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Quillen connection and the uniformization of Riemann surfaces

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Abstract
The Quillen connection on $\mathcal{L} \longrightarrow \mathcal{M}_g$, where $\mathcal{L}^*$ is the Hodge line bundle over the moduli stack of smooth complex projective curves curves $\mathcal{M}_g$, $g \geq 5$, is uniquely determined by the condition that its curvature is the Weil–Petersson form on $\mathcal{M}_g$. The bundle of holomorphic connections on $\mathcal{L}$ has a unique holomorphic isomorphism with the bundle on $\mathcal{M}_g$ given by the moduli stack of projective structures. This isomorphism takes the C^∞ section of the first bundle given by the Quillen connection on $\mathcal{L}$ to the C^∞ section of the second bundle given by the uniformization theorem. Therefore, any one of these two sections determines the other uniquely.

Keywords Uniformization · Projective structure · Quillen connection · Torsor

Mathematics Subject Classification 30F10 · 14H15 · 53C07 · 53B10

1 Introduction

A holomorphic Ω^1_Z–torsor over a complex manifold Z is a holomorphic fiber bundle $\mathcal{E}$ over Z on which the holomorphic cotangent bundle Ω^1_Z, considered as a bundle of groups over Z, acts satisfying the condition that the map

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$$\mathcal{E} \times_Z \Omega^1_Z \longrightarrow \mathcal{E} \times_Z \mathcal{E},$$

constructed using the action map $\mathcal{E} \times_Z \Omega^1_Z \longrightarrow \mathcal{E}$ and the identity map of $\mathcal{E}$, is a biholomorphism. This notion of a holomorphic torsor extends to smooth Deligne–Mumford stacks.

Here we investigate two different natural holomorphic $\Omega^1_{\mathcal{M}_g}$–torsors over the moduli stack $\mathcal{M}_g$ of irreducible smooth complex projective curves of genus g (throughout we assume that $g \geq 5$). The first $\Omega^1_{\mathcal{M}_g}$–torsor is given by the moduli stack $\mathcal{P}_g$ of genus g compact connected Riemann surfaces equipped with a projective structure. We recall that the projective structures on a Riemann surface X is an affine space modeled on the complex vector space $H^0(X, K_X^{\otimes 2}) = (\Omega^1_{\mathcal{M}_g})_X$. The second $\Omega^1_{\mathcal{M}_g}$–torsor is given by the sheaf of holomorphic connections $\mathrm{Conn}(\mathcal{L})$ on the dual $\mathcal{L}$ of the Hodge line bundle on the moduli space $\mathcal{M}_g$. We recall that the space of holomorphic connections on $\mathcal{L}\big|_U$, where $U \subset \mathcal{M}_g$ is an affine open subset, is an affine space modeled on the complex vector space $H^0(U, \Omega^1_U)$.

The uniformization theorem gives a C^∞ section

$$\Psi : \mathcal{M}_g \longrightarrow \mathcal{P}_g .$$

On the other hand, the holomorphic line bundle $\mathcal{L}$ has a complex connection associated to the Quillen metric on it. The Quillen metric is constructed using the eigenvalues of the Laplacian acting on the functions on Riemann surfaces. This Quillen connection is uniquely determined, among all complex connections on $\mathcal{L}$, by the property that its curvature is

$$\frac{\sqrt{-1}}{6\pi}\omega_{WP} ,$$

where ω_{WP} is the Weil–Petersson form on $\mathcal{M}_g$ (see Corollary 2.2). We recall that the Weil-Petersson form on the moduli space is constructed using the uniformization of the Riemann surfaces.

We construct from $\mathrm{Conn}(\mathcal{L})$ a new holomorphic $\Omega^1_{\mathcal{M}_g}$–torsor simply by scaling the action of $\Omega^1_{\mathcal{M}_g}$. More precisely, if

$$A : \mathrm{Conn}(\mathcal{L}) \times_{\mathcal{M}_g} \Omega^1_{\mathcal{M}_g} \longrightarrow \mathrm{Conn}(\mathcal{L})$$

is the action of $\Omega^1_{\mathcal{M}_g}$ on $\mathrm{Conn}(\mathcal{L})$, then define the following new holomorphic action of $\Omega^1_{\mathcal{M}_g}$ on the same holomorphic fiber bundle $\mathrm{Conn}(\mathcal{L})$:

$$A^t : \mathrm{Conn}(\mathcal{L}) \times_{\mathcal{M}_g} \Omega^1_{\mathcal{M}_g} \longrightarrow \mathrm{Conn}(\mathcal{L}), \ (z, v) \longmapsto A\left(z, \frac{\sqrt{-1}}{6\pi} \cdot v\right).$$

The resulting holomorphic $\Omega^1_{\mathcal{M}_g}$–torsor $(\mathrm{Conn}(\mathcal{L}), A^t)$ will be denoted by $\mathrm{Conn}^t(\mathcal{L})$.

