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Differential geometry of $\mathsf{SO}^*(2n)$-type structures

Ioannis Chrysikos[1] · Jan Gregorovič[1] · Henrik Winther[2]

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Abstract
We study $4n$-dimensional smooth manifolds admitting a $\mathsf{SO}^*(2n)$- or a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure, where $\mathsf{SO}^*(2n)$ is the quaternionic real form of $\mathsf{SO}(2n, \mathbb{C})$. We show that such G-structures, called almost hypercomplex/quaternionic skew-Hermitian structures, form the symplectic analogue of the better known almost hypercomplex/quaternionic-Hermitian structures (hH/qH for short). We present several equivalent definitions of $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures in terms of almost symplectic forms compatible with an almost hypercomplex/quaternionic structure, a quaternionic skew-Hermitian form, or a symmetric 4-tensor, the latter establishing the counterpart of the fundamental 4-form in almost hH/qH geometries. The intrinsic torsion of such structures is presented in terms of Salamon's EH-formalism, and the algebraic types of the corresponding geometries are classified. We construct explicit adapted connections to our G-structures and specify certain normalization conditions, under which these connections become minimal. Finally, we present the classification of symmetric spaces K/L with K semisimple admitting an invariant torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure. This paper is the first in a series aiming at the description of the differential geometry of $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures.

Keywords Quaternionic real form · Almost hypercomplex structures · Almost quaternionic structures · Almost hypercomplex skew-Hermitian structures · Almost quaternionic skew-Hermitian structures · Skew-Hermitian quaternionic forms · Scalar 2-forms · Intrinsic torsion

✉ Ioannis Chrysikos
chrysikosmaths@gmail.com

Jan Gregorovič
jan.gregorovic@seznam.cz

Henrik Winther
winther@math.muni.cz

[1] Faculty of Science, University of Hradec Králové, Rokitanskeho 62, 500 03 Hradec Králové, Czech Republic

[2] Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

1 Introduction

This article is the first in a series studying $4n$-dimensional manifolds M ($n > 1$) admitting a reduction of the frame bundle to the Lie subgroups $\mathsf{SO}^*(2n)$, or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ of $\mathsf{GL}(4n, \mathbb{R})$. Here, $\mathsf{SO}^*(2n)$ denotes the quaternionic real form $\mathsf{SO}(2n, \mathbb{C})$, and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ denotes the Lie group $\mathsf{SO}^*(2n) \times_{\mathbb{Z}_2} \mathsf{Sp}(1)$. Such structures lie inside of the realm of almost hypercomplex and almost quaternionic geometries, respectively. The aim of this note is to highlight them as the symplectic analogue of the well-known almost hypercomplex-Hermitian (hH) structures, almost quaternionic-Hermitian (qH) structures and their pseudo-Riemannian counterparts, which have been examined by many leaders in differential geometry, see for example [2, 4, 10, 20, 36, 40].

Recall that given an almost hypercomplex manifold (M, H), a pseudo-Riemannian metric g, which is H-Hermitian, corresponds to a reduction to $\mathsf{Sp}(p,q) \subset \mathsf{GL}(n, \mathbb{H})$. Similarly, given an almost quaternionic manifold (M, Q) a pseudo-Riemannian metric g, which is Q-Hermitian, corresponds to a reduction to $\mathsf{Sp}(p,q)\,\mathsf{Sp}(1) \subset \mathsf{GL}(n, \mathbb{H})\,\mathsf{Sp}(1)$. Note that these pseudo-Riemannian metrics exist only if certain topological conditions are satisfied. Here, as usual, we have interpreted an almost hypercomplex structure $H = \{J_a : a = 1, 2, 3\}$ as a $\mathsf{GL}(n, \mathbb{H})$-structure, and an almost quaternionic structure $Q \subset \mathsf{End}(TM)$ as a $\mathsf{GL}(n, \mathbb{H})\,\mathsf{Sp}(1)$-structure, respectively. After fixing a $\mathsf{GL}(n, \mathbb{H})$- or a $\mathsf{GL}(n, \mathbb{H})\,\mathsf{Sp}(1)$-structure, Riemannian metrics of the above type always exist and correspond to the reductions induced by the maximal compact subgroups, i.e., the inclusions $\mathsf{Sp}(n) \subset \mathsf{GL}(n, \mathbb{H})$ and $\mathsf{Sp}(n)\,\mathsf{Sp}(1) \subset \mathsf{GL}(n, \mathbb{H})\,\mathsf{Sp}(1)$, respectively.

By comparison, the structures that we treat in this article arise from almost symplectic forms ω which are H-Hermitian, respectively, Q-Hermitian. Hence, it is natural to refer to such G-structures by the terms almost hypercomplex skew-Hermitian structures, denoted by (H, ω), and almost quaternionic skew-Hermitian structures, denoted by (Q, ω), respectively, a terminology which is also motivated by the discussion in [25] of the eight types of inner product spaces. It is also convenient to refer to such non-degenerate $\mathsf{Sp}(1)$-invariant real-valued 2-forms by the term scalar 2-forms. To provide some further motivation behind our considerations, recall that for an almost quaternionic manifold (M, Q) the space $\Lambda^2 T^*M$ of antisymmetric bilinear 2-forms admits the following $\mathsf{GL}(n, \mathbb{H})\,\mathsf{Sp}(1)$-equivariant decomposition into irreducible submodules

$$\Lambda^2 T_x^*M = \Lambda^2_{\mathbb{Im}(\mathbb{H})} T_x^*M \oplus \Lambda^2_{\mathbb{Re}(\mathbb{H})} T_x^*M\,, \qquad \forall\, x \in M\,.$$

In terms of bundles and the EH-formalism of Salamon (see [35]), we may identify $\Lambda^2_{\mathbb{Im}(\mathbb{H})} T_x^*M \cong [\Lambda^2 \mathsf{E}]^* \otimes [S^2 \mathsf{H}]^*$ and this is the module where the (local) Kähler forms in almost qH geometry take values. Hence, nowadays there is a rich variety of works related to such 2-forms, see for example [9, 10, 20] and the references therein. The second module $\Lambda^2_{\mathbb{Re}(\mathbb{H})} T_x^*M \cong [S^2 \mathsf{E}]^* \otimes [\Lambda^2 \mathsf{H}]^*$ is spanned by the scalar 2-forms and has not received the same attention yet. Under the $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-action, it takes the form $\Lambda^2_{\mathbb{Re}(\mathbb{H})} T_x^*M \cong [S^2_0 \mathsf{E}]^* \oplus \langle \omega_0 \rangle$, where ω_0 is the standard scalar 2-form on $[\mathsf{EH}] \cong T_xM$. In this paper, we will use the notion of scalar 2-forms to facilitate a systematic treatment of the geometries under examination.

From the viewpoint of holonomy theory, recall that by a classical result of Hano and Ozeki [24], a connected and simply connected smooth manifold M^{4n} can always be equipped with an affine connection with torsion, whose holonomy group will coincide with $\mathsf{SO}^*(2n)$ (or with $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$). Therefore, there are proper examples of manifolds with non-integrable $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, which make the examination of such geometries a reasonable task. In contrast, for $\mathsf{SO}^*(2n)$ it is known by Bryant [8] that torsion-free affine connections with (irreducible) full holonomy group $\mathsf{SO}^*(2n)$ cannot exist. This is because Berger's first criterion fails for the Lie algebra $\mathfrak{so}^*(2n)$. On the other hand, there are well-known constructions of torsion-free connections with prescribed symplectic holonomy (see [7, 16, 17]) and in particular there exist torsion-free connections with full holonomy $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ (see [8, 11, 30, 38, 39]). In other words, $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ is a real non-symmetric Berger subgroup and appears in the list of exotic holonomies, see [8] and see also [30, Table 3]. In addition to these advances, Čap and Salač [14, 15] have recently discussed special symplectic connections within the more general framework of parabolic conformally symplectic structures.

Since much of our attention has been attracted by the general non-integrable case of $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, the approach in this paper differs from those which are mainly devoted to the torsion-free case ([8, 11, 30, 38, 39]). In particular, a main contribution of this work is based on the establishment of the local geometry of $\mathsf{SO}^*(2n)$- or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, via a geometric approach based on defining tensor fields, adapted frames, adapted connections, intrinsic torsion modules, minimal connections and normalization conditions. This method allows us to proceed systematically with a differential-geometric treatment of manifolds carrying such structures and highlight some parts of their intrinsic geometry.

Let us summarize some basic properties of our G-structures, by fixing an almost hypercomplex skew-Hermitian manifold $(M, H=\{I,J,K\}, \omega)$. We show that such a manifold admits three pseudo-metric tensors g_I, g_J, g_K, which are of signature $(2n, 2n)$ (but not of Norden type). It turns out that any of g_I, g_J, g_K, or their linear combination, provides an embedding $\mathsf{SO}^*(2n)\subset \mathsf{U}(n,n)\subset \mathsf{SO}(2n,2n)$. However, there is no natural way to pick a unique compatible metric among these embeddings, and in particular for an almost quaternionic skew-Hermitian structure (Q,ω), these metrics exist only locally. We also introduce a symmetric 4-tensor Φ, called the fundamental 4-tensor, given by

$$\Phi := g_I\odot g_I + g_J\odot g_J + g_K\odot g_K = \mathsf{Sym}(g_I\otimes g_I + g_J\otimes g_J + g_K\otimes g_K)\,.$$

For an almost quaternionic skew-Hermitian structure (Q,ω), we show that the fundamental tensor Φ is globally defined; hence, it forms the analogue of the fundamental 4-form on an almost qH manifold. Note that Φ provides an equivalent definition of almost quaternionic skew-Hermitian structures, while a similar characterization occurs also in terms of a quaternionic skew-Hermitian form h on M, defined by

$$h := \omega\,\mathsf{id}_{TM} + g_I I + g_J J + g_K K \in \Gamma\big(T^*M\otimes T^*M\otimes \mathsf{End}(TM)\big)\,.$$

These tensors also occur on an almost hypercomplex skew-Hermitian manifold (M,H,ω), although we prove that they are both stabilized by the larger group $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$. They are important since they have analogous applications as the fundamental 4-form on almost hH/qH geometries, a fact which we thoroughly investigate in the second part of this series. In this paper, we use the Obata connection ∇^H related to an almost hypercomplex structure, or an Oproiu connection ∇^Q related to an almost quaternionic structure Q, to present adapted $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-connections, denoted by $\nabla^{H,\omega}$ and $\nabla^{Q,\omega}$, respectively.

With the aim of exploring the underlying geometries by using these adapted connections, we proceed by presenting the intrinsic torsion of $\mathsf{SO}^*(2n)$- or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ -structures. This description is given in a convenient way, in terms of the EH-formalism of Salamon. Therefore, we compute the (second) Spencer cohomology $\mathcal{H}^{0,2}(\mathfrak{so}^*(2n))$ and $\mathcal{H}^{0,2}(\mathfrak{so}^*(2n)\oplus\mathfrak{sp}(1))$ associated to the Lie algebras $\mathfrak{so}^*(2n)$ and $\mathfrak{so}^*(2n)\oplus\mathfrak{sp}(1)$, respectively, and present the number of algebraic types of such geometric structures. For $n > 3$ and for $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ we obtain five pure types $\mathcal{X}_i$ $(i = 1, \dots, 5)$ and a totality of 2^5 algebraic types. The case for $\mathsf{SO}^*(2n)$ is rather complicated, due to the appearance of some multiplicities. However, we prove that for $n > 3$ the number of algebraic types of $\mathsf{SO}^*(2n)$-geometries is equal to 2^{10}, and moreover, we specify seven special $\mathsf{Sp}(1)$-invariant classes $\mathcal{X}_1, \dots, \mathcal{X}_7$, determined in terms of $\mathsf{Sp}(1)$-invariant conditions. For the low-dimensional cases $n = 2, 3$, we finally show that both structures under investigation include some extra algebraic types. We also obtain a characterization of geometries with intrinsic torsion a 3-form, or with intrinsic torsion of vectorial type.

Another contribution of this first part is the explicit description of certain normalization conditions, which allow us to regard the adapted connections $\nabla^{H,\omega}$ and $\nabla^{Q,\omega}$ as minimal connections for our structures. This description is based on our intrinsic torsion decompositions and the theory that we establish about these two adapted connections. We then rely on these minimal connections to answer the question of equivalence of $\mathsf{SO}^*(2n)$- or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures. Note that in the generic case this is a non-trivial task due to the multiplicities appearing in the torsion decomposition into irreducible submodules.

A final contribution of this work is the classification of symmetric spaces K/L with K semisimple, admitting a K-invariant torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure. To obtain this classification, we are based on previous results obtained by the second author in [22] and on the classification of pseudo-Wolf spaces given by Alekseevsky and Cortés in [3]. We prove that only the following three series of symmetric spaces admit such an invariant torsion-free structure:

$$\mathsf{SO}^*(2n+2)/\,\mathsf{SO}^*(2n)\mathsf{U}(1)\,, \qquad \mathsf{SU}(2+p,q)/(\,\mathsf{SU}(2)\,\mathsf{SU}(p,q)\mathsf{U}(1))$$
$$\mathsf{SL}(n+1,\mathbb{H})/(\,\mathsf{GL}(1,\mathbb{H})\,\mathsf{SL}(n,\mathbb{H}))\,.$$

Note that the last two coset spaces belong to the list of pseudo-Wolf spaces, and moreover, the second family for $q = 0$ gives rise to the compact Wolf space $\mathsf{SU}(2+p)/\mathsf{S}(\mathsf{U}(2)\times\mathsf{U}(p))$.

Let us now briefly introduce the main applications of the results obtained in this paper, which are presented in the second part [19] of this series. There, we

- Use the algebraic types $\mathcal{X}_{i_1\dots i_j}$ of the intrinsic torsion to derive 1st-order integrability conditions corresponding to $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, for the underlying almost hypercomplex, quaternionic or symplectic geometries, respectively;
- Focus on the fundamental tensor Φ and examine its interaction with distinguished connections, such as the Obata connection ∇^H, or the unimodular Oproiu connection $\nabla^{Q,\mathrm{vol}}$, or an arbitrary almost symplectic connection ∇^ω, where ω is the scalar 2-form. This allows us to provide further geometric interpretations of some classes $\mathcal{X}_{i_1\dots i_j}$;
- Provide some general constructions of such geometries and in particular illustrate many types of non-integrable $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures via explicit examples.
- Describe certain topological conditions which constrain the existence of such G-structures, and introduce the related "spin structures."

The content of our further investigation, which includes a description of the curvature invariants related to the G-structures under examination, twistor constructions and other open tasks, is summarized in the last section of [19]. We plan to resolve some of these open problems in the third part of this series.

The structure of the paper is given as follows. In Sect. 2, we recall some generalities of the Lie groups $\mathsf{SO}^*(2n)$ and $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$ and next introduce the reader to the world of linear $\mathsf{SO}^*(2n)$-structures and linear $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-structures. Based on the EH-formalism, we develop a theoretical framework for linear G-structures of this type, which allows us to derive the corresponding defining tensors explicitly, introduce the associated adapted bases and provide some details of the symplectic point of view. In the Appendix A, we provide an alternative description of linear $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-structures in terms of right quaternionic vector spaces. In Sect. 3, we introduce $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-structures on smooth manifolds and describe many of their basic features, as well the fundamental 4-tensor Φ and the quaternionic skew-Hermitian form h. In Sect. 4, we present the construction of the corresponding adapted connections, and analyze the associated intrinsic torsion modules and their related decompositions into irreducible submodules. In the following section, we present minimal connections and the corresponding normalization conditions, and solve the equivalence problem. Finally, in Sect. 6 we discuss torsion-free examples. We also present a few remarks about special symplectic holonomy.

2 Linear $\mathsf{SO}^*(2n)$-structures and linear $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-structures

In this section, we introduce the notion of linear hypercomplex skew-Hermitian structures and linear quaternionic skew-Hermitian structures. These are linear geometric structures determined by the action of the Lie groups $\mathsf{SO}^*(2n)$ and $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$, respectively, on some $4n$-dimensional vector space V. We will show that similarly to the case of $\mathsf{Sp}\,(n)$- or $\mathsf{Sp}\,(n)\,\mathsf{Sp}\,(1)$-structures, there are many invariant tensors that can be used to define such structures.

Below we shall mainly use the EH-formalism of Salamon ([35, 36]), instead of an arbitrary quaternionic vector space. However, before we begin with preliminaries about these groups and their representations, it is convenient to include a short summary of some classical definitions.

Definition 2.1 (1) A linear hypercomplex structure H on V is a triple $\{I, J, K\}$ of linear complex structures $I, J, K \in \mathsf{End}\,(V)$ satisfying the quaternionic relations, i.e., $I^2 = J^2 = K^2 = -\,\mathsf{id} = IJK$.

(2) A linear quaternionic structure Q on V is a 3-dimensional subspace of $\mathsf{End}\,(V)$ spanned by an arbitrary linear hypercomplex structure H, i.e., $Q = \langle H\rangle$. In this case, H is called an admissible basis of Q and it is easy to see that any two admissible bases of Q are related by an element in $\mathsf{SO}(3)$.

(3) For the algebra $\mathbb{H}$ of quaternions, we will denote by $\mathbb{Re}(\mathbb{H}) := \mathbb{R}$ (respectively, $\mathbb{Im}(\mathbb{H}) := \mathfrak{sp}(1)$) its real (respectively, imaginary) part. We have the corresponding Lie group decomposition $\mathbb{H}^\times = \mathbb{R}^\times\,\mathsf{Sp}\,(1)$, whereas usual we set $\mathbb{H}^\times := \mathbb{H}\backslash\{0\}$ and $\mathbb{R}^\times := \mathbb{R}\backslash\{0\}$. A choice of an admissible basis $H = \{I, J, K\}$ of a linear quaternionic structure Q on V provides an isomorphism $\mathfrak{sp}(1) \cong Q$, and moreover the diffeomorphisms $\mathsf{Sp}\,(1) \cong \{\mu_0\,\mathsf{id}_V + \mu_1 I + \mu_2 J + \mu_3 K \,:\, \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 = 1\} \cong \mathrm{S}^3$ and

$$\mathsf{Sp}\,(1) \cap \mathfrak{sp}(1) \cong \mathrm{S}\,(Q) := \{\mu_1 I + \mu_2 J + \mu_3 K \,:\, \mu_1^2 + \mu_2^2 + \mu_3^2 = 1\}\,.$$

So $\mathrm{S}(Q)$ is the space of complex structures in Q, i.e., the space of endomorphisms $\mathrm{J} \in Q$ such that $\mathrm{J}^2 = -\mathrm{id}$. Note that $\mathrm{S}(Q) \cong \mathrm{S}^2$, where S^n will denote the n-sphere.

(4) Given a linear complex structure $J \in \mathrm{End}(V)$, a real-valued bilinear form f on V is called Hermitian if $f(Jx, Jy) = f(x, y)$ for any $x, y \in V$.

(5) Given a linear hypercomplex structure $H = \{J_a : a = 1, 2, 3\}$ on V, a real-valued bilinear form f on V is called Hermitian with respect to H (H-Hermitian for short), if f is a Hermitian bilinear form with respect to J_a for any $a = 1, 2, 3$, and Hermitian with respect to Q (Q-Hermitian for short), if $f(\mathrm{J}x, \mathrm{J}y) = f(x, y)$ for any $\mathrm{J} \in \mathrm{S}(Q)$ and $x, y \in V$.

(6) Let V be a vector space with a linear quaternionic structure $Q = \langle H \rangle$, where H is an admissible basis providing the isomorphism $\mathbb{H} \cong \mathbb{R} \oplus Q$. Traditionally, a quaternionic skew-Hermitian form is defined to be a $\mathbb{R}$-bilinear map $h : V \times V \to \mathbb{H}$, satisfying

$$h(xp, yq) = ph(x, y)\bar{q}, \quad h(x, y) = -\overline{h(y, x)}, \quad \forall\, x, y \in V,\ p, q \in \mathbb{H}.$$

Note that the first condition actually says that the form $h : V \times V \to \mathbb{H}$ does *not* depend on the admissible basis H providing the isomorphism $\mathbb{H} \cong \mathbb{R} \oplus Q$. In particular, h is a sesquilinear form and as we will show later (see Proposition 2.15), up to (quaternionic) linear automorphisms, any finite-dimensional quaternionic vector space admits a unique non-degenerate quaternionic skew-Hermitian form, see also [25, Chapter 2].

2.1 Preliminaries about the Lie group $\mathrm{SO}^*(2n)$

We are mainly interested in cases with $n \geq 2$, and then, the Lie group $\mathrm{SO}^*(2n)$ is a non-compact real form of $\mathrm{SO}(2n, \mathbb{C})$, of real dimension $n(2n-1)$, which is semisimple for $n = 2$ and simple for $n \geq 3$ (see [26]). We shall refer to $\mathrm{SO}^*(2n)$ by the term quaternionic real form. Be aware that there exist many different notations for the Lie group $\mathrm{SO}^*(2n)$, for example $\mathrm{SO}(n, \mathbb{H})$ in [8, 38, 39], $\mathrm{U}^*_n(\mathbb{H})$ in [34], or $\mathrm{Sk}(n, \mathbb{H})$ in [25]. Traditionally, $\mathrm{SO}^*(2n)$ is defined as the stabilizer of a quaternionic skew-Hermitian form h, as above, an interpretation that we will discuss in detail below. If we view $\mathrm{SO}(2n, \mathbb{C})$ as the Lie group of complex linear transformations preserving the standard complex Euclidean metric on $\mathsf{E} := \mathbb{C}^{2n}$, then the real form $\mathrm{SO}^*(2n)$ of $\mathrm{SO}(2n, \mathbb{C})$ is the fixed point set of the following involution:

$$\sigma(\phi) = \begin{pmatrix} 0 & -\mathrm{id}_{\mathbb{C}^n} \\ \mathrm{id}_{\mathbb{C}^n} & 0 \end{pmatrix}^{-1} (\bar{\phi}^t)^{-1} \begin{pmatrix} 0 & -\mathrm{id}_{\mathbb{C}^n} \\ \mathrm{id}_{\mathbb{C}^n} & 0 \end{pmatrix},$$

for $\phi \in \mathrm{SO}(2n, \mathbb{C})$, i.e.,

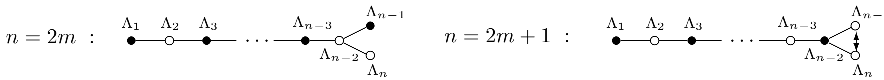
$$\mathrm{SO}^*(2n) = \{\phi \in \mathrm{SO}(2n, \mathbb{C}) : \sigma(\phi) = \phi\}.$$

In this way, E becomes the standard representation of $\mathrm{SO}^*(2n)$. The Lie algebra $\mathfrak{so}^*(2n)$ of $\mathrm{SO}^*(2n)$ is represented by the following endomorphisms of E:

$$\begin{pmatrix} Z_1 & -Z_2 \\ \overline{Z_2} & \overline{Z_1} \end{pmatrix},$$

for Z_1, Z_2 complex $(n \times n)$-matrices satisfying $Z_1^t = -Z_1$ and $Z_2^t = \overline{Z_2}$, see [26]. There are other presentations of $\mathfrak{so}^*(2n)$ related to identifications of E as a right quaternionic vector space, which we review in Appendix A (see Proposition A. 9).

Recall now that $\mathsf{SO}^*(2n)$ is connected, with $\pi_1(\mathsf{SO}^*(2n)) = \mathbb{Z}$. Therefore, $\mathsf{SO}^*(2n)$ is not simply connected and there are further Lie groups with Lie algebra $\mathfrak{so}^*(2n)$, which we describe in [19, Appendix A]. In fact, since the restricted root system of $\mathfrak{so}^*(2n)$ depends on the parity of the quaternionic dimension n, there are two related Satake diagrams, which we present below (a detailed exposition of real forms and Satake diagrams can be found in Onishchik's book [34], see also [13, pp. 214-223]).



Therefore, for $n = 2, 3, 4$ there are special isomorphisms with classical Lie algebras, and we present the related details in the second part of this work. However, since E is not the standard representation of these classical Lie algebras, this viewpoint does not relate these geometric structures with geometric structures which have been studied earlier.

2.2 The EH-formalism adapted to $\mathsf{SO}^*(2n)$ and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$

The most convenient way to visualize a linear quaternionic structure is by using the EH -formalism of Salamon [35, 36]. The EH-formalism is usually used for the standard representation E of $\mathsf{GL}(n, \mathbb{H})$ or $\mathsf{Sp}(n)$ (or $\mathsf{Sp}(p, q)$), and we adapt this formalism to $\mathsf{SO}^*(2n)$. So, let us denote by H the standard representation of $\mathsf{Sp}(1)$ on $\mathbb{C}^2$. This is of quaternionic type, and the same holds for the $\mathsf{SO}^*(2n)$-representation E. Thus, the maps defined by

$$\epsilon_{\mathsf{E}} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix}, \quad \epsilon_{\mathsf{H}} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix},$$

for $a, b \in \mathbb{C}^n$, $a, b \in \mathbb{C}$, respectively, are complex anti-linear involutions of E and H, which commute with the actions of $\mathsf{SO}^*(2n)$ and $\mathsf{Sp}(1)$, respectively. Let us also denote the group $\mathsf{SO}^*(2n) \times_{\mathbb{Z}_2} \mathsf{Sp}(1) = (\mathsf{SO}^*(2n) \times \mathsf{Sp}(1))/\mathbb{Z}_2$ by

$$\mathsf{SO}^*(2n)\,\mathsf{Sp}(1),$$

which is the image of the product $\mathsf{SO}^*(2n) \times \mathsf{Sp}(1)$ in $\mathsf{End}(\mathsf{E} \otimes_{\mathbb{C}} \mathsf{H})$, via the tensor product representation. The standard quaternionic representation of $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ is the real form $[\mathsf{EH}]$ inside $\mathsf{E} \otimes_{\mathbb{C}} \mathsf{H}$, fixed by the real structure $\epsilon_{\mathsf{E}} \otimes \epsilon_{\mathsf{H}}$. Note that the EH-formalism extends this description to all tensor products of E and H, where products of ϵ_{E} and ϵ_{H} still define a real structure, for which we shall maintain the same notation [ ].

