Annali di Matematica Pura ed Applicata (1923 -) (2022) 201:2837–2845
https://doi.org/10.1007/s10231-022-01221-x

A non compact Krylov–Bogolioubov theorem

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Received: 16 November 2021 / Accepted: 22 April 2022 / Published online: 19 May 2022

Abstract
We show that each locally uniformly Lipschitz and minimal action of a locally compact group G on a locally compact metrisable space X admits an invariant Radon measure. We deduce a Krylov–Bogolioubov type theorem for flows on non compact metric spaces.

Keywords Krylov–Bogolioubov theorem · Invariant measure · Minimal action · Locally uniformly Lipschitz

Mathematics Subject Classification 28C10 · 37B05

1 Introduction

The objective of this short article is to give a new existence result for invariant Radon measures for locally compact transformation groups (i.e., for the action of a locally compact, Hausdorff topological group on a locally compact metrisable space; see Theorem 3). We need to impose some (rather natural) conditions on the action, which correspond to certain previously known conditions in the classical case of a space with a single transformation. The conditions are of two natures: The first one is a continuity type condition (related directly to the metric, but preserved if one switches to an equivalent metric with proper constants). The other one is a transitivity type condition, called topological transitivity, which is the same as the requirement that all orbits are dense.

The existence of Radon invariant measures for group actions on locally compact spaces can be viewed as natural (possibly infinite volume) version of the Krylov–Bogolioubov theorem. This has attracted renewed attention recently following the work of Kellerhals et.al., linking this question to the notion of superamenability of the acting group [13].

The result is compared and contrasted to the existing results by Fomin, Rosenblatt and Halmos (see section 2).

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2 Dynamics

Ergodic sets were introduced by Krylov and Bogolioubov in 1937 [14]. This was part of their study of compact dynamical systems, which we briefly review here based on the elegant paper of Oxtoby [17].

2.1 Compact dynamics

If X is a set and $p \in X$, $f : X \to \mathbb{R}$ is a function and $T : X \to X$ is a bijection,

$$M(f,p,k) = f_k(p) := \frac{1}{k}\sum_{i=1}^{k} f(T^i p) \quad (k = 1, 2, \cdots)$$

and

$$M(f,p) = f^*(p) := \lim_{k\to\infty} M(f,p,k),$$

when the limit exists.

When (X, μ) is a probability measure space and T is measure preserving bijective transformation on X, the celebrated ergodic theorem states as follows.

Theorem 1 (i) (Birkhoff ergodic theorem) If $f \in L^1(X,\mu)$ then the sequence $(f_k(p))$ converges a.e. to a function $f^* \in L^1(X,\mu)$ satisfying $f^*(Tp) = f^*(p)$ and $\int_X f d\mu = \int_X f^* d\mu$.

(ii) (mean ergodic theorem) If $f \in L^q(X,\mu)$ $(1 \le q < \infty)$ then $\|f_k - f^*\|_q \to 0$ as $k \to \infty$.

Now if X is a compact metric space and T is a bijective homeomorphism of X, any finite Radon measures μ on X correspond to positive linear functionals on $C(X)$. For $p \in X$ one can find (by a diagonal process) an increasing sequence (k_i) of positive integers such that $I(f) := \lim_{i\to\infty} M(f,p,k_i)$ exists for a countable dense set of functions f in $C(X)$. This extends to an invariant positive linear functional on $C(X)$ with value 1 at the constant function 1. Let μ be the corresponding invariant Radon probability measure on X. If $K \subseteq X$ is closed such that $M(\chi_K, p)$ is not identically zero, $\alpha := \limsup_{i\to\infty} M(\chi_K, p, k_i) > 0$, for some $p \in X$ and some increasing sequence (k_i) of positive integers. Therefore, $I(f) \ge \alpha$ for $f \ge \chi_K$, thus $\mu(K) \ge \alpha$. This is the content of the classical Krylov–Bogolioubov result for compact systems [14].

Theorem 2 [*Krylov–Bogolioubov theorem*] *Any compact system* (X, T) *admits an invariant probability Radon measure* μ. *If* K *is a closed subset of* X, *then either* $\mu(K) > 0$ *for some invariant measure* μ *or else* $M(\chi_K, p) = 0$ *for every* $p \in X$.