Let

$$\Phi : \mathcal{M}_g \longrightarrow \mathrm{Conn}^t(\mathcal{L}) = \mathrm{Conn}(\mathcal{L})$$

be the C^∞ section given by the Quillen connection on $\mathcal{L}$.

We prove the following (see Theorem 3.1):

Theorem 1.1

1. *There is exactly one holomorphic isomorphism between the* $\Omega^1_{\mathcal{M}_g}$*–torsors* $\mathrm{Conn}^t(\mathcal{L})$ *and* $\mathcal{P}_g$.
2. *The holomorphic isomorphism between the* $\Omega^1_{\mathcal{M}_g}$*–torsors* $\mathrm{Conn}^t(\mathcal{L})$ *and* $\mathcal{P}_g$ *takes the above section* Φ *of* $\mathrm{Conn}^t(\mathcal{L})$ *to the section* Ψ *of* $\mathcal{P}_g$ *given by the uniformization theorem.*

We note that from Theorem 1.1 it follows that each of the sections Φ and Ψ determines the other uniquely.

2 Quillen metric on a line bundle

For $g \geq 5$, let $\mathcal{M}_g$ denote the moduli stack of smooth complex projective curves of genus g. It is an irreducible smooth complex quasiprojective orbifold of dimension $3(g-1)$ [7]. Moreover, $\mathcal{M}_g$ has a natural Kähler structure

$$\omega_{WP} \in C^\infty(\mathcal{M}_g, \Omega^{1,1}_{\mathcal{M}_g}) \tag{2.1}$$

which is known as the *Weil–Petersson form*.

A torsor over $\mathcal{M}_g$ for the holomorphic cotangent bundle $\Omega^1_{\mathcal{M}_g}$ is a fiber bundle

$$\mathcal{E} \longrightarrow \mathcal{M}_g$$

together with a morphism

$$\varpi : \mathcal{E} \times_{\mathcal{M}_g} \Omega^1_{\mathcal{M}_g} \longrightarrow \mathcal{E}$$

such that

- ϖ_X is an action of the vector space $(\Omega^1_{\mathcal{M}_g})_X$ on the fiber $\mathcal{E}_X$ for every $X \in \mathcal{M}_g$, and
- the map of fiber products

  $$\mathcal{E} \times_{\mathcal{M}_g} \Omega^1_{\mathcal{M}_g} \longrightarrow \mathcal{E} \times_{\mathcal{M}_g} \mathcal{E}, \quad (e, v) \longmapsto (e, \varpi(e, v))$$

  is an isomorphism.

Let

$$\varphi : \mathcal{C}_g \longrightarrow \mathcal{M}_g \tag{2.2}$$

be the universal curve. The line bundle on $\mathcal{M}_g$

$$\mathcal{L} := \det R^1\varphi_*\mathcal{O}_{\mathcal{C}_g} = \bigwedge^g R^1\varphi_*\mathcal{O}_{\mathcal{C}_g}$$

is a generator of $\mathrm{Pic}(\mathcal{M}_g) = \mathbb{Z}$ [1, p. 154, Theorem 1].

Let $\mathrm{Conn}(\mathcal{L}) \longrightarrow \mathcal{M}_g$ be the holomorphic fiber bundle given by the sheaf of holomorphic connections on $\mathcal{L}$. We will briefly recall the construction of $\mathrm{Conn}(\mathcal{L})$. Consider the Atiyah exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}_g} = \mathrm{Diff}^0(\mathcal{L}, \mathcal{L}) \longrightarrow \mathrm{At}(\mathcal{L}) := \mathrm{Diff}^1(\mathcal{L}, \mathcal{L}) \xrightarrow{p_0} T\mathcal{M}_g \longrightarrow 0, \tag{2.3}$$

where $\mathrm{Diff}^i(\mathcal{L}, \mathcal{L})$ is the holomorphic vector bundle over $\mathcal{M}_g$ corresponding to the sheaf of holomorphic differential operators of order i from $\mathcal{L}$ to itself, $T\mathcal{M}_g$ is the holomorphic tangent bundle of $\mathcal{M}_g$, and p_0 is the symbol map. For any open subset $U \subset \mathcal{M}_g$, giving a holomorphic connection on $\mathcal{L}\big|_U$ is equivalent to giving a holomorphic splitting of (2.3) over U [2]. Let

$$0 \longrightarrow \Omega^1_{\mathcal{M}_g} \longrightarrow \mathrm{At}(\mathcal{L})^* \stackrel{\beta}{\longrightarrow} \mathcal{O}_{\mathcal{M}_g} \longrightarrow 0 \tag{2.4}$$

be the dual of the sequence in (2.3). Let $\mathbf{1}_{\mathcal{M}_g} : \mathcal{M}_g \longrightarrow \mathcal{O}_{\mathcal{M}_g}$ denote the section given by the constant function 1 on $\mathcal{M}_g$. Then

$$\mathrm{At}(\mathcal{L})^* \supset \beta^{-1}(\mathbf{1}_{\mathcal{M}_g}(\mathcal{M}_g)) =: \mathrm{Conn}(\mathcal{L}) \stackrel{\phi}{\longrightarrow} \mathcal{M}_g \tag{2.5}$$