Now, since E is of quaternionic type, we have

$$N_{\mathfrak{gl}(\mathsf{E})}(\mathfrak{so}^*(2n)) = \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}\,, \quad C_{\mathfrak{gl}(\mathsf{E})}(\mathfrak{so}^*(2n)) = \mathfrak{sp}(1) \oplus \mathbb{R}\,.$$

In particular, the $\mathfrak{sp}(1)$-action commutes with $\mathsf{SO}^*(2n)$ and defines a linear quaternionic structure on $[\mathsf{EH}]$, i.e., there is admissible basis $H = \{J_1, J_2, J_3\}$ of $\mathfrak{sp}(1)$ such that

$$J_1^2 = J_2^2 = J_3^2 = -\,\mathrm{Id} = J_1 J_2 J_3\,.$$

In this series of papers, we will maintain the following conventions.

Conventions. (i) Q_0 will denote the quaternionic structure on $[\mathsf{E}\,\mathsf{H}]$, defined by the $\mathfrak{sp}(1)$ -action.

(ii) The notation $R(\alpha)$ will encode a complex irreducible module corresponding to the highest weight α, for some of the Lie algebras of $\mathsf{SO}^*(2n)$ or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$. The letter θ will be used for the fundamental weight of $\mathfrak{sp}(1)$, and π_k, $k=1,\dots,2n$ will be the fundamental weights of $\mathfrak{so}^*(2n)$. For instance, for $n>2$ we have $\mathsf{EH}=R(\pi_1)\otimes_{\mathbb{C}} R(\theta)$. It is also useful to mention the following $\mathsf{SO}^*(2n)$-equivariant isomorphisms

$$
\begin{aligned}
R(\pi_k) =& \Lambda^k\,\mathsf{E}\ \text{ for } k<n-1\,,\\
R(k\pi_1) =& S^k_0\,\mathsf{E}\ \cong S^k\,\mathsf{E}\,/S^{k-2}\,\mathsf{E}\ .
\end{aligned}
$$

Here, $S^k_0\,\mathsf{E}$ denotes the trace free part of $S^k\,\mathsf{E}$, for any $k>0$, see next section for more details. For $k=$ even, the complex representation $R(k\pi_1)$ is of real type, and for $k=$ odd we see that $R(k\pi_1)$ is of quaternionic type, similarly to the $\mathsf{Sp}\,(n)$-action (or $\mathsf{Sp}\,(p,q)$-action). Note, however, that $\Lambda^k\,\mathsf{E}$ is only irreducible for $\mathsf{SO}^*(2n)$, but reducible in the $\mathsf{Sp}\,(n)$-case. For modules appearing in low-dimensional cases, we have some exceptions which are indicated in Table 1, together with the corresponding dimensions (see also [21]).

Let us now recall the following basic result, which is useful for our considerations (see also [36]).

Lemma 2.2 (1) *The module* $[\,\mathsf{H}\ \mathsf{H}\,]^*$ *admits a* 1*-dimensional* $\mathsf{Sp}\,(1)$*-submodule* $[\Lambda^2\,\mathsf{H}\,]^*$ *spanned by a real form of the complex-valued volume form on* H. *The latter is defined by* $\omega_{\mathsf{H}}(\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}) := \frac{1}{2}(ad-bc)$, *for* $a,b,c,d\in\mathbb{C}$, *and satisfies* $\omega_{\mathsf{H}}(\epsilon_{\mathsf{H}}x,\epsilon_{\mathsf{H}}y)=\overline{\omega_{\mathsf{H}}(x,y)}$. *In particular*, ω_{H} *establishes an isomorphism* $\mathsf{H}^*\cong\mathsf{H}$.

(2) The linear quaternionic structure Q_0 on $[\mathsf{E}\,\mathsf{H}]$ is isomorphic to $[S^2\,\mathsf{H}\,]^*$, that is $[S^2\,\mathsf{H}\,]^*\cong_{\omega_{\mathsf{H}}} Q_0=\mathfrak{sp}(1)$. Moreover, $[\,\mathsf{H}\ \mathsf{H}\,]^*\cong_{\omega_{\mathsf{H}}}[\,\mathsf{H}\ \mathsf{H}^*]=\mathbb{R}\oplus Q_0=\mathbb{R}\oplus\mathfrak{sp}(1)$, and a choice of an admissible basis for Q_0 provides an isomorphism $[\,\mathsf{H}\ \mathsf{H}\,]^*\cong\mathbb{H}$.

Table 1 Some $\mathsf{SO}^*(2n)$-modules for low quaternionic dimensions

	$n>4$	$n=4$	$n=3$	$n=2$
E	$R(\pi_1)$	$R(\pi_1)$	$R(\pi_1)$	$R(\pi_1+\pi_2)$
$\dim_{\mathbb{C}}$	$2n$	8	6	4
$\Lambda^2\,\mathsf{E}$	$R(\pi_2)$	$R(\pi_2)$	$R(\pi_2+\pi_3)$	$R(2\pi_1)\oplus R(2\pi_2)$
$\dim_{\mathbb{C}}$	$n(2n-1)$	28	15	6
$S^2_0\,\mathsf{E}$	$R(2\pi_1)$	$R(2\pi_1)$	$R(2\pi_1)$	$R(2\pi_1+2\pi_2)$
$\dim_{\mathbb{C}}$	$2n^2+n-1$	35	20	9
K	$R(\pi_1+\pi_2)$	$R(\pi_1+\pi_2)$	$R(\pi_1+\pi_2+\pi_3)$	$R(\pi_1+3\pi_2)\oplus R(3\pi_1+\pi_2)$
$\dim_{\mathbb{C}}$	$\frac{8}{3}(n^3-n)$	160	64	16
$\Lambda^3\,\mathsf{E}$	$R(\pi_3)$	$R(\pi_3+\pi_4)$	$R(2\pi_2)\oplus R(2\pi_3)$	$R(\pi_1+\pi_2)$
$\dim_{\mathbb{C}}$	$\frac{2n}{3}(2n-1)(n-1)$	56	20	4
$S^3_0\,\mathsf{E}$	$R(3\pi_1)$	$R(3\pi_1)$	$R(3\pi_1)$	$R(3\pi_1\oplus 3\pi_2)$
$\dim_{\mathbb{C}}$	$\frac{2n}{3}(2n-1)(n+2)$	112	50	16

(3) *The module* $[S^2\,\mathsf{E}\,]^*$ *admits a* 1*-dimensional* $\mathsf{SO}^*(2n)$*-submodule spanned by a real form of the complex-valued metric on* E*. The latter is defined by* $g_{\mathsf{E}}(\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}) := a^tc + b^td$*, for* $a,b,c,d\in\mathbb{C}^n$*, and satisfies* $g_{\mathsf{E}}(\epsilon_{\mathsf{E}}x,\epsilon_{\mathsf{E}}y)=\overline{g_{\mathsf{E}}(x,y)}$*. In particular,* g_{E} *establishes an isomorphism* $\mathsf{E}^*\cong\mathsf{E}$.

Let us point out some further similarities and differences between the EH-formalism adapted to $\mathsf{GL}(n,\mathbb{H})$- or $\mathsf{Sp}(p,q)$-actions in comparison to $\mathsf{SO}^*(2n)$-actions.

- The $\mathsf{SO}^*(2n)$-action can be naturally extended to a $\mathsf{GL}(n,\mathbb{H})$-action; however, the real form of $g_{\mathsf{E}}\in S^2\,\mathsf{E}^*$ will be no longer invariant. Thus in this case, there is no canonical isomorphism $\mathsf{E}^*\cong\mathsf{E}$. Nevertheless, independently of the actions, non-degenerate elements of $[S^2\,\mathsf{E}\,]^*$ induce skew-symmetric $\mathbb{R}$-bilinear forms (2-forms) on $[\mathsf{E}\,\mathsf{H}]$. Under the $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-action, we will study these 2-forms in a great detail in next subsection.
- The invariant tensor of the $\mathsf{Sp}(p,q)$-action providing the isomorphism $\mathsf{E}^*\cong\mathsf{E}$, is an element of $\Lambda^2\,\mathsf{E}^*$ instead of $S^2\,\mathsf{E}^*$, which we may denote by ω_{E}, see also Remark 2.8. In general, independently of the actions, non-degenerate elements of $[\Lambda^2\,\mathsf{E}\,]^*$ induce qH pseudo-Euclidean metrics on $[\mathsf{E}\,\mathsf{H}]$.
- When decomposing low order tensor products of E with respect to $\mathsf{Sp}(p,q)$, it is customary to introduce the $\mathsf{Sp}(p,q)$-module K, see [41]. However, note that whenever we decompose low order tensor products of E with respect to $\mathsf{SO}^*(2n)$, the $\mathsf{SO}^*(2n)$-module K represents a different submodule in the corresponding decomposition. Nevertheless, K has the same dimension as in the $\mathsf{Sp}(p,q)$-case and also the same expression in terms of fundamental weights (but for different Lie algebras).

2.3 Invariants of actions of $\mathsf{SO}^*(2n)$ and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$

Let us now discuss important representations of $\mathsf{SO}^*(2n)$ and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ for $n>1$, in detail. We first treat the representations indicated in the previous section, and in particular in the above Lemma 2.2.

Lemma 2.3 *The module* $[\mathsf{E}\,\mathsf{H}]$ *consists of the following elements of* $\mathsf{E}\otimes_{\mathbb{C}}\mathsf{H}$:

$$a := \begin{pmatrix}a\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\\bar a\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix},$$
$$bj := \begin{pmatrix}0\\-b\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\bar b\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix},$$

for $a,b\in\mathbb{C}^n$*. The linear quaternionic structure* Q_0 *on* $[\mathsf{EH}]$ *can be defined by the following admissible basis* $H_0=\{\mathcal{J}_a\in\mathsf{End}([\mathsf{EH}]) : a=1,2,3\}$:

$$\mathcal{J}_1 = i\,\mathsf{id}_{\mathbb{C}^{2n}}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \mathcal{J}_2 = \mathsf{id}_{\mathbb{C}^{2n}}\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix},\quad \mathcal{J}_3 = i\,\mathsf{id}_{\mathbb{C}^{2n}}\otimes\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$$

such that

$$(a_1 + a_2\mathcal{J}_1 + a_3\mathcal{J}_2 + a_4\mathcal{J}_3)(a + bj) = (a_1 + a_2 i + a_3 j + a_4 k)(a + bj)$$

for $a, b \in \mathbb{C}^n$, where on the right-hand side we view $a + bj \in [\mathsf{EH}]$ as an element of $\mathbb{H}^n$.

Remark 2.4 Let us emphasize that the identification with $\mathbb{H}^n$ is possible only after the choice of an admissible basis, and that in general $a + bj \in [\mathsf{EH}]$ is just notation. However, there is also an identification $\bar{a} - bj \in [\mathsf{EH}]$ with $a + bj \in \mathbb{H}^n$ as right quaternionic vector spaces, and for convenience the details related to the view of $\mathbb{H}^n$ as a left or right quaternionic vector space are described in Appendix A. So, let us focus on $[\mathsf{E\,H}]$ from now on.

Proof It is a simple observation that the real structure on $[\mathsf{EH}]$ defined by $\epsilon_{\mathsf{E}} \otimes \epsilon_{\mathsf{H}}$ fixes the elements a, bj, and that the real dimension is $4n$, as it is expected. Now, the chosen representation of $\mathsf{Sp}(1)$ on H suggests using the following elements of $\mathsf{Sp}(1) \cap \mathfrak{sp}(1) = S(Q)$ to define an admissible basis $\{\mathcal{J}_a : a = 1, 2, 3\}$:

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

Since we can always move i from H to E, it is simple computation to check that such endomorphisms preserve $[\mathsf{EH}]$ and take the claimed form $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$. Also, it is not hard to prove that the elements in $H_0 := \{\mathcal{J}_a, a = 1, 2, 3\}$ correspond to the left multiplication by i, j, k, respectively, when we view $a + bj$ as an element of $\mathbb{H}^n$. □

Definition 2.5 We shall refer to the admissible basis H_0 of the linear quaternionic structure Q_0 on $[\mathsf{EH}]$ presented in Lemma 2.3 be the term standard admissible basis of Q_0 on $[\mathsf{EH}]$.

Next, we prove that the elements $g_{\mathsf{E}}, \omega_{\mathsf{H}}$ introduced before, can be combined into a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-invariant linear symplectic form on $[\mathsf{EH}]$.

Proposition 2.6 *The expression $\omega_0 := g_{\mathsf{E}} \otimes \omega_{\mathsf{H}}$ defines a non-degenerate $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-invariant real 2-form on $[\mathsf{EH}]$, explicitly given by*

$$\omega_0(a + bj, c + dj) = \mathbb{Re}(a^t\bar{d} - b^t\bar{c}), \quad \forall\, a + bj,\ c + dj \in [\mathsf{EH}].$$

Proof We compute

$$\begin{aligned}\omega_0(a + bj, c + dj) :=& g_{\mathsf{E}} \otimes \omega_{\mathsf{H}}\left(\begin{pmatrix} a \\ -b \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{b} \\ \bar{a} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} c \\ -d \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \bar{d} \\ \bar{c} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ =& \frac{1}{2}(a^t\bar{d} - b^t\bar{c} + \bar{a}^t d - \bar{b}^t c) = \mathbb{Re}(a^t\bar{d} - b^t\bar{c}).\end{aligned}$$

This is clearly a 2-form, which is $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-invariant and non-degenerate by definition. □

Let us look for further invariants. We start with the space $[\mathsf{EH}]^* \otimes_{\mathbb{R}} [\mathsf{EH}]^*$ of real-valued bilinear forms on $[\mathsf{EH}]$, for which we have the following equivariant decompositions into submodules, which are irreducible for $n > 2$.

Proposition 2.7

$$
\begin{aligned}
\Lambda^2[\mathrm{EH}]^* &\cong_{\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)} [\Lambda^2 \mathrm{E}]^* \otimes_{\mathbb{R}} \mathfrak{sp}(1) \oplus [S^2 \mathrm{E}]^* \otimes_{\mathbb{R}} [\Lambda^2 \mathrm{H}]^*\,,\\
&\cong_{\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)} [\Lambda^2 \mathrm{E}]^* \otimes_{\mathbb{R}} \mathfrak{sp}(1) \oplus [S^2_0 \mathrm{E}]^* \oplus \langle \omega_0 \rangle\,,\\
&\cong_{\mathrm{GL}(n,\mathbb{H})} 3[\Lambda^2 \mathrm{E}]^* \oplus [S^2 \mathrm{E}]^*\,,\\
&\cong_{\mathrm{SO}^*(2n)} 3[\Lambda^2 \mathrm{E}]^* \oplus [S^2_0 \mathrm{E}]^* \oplus \langle \omega_0 \rangle\,.\\
S^2[\mathrm{EH}]^* &\cong_{\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)} [\Lambda^2 \mathrm{E}]^* \otimes_{\mathbb{R}} [\Lambda^2 \mathrm{H}]^* \oplus [S^2 \mathrm{E}]^* \otimes_{\mathbb{R}} \mathfrak{sp}(1)\,,\\
&\cong_{\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)} [\Lambda^2 \mathrm{E}]^* \oplus [S^2_0 \mathrm{E}]^* \otimes_{\mathbb{R}} \mathfrak{sp}(1) \oplus \langle \omega_0 \rangle \otimes_{\mathbb{R}} \mathfrak{sp}(1)\,,\\
&\cong_{\mathrm{GL}(n,\mathbb{H})} [\Lambda^2 \mathrm{E}]^* \oplus 3[S^2 \mathrm{E}]^*\,,\\
&\cong_{\mathrm{SO}^*(2n)} [\Lambda^2 \mathrm{E}]^* \oplus 3[S^2_0 \mathrm{E}]^* \oplus 3\mathbb{R}\,.
\end{aligned}
$$

Remark 2.8 (1) These results are obtained by applying basic representation theory, see for example [21], or the summary given in [18]. The reader may feel more familiar with the decompositions for the $\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$- and $\mathrm{GL}(n,\mathbb{H})$-actions, see for example [10, 36, 40].

(2) The $\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$-module $[\Lambda^2 \mathrm{E}]^* \subset S^2[\mathrm{EH}]^*$ is spanned by the qH pseudo-Euclidean metrics on $[\mathrm{EH}]$ of signature $(4p, 4q)$, which have a standard representative given by (see [3, 36])

$$(\,,\,)_0 := \omega_{\mathrm{E}} \otimes \omega_{\mathrm{H}}\,.$$

At the same time, $[\Lambda^2 \mathrm{E}]^*$ can be viewed as the $\mathrm{GL}(n,\mathbb{H})$-module of the hH pseudo-Euclidean metrics on $[\mathrm{EH}]$ of signature $(4p, 4q)$. The corresponding fundamental 2-forms (skew-symmetric non-degenerate $\mathbb{R}$-valued bilinear forms)

$$\omega_a(\cdot\,,\cdot) := (\cdot\,, \mathcal{J}_a \cdot)_0\,, \quad a = 1, 2, 3,$$

where $H_0 = \{\mathcal{J}_a : a = 1, 2, 3\}$ is the standard admissible basis, are three distinguished elements of the $\mathrm{GL}(n,\mathbb{H})$-module $3[\Lambda^2 \mathrm{E}]^*$, inducing a $\mathrm{Sp}(1)$-submodule of the $\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$-module $[\Lambda^2 \mathrm{E}]^* \otimes_{\mathbb{R}} [S^2 \mathrm{H}]^*$ (independently of the choice of an admissible basis).

(3) The irreducible $\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$-module $[S^2 \mathrm{E}]^* \otimes_{\mathbb{R}} [\Lambda^2 \mathrm{H}]^* \subset \Lambda^2[\mathrm{EH}]^*$, which is also an irreducible $\mathrm{GL}(n,\mathbb{H})$-module, consists of 2-forms of a distinguished type, playing an important role in our considerations related to linear $\mathrm{SO}^*(2n)$- or $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$-structures. For such 2-forms, we shall use the following terminology.

Definition 2.9 (1) Consider $[\mathrm{EH}]$ with the standard admissible basis H_0. Then, a 2-form ω on $[\mathrm{EH}]$ is called a scalar 2-form (with respect to H_0), if ω is non-degenerate and H_0-Hermitian.

(2) Consider $[\mathrm{EH}]$ with the standard linear quaternionic structure Q_0. Then, a 2-form ω on $[\mathrm{EH}]$ is called a scalar 2-form (with respect to Q_0), if ω is non-degenerate and Q_0-Hermitian.

Next we shall refer to the scalar 2-form ω_0 on $[\mathrm{EH}]$ introduced in Proposition 2.6 via the term standard scalar 2-form on $[\mathrm{EH}]$ (with respect to $Q_0 = \langle H_0 \rangle$). As an immediate consequence of Proposition 2.6, we conclude that

$$\mathrm{Aut}\big(H_0, \omega_0\big) = \mathrm{SO}^*(2n)\,, \quad \mathrm{Aut}\big(Q_0 = \langle H_0 \rangle, \omega_0\big) = \mathrm{SO}^*(2n)\,\mathrm{Sp}(1)\,,$$

respectively. Let us now provide a useful characterization of scalar 2-forms.

Proposition 2.10 *The following conditions are equivalent for a non-degenerate* 2*-form* ω *on* [EH]:

(1) ω is conjugated to ω_0 by an element of $\mathsf{GL}(n, \mathbb{H})$.
(2) $\omega \in [S^2\,\mathsf{E}\,]^* = [S^2_0\,\mathsf{E}\,]^* \oplus \langle \omega_0 \rangle$.
(3) ω is $\mathsf{Sp}(1)$-invariant, i.e., $\omega(A\cdot, A\cdot) = \omega(\cdot, \cdot)$, for all $A \in \mathsf{Sp}(1)$.
(4) $\omega(A\cdot, \cdot) + \omega(\cdot, A\cdot) = 0$, for all $A \in \mathfrak{sp}(1)$.
(5) $\omega(\mathsf{J}\cdot, \mathsf{J}\cdot) = \omega(\cdot, \cdot)$, for all $\mathsf{J} \in \mathsf{S}(Q_0)$, which means that ω is scalar.
(6) $\omega(\mathsf{J}\cdot, \cdot) + \omega(\cdot, \mathsf{J}\cdot) = 0$, for all $\mathsf{J} \in \mathsf{S}(Q_0)$.
(7) $\omega(J_a\cdot, J_a\cdot) = \omega(\cdot, \cdot)$, for any admissible basis $H = \{J_a : a = 1, 2, 3\}$ of Q_0.
(8) $\omega(J_a\cdot, \cdot) + \omega(\cdot, J_a\cdot) = 0$, for any admissible basis $H = \{J_a : a = 1, 2, 3\}$ of Q_0.

Proof Clearly, (2) is equivalent to (3), because $[\Lambda^2\,\mathsf{H}\,]^*$ is a trivial $\mathsf{Sp}(1)$-module and $[S^2\,\mathsf{H}\,]^* \cong \mathfrak{sp}(1)$ is not. Now, we may express the qH pseudo-Euclidean metric $(\ ,\ )_0$ on [EH] as

$$(a + bj, c + dj)_0 = \mathbb{Re}(a^t \bar{c} + b^t \bar{d})\,.$$

It follows that $(x, Ay)_0 = (Ay, x)_0 = -(y, Ax)_0$, for any $A \in \mathfrak{sp}(n)$ and $x, y \in$ [EH]. Since $[S^2\,\mathsf{E}\,]^* \cong \mathfrak{sp}(n)$, the scalar 2-forms correspond to invertible elements of $\mathfrak{sp}(n)$. Moreover, the $\mathsf{Sp}(1)$-action on the 2-form $\omega(\cdot, \cdot) := (\cdot, A\cdot)_0$ commutes with $A \in \mathfrak{sp}(n)$. Therefore, the claims (2)–(8) are equivalent due to the usual properties of $(\ ,\ )_0$. Now, because ω_0 is scalar, and the space $[S^2\,\mathsf{E}\,]^* \otimes_{\mathbb{R}} [\Lambda^2\,\mathsf{H}\,]^*$ is $\mathsf{GL}(n, \mathbb{H})$-invariant, it remains to show that for some invertible $A \in \mathfrak{sp}(n)$, the expression $(\cdot, A\cdot)_0$ is conjugated to ω_0. Indeed, for $B \in \mathsf{GL}(n, \mathbb{H})$, we get

$$\omega(B\cdot, B\cdot) = (B\cdot, AB\cdot)_0 = (\cdot, B^*AB\cdot)_0\,,$$

where B^* is the conjugate transpose. Also, for $B \in \mathsf{Sp}(n)$ we see that $B^*AB = B^{-1}AB$, and it is known that every element $A \in \mathfrak{sp}(n)$ is conjugated to a diagonal purely imaginary quaternionic matrix (in a maximal torus in $\mathfrak{sp}(n)$). Now, since for any $a \in \mathfrak{sp}(1)$ there is $b \in \mathsf{GL}(1, \mathbb{H})$, such that $\bar{b}ab = j \in \mathbb{H}$, we may find $B \in \mathsf{GL}(n, \mathbb{H})$ such that

$$\omega(Bx, By) = (x, j\,\mathsf{id}_{[EH]}y)_0\,.$$

But then for $x = a + bj, y = c + dj \in$ [EH] we obtain $j\,\mathsf{id}_{[EH]}y = d - cj$, and thus

$$\omega(Bx, By) = (a + bj, d - cj)_0 = \mathbb{Re}(a^t \bar{d} - b^t \bar{c}) = \omega_0(a + bj, c + dj)\,.$$

□

Let us now focus on invariant symmetric bilinear forms on [EH] induced by the module

$$\langle \omega_0 \rangle \otimes_{\mathbb{R}} \mathfrak{sp}(1) \subset S^2[\mathsf{EH}]^*\,.$$

This module is trivial under the $\mathsf{SO}^*(2n)$-action, but non-trivial for the $\mathsf{Sp}(1)$-action. Let us show how these symmetric bilinear forms can provide three $\mathsf{SO}^*(2n)$-invariant pseudo-Euclidean metrics, which form an analogue of the usual fundamental (Kähler) 2-forms arising in the (linear) hH/qH setting.

Proposition 2.11 (1) *The vector space* [EH] *admits three pseudo-Euclidean metrics* $g_a(\cdot,\cdot) := \omega_0(\cdot, \mathcal{J}_a\cdot)$ *of signature* $(2n, 2n)$, *satisfying*

$$g_a(\mathcal{J}_a\cdot, \mathcal{J}_a\cdot) = g_a(\cdot,\cdot), \quad \forall\, a = 1,2,3,$$

where $H_0 = \{\mathcal{J}_a : a = 1,2,3\}$ *is the standard admissible basis and* ω_0 *is the standard scalar 2-form on* [EH].

(2) *Assume that* $H = \{I, J, K\}$ *is an admissible basis of* Q_0 *and that* ω *is a scalar 2-form on* [EH] *with respect to* Q_0. *Then, the elements* $g_I = \omega(\cdot, I\cdot)$, $g_J = \omega(\cdot, J\cdot)$ *and* $g_K = \omega(\cdot, K\cdot)$ *are simultaneously conjugated to* g_1, g_2, g_3 *by an element in* $\mathsf{GL}(n,\mathbb{H})\,\mathsf{Sp}(1)$, *and thus they have the same properties.*

(3) *For any* $\mathsf{J} \in \mathsf{S}(Q_0)$, *the tensor*

$$\langle\cdot,\cdot\rangle_{\mathsf{J}} := \omega_0(\cdot, \mathsf{J}\cdot) = g_{\mathsf{E}} \otimes \mathsf{J} \in \langle\omega_0\rangle \otimes_{\mathbb{R}} \mathfrak{sp}(1)$$

is a pseudo-Euclidean metric of signature $(2n, 2n)$ *satisfying* $\langle \mathsf{J}x, \mathsf{J}y\rangle_{\mathsf{J}} = \langle x, y\rangle_{\mathsf{J}}$, *for all* $x, y \in$ [EH].

