept of subprojectivity is relatively closely linked to that of null-homotopy of morphisms. Therefore, what we intend in the two main results of this paper (Theorems 1 and 2) is to deepen the understanding of this relationship. Namely, in Theorem 1, we prove that if $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ ($\mathscr{C}(\mathscr{A})$ is the category of complexes of $\mathscr{A}$) if and only if $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], K) = 0$ ($\mathscr{K}(\mathscr{A})$ is the homotopy category of $\mathscr{C}(\mathscr{A})$) for every short exact sequence of complexes $0 \to K \to P \to N \to 0$ with P projective. The proof of this theorem is based on a new characterization of the subprojectivity of an object in any abelian category with enough projectives in terms of the splitting of some particular short exact sequences (Proposition 1).

The second main result of the paper (Theorem 2) assures that for any two complexes M and N with $N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$, the conditions $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M,N) = 0$ are equivalent. This time, the idea is based on a new characterization of subprojectivity in terms of factorizations by contractible complexes (Proposition 4).

Theorem 2 allows us to determine exactly when a complex N is in the subprojectivity domain of all the shifts $M[n]$ of a given complex M (Proposition 6), which, at the same time helps in characterizing subprojectivity domains of complexes of the form $\oplus_{n\in\mathbb{Z}}\overline{M}[n]$ (Proposition 7) and of the form $\oplus_{n\in\mathbb{Z}}\underline{M}[n]$ (Proposition 8) for a given object M. A particular case of Proposition 8 typifies exact complexes in terms of subprojectivity in the following sense: N is exact if and only if $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P}[n])$ for every $n \in \mathbb{Z}$, where P is a projective generator of $\mathscr{A}$ (Corollary 1). Motivated by this result, we asked whether subprojectivity can measure the exactness of a complex N at each N_i. In fact, we prove that, for any complex N and any $n \in \mathbb{Z}$, $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(\underline{P}[n])$ if and only if $H_n(N) = 0$ (see Proposition 9). This result allows us to answer two interesting questions. Namely, we provide an example showing that the subprojectivity domains are not closed under kernel of epimorphisms (see Example 2). And, we give an example showing that the equivalence of Theorem 1 mentioned above does not hold in general if we replace the condition "P is projective" with $P \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ (see Remark 1 and Example 3). The necessity and the importance of the conditions given in the main Theorems 1 and 2 are deeply discussed in Propositions 2 and 5, respectively, and Example 1. It is worth noting that semisimple categories (in the sense that every object is projective) are also characterized in terms of subprojectivity. In fact, this was a consequence of the study of the condition "$N_{n+1} \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$" assumed in Theorem 2. Namely, we prove that the category $\mathscr{A}$ must be semisimple when this condition implies the condition $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ for every two complexes M and N (Proposition 5). All this is done in Section 3.

Finally, Section 4 is devoted to some applications. Namely, we give, as consequences of Theorem 1, some new characterizations of some classical rings. In Proposition 10, we characterize left hereditary rings in terms of subprojectivity as those rings for which every subcomplex of a DG-projective complex is DG-projective. Furthermore, we do it without the condition "Every exact complex of projective modules is projective" needed in [5, Proposition 2.3].

Following the same context, subprojectivity also makes it possible to characterize rings of weak global dimension at most 1, and using subprojectivity domains we prove that these rings are the ones over which subcomplexes of DG-flat complexes are always also DG-flat (Proposition 11). As a consequence, left semi-hereditary rings are also characterized in terms of subprojectivity (Corollary 2).

2 Preliminaries

In this section, we fix some notations from [6] and recall some definitions and basic results that will be used throughout this article.

Recall that, for two objects M and N of $\mathscr{A}$, M is said to be N-subprojective if for every epimorphism $g : B \to N$ and every morphism $f : M \to N$, there exists a morphism

$h : M \to B$ such that $gh = f$. Equivalently, M is N-subprojective if and only if every morphism $M \to N$ factors through a projective object ([4, Proposition 2.7]). The subprojectivity domain of any object M is defined as:

$$\underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M) = \{N \in \mathscr{A}; M \text{ is } N\text{-subprojective }\}.$$

By a complex X of objects of $\mathscr{A}$, we mean a sequence of objects and morphisms

$$\cdots \longrightarrow X_2 \overset{d_2}{\rightarrow} X_1 \overset{d_1}{\rightarrow} X_0 \overset{d_0}{\rightarrow} X_{-1} \overset{d_{-1}}{\rightarrow} X_{-2} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. If $\text{Im } d_{n+1} = \ker d_n$ for all $n \in \mathbb{Z}$, then we say that X is exact, and given an object M of $\mathscr{A}$, X is said to be $\text{Hom}_{\mathscr{A}}(M, -)$-exact if the complex of abelian groups $\text{Hom}_{\mathscr{A}}(M, N)$ is exact. We denote by $\epsilon_n^X : X_n \to \text{Im } d_n$ the canonical epimorphism and by $\mu_n^X : \text{Ker}(d_{n-1}) \to X_{n-1}$ the canonical monomorphism.

The n^{th} boundary (respectively, cycle, homology) of a complex X is defined as $\text{Im } d_{n+1}^X$ (respectively, $\text{Ker } d_n^X$, $\text{Ker } d_n^X / \text{Im } d_{n+1}^X$), and it is denoted by $B_n(X)$ (respectively, $Z_n(X)$, $H_n(X)$).

Throughout the paper, we use the following particular kind of complexes:

Disc complex. Given an object M, we denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \overset{\text{id}_M}{\rightarrow} M \longrightarrow 0 \longrightarrow \cdots$$

with all terms 0 except M in the degrees 1 and 0.

Sphere complex. Also, for an object M, we denote by $\underline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

with all terms 0 except M in the degree 0.

Shift complex. Let X be a complex with differential d^X and fix an integer n. We denote by $X[n]$ the complex consisting of X_{i-n} in degree i with differential $(-1)^n d_{i-n}^X$.