where β is the projection in (2.4). From (2.4) it follows immediately that $\mathrm{Conn}(\mathcal{L})$ is a holomorphic torsor over $\mathcal{M}_g$ for the holomorphic cotangent bundle $\Omega^1_{\mathcal{M}_g}$. In particular, for any $X \in \mathcal{M}_g$ the complex vector space $(\Omega^1_{\mathcal{M}_g})_X$ acts freely transitively on the fiber of $\mathrm{Conn}(\mathcal{L})$ over X. Let

$$A : \mathrm{Conn}(\mathcal{L}) \times_{\mathcal{M}_g} \Omega^1_{\mathcal{M}_g} \longrightarrow \mathrm{Conn}(\mathcal{L}) \tag{2.6}$$

be the holomorphic map giving the torsor structure. For any C^∞ (1, 0)-form η on $\mathcal{M}_g$, let

$$A_\eta : \mathrm{Conn}(\mathcal{L}) \longrightarrow \mathrm{Conn}(\mathcal{L}), \quad z \longmapsto A(z, \eta(\phi(z))) \tag{2.7}$$

be the C^∞ automorphism of $\mathrm{Conn}(\mathcal{L})$ over $\mathcal{M}_g$, where ϕ and A are the maps in (2.5) and (2.6) respectively.

A *complex connection* on $\mathcal{L}$ is a C^∞ connection ∇ on $\mathcal{L}$ such that the (0, 1)-component $\nabla^{0,1}$ of ∇ coincides with the Dolbeault operator on $\mathcal{L}$ that defines the holomorphic structure of $\mathcal{L}$. The space of complex connections on $\mathcal{L}$ is in a natural bijection with the space of C^∞ sections $\mathcal{M}_g \longrightarrow \mathrm{Conn}(\mathcal{L})$ of the projection ϕ in (2.5). There is a tautological holomorphic connection D^0 on the line bundle $\phi^*\mathcal{L}$, whose curvature $\Theta = \mathrm{Curv}(D^0)$ is a holomorphic symplectic form on $\mathrm{Conn}(\mathcal{L})$ (see [4, p. 372, Proposition 3.3]). Any complex connection ∇ on $\mathcal{L}$ satisfies the equation

$$\nabla = f_\nabla^* D^0 ,$$

where $f_\nabla : \mathcal{M}_g \longrightarrow \mathrm{Conn}(\mathcal{L})$ is the C^∞ section corresponding to ∇. Consequently, the curvature $\mathrm{Curv}(\nabla)$ of ∇ satisfies the equation

$$\mathrm{Curv}(\nabla) = f_\nabla^* \Theta . \tag{2.8}$$

We also have

$$A_\eta^* \Theta = \Theta + d\eta, \tag{2.9}$$

where A_η is the map in (2.7).

Given a Hermitian structure h_1 on $\mathcal{L}$, there is a unique complex connection on $\mathcal{L}$ that preserves h_1 [12, p. 11, Proposition 4.9]; it is known as the *Chern connection*.

Equip the family of Riemann surfaces $\mathcal{C}_g$ in (2.2) with the relative Poincaré metric. Also, equip $\mathcal{O}_{\mathcal{C}_g}$ with the trivial (constant) Hermitian structure; the pointwise norm of the constant section with value c is the constant $|c|$. These two together produce a Hermitian structure h_Q on $\mathcal{L}$, which is known as the Quillen metric [3, 14]. Let

$$\mathrm{Curv}(\nabla^Q) \in C^\infty(\mathcal{M}_g, \Omega^{1,1}_{\mathcal{M}_g})$$

be the curvature of the Chern connection ∇^Q on $\mathcal{L}$ for the Hermitian structure h_Q; this ∇^Q is known as the *Quillen connection*. Then

$$\mathrm{Curv}(\nabla^Q) = \frac{\sqrt{-1}}{6\pi}\omega_{WP}, \tag{2.10}$$

where ω_{WP} is the Kähler form in (2.1) [15, p. 184, Theorem 2]; a much more general result is proved in [3, p. 51, Theorem 0.1] from which (2.10) follows immediately. Let

$$\Phi : \mathcal{M}_g \longrightarrow \mathrm{Conn}(\mathcal{L}) \tag{2.11}$$

be the C^∞ section of the projection ϕ in (2.5) given by the above Quillen connection ∇^Q.

Lemma 2.1 *There is exactly one complex connection* ∇ *on* $\mathcal{L}$ *such that the curvature* $\mathrm{Curv}(\nabla)$ *of* ∇ *satisfies the equation*

$$\mathrm{Curv}(\nabla) = = \frac{\sqrt{-1}}{6\pi}\omega_{WP}.$$

Proof From (2.10) we know that the Quillen connection ∇^Q satisfies this equation. Let ∇ be another connection on $\mathcal{L}$ satisfying this condition. Consider the C^∞ (1, 0)-form $\eta_0 = \nabla^Q - \nabla$ on $\mathcal{M}_g$. From (2.9) and (2.8) it follows that $d\eta_0 = \mathrm{Curv}(\nabla^Q) - \mathrm{Curv}(\nabla) = 0$
.