(4) *For any* $\mathsf{J} \in \mathsf{S}(Q_0)$, *the tensor*

$$g_{\mathsf{J}}(\cdot,\cdot) := \omega(\cdot, \mathsf{J}\cdot) \in [S^2\,\mathsf{E}]^* \otimes \mathfrak{sp}(1)$$

is conjugated to $\langle\cdot,\cdot\rangle_{\mathsf{J}}$ *by an element of* $\mathsf{GL}(n,\mathbb{H})$, *and thus has the same properties, too. Note that* $\langle\cdot,\cdot\rangle_{\mathcal{J}_a} = g_a(\cdot,\cdot)$, *for* $a = 1,2,3$.

Proof It is not hard to check that $g_{\mathsf{J}}(\mathsf{J}\cdot, \mathsf{J}\cdot) = \omega(\cdot, -\mathsf{J}^3\cdot) = g_{\mathsf{J}}(\cdot,\cdot)$, for $\mathsf{J} \in \mathsf{S}(Q_0)$. Moreover, since ω is conjugated to ω_0 by an element in $\mathsf{GL}(n,\mathbb{H})$, it suffices to prove the claims for $\langle\cdot,\cdot\rangle_{\mathsf{J}}$. By using the standard admissible basis $\{\mathcal{J}_a : a = 1,2,3\}$ on [EH] we may consider the linear complex structure $\mathsf{J} = \mu_1\mathcal{J}_1 + \mu_2\mathcal{J}_2 + \mu_3\mathcal{J}_3$, $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$, with $\sum_{a=1}^3 \mu_a^2 = 1$. Then, we obtain $\langle\cdot,\cdot\rangle_{\mathsf{J}} = \sum_{a=1}^3 \mu_a \langle\cdot,\cdot\rangle_{\mathcal{J}_a} = \sum_{a=1}^3 \mu_a g_a$, where

$$\begin{aligned}
g_1(a+bj, c+dj) &= \langle a+bj, c+dj\rangle_{\mathcal{J}_1} = \mathbb{R}\mathrm{e}(-ia^t\bar{d} + ib^t\bar{c}) = \mathbb{I}\mathrm{m}(a^t\bar{d} - b^t\bar{c}),\\
g_2(a+bj, c+dj) &= \langle a+bj, c+dj\rangle_{\mathcal{J}_2} = \omega_0(a+bj, -\bar{d} + \bar{c}j) = \mathbb{R}\mathrm{e}(a^tc + b^td),\\
g_3(a+bj, c+dj) &= \langle a+bj, c+dj\rangle_{\mathcal{J}_3} = \mathbb{I}\mathrm{m}(a^tc + b^td)
\end{aligned}$$

are split signature $(2n, 2n)$ metrics. To check the signature, we proceed as follows: If e_ℓ is ℓ-th vector of standard basis of $\mathbb{C}^n$, then $\langle\cdot,\cdot\rangle_{\mathsf{J}}$ restricted to $e_\ell, \mathcal{J}_1e_\ell, \mathcal{J}_2e_\ell, \mathcal{J}_3e_\ell$ takes the form

$$\begin{pmatrix} \mu_2 & \mu_3 & 0 & -\mu_1\\ \mu_3 & -\mu_2 & \mu_1 & 0\\ 0 & \mu_1 & \mu_2 & \mu_3\\ -\mu_1 & 0 & \mu_3 & -\mu_2 \end{pmatrix}.$$

This matrix has four eigenvalues: Two of them are given by $\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}=1$, and the other by $-\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}=-1$. Since the 4-dimensional subspaces for different basis vectors are clearly orthogonal to each other, it follows that the signature is $(2n, 2n)$. This completes the proof. □

Remark 2.12 (1) Let us remark that although the above pseudo-Euclidean metrics g_a have signature $(2n, 2n)$, they are *not* Norden metrics, since none of them is H_0-Hermitian. In particular, each g_a is $\mathcal{J}_a$-Hermitian, but not Hermitian for $\mathcal{J}_b$ with $b \neq a$. As a consequence of the first observation, there is an embedding

$$\mathsf{SO}^*(2n) \subset \mathsf{U}(n,n) \subset \mathsf{SO}(2n,2n)\,.$$

The same conclusion applies for g_{J} (and $\langle\cdot,\cdot\rangle_{\mathsf{J}}$). We mention that $\mathsf{Aut}\left(H_0, g_i\right) = \mathsf{SO}^*(2n)$, but

$$\mathsf{Aut}\left(Q_0=\langle H_0\rangle, g_i\right) = \mathsf{SO}^*(2n)\mathsf{U}(1) \ltimes \mathbb{Z}_2 \subset \mathsf{GL}\,(n,\,\mathsf{H})\,\mathsf{Sp}\,(1)\,,$$

for any $i=1,2,3$, and the same applies for the stabilizer of $\langle\cdot\,,\cdot\rangle_{\mathsf{J}}$ in $\mathsf{GL}\,(n,\,\mathsf{H})\,\mathsf{Sp}\,(1)$. Here, one should view $\mathsf{U}(1)$ as the stabilizer of $\mathsf{J} \in S(Q)$, and observe that $\mathbb{Z}_2$ acts as $-\,\mathsf{id}$ on J and ω_0.

(2) Finally, let us also mention that the condition $-g_{J_a}(J_a x, J_a y) + g_{J_a}(x,y) = 0$, for any $a=1,2,3$, $x,y \in [\mathsf{EH}]$, and any admissible basis $H=\{J_a : a=1,2,3\}$ of Q_0, is equivalent to the final condition posed in Proposition 2.10.

The Killing form of $\mathfrak{sp}(1)$ provides a trivial $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-invariant submodule in

$$([S^2\,\mathsf{E}\,]^* \otimes \mathfrak{sp}(1)) \otimes \mathfrak{sp}(1) \subset S^2[\mathsf{EH}]^* \otimes \mathsf{GL}\,([\mathsf{EH}])\,.$$

We show that this allows us to encode the data $\{\omega_0, g_1, g_2, g_3\}$ described above, into a single quaternionic skew-Hermitian form on $[\mathsf{EH}]$, which we may denote by

$$h \in [\mathsf{EH}]^* \otimes_{\mathbb{R}} [\mathsf{EH}]^* \otimes_{\mathbb{R}} \mathsf{GL}\,([\mathsf{EH}])\,.$$

Definition 2.13 A $\mathbb{R}$-bilinear form which is valued in endomorphisms of $[\mathsf{EH}]$, that is an element $h \in [\mathsf{EH}]^* \otimes_{\mathbb{R}} [\mathsf{EH}]^* \otimes_{\mathbb{R}} \mathsf{GL}\,([\mathsf{EH}])$, is said to be quaternionic skew-Hermitian if the following three conditions are satisfied:

- $\mathbb{Re}(h)(x,y) := \frac{1}{2}(h(x,y) - h(y,x)) \in \mathbb{R}\cdot \mathsf{id}$;
- $\mathbb{Im}(h)(x,y) := \frac{1}{2}(h(x,y) + h(y,x)) \in Q_0 = \mathfrak{sp}(1)$;
- $h(\mathsf{J}\,\cdot\,,\cdot) = \mathsf{J} \circ h(\cdot\,,\cdot)$,

for all $x, y \in [\mathsf{EH}]$ and $\mathsf{J} \in \mathsf{S}\,(Q_0)$. We call $\mathbb{Re}(h), \mathbb{Im}(h)$ the real, respectively imaginary part of h.

Remark 2.14 The standard admissible basis H_0 on $[\mathsf{EH}]$ provides the identification $\mathbb{H} \cong \mathbb{R} \oplus Q_0$, and the second condition in the more traditional Definition 2.1 is clearly equivalent to the first two conditions in Definition 2.13. Finally, the third condition in Definition 2.13 is clearly equivalent to the first condition in Definition 2.1 by $\mathbb{R}$-bilinearity of h.

Proposition 2.15 *There is a unique* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-invariant trivial submodule in* $S^2[\mathsf{EH}]^* \otimes \mathfrak{sp}(1)$ *which provides the following imaginary part of the* standard quaternionic skew-Hermitian form $h_0 = \mathbb{Re}(h_0) + \mathbb{Im}(h_0) = g_{\mathsf{E}} \otimes \mathrm{id}_{[\mathsf{H}\,\mathsf{H}^*]} \in [\mathsf{EH}]^* \otimes_{\mathbb{R}} [\mathsf{EH}]^* \otimes_{\mathbb{R}} \mathsf{GL}([\mathsf{EH}])$ *on* $[\mathsf{EH}]$,

$$\mathbb{Im}(h_0)(\cdot\,,\cdot) := \sum_{a=1}^{3} g_a(\cdot\,,\cdot)\mathcal{J}_a\,, \tag{2.1}$$

where $H_0 = \{\mathcal{J}_a : a = 1,2,3\}$ *is the standard admissible basis and* $g_a, a = 1,2,3$ *are defined in Proposition 2.11. Moreover,*

$$\mathbb{Re}(h_0)(x,y) := \omega_0(x,y) \otimes \mathrm{id}$$

is the real part of h_0 *and the stabilizer in* $\mathsf{GL}([\mathsf{EH}])$ *of* h_0 *is the Lie group* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*, in particular*

$$\mathsf{Aut}(h_0) = \mathsf{SO}^*(2n)\,\mathsf{Sp}(1)\,.$$

Finally, h_0 *is equivalent to the linear quaternionic structure* $Q_0 = \langle H_0 \rangle$ *and the scalar 2-form* ω_0 *on* $[\mathsf{EH}]$*, while any quaternionic skew-Hermitian form* h *on* $[\mathsf{EH}]$ *is conjugated to* h_0 *by an element in* $\mathsf{GL}(n,\mathbb{H})$*.*

Proof Observe first that there is a non-degenerate skew $\mathbb{C}$-Hermitian form h on $[\mathsf{EH}]$, defined by

$$h(a+bj, c+dj) := \omega_0(a+bj, c+dj) + \langle a+bj, c+dj\rangle_{\mathcal{J}_1}\mathcal{J}_1 = a^t\bar{d} - b^t\bar{c}\,.$$

Moreover, we see that

$$\begin{aligned}\langle a+bj, c+dj\rangle_{\mathcal{J}_2}\mathcal{J}_2 &= \omega_0(a+bj, -\bar{d}+\bar{c}j)\mathcal{J}_2 = \mathbb{Re}(a^t c + b^t d)\mathcal{J}_2\,,\\ \langle a+bj, c+dj\rangle_{\mathcal{J}_3}\mathcal{J}_3 &= \langle a+bj, -\bar{d}+\bar{c}j\rangle_{\mathcal{J}_3}\mathcal{J}_3 = \mathbb{Im}(a^t c + b^t d)\mathcal{J}_3\,,\end{aligned}$$

and therefore, $\langle x,y\rangle_{\mathcal{J}_2}\mathcal{J}_2 + \langle x,y\rangle_{\mathcal{J}_3}\mathcal{J}_3 = g_{\mathsf{E}}(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix})\mathcal{J}_2$. Define now

$$h_0 := g_{\mathsf{E}} \otimes \mathrm{id}_{[\mathsf{H}\,\mathsf{H}^*]} : [\mathsf{EH}] \otimes [\mathsf{E}\,\mathsf{H}] \to [\mathsf{H}\,\mathsf{H}^*] \cong \mathbb{H}\,,$$

where we view $[\mathsf{H}\,\mathsf{H}^*] \subset \mathsf{H} \otimes_{\mathbb{C}} \mathsf{H}$ as a 1-dimensional left quaternionic vector space via the map

$$p = p_1 + p_2 j \mapsto p_1\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} - \bar{p_1}\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + p_2\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \bar{p_2}\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}.$$

Then, for any $a+bj, c+dj \in [\mathsf{E}\,\mathsf{H}]$, we obtain that

$$\begin{aligned}h_0(a+bj, c+dj) &= (a^t\bar{d} - b^t\bar{c}) \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + (\bar{b}^t c - \bar{a}^t d) \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\\ &\quad + (a^t c + b^t d) \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + (\bar{b}^t\bar{d} + \bar{a}^t\bar{c}) \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\\ &= a^t\bar{d} - b^t\bar{c} + (a^t c + b^t d)\mathcal{J}_2\\ &= h(a+bj, c+dj) + g_{\mathsf{E}}(a+bj, c+dj)\mathcal{J}_2\,.\end{aligned}$$

This shows that the definition of h_0 is independent of the choice of an admissible hypercomplex basis. Moreover, by Definition 2.1 it remains to check the following:

$$h_0(c+dj,a+bj) = -\overline{h(a+bj,c+dj)} + g_{\mathsf{E}}(a+bj,c+dj)\mathcal{J}_2 = -\overline{h_0(a+bj,c+dj)}\,,$$
$$h_0(p(a+bj),c+dj) = p_1h_0(a+bj,c+dj) + p_2h_0(-\bar{b}+\bar{a}j,c+dj) = p_1h_0(a+bj,c+dj) + p_2j(-j)(-\bar{b}^t\bar{d} - \bar{a}^t\bar{c} + (-\bar{b}^tc + \bar{a}^td)j) = ph_0(a+bj,c+dj)\,,$$

for any $p = p_1 + p_2j \in \mathbb{H}$. Consequently, h_0 is a quaternionic skew-Hermitian form on $[\mathsf{EH}]$.

To conclude the proof, note that $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$ is clearly contained in the stabilizer of h_0. In addition, the linear quaternionic structure Q_0 is spanned by the imaginary part $\mathbb{Im}(h_0)(x,y)$, for $x,y \in [\mathsf{EH}]$, and thus the stabilizer of h_0 is contained in $\mathsf{GL}\,(n,\mathbb{H})\,\mathsf{Sp}\,(1)$. On the other hand, the real part of $h_0(x,y)$ recovers the scalar 2-form ω_0, whose stabilizer inside $\mathsf{GL}\,(n,\mathbb{H})\,\mathsf{Sp}\,(1)$ coincides with the Lie group $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$. Thus, the last claim follows, since for any quaternionic skew-Hermitian form h on $[\mathsf{EH}]$ we have

$$\omega = \mathbb{Re}(h)\,, \quad \mathbb{Im}(h) = \sum_{a=1}^{3}\omega(\cdot,\mathcal{J}_a\cdot)\mathcal{J}_a = \sum_{a=1}^{3} g_{\mathcal{J}_a}(\cdot,\cdot)\mathcal{J}_a\,,$$

where $\omega, g_{\mathcal{J}_a}$ are simultaneously conjugated to ω_0, g_a by an element in $\mathsf{GL}\,(n,\mathbb{H})$. □

Again the Killing form of $\mathfrak{sp}(1)$ provides a trivial submodule in $S^2(\langle\omega_0\rangle \otimes_{\mathbb{R}} \mathfrak{sp}(1))$, and thus a trivial $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-invariant submodule of $S^4[\mathsf{EH}]^*$. Next we will use Proposition 2.15 to prove that this tensor provides an analogue of the fundamental 4-form $\Omega_0 = \sum_a \omega_a \wedge \omega_a$, appearing in the theory of hH/qH structures.

Proposition 2.16 *There is a unique* $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$*-invariant trivial submodule in* $S^4[\mathsf{EH}]^*$, *spanned by the totally symmetric* 4*-tensor*

$$\Phi_0 := g_1 \odot g_1 + g_2 \odot g_2 + g_3 \odot g_3\,, \tag{2.2}$$

where $\odot$ *denotes the symmetrized tensor product and* $g_a, a := 1,2,3$ *are defined in Proposition 2.11. Moreover, the* complete symmetrization Sym *of* $\omega_0(\cdot\,,\mathbb{Im}(h_0)\cdot)$, *where* $\mathbb{Im}(h_0)$ *is defined by* (2.1), *satisfies the relation*

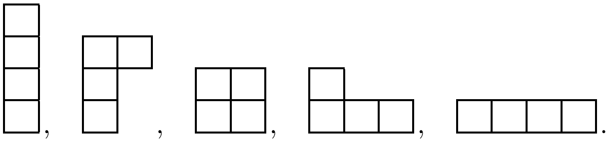
$$\Phi_0 = \mathsf{Sym}\big(\omega_0(\cdot\,,\mathbb{Im}(h_0)\cdot)\big).$$

Thus, the stabilizer of Φ_0 *in* $\mathsf{GL}\,([\mathsf{EH}])$ *is* $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$, *i.e.*, $\mathsf{Aut}\,(\Phi_0) = \mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$.

Proof The computation of the dimension of the space of invariant symmetric 4-tensors requires deeper results from representation theory, which we avoid to review in detail and refer to [21]. Recall that θ is the fundamental weight of $\mathfrak{sp}(1)$, and $\mathsf{H} = R(\theta)$. The following equality is a special case of an equivariant isomorphism which holds for any tensor product of Lie algebra modules (see [21]):

$$S^4(\mathsf{EH}) = \sum_{Y\in\mathrm{Young}(4)} Y(\mathsf{E}) \otimes Y(\mathsf{H})\,, \tag{2.3}$$

where in general Young(n) denotes the set of plethysms associated to Young diagrams with n boxes. For $n = 4$, there are the following five Young diagrams:



Then, with respect to $\mathsf{Sp}(1)$ we obtain the following:

- $(4)R(\theta) = R(4\theta)$;
- $(3,1)R(\theta) = R(2\theta)$;
- $(2,2)R(\theta) = R(0)$, the trivial representation;
- $(2,1,1)R(\theta) = \{0\}$, 0 dimensional;
- $(1,1,1,1)R(\theta) = \{0\}$, 0 dimensional.

This shows that any trivial $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-invariant subspace of $S^4(\mathsf{E}\,\mathsf{H})$ must be contained in the summand $(2,2)\,\mathsf{E} \otimes (2,2)\,\mathsf{H}$, and the dimension is equal to the dimension of $\mathsf{SO}(2n,\mathbb{C})$-invariant subspaces in $(2,2)\,\mathsf{E}$. In particular, the dimension of the space of invariants is one, which yields the tensor Φ_0 (this claim is valid also for the low-dimensional cases included in Table 1). Indeed, the space $\langle g_1, g_2, g_3\rangle \subset S^2[\mathsf{EH}]^*$ is $\mathsf{SO}^*(2n)$-trivial, but $\mathsf{Sp}(1)$-invariant, and equivariantly isomorphic to the space of imaginary quaternions $\mathbb{Im}(\mathbb{H})$, equipped with the standard admissible basis H_0. Since the latter space is self-dual, and has an invariant inner product given by the sum of squares of the admissible basis, the tensor Φ_0 given by formula (2.2) is also invariant and thus spans the invariant subspace.

Finally, it is a simple observation that $\Phi_0 = \mathsf{Sym}\big(\omega_0(\cdot\,, \mathbb{Im}(h_0)\cdot)\big)$, where Sym is the operator of complete symmetrization, thus by Proposition 2.15 the stabilizer of Φ_0 in $\mathsf{GL}([\mathsf{EH}])$ must contain the Lie group $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$, that is $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1) \subseteq \mathsf{Aut}(\Phi_0)$. We will prove also the other inclusion at an infinitesimal level. First, under the $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-action we see by Proposition 2.7 that $\mathsf{End}([\mathsf{EH}])$ decomposes as follows:

$$\mathsf{End}([\mathsf{EH}]) \cong \mathbb{R}\cdot \mathsf{Id} \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n) \oplus \frac{\mathfrak{sl}(n,\mathbb{H})}{\mathfrak{so}^*(2n)} \oplus \frac{\mathfrak{sp}(\omega_0)}{\mathfrak{so}^*(2n)\oplus\mathfrak{sp}(1)} \oplus [\Lambda^2\,\mathsf{E}\,S^2\,\mathsf{H}\,]^*\,,$$

where $\mathbb{R}\cdot\mathsf{id} \cong \langle\omega_0\rangle$ and

$$\mathfrak{sp}(1) \cong [S^2\,\mathsf{H}\,]^*\,,\quad \mathfrak{so}^*(2n) \cong [\Lambda^2\,\mathsf{E}\,]^*\,,\quad \frac{\mathfrak{sl}(n,\mathbb{H})}{\mathfrak{so}^*(2n)} \cong [S^2_0\,\mathsf{E}\,]^*\,,$$
$$\frac{\mathfrak{sp}(\omega_0)}{\mathfrak{so}^*(2n)\oplus\mathfrak{sp}(1)} \cong [S^2_0\,\mathsf{E}\,]^* \otimes \mathfrak{sp}(1)\,.$$

Note that for $n > 2$ the above decomposition can be read in terms of irreducible submodules. The Lie algebra of $\mathsf{Aut}(\Phi_0)$ is a proper submodule of the above. If an element of $\mathfrak{sp}(\omega_0)$ or $\mathfrak{gl}(n,\mathbb{H})$ preserves Φ_0, then this element should belong to $\mathfrak{so}^*(2n)\oplus\mathfrak{sp}(1)$, since none of the algebras $\mathfrak{sp}(\omega_0)$ or $\mathfrak{gl}(n,\mathbb{H})$ preserves a symmetric 4-tensor. On the other hand, by Remark 2.8 we can express the pure tensors A in the final submodule $[\Lambda^2\,\mathsf{E}\,S^2\,\mathsf{H}\,]^*$ as

$$\omega_0(Ax, y) = \rho_A(x, Jy)\,,$$

for some qH pseudo-Euclidean metric ρ_A depending on A, and some almost complex structure J belonging to an admissible basis H of Q_0. Then, we can compute the action $A \cdot \Phi_0$. To do so, we need the action of A on g_I, g_J, g_K. Based on the fact that ω_0 is scalar and by using Proposition 2.10, we deduce that

$$\begin{aligned}(A \cdot g_I)(x, y) &= -\, g_I(Ax, y) - g_I(x, Ay) = -\omega_0(Ax, Iy) - \omega_0(Ay, Ix)\\ &= -\, \rho_A(x, JIy) - \rho_A(y, JIx) = 0\,,\\ (A \cdot g_J)(x, y) &= -\, g_J(Ax, y) - g_J(x, Ay) = -\omega_0(Ax, Jy) - \omega_0(Ay, Jx)\\ &= -\, \rho_A(x, J^2y) - \rho_A(y, J^2x) = 2\rho_A(x, y)\,,\\ (A \cdot g_K)(x, y) &= -\, g_K(Ax, y) - g_K(x, Ay) = -\omega_0(Ax, Ky) - \omega_0(Ay, Kx)\\ &= -\, \rho_A(x, JKy) - \rho_A(y, JKx) = 0\,.\end{aligned}$$

Thus, by the definition of Φ_0 we finally obtain

$$A \cdot \Phi_0 = 4\rho_A \odot g_J\,,$$

which never vanishes. In particular, for linear independent pure tensors A we see that also the corresponding qH pseudo-Euclidean metrics ρ_A are linear independent. Thus, together with the previous inclusion we deduce that the Lie algebra of $\mathsf{Aut}(\Phi_0)$ coincides with $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$. □

2.4 Linear $\mathrm{SO}^*(2n)$-structures and linear $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$-structures

Let $H = \{J_1, J_2, J_3\}$ be a linear hypercomplex structure on $4n$-dimensional vector space V, or let $H = \{J_1, J_2, J_3\}$ be an admissible basis of a linear quaternionic structure Q on V. Next we will show that the basis defined below provides the identification with the EH-formalism.

Definition 2.17 We say that a basis $e_1, \dots, e_{2n}, f_1, \dots, f_{2n}$ of V is adapted to H if

$$J_1(e_c) = e_{c+n}\,, \quad J_2(e_c) = f_c\,, \quad J_3(e_c) = f_{c+n}$$

for $c = 1, \dots n$. Let us also use the notation

$$a = (u_1 + iu_{n+1}, \dots, u_n + iu_{2n})^t\,, \quad bj = (v_1 + iv_{n+1}, \dots, v_n + iv_{2n})^t$$

for the coordinates $(u_1, \dots, u_{2n}, v_1, \dots, v_{2n})^t$ in the basis which provides the isomorphism $V \cong [\mathsf{EH}]$.

We should mention that such a basis is not an *admissible basis to H* in terms of [4, Def. 1.4] (see also the appendix, Section A).

Example 2.18 For $n = 2$, assume that e_1, e_2 are nonzero vectors in $\mathbb{H}^2$ for which the quaternionic lines $\mathbb{H} \cdot e_1$ and $\mathbb{H} \cdot e_2$ do not coincide. If H is the linear hypercomplex structure induced by left multiplication via i, j, k, then the basis adapted to H is given by $\{e_1, e_2, ie_1, ie_2, je_1, je_2, ke_1, ke_2\}$.

Proposition 2.19 *Let $H = \{J_1, J_2, J_3\}$ be a linear hypercomplex structure on $4n$-dimensional vector space V, or let $H = \{J_1, J_2, J_3\}$ be an admissible basis of a linear quaternionic structure Q on V. Then, there is a basis adapted to H, such that*

$$V \cong [\mathsf{EH}]\,,$$

and under this isomorphism we get the identification $H = H_0 = \{\mathcal{J}_a : a = 1, 2, 3\}$, where H_0 is the standard admissible basis of Q_0 on [EH].

Proof Clearly, there is an n-tuple of linearly independent vectors $e_1, \dots, e_n$ such that

$$e_{c+n} := J_1(e_c)\,, \quad f_c := J_2(e_c)\,, \quad f_{c+n} := J_3(e_c)\,,$$

are all linearly independent, which means that there is a basis adapted to H in the above terms. It is clear that under the isomorphism $V \cong [\mathsf{EH}]$ provided by this basis, we have

$$\begin{aligned}(a_1 + a_2J_1 + a_3J_2 + a_4J_3)(a + bj) &= (a_1 + a_2i + a_3j + a_4k)(a + bj)\\ &= (a_1 + a_2\mathcal{J}_1 + a_3\mathcal{J}_2 + a_4\mathcal{J}_3)(a + bj)\end{aligned}$$

and thus $H = H_0$. This proves our assertion. □

Proposition 2.15 in combination with the above construction motivates us to proceed with the following definitions.