Now, by a morphism of complexes $f : X \to Y$, we mean a family of morphisms $f_n : X_n \to Y_n$ such that $d_n^Y f_n = f_{n-1} d_n^X$ for all $n \in \mathbb{Z}$. The category of complexes of $\mathscr{A}$ will be denoted by $\mathscr{C}(\mathscr{A})$. In particular, the category of complexes of modules over the ring R will be denoted by $\mathscr{C}(R)$.

A morphism of complexes $f : X \to Y$ is said to be null-homotopic if, for all $n \in \mathbb{Z}$, there exist morphisms $s_n : X_n \to Y_{n+1}$ such that for any n we have $f_n = d_{n+1}^Y s_n + s_{n-1} d_n^X$, and then we say that f is null-homotopic by s. For a complex X, id_X is null-homotopic if and only if X is of the form $\oplus_{n\in\mathbb{Z}} \overline{M_n}[n]$ for some family of objects M_n. A complex of this special type is called contractible.

Two morphisms of complexes f and g are homotopic, $f \sim g$ in symbols, if $f - g$ is null-homotopic. The relation $f \sim g$ is an equivalence relation. The homotopy category $\mathscr{K}(\mathscr{A})$ is defined as the one having the same objects as $\mathscr{C}(\mathscr{A})$, and which morphisms are homotopy equivalence classes of morphisms in $\mathscr{C}(\mathscr{A})$.

For complexes X and Y, we let $\text{Hom}^{\bullet}(X, Y)$ denote the complex of abelian groups with

$$\mathrm{Hom}^{\bullet}(X,Y)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{A}}(X_i, Y_{i+n})$$

and

$$d_n^{\mathrm{Hom}^{\bullet}(X,Y)}(\psi) = (d_{i+n}^{Y}\psi_i - (-1)^n \psi_{i-1} d_i^{X})_{i\in\mathbb{Z}}.$$

Note that for every $n \in \mathbb{Z}$,

$$Z_n(\mathrm{Hom}^{\bullet}(X,Y)) = \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(X[n],Y) = \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(X,Y[-n])$$

and

$$H_n(\mathrm{Hom}^{\bullet}(X,Y)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(X[n],Y) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(X,Y[-n]).$$

For every complex X, $\mathrm{Hom}^{\bullet}(X,-)$ is a left exact functor from the category of complexes of $\mathscr{A}$ to the category of complexes of abelian groups.

Recall from [7, Proposition 2.3.6] that a complex P is projective as an object of $\mathscr{C}(\mathscr{A})$ if and only if it is contractible of projectives if and only if it is exact and the cycles $Z_i(P)$ are projective in $\mathscr{A}$. In particular, each P_i is projective in $\mathscr{A}$.

3 Subprojectivity and null-homotopy

As mentioned in the introduction, subprojectivity of complexes is closely related to null-homotopy of morphisms of complexes and kernels of epimorphisms. The aim of this section is to deepen the understanding of this relationship.

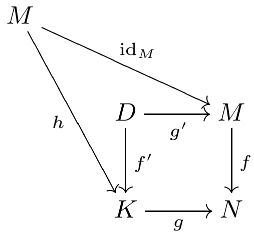
We start with a new characterization of subprojectivity in terms of splitting short exact sequences which will be considered somehow as the subprojectivity analogue of the classical characterization of projectivity.

We fix the following notation: the pullback of two morphisms $g : C \to B$ and $f : A \to B$ will be denoted by (D, g', f').

Proposition 1 *Let $\mathscr{A}$ be an abelian category with enough projectives. If M and N are two objects of $\mathscr{A}$, the following conditions are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$.
2. *For every epimorphism $g : K \to N$ and every morphism $f : M \to N$, the epimorphism $g' : D \to M$ given by the pullback (D, g', f') of g and f, splits.*
3. *There exists an epimorphism $g : P \to N$ with P projective such that for every morphism $f : M \to N$, the epimorphism $g' : D \to M$ given by the pullback (D, g', f') of g and f, splits.*
4. *There exists an epimorphism $g : P \to N$ with $P \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M)$ such that for every morphism $f : M \to N$, the epimorphism $g' : D \to M$ given by the pullback (D, g', f') of g and f, splits.*

Proof 1. $\Rightarrow$ 2. Let $g : K \to N$ be an epimorphism, $f : M \to N$ be a morphism and (D, g', f') be their pullback. Since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$, there exists a morphism $h : M \to K$ such that the following diagram commutes

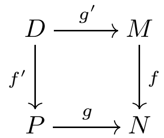


Then, by the universal property of pullbacks, there exists a morphism $k : M \to D$ such that $g'k = id_M$. Hence g' splits, as desired.

2. $\Rightarrow$ 3. This is clear since the category $\mathscr{A}$ is supposed to have enough projectives.

3. $\Rightarrow$ 4. This is clear since every projective object belongs to $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.

4. $\Rightarrow$ 1. Let $g : P \to N$ be the epimorphism of statement 4., $f : M \to N$ be a morphism and (D, g', f') their pullback



Then, by assumption, there exists a morphism $h : M \to D$ such that $g'h = id_M$, hence $f = fg'h = gf'h$. Therefore, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ (see [4, Proposition 2.2]). □

The following two lemmas will be useful in the proof of Theorem 1.

Lemma 1 *For two complexes M and N with $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$,* $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M, N) = 0$.

Proof Let $f \in \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(M, N)$, then there exist two morphisms $\alpha : P \to N$ and $\beta : M \to P$ such that P is projective and $f = \alpha\beta$ (see [4, Proposition 2.7]). Now, id_P is null-homotopic since P is contractible; thus, the composition $\alpha \mathrm{id}_P \beta$ is null-homotopic. Therefore, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M, N) = 0$. □

Lemma 2 *If (D, g', f') is the pullback of two morphisms of complexes $g : C \to B$ and $f : A \to B$, then (D_n, g'_n, f'_n) is the pullback of $g_n : C_n \to B_n$ and $f_n : A_n \to B_n$ for every $n \in \mathbb{Z}$.*

Proof Let $\alpha : X \to A_n$ and $\beta : X \to C_n$ be two morphisms of $\mathscr{A}$ such that $f_n\alpha = g_n\beta$ and consider the two morphisms of complexes $\overline{\alpha} : \overline{X}[n-1] \to A$ and $\overline{\beta} : \overline{X}[n-1] \to C$ induced by α and β, respectively. It is straightforward to verify that $f\overline{\alpha} = g\overline{\beta}$, so there exists a unique morphism of complexes $h : \overline{X}[n-1] \to D$ such that $g'h = \overline{\alpha}$ and $f'h = \overline{\beta}$. Then, $g'_n h_n = \alpha$ and $f'_n h_n = \beta$.