It is known that $\mathcal{M}_g$ does not admit any nonzero closed (1, 0)-form (see [13, p. 228, Theorem 2], [10, Lemma 1.1]). In fact, $\mathcal{M}_g$ does not admit any nonzero holomorphic 1-form [8, Theorem 3.1]; recall that $g \geq 5$. So we have $\eta_0 = 0$, and hence $\nabla^Q = \nabla$. □

Lemma 2.1 has the following immediate consequence:

Corollary 2.2 *The curvature equation* (2.10) *uniquely determines the Quillen connection* ∇^Q *among the space of all complex connections on* $\mathcal{L}$.

2.1 Projective structures and uniformization

Take any smooth complex projective curve X. A holomorphic coordinate chart on X is a pair of the form (U, f), where $U \subset X$ is an analytic open subset and $f : U \longrightarrow \mathbb{CP}^1$ is a holomorphic embedding. A holomorphic coordinate atlas on X is a collection of coordinate charts $\{(U_i, f_i)\}_{i\in I}$ such that

$$X = \bigcup_{i\in I} U_i.$$

A projective structure on X is given by a holomorphic coordinate atlas $\{(U_i, f_i)\}_{i\in I}$ satisfying the condition that for every $i, j \in I \times I$ with $U_i \bigcap U_j \neq \emptyset$, and every connected component $V_c \subset U_i \bigcap U_j$, there is an element $A^c_{j,i} \in \mathrm{Aut}(\mathbb{CP}^1) = \mathrm{PGL}(2, \mathbb{C})$ such that the map $(f_j \circ f_i^{-1})|_{f_i(V_c)}$ is the restriction of the automorphism $A^c_{j,i}$ of $\mathbb{CP}^1$ to the open subset $f_i(V_c)$. Two holomorphic coordinate atlases $\{(U_i, f_i)\}_{i\in I}$ and $\{(U_i, f_i)\}_{i\in I'}$ satisfying the above condition are called *equivalent* if their union $\{(U_i, f_i)\}_{i\in I\cup I'}$ also satisfies the above condition. A *projective structure* on X is an equivalence class of holomorphic coordinate atlases satisfying the above condition (see [9]).

Take the extension E

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow TX \longrightarrow 0$$

corresponding to $1 \in H^1(X, K_X) = \mathbb{C}$. Note that there is exactly one nontrivial extension of TX by $\mathcal{O}_X$ up to the scalings of $\mathcal{O}_X$. Giving a projective structure on X is equivalent to giving a holomorphic connection on the projective bundle $\mathbb{P}(E)$. More precisely, projective structures on X are identified with the quotient of the space of all holomorphic connections on $\mathbb{P}(E)$ by the group of all holomorphic automorphisms of $\mathbb{P}(E)$ [5, 9]. From this it follows that the space of all projective structures on X is an affine space modeled on $H^0(X, K_X^{\otimes 2}) = (\Omega^1_{\mathcal{M}_g})_X$; see [9, 5]. Let $\mathcal{P}_g$ denote the space of all pairs (X, ρ), where $X \in \mathcal{M}_g$ and ρ is a projective structure on X. From the above description of projective structures on X in terms of the holomorphic connections on $\mathbb{P}(E)$ it follows that $\mathcal{P}_g$ has a natural structure of a Deligne–Mumford stack. Let

$$\psi : \mathcal{P}_g \longrightarrow \mathcal{M}_g \tag{2.12}$$

be the natural projection. We note that $\mathcal{P}_g$ is a holomorphic torsor over $\mathcal{M}_g$ for the cotangent bundle $\Omega^1_{\mathcal{M}_g}$.

Every Riemann surface admits a projective structure. In fact, the uniformization theorem produces a projective structure, because the automorphism groups of $\mathbb{C}$, $\mathbb{CP}^1$ and the upper-half plane $\mathbb{H}$ are all contained in $\mathrm{PGL}(2, \mathbb{C})$. Consequently, the uniformization theorem produces a C^∞ section

$$\Psi : \mathcal{M}_g \longrightarrow \mathcal{P}_g \tag{2.13}$$

of the projection ψ in (2.12). (See [6] for another canonical section of $\mathcal{P}_g$.)

3 Holomorphic isomorphism of torsors

We will construct a new holomorphic $\Omega^1_{\mathcal{M}_g}$–torsor from $\mathrm{Conn}(\mathcal{L})$ in (2.5) by simply scaling the action of $\Omega^1_{\mathcal{M}_g}$, while keeping the holomorphic fiber bundle unchanged. Define

$$A^t : \mathrm{Conn}(\mathcal{L}) \times_{\mathcal{M}_g} \Omega^1_{\mathcal{M}_g} \longrightarrow \mathrm{Conn}(\mathcal{L}), \quad (z, v) \longmapsto A\left(z, \frac{\sqrt{-1}}{6\pi} \cdot v\right), \tag{3.1}$$

where A is the map in (2.6); the map A^t is holomorphic because A is so. The resulting holomorphic $\Omega^1_{\mathcal{M}_g}$–torsor $(\mathrm{Conn}(\mathcal{L}), A^t)$ will be denoted by $\mathrm{Conn}^t(\mathcal{L})$. This $\mathrm{Conn}^t(\mathcal{L})$ can be interpreted as the bundle of connections on the (nonexistent) line bundle $\mathcal{L}^{\otimes \frac{\sqrt{-1}}{6\pi}}$.