Definition 2.20 Let V be a $4n$-dimensional real vector space. A pair (h, H) consisting of an element $h \in V^* \otimes V^* \otimes \mathsf{GL}(V)$ and a linear hypercomplex structure $H = \{J_a \in \mathsf{End}(V) : a = 1, 2, 3\}$, is said to be a linear hypercomplex skew-Hermitian structure on V (linear hs-H structure for short), if the following conditions are satisfied:

(1) The real part $\mathbb{Re}(h)(x, y) := \frac{1}{2}(h(x, y) - h(y, x))$ of h satisfies $\mathbb{Re}(h)(x, y) = \omega(x, y) \cdot \mathsf{id}$, for all $x, y \in V$ and some non-degenerate 2-form ω on V.
(2) h is a quaternionic skew-Hermitian form with respect to H, that is, ω is a scalar 2-form and

$$h(x, y) = \omega(x, y)\,\mathsf{id} + \sum_{a=1}^{3} g_{J_a}(x, y)J_a$$

holds for all $x, y \in V$.

Remark 2.21 Let us also emphasize that the linear quaternionic structure generated by H can be equivalently obtained by the image of the imaginary part of h, defined by

$$\mathbb{Im}(h) := \frac{1}{2}(h(x,y) + h(y,x)), \quad \forall\, x, y \in V.$$

Definition 2.22 Let V be a $4n$-dimensional real vector space. An element $h \in V^* \otimes V^* \otimes \mathsf{GL}(V)$ is said to be a linear quaternionic skew-Hermitian structure (linear qs-H structure for short), if the following conditions are satisfied:

(1) The real part $\mathbb{Re}(h)$ of h satisfies $\mathbb{Re}(h)(x,y) = \omega(x,y)\cdot \mathsf{id}$, for all $x, y \in V$ and some non-degenerate 2-form ω on V.
(2) The imaginary part $\mathbb{Im}(h)$ of h induces a linear quaternionic structure Q on V.
(3) h is a quaternionic skew-Hermitian form, that is ω is a scalar 2-form and

$$h(x,y) = \omega(x,y)\,\mathsf{id} + \sum_{a=1}^{3} g_{J_a}(x,y)J_a$$

holds for any admissible basis $H = \{J_1, J_2, J_3\}$ of Q and for all $x, y \in V$.

Often, we shall call such an admissible basis H of Q also an admissible basis of the linear quaternionic skew-Hermitian structure $(h, Q = \langle H\rangle)$.

Let us now specify the bases that allow us to relate linear hs-H and qs-H structures, as defined above, with the results from the previous subsection.

Proposition 2.23 *Let $(h, H = \{J_1, J_2, J_3\})$ be a linear hypercomplex skew-Hermitian structure on V, or let $H = \{J_1, J_2, J_3\}$ be an admissible basis of a linear quaternionic skew-Hermitian structure $(h, Q = \langle H\rangle)$. Set $\omega := \mathbb{Re}(h)$. Then, there is a symplectic basis of ω adapted to H, that is*

$$\omega(e_r, e_s) = 0, \quad \omega(f_r, f_s) = 0, \quad \omega(e_r, f_r) = 1, \quad \omega(e_r, f_s) = 0, (r \neq s)$$

for $1 \leq r \leq 2n$ and $1 \leq s \leq 2n$, and

$$J_1(e_c) = e_{c+n}, \quad J_2(e_c) = f_c, \quad J_3(e_c) = f_{c+n},$$

for $1 \leq c \leq n$, respectively. Moreover, under the isomorphism $V \cong [\mathsf{EH}]$ provided by the basis adapted to H, we have $\omega = \omega_0$, $H = H_0$, where ω_0 is the standard scalar 2-form and H_0 is the standard admissible basis on $[\mathsf{EH}]$, respectively. In particular, the following claims hold:

(1) A linear hypercomplex skew-Hermitian structure on V is equivalent to a pair (H, ω), consisting of a linear hypercomplex structure $H = \{J_1, J_2, J_3\}$ and a scalar 2-form ω, both defined on V. Equivalently, a linear hypercomplex skew-Hermitian structure on V is a $\mathsf{SO}^*(2n)$-structure on V.
(2) A linear quaternionic skew-Hermitian structure on V is equivalent to a pair (Q, ω), consisting of a linear quaternionic structure Q and a scalar 2-form ω, both defined on V. Equivalently, a linear quaternionic skew-Hermitian structure on V is a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure on V.

Proof The definition of a linear hs-H structure or a linear qs-H structure, provides pairs (H, ω) and (Q, ω), respectively, with the claimed properties. Picking an admissible basis for Q reduces us to the situation of a pair $(H = \{J_1, J_2, J_3\}, \omega)$. Due to Proposition 2.19, there is a basis $e'_1, \dots, e'_{2n}, f'_1, \dots, f'_{2n}$ adapted to H, and ω is a scalar 2-form on [EH] under the isomorphism $V \cong$ [EH]. We also know by Proposition 2.10 that ω is conjugated to ω_0 by an element $B \in \mathsf{GL}(n, \mathbb{H})$. This provides a basis $e_1, \dots, e_{2n}, f_1, \dots, f_{2n}$ of V, such that (after the change of coordinates) $\omega = \omega_0$. It is a simple observation that ω_0 is the standard symplectic form in these coordinates, and thus $e_1, \dots, e_{2n}, f_1, \dots, f_{2n}$ is a symplectic basis adapted to the linear hypercomplex structure H on V. This is because the action of B commutes with the action of H. In particular, the standard quaternionic skew-Hermitian form h_0 introduced in Proposition 2.15 defines a linear hs-H structure, and a linear qs-H structure on V. By the last claim in Proposition 2.15, if we start with a linear hs-H or qs-H structure, then we just obtain its coordinates in the basis $e_1, \dots, e_{2n}, f_1, \dots, f_{2n}$ of V, and therefore all claims (1) and (2) must hold (see the next section for more details on G-structures). □

Having discussed many alternative ways to define the particular types of linear structures that we are interested in, it is convenient to summarize their differences from the well-known linear hH/qH structures, which we encode below in Table 2.

Proposition 2.23 is a powerful tool which we will often apply when we examine $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures on manifolds. Moreover, it motivates us to introduce the following

Definition 2.24 Let $(h, H = \{J_1, J_2, J_3\})$ be a linear hs-H structure on V or let $H = \{J_1, J_2, J_3\}$ be an admissible basis of a linear quaternionic skew-Hermitian form h. We say that the symplectic basis adapted to H by Proposition 2.23 is a skew-Hermitian basis of the linear hs-H or linear qs-H structure, respectively.

Example 2.25 By Example 2.18, we can consider $\mathbb{H}^2$ endowed with H given by the left quaternionic multiplication, and an adapted basis to H given by $\mathfrak{B} := \{e_1, e_2, ie_1, ie_2, je_1, je_2, ke_1, ke_2\}$. Let ω be the bilinear form on $\mathbb{H}^2$ defined by

$$\omega(x, y) = \frac{1}{2}(x^t j \bar{y} - y^t j \bar{x}), \quad \forall\, x, y \in \mathbb{H}^2,$$

where $\bar{x}$ denotes the quaternionic conjugate. Then, ω is a scalar 2-form with respect to H and the basis $\mathfrak{B}$ is a skew-Hermitian basis of the linear hs-H structure (h, H). Here, the linear quaternionic skew-Hermitian form h is induced by ω and H and so it takes form $h(x, y) = x^t j \bar{y}$, see also Corollary A. 1 in the appendix.

Table 2 hH/qH linear structures versus hs-H/qs-H linear structures

Linear G-structure	Initial data	Tensors	Fundam. tensor	Stabilizer G
hH	$\big((\ ,\ )_0 = \omega_{\mathsf{E}} \otimes \omega_{\mathsf{H}}, H_0\big)$	$\omega_a(\cdot,\cdot) = (\cdot, J_a \cdot)_0$	$\Omega_0 \in (\Lambda^4[\mathsf{EH}]^*)^G$	$\mathsf{Sp}(n)$
hs-H	$\big(\omega_0 = g_{\mathsf{E}} \otimes \omega_{\mathsf{H}}, H_0\big)$	$g_a(\cdot,\cdot) = \omega_0(\cdot, J_a \cdot)$	$\Phi_0 \in (S^4[\mathsf{EH}]^*)^G$	$\mathsf{SO}^*(2n)$
qH	$\big((\ ,\ )_0, Q_0 = \langle H_0 \rangle\big)$	$\omega_a(\cdot,\cdot) = (\cdot, J_a \cdot)_0$	$\Omega_0 \in (\Lambda^4[\mathsf{EH}]^*)^G$	$\mathsf{Sp}(n)\,\mathsf{Sp}(1)$
qs-H	$\big(\omega_0, Q_0 = \langle H_0 \rangle\big)$	$g_a(\cdot,\cdot) = \omega_0(\cdot, J_a \cdot)$	$\Phi_0 \in (S^4[\mathsf{EH}]^*)^G$	$\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$

Note that in this example, we have chosen ω is such a way that the adapted basis to H is the same with the skew-Hermitian basis of the linear hs-H structure (h, H). However, this is not the generic case, and we should emphasize that in general an explicit transition between a basis adapted to H and a skew-Hermitian basis, can be carried out by a generalization of the Gram-Schmidt orthogonalization process. Since the action of H identifies V with a *left* quaternionic vector space, these "transitions" between the bases can *not* be realized by left multiplication by quaternionic matrices. Thus, we will postpone the explicit construction to the appendix, where bases that provide an identification of V with a right quaternionic vector space are specified.

2.5 The symplectic viewpoint

In Sect. 2.4, we started by fixing a linear hypercomplex structure H on V, and by using bases adapted to H, we obtained the identification $V \cong [\mathsf{EH}]$. This enabled a convenient description of linear hs-H and qs-H structures in terms of the EH-formalism. In this section, we shall adopt the opposite point of view. This means that we will fix a linear symplectic form ω on V (i.e., a non-degenerate 2-form on V) and a symplectic basis, to get an identification $V \cong \mathbb{R}^{4n}$. This procedure will allow us to examine linear hs-H and qs-H structures from a symplectic point of view, which can be analyzed in terms of the standard symplectic form $\omega_{\mathsf{stn}}(x, y) = x^t S_0 y$ on $\mathbb{R}^{4n}$. Here, as usual, S_0 is the matrix defined by

$$S_0 := \begin{pmatrix} 0 & \mathrm{Id}_{2n} \\ -\mathrm{Id}_{2n} & 0 \end{pmatrix} .$$

With this goal in mind, it is convenient to recall first the notion of the so-called symplectic twistor space attached to a symplectic vector space, see also [12].

Definition 2.26 The symplectic twistor space of $(\mathbb{R}^{4n}, \omega_{\mathsf{stn}})$ of signature (p, q) is the set of all linear complex structures compatible with ω_{stn}, that is

- $J^2 = -\mathrm{id}_{\mathbb{R}^{4n}}$;
- $\omega_{\mathsf{stn}}(Jx, Jy) = \omega_{\mathsf{stn}}(x, y)$, for any $x, y \in \mathbb{R}^{4n}$;
- the pseudo-Euclidean Hermitian metric $g_J(\cdot\,,\cdot) := \omega_{st}(\cdot\,, J\cdot)$ has signature (p, q);

We can now describe the twistor space in the following way:

Lemma 2.27 *The union of all symplectic twistor spaces for all signatures coincides with the space* $\mathsf{Sp}(4n, \mathbb{R}) \cap \mathfrak{sp}(4n, \mathbb{R})$, *and in these terms the following claims hold*:

(1) *The adjoint orbits of* $\mathsf{Sp}(4n, \mathbb{R})$ *in* $\mathsf{Sp}(4n, \mathbb{R}) \cap \mathfrak{sp}(4n, \mathbb{R})$ *are uniquely characterized by the signature* $(4n - 2q, 2q)$ *of the metric*

$$g_J(x, y) = \omega_{\mathsf{stn}}(x, Jy) = x^t S_0 J y ,$$

that is, the stabilizer in $\mathsf{Sp}(4n, \mathbb{R})$ *of a point in an orbit with signature* $(4n - 2q, 2q)$ *is the Lie group* $\mathsf{U}(2n - q, q)$.

(2) *If* $I, J, K \in \mathsf{Sp}(4n, \mathbb{R}) \cap \mathfrak{sp}(4n, \mathbb{R})$ *define a linear hypercomplex or a linear quaternionic structure on* $\mathbb{R}^{4n}$, *then* $I, J, K \in \mathsf{Sp}(4n, \mathbb{R})/\mathsf{U}(n, n)$, *i.e., they are elements of the symplectic twistor space of signature* (n, n).

Proof By definition, any $J \in \mathsf{Sp}(4n, \mathbb{R}) \cap \mathfrak{sp}(4n, \mathbb{R})$ satisfies

$$\omega_{\mathsf{stn}}(Jx, Jy) = \omega_{\mathsf{stn}}(x, y)$$

and

$$\omega_{\mathsf{stn}}(Jx, y) + \omega_{\mathsf{stn}}(x, Jy) = 0,$$

for any $x, y \in \mathbb{R}^{4n}$. Thus $\omega_{\mathsf{stn}}(x, y) = \omega_{\mathsf{stn}}(Jx, Jy) = -\omega_{\mathsf{stn}}(x, J^2 y)$. Since ω_{stn} is non-degenerate, this implies $J^2 = -\mathsf{id}$. Conversely, if J is such that $\omega_{\mathsf{stn}}(Jx, Jy) = \omega_{\mathsf{stn}}(x, y)$ and $J^2 = -\mathsf{id}$, then

$$\omega_{\mathsf{stn}}(Jx, y) + \omega_{\mathsf{stn}}(x, Jy) = \omega_{\mathsf{stn}}(J^2 x, Jy) + \omega_{\mathsf{stn}}(x, Jy) = 0\,.$$

This proves our initial claim. Now, to prove 1) note that the adjoint orbits of $\mathsf{Sp}(4n, \mathbb{R})$ in $\mathfrak{sp}(4n, \mathbb{R})$ are well known and the representatives J of orbits satisfying $J^2 = -\mathsf{id}$ are given by

$$J_{2n-q,q} := \begin{pmatrix} 0 & 0 & \mathtt{Id}_{2n-q} & 0 \\ 0 & 0 & 0 & -\mathtt{Id}_{q} \\ -\mathtt{Id}_{2n-q} & 0 & 0 & 0 \\ 0 & \mathtt{Id}_{q} & 0 & 0 \end{pmatrix}.$$

Thus, the pseudo-Euclidean metric defined by $\omega_{\mathsf{stn}}(x, J_{2n-q,q} y) = x^t S_0 J_{2n-q,q} y$ has signature $(4n - 2q, 2q)$. Finally, we should mention that the assertion 2) is a consequence of Proposition 2.11. □

The proof of Lemma 2.27 suggests defining the following operation:

Definition 2.28 We say that $A^T \in \mathsf{GL}(\mathbb{R}^{4n})$ is the symplectic transpose of $A \in \mathsf{GL}(\mathbb{R}^{4n})$ if

$$\omega_{\mathsf{stn}}(A^T x, y) = -\omega_{\mathsf{stn}}(x, Ay),$$

holds for all $x, y \in \mathbb{R}^{4n}$.

We shall now prove that the symplectic transpose always exists.

Lemma 2.29 *Let* $\mathscr{B}_{\mathsf{stn}}$ *be the symplectic basis on* $\mathbb{R}^{4n}$ *such that* ω_{stn} *is given by* $\omega_{\mathsf{stn}}(x, y) = x^t S_0 y$. *Then,* $A^T = S_0 A^t S_0$, *where* A^t *is the usual transpose of* A *in the coordinates of the symplectic basis.*

Proof By a direct computation, we obtain

$$\omega_{\mathsf{stn}}(Ay, x) = y^t A^t S_0 x = y^t S_0 (S_0^{-1} A^t S_0) x = \omega_{\mathsf{stn}}(y, S_0^{-1} A^t S_0 x) = -\omega_{\mathsf{stn}}(-S_0 A^t S_0 x, y)$$

and by the non-degeneracy of ω_{stn}, it follows that $A^T = S_0 A^t S_0$. □

As a corollary, we deduce the following:

Corollary 2.30 *The 2-form ω_{stn} is Hermitian with respect to a linear complex structure J on $\mathbb{R}^{4n}$, if and only if $J^T = J$.*

Proof Assume that ω_{stn} is J-Hermitian, i.e., $\omega_{\mathsf{stn}}(Jx, Jy) = \omega_{\mathsf{stn}}(x, y)$ for any $x, y \in \mathbb{R}^{4n}$. Then, by replacing x by Jx and by definition of J^T we obtain

$$\omega_{\mathsf{stn}}(J^2x, Jy) = \omega_{\mathsf{stn}}(Jx, y) = -\omega_{\mathsf{stn}}(y, Jx) = \omega_{\mathsf{stn}}(J^T y, x)\,.$$

Since also $\omega_{\mathsf{stn}}(J^2x, Jy) = -\omega_{\mathsf{stn}}(x, Jy) = \omega_{\mathsf{stn}}(Jy, x)$, we finally obtain $\omega_{\mathsf{stn}}(Jy, x) = \omega_{\mathsf{stn}}(J^T y, x)$, that is $J = J^T$. The converse is treated similarly. □

Let us now link the above description with the structures that we are interested in. So, assume that $(\mathbb{R}^{4n}, \omega_{\mathsf{stn}})$ is endowed with a linear quaternionic structure $Q \subset \mathsf{End}(\mathbb{R}^{4n})$, for which ω_{stn} is scalar, that is ω_{stn} is Q-Hermitian in terms of the Definition 2.1 (see also Proposition 2.10). Then, by the above description it follows that

Corollary 2.31 *The 2-sphere $S(Q) = \mathsf{Sp}(1) \cap \mathfrak{sp}(1)$ associated to a linear quaternionic structure Q is a subspace of the $\mathsf{Sp}(4n, \mathbb{R})/\mathsf{U}(n, n)$-orbit.*

Let us now characterize the space of such linear quaternionic structures Q.

Lemma 2.32 *Let $(\mathbb{R}^{4n}, \omega_{\mathsf{stn}})$ be the standard symplectic vector space. Then, the following hold:*

(1) *Let H be a linear hypercomplex structure on $\mathbb{R}^{4n}$ such that the corresponding symplectic bases are adapted to H, in terms of Definition 2.17 and let $f : \mathbb{R}^{4n} \to [\mathsf{EH}]$ be the induced isomorphism. This defines a surjective map from the space of symplectic bases onto the space of all linear hypercomplex structures H, and moreover onto the space of all linear quaternionic structures Q on $\mathbb{R}^{4n}$, such that*

$$\omega_{\mathsf{stn}} = f^*\omega_0\,, \quad H = f^*H_0\,, \quad Q = f^*Q_0\,.$$

In particular, the pairs $(\omega_{\mathsf{stn}}, H)$ and $(\omega_{\mathsf{stn}}, Q)$ are linear hs-H/qs-H structures, respectively.

(2) *Two symplectic bases of $\mathbb{R}^{4n}$ define the same linear hs-H structure, if and only if the transition matrix between them is an element of $\mathsf{SO}^*(2n)$, and they define the same linear qs-H structure, if and only if the transition matrix between them is an element of $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$.*

Proof By the existence result for bases adapted to the linear hypercomplex structure H (see Proposition 2.19), the surjectivity follows. The claims about the stabilizers follow by Proposition 2.15. □

3 Almost hypercomplex/quaternionic skew-Hermitian structures

The description of the most basic features of linear $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures given in the previous section, enables us to conveniently investigate $\mathsf{SO}^*(2n)$-type structures on smooth manifolds. For the convenience of the reader, let us begin by refreshing a few basic facts from the theory of G-structures (for more details see [6, 28, 37, 38]).

Let us fix, once and for all, a connected $4n$-dimensional smooth manifold M and some reference $4n$-dimensional real vector space V (which we will use as a model of T_xM). The frame bundle $\mathcal{F} = \mathcal{F}(M)$ of M consists of all linear isomorphisms between the tangent space T_xM of M at $x \in M$ and V, which we view as co-frames $u : T_xM \to V$. The frame bundle $\mathcal{F}$ is a principal $\mathsf{GL}(V)$-bundle over M. A G-structure on M is defined to be a reduction of the frame bundle to a closed Lie subgroup $G \subset \mathsf{GL}(V)$, i.e., a sub-bundle $\mathcal{P} \subset \mathcal{F}(M)$ with structure group G.

Let $\pi : \mathcal{P} \to M$ be a G-structure on M and let $\vartheta \in \Omega^1(\mathcal{P}, V)$ be the tautological 1-form on $\mathcal{P}$, defined by $\vartheta(X) = u(\pi_*(X))$ for any co-frame $u \in \mathcal{P}$ and $X \in T_u\mathcal{P}$. The tautological form is strictly horizontal, in the sense that the kernel of ϑ coincides with the vertical sub-bundle $T^{\mathsf{ver}}\mathcal{P}$ of the tangent bundle $T\mathcal{P}$, and G-equivariant, i.e., $r_a^*\vartheta = a^{-1}\vartheta$ for any $a \in G$, where r_a denotes the right translation by an element $a \in G$. Such 1-forms may characterize a G-structure, in particular under our assumptions, a G-structure on M is equivalently defined to be a principal G-bundle $\pi : \mathcal{P} \to M$ over M together with a 1-form $\vartheta \in \Omega^1(P, V)$ such that $\ker\vartheta = \ker \mathrm{d}\pi = T^{\mathsf{ver}}\mathcal{P}$ and $r_a^*\vartheta = a^{-1}\vartheta$, for any $a \in G$. This definition enables generalizing the notion of G-structures to the case where $\rho : G \to \mathsf{GL}(V)$ is a covering of a closed subgroup of $\mathsf{GL}(V)$ (like the case of spin structures), by assuming $r_a^*\vartheta = \rho(a)^{-1}\vartheta$.

Let us now recall the following examples of G-structures which we will use frequently below.

Definition 3.1 (1) An almost hypercomplex structure on M is a G-structure with $G = \mathsf{GL}(n, \mathbb{H})$. This mean that M admits a triple $H = \{J_a : a = 1, 2, 3\}$ of smooth endomorphisms $J_a \in \mathsf{End}(TM)$ satisfying the quaternionic identity $J_1^2 = J_2^2 = J_3^2 = -\mathsf{Id} = J_1J_2J_3$. Any almost hypercomplex structure H induces a linear hypercomplex structure H_x at each T_xM, which establishes a linear isomorphism $(T_xM, H_x) \cong ([\mathsf{EH}], H_0)$. Such a pair (M, H) is said to be an almost hypercomplex manifold. Note that $\mathsf{Aut}(H_0) \cong \mathsf{GL}(n, \mathbb{H})$, and in this case the reduction bundle of the frame bundle of M consists of all bases of T_xM adapted to H_x. Such a basis induces a linear hypercomplex isomorphism $u : T_xM \to [\mathsf{EH}]$.

(2) An almost quaternionic structure on M is a G-structure with $G = \mathsf{GL}(n, \mathbb{H})\,\mathsf{Sp}(1)$. This means that M admits a rank-3 smooth sub-bundle $Q \subset \mathsf{End}(TM) \cong T^*M \otimes TM$ which is locally generated by an almost hypercomplex structure $H = \{J_a : a = 1, 2, 3\}$. Such a locally defined triple H is called a (local) admissible frame of Q, and the pair (M, Q) is called an almost quaternionic manifold. Note that $\mathsf{Aut}(Q_0) \cong \mathsf{GL}(n, \mathbb{H})\,\mathsf{Sp}(1)$, and in this case the reduction bundle of the frame bundle of M consists of all bases of T_xM adapted to some admissible basis H_x of Q_x. Such a basis induces a linear quaternionic isomorphism $u : T_xM \to [\mathsf{EH}]$.

(3) An almost symplectic structure on M is a G-structure with $G = \mathsf{Sp}(4n, \mathbb{R})$. This means that M admits a non-degenerate 2-form ω, called an almost symplectic

form. Such a pair (M,ω) is referred to as an almost symplectic manifold. Note that $\mathsf{Aut}(\omega_0)\cong \mathsf{Sp}(4n,\mathbb{R})$, and in this case the reduction of $\mathcal{F}(M)$ to $\mathsf{Sp}(2n,\mathbb{R})$ consist all symplectic bases of (T_xM,ω_x). Similarly, such a basis induces a linear symplectomorphism $u : T_xM \to \mathbb{R}^{4n}$, where we consider $\mathbb{R}^{4n}$ as endowed with the standard linear symplectic form ω_{stn}.

Let us also recall the following operators, which are naturally defined on any almost symplectic manifold (M,ω).

Definition 3.2 Let $L_X \in \mathsf{End}(TM)$ be an endomorphism on an almost symplectic manifold (M,ω), induced by a vector-valued smooth 2-form L on M, that is $L_X = L(X,\cdot)$ for any $X\in\Gamma(TM)$. Then:

(1) The symplectic transpose L_X^T of L_X with respect to ω is defined by

$$\omega(L_X^T Y, Z) + \omega(Y, L_X Z) = 0\,, \quad \forall\, X, Y, Z \in \Gamma(TM)\,.$$

(2) The operator of symmetrization/antisymmetrization of L_X with respect to the symplectic transpose L_X^T, is, respectively, defined by

$$Sym(L_X) := \frac{1}{2}\big(L_X + L_X^T\big)\,, \quad Asym(L_X) := \frac{1}{2}\big(L_X - L_X^T\big)\,.$$

A remark of caution: Below we shall use the EH-formalism, where it is appropriate to emphasize on the role of topology of the smooth manifold M admitting an almost quaternionic structure $Q \subset \mathsf{End}(TM)$. This is because not everything from EH-formalism extends globally to a manifold setting, see also [29, 35, 36]. Recall first that the quaternionic structure Q is naturally identified with an associated bundle over M with fiber $[S^2\,\mathsf{H}]^*$, via the canonical section ω_{H} of the associated bundle with fiber $[\Lambda^2\,\mathsf{H}]^*$. However, since $\mathsf{GL}(n,\mathbb{H})\,\mathsf{Sp}(1)$ is a quotient of $\mathsf{GL}(n,\mathbb{H})\times\mathsf{Sp}(1)$, the bundle analogies of the modules E and H are not necessarily globally defined over M. In the second part of this work we describe the analogous result for $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, see the appendix in [19]. Obviously, another obstruction is the global trivializability of Q. Hence, an admissible frame $H=\{I,J,K\}$ of Q, or the vector bundles associated to a $\mathsf{GL}(n,\mathbb{H})\,\mathsf{Sp}(1)$ -structure via the representations E and H, may not exist globally. However, note that the projectivization $\mathbb{P}(\mathsf{H})$ of H globally exists and provides us with the twistor bundle (or unit sphere bundle) $Z\to M$ associated to any almost quaternionic manifold (M, Q). As an example, note that there are manifolds (e.g., the quaternionic projective space), which cannot carry a $\mathsf{GL}(n,\mathbb{H})$-structure, but admit a $\mathsf{GL}(n,\mathbb{H})\,\mathsf{Sp}(1)$-structure. We partially examine the topology of $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures in [19].