2.2 Non compact dynamics

In the non compact case, if T is a homeomorphism of a complete separable metric space X, the system (X, T) need not admit a finite invariant Borel measure (e.g., for $X = \mathbb{R}$ and

$T(x) = x + 1$.) However, if such a measure exists, the other results of Krylov and Bogolioubov remain true up to appropriate modification, as observed by Fomin [3] (a generalization in another direction is given by Yoshida [26].)

A Borel system is a pair (Y, T) where Y is a Borel subset of a complete separable metric space X, and T is a homeomorphism of Y onto itself. By a result of Urysohn, we may embed Y into the Hilbert cube (which is a compact metrisable space), (Y, T) is homeomorphic to a subsystem of the compact system (X, S), where X is a countable infinite Cartesian product of Hilbert cubes, and S is the forward shift. Let $Y' \subseteq X$ be the image of Y under this homeomorphism, then the invariant finite Borel measures of (Y, T) correspond to invariant probability measures μ of (X, S) for which $\mu(X \backslash Y') = 0$. It follows that the system (Y, T) admits no finite invariant Borel measure iff $M(\chi_K, p) = 0$, for every compact set $K \subseteq Y$ and every point $p \in Y$. According to Fomin [3], quasi-regular points p of the Borel system (Y, T) should be those points for which the mean value $M(f, p)$ exists for each bounded continuous function $f \in C_b(Y)$ such that for every $\varepsilon > 0$ there is a compact set $K \subseteq Y$ with $M(\chi_K, p) > 1 - \varepsilon$. For the corresponding subsystem (Y', S) a point $p \in Y'$ is quasi-regular (in the sense of Fomin) iff p is quasi-regular in (X, S) (in the sense of Krylov–Bogolioubov) and $\mu_p(Y') = 1$.

The set $Q' \subseteq X$ of quasi-regular points of (X, S) for which $\mu_p(Y') = 1$ is Borel, and so is the set $Q \subseteq Y$ of quasi-regular points of (Y, T). If E is an ergodic set of (X, S), with ergodic measure μ, then $\mu(E \cap Y') = 0$ or 1, In particular $Y' \backslash Q'$ has invariant measure zero in (X, S), therefore Q has invariant measure one in (Y, T). Now the points of density, transitive and regular points could be defined as above and similar results for compact systems also hold for any Borel system.

The drawback in the results such as those of Fomin is that all these are the case only when the Borel system admits at least one finite invariant measure. The main objective of this paper is to go beyond this by studying Borel systems with not necessarily finite invariant measures.

Certain comparable results for actions on abstract sets are obtained by J. Rosenblatt. If a finitely generated group G with polynomial growth acts on a set X, then for a given nonempty subset A of X there exists a finitely-additive G-invariant positive extended real-valued measure μ defined on all subsets of X with $\mu(A) = 1$ [18]. Here the drawback is that slow growth conditions as above impose amenability type conditions on the group. If G contains a free subsemigroup S on two generators, then G has exponential growth and there does not exist a measure as above even for G acting on itself by translation [18, Theorem 4.6].

2.3 σ-Compact dynamics

In locally compact dynamics, one might single out the σ-compact (and locally compact) case, as in this case a Radon measure is σ-finite, and a σ-finite invariant measure happen to behave like a finite one. Halmos in [8] investigates the following problem: Given a measurable transformation T on a measure space (X, m), find a T-invariant measure m^* with $m \ll m^*$. Halmos finds conditions under which such a σ-finite invariant measure exists, provided that the original measure is σ-finite, argues that only the σ-finite case is interesting, and conjectures that these conditions are essentially needed (i.e., the invariant measure fails to exist in general). Halmos requires the σ-boundedness of the transformation, which is something hard to check. This should be clear from the two examples, proposed by Halmos as candidates for completely unbounded transformations [8, page 754].

3 Transformation groups

In this section we present the main result of the paper.