The C^∞ section of $\mathrm{Conn}^t(\mathcal{L})$ given by the section Φ (in (2.11)) of $\mathrm{Conn}(\mathcal{L})$ will also be denoted by Φ. Since the two holomorphic fiber bundles $\mathrm{Conn}^t(\mathcal{L})$ and $\mathrm{Conn}(\mathcal{L})$ coincide, this should not cause any confusion. For the same reason the projection of $\text{Conn}^t(\mathcal{L})$ to $\mathcal{M}_g$ will be denoted by ϕ (as in (2.5)).

A C^∞ *isomorphism* between the $\Omega^1_{\mathcal{M}_g}$–torsors $\text{Conn}^t(\mathcal{L})$ and $\mathcal{P}_g$ (constructed in (2.12)) is a diffeomorphism

$$F : \text{Conn}^t(\mathcal{L}) \longrightarrow \mathcal{P}_g$$

such that

1. $\psi \circ F = \phi$, where ψ is the projection in (2.12), and
2. $F(c+w) = F(c) + w$, for all $c \in \text{Conn}^t(\mathcal{L})_X$, $w \in (\Omega^1_{\mathcal{M}_g})_X$ and $X \in \mathcal{M}_g$.

A *holomorphic isomorphism* between the $\Omega^1_{\mathcal{M}_g}$–torsors $\text{Conn}^t(\mathcal{L})$ and $\mathcal{P}_g$ is a C^∞ isomorphism F as above satisfying the condition that F is a biholomorphism.

Theorem 3.1

1. *There is exactly one holomorphic isomorphism between the* $\Omega^1_{\mathcal{M}_g}$*–torsors* $\text{Conn}^t(\mathcal{L})$ *and* $\mathcal{P}_g$.
2. *The holomorphic isomorphism between the* $\Omega^1_{\mathcal{M}_g}$*–torsors* $\text{Conn}^t(\mathcal{L})$ *and* $\mathcal{P}_g$ *takes the section* Φ *in* (2.11) *to the section* Ψ *in* (2.13).

Proof We will first prove that there is at most one holomorphic isomorphism between the two $\Omega^1_{\mathcal{M}_g}$–torsors $\text{Conn}^t(\mathcal{L})$ and $\mathcal{P}_g$. To prove this, for $i = 1, 2$, let

$$F_i : \text{Conn}^t(\mathcal{L}) \longrightarrow \mathcal{P}_g$$

be a holomorphic isomorphism. Consider the difference

$$F_1 - F_2 : \text{Conn}^t(\mathcal{L}) \longrightarrow \Omega^1_{\mathcal{M}_g}, \quad c \longmapsto F_1(c) - F_2(c)$$

defined using the $\Omega^1_{\mathcal{M}_g}$–torsor structure on $\mathcal{P}_g$. Since $F_i(c+w) = F_i(c) + w$ for all $c \in \text{Conn}^t(\mathcal{L})_X$, $w \in (\Omega^1_{\mathcal{M}_g})_X$ and $X \in \mathcal{M}_g$, we conclude that

$$(F_1 - F_2)(c+w) = (F_1 - F_2)(c).$$

Consequently, $F_1 - F_2$ descends to a holomorphic 1-form on $\mathcal{M}_g$. But there is no nonzero holomorphic 1-form on $\mathcal{M}_g$ [8, Theorem 3.1]; recall that $g \geq 5$. This implies that $F_1 = F_2$. In other words, there is at most one holomorphic isomorphism between the two $\Omega^1_{\mathcal{M}_g}$–torsors $\text{Conn}^t(\mathcal{L})$ and $\mathcal{P}_g$.

We will now construct a C^∞ isomorphism $\mathbb{F}$ between the two $\Omega^1_{\mathcal{M}_g}$–torsors $\text{Conn}^t(\mathcal{L})$ and $\mathcal{P}_g$. Take any $X \in \mathcal{M}_g$ and any $c \in \phi^{-1}(X) \subset \text{Conn}^t(\mathcal{L}) = \text{Conn}(\mathcal{L})$, where ϕ as before is the projection of $\text{Conn}^t(\mathcal{L})$ to $\mathcal{M}_g$. So $c = \Phi(X) + w$, where Φ is the section in (2.11) and $w \in (\Omega^1_{\mathcal{M}_g})_X$. Now define

$$\mathbb{F}(c) = \Psi(X) + w,$$

where Ψ is the section in (2.13). This produces a map

$$\mathbb{F} : \mathrm{Conn}^t(\mathcal{L}) \longrightarrow \mathcal{P}_g . \tag{3.2}$$

It is straightforward to check that this map $\mathbb{F}$ is a C^∞ isomorphism between the two $\Omega^1_{\mathcal{M}_g}$–torsors $\mathrm{Conn}^t(\mathcal{L})$ and $\mathcal{P}_g$.