3.1 Scalar 2-forms

Let us now introduce $\mathsf{SO}^*(2n)$-type structures on smooth manifolds. This topic will constitute the core of this article. From now on, we may fix $V=[\mathsf{EH}]$ and assume that $n>1$. We begin with the following definition, as the analogue of Definition 2.9 of scalar 2-forms on manifolds.

Definition 3.3 Let M be a smooth connected manifold.

(1) Assume that M admits an almost hypercomplex structure $H = \{J_a : a = 1, 2, 3\}$. Then, a smooth 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$ is called H-Hermitian, if ω_x is Hermitian with respect to $H_x = \{(J_a)_x : a = 1, 2, 3\}$ in terms of Definition 2.1, for any $x \in M$. An everywhere non-degenerate H-Hermitian 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$ will be called a scalar 2-form (with respect to H) on M.

(2) Assume that M admits an almost quaternionic structure Q. Then, a smooth 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$ is called Q-Hermitian, if ω_x is Hermitian with respect to Q_x in terms of Definition 2.1, for any $x \in M$. An everywhere non-degenerate Q-Hermitian 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$ will be called a scalar 2-form (with respect to Q) on M.

As a consequence of Proposition 2.23, we may now pose the following characterization of smooth scalar 2-forms in the manifold setting.

Corollary 3.4 (1) *Let $(M, H = \{J_a : a = 1, 2, 3\})$ be an almost hypercomplex manifold. Then, a real-valued smooth 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$ is a smooth scalar 2-form, if and only if there is a linear hypercomplex skew-Hermitian structure (H_x, h_x) on T_xM for any $x \in M$ such that $\omega_x = \mathbb{Re}(h_x)$, i.e., ω_x is a scalar 2-form on T_xM (with respect to H_x), for any $x \in M$.*

(2) *textsfLet (M, Q) be an almost quaternionic manifold. Then, a real-valued smooth 2-form $\omega \in \Gamma(\Lambda^2 T^*M)$ is a smooth scalar 2-form, if and only if there is a linear quaternionic skew-Hermitian structure h_x on T_xM for any $x \in M$ such that $\omega_x = \mathbb{Re}(h_x)$ and Q_x is the quaternionic structure induced by $\mathbb{Im}(h_x)$, i.e., ω_x is a scalar 2-form on T_xM (with respect to Q_x), for any $x \in M$.*

(3) *Let $(M, H = \{J_a : a = 1, 2, 3\})$ or (M, Q) be as above. Then, the set of smooth scalar 2-forms on M defines a sub-bundle of $\Lambda^2 T^*M$, which we denote by $\Lambda^2_{\mathsf{sc}} T^*M$.*

Remark 3.5 Note that $\Lambda^2_{\mathsf{sc}} T^*M \subset \Lambda^2_{\mathbb{Re}(\mathbb{H})} T^*M$ is *not* a vector sub-bundle of $\Lambda^2 T^*M$, due to the requirement that smooth scalar 2-forms are non-degenerate. The reader may consult Proposition 2.10 to derive further equivalent characterizations of scalar 2-forms on smooth manifolds.

We can now proceed by introducing the geometric structures that we are mainly interested in.

Definition 3.6 (1a) An almost hypercomplex skew-Hermitian structure (H, ω) on a $4n$-dimensional manifold M (almost hs-H structure for short) consists of an almost hypercomplex structure $H = \{J_a : a = 1, 2, 3\}$ and a smooth scalar 2-form $\omega \in \Gamma(\Lambda^2_{\mathsf{sc}} T^*M)$ (with respect to H). A manifold M endowed with an almost hypercomplex skew-Hermitian structure will be referred to as an almost hypercomplex skew-Hermitian manifold (almost hs-H manifold for short), and denoted by (M, H, ω).

(1b) A hypercomplex symplectomorphism $f : (M, H, \omega) \to (\hat{M}, \hat{H}, \hat{\omega})$ between two almost hypercomplex skew-Hermitian manifolds is a diffeomorphism $f : M \to \hat{M}$ satisfying $H = f^*\hat{H}$ and $\omega = f^*\hat{\omega}$.

(2a) An almost quaternionic skew-Hermitian structure (Q, ω) on a $4n$-dimensional manifold M (almost qs-H structure for short) consists of an almost quaternionic structure $Q \subset \mathsf{End}\,(TM)$ and a smooth scalar 2-form $\omega \in \Gamma(\Lambda^2_{\mathsf{sc}} T^*M)$ (with respect to Q). A manifold M endowed with an almost quaternionic skew-Hermitian structure will be referred to as an almost quaternionic skew-Hermitian manifold (almost qs-H manifold for short), and denoted by (M, Q, ω).

(2b) A quaternionic symplectomorphism $f : (M, Q, \omega) \to (\hat{M}, \hat{Q}, \hat{\omega})$ between two almost quaternionic skew-Hermitian manifolds is a diffeomorphism $f : M \to \hat{M}$ satisfying $Q = f^*\hat{Q}$ and $\omega = f^*\hat{\omega}$.

From this definition, it is obvious that

- An almost hs-H manifold (M, H, ω) is already an almost hypercomplex manifold (M, H).
- An almost qs-H manifold (M, Q, ω) is already an almost quaternionic manifold (M, Q).

Therefore, the structures that we treat are special subclasses of almost hypercomplex/quaternionic structures, endowed with a bit more structure provided by the smooth scalar 2-form ω. In particular, they form the symplectic counterparts of almost hH structures and almost qH structures, respectively, which are almost hypercomplex/quaternionic structures endowed with a (pseudo)-Riemannian metric which is Hermitian with respect to H and Q, respectively. On the other side,

- Any almost hs-H manifold (M, H, ω) or any almost qs-H manifold (M, Q, ω) is already an almost symplectic manifold (M, ω).

Hence, one may start with an almost symplectic structure ω and look for "compatible" almost hypercomplex/quaternionic structures, in the sense that we require ω to be a scalar 2-form with respect to such an almost hypercomplex/quaternionic structure.

Next we describe the bundle reductions corresponding to such G-structures. By Proposition 2.23, we obtain the following characterization.

Proposition 3.7 (1) *An* almost hs-H manifold *is a* $4n$*-dimensional connected manifold* M, *whose frame bundle* $\mathcal{F} = \mathcal{F}(M)$ *admits a reduction to* $\mathsf{SO}^*(2n) \subset \mathsf{GL}\,([\mathsf{EH}])$, *namely* $\mathcal{P} = \sqcup_{x\in M}\mathcal{P}_x$, *where*

$$\mathcal{P}_x := \left\{u := (e_1, \dots, e_{2n}, f_1, \dots, f_{2n}) : u \text{ skew-Hermitian basis of } (T_xM, H_x, \omega_x)\right\}.$$

Thus, $\mathcal{P}$ *is a principal* $\mathsf{SO}^*(2n)$*-bundle over* M, *and we can identify*

$$\mathcal{F} = \mathcal{P} \times_{\mathsf{SO}^*(2n)} \mathsf{GL}\,([\mathsf{EH}]), \quad \text{and} \quad TM = \mathcal{F} \times_{\mathsf{GL}\,(V)} V \cong \mathcal{P} \times_{\mathsf{SO}^*(2n)} [\mathsf{EH}].$$

(2) *The set of* $\mathsf{SO}^*(2n)$*-structures on* M *coincides with the space of sections of the quotient bundle*

$$\mathcal{F}/\,\mathsf{SO}^*(2n) = \mathcal{F} \times_{\mathsf{GL}\,(V)} \big(\mathsf{GL}\,(V)/\,\mathsf{SO}^*(2n)\big)$$

with typical fiber isomorphic to the space $\mathsf{GL}\,([\mathsf{EH}])/\,\mathsf{SO}^*(2n)$.

Observe that the existence of global sections of $\mathcal{F}/\,\mathsf{SO}^*(2n)$ is purely topological in nature.

Example 3.8 Consider $M = [\mathsf{EH}]$. Then, M is an almost hs-H manifold, and the space of almost hs-H-structures coincides with the space of functions from $[\mathsf{EH}]$ to $\mathsf{GL}\,([\mathsf{EH}])/\,\mathsf{SO}^*(2n)$. At $x \in M$ such a function describes a linear transformation from (H_x, ω_x) to (H_0, ω_0).

Similarly, by the discussion in Sect. 2.4 and Proposition 2.23 we obtain an analogous statement for almost qs-H structures.

Proposition 3.9 (1) *An* almost qs-H manifold *is a* $4n$*-dimensional connected manifold* M*, whose frame bundle* $\mathcal{F} = \mathcal{F}(M)$ *admits a reduction to* $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1) \subset \mathsf{GL}\,([\mathsf{EH}])$*, namely* $\mathcal{Q} = \sqcup_{x\in M} \mathcal{Q}_x \subset \mathcal{F}$*, where*

$$\mathcal{Q}_x := \left\{ \begin{array}{ll} u := (e_1, \dots, e_{2n}, f_1, \dots, f_{2n}) & : u \text{ skew-Hermitian basis of } (T_xM, Q_x = \langle H_x\rangle, \omega_x) \\ & \qquad\qquad \text{for all admissible bases } H_x \text{ of } Q_x \end{array} \right\}.$$

Thus, $\mathcal{Q}$ *is a principal* $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$*-bundle over* M*, and we can identify*

$$\mathcal{F} = \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)} \mathsf{GL}\,([\mathsf{EH}]), \quad \text{and} \quad TM = \mathcal{F} \times_{\mathsf{GL}\,(V)} V \cong \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)} [\mathsf{EH}]\,.$$

(2) *The set of* $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$*-structures on* M *coincides with the space of sections of the quotient bundle*

$$\mathcal{F}/\,\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1) = \mathcal{F} \times_{\mathsf{GL}\,(V)} \big(\mathsf{GL}\,(V)/\,\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)\big)$$

with typical fiber isomorphic to the space $\mathsf{GL}\,([\mathsf{EH}])/\,\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$.

Definition 3.10 (1) Let (M, H, ω) be an almost hs-H manifold. A (local) section of the principal $\mathsf{SO}^*(2n)$-bundle $\mathcal{P} \to M$ given in Proposition 3.7 is said to be a (local) skew-Hermitian frame.

(2) Let (M, Q, ω) be an almost qs-H manifold. A (local) section of the principal $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-bundle $\mathcal{Q} \to M$ given in Proposition 3.9 is said to be a (local) skew-Hermitian frame with respect to some local admissible frame H of Q. Note that the local admissible frame H is uniquely determined by the corresponding skew-Hermitian frame.

According to Proposition 2.10, the $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-module of linear scalar 2-forms on $[\mathsf{EH}]$, denoted by $\Lambda^2_{\mathsf{sc}}[\mathsf{EH}]^*$, is the set of non-degenerate elements inside $[S^2\,\mathsf{E}\,]^* = [S^2_0\,\mathsf{E}\,]^* \oplus \langle\omega_0\rangle$. Hence, given an almost quaternionic skew-Hermitian manifold (M, Q, ω) with reduction $\mathcal{Q} \subset \mathcal{F}$ described in Proposition 3.9, the scalar 2-form ω or any other smooth scalar 2-form can be viewed as a smooth section of the following associated bundle:

$$\begin{aligned}\Lambda^2_{\mathsf{sc}} T^*M =& \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)} \Lambda^2_{\mathsf{sc}}[\mathsf{E}\,\mathsf{H}]^* \\ =& \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)} \Lambda^2_{\mathsf{sc},0}[\mathsf{EH}]^* \oplus \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)} \mathbb{R}^{\times}\omega_0 \\ =& \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)} \Lambda^2_{\mathsf{sc},0}[\mathsf{E}\,\mathsf{H}]^* \oplus \mathcal{L}_{\omega_0} .\end{aligned}$$

Here, the notation $\Lambda^2_{\mathsf{sc},0}[\mathsf{EH}]^*$ encodes the non-degenerate elements in $[S^2_0\,\mathsf{E}\,]^*$, and

$$\mathcal{L}_{\omega_0} = \mathcal{Q} \times_{\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)} \mathbb{R}^{\times}\omega_0$$

is a line bundle without zero sections. Due to Propositions 2.10 and 3.7, the reader can describe a similar decomposition of the space of scalar 2-forms $\Lambda^2_{\mathsf{sc}} T^*M$ associated to an almost hs-H manifold (M, H, ω).

Remark 3.11 The line bundle $\mathcal{L}_{\omega_0}$ defined above defines the (almost) conformal symplectic version of $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-structures, which was thoroughly discussed by Salač and Čap [14]. For a qs-H manifold (M, Q, ω), the scalar 2-form ω defines a global section of $\mathcal{L}_{\omega_0}$, which does not have to exist globally in the conformal symplectic setting (in this case the corresponding structure group is the Lie group $\mathsf{SO}^*(2n)\,\mathsf{GL}\,(1, \mathbb{H})$). This is responsible for the differences between our results and the results in [14].

Finally, note that locally an almost qs-H structure on a manifold M can be understood in terms of an almost hs-H structure, although globally the situation differs. Of course, this establishes an analogue with the local relation of almost quaternionic structures and almost hypercomplex structures. Let us summarize this phenomenon as follows:

Proposition 3.12 *Let (M, Q, ω) be an* almost qs-H manifold *and let $x \in M$ be some point of M. Then, there exists an open neighborhood $U \subset M$ of x and an* almost hs-H structure *(H, ω') defined on U, such that*

$$\omega|_U = \omega', \quad Q|_U = \langle H \rangle.$$

Explicit constructions providing examples of the structures introduced above are analyzed in the second part of this work, in particular see Sections 3, 4 and 5 in [19]. In the final section of this first part, we have collected details related to torsion-free (or integrable) examples (see below Sect. 4 for adapted connections and intrinsic torsion).

3.2 The quaternionic skew-Hermitian form and the fundamental 4-tensor

Since for $n > 1$ the Lie group $\mathsf{SO}^*(2n)$ is non-compact, in principle there is *no* underlying Riemannian metric structure on a manifold M with a $\mathsf{SO}^*(2n)$-structure, and similarly for G-structures with $G = \mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$. However, given any almost hs-H manifold $(M, H = \{I, J, K\}, \omega)$ by Propositions 2.11 and 2.23 we may introduce three pseudo-Riemannian metrics of signature $(2n, 2n)$, defined by

$$g_I(X, Y) := \omega(X, IY), \quad g_J(X, Y) := \omega(X, JY), \quad g_K(X, Y) := \omega(X, KY),$$

for any $X, Y \in \Gamma(TM)$. Note that these metrics are $\mathrm{SO}^*(2n)$-invariant, as it is claimed in Remark 2.12. The above tensors are also obtained in the case of an almost qs-H manifold (M, Q, ω), where $H = \{I, J, K\}$ is a local admissible frame of Q, but they are only locally defined. In particular, for any local section $\mathsf{J} \in \Gamma(Z)$ the tensor $g_{\mathsf{J}}(X, Y) := \omega(X, \mathsf{J}Y)$ is a locally defined tensor. Nevertheless, via the assignment $\mathsf{J} \mapsto g_{\mathsf{J}}$ we obtain a global embedding of the twistor bundle $Z \to M$ into S^2T^*M.

Let us now construct some globally defined tensors, which allow us to provide alternative definitions of the structures that we are interested in. Indeed, by Proposition 2.15 on an almost qs-H manifold (M, Q, ω) we obtain a globally defined tensor h, given by

$$h := \omega\, \mathrm{id}_{TM} + g_I I + g_J J + g_K K \in \Gamma\big(T^*M \otimes T^*M \otimes \mathrm{End}(TM)\big)\,, \tag{3.1}$$

where $\{I, J, K\}$ is an arbitrary local admissible frame of Q. In particular,

$$h(X, Y)Z = \omega(X, Y)Z + +g_I(X, Y)IZ + g_J(X, Y)JZ + g_K(X, Y)KZ$$

for any $X, Y, Z \in \Gamma(TM)$. Since each (T_xM, h_x) is a vector space with a linear qs-H structure as defined in Definition 2.22, we shall refer to h by the term quaternionic skew-Hermitian form associated to the almost qs-H structure (Q, ω). Observe that h is defined even for the case where (M, H, ω) is an almost hs-H manifold. However, note that h is actually stabilized by the larger group $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$, and we may pose the following characterization:

Corollary 3.13 *A* $4n$*-dimensional connected smooth manifold* M *admits a* $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$*-structure, if and only if admits a smooth* $(1, 3)$*-tensor* h *which in a local frame of* TM *is given by the tensor* h_0 *of Proposition* 2.15.

Of course, this corollary may serve as an alternative way to define $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$-structures. Similarly, by Proposition 2.16 on (M, Q, ω), we obtain a globally defined 4-tensor Φ, given by

$$\Phi := g_I \odot g_I + g_J \odot g_J + g_K \odot g_K = \mathrm{Sym}(g_I \otimes g_I + g_J \otimes g_J + g_K \otimes g_K) \in \Gamma\big(S^4T^*M\big)\,, \tag{3.2}$$

where $\mathrm{Sym} : \mathcal{T}^4T^*M \to S^4T^*M$ denotes the operator of complete symmetrization at the bundle level, and $\{I, J, K\}$ is an arbitrary local admissible frame of Q. We call Φ the fundamental 4-tensor field associated to the almost qs-H structure (Q, ω). Again, Φ is defined even for the case where (M, H, ω) is an almost hs-H manifold. However, similarly to h above, note that Φ is actually stabilized by the larger group $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$, so similarly we get the following

Corollary 3.14 *A* $4n$*-dimensional connected smooth manifold* M *admits a* $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$*-structure, if and only if admits a symmetric* 4*-tensor* Φ *which in a local frame of* TM *is given by the tensor* Φ_0 *of Proposition* 2.16.

Corollary 3.14, just like Corollary 3.13, can be used to define $\mathrm{SO}^*(2n)\,\mathrm{Sp}(1)$-structures in an alternative way, via a global symmetric 4-tensor. Moreover, since Φ satisfies

$$\Phi(X,Y,Z,W) = \frac{1}{24}\sum_{\sigma\in S^4}\Big(g_I(X_{\sigma(1)},Y_{\sigma(2)})g_I(Z_{\sigma(3)},W_{\sigma(4)}) + g_J(X_{\sigma(1)},Y_{\sigma(2)})g_I(Z_{\sigma(3)},W_{\sigma(4)}) + g_K(X_{\sigma(1)},Y_{\sigma(2)})g_K(Z_{\sigma(3)},W_{\sigma(4)})\Big)$$

for any $X, Y, Z, W \in \Gamma(TM)$, we deduce that

Lemma 3.15 *The fundamental tensor field* Φ *satisfies the identities*:

$$\begin{aligned}\Phi(X,Y,Z,W) =& \frac{1}{3}\sum_{I\in H}\Big(\mathfrak{S}_{Y,Z,W}\, g_I(X,Y)g_I(Z,W)\Big)\\ =& \frac{1}{3}\Big(g_I(X,Y)g_I(Z,W) + g_I(X,Z)g_I(Y,W) + g_I(X,W)g_I(Y,Z)\\ &+ g_J(X,Y)g_J(Z,W) + g_J(X,Z)g_J(Y,W) + g_J(X,W)g_J(Y,Z)\\ &+ g_K(X,Y)g_K(Z,W) + g_K(X,Z)g_K(Y,W) + g_K(X,W)g_K(Y,Z)\Big),\end{aligned}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

4 Intrinsic torsion of $\mathsf{SO}^*(2n)$-structures and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures

4.1 Generalities on adapted connections to *G*-structures

Let us maintain the notation from the introduction in Sect. 3, and assume that M is a connected manifold, but not necessarily $4n$-dimensional, and that $G \subset \mathsf{GL}(V)$ is a closed subgroup. Recall that the torsion of a linear connection ∇ on M is a smooth section of the torsion bundle $\mathsf{Tor}(M) := \Lambda^2 T^*M \otimes TM$ defined by

$$T^{\nabla}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y], \quad X, Y \in \Gamma(TM).$$

If T^{∇} vanishes identically, then ∇ is said to be torsion-free.

Let $\pi : \mathcal{P} \to M$ be a G-structure on M with tautological 1-form ϑ. A linear connection ∇ is called adapted to $\mathcal{P} \subset \mathcal{F}$, or simply a G-connection, when the corresponding connection on the frame bundle $\mathcal{F}$ of M reduces to $\mathcal{P}$. Since the Lie algebra $\mathfrak{g}$ of G can be identified with a subalgebra of $\mathfrak{gl}(V) = \mathsf{End}(V)$, one can show that the space of G-principal connections on $\mathcal{P}$ is a space modeled on the space of smooth sections of the associated bundle $\mathcal{P}\times_G (V^*\otimes\mathfrak{g}) = T^*M \otimes \mathfrak{g}_{\mathcal{P}}$, where $\mathfrak{g}_{\mathcal{P}}$ is the adjoint bundle. Each G-principal connections on $\mathcal{P}$ is defined by a connection 1-form $\gamma : T\mathcal{P} \to \mathfrak{g}$ and induces a G-connection ∇^{γ} on TM. In particular, there is bijective correspondence between G-connections and G-principal connections on $\mathcal{P}$. For any adapted connection $\nabla = \nabla^{\gamma}$ corresponding to a connection 1-form $\gamma : T\mathcal{P} \to \mathfrak{g}$, we define its torsion form Θ^{γ}, which is the vector-valued 2-form on $\mathcal{P}$ defined by the structure equation $\Theta^{\gamma} = \mathrm{d}\,\vartheta + \gamma \wedge \vartheta$. When the torsion form Θ^{γ} vanishes, the G-connection corresponding to the connection 1-form γ is said to be torsion-free, which is in a line with the definition above. Indeed, Θ^{γ} is G-equivariant and strictly horizontal, so it induces a smooth G-equivariant torsion function $t^{\gamma} : \mathcal{P} \to \Lambda^2 V^* \otimes V$, which assigns to a co-frame $u \in \mathcal{P}$ the coordinates $t^{\gamma}(u) = u(T^{\nabla})$ of the torsion T^{∇} in $\Lambda^2 V^* \otimes V$.

Definition 4.1 A G-structure $\mathcal{P} \subset \mathcal{F}$ is called a torsion-free G-structure, or 1-integrable, when it admits a torsion-free adapted connection.

Let us fix a G-structure $\pi : \mathcal{P} \to M$ on M as above. Recall that the first prolongation of the Lie algebra $\mathfrak{g}$ of G is defined by

$$\mathfrak{g}^{(1)} := (V^* \otimes \mathfrak{g}) \cap (S^2 V^* \otimes V) = \{\alpha \in V^* \otimes \mathfrak{g} : \alpha(x)y = \alpha(y)x, \ \forall\, x, y \in V\} \subset \mathsf{Hom}\,(V, \mathfrak{g}).$$

Note that for any Lie subalgebra $\mathfrak{g} \subset \mathsf{End}\,(V)$, we may consider the G-equivariant map

$$\delta : V^* \otimes \mathfrak{g} \to \Lambda^2 V^* \otimes V, \quad \delta(\alpha)(x, y) := \alpha(x)y - \alpha(y)x,$$

with $\alpha \in V^* \otimes \mathfrak{g}$ and $x, y \in V$. This is the Spencer operator of alternation, which is actually one of the boundary maps of the Spencer complex of $\mathfrak{g} \subset \mathsf{End}\,(V)$, also called Spencer differential. It fits into the following exact sequence,

$$\begin{aligned} 0 &\longrightarrow \mathsf{Ker}\,\delta \cong \mathfrak{g}^{(1)} \longrightarrow V^* \otimes \mathfrak{g} \\ &\cong \mathsf{Hom}\,(V, \mathfrak{g}) \xrightarrow{\delta} \Lambda^2 V^* \otimes V \cong \mathsf{Hom}\,(\Lambda^2 V, V) \longrightarrow \mathsf{Coker}(\delta) \cong \mathcal{H}(\mathfrak{g}) \longrightarrow 0 \end{aligned}$$

where we denote by $\mathcal{H}(\mathfrak{g}) \equiv \mathcal{H}^{0,2}(\mathfrak{g})$ the following Spencer cohomology of $\mathfrak{g}$:

$$\mathcal{H}(\mathfrak{g}) := \mathsf{Hom}\,(\Lambda^2 V, V)/\,\mathsf{Im}\,(\delta) = \Lambda^2 V^* \otimes V/\,\mathsf{Im}\,(\delta).$$

Let us consider the vector bundle $\mathscr{H}(\mathfrak{g}) := \mathcal{P} \times_G \mathcal{H}(\mathfrak{g})$, called the intrinsic torsion bundle over M, and maintain the same notation for the bundle map

$$\delta : T^*M \otimes \mathfrak{g}_{\mathcal{P}} \to \mathsf{Tor}\,(M),$$

induced by the Spencer operator $\delta : V^* \otimes \mathfrak{g} \to \Lambda^2 V^* \otimes V$. There is a natural projection from $\mathsf{Tor}\,(M)$ to $\mathscr{H}(\mathfrak{g})$ which we shall denote by $p : \mathsf{Tor}\,(M) \to \mathscr{H}(\mathfrak{g})$. In these terms, we have an isomorphism

$$\mathscr{H}(\mathfrak{g}) \cong \mathsf{Tor}(M)/\,\mathsf{Im}\,(\delta) = \Lambda^2 T^*M \otimes TM/\delta(T^*M \otimes \mathfrak{g}_{\mathcal{P}}),$$

where similarly we maintain the same notation for $\mathsf{Im}\,(\delta) \subset \Lambda^2 V^* \otimes V$ and the corresponding sub-bundle induced in $\mathsf{Tor}(M)$. It is not hard to see that the projection of the torsion T^∇ via p to this quotient bundle is the same for all G-connections ∇, and it only depends on the specific G-structure. Hence, one obtains a well-defined section $\tau := p(T^\nabla) \in \Gamma(\mathscr{H}(\mathfrak{g}))$ of $\mathscr{H}(\mathfrak{g})$, which is an invariant of 1st-order structures with structure group G, called the intrinsic torsion of $\mathcal{P}$.