3.1 Actions on metric spaces

Let G be a locally compact group and (X, d) be a metric space. A continuous *action* of G on X is a continuous map: $G \times X \to X;\ (t,x) \mapsto t \cdot x$, satisfying

$$t \cdot (s \cdot x) = ts \cdot x \ \ (s,t \in G, x \in X).$$

We say that the action is *isometric* if $d(t \cdot x, t \cdot y) = d(x,y)$, for each $t \in G$ and $x, y \in X$ and *isometrisable* if there is an equivalent metric for which the action is isometric [1]. The action is called locally Lipschitz at $x_0 \in X$ if for each $t \in G$, there is $\delta = \delta(t) > 0$ and a constant $C = C(t) > 0$ such that $d(t \cdot x, t \cdot x_0) \le Cd(x, x_0)$, whenever $d(x, x_0) < \delta$. When the constants δ, C are independent of t, we say that the action is locally uniformly Lipschitz at x_0. The action is called *locally Lipschitz* (**LL**) if it is locally Lipschitz at each x_0, and *locally uniformly Lipschitz* (**LUL**) if it is locally uniformly Lipschitz at each x_0. These notions are analogs of the same notions for classical dynamics [2, 24]. It turns out that that the latter condition is too strong for our purposes and hardly holds in classical dynamics. The former is much weaker notion and suffices for controlling the behavior of the change of diameter of the subsets of compact sets under the dynamics and holds under quite natural conditions for the classical case.

We say that the action is *transitive* if for each $x, y \in X$, there is $t \in G$ with $t \cdot x = y$; and *minimal* if for each $x \in X$, each open set $U \subseteq X$, there is $t \in G$ with $t \cdot x \in U$. The latter condition is clearly equivalent to having a dense orbit at each point. Finally, the action is *topologically transitive* if for every nonempty open subsets U and V of X, there is $t \in G$ such that $(t \cdot U) \cap V \neq \emptyset$. When G is countable and X is a Polish space, this is equivalent to the existence of a dense orbit. For flows on compact metrisable spaces, it is also known that topological transitivity is equivalent to the average shadowing property (possibly with respect to some other equivalent metric) [20, Theorem 2]. Finally, the action is called *uniformly topologically equicontinuous* if for any $y \in X$ and open subset V containing y there is an open subset U containing y with $U \subseteq V$ such that, for each $x \in X$ there exists an open subset W containing x satisfying $t \cdot W \subseteq V$ whenever $t \cdot W \cap U$ is non-empty, for all $t \in G$ [1]. This is a uniform version of a classical notion defined by Royden [19], which is equivalent to isometrisability of the action when X is second countable and metrisable [1, Theorem 2.10].

A Borel measure μ on X is *invariant* if $\mu(t \cdot E) = \mu(E)$, for each $t \in G$ and each Borel set $E \subseteq X$; and *quasi-invariant* if $\mu(E) = 0$ iff $\mu(t \cdot E) = 0$, for each $t \in G$ and each Borel set $E \subseteq X$. The main result of this paper now could be phrased as follows.

Theorem 3 (Invariant measure of group actions) *Let G be a locally compact group acting continuously on a locally compact metrisable space X. If the action is locally uniformly Lipschitz at some point $x_0 \in X$ and minimal then there is a* (*non zero*) *invariant Radon measure μ on X.*

In particular, if X is also second countable, the invariant measure exists when the action is uniformly topologically equicontinuous and minimal. Note also that by Urysohn metrisation theorem, X is metrisable if it is second countable and regular (i.e., satisfies separation axiom T_3). Also note that there are non transitive actions which are minimal. A concrete example is the irrational rotation on the unit circle $\mathbb{S}^1$, which is not transitive, but has dense orbits, and so is minimal.

3.2 Flows on non compact metric spaces

Let T be a homeomorphism which is locally bi-Lipschitz on (X, d), that is, each point $x \in X$ has a neighborhood $B_\delta(x)$ such that the restrictions of T and T^{-1} to $B_\delta(x)$ are Lipschitz (see for instance, [15, 21]), then the corresponding $\mathbb{Z}$-action on X is locally Lipschitz: for each $x_0 \in X$, there are constants $\delta > 0$ and $C > 1$ such that for each $n \in \mathbb{Z}$ we have $d(T^n x, T^n x_0) \le C^{|n|} d(x, x_0)$, whenever $d(x, x_0) < \delta / C^{|n|}$. It is therefore natural to call the transformation T *locally uniformly bi-Lipschitz* if the corresponding $\mathbb{Z}$-action on X is locally Lipschitz is locally uniformly Lipschitz. The following result now immediately follows from our main theorem.