It is evident that $\mathbb{F}(\Phi(\mathcal{M}_g)) = \Psi(\mathcal{M}_g)$.

To complete the proof we need to show that $\mathbb{F}$ is a biholomorphism.

The real tangent bundles of $\mathrm{Conn}^t(\mathcal{L})$ and $\mathcal{P}_g$ will be denoted by $T^{\mathbb{R}}\mathrm{Conn}^t(\mathcal{L})$ and $T^{\mathbb{R}}\mathcal{P}_g$ respectively. Let

$$d\mathbb{F} : T^{\mathbb{R}}\mathrm{Conn}^t(\mathcal{L}) \longrightarrow \mathbb{F}^*T^{\mathbb{R}}\mathcal{P}_g$$

be the differential of the map $\mathbb{F}$ in (3.2). Let J_C (respectively, J_P) be the almost complex structure on $\mathrm{Conn}^t(\mathcal{L})$ (respectively, $\mathcal{P}_g$). Since $\mathbb{F}$ is a diffeomorphism, to prove that $\mathbb{F}$ is a biholomorphism it is enough to show that

$$d\mathbb{F}\circ J_C = (\mathbb{F}^*J_P)\circ(d\mathbb{F}) \tag{3.3}$$

as maps from $T^{\mathbb{R}}\mathrm{Conn}^t(\mathcal{L})$ to $\mathbb{F}^*T^{\mathbb{R}}\mathcal{P}_g$; the automorphism of $\mathbb{F}^*T^{\mathbb{R}}\mathcal{P}_g$ given by the automorphism J_P of $T^{\mathbb{R}}\mathcal{P}_g$ is denoted by $\mathbb{F}^*J_P$.

Take any point $X \in \mathcal{M}_g$. The restriction of $\mathbb{F}$ to $\phi^{-1}(X) \subset \mathrm{Conn}^t(\mathcal{L})$ is a biholomorphism with $\psi^{-1}(X)$, where ψ is the map in (2.12). More precisely, this restriction is an isomorphism of affine spaces modeled on the vector space $(\Omega^1_{\mathcal{M}_g})_X$. Therefore, the equation in (3.3) holds for the subbundle of $T^{\mathbb{R}}\mathrm{Conn}^t(\mathcal{L})$ given by the relative tangent bundle for the projection ϕ to $\mathcal{M}_g$.

For convenience, the image in $\mathrm{Conn}^t(\mathcal{L})$ of the map Φ (see (2.11)) will be denoted by Y. Let

$$\iota : Y \hookrightarrow \mathrm{Conn}^t(\mathcal{L}) \tag{3.4}$$

be the inclusion map. Since Φ is just a C^∞ section, this Y does not inherit any complex structure from $\mathrm{Conn}^t(\mathcal{L})$. Note that Y can be given a complex structure, because the restriction of the projection ϕ (see (2.5)) to Y is a diffeomorphism of Y with $\mathcal{M}_g$, so the complex structure on $\mathcal{M}_g$ produces a complex structure on Y. It should be clarified that for this complex structure on Y the inclusion map ι in (3.4) is not holomorphic, because the section Φ is not holomorphic.

Using the differential $d\iota : T^{\mathbb{R}}Y \longrightarrow \iota^*T^{\mathbb{R}}\mathrm{Conn}^t(\mathcal{L})$ of the embedding ι in (3.4) the tangent bundle $T^{\mathbb{R}}Y$ is realized as a C^∞ subbundle of $\iota^*T^{\mathbb{R}}\mathrm{Conn}^t(\mathcal{L})$. So we have

$$T^{\mathbb{R}}Y \subset \iota^*T^{\mathbb{R}}\mathrm{Conn}^t(\mathcal{L}) \subset T^{\mathbb{R}}\mathrm{Conn}(\mathcal{L}) . \tag{3.5}$$

To prove (3.3), take any point $\gamma \in \mathrm{Conn}^t(\mathcal{L})$, and any tangent vector

$$v \in T^{\mathbb{R}}_\gamma\mathrm{Conn}^t(\mathcal{L}) \tag{3.6}$$

at γ. We noted earlier that (3.3) holds for the relative tangent bundle for the projection ϕ to $\mathcal{M}_g$. So we assume that v is not vertical for the projection ϕ.