Remark 4.2 For a given G-structure $\mathcal{P}$ on M, the intrinsic torsion measures the obstruction to the existence of adapted torsion-free connections. In particular, it vanishes if and only if M admits a torsion-free adapted connection, i.e., $\mathcal{P}$ is a torsion-free G-structure.

Suppose now that there exists a G-invariant complementary space $\mathcal{D} = \mathcal{D}(\mathfrak{g})$ of $\mathsf{Im}\,(\delta)$ inside $\Lambda^2 V^* \otimes V$ which gives rise to the direct sum decomposition

$$\Lambda^2 V^* \otimes V = \mathsf{Im}\,(\delta) \oplus \mathcal{D}(\mathfrak{g}). \tag{4.1}$$

Usually, one refers to $\mathcal{D}(\mathfrak{g})$ as a normalization condition, and it is easy to prove that such a normalization condition always exists for reductive $G \subset \mathsf{GL}(V)$. However, in general there is neither any natural way of fixing such a complement, nor is it necessarily unique.

A normalization condition $\mathcal{D}(\mathfrak{g})$ for a given G-structure on M determines a smooth subbundle of $\mathsf{Tor}(M)$, which we denote by $\mathscr{D}(\mathfrak{g})$. This is isomorphic with the associated vector bundle $\mathcal{P} \times_G \mathcal{D}(\mathfrak{g})$. Then, the splitting (4.1) induces the following bundle decomposition

$$\mathsf{Tor}(M) = \mathsf{Im}(\delta) \oplus \mathscr{D}(\mathfrak{g}).$$

In this case, a G-connection $\nabla = \nabla^\gamma$ is called minimal with respect to the normalization condition $\mathcal{D}(\mathfrak{g})$ (or simply a $\mathcal{D}$-connection if there is no matter of confusion), if T^∇ is smooth section of $\mathscr{D}(\mathfrak{g})$, i.e., its torsion function t^γ takes values in $\mathcal{D}(\mathfrak{g})$. Moreover, one can show that

Corollary 4.3 *Let $\pi : \mathcal{P} \to M$ be a G-structure on a smooth manifold M, where $G \subset \mathsf{GL}(V)$ is a closed subgroup, and let $\mathcal{D}(\mathfrak{g})$ be a normalization condition. Then, the space of all $\mathcal{D}$-connections is an affine space modeled on smooth sections of the associated bundle $\mathcal{P} \times_G \mathfrak{g}^{(1)}$. Hence, whenever the first prolongation $\mathfrak{g}^{(1)}$ is trivial, then the principal G-bundle $\mathcal{P} \subset \mathcal{F}$ admits a unique (up to choice of a normalization condition) minimal connection.*

4.2 Adapted connections to $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures

For the remainder of this section, we shall discuss adapted connections to $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures. It is well known which linear connections preserve an almost symplectic, an almost hypercomplex, or an almost quaternionic structure, separately, see [4, 42] and the references therein, and see also below. Given an almost hs-H manifold (M, H, ω), we want to specify a linear connection $\nabla^{H,\omega}$ on M which preserves the pair (H, ω), that is

$$\nabla^{H,\omega}\omega = 0, \qquad \nabla^{H,\omega}H = 0.$$

It is easy to prove that these conditions are equivalent to the following relations

$$\begin{aligned}\nabla^{H,\omega}_X\omega(Y,Z) &= \omega(\nabla^{H,\omega}_X Y, Z) + \omega(Y, \nabla^{H,\omega}_X Z),\\ \nabla^{H,\omega}_X J_a(Y) &= J_a(\nabla^{H,\omega}_X Y), \quad a = 1,2,3,\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, where $H = \{J_a : a = 1,2,3\}$. Such a connection is a $\mathsf{SO}^*(2n)$-connection in the above terms, and one may refer to $\nabla^{H,\omega}$ by the term almost hypercomplex skew-Hermitian connection. When $\nabla^{H,\omega}$ is torsion free, then it is said to be a hypercomplex skew-Hermitian connection.

Similarly, given an almost qs-H manifold (M, Q, ω), we want to specify a linear connection $\nabla^{Q,\omega}$ on M which preserves the pair (Q, ω), that is

$$\nabla^{Q,\omega}\omega = 0, \quad \text{and} \quad \nabla^{Q,\omega}_X\sigma \in \Gamma(Q),$$

for any smooth vector field $X \in \Gamma(TM)$ and smooth section $\sigma \in \Gamma(Q)$. Here, the second condition is equivalent to say that (see for example [4])

$$\nabla^{Q,\omega}_X J_a = \varphi_c(X)J_b - \varphi_b(X)J_c\,, \quad \forall\, X \in \Gamma(TM),\ a = 1,2,3\,,$$

for any cyclic permutation (a, b, c) of (1, 2, 3), where $\{J_a : a = 1,2,3\}$ is a local admissible basis of Q and φ_a are local 1-forms for any $a = 1,2,3$. Such a connection is a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-connection and one may call $\nabla^{Q,\omega}$ an almost quaternionic skew-Hermitian connection. When $\nabla^{Q,\omega}$ is torsion free, then it is said to be a quaternionic skew-Hermitian connection.

For the above goal, it is convenient to start with a *unique* connection that preserves part of the structure and modify it, to preserve all of it. The other connections differ from this connection by an endomorphism valued 1-form with values in $\mathfrak{so}^*(2n)$, and $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$, respectively. We should mention that on any almost hs-H manifold (M, H, ω) or any almost qs-H manifold (M^{4n}, Q, ω) we may define an orientation induced by the scalar 2-form ω. This is given by the globally defined volume form

$$\mathsf{vol} := \omega^{2n} := \underbrace{\omega \wedge \cdots \wedge \omega}_{2n-\text{times}}\,.$$

Hence, we get the following as an immediate corollary.

Corollary 4.4 *Any* almost hs-H/qs-H structure *is a* unimodular structure *in terms of* [4].

We shall make use of this corollary especially for $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, because among all the Oproiu connections ∇^Q there is a *unique* unimodular Oproiu connection $\nabla^{Q,\mathsf{vol}}$ associated to the pair (Q, vol), see below for more details.

A further goal is to determine explicitly normalization conditions which establish $\nabla^{H,\omega}, \nabla^{Q,\omega}$ as minimal connections. For this task, it is useful to proceed with a detailed description of the torsion corresponding to such structures, and in particular of their intrinsic torsion, which we present in Sect. 4.3, while minimal connections are presented in Sect. 5.

We begin with details about the first prolongation of $\mathsf{SO}^*(2n)$ and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$. It is known by results of Cartan and others (see for example [8, p. 113] and [4]), that for $\mathfrak{g} = \mathfrak{so}^*(2n)$, $\mathfrak{g} = \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{H})$ and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1)$ the first prolongation $\mathfrak{g}^{(1)}$ is trivial, $\mathfrak{g}^{(1)} = \{0\}$. However, next we demonstrate this result for $\mathfrak{so}^*(2n)$. In particular, we will provide a proof of the statement $\ker(\delta) = \mathfrak{so}^*(2n)^{(1)} = \{0\}$, based explicitly on the geometric properties of the tensor fields defining a $\mathsf{SO}^*(2n)$-structure, proving in this way also compatibility of our new definitions with previous considerations by other authors.

Lemma 4.5 *Let* (H, ω) *be an* almost hs-H structure, *or let* (Q, ω) *be an* almost qs-H structure *on a* $4n$*-dimensional manifold* M. *Let* $\mathfrak{g}$ *be one of the Lie algebras* $\mathfrak{so}^*(2n)$ *or* $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$. *Then, the Spencer operator of alternation*

$$\delta : \mathsf{Hom}([\mathsf{EH}], \mathfrak{g}) = [\mathsf{EH}]^* \otimes \mathfrak{g} \to \mathsf{Hom}(\Lambda^2[\mathsf{EH}], [\mathsf{EH}]) = \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$$

is injective, $\mathfrak{g}^{(1)} = \ker(\delta) = \{0\}$.

Proof As it is mentioned above, we shall prove the statement for $\mathfrak{g} = \mathfrak{so}^*(2n)$ only. Of course, the vanishing of $\mathfrak{g}^{(1)}$ is a direct consequence of the embedding of $\mathfrak{g} = \mathfrak{so}^*(2n)$ in $\mathfrak{so}(p, q)$ in combination with the relation $\mathfrak{so}(p, q)^{(1)} = \{0\}$, see [8, p. 113]. To provide an alternative proof, let $\alpha : [\mathsf{EH}] \to \mathfrak{so}^*(2n)$ belonging to the kernel of δ, that is $\alpha_X Y = \alpha_Y X$. Since $\alpha_X \in \mathfrak{so}^*(2n)$, we have

$$\alpha_{J_a X} Y = J_a(\alpha_X Y), \quad \text{for any } J_a \in H, \ (a = 1, 2, 3), \tag{4.2}$$

$$\omega(\alpha_X Y, Z) = \omega(Y, \alpha_X Z), \tag{4.3}$$

for any three vectors $X, Y, Z \in [\mathsf{EH}]$. Since $\alpha_X Y = \alpha_Y X$ for any X, Y, by (4.2) we obtain

$$J_a(\alpha_Y X) = J_a(\alpha_X Y) = \alpha_{J_a X} Y = \alpha_Y(J_a X), \quad \forall\, a = 1, 2, 3,$$

or in other words (4.2) is equivalent to $\alpha_X(J_a Y) = J_a(\alpha_X Y)$, for any $a = 1, 2, 3$, and $X, Y \in [\mathsf{EH}]$. Moreover, for any triple $I, J, K = IJ$, where $I, J \in H$ are two anticommuting almost complex structures, we infer that $I(\alpha_X JY) = \alpha_{IX}(JY) = \alpha_{JY}(IX)$. To see this, in the relation $\alpha_{IX} Y = I\alpha_X Y$, replace X by IX and Y by JY, and next multiply both sides of the relation by I. By combining these relations, it is now easy to prove that $K(\alpha_X Y) = 0$, which obviously implies the vanishing of α. On the other hand, for some $\alpha \in \ker(\delta)$, by Proposition 2.10 and the non-degeneracy of ω, we also see that (4.3) is equivalent to the condition $\alpha_Z(J_a Y) = \alpha_Y(J_a Z)$, or equivalent to $\alpha_{J_a Y} Z = \alpha_Y(J_a Z)$ for any $a = 1, 2, 3$, and hence it coincides with the first condition (4.2). □

Lemma 4.5 has several direct but important consequences, which we summarize in a corollary.

Corollary 4.6 *Let M be a $4n$-dimensional connected smooth manifold. Then,*

(1) *An adapted connection ∇ to a $\mathsf{SO}^*(2n)$-structure, or to a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure on M, or, respectively, a $\mathsf{SO}^*(2n)$- or a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-connection, is entirely determined by its torsion T^{∇}.*

(2) *A torsion-free $\mathsf{SO}^*(2n)$-connection, or $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-connection, if any, is unique.*

(3) *Let $\mathfrak{g}$ be one of the Lie algebras $\mathfrak{so}^*(2n)$ or $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$, and let $\mathcal{D} = \mathcal{D}(\mathfrak{g})$ be a normalization condition related to such a G-structure on M, where G denotes the connected Lie group corresponding to $\mathfrak{g}$. Then, a $\mathcal{D}$-connection, or in other words a minimal connection of such a G-structure on M with respect to the normalization condition $\mathcal{D}(\mathfrak{g})$, is unique.*

We should emphasize on the fact that similarly to the case of almost hypercomplex structures ([23]), such normalization conditions $\mathcal{D}(\mathfrak{g})$ are not unique (due to multiplicities of the involved representations). This provides a certain difficulty to the description of the (local) geometry associated to $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures on $4n$-dimensional smooth manifolds.

Let $(M, H = \{J_a\}, \omega)$ be an almost hs-H manifold. By Obata [33], it is known that there is a *unique minimal* affine connection ∇^H with respect to a certain normalization condition, preserving the almost hypercomplex structure $H = \{J_a : a = 1, 2, 3\}$, that is $\nabla^H J_a = 0$ for any $a = 1, 2, 3$. We will refer to this connection as the Obata connection.

Proposition 4.7 *Let* $\nabla_X Y = \nabla^H_X Y + A_X Y = \nabla^H_X Y + A(X,Y)$ *be a linear connection on an* almost hs-H manifold (M,H,ω), *where* ∇^H *is the Obata connection and* A *is a smooth tensor field on* M *of type* $(1,2)$. *Then,* ∇ *satisfies the conditions*

$$\nabla\omega = 0\,, \quad \text{and} \quad \nabla J_a = 0\,, \quad \forall\, a = 1,2,3\,,$$

if and only if the following two relations hold for any $X,Y,Z \in \Gamma(TM)$:

$$(\nabla^H_X\omega)(Y,Z) = \omega\big(A(X,Y),Z\big) + \omega\big(Y,A(X,Z)\big)\,, \tag{4.4}$$

$$A(X,J_aY) = J_a\big(A(X,Y)\big)\,,\ \forall\, a = 1,2,3\,. \tag{4.5}$$

In particular, if $\nabla^{H,\omega}$ *is an almost hypercomplex skew-Hermitian connection and* A *is a tensor field on* M *of type* $(1,2)$, *then* $\nabla^{H,\omega,A}_X Y = \nabla^{H,\omega}_X Y + A(X,Y)$ *is an almost hypercomplex skew-Hermitian connection, if and only if* A *has values in* $[\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n)$.

Proof The proof is easy and left to the reader. □

Let us now find some particular A depending only on the Obata connection ∇^H and the almost symplectic form ω, to define the connection $\nabla^{H,\omega}$.

Theorem 4.8 *Let* $(M, H = \{J_a : a = 1,2,3\}, \omega)$ *be an* almost hs-H manifold *endowed with the Obata connection* ∇^H. *Then, the connection* $\nabla^{H,\omega} := \nabla^H + A$, *where the tensor field* A *of type* $(1,2)$ *is defined by*

$$\omega\big(A(X,Y),Z\big) = \frac{1}{2}(\nabla^H_X\omega)(Y,Z)$$

for any $X,Y,Z \in \Gamma(TM)$, *is an almost hypercomplex skew-Hermitian connection. In particular, the tensor* $\omega\big(A(\cdot\,,\cdot),\cdot\big)$ *of type* $(0,3)$ *takes values in* $[\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*$. *Moreover, the torsion of* $\nabla^{H,\omega}$ *takes the form* $T^H + \delta(A)$, *where* T^H *is the torsion of* ∇^H.

Proof When A is defined as above, then the condition (4.4) is satisfied, hence $\nabla^{H,\omega}\omega = 0$. We will show that also $\nabla^{H,\omega}J_a = 0$ for any $a = 1,2,3$. By Proposition 4.10 this is equivalent to the relation (4.5). Notice that after applying ω on (4.5) it yields the relation

$$\omega\big(A(X,J_aY),Z\big) = \omega\big(J_a\big(A(X,Y)\big),Z\big) = -\omega\big(A(X,Y),J_aZ\big)\,, \quad \forall X,Y,Z \in \Gamma(TM)\,,$$

where the second equality occurs via the identity

$$\omega(J_aX,Y) + \omega(X,J_aY) = 0, \quad \forall X,Y \in \Gamma(TM)\,,$$

see Proposition 2.10. In particular, it turns out that the relation

$$\omega\big(A(X,J_aY),Z\big) + \omega\big(A(X,Y),J_aZ\big) = 0\,, \quad \forall X,Y,Z \in \Gamma(TM)\,,$$

is equivalent to say that the tensor $\omega\big(A(\cdot\,,\cdot),\cdot\big)$ of type $(0,3)$ is H-Hermitian with respect to the last two indices. However, the Obata connection ∇^H is a $\mathsf{GL}\,(n,\mathbb{H})$-connection, and the space $[S^2\,\mathsf{E}\,]^*$ of scalar 2-forms is a $\mathsf{GL}\,(n,\mathbb{H})$-submodule of the 2-tensors that are H-Hermitian. Hence, $\omega\big(A(\cdot\,,\cdot),\cdot\big)$ takes values in $[\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*$, and we conclude that the claim holds. □

Proposition 4.9 *An almost hypercomplex skew-Hermitian connection* ∇ *on an* almost hs-H manifold $(M, H = \{I, J, K\}, \omega)$ *satisfies*,

$$\nabla g_I = \nabla g_J = \nabla g_K = \nabla h = \nabla \Phi = 0\,.$$

Hence, it is a metric connection with respect to any of the three pseudo-Riemannian metrics g_I, g_J, g_K *and preserves the quaternionic skew-Hermitian form h and the fundamental* 4-*tensor* Φ.

Proof Consider for example g_I. It coincides with a contraction of the composition of two ∇ -parallel tensor fields, namely ω and I. Hence, g_I must be parallel, which also occurs by a straightforward computation. The other claims are treated similarly. □

We now proceed with adapted connections to $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-structures. Given an almost qs-H manifold (M^{4n}, Q, ω) with $n > 1$, there exists a class of Oproiu connections ∇^Q preserving Q, that is $\nabla^Q_X \sigma \in \Gamma(Q)$ for all $X \in \Gamma(TM)$ and all smooth sections $\sigma \in \Gamma(Q)$. Moreover, there is a unique Oproiu connection $\nabla^{Q,\,\mathsf{vol}}$ preserving Q and the volume form $\mathsf{vol} = \omega^{2n}$ induced by ω, that is

$$\nabla^{Q,\,\mathsf{vol}}\,\mathsf{vol} = 0\,, \quad \nabla^{Q,\,\mathsf{vol}}_X \sigma \in \Gamma(Q), \quad \forall\, X \in \Gamma(TM), \quad \forall\, \sigma \in \Gamma(Q)\,.$$

This connection is the so-called unimodular Oproiu connection for the pair (Q, vol). Recall that an Oproiu connection for an almost quaternionic structure Q is a minimal connection for Q, see the seminal works of Oproiu [31, 32] and see also [4] for more details on $\nabla^{Q,\,\mathsf{vol}}$.

Proposition 4.10 *Let* $\nabla_X Y = \nabla^Q_X Y + A_X Y = \nabla^Q_X Y + A(X, Y)$ *be a linear connection on an* almost qs-H manifold (M, Q, ω), *where* ∇^Q *is any Oproiu connection, and A is a smooth tensor field on M of type* (1, 2). *Then,* ∇ *satisfies the conditions*

$$\nabla\omega = 0, \quad \text{and} \quad \nabla_X \sigma \in \Gamma(Q),$$

for any vector field $X \in \Gamma(TM)$ *and section* $\sigma \in \Gamma(Q)$, *if and only if any* $X, Y, Z \in \Gamma(TM)$ *satisfy the following two relations*

$$(\nabla^Q_X \omega)(Y, Z) = \omega\big(A(X, Y), Z\big) + \omega\big(Y, A(X, Z)\big)\,, \tag{4.6}$$

$$A(X, J_a Y) - J_a\big(A(X, Y)\big) = \varphi^A_c(X) J_b(Y) - \varphi^A_b(X) J_c(Y)\,, \tag{4.7}$$

for any cyclic permutation $(a,\, b,\, c)$ *of* $(1,\, 2,\, 3)$, *any admissible basis* $H = \{J_a\}$ *and some* (*local*) 1-*forms* φ^A_a *for any* $a = 1, 2, 3$.

In particular, if $\nabla^{Q,\omega}$ *is an almost quaternionic skew-Hermitian connection and A is a smooth tensor field on M of type* (1, 2), *then* $\nabla^{Q,\omega,A}_X Y = \nabla^{Q,\omega}_X Y + A(X, Y)$ *is an almost quaternionic skew-Hermitian connection, if and only if A takes values in* $[\mathsf{EH}]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$.

Proof The claim follows since only the part of A belonging to $[\mathsf{EH}]^* \otimes \mathfrak{sp}(1)$ acts non-trivially on the local admissible basis $\{J_a : a = 1, 2, 3\}$, and at the same time preserves

$\{J_a : a = 1,2,3\}$ in $\Gamma(Q)$. This provides the stated formula (4.7), and we leave the further details to the reader. □

Next we will construct an almost quaternionic skew-Hermitian connection, as in Theorem 4.8, by using a tensor field A of type (1, 2) defined via the relation

$$\omega\big(A(X,Y),Z\big) = \frac{1}{2}(\nabla^Q_X\omega)(Y,Z), \quad X,Y,Z \in \Gamma(TM). \tag{4.8}$$

To do so, we benefit from the fact that the space $[S^2\,\mathsf{E}\,]^*$ of scalar 2-forms is a $\mathsf{GL}\,(n,\mathbb{H})\,\mathsf{Sp}\,(1)$-module and ∇^Q is an $\mathsf{GL}\,(n,\mathbb{H})\,\mathsf{Sp}\,(1)$-connection. Thus, again $\omega\big(A(\cdot\,,\cdot),\cdot\big)$ has values in $[\mathsf{EH}]^*\otimes[S^2\,\mathsf{E}\,]^*$, and consequently, the conditions (4.6) and (4.7) must be satisfied by similar arguments as in the proof of Theorem 4.8. However, to complete our construction, we need to overcome the following

Challenge: *Although A is determined uniquely by* (4.8), *it depends on the choice of Oproiu connection and thus the space spanned by δA is not complementary to* $\delta([\mathsf{EH}]^*\otimes(\mathfrak{so}^*(2n)\oplus\mathfrak{sp}(1))$.

For a better visualization of this problem, we need to consider the following four components isomorphic to $[\mathsf{E\,H}]^*$ (see also [4, p. 215] but be aware of slightly different conventions):

(A) The component spanned by $\zeta\otimes\mathrm{id}$ for $\zeta\in\Gamma(T^*M)$, with values in $[\mathsf{EH}]^*\otimes[S^2\,\mathsf{E}\,]^*\subset[\mathsf{EH}]^*\otimes\mathfrak{gl}(n,\mathbb{H})$.

(B) The component spanned by the projection

$$\pi_A(\omega\otimes Z)(X,Y) := Asym\Big(\pi_{1,1}(\omega(X,.)\otimes Z)\Big)Y = \frac{1}{8}\Big(\omega(X,Y)Z-\omega(X,Z)Y - \sum_a g_{J_a}(X,Y)J_aZ + \sum_a g_{J_a}(X,Z)J_aY\Big)$$

of $\omega\otimes Z$ for $X,Y,Z\in\Gamma(TM)$, with values in $[\mathsf{EH}]^*\otimes[S^2\,\mathsf{E}\,]^*$, where $\pi_{1,1}$ is the usual projection (see for example [4, p. 214] or [13, p. 395])

$$\pi_{1,1} : \mathfrak{gl}([\mathsf{EH}])\to\mathfrak{gl}(n,\mathbb{H}), \quad \pi_{1,1}(\omega(X,.)\otimes Z)Y := \frac{1}{4}\Big(\omega(X,Y)Z-\sum_a g_{J_a}(X,Y)J_aZ\Big).$$

Here, *Asym* denotes the antisymmetrization with respect to the symplectic transpose (see Definition 3.2).

(C) The component spanned by the projection

$$\pi_S(\omega\otimes Z)(X,Y) := Sym\Big(\pi_{1,1}(\omega(X,.)\otimes Z)\Big)Y = \frac{1}{8}\Big(\omega(X,Y)Z+\omega(X,Z)Y - \sum_a g_{J_a}(X,Y)J_aZ - \sum_a g_{J_a}(X,Z)J_aY\Big)$$

of $\omega\otimes Z$ for $X,Y,Z\in\Gamma(TM)$, with values in $[\mathsf{EH}]^*\otimes\mathfrak{so}^*(2n)$, where *Sym* denotes the symmetrization with respect to the symplectic transpose (see Definition 3.2).

(D) The component locally spanned by $\sum_{a=1}^3\zeta\circ J_a\otimes J_a$ with values in $[\mathsf{EH}]^*\otimes\mathfrak{sp}(1)$ for some $\zeta\in\Gamma(T^*M)$ and some local admissible basis $H=\{J_1,J_2,J_3\}$.

We also need to consider the following traces $\mathsf{Tr}_i : \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] \to [\mathsf{EH}]^*$ for $i = 1, \dots, 4$:

(i) $\mathsf{Tr}_1(A)(X) := \mathsf{Tr}(A(\cdot, X))$;
(ii) $\mathsf{Tr}_2(A)(X) := \mathsf{Tr}(A(X, \cdot))$;
(iii) $\mathsf{Tr}_3(A)(X) := \mathsf{Tr}(A_X^T)$, where A_X^T is the symplectic transpose of $A_X := A(X, \cdot)$;
(iv) $\mathsf{Tr}_4(A)(X) := \mathsf{Tr}(\mathsf{J}A(\mathsf{J}X, \cdot))$, for $\mathsf{J} \in \mathsf{S}(Q)$.

Clearly, the components (A) and (B) are parts of the tensor A, and the components (C) and (D) are in $[\mathsf{EH}]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$. Hence, to assert our claim, it suffices to show that δ is *not* injective on these four components.

Lemma 4.11 *Let us set $X^T(Y) := \omega(X, Y)$ for any $X, Y \in \Gamma(TM)$ and consider the tensor field*

$$A := \zeta_1 \otimes \mathsf{id} + \pi_A(\omega \otimes Z_2) + \pi_S(\omega \otimes Z_3) + \sum_{a=1}^{3} \zeta_4 \circ J_a \otimes J_a$$

for $\zeta_1, \zeta_4 \in \Gamma(T^*M)$ and $Z_2, Z_3 \in \Gamma(TM)$, given by the above components (A), (B), (C) *and* (D). *Then, the traces $\mathsf{Tr}_i(A) \in \Gamma(T^*M)$ for $i = 1, \dots, 4$ do not depend on the choice of* $\mathsf{J} \in \Gamma(Z)$, *and moreover, the following holds*:

α) $\delta(A) = 0$, if and only if $\zeta_1 = -\zeta_4 = -\frac{1}{4}Z_2^T = \frac{1}{4}Z_3^T$.