The unicity of $h_n : X \to D_n$ comes from the unicity of h. □

Now, we give the first main result of the paper.

Theorem 1 *Let M and N be two complexes such that* $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ *for every* $n \in \mathbb{Z}$. *Then, the following statements are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$.
2. *For every short exact sequence* $0 \to K \to P \to N \to 0$ *with P projective, the equation* $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], K) = 0$ *holds.*
3. *There exists a short exact sequence* $0 \to K \to P \to N \to 0$ *with P projective such that* $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], K) = 0$.
4. *There exists a short exact sequence* $0 \to K \to P \to N \to 0$ *with* $P \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ *such that* $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], K) = 0$.

Proof 1. ⇒ 2. Let $0 \to K \to P \to N \to 0$ be a short exact sequence with P projective and consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_0(\mathrm{Hom}^\bullet(M,K)) & \to & \mathrm{Hom}^\bullet(M,K)_0 & \to & B_{-1}(\mathrm{Hom}^\bullet(M,K)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_0(\mathrm{Hom}^\bullet(M,P)) & \to & \mathrm{Hom}^\bullet(M,P)_0 & \to & B_{-1}(\mathrm{Hom}^\bullet(M,P)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_0(\mathrm{Hom}^\bullet(M,N)) & \to & \mathrm{Hom}^\bullet(M,N)_0 & \to & B_{-1}(\mathrm{Hom}^\bullet(M,N)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The first and second columns are exact since $N \in \underline{\mathfrak{Pr}}_{\mathscr{C}(\mathscr{A})}^{-1}(M)$ and $N_n \in \underline{\mathfrak{Pr}}_{\mathscr{A}}^{-1}(M_n)$ for every $n \in \mathbb{Z}$, respectively. Hence, the third column is also exact.

Now, applying the Snake Lemma to the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B_{-1}(\mathrm{Hom}^\bullet(M,K)) & \to & B_{-1}(\mathrm{Hom}^\bullet(M,P)) & \to & B_{-1}(\mathrm{Hom}^\bullet(M,N)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_{-1}(\mathrm{Hom}^\bullet(M,K)) & \to & Z_{-1}(\mathrm{Hom}^\bullet(M,P)) & \to & Z_{-1}(\mathrm{Hom}^\bullet(M,N)) & &
\end{array}$$

we get the exact sequence

$$0 \to H_{-1}(\mathrm{Hom}^\bullet(M,K)) \to H_{-1}(\mathrm{Hom}^\bullet(M,P)) \to H_{-1}(\mathrm{Hom}^\bullet(M,N)) \ .$$

Since P is projective as a complex, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], P) = 0$ by Lemma 1. Then

$$H_{-1}(\mathrm{Hom}^{\bullet}(M, P)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], P) = 0.$$

Thus, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], K) = H_{-1}(\mathrm{Hom}^{\bullet}(M, K)) = 0$.

2. $\Rightarrow$ 3. Clear since the category of complexes has enough projectives.

3. $\Rightarrow$ 4. This is clear since every projective complex belongs to $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.

4. $\Rightarrow$ 1. Let $0 \to K \to P \to N \to 0$ be the short exact sequence of statement 4., $f : M \to N$ be any morphism of complexes and consider the following pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

For every $n \in \mathbb{Z}$, D_n is a pullback by Lemma 2, so by assumption and Proposition 1 the short exact sequence $0 \to K \to D \to M \to 0$ is degreewise splits. Then, this sequence is equivalent to a short exact sequence $0 \to K \to M(g) \to M \to 0$ being $M(g)$ the mapping cone of a morphism $g : M[-1] \to K$, but $g : M[-1] \to K$ is null-homotopic by assumption so $0 \to K \to M(g) \to M \to 0$ splits (using the same arguments of [8, Proposition 3.3.2]). Therefore, the sequence $0 \to K \to D \to M \to 0$ splits too and then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ by Proposition 1. □

Remark 1 It is natural to ask whether, as in the case of exact sequences $0 \to K \to P \to N \to 0$ with P projective, the statements of Theorem 1 are equivalent to the following: "For every short exact sequence $0 \to K \to P \to N \to 0$ with $P \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$, the equation $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], K) = 0$ holds". We will see in Example 3 that they are not equivalent.

Given two complexes M and N, it is natural to ask if $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ is sufficient to get that, for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$. This is not true in general. Indeed, we can always consider, over a non-semisimple ring R, two modules X and Y with $Y \notin \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(X)$, while it is clear that we always have $\underline{Y} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\overline{X})$ since every morphism $\overline{X} \to \underline{Y}$ is zero. Nevertheless, the answer to the question would be positive if we assume, furthermore, that N belongs to $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1])$.

Proposition 2 *Let M and N be two complexes such that*

$$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1]) \bigcap \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M).$$

Then, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ *for every* $n \in \mathbb{Z}$.

Proof Let P be a projective complex and $P \to N$ be an epimorphism of complexes. Since $N, P \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1])$, $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], P) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[-1], N) = 0$ by Lemma 1. So, the horizontal maps of the following commutative diagram are isomorphisms

$$\begin{array}{ccc} B_{-1}(\mathrm{Hom}^\bullet(M,P)) & \longrightarrow & Z_{-1}(\mathrm{Hom}^\bullet(M,P)) \\ \downarrow & & \downarrow \\ B_{-1}(\mathrm{Hom}^\bullet(M,N)) & \longrightarrow & Z_{-1}(\mathrm{Hom}^\bullet(M,N)) \end{array}$$

The morphism

$$Z_{-1}(\mathrm{Hom}^\bullet(M,P)) \to Z_{-1}(\mathrm{Hom}^\bullet(M,N))$$

coincides with $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(M[-1], P) \to \mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(M[-1], N)$, and it is epic since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[-1])$, so the map $B_{-1}(\mathrm{Hom}^\bullet(M,P)) \to B_{-1}(\mathrm{Hom}^\bullet(M,N))$ must also be epic.