Corollary 1 (invariant measure of flows) *Let (X, d) be a locally compact metrisable space and $T : X \to X$ be a locally uniformly bi-Lipschitz homeomorphism with dense orbits. Then there is a non zero T-invariant Radon measure μ on X.*

4 Proof of the main result

The proof of the main result is an adaptation of the proof of the existence of Haar measures of locally compact groups by Alfréd Haar [10] in the second countable case (extended to the general case by André Weil [25]). The proof is by Haar ratio method which is adapted to dynamical setting by replacing "local basis at the identity" (in the group case) by "sets of vanishing diameter" (in the metric space on which group is acting).

Proof of Theorem 3 Given $f, \varphi \in C_c^+(X)$, with φ non zero, put

$$(f : \varphi) := \inf\{\sum_{i=1}^{n} c_i : c_1, \cdots, c_n \in \mathbb{R}^+, f \le \sum_{i=1}^{n} c_i L_{t_i}\varphi, \text{ for some } t_1, \cdots, t_n \in G\}$$

where $L_t\varphi(x) = \varphi(t^{-1} \cdot x)$, for $t \in G$ and $x \in X$. To see that the above set is never empty, use the fact that the action is topologically transitive for $K := \operatorname{supp}(f)$ and $U := \{x \in X : \varphi(x) > \frac{1}{2}\|\varphi\|_\infty\}$ to find $t_1, \cdots, t_n \in G$ with $K \subseteq t_1 \cdot U \cup \cdots \cup t_n \cdot U$ and observe that

$$f(x) \le \sum_{i=1}^{n} \frac{2\|f\|_\infty}{\|\varphi\|_\infty} \varphi(t_i^{-1} \cdot x), \quad (x \in K).$$

It is straightforward to check that $(f : \varphi)$ satisfies the following properties:

(*i*) $(f : \varphi) = (L_t f : \varphi)$,

(*ii*) $(f_1 + f_2 : \varphi) \le (f_1 : \varphi) + (f_2 : \varphi)$ and $(cf : \varphi) = c(f : \varphi)$,

(*iii*) $(f_1 : \varphi) \le (f_2 : \varphi)$ whenever $f_1 \le f_2$,

(*iv*) $(f : \varphi) \le (f : \psi)(\psi : \varphi)$,

(*v*) $(f : \varphi) \ge \frac{\|f\|_\infty}{\|\varphi\|_\infty}$, for each $f, f_1, f_2, \varphi, \psi \in C_c^+(X)$ and $c \geq 0$. Fix a non zero function $f_0 \in C_c^+(X)$ and put

$$I_\varphi(f) := \frac{(f : \varphi)}{(f_0 : \varphi)} \quad (f, \varphi \in C_c^+(X)).$$

Then by (iv),

$$(f_0 : f)^{-1} \leq I_\varphi(f) \leq (f : f_0).$$

Let $f_1, f_2 \in C_c^+(X)$ and $\varepsilon > 0$. Choose $g \in C_c^+(X)$ with $g = 1$ on $\mathrm{supp}(f_1 + f_2)$ and given $\gamma > 0$ put $h = f_1 + f_2 + \gamma g$ and $h_i = f_i / h$ (with the convention that h_i is defined to be zero at points where f_i is zero) for $i = 1, 2$. Since $h_i \in C_c^+(X)$ and X is a metric space, h_i is uniformly continuous and there is $\delta_1 > 0$ with $|h_i(x) - h_i(y)| < \gamma$ $(i = 1, 2)$, whenever $d(x, y) < \delta_1$.

By the characteristic property of infimum, there are constants $c_1, \cdots, c_n$ and finite subset $K = \{t_1, \cdots, t_n\} \subseteq G$ such that $h \leq \sum_{j=1}^n c_j L_{t_j}\varphi$, and $\sum_{j=1}^n c_j \leq (h : \varphi) + \gamma$. By (**LUL**) condition at x_0, there are constants $\delta_2 > 0$ and $C > 0$ such that

$$d(t_j \cdot x, t_j \cdot x_0) \leq C d(x, x_0),$$

whenever $d(x, x_0) < \delta_2$. Put $\delta = \min(\delta_1, \delta_2)$ and assume further that