β) $\omega(\delta(A), \cdot) \in \Gamma(\Lambda^3 T^*M)$, if and only if

$$\mathsf{Tr}_1(A) + \mathsf{Tr}_3(A) = 0, \quad \text{and} \quad \mathsf{Tr}_4(A) = 0,$$

which is equivalent to say that $Z_3^T = -\frac{1}{3}Z_2^T + \frac{8}{3}\zeta_1$ and $\zeta_4 = \frac{1}{6}Z_2^T - \frac{1}{3}\zeta_1$.

γ) $A \in \Gamma([\mathsf{EH}]^* \otimes (\mathfrak{sl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1)))$, if and only if $\mathsf{Tr}_2(A)$ vanishes, which is equivalent to say that $Z_2^T = 4n\zeta_1$.

Proof We can directly compute the traces of A and obtain the following matrix equality, which is independent of the choice of $\mathsf{J} \in \Gamma(Z)$:

$$\begin{pmatrix} \mathsf{Tr}_1(A) \\ \mathsf{Tr}_2(A) \\ \mathsf{Tr}_3(A) \\ \mathsf{Tr}_4(A) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2n+1}{4} & \frac{2n-1}{4} & -3 \\ 4n & -1 & 0 & 0 \\ -1 & \frac{2n+1}{4} & \frac{2n-1}{4} & -3 \\ 0 & 0 & 0 & 4n \end{pmatrix} \begin{pmatrix} \zeta_1 \\ Z_2^T \\ Z_3^T \\ \zeta_4 \end{pmatrix}.$$

Note that the determinant of the matrix is $2n(n+1)(2n-1)^2$, hence the matrix is invertible. Moreover, $\mathsf{Tr}_1(\delta(A)) = -\mathsf{Tr}_2(\delta(A))$ and we obtain

$$\begin{pmatrix} \mathsf{Tr}_1(\delta(A)) \\ \mathsf{Tr}_3(\delta(A)) \\ \mathsf{Tr}_4(\delta(A)) \end{pmatrix} = \begin{pmatrix} 1-4n & -\frac{2n-3}{4} & \frac{2n-1}{4} & -3 \\ -2 & \frac{2n+1}{2} & \frac{2n-1}{2} & -6 \\ 1 & -\frac{2n+1}{4} & \frac{2n-1}{4} & 4n+1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ Z_2^T \\ Z_3^T \\ \zeta_4 \end{pmatrix}, \tag{4.9}$$

which provides the claimed kernel of δ. On the other hand, on an element ϕ of $\Gamma(\Lambda^3 T^*M)$ given by the complete antisymmetrization of $2\omega \otimes \zeta$ we see that

$$\mathsf{Tr}_1(\phi) = -\frac{4}{3}(2n-1)\zeta\,, \quad \mathsf{Tr}_3(\phi) = \frac{4}{3}(2n-1)\zeta\,, \quad \mathsf{Tr}_4(\phi) = 0\,.$$

In this way, we obtain the claimed condition for $\omega(\delta(A),\cdot) \in \Gamma(\Lambda^3 T^*M)$. Since $\pi_S(\omega \otimes Z_3) + \sum_{a=1}^3 \zeta_4 \circ J_a \otimes J_a$ has values in $[\mathsf{EH}]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$, we need to characterize when $\zeta_1 \otimes \mathsf{id} + \pi_A(\omega \otimes Z_2)$ has values in $[\mathsf{EH}]^* \otimes (\mathfrak{sl}(n,\mathbb{H}) \oplus \mathfrak{sp}(1))$, which is encoded by the vanishing of Tr_2. This is because $A \cdot \mathsf{vol} = \mathsf{Tr}_2(A)\,\mathsf{vol}$. This completes the proof. □

Now, we are able to define the connection $\nabla^{Q,\omega}$ explicitly.

Theorem 4.12 *Let (M,ω,Q) be an* almost qs-H manifold *endowed with any Oproiu connection ∇^Q, and let us denote its torsion by T^Q. Let A be the $(1,2)$-tensor field on M defined by*

$$\omega\big(A(X,Y),Z\big) = \frac{1}{2}(\nabla^Q_X\omega)(Y,Z)\,, \quad \forall\, X,Y,Z \in \Gamma(TM)\,,$$

and set $Z_3^T := \frac{\mathsf{Tr}_2(A)}{n+1}$, $\zeta_4 := -\frac{\mathsf{Tr}_2(A)}{4(n+1)}$. Then, the connection

$$\nabla^{Q,\omega} := \nabla^Q + A + \pi_S(\omega \otimes Z_3) + \sum_{a=1}^3 \zeta_4 \circ J_a \otimes J_a$$

is an almost quaternionic skew-Hermitian connection with the following property: The only component of its torsion $T^{Q,\omega} = T^Q + \delta(A + \pi_S(\omega \otimes Z_3) + \sum_{a=1}^3 \zeta_4 \circ J_a \otimes J_a)$ isomorphic to $[\mathsf{EH}]^*$, *is contained in* $\mathsf{Ker}(2\,\mathsf{Tr}_1 + \mathsf{Tr}_3) \cap \mathsf{Ker}(\mathsf{Tr}_1 - \mathsf{Tr}_4) \subset \Gamma(\mathsf{Tor}(M))$. *In particular, for the unimodular Oproiu connection $\nabla^{Q,\mathsf{vol}}$ and a tensor A^{vol} defined by $\omega\big(A^{\mathsf{vol}}(X,Y),Z\big) = \frac{1}{2}(\nabla^{Q,\mathsf{vol}}_X\omega)(Y,Z)$, we obtain*

$$\nabla^{Q,\omega} = \nabla^{Q,\mathsf{vol}} + A^{\mathsf{vol}}\,.$$

Proof We know that $\omega\big(A(\cdot\,,\cdot),\cdot\big)$ has values in $[\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*$, while $\pi_S(\omega \otimes Z_3) + \sum_{a=1}^3 \zeta_4 \circ J_a \otimes J_a$ has values in $[\mathsf{EH}]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$. Thus, since A satisfies the condition (4.6) by definition, it follows that $A + \pi_S(\omega \otimes Z_3) + \sum_{a=1}^3 \zeta_4 \circ J_a \otimes J_a$ will satisfy the same relation. Moreover, for the same reasons as in Theorem 4.8 we conclude that the relation (4.7) is valid for the tensor field $A + \pi_S(\omega \otimes Z_3) + \sum_{a=1}^3 \zeta_4 \circ J_a \otimes J_a$. Hence, Proposition 4.10 is satisfied, and consequently $\nabla^{Q,\omega}$ is an almost quaternionic skew-Hermitian connection. Next, we need to show that $\nabla^{Q,\omega}$ does not depend on the choice of ∇^Q. By Lemma 4.5 and Corollary 4.6, it suffices to show that $T^{Q,\omega}$ does not depend on the choice of ∇^Q. However, it is well known that all Oproiu connections have the same torsion T^Q and hence $T^{Q,\omega} = T^Q + \delta\big(A + \pi_S(\omega \otimes Z_3) + \sum_{a=1}^3 \zeta_4 \circ J_a \otimes J_a\big)$. Since the difference of two Oproiu connections belongs to the kernel $\mathsf{Ker}(\delta)$ described in Lemma 4.11, we conclude that such an element in the kernel of δ will change A by $-\zeta_1 \otimes \mathsf{id} + \pi_A(\omega \otimes 4\zeta_1^T)$ for some 1-form ζ_1. We compute

$$\mathsf{Tr}_2(-\zeta_1 \otimes \mathsf{id} + \pi_A(\omega \otimes 4\zeta_1^T)) = -4(n+1)\zeta_1\,,$$

and hence we conclude that Z_3 will change by $-4\zeta_1^T$ and ζ_4 will change by ζ_1. Altogether, this change takes the form $-\zeta_1 \otimes \mathsf{id} + \pi_A(\omega \otimes 4\zeta_1^T) - 4\pi_S(\omega \otimes \zeta_1^T) + \sum_{a=1}^3 \zeta_1 \circ J_a \otimes J_a$, and hence by Lemma 4.11 we deduce that it belongs to $\mathsf{Ker}(\delta)$. This shows that $T^{Q,\omega}$ is independent of the Oproiu connection. Now, by using the formula (4.9) we deduce that the component of the torsion $T^{Q,\omega}$ isomorphic to $[\mathsf{EH}]^*$ belongs to $\mathsf{Ker}(2\,\mathsf{Tr}_1 + \mathsf{Tr}_3) \cap \mathsf{Ker}(\mathsf{Tr}_1 - \mathsf{Tr}_4)$. Finally, since the unimodular Oproiu connection is a $\mathsf{SL}(n, \mathbb{H})\,\mathsf{Sp}(1)$-connection, by Lemma 4.11 the corresponding tensor field A^{vol} satisfies $\mathsf{Tr}_2(A^{\mathsf{vol}}) = 0$ and the last claim follows, because then we have $Z_3^T = \zeta_4 = 0$. □

Proposition 4.13 *An almost quaternionic skew-Hermitian connection* ∇ *on an* almost qs-H manifold (M, Q, ω) *satisfies*,

$$\nabla g_{J_a} = \varphi_c \otimes g_{J_b} - \varphi_b \otimes g_{J_c}\,,$$
$$\nabla h = \nabla \Phi = 0\,,$$

for any cyclic permutation (a, b, c) *of* $(1, 2, 3)$ *and some local* 1*-forms* φ_a $(a = 1, 2, 3)$ *on* M, *depending on a local admissible frame* $\{J_1, J_2, J_3\}$ *of* Q. *Hence*, ∇ *is not* (*necessarily*) *a metric connection with respect to any of the three pseudo-Riemannian metrics* g_{J_a} *for* $a = 1, 2, 3$, *but it preserves the quaternionic skew-Hermitian form* h *and the fundamental* 4*-tensor* Φ.

Proof Since ω is ∇-parallel, the covariant derivatives ∇g_{J_a} are induced by the covariant derivatives of the local admissible frame, which are given by the claimed action of elements of $[\,\mathsf{E}\;\mathsf{H}\,]^* \otimes \mathfrak{sp}(1)$. Since the Lie algebra $\mathfrak{sp}(1)$ acts trivially on h and Φ, we get the other claims directly. □

Let us consider the adapted connections $\nabla^{H,\omega}$ and $\nabla^{Q,\omega}$ constructed in Theorems 4.8 and 4.12, respectively. Due to Propositions 4.9 and 4.13, it makes sense to consider the covariant derivatives $\nabla^H \Phi$ and $\nabla^Q \Phi$, respectively. It follows that these covariant derivatives take values in modules which are naturally isomorphic to submodules of the intrinsic torsion corresponding to $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures on a $4n$-dimensional manifold M, respectively. These modules are related to 1st-order integrability conditions which we analyze in detail in the second part of this work, together with $T^{Q,\omega}$ (see Sections 1 and 2 in [19]). The covariant derivative $\nabla^\omega \Phi$, where ∇^ω is any almost symplectic connection on M, can be used similarly, while obviously a similar idea can be carried out via the quaternionic skew-Hermitian form h and the covariant derivatives $\nabla^H h$, $\nabla^{Q,\mathsf{vol}} h$, and $\nabla^\omega h$, respectively.

4.3 Decomposition of the space of torsion tensors and intrinsic torsion

Next we present the decomposition of the module $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$ into submodules with respect to $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$- and $\mathsf{SO}^*(2n)$-action, respectively.

Proposition 4.14 *Let* (M, Q, ω) *be an* almost qs-H manifold. *Then*, *the following* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-equivariant decompositions hold* (*and should be read according to the conventions given in Sect*. 2.2):

$$\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] \cong [(\Lambda^3\mathsf{E} \oplus \mathsf{K} \oplus \mathsf{E}) \otimes S^3\mathsf{H}]^* \oplus [(\Lambda^3\mathsf{E} \oplus 2\mathsf{K} \oplus 3\mathsf{E} \oplus S^3_0\mathsf{E}) \otimes \mathsf{H}]^*,$$
$$\delta([\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n)) \cong [(\Lambda^3\mathsf{E} \oplus \mathsf{K} \oplus \mathsf{E}) \otimes \mathsf{H}]^*,$$
$$\delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(1)) \cong [\mathsf{E} \otimes (S^3\mathsf{H} \oplus \mathsf{H})]^*.$$

All the components in the decompositions are irreducible as $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-representations, with the exception of* K *for* $n = 2$ *and of* $\Lambda^3\mathsf{E}$ *for* $n = 3$. *Moreover, all these* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-modules are non-equivalent, apart from the stated multiplicities, and the isomorphism* $\Lambda^3\mathsf{E} \cong \mathsf{E}$ *for* $n = 2$.

Proof In terms of the EH-formalism, EH is the complex tensor product $\mathsf{E} \otimes_{\mathbb{C}} R(\theta)$, and it is irreducible as a complex $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$-representation. We now compute the decomposition of $\Lambda^2[\mathsf{EH}]^*$ via complexification (note that all $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$-modules that appear below have real type, so this is identical to the real decomposition). We obtain

$$\begin{aligned}\Lambda^2\mathsf{EH} &= \big(\Lambda^2\mathsf{E} \otimes S^2R(\theta)\big) \oplus \big(S^2\mathsf{E} \otimes \Lambda^2R(\theta)\big)\\ &= \big(\Lambda^2\mathsf{E} \otimes S^2\mathsf{H}\big) \oplus \big(S^2_0\mathsf{E} \oplus R(0)\big).\end{aligned}$$

As a consequence, we deduce that

$$\begin{aligned}\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] &= [\Lambda^2\mathsf{E} \otimes S^2\mathsf{H} \oplus S^2_0\mathsf{E} \oplus R(0)]^* \otimes [\mathsf{EH}]\\ &= [\big(\Lambda^2\mathsf{E} \otimes \mathsf{E}\big) \otimes \big(S^2R(\theta) \otimes R(\theta)\big)]^*\\ &\quad \oplus [\big(S^2_0\mathsf{E} \otimes \mathsf{E}\big) \otimes R(\theta)]^* \oplus [\mathsf{EH}]^*\\ &= [(\Lambda^3\mathsf{E} \oplus \mathsf{K} \oplus \mathsf{E}) \otimes S^3\mathsf{H}]^* \oplus [(\Lambda^3\mathsf{E} \oplus \mathsf{K} \oplus \mathsf{E}) \otimes \mathsf{H}]^*\\ &\quad \oplus [(S^3_0\mathsf{E} \oplus \mathsf{K} \oplus \mathsf{E}) \otimes \mathsf{H}]^* \oplus [\mathsf{EH}]^*.\end{aligned}$$

This gives rise to the first result. We also compute

$$[\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n) = [\mathsf{E} \otimes R(\theta) \otimes \Lambda^2\mathsf{E}]^* = [(\Lambda^3\mathsf{E} \oplus \mathsf{K} \oplus \mathsf{E}) \otimes \mathsf{H}]^*,$$
$$[\mathsf{EH}]^* \otimes \mathfrak{sp}(1) = [\mathsf{E} \otimes R(\theta) \otimes R(2\theta)]^* = [\mathsf{E} \otimes (R(3\theta) \oplus R(\theta))]^* = [\mathsf{E} \otimes (S^3\mathsf{H} \oplus \mathsf{H})]^*.$$

Since by Lemma 4.5 for $\mathfrak{g} = \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ or $\mathfrak{g} = \mathfrak{so}^*(2n)$, the Spencer differential δ is injective, our final statement easily follows by the above decomposition. □

As a corollary (in combination with the results in Table 1), we obtain the number of algebraic types depending on the irreducible intrinsic torsion modules of $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures on a $4n$-dimensional smooth manifold M ($n > 1$).

Corollary 4.15 *For* $n > 3$, *the intrinsic torsion module corresponding to* $\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)$ *admits the following* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-equivariant decomposition into irreducible submodules*:

$$\begin{aligned}\mathcal{H}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) &\cong \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \oplus \mathcal{X}_5\\ &= [\mathsf{K}S^3\mathsf{H}]^* \oplus [\Lambda^3\mathsf{E}S^3\mathsf{H}]^* \oplus [\mathsf{K}\,\mathsf{H}]^* \oplus [\mathsf{EH}]^* \oplus [S^3_0\mathsf{EH}]^*,\end{aligned}$$

where $\mathcal{X}_3 \oplus \mathcal{X}_4 = [\mathsf{K}\,\mathsf{H}]^* \oplus [\mathsf{EH}]^* \subset \Lambda^3[\mathsf{EH}]^*$. *Consequently, for* $n > 3$ *there exist five main types of* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-structures*, $\mathcal{X}_1, \ldots, \mathcal{X}_5$, *defined as above, and up to* $2^5 = 32$ *algebraic types of* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-geometries.*

- For $n = 3$, $\mathcal{X}_2$ decomposes into two irreducible $\mathsf{SO}^*(6)\,\mathsf{Sp}(1)$-modules, namely

$$\mathcal{X}_2 = \mathcal{X}_2^+ \oplus \mathcal{X}_2^- \cong [R(2\pi_2)S^3\,\mathsf{H}]^* \oplus [R(2\pi_3)S^3\,\mathsf{H}]^* . \tag{4.10}$$

- For $n = 2$, both $\mathcal{X}_1$ and $\mathcal{X}_3$ decompose into two irreducible $\mathsf{SO}^*(4)\,\mathsf{Sp}(1)$-modules, namely

$$\begin{cases} \mathcal{X}_1 = \mathcal{X}_1^+ \oplus \mathcal{X}_1^- \cong [R(\pi_1 + 3\pi_2)S^3\,\mathsf{H}]^* \oplus [R(3\pi_1 + \pi_2)S^3\,\mathsf{H}]^* , \\ \mathcal{X}_3 = \mathcal{X}_3^+ \oplus \mathcal{X}_3^- \cong [R(\pi_1 + 3\pi_2)\,\mathsf{H}]^* \oplus [R(3\pi_1 + \pi_2)\,\mathsf{H}]^* . \end{cases} \tag{4.11}$$

Consequently, there exist up to 2^6 algebraic types of $\mathsf{SO}^*(6)\,\mathsf{Sp}(1)$-geometries, and up to 2^7 algebraic types of $\mathsf{SO}^*(4)\,\mathsf{Sp}(1)$-geometries.

As it is customary to the theory of G-structures, we split the geometries into classes with respect to the algebraic types.

Definition 4.16 For $n > 3$, a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure is said to be of pure type $\mathcal{X}_i$ for $i = 1, \dots, 5$, if the intrinsic torsion takes values in the corresponding irreducible submodule. Similarly, we say that a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure is of mixed type $\mathcal{X}_{i_1 \dots i_j}$ for $1 \leq i_1 < \dots < i_j \leq 5$, if the intrinsic torsion takes values in the module $\mathcal{X}_{i_1} \oplus \cdots \oplus \mathcal{X}_{i_j}$. For instance, mixed type $\mathcal{X}_{135}$ means that the intrinsic torsion takes values in $\mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_5$. Similar notations are adapted for $n = 2, 3$.

Let us discuss now the case of $\mathsf{SO}^*(2n)$. Assume that W is some $\mathsf{SO}^*(2n)$-module. Then, we obtain the following branching from $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-modules to $\mathsf{SO}^*(2n)$-modules:

$$[W \otimes \mathsf{H}]^* \cong W^* , \quad [W \otimes S^3\,\mathsf{H}]^* \cong 2W^* .$$

Therefore, the following occurs as a simple corollary of Proposition 4.14, where for the low-dimensional cases we again rely on Table 1.

Corollary 4.17 1) *As* $\mathsf{SO}^*(2n)$*-modules*, $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$ *and the image* $\delta([\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n))$ *admit the following* $\mathsf{SO}^*(2n)$*-equivariant (irreducible for* $n > 3$*) decompositions*:

$$\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] \cong 3\Lambda^3\,\mathsf{E}^* \oplus 4\,\mathsf{K}^* \oplus 5\,\mathsf{E}^* \oplus S^3_0\,\mathsf{E}^* ,$$
$$\delta([\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n)) \cong \Lambda^3\,\mathsf{E}^* \oplus \mathsf{K}^* \oplus \mathsf{E}^* .$$

All the components in the decompositions are irreducible as $\mathsf{SO}^*(2n)$*-representations, with the exception of* K *for* $n = 2$ *and of* $\Lambda^3\,\mathsf{E}$ *for* $n = 3$. *As a consequence, in terms of* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-modules we have the decomposition*

$$\begin{aligned}\mathcal{H}(\mathfrak{so}^*(2n)) \cong\;& \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 \oplus \mathcal{X}_5 \oplus \mathcal{X}_6 \oplus \mathcal{X}_7 \\ =\;& [\mathsf{K}S^3\,\mathsf{H}]^* \oplus [\Lambda^3\,\mathsf{E}S^3\,\mathsf{H}]^* \oplus [\mathsf{K}\,\mathsf{H}]^* \oplus [\mathsf{EH}]^* \oplus [S^3_0\mathsf{EH}]^* \oplus [\mathsf{E}S^3\,\mathsf{H}]^* \oplus [\mathsf{EH}]^*\end{aligned}$$

where $\mathcal{X}_3 \oplus \mathcal{X}_4 = [\mathsf{K}\,\mathsf{H}]^* \oplus [\mathsf{EH}]^* \subset \Lambda^3[\mathsf{EH}]^*$, $\mathcal{X}_6 = [\mathsf{E}S^3\,\mathsf{H}]^* \subset \delta\big([\mathsf{EH}]^* \otimes \mathfrak{sp}(1)\big)$,

$$\mathcal{X}_7 = [\mathsf{EH}]^* = \Big\{ \delta\Big(\sum_{a=1}^{3} \zeta \circ J_a \otimes J_a \Big) \in \delta\big([\mathsf{EH}]^* \otimes \mathfrak{sp}(1)\big) \;:\; \zeta \in [\mathsf{EH}]^* \Big\} ,$$

and $H = \{J_a : a = 1, 2, 3\}$ *denotes the corresponding almost hypercomplex structure. On the other hand, in terms of* $\mathsf{SO}^*(2n)$*-modules we get the decomposition*

$$\mathcal{H}(\mathfrak{so}^*(2n)) \cong 2\Lambda^3 \mathsf{E}^* \oplus 3\,\mathsf{K}^* \oplus 4\,\mathsf{E}^* \oplus S^3_0 \mathsf{E}^* . \tag{4.12}$$

2) *Consequently, for* $n > 3$ *there exist up to* 2^7 $\mathsf{Sp}(1)$*-invariant algebraic types of* $\mathsf{SO}^*(2n)$*-geometries, and totally up to* 2^{10} *algebraic types of* $\mathsf{SO}^*(2n)$*-geometries.*

Moreover, for $n = 2, 3$ *the following holds:*

- For $n = 2$, there exist up to 2^9 $\mathsf{Sp}(1)$-invariant algebraic types of $\mathsf{SO}^*(4)$-geometries, since in this case the modules $\mathcal{X}_1, \mathcal{X}_3$ decomposes as in (4.11).
- For $n = 3$, there exists up to 2^8 $\mathsf{Sp}(1)$-invariant algebraic types of $\mathsf{SO}^*(6)$-geometries, since in this case the module $\mathcal{X}_2$ decomposes as in (4.10).

Remark 4.18 (1) Due to the multiplicities appearing in the decomposition (4.12) of $\mathcal{H}(\mathfrak{so}^*(2n))$, a definition similar with the Definition 4.16 requires precise projections to the individual irreducible factors. In general, such projections are not unique and their construction is a very complicated task, which requires a further study of $\mathsf{SO}^*(2n)$-structures. However, we can successfully use the $\mathsf{Sp}(1)$-invariant algebraic types to describe some distinguished classes of $\mathsf{SO}^*(2n)$-structures, characterized by $\mathsf{Sp}(1)$-invariant conditions. This procedure is analyzed in Sections 1 and 2 of [19]. Note also that a precise geometric characterization of the pure modules $\mathcal{X}_i$ $(i = 1, \dots, 7)$ requires deeper investigation of the corresponding Bianchi identities, which we do not present in this first part.

(2) The decompositions of $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$ under $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ and $\mathsf{SO}^*(2n)$, given, respectively, in Proposition 4.14 and Corollary 4.17, can be viewed as the counterpart to Cartan's decomposition of the space of torsion tensors corresponding to metric connections, see, e.g., [18] and the references therein. For $G = \mathsf{Sp}(2n, \mathbb{R})$ and the torsion of almost symplectic connections, analogous decompositions have been recently presented in [1].

5 Minimality of adapted connections to $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures

5.1 Minimal connections

Corollaries 4.15 and 4.17 need to be studied in a greater detail, since neither the posed isomorphisms, nor the way that the image of the Spencer differential δ sits inside $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$, are obvious. Both these tasks occur due to the involved multiplicities appearing in the decompositions presented in Corollary 4.15 and Corollary 4.17, respectively. As a consequence

Lemma 5.1 *For* $\mathsf{SO}^*(2n)$*- or* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$*-structures, there are many possible invariant normalization conditions, and thus different classes of minimal adapted connections.*

Example 5.2 For example, our choice of the modules $\mathcal{X}_1, \dots, \mathcal{X}_7$ provides a particular normalization condition for $\mathsf{SO}^*(2n)$-structures. However, $\nabla^{H,\omega}$ is not a minimal connection with respect to this normalization condition.

Our goal below is to provide the normalization conditions which establish both of our connections $\nabla^{H,\omega}$ and $\nabla^{Q,\omega}$ defined in Sect. 4.2, as minimal. In order to do this, we will combine results of Sects. 4.2 and 4.3 with certain results from [4, 5, 23], which we recall.