Now, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_0(\mathrm{Hom}^\bullet(M,P)) & \longrightarrow & \mathrm{Hom}^\bullet(M,P)_0 & \longrightarrow & B_{-1}(\mathrm{Hom}^\bullet(M,P)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_0(\mathrm{Hom}^\bullet(M,N)) & \longrightarrow & \mathrm{Hom}^\bullet(M,N)_0 & \longrightarrow & B_{-1}(\mathrm{Hom}^\bullet(M,N)) & \longrightarrow & 0 \end{array}$$

The map $Z_0(\mathrm{Hom}^\bullet(M,P)) \to Z_0(\mathrm{Hom}^\bullet(M,N))$ is epic since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$, so again $\mathrm{Hom}^\bullet(M,P)_0 \to \mathrm{Hom}^\bullet(M,N)_0$ is epic so we see that every morphism $M_n \to N_n$ factors through P_n for every $n \in \mathbb{Z}$. □

Though the fact that a complex N belongs to the subprojectivity domain of another complex M does not imply that the components of N necessarily belong to the subprojectivity domains of the components of M, the answer is completely different if we ask about cycles of N instead of components of N. We can see this in the following result.

Lemma 3 *Let $n \in \mathbb{Z}$, N be a complex and M be an object of $\mathscr{A}$. If $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M}[n])$, then $Z_n(N) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$.*

Proof Let $f : M \to Z_n(N)$ be any morphism of $\mathscr{A}$ and $\underline{f} : \underline{M}[n] \to N$ be the induced morphism of complexes. By assumption $\underline{f}$ factors as

$$\underline{M}[n] \xrightarrow[\alpha]{} P \xrightarrow[\beta]{} N, \qquad \beta\alpha = \underline{f}$$

for some projective complex P. Then, $d_n^P \alpha_n = 0$, so there exists a morphism $h : M \to Z_n(P)$ such that $\mu_n^P h = \alpha_n$.

On the other side, the morphism β induces a morphism $g : Z_n(P) \to Z_n(N)$ such that $\mu_n^N g = \beta_n \mu_n^P$. Then, we have

$$\mu_n^N gh = \beta_n \mu_n^P h = \beta_n \alpha_n = \underline{f}_n = \mu_n^N f,$$

that is, $f = gh$, so f factors through the projective object $Z_n(P)$. □

Another natural question at this point is whether the inverse implication of Proposition 2 is true or not. Namely, given two complexes M and N, is the condition "$N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$", sufficient to assure that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$? Again, this is not true in general since, for instance, for exact complexes of modules it only holds over left hereditary rings (see Proposition 10).

We have studied so far the relation between subprojectivity and null-homotopic morphisms involving kernels of epimorphisms. We will now see that this relation can also be described without considering such kernels (Theorem 2).

We start by characterizing when a contractible complex holds in the subprojectivity domain of another complex. We need the following lemma.

Lemma 4 *Let M be a complex, N be an object of $\mathscr{A}$ and $n \in \mathbb{Z}$. Then, $\overline{N}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ if and only if $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$.*

Proof Suppose that $\overline{N}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ and let $f : M_n \to N$ be a morphism in $\mathscr{A}$. The induced morphism $\overline{f} : M \to \overline{N}[n]$ (that is, $\overline{f}_n = f$) factors through a projective complex P by the hypothesis, so f factors through the projective object P_n.

Conversely, let $f : M \to \overline{N}[n]$ be a morphism of complexes. Since $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$, the morphism f_n factors as

$$M_n \xrightarrow[\alpha]{} P \xrightarrow[\beta]{} N, \qquad f_n = \beta\alpha$$

for some projective object P of $\mathscr{A}$. Then, if we let $g : M \to \overline{P}[n]$ be the morphism of complexes with $g_n = \alpha$ and $g_{n+1} = \alpha d^M_{n+1}$, and $h : \overline{P}[n] \to \overline{N}[n]$ be the morphism of complexes with $h_n = h_{n+1} = \beta$, we clearly get that $f = hg$, hence $\overline{N}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$. □

Proposition 3 *Let M be a complex and $(N_n)_{n\in\mathbb{Z}}$ be a family of objects of $\mathscr{A}$. Then, $\oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ if and only if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$.*

Proof If $\oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ then $\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ for every $n \in \mathbb{Z}$ since $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ is closed under direct summands (see [4, Proposition 3.1]). Then, by Lemma 4 we get that for every $n \in \mathbb{Z}$, $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$.

Conversely, if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$ then $\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ for every $n \in \mathbb{Z}$ again by Lemma 4.

Now, let $f : M \to \oplus_{n\in\mathbb{Z}}\overline{N_n}[n]$ be a morphism of complexes and, for every m, choose an epimorphism $g^m : \overline{P_m}[m] \to \overline{N_m}[m]$ with P_m a projective object of $\mathscr{A}$.

If we let

$$\pi^m : \oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \to \overline{N_m}[m]$$

be the projection morphism, for any m there exists a morphism $h^m : M \to \overline{P_m}[m]$ such that $\pi^m f = g^m h^m$.

But $\oplus_{n\in\mathbb{Z}}\overline{P_n}[n]$ coincides with $\prod_{n\in\mathbb{Z}}\overline{P_n}[n]$, so if we call

$$\pi'^m : \oplus_{n\in\mathbb{Z}}\overline{P_n}[n] \to \overline{P_m}[m]$$

the projection morphism, we get a morphism $h : M \to \oplus_{n\in\mathbb{Z}}\overline{P_n}[n]$ such that $\pi'^m h = h^m$ for every m.