$$\mathrm{diam}(\mathrm{supp}(\varphi)) < \min\{\delta, \delta / C\}.$$

Then for each $x \in \mathrm{supp}(h)$,

$$\begin{aligned} f_i(x) = h(x) h_i(x) &\leq \sum_{j=1}^n c_j \varphi(t_j^{-1} \cdot x) h_i(x) \\ &\leq \sum_{j=1}^n c_j \varphi(t_j^{-1} \cdot x) \big(h_i(t_j \cdot x_0) + \gamma\big), \end{aligned}$$

where the second inequality follows from the fact that if $t_j^{-1} \cdot x \in \mathrm{supp}(\varphi)$ then $d(t_j^{-1} \cdot x, x_0) \leq \mathrm{diam}(\mathrm{supp}(\varphi)) < \min\{\delta, \delta / C\} \leq \delta_2$, and so

$$d(x, t_j \cdot x_0) \leq C d(t_j^{-1} \cdot x, x_0) < \delta \leq \delta_1.$$

Therefore,

$$\begin{aligned} (f_1 : \varphi) + (f_2 : \varphi) &\leq \sum_{j=1}^n c_j \big(h_1(t_j \cdot x_0) + h_2(t_j \cdot x_0) + 2\gamma\big) \\ &\leq \sum_{j=1}^n c_j (1 + 2\gamma) \\ &\leq (1 + 2\gamma)[(h : \varphi) + \gamma], \end{aligned}$$

thus, by (ii),

$$
\begin{aligned}
I_\varphi(f_1) + I_\varphi(f_2) &\le (1+2\gamma)(I_\varphi(h)+\gamma)\\
&\le (1+2\gamma)\big(I_\varphi(f_1+f_2)+\gamma I_\varphi(g)+\gamma\big)\\
&\le I_\varphi(f_1+f_2)+\varepsilon,
\end{aligned}
$$

for γ small enough.

For $f \in C_c^+(X)$ consider the closed interval $\Omega_f := [(f_0 : f)^{-1}, (f : f_0)]$ and the compact Hausdorff space $\Omega = \prod_f \Omega_f$ (in the product topology). The elements of Ω are choice functions, and in particular, Ω includes the maps I_φ, for each non zero $\varphi \in C_c^+(X)$. Next, the closure $L_{\delta,C}$ of the set

$$\{I_\varphi : \operatorname{diam}(\operatorname{supp}(\varphi)) < \min\{\delta, \delta/C\}\} \subseteq \Omega,$$

is compact, for each $\delta, C > 0$. Then the family $\mathfrak{F} := \{L_{\delta,C} : \delta > 0, C > o\}$ has finite intersection property, since

$$\bigcap_{i=1}^{n} L_{\delta_i,C_i} \supseteq L_{\delta,C}, \text{ for } \delta = \min_i \delta_i,\ C = \max_i C_i.$$

It follows that the whole family has a non-empty intersection, from which we may choose an element I. For each open neighborhood V of I in Ω and each $\delta, C > 0$, there is $\varphi \in C_c^+(X)$ with $\operatorname{diam}(\operatorname{supp}(\varphi)) < \min\{\delta, \delta/C\}$ such that $I_\varphi \in V$. This in particular holds for elements of the local basis at I of the form

$$V(\varepsilon, n, f_1, \cdots, f_n) := \{J \in \Omega : |J(f_i) - I(f_i)| < \varepsilon,\ (1 \le i \le n)\}.$$

Given $f_1, f_2 \in C_c^+(X)$, use the above observation for $V(\varepsilon, 3, f_1, f_2, f_1 + f_2)$ and a standard triangle inequality argument to get $I(f_1+f_2) \le I(f_1) + I(f_2) \le I(f_1+f_2) + 4\varepsilon$. Tending ε to zero, we get $I(f_1+f_2) = I(f_1) + I(f_2)$. Also clearly, $I(L_t f) = I(f)$, for each $t \in G$ and $f \in C_c^+(X)$. Finally, for $f \in C_c(X, \mathbb{R})$ let $f = f^+ - f^-$ be the unique decomposition of f into difference of two functions in $C_c^+(X)$ with $f^+ f^- = 0$, then $I(f) := I(f^+) - I(f^-)$ clearly defines a well-defined $\mathbb{R}$-linear, positive linear functional on $C_c(X, \mathbb{R})$, which is also non zero, as $I_\varphi(f_0) = 1$, for the fixed function above (used in the process of defining I_φ). Finally, I extends to a complex valued positive linear functional on $C_c(X)$, and by Riesz representation theorem, there is a unique (not necessarily finite) positive Radon measure μ on X with

$$I(f) = \int_X f d\mu,\ \ (f \in C_c(X)).$$

The fact that I is translation invariant and a standard argument based on Urysohn lemma gives translation invariance of μ. □