Denote $\phi(\gamma) \in \mathcal{M}_g$ by z, and also denote $\Phi(z) \in \mathrm{Conn}^t(\mathcal{L})$ by δ. So we have

$$w_0 := \gamma - \delta \in (\Omega^1_{\mathcal{M}_g})_z \tag{3.7}$$

using the $\Omega^1_{\mathcal{M}_g}$–torsor structure of $\text{Conn}^t(\mathcal{L})$ (see (3.1)). Let $\widetilde{v} \in T^{\mathbb{R}}_z\mathcal{M}_g$ be the image of v in (3.6) by the differential $d\phi$ of ϕ. Let

$$u \in T^{\mathbb{R}}_{\delta}Y \tag{3.8}$$

be the image of $\widetilde{v}$ by the differential $d\Phi$ of Φ.

Take a holomorphic 1-form w defined on some analytic neighborhood U of $z \in \mathcal{M}_g$. Then w defines a biholomorphism

$$\mathcal{T}_w \,:\, \phi^{-1}(U) \longrightarrow \phi^{-1}(U)\,, \;\; \alpha \longmapsto \alpha + w(\phi(\alpha))\,; \tag{3.9}$$

here the $\Omega^1_{\mathcal{M}_g}$–torsor structure of $\text{Conn}^t(\mathcal{L})$ is used. Now choose the 1-form w such that

- $w(z) = w_0$; see (3.7) (this condition is clearly equivalent to the condition that $\mathcal{T}_w(\delta) = \gamma$), and
- the differential $d\mathcal{T}_w$ of $\mathcal{T}_w$ takes u in (3.8) to v (see (3.6)).

Note that since v is not vertical for the projection ϕ, such a 1-form w exists. We have the following biholomorphism of $\psi^{-1}(U) \subset \mathcal{P}_g$, where ψ is the projection in (2.12):

$$\mathcal{T}^1_w \,:\, \psi^{-1}(U) \longrightarrow \psi^{-1}(U)\,, \;\; \alpha \longmapsto \alpha + w(\psi(\alpha)).$$

From the construction of $\mathbb{F}$ in (3.2) it follows immediately that

$$\mathbb{F}\circ\mathcal{T}_w = \mathcal{T}^1_w\circ\mathbb{F} \tag{3.10}$$

as maps from $\phi^{-1}(U) \subset \text{Conn}^t(\mathcal{L})$ to $\psi^{-1}(U) \subset \mathcal{P}_g$.

Since $\mathcal{T}_w$ (respectively, $\mathcal{T}^1_w$) is a biholomorphism, its differential $d\mathcal{T}_w$ (respectively, $d\mathcal{T}^1_w$) preserves the almost complex structure J_C (respectively, J_P) on $\phi^{-1}(U)$ (respectively, $\psi^{-1}(U)$). Therefore, from (3.10) we conclude the following:

Take any point $\mu \in \phi^{-1}(U) \subset \text{Conn}^t(\mathcal{L})$. Then (3.3) holds for all vectors in the tangent space $T^{\mathbb{R}}_{\mu}\text{Conn}^t(\mathcal{L})$ if and only if (3.3) holds for all vectors in the tangent space $T^{\mathbb{R}}_{\mu+w(\phi(\mu))}\text{Conn}^t(\mathcal{L})$, where w is the above holomorphic 1-form. More precisely, (3.3) holds for a tangent vector $v_0 \in T^{\mathbb{R}}_{\mu}\text{Conn}^t(\mathcal{L})$ if and only if (3.3) holds for

$$(d\mathcal{T}_w)(v_0) \in T^{\mathbb{R}}_{\mu+w(\phi(\mu))}\text{Conn}^t(\mathcal{L})\,,$$

where $d\mathcal{T}_w \,:\, T^{\mathbb{R}}\phi^{-1}(U) \longrightarrow T^{\mathbb{R}}\phi^{-1}(U)$ is the differential of the map $\mathcal{T}_w$ in (3.9).

Setting $\mu = \delta$ and $v_0 = u$ (see (3.7) and (3.8)) in the above statement we obtain that (3.3) holds for $u \in T^{\mathbb{R}}_{\delta}Y \subset T_{\delta}\text{Conn}^t(\mathcal{L})$ if and only if (3.3) holds for $v \in T_{\gamma}\text{Conn}^t(\mathcal{L})$ in (3.6).

Consequently, to prove (3.3) it suffices to establish it for all tangent vectors in the subspace $T^{\mathbb{R}}Y$ in (3.5).

Let $q \,:\, \mathcal{V} \longrightarrow \mathcal{M}_g$ be a holomorphic $\Omega^1_{\mathcal{M}_g}$–torsor on $\mathcal{M}_g$, and let $S \,:\, \mathcal{M}_g \longrightarrow \mathcal{V}$ be a C^∞ section of $\mathcal{V}$. From these we will construct a C^∞ (1, 1)-form on $\mathcal{M}_g$. The almost complex structures on $\mathcal{V}$ and $\mathcal{M}_g$ will be denoted by J_V and J_M respectively. Let

$$dS \,:\, T^{\mathbb{R}}\mathcal{M}_g \longrightarrow T^{\mathbb{R}}\mathcal{V}$$

be the differential of the map S. Take any $X \in \mathcal{M}_g$ and any $v \in T^{\mathbb{R}}_X\mathcal{M}_g$. Define

$$\widetilde{S}(v) := dS(J_M(v)) - (J_V \circ dS)(v) \in T^{\mathbb{R}}_{S(X)}\mathcal{V}\,.$$