Therefore, for every $m \in \mathbb{Z}$ we have

$$\pi^m f = g^m h^m = g^m \pi'^m h = \pi^m(\oplus g^n)h$$

so we see that $f = (\oplus g^n)h$. This means that f factors through the projective complex $\oplus_{n\in\mathbb{Z}}\overline{P_n}[n]$ and so that $\oplus_{n\in\mathbb{Z}}\overline{N_n}[n] \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$. □

The following result characterizes subprojectivity in terms of factorization of morphisms through contractible complexes and through complexes in subprojectivity domains.

Proposition 4 *Let M and N be two complexes. The following conditions are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.
2. *Every morphism* $M \to N$ *factors through a complex of* $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.
3. *Every morphism* $M \to N$ *factors through a contractible complex* $\oplus_{n\in\mathbb{Z}}\overline{X_n}[n]$ *such that* $X_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ *for every* $n \in \mathbb{Z}$.

Proof 1. ⇒ 2. This is clear since every projective complex holds in $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.

2. ⇒ 1. Let $f : M \to N$ be a morphism of complexes. By the hypothesis there exist two morphisms of complexes $\alpha : M \to L$ and $\beta : L \to N$ such that $f = \beta\alpha$ and that $L \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$. But then, $\alpha : M \to L$ factors through a projective complex P so f factors through P.

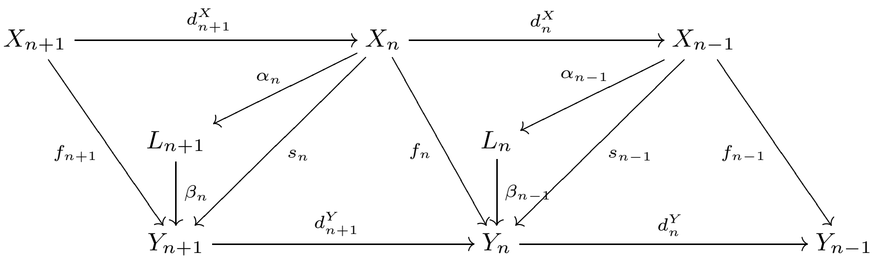
1. ⇒ 3. Clear since every projective complex is a contractible complex of projective objects of $\mathscr{A}$.

3. ⇒ 2. Apply Proposition 3. □

Notice that conditions 1. and 2. of Proposition 4 are equivalent in any abelian category with enough projectives.

Lemma 5 *Let* $f : X \to Y$ *be a null-homotopic morphism of complexes by a morphism s. If every morphism* $s_n : X_n \to Y_{n+1}$ *of* $\mathscr{A}$ *factors through an object* L_{n+1}*, then* $f : X \to Y$ *factors through the contractible complex* $\oplus_{n\in\mathbb{Z}}\overline{L_{n+1}}[n]$*. In particular,* $f : X \to Y$ *factors through the contractible complex* $\oplus_{n\in\mathbb{Z}}\overline{Y_{n+1}}[n]$.

Proof Suppose that for any n there exist two morphisms $\alpha_n : X_n \to L_{n+1}$ and $\beta_n : L_{n+1} \to Y_{n+1}$ such that $s_n = \beta_n\alpha_n$. Then, we have the situation



For every $n \in \mathbb{Z}$, let $p^1_{n+1} : L_{n+1} \oplus L_n \to L_{n+1}$ and $p^2_n : L_{n+1} \oplus L_n \to L_n$ be the canonical projections, and $k^1_{n+1} : L_{n+1} \to L_{n+1} \oplus L_n$ and $k^2_n : L_n \to L_{n+1} \oplus L_n$ be the canonical injections. Now, call Z the complex $\oplus_{n\in\mathbb{Z}}\overline{L_{n+1}}[n]$ and consider, for every $n \in \mathbb{Z}$, the two morphisms of $\mathscr{A}$ $h_n : L_{n+1} \oplus L_n \to Y_n$ given by $h_n = d^Y_{n+1}\beta_n p^1_{n+1} + \beta_{n-1}p^2_n$, and $g_n : X_n \to L_{n+1} \oplus L_n$ given by $g_n = (\alpha_n, \alpha_{n-1}d^X_n)$. We claim that both $h : Z \to Y$ and $g : X \to Z$ are morphisms of complexes.

For any $n \in \mathbb{Z}$, we have $d^Y_n h_n = d^Y_n(d^Y_{n+1}\beta_n p^1_{n+1} + \beta_{n-1}p^2_n) = d^Y_n\beta_{n-1}p^2_n$, and $h_{n-1}d^Z_n = (d^Y_n\beta_{n-1}p^1_n + \beta_{n-2}p^2_{n-1})k^1_n p^2_n = d^Y_n\beta_{n-1}P^1_n k^1_n p^2_n = d^Y_n\beta_{n-1}p^2_n$, so h is a morphism of complexes, and for any $n \in \mathbb{Z}$ we have

$$g_{n-1}d^X_n = (\alpha_{n-1}, \alpha_{n-2}d^X_{n-1})d^X_n = (\alpha_{n-1}d^X_n, 0) = d^Z_n(\alpha_n, \alpha_{n-1}d^X_n) = d^Z_n g_n,$$

so g is also a morphism of complexes.

Now we see that $f = hg$ since for any $n \in \mathbb{Z}$ we have

$$h_n g_n = d^Y_{n+1}\beta_n\alpha_n + \beta_{n-1}\alpha_{n-1}d^X_n = d^Y_{n+1}s_n + s_{n-1}d^X_n = f_n.$$

Therefore, $f : X \to Y$ factors through the contractible complex $Z = \oplus_{n\in\mathbb{Z}}\overline{L_{n+1}}[n]$. □

Theorem 2 *Let M and N be two complexes such that $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$. Then, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ if and only if* $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M, N) = 0$.

Proof By Lemma 1.

Conversely, if $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M, N) = 0$ then, by Lemma 5, every morphism $M \to N$ factors through the contractible complex $\oplus_{n\in\mathbb{Z}}\overline{N_{n+1}}[n]$, so $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ by Proposition 4. □

The following example shows that the condition $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$ in Theorem 2 cannot be removed in general.

Example 1 Let X be any non-projective module and choose any other module Y out of the subprojectivity domain of X (such modules exist over any non-semisimple ring). It is clear that $\mathrm{Hom}_{\mathscr{K}(R)}(\overline{X}, \overline{Y}) = 0$ and, by Lemma 4, that $\overline{Y} \notin \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\overline{X})$.