Let $(H = \{J_a : a = 1,2,3\}, \omega)$ be an almost hs-H structure on a smooth manifold M, or let (Q, ω) be an almost qs-H structure on M for which H provides a local admissible frame. Next it is again convenient to work at an algebraic level, in terms of the EH-formalism. Since ω is a scalar 2-form with respect to H, the space $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$ admits several distinguished equivariant projections, which induce projections on the space of sections of the induced vector bundles. First, we may associate to any $J_a \in H$ the subspace

$$\mathscr{C}_{J_a} := \left\{\phi \in \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] : \phi(J_a X, Y) = \phi(X, J_a Y) = -J_a \phi(X,Y)\right\},$$

which is isomorphic to $\delta([\mathsf{EH}]^* \otimes \mathfrak{gl}(2n, \mathbb{C}))$. We have a $\mathsf{GL}(2n, \mathbb{C})$-equivariant projection $\pi_{J_a} : \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] \longrightarrow \mathscr{C}_{J_a}$ defined by

$$\pi_{J_a}(\phi)(X,Y) := \frac{1}{4}\Big(\phi(X,Y) + J_a\big(\phi(J_a X, Y) + \phi(X, J_a Y)\big) - \phi(J_a X, J_a Y)\Big), \quad a = 1,2,3,$$

for any $X, Y, Z \in [\mathsf{EH}]$. By setting

$$\pi_H := \frac{2}{3}(\pi_{J_1} + \pi_{J_2} + \pi_{J_3}) = \frac{2}{3}\sum_{a=1}^{3} \pi_{J_a},$$

we obtain a $\mathsf{GL}(n, \mathbb{H})$-equivariant projection, i.e., $\pi_H^2 = \pi_H$, see [5, p. 420]. In full terms

$$\begin{aligned}\pi_H(\phi)(X,Y) =& \frac{1}{6}\Big(3\phi(X,Y) - \phi(IX,IY) - \phi(JX,JY) - \phi(KX,KY) + I\phi(X,IY) \\ &+ I\phi(IX,Y) + J\phi(X,JY) + J\phi(JX,Y) + K\phi(X,KY) + K\phi(KX,Y)\Big),\end{aligned}$$

for any $\phi \in \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$, where we have assumed that $H = \{I, J, K\}$. In fact, π_H is $\mathsf{GL}(n, \mathbb{H})\mathsf{Sp}(1)$-equivariant. Then, we get the following $\mathsf{GL}(n, \mathbb{H})$-equivariant isomorphisms

$$\mathsf{Ker}(\pi_H) \cong \cap_{J_a \in S^2} \mathsf{Ker}(\pi_{J_a}) \cong \delta([\mathsf{EH}]^* \otimes \mathfrak{gl}(n, \mathbb{H})),$$

which are also $\mathsf{GL}(n, \mathbb{H})\mathsf{Sp}(1)$-equivariant. It follows that for a $\mathsf{GL}(n, \mathbb{H})$-structure $H = \{I, J, K\}$ the intrinsic torsion module $\mathcal{H}(\mathfrak{gl}(n, \mathbb{H}))$ coincides with $\mathsf{Im}(\pi_H)$, and moreover, the intrinsic torsion itself is expressed by an appropriate linear combination of the Nijenhuis tensors corresponding to two anticommuting elements $I, J \in H$. In particular, the torsion T^H of the Obata connection ∇^H associated to H satisfies $T^H = \pi_H(T^H)$. The torsion T^Q of ∇^Q is also in the image of π_H and in addition it satisfies the condition (see [4])

$$\mathsf{Tr}_4(T^Q_X) = \mathsf{Tr}(\mathsf{J} \circ T^Q_{\mathsf{J}X}) = 0,$$

for all $\mathsf{J} \in S(Q) \cong S^2$. We are now ready to pose our main theorem related to minimal connections.

Theorem 5.3 (1) *The connection $\nabla^{H,\omega}$ defined in Theorem 4.8 is the unique almost hypercomplex skew-Hermitian connection with torsion in the module*

$$\mathsf{Im}\,(\pi_H) \oplus \delta([\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*) \cong (\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_6) \oplus (\mathcal{X}_3 \oplus \mathcal{X}_4 \oplus \mathcal{X}_5 \oplus \mathcal{X}_7) \quad (5.1)$$

$$\cong (2\Lambda^3\,\mathsf{E}^* \oplus 2\,\mathsf{K}^* \oplus 2\,\mathsf{E}^*) \oplus (\mathsf{K}^* \oplus 2\,\mathsf{E}^* \oplus S^3_0\,\mathsf{E}^*), \quad (5.2)$$

which is complementary to $\delta\big([\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n)\big)$, that is $\nabla^{H,\omega}$ is the unique minimal ($\mathcal{D}$-connection) for the normalization condition

$$\mathcal{D}(\mathfrak{so}^*(2n)) := \mathsf{Im}\,(\pi_H) \oplus \delta([\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*).$$

Note that the first decomposition (5.1) is given in terms of $\mathsf{SO}^*(2n)\,\mathsf{Sp}\,(1)$-modules, while (5.2) should be read in terms of $\mathsf{SO}^*(2n)$-modules.

(2 *Let us define $\mathcal{D}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$ as the submodule of $\mathsf{Im}\,(\pi_H) \oplus \delta\big([\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*\big)$ which is in the kernel of $(2\,\mathsf{Tr}_1 + \mathsf{Tr}_3)$ and $(\mathsf{Tr}_1 - \mathsf{Tr}_4)$, for all $\mathsf{J} \in \mathsf{S}\,(Q)$. Then, the connection $\nabla^{Q,\omega}$ defined in Theorem 4.12 is the unique almost quaternionic skew-Hermitian connection with torsion in the module*

$$\mathcal{D}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1)) \cong [\,\mathsf{K}\,S^3\,\mathsf{H}\,]^* \oplus [\Lambda^3\,\mathsf{E}\,S^3\,\mathsf{H}\,]^* \oplus [\,\mathsf{K}\,\mathsf{H}\,]^* \oplus [\mathsf{EH}]^* \oplus [S^3_0\mathsf{EH}]^*,$$

which is complementary to $\delta\big([\mathsf{EH}]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))\big)$, that is $\nabla^{Q,\omega}$ is the unique minimal ($\mathcal{D}$-connection) for the normalization condition $\mathcal{D}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$.

Proof A part of the uniqueness claim is related to the vanishing of $\mathfrak{g}^{(1)}$ for

$$\mathfrak{g} \in \{\mathfrak{so}^*(2n), \mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1), \mathfrak{gl}(n,\mathbb{H}), \mathfrak{sl}(n,\mathbb{H}) \oplus \mathfrak{sp}(1)\},$$

as we have mentioned before. Also, by the previous discussion, we know that $\mathsf{Im}\,(\pi_H)$ is complementary to the image $\delta\big([\mathsf{EH}]^* \otimes \mathfrak{gl}(n,\mathbb{H})\big)$. Now, $[\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*$ is clearly a complementary subspace to $[\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n)$ in $[\mathsf{EH}]^* \otimes \mathfrak{gl}(n,\mathbb{H})$. Since $\mathfrak{g}^{(1)}$ vanishes, δ is injective and so the first claim follows in combination with Corollary 4.17. For the second assertion, by Lemma 4.11 we know that

$$\delta([\mathsf{EH}]^* \otimes [S^2\,\mathsf{E}\,]^*)$$

is *not* a complementary subspace of $\delta\big([\mathsf{EH}]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))\big)$. On the other hand, Theorem 4.12 implies that the connection $\nabla^{Q,\omega}$ is the unique almost quaternionic skew-Hermitian connection with torsion in the module $\mathcal{D}(\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))$. Since its torsion component isomorphic to $[\mathsf{EH}]$ is by Lemma 4.11 complementary to $\delta\big([\mathsf{EH}]^* \otimes (\mathfrak{so}^*(2n) \oplus \mathfrak{sp}(1))\big)$, we get the second claim in combination with Corollary 4.15. □

Let us now recall that the torsion of almost symplectic connections is normalized to be an element of $\Lambda^3[\mathsf{EH}]^*$, which is complementary to $\delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n,\mathbb{R}))$ in the space of torsion tensors $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$ (after raising the last index using ω). In particular, $\Lambda^3[\mathsf{EH}]^*$ sits inside $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$ and one can introduce the following $\mathsf{Sp}\,(4n,\mathbb{R})$-equivariant projection

$$\pi_\omega : \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] \longrightarrow \Lambda^3[\mathsf{EH}]^*, \quad \pi_\omega(\phi)(X,Y,Z) := \frac{1}{3}\mathfrak{S}_{X,Y,Z}\,\omega\big(\phi(X,Y),Z\big),$$

where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum over $X, Y, Z \in [\mathsf{EH}]$. Next we shall prove that the decomposition $\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_7$ described in Corollary 4.17 for $\mathsf{SO}^*(2n)$-structures, is compatible with the decomposition $\Lambda^3[\mathsf{EH}]^* \oplus \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n, \mathbb{R}))$, and we shall analyze some further properties of these special $\mathsf{Sp}(1)$-invariant intrinsic torsion modules.

To do so, initially we need to do some preparatory work and prove a preliminary lemma. Next we shall denote the "lowering operator" by

$$\ell_3 : \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}] \longrightarrow \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]^* \subset \otimes^3[\mathsf{EH}]^*, \quad \ell_3(\phi)(X, Y, Z) := \omega\big(\phi(X, Y), Z\big),$$

for any $\phi \in \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$. The left inverse of ℓ_3 is the "raising operator"

$$\ell_3^{-1} : \otimes^3[\mathsf{EH}]^* \longrightarrow \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}],$$

defined as follows:

$$\otimes^3[\mathsf{EH}]^* \ni \Theta \longmapsto \ell_3^{-1}(\Theta) \in \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}], \quad \ell_3^{-1}(\Theta)(X, Y) := \phi(X, Y),$$

where $X, Y \in [\mathsf{EH}]$ and $\phi \in \Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]$ is a vector-valued 2-form with $\Theta = \ell_3(\phi)$. Passing to the level of bundles, we can prove that

Lemma 5.4 *Let (M, ω) be an* almost symplectic manifold. *Then, for any $\phi \in \Gamma(\mathsf{Tor}(M))$ we have $\ell_3^{-1}\big(\pi_\omega(\phi)\big) = \mathsf{Alt}_\phi$, where $\mathsf{Alt}_\phi \in \Gamma(\mathsf{Tor}(M))$ is the operator given by*

$$\mathsf{Alt}_\phi(X, Y) = \frac{1}{3}(\phi_X Y - \phi_X^T Y + \phi_Y^T X). \tag{5.3}$$

Here, for the vector-valued 2-form ϕ, and for any $X \in \Gamma(TM)$, we denote by $\phi_X \in \mathsf{End}(TM)$ the induced endomorphism with $\phi_X Y = \phi(X, Y)$, and by ϕ_X^T its symplectic transpose with respect to ω.

Proof It is sufficient to prove that

$$\pi_\omega(\phi)(X, Y, Z) = \omega\big(\mathsf{Alt}_\phi(X, Y), Z\big), \tag{5.4}$$

where Alt_ϕ is given by (5.3). Recall that the symplectic transpose ϕ_X^T is defined via the relation $\omega(\phi_X^T Y, Z) = -\omega(Y, \phi_X Z)$, for any $X, Y, Z \in \Gamma(TM)$. Since any $\phi \in \Lambda^2 T^*M \otimes TM$ satisfies $\phi_X Y = -\phi_Y X$, by the definition of π_ω we see that

$$\begin{aligned} 3\pi_\omega(\phi)(X, Y, Z) =& \omega\big(\phi(X, Y), Z\big) + \omega\big(\phi(Y, Z), X\big) + \omega\big(\phi(Z, X), Y\big) \\ =& \omega(\phi_X Y, Z) - \omega(X, \phi_Y Z) + \omega(Y, \phi_X Z) \\ =& \omega(\phi_X Y, Z) + \omega(\phi_Y^T X, Z) - \omega(\phi_X^T Y, Z) \\ =& \omega(\phi_X Y - \phi_X^T Y + \phi_Y^T X, Z), \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, which proves (5.4). Then, by the definition of ℓ_3^{-1} it follows that

$$\ell_3^{-1}\big(\pi_\omega(\phi)\big)(X, Y) = \mathsf{Alt}_\phi(X, Y) = \frac{1}{3}(\phi_X Y - \phi_X^T Y + \phi_Y^T X).$$

□

Now we are able to proceed with a proof of the claims pronounced above.

Proposition 5.5 (1) *The decomposition* $\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_7$ *is compatible with the decomposition* $\Lambda^3[\mathsf{EH}]^* \oplus \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n, \mathbb{R}))$. *In particular*, $\mathcal{X}_{234} = \Lambda^3[\mathsf{EH}]^*$ *and* $\mathcal{X}_{1567} \subset \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n, \mathbb{R}))$.

(2) *The torsion components* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_5$ *and* $\mathcal{X}_6$ *are independent of the normalization condition and coincide with the following torsion components of the connection* $\nabla^{H,\omega}$ *introduced in Theorem 4.8*: $[\mathsf{K} S^3 \mathsf{H}]^*$, $[\Lambda^3 \mathsf{E} S^3 \mathsf{H}]^*$, $[S^3_0\mathsf{EH}]^*$ *and* $[\mathsf{E} S^3 \mathsf{H}]^*$, *respectively. Moreover, there are the following* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-*equivariant maps*

$$\pi_3 : [\mathsf{K}\,\mathsf{H}]^* \to \mathcal{X}_3\,, \quad \pi_4 : 2[\mathsf{EH}]^* \to \mathcal{X}_4\,, \quad \pi_7 : 2[\mathsf{EH}]^* \to \mathcal{X}_7\,,$$

explicitly defined as

$$\pi_3(\phi) := \mathsf{Alt}_\phi \in \mathcal{X}_3 \subset \Lambda^3[\mathsf{EH}]^*\,,$$
$$\pi_4\Big(\delta\big(\zeta_1 \otimes \mathsf{id} + \pi_A(\omega \otimes Z_2)\big)\Big) := \mathsf{Alt}_{(2\zeta_1 + \frac{1}{2}Z_2^T)\otimes\omega} \in \mathcal{X}_4 \subset \Lambda^3[\mathsf{EH}]^*\,,$$
$$\pi_7\Big(\delta\big(\zeta_1 \otimes \mathsf{id} + \pi_A(\omega \otimes Z_2)\big)\Big) := \delta\Big(\sum_{a=1}^{3}(\frac{1}{3}\zeta_1 - \frac{1}{6}Z_2^T)\circ J_a \otimes J_a\Big) \in \mathcal{X}_7 \subset \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n, \mathbb{R}))\,,$$

where $\zeta_1, Z_2^T \in [\mathsf{EH}]^*$, $[\mathsf{K}\,\mathsf{H}]^*$ *and* $2[\mathsf{EH}]^*$ *are torsion components of* $\nabla^{H,\omega}$, *and* Alt *is the operator introduced in Lemma 5.4. In particular*, π_3 *is an isomorphism*, π_4, π_7 *are surjections, and the sum* $\pi_4 \oplus \pi_7$ *is also an isomorphism.*

() *The torsion components* $\mathcal{X}_1, \mathcal{X}_2$ *and* $\mathcal{X}_5$ *are independent of the normalization condition and coincide with the following torsion components of the connection* $\nabla^{Q,\omega}$ *introduced in Theorem 4.12*: $[\mathsf{K} S^3 \mathsf{H}]^*$, $[\Lambda^3 \mathsf{E} S^3 \mathsf{H}]^*$, *and* $[S^3_0\mathsf{EH}]^*$, *respectively. Moreover, there are the following* $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-*equivariant isomorphisms*

$$\pi_3 : [\mathsf{K}\,\mathsf{H}]^* \to \mathcal{X}_3\,, \quad \pi_4 : [\mathsf{EH}]^* \to \mathcal{X}_4\,,$$

where π_3, π_4 *are given as above, and* $[\mathsf{K}\,\mathsf{H}]^*$ *and* $[\mathsf{EH}]^*$ *are torsion components of* $\nabla^{Q,\omega}$.

Proof First we need to compute $\Lambda^3[\mathsf{EH}]^*$. This is of real type, and can be computed via the following $\big(\mathfrak{gl}([\mathsf{E}]) \oplus \mathfrak{gl}([\mathsf{H}])\big)$-invariant decomposition

$$\Lambda^3\mathsf{EH} = \bigoplus_{Y \in \mathrm{Young}(3)} Y(\mathsf{E})Y^t(\mathsf{H})\,,$$

whereas before Young(3) denotes the set of plethysms associated to Young diagrams with 3 boxes, and Y^t is the diagram Y transposed. This is straightforward to evaluate and yields the following $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-equivariant decomposition

$$\Lambda^3[\mathsf{EH}]^* \cong [\Lambda^3 \mathsf{E} S^3 \mathsf{H}]^* \oplus [\mathsf{K}\,\mathsf{H}]^* \oplus [\mathsf{EH}]^*\,. \tag{5.5}$$

Hence, by Corollaries 4.15 and 4.17 we obtain the assertion $\Lambda^3[\mathsf{EH}]^* \cong \mathcal{X}_{234}$. Next, identifying $[\mathsf{EH}]$ with the standard $\mathfrak{sp}(4n, \mathbb{R})$-module, we see that the branching of the $\mathfrak{sp}(4n, \mathbb{R})$-module $\delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n, \mathbb{R}))$ to $\mathfrak{sp}(1) \oplus \mathfrak{so}^*(2n)$ is isomorphic to the quotient

$$\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]^* / \Lambda^3[\mathsf{EH}]^*\,,$$

as abstract modules. Then, by using the decomposition from Proposition 4.14, as well as the computation above, we obtain the following equivariant isomorphism

$$\delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n,\mathbb{R})) \cong [(\mathsf{K} \oplus \mathsf{E}) \otimes S^3\mathsf{H}]^* \oplus [(\Lambda^3\mathsf{E} \oplus \mathsf{K} \oplus 2\mathsf{E} \oplus S^3_0\mathsf{E}) \otimes \mathsf{H}]^* .$$

This still leaves the question of embedding into $\Lambda^2[\mathsf{EH}]^* \otimes [\mathsf{EH}]^*$. We can uniquely recognize the modules with multiplicity one in this embedding, independently of any $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-equivariant choice of embedding. Thus, the relation between the submodules of torsion tensors in $2[\mathsf{K}\,\mathsf{H}]^* \oplus 3[\mathsf{EH}]^*$, $\mathcal{X}_{347}$ and $\Lambda^3[\mathsf{EH}]^* \oplus \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n,\mathbb{R}))$ remains to be clarified. However, by definition of $\mathcal{X}_{347}$ in Corollaries 4.15 and 4.17, these modules are compatible with the decomposition $\Lambda^3[\mathsf{EH}]^* \oplus \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n,\mathbb{R}))$. This completes the proof of the first assertion.

Now, having the explicit formula of Alt_ϕ by Lemma 5.4 we see that

$$\mathsf{Ker}\,(\mathsf{Alt}_\phi|_{2[\mathsf{K}\,\mathsf{H}]^*}) = [\mathsf{K}\,\mathsf{H}]^* \subset \delta([\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n)) \subset \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(4n,\mathbb{R}))\,,$$

which follows by counting multiplicities. Hence, π_3 is a well-defined isomorphism between the spaces presented in the second and third claim, respectively. Be aware, however, that π_3 is not the identity map.

Finally, we have three parametrizations of $3[\mathsf{EH}]^*$ by Lemma 4.11. The first one is given by the torsion components $2[\mathsf{EH}]^*$ of $\nabla^{H,\omega}$ and by $[\mathsf{EH}]^* \subset \delta([\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n))$. The second one consists of the torsion component $\mathcal{X}_4$ of $\nabla^{Q,\omega}$, and of $\mathcal{X}_7 \oplus [\mathsf{EH}]^* \subset \delta([\mathsf{EH}]^* \otimes \mathfrak{sp}(1)) \oplus \delta([\mathsf{EH}]^* \otimes \mathfrak{so}^*(2n))$. The third one is provided by the traces Tr_1, Tr_3, Tr_4. The transition matrix from the first parametrization to the third one can be immediately deduced from the formula (4.9). The transition matrix from the second parametrization to the third one can be also deduced from the formula (4.9), and the explicit computation of the traces presented at the end of the proof of Lemma 4.11. Finally, the composition of the endomorphisms corresponding to these two matrices provides the claimed formulas for π_4 and π_7. This finishes the proof of the second and third claim.

□

Remark 5.6 The projections π_3, π_4, π_7 from Proposition 5.5 provide the difference of the torsions of the two minimal connections with respect to our normalization conditions from Theorem 5.3, and the normalization condition given by the modules $\mathcal{X}_1, \ldots, \mathcal{X}_7$, respectively. Therefore, by the inverse of δ one obtains the difference of the corresponding minimal connections. However, the formula for inverse of δ is too complicated to be presented here.

Finally, as a conclusion of the above results we obtain the following.

Corollary 5.7 (1) *Let (H,ω) be a $\mathsf{SO}^*(2n)$-structure. Then, the corresponding intrinsic torsion is a 3-form if and only if (H,ω) is of type $\mathcal{X}_{234}$, and it is of vectorial type (i.e., it is defined by a non-trivial vector field on M) if and only if (H,ω) is of type $\mathcal{X}_{47}$.*

(2) *Let (Q,ω) be a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure. Then, the corresponding intrinsic torsion is a 3-form if and only if (Q,ω) is of type $\mathcal{X}_{234}$, and it is of vectorial type if and only if (Q,ω) is of type $\mathcal{X}_4$.*

Note that for $n=2,3$ the module $\mathcal{X}_{234}$ decomposes into further irreducible submodules.

5.2 Symplectomorphisms that are affine maps of minimal connections

Since the first prolongation of our G-structures $G \in \{\mathsf{SO}^*(2n), \mathsf{SO}^*(2n)\mathsf{Sp}(1)\}$ vanishes, the general theory of G-structures (see [28]) provides several important assertions about the hypercomplex/quaternionic symplectomorphisms. Such conclusions occur due to the uniqueness of the minimal almost hypercomplex skew-Hermitian connection $\nabla^{H,\omega}$ (respectively, minimal almost quaternionic skew-Hermitian connection $\nabla^{Q,\omega}$), with respect to certain normalization conditions described in Theorem 5.3. In particular:

Proposition 5.8 (1) *The hypercomplex symplectomorphisms between two* almost hs-H manifolds (M, H, ω) *and* $(\hat{M}, \hat{H}, \hat{\omega})$ *are those affine transformations between* $(M, \nabla^{H,\omega})$ *and* $(\hat{M}, \nabla^{\hat{H},\hat{\omega}})$, *satisfying the relations*

$$f^*\hat{H}_{f(x)} = H_x\,, \quad f^*\hat{\omega}_{f(x)} = \omega_x\,, \quad x \in M\,.$$

(2) *The quaternionic symplectomorphisms between two* almost qs-H manifolds (M, Q, ω) *and* $(\hat{M}, \hat{Q}, \hat{\omega})$ *are those affine transformations between* $(M, \nabla^{Q,\omega})$ *and* $(\hat{M}, \nabla^{\hat{Q},\hat{\omega}})$, *satisfying the relations*

$$f^*\hat{Q}_{f(x)} = Q_x\,, \quad f^*\hat{\omega}_{f(x)} = \omega_x\,, \quad x \in M\,.$$

(3) *If two hypercomplex/quaternionic symplectomorphisms* $f_1, f_2 : M \to \hat{M}$ *satisfy* $j^1_x f_1 = j^1_x f_2$ *for some* $x \in M$, *then* $f_1 = f_2$, *where in general* $j^1_x f$ *denotes the first jet at* $x \in M$ *of a smooth function* $f : M \to \hat{M}$.

Proof Since the hypercomplex/quaternionic symplectomorphisms map minimal connections to minimal connections, the first two claims are consequences of the uniqueness of such connections, see Corollary 4.6. We leave the details of the remaining assertion to the reader. □

This result has the following classical consequences.

Corollary 5.9 (1) *The group of hypercomplex symplectomorphisms of a* $4n$*-dimensional* almost hs-H manifold (M, H, ω) *is a Lie group of dimension less than or equal to* $2n^2 + 3n$.

(2) *The group of quaternionic symplectomorphisms of a* $4n$*-dimensional* almost qs-H manifold (M, Q, ω) *is a Lie group of dimension less than or equal to* $2n^2 + 3n + 3$.

Proof By Lemma 4.5, a Lie algebra $\mathfrak{g} \in \{\mathfrak{so}^*(2n), \mathfrak{so}^*(2n)\mathfrak{sp}(1)\}$ has vanishing first prolongation. Since the dimension of the automorphism group of a G-structure with $\mathfrak{g}^{(1)}$ trivial is bounded by $\dim_{\mathbb{R}} M + \dim_{\mathbb{R}} \mathfrak{g}$, see [28], the claim follows. □

6 Torsion-free examples

In this final section of this article, we will focus on torsion-free examples. In particular, based on certain conclusions presented in the articles [3] and [22], we shall present the classification of symmetric spaces K/L admitting invariant torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, under the assumption that K is semisimple. Moreover, we recall a construction from the theory of special symplectic connections. Note that many examples with torsion which realize some of the types $\mathcal{X}_{i_1\ldots i_j}$ introduced in this article, are described in [19]. There we illustrate non-integrable $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures in terms of both homogeneous and non-homogeneous geometries, and other constructions arising for example by using the bundle of Weyl structures, and more.

6.1 Semisimple symmetric spaces

For $\mathsf{SO}^*(2n)$- and $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structures, a natural source where one may initially look for integrable examples is the category of symmetric spaces (see [26] for the theory of symmetric spaces). By the results in [22], it follows that there are no semisimple symmetric spaces with an invariant almost hypercomplex skew-Hermitian structure. However, let us consider the symmetric space

$$M = K/L = \mathsf{SO}^*(2n+2)/\mathsf{SO}^*(2n)\mathsf{U}(1)\,.$$

By [22], it is known that M carries a $\mathsf{SO}^*(2n+2)$-invariant quaternionic structure, although it is *not* a pseudo-Wolf space. Let us denote by $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}$ the corresponding Cartan decomposition and by $\chi : K \to \mathsf{Aut}(\mathfrak{m})$ the isotropy representation, where as usual we identify $\mathfrak{m} \cong T_{eL}K