Since q is holomorphic, and $q \circ S = \mathrm{Id}_{\mathcal{M}_g}$, it can be shown that the tangent vector $\widetilde{S}(v)$ is vertical for the projection q. Indeed, we have

$$dq(dS(J_M(v)) - (J_V \circ dS)(v)) = dq(dS(J_M(v))) - dq((J_V \circ dS)(v))$$

$$= d\mathrm{Id}_{\mathcal{M}_g}(J_M(v)) - J_M(dq((dS)(v))) = J_M(v) - J_M(v) = 0\,.$$

So $\widetilde{S}(v)$ is vertical for the projection q. On the other hand, using the $\Omega^1_{\mathcal{M}_g}$–torsor structure of $\mathcal{V}$, the vertical tangent space at $S(X) \in \mathcal{V}$ is identified with $(\Omega^1_{\mathcal{M}_g})_X$. Also, $T^{\mathbb{R}}_X\mathcal{M}_g$ is identified with the real vector space underlying $(\Omega^{0,1}_{\mathcal{M}_g})_X = (\overline{\Omega}^1_{\mathcal{M}_g})_X$. Using these we have

$$\widetilde{S} \in C^\infty(\mathcal{M}_g, \Omega^{1,1}_{\mathcal{M}_g})$$

[11]. Note that S is a holomorphic section if and only if $\widetilde{S} = 0$. This form $\widetilde{S}$ is called the *obstruction* for S to be holomorphic (see [11]).

The obstruction for the section Ψ in (2.13) to be holomorphic is the Weil–Petersson form ω_{WP} in (2.1) [11, p. 214, Theorem 1.7], [16]. On the other hand, the obstruction for the section Φ of Conn($\mathcal{L}$) in (2.11) to be holomorphic is the (1, 1)-component of the curvature of the connection on $\mathcal{L}$ corresponding to Φ. From (2.10) we know that this curvature itself is of type (1, 1) and it is $\frac{\sqrt{-1}}{6\pi}\omega_{WP}$. Now from (3.1) we conclude that the the obstruction for the section Φ of Conn′($\mathcal{L}$) to be holomorphic is the Weil–Petersson form ω_{WP}. Comparing the obstructions for the sections Φ and Ψ we conclude that (3.3) holds for all tangent vectors in the subspace $T^{\mathbb{R}}Y$ in (3.5), because the two obstructions coincide. This completes the proof. □

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References

1. Arbarello, E., Cornalba, M.: The Picard groups of the moduli spaces of curves. Topology **26**, 153–171 (1987)
2. Atiyah, M.F.: Complex analytic connections in fibre bundles. Trans. Amer. Math. Soc. **85**, 181–207 (1957)
3. Bismut, J.-M., Gillet, H., Soulé, C.: Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion. Comm. Math. Phys. **115**, 49–78 (1988)
4. Biswas, I., Hurtubise, J., Stasheff, J.: A construction of a universal connection. Forum Math. **24**, 365–378 (2012)
5. Biswas,I., Raina,A. K.: Projective structures on a Riemann surface, II, Inter. Math. Res. Not. (1999), no. 13, 685–716
6. Biswas, I., Colombo, E., Frediani, P., Pirola, G.P.: A Hodge theoretic projective structure on compact Riemann surfaces. Jour. Math. Pures Appl. **149**, 1–27 (2021)

7. Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math. **36**, 75–109 (1969)
8. Favale,F. F., Pirola,G. P., Torelli,S.: Holomorphic 1-forms on the moduli space of curves, arXiv:2009.10490
9. Gunning, R.C.: On uniformization of complex manifolds: the role of connections. Princeton Univ, Press (1978)
10. Harer, J.: The second homology group of the mapping class group of an orientable surface. Invent. Math. **72**, 221–239 (1983)
11. Ivanov, N.V.: Projective structures, flat bundles and Kähler metrics on moduli spaces. Math. USSR-Sb. **61**, 211–224 (1988)
12. Kobayashi,S.: *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15. Kanô Memorial Lectures, 5. Princeton University Press, Princeton, NJ; Princeton University Press, Princeton, NJ, 1987
13. Mumford, D.: Abelian quotients of the Teichmüller modular group. Jour. Analyse Math. **18**, 227–244 (1967)
14. Quillen, D.G.: Determinants of Cauchy-Riemann operators on Riemann surfaces. Funct. Anal. Appl. **19**, 31–34 (1985)
15. Zograf, P.G., Takhtadzhyan, L.A.: A local index theorem for families of $\bar{\partial}$-operators on Riemann surfaces. Russian Math. Surv. **42–6**, 169–190 (1987)
16. Zograf, P.G., Takhtadzhyan, L.A.: On the uniformization of Riemann surfaces and on the Weil-Petersson metric on the Teichmüller and Schottky spaces. Math. USSR-Sb. **60**, 297–313 (1988)