Given two complexes M and N, it is clear that the condition "$N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$" is not enough in general to get $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$. For instance, if $\mathscr{A}$ is semisimple (in the sense that every object is projective) and M is not exact (so M is not a projective complex), then for sure we can find complexes not in $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.

In the following result, we prove that this condition suffices for exact complexes if and only if $\mathscr{A}$ is semisimple.

Proposition 5 *The following conditions are equivalent.*

1. *$\mathscr{A}$ is semisimple.*
2. *For every complex M and every exact complex N, if $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$.*
3. *For every object M of $\mathscr{A}$ and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M}[n])$.*

Proof 1. ⇒ 2. Every exact complex N is projective so $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ for every complex M.

2. ⇒ 3. Clear.

3. ⇒ 1. Let M be an object of $\mathscr{A}$ and $\mathcal{P}$ be a projective resolution of M. Then, $\mathcal{P}_1 \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(\underline{M})$ and so $\mathcal{P} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M})$ by assumption. Then, by Lemma 3, $M = Z_0(\mathcal{P}) \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$. This means that M is projective and therefore that $\mathscr{A}$ is semisimple. □

Given two complexes M and N, it is natural to ask whether $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ implies that $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$. This is not true in general. For instance, in the category of R-modules, if we take any non-projective module X and choose any other module Y out of the subprojectivity domain of X (such modules exist over any non-semisimple ring). Then, the complex $\underline{Y}[2]$ belongs to $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(X)$ since $\mathrm{Hom}_{\mathscr{C}(R)}(\overline{X}, \underline{Y}[2]) = 0$, but $\underline{Y}[2]_2 = Y \notin \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(\overline{X})$.

However, if we add the condition "$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[1])$", then Proposition 2 says that $N_{n+1} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_n)$ for every $n \in \mathbb{Z}$.

Inspired by Proposition 2, we give the following result.

Proposition 6 *Let M and N be two complexes. The following statements are equivalent.*

1. *$N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M[n])$ for every $n \in \mathbb{Z}$.*
2. *For every $i, j \in \mathbb{Z}$, $N_i \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M_j)$, and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(M[n], N) = 0$ for every $n \in \mathbb{Z}$.*

Proof Apply Proposition 2 and Theorem 2. □

Now, we give some applications of Proposition 6. Namely, given any object M of $\mathscr{A}$, Proposition 6 can be used to study the subprojectivity domain of the complexes $\bigoplus_{n\in\mathbb{Z}}\overline{M}[n]$ (Proposition 7) and $\bigoplus_{n\in\mathbb{Z}}\underline{M}[n]$ (Proposition 8).

Proposition 7 *Let N be a complex and M be an object of $\mathscr{A}$. The following statements are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\overline{M}[n])$.
2. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\overline{M}[n])$ *for every* $n \in \mathbb{Z}$.
3. $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ *for every* $n \in \mathbb{Z}$.

Proof 1. ⇔ 2. Clear by [4, Proposition 2.16].
2. ⇔ 3. Clear by Proposition 6 since $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\overline{M}[n], N) = 0$ for every $n \in \mathbb{Z}$. □

Proposition 8 *Let N be a complex and M be an object of* $\mathscr{A}$. *The following statements are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\underline{M}[n])$.
2. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M}[n])$ *for every* $n \in \mathbb{Z}$.
3. *N is* $\mathrm{Hom}_{\mathscr{A}}(M,-)$*-exact and* $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ *for every* $n \in \mathbb{Z}$.

Proof 1. ⇔ 2. Clear by [4, Proposition 2.16].
2. ⇒ 3. By Proposition 6 we know that $N_n \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{A}}(M)$ for every $n \in \mathbb{Z}$ and that $H_n(\mathrm{Hom}^{\bullet}(\underline{M}, N)) = \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{M}[n], N) = 0$. But $\mathrm{Hom}^{\bullet}(\underline{M}, N) \cong \mathrm{Hom}_{\mathscr{A}}(M, N)$ so $\mathrm{Hom}_{\mathscr{A}}(M, N)$ is exact.
3. ⇒ 2. Let $n \in \mathbb{Z}$ and $f : \underline{M}[n] \to N$ be a morphism of complexes. Since $d^N_n f_n = 0$ we get that $f_n \in \mathrm{Ker}(\mathrm{Hom}_{\mathscr{A}}(M, d^N_n)) = \mathrm{Im}\,(\mathrm{Hom}_{\mathscr{A}}(M, d^N_{n+1}))$, so there exists a morphism $g : M \to N_{n+1}$ such that $d^N_{n+1} g = f_n$. Thus, f is null-homotopic and $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{M}[n], N) = 0$ for every $n \in \mathbb{Z}$. Proposition 6 says then that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{M}[n])$ for every $n \in \mathbb{Z}$. □

From now on we will assume in this section that $\mathscr{A}$ has a projective generator P.

If we let $M = P$ in Proposition 8, then the condition "N is $\mathrm{Hom}_{\mathscr{A}}(P,-)$-exact" means that N is exact (since P preserves and reflects exactness by its definition). This leads to the following characterization of exact complexes in terms of subprojectivity.

Corollary 1 *Let P be a projective generator of* $\mathscr{A}$ *and N be a complex. The following assertions are equivalent.*

1. *N is exact.*
2. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\oplus_{n\in\mathbb{Z}}\underline{P}[n])$.
3. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P}[n])$ *for every* $n \in \mathbb{Z}$.

There is now a natural question which comes to mind after Corollary 1: we have described, for the projective generator P, how the subprojectivity domain of the set of complexes $\{\underline{P}[n], n \in \mathbb{Z}\}$ is, so, what about the subprojectivity domain of each of the complexes $\underline{P}[n]$? Can we describe them as well?

Given a complex N, we know, by Theorem 2, that $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P}[n])$ if and only if $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n], N) = 0$. But,

$$\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n], N) = H_n(\mathrm{Hom}^{\bullet}(\underline{P}, N)) = H_n(\mathrm{Hom}_{\mathscr{A}}(P, N)).$$

So, the condition $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n], N) = 0$ is equivalent to $H_n(N) = 0$ since P is a projective generator of $\mathscr{A}$. We state this fact in the following proposition.