It is worth nothing that the conditions of the above theorem are enough for the existence of an invariant measure, but by no means necessary. We adapt an example given by Steinlage which is locally Lipschitz (**LL**) but not locally uniformly Lipschitz (**LUL**), but has an invariant measure [22, Example 4.7]. The space X in this example is the space of "oriented lines" in the Euclidean space $\mathbb{R}^3$ and the group G is the group of solid motions of $\mathbb{R}^3$, identified with $O(3) \ltimes \mathbb{R}^3$. It is shown in [23, Theorem 5.5] that X is homeomorphic to the homogeneous space G/H, consisting of *right* cosets of H, for the subgroup H of motions keeping the z-axis (upward oriented) fixed. Since both G and H are unimodular, and the existence of invariant measure is concluded by Weil's theorem (see also, [22]). It is also shown that G has a *left* translation invariant metric, which then a metric on G/H by [11, 8.14(b)] (c.f., proof of [23, Lemma 5.2]). The problem is that, as one can directly check, the metric on G is not *right* translation invariant. However, the action of G on itself by right translation satisfies (**LL**): given $(A, a), (B, b), (C, c) \in G := O(3) \ltimes \mathbb{R}^3$,

$$\begin{aligned} d((A,a)(C,c),(B,b)(C,c))^2 &= d((AC, a + Ac), (BC, b + Bc))^2 \\ &= \|AC - BC\|^2 + |(a + Ac) - (b + Bc)|^2 \\ &\leq (1 + |c|^2)\|A - B\|^2 + |a - b|^2 \\ &\leq (1 + |c|^2) d((A,a),(B,b))^2 \end{aligned}$$

Now since G acts on G/H by right translation and

$$d(Hx, Hy) := \inf_{h_1, h_2 \in H} d(h_1 x, h_2 y),$$

the action of G on G/H also satisfies (**LL**). Indeed, given $(C, c) := g \in G$ and distinct points $Hx, Hy \in G/H$, by the characteristic property of infimum, we may choose $h_1, h_2 \in H$ with $d(h_1x, h_2y) < 2d(Hx, Hy)$. By the above calculation with $(A, a) := x, (B, b) := y$,

$$\begin{aligned} d(Hxg, Hyg) &\leq d(h_1xg, h_2yg) \\ &\leq \sqrt{1 + |c|^2} d(h_1x, h_2y) \\ &< 2\sqrt{1 + |c|^2} d(Hx, Hy), \end{aligned}$$

that is, (**LL**) holds for the action of G on G/H by right translation.

To see that this example does not satisfy (**LUL**), one should observe that (**LUL**) implies the property of *separation of compact* (**SC**) defined by Steinlage as follows: given compact disjoint sets B and C in X, there is an open set O in X such that for no $t \in G$, $t \cdot O$ could meet both B and C. When X is a metric space and $d := \operatorname{dist}(B, C) > 0$, then for an open set $O \subseteq X$ with $\operatorname{diam}(O) < \min(\delta, d/C)$, where $\delta, C > 0$ are the constants in (**LUL**) condition, if there is $t \in G$ such that tO meets both B and C (say at x and y, respectively), then

$$d(t^{-1} \cdot x, t^{-1} \cdot y) \leq \operatorname{diam}(O) < \delta,$$

thus by (**LUL**) condition,

$$d(x, y) \leq Cd(t^{-1} \cdot x, t^{-1} \cdot y) \leq \operatorname{diam}(O) < d = \operatorname{dist}(B, C),$$

which contradicts the fact that $x \in B, y \in C$. Now it is shown by Steinlage in [22, Example 4.7] that the above example does not satisfy (**SC**), so it cannot satisfy (**LUL**) by the above observation.

Note that in the above argument showing that (**LUL**)⇒(**SC**), we used the fact that the action is locally uniformly Lipschitz at *every* point, whereas in the above theorem, we only assume that the action is locally uniformly Lipschitz at *some* point, so our main result does not follow from that of Steinlage in [22, Lemma 4.3].

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