Proposition 9 *Let P be a projective generator of $\mathscr{A}$, N be a complex, and $n \in \mathbb{Z}$. The following assertions are equivalent.*

1. $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P}[n])$.
2. $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[n], N) = 0$.
3. $H_n(N) = 0$.

Now, with Proposition 9 in hand, it is easy to see that subprojectivity domains are not closed under kernels of epimorphisms in general.

Example 2 Consider the short exact sequence of complexes

$$0 \to \underline{P} \to \overline{P} \to \underline{P}[1] \to 0,$$

where P is the projective generator of $\mathscr{A}$. It is clear by Proposition 9 that $\underline{P}[1]$ and $\overline{P}$ both hold in $\underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P})$, but $\underline{P}$ does not. Therefore, the subprojectivity domain of $\underline{P}$ is not closed under kernels of epimorphisms.

Moreover, Proposition 9 helps us to answer a question raised in Remark 1. Precisely, it is understood by the equivalence ($1 \Leftrightarrow 4$) in Theorem 1 that the second assertion remains equivalent to the first assertion even if we replace the condition "Q is projective" with $Q \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$. However, this fact does not hold true. Namely, the following example shows that if we replace "Q is projective" with $Q \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(M)$ in assertion 2, the equivalent does not hold.

Example 3 Let $0 \to N_3 \to N_2 \to N_1 \to 0$ be a short exact sequence in $\mathscr{A}$ such that $N_3 \neq 0$ and let $X_i := \overline{N_i} \oplus N_i[-1]$ for $i \in \{1, 2, 3\}$. Then, we have an induced exact sequence of complexes $0 \to X_3 \to X_2 \to X_1 \to 0$.

Moreover, we see that for $i \in \{1, 2, 3\}$ it holds that $H_0(X_i) = H_0(\overline{N_i}) \oplus H_0(N_i[-1]) = 0$ and that $H_{-1}(X_i) = H_{-1}(\overline{N_i}) \oplus H_{-1}(N_i[-1]) = H_{-1}(N_i[-1]) = N_i$. Thus, we can assert that $N_1, N_2 \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(\mathscr{A})}(\underline{P})$ and that $\mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(\underline{P}[-1], X_3) \neq 0$ where P is the projective generator of $\mathscr{A}$ (see Proposition 9).

4 Applications

Recall that a the ring R is said to be left hereditary if any left R-submodule of a projective left R-module is projective. Recall also that a complex P is said to be DG-projective if its components are projective and $\mathrm{Hom}^{\bullet}(P, E)$ is exact for every exact complex E. In [5, Proposition 2.3] it is proved that, under certain conditions, a ring is left hereditary if and only if every subcomplex of a DG-projective complex is DG-projective. Among these conditions, the authors included: "Every exact complex of projective modules is projective". In this section, using the properties of subprojectivity domains, we will show that the latter equivalence holds without the mentioned assumption.

Proposition 10 *For any ring R, the following statements are equivalent.*

1. *R is left hereditary.*
2. *For every complex M and every exact complex N, if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.*
3. *For every module M and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{M}[n])$.*
4. *Every subcomplex of a DG-projective complex is DG-projective.*

Proof 1. ⇒ 2. If $0 \to K \to P \to N \to 0$ is a short exact sequence of complexes with P projective, then K is exact (P and N are exact) and all cycles $Z_n(K)$ are projective by 1. Therefore, K is projective and then, by Lemma 1, $\mathrm{Hom}_{\mathscr{K}(R)}(M[-1], K) = 0$, so $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$ by Theorem 1.

2. ⇒ 3. Clear.

3. ⇒ 1. Let Q be a projective module and Y be any submodule of Q. Let us prove that $\underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(Y) = R{-}\mathrm{Mod}$. For let X be a module and consider the exact complex

$$\mathcal{C} : \cdots \to 0 \to X \to E(X) \to C \to 0 \to \cdots$$

($E(X)$ in the 0-position). By [9, Lemma 2.2] $E(X) \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(Y)$. So we get that $\mathcal{C} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{Y})$. Then, $X \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(Y)$ by Lemma 3.

1. ⇒ 4. Let P be a DG-projective complex and Q a subcomplex of P. Then, every module Q_n is projective by condition 1.

Now, let E be an exact complex and let us prove that $\mathrm{Hom}^\bullet(Q, E)$ is exact.

Let $0 \to E \to I \to C \to 0$ be a short exact sequence of complexes with I injective. Since every module Q_n is projective we get that for every $n \in \mathbb{Z}$, $\mathrm{Hom}^\bullet(Q, I)_n \to \mathrm{Hom}^\bullet(Q, C)_n$ is epic, and for every $i, j \in \mathbb{Z}$, $C_i \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(Q_j)$. Then, by condition 2. we get that $C \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(Q[n])$ for every $n \in \mathbb{Z}$ (C is exact since I and E are exact), so for every $n \in \mathbb{Z}$, $Z_n(\mathrm{Hom}^\bullet(Q, I)) \to Z_n(\mathrm{Hom}^\bullet(Q, C))$ is epic. Therefore, for every $n \in \mathbb{Z}$ the two first columns of the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_n(\mathrm{Hom}^\bullet(Q,E)) & \to & \mathrm{Hom}^\bullet(Q,E)_n & \to & B_{n-1}(\mathrm{Hom}^\bullet(Q,E)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_n(\mathrm{Hom}^\bullet(Q,I)) & \to & \mathrm{Hom}^\bullet(Q,I)_n & \to & B_{n-1}(\mathrm{Hom}^\bullet(Q,I)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Z_n(\mathrm{Hom}^\bullet(Q,C)) & \to & \mathrm{Hom}^\bullet(Q,C)_n & \to & B_{n-1}(\mathrm{Hom}^\bullet(Q,C)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

are exact, so the third is also exact.

Now consider, for every $n \in \mathbb{Z}$, the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B_n(\mathrm{Hom}^\bullet(Q,E)) & \to & Z_n(\mathrm{Hom}^\bullet(Q,E)) & \to & H_n(\mathrm{Hom}^\bullet(Q,E)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B_n(\mathrm{Hom}^\bullet(Q,I)) & \to & Z_n(\mathrm{Hom}^\bullet(Q,I)) & \to & H_n(\mathrm{Hom}^\bullet(Q,I)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B_n(\mathrm{Hom}^\bullet(Q,C)) & \to & Z_n(\mathrm{Hom}^\bullet(Q,C)) & \to & H_n(\mathrm{Hom}^\bullet(Q,C)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The first and second columns are exact, so the third one is also exact. But, for every $n \in \mathbb{Z}$, $H_n(\mathrm{Hom}^\bullet(Q,I)) = \mathrm{Hom}_{\mathscr{K}(R)}(Q[n],I) = 0$ since I is contractible.

4. $\Rightarrow$ 1. Let Q be a projective module and Y a submodule of Q. Since $\underline{Y}$ is a subcomplex of the DG-projective complex $\underline{Q}$, $\underline{Y}$ must be DG-projective by assumption, so Y is projective. □

It is a well-known fact that a ring is left semi-hereditary if and only if it is left coherent and every submodule of a flat module is flat (i.e., the weak global dimension of the ring is at most 1). Using subprojectivity we can prove a similar result in the categories of complexes. Namely, a ring is left semi-hereditary if and only if it is left coherent and every subcomplex of a DG-flat complex is DG-flat (Corollary 2). This is so because rings for which subcomplexes of DG-flat complexes are DG-flat are precisely those of weak global dimension at most 1 (Proposition 11).

We first recall that a complex is finitely presented if it is bounded and has finitely presented components (see [10, Lemma 4.1.1]). Recall also that the subprojectivity domain of the class of all finitely presented complexes (respectively, modules) is the class of all flat complexes (respectively, modules) (see [4, Proposition 2.18]). Finally, recall that a complex F is said to be DG-flat if F_n is flat for every $n \in \mathbb{Z}$ and the complex $E \otimes^\bullet F$ is exact for any exact complex E of right R-modules (see [11]).

Proposition 11 *For any ring R, the following assertions are equivalent.*

1. *The weak global dimension of R is at most* 1.
2. *For every finitely presented complex M and every exact complex N, if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.*
3. *For every finitely presented module M and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R\text{-Mod}}(M)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{M}[n])$.*
4. *Every subcomplex of a DG-flat complex is DG-flat.*

Proof 1. $\Rightarrow$ 2. Consider a short exact sequence of complexes $0 \to K \to P \to N \to 0$ with P projective. Since all cycles $Z_n(P)$ are projective, every cycle $Z_n(K)$ is flat by assumption.

Then, K is flat (K is exact since P and N are), so $K \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M[-1])$ and hence $\mathrm{Hom}_{\mathscr{K}(R)}(M[-1], K) = 0$ by Lemma 1. Therefore, $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$ by Theorem 1.

2. ⇒ 3. Clear.

3. ⇒ 1. Let X be a submodule of a flat module F. Let us prove that $X \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M)$ for every finitely presented module M. For let M be a finitely presented module and consider the exact complex

$$\mathcal{F} : \cdots \to 0 \to X \to F \to C \to 0 \to \cdots$$

with F in the 0-position.

Since $F \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M)$ we have that $\mathcal{F} \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{M})$ by assumption, and then $X \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M)$ by Lemma 3.

1. ⇒ 4. Let F be a DG-flat complex, N be a subcomplex of F and $P \to N$ be an epic quasi-isomorphism with P DG-projective. To prove that N is DG-flat it is sufficient to prove that for every finitely presented complex M, $\mathrm{Hom}_{\mathscr{C}(R)}(M, P) \to \mathrm{Hom}_{\mathscr{C}(R)}(M, N)$ is epic (see [12, Proposition 6.2]). For let $f : M \to N$ be a morphism of complexes with M finitely presented and consider the following pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & D & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & E & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Every module N_n is flat by 1, so $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M_n)$ for every $n \in \mathbb{Z}$, and hence the short exact sequence $0 \to E \to D \to M \to 0$ splits at the module level by Proposition 1 since for every $n \in \mathbb{Z}$, D_n is a pullback (see Lemma 2). Then, the sequence $0 \to E \to D \to M \to 0$ is equivalent to a short exact sequence $0 \to E \to M(g) \to M \to 0$ where $M(g)$ is the mapping cone of a morphism $g : M[-1] \to E$ (see [8, Section 3.3]).

Now, every module E_n is flat by condition 1. So, $E_n \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M_{n+1})$ for every $n \in \mathbb{Z}$. Thus, $E \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M[-1])$ by condition 2 and then by Lemma 1 we get that $\mathrm{Hom}_{\mathscr{K}(R)}(M[-1], E) = 0$.

In particular, $g : M[-1] \to E$ is null-homotopic so the sequence $0 \to E \to M(g) \to M \to 0$ splits (see [8, Proposition 3.3.2])and then the sequence $0 \to E \to D \to M \to 0$ splits. Therefore, f clearly factors through $P \to N$.

4. ⇒ 1. Let F be a flat module and Y a submodule of F. Then, $\underline{Y}$ is a subcomplex of the DG-flat complex $\underline{F}$, so $\underline{Y}$ is also DG-flat by assumption and therefore Y is flat. □

Corollary 2 *For any ring R the following statements are equivalent.*

1. *R is left semi-hereditary.*
2. *R is left coherent and for every finitely presented complex M and every exact complex N, if $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M_n)$ for every $n \in \mathbb{Z}$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(M)$.*
3. *R is left coherent and for every finitely presented module M and every exact complex N, if there exists $n \in \mathbb{Z}$ such that $N_n \in \underline{\mathfrak{Pr}}^{-1}_{R-\mathrm{Mod}}(M)$, then $N \in \underline{\mathfrak{Pr}}^{-1}_{\mathscr{C}(R)}(\underline{M}[n])$.*
4. *R is left coherent and every subcomplex of a DG-flat complex is DG-flat.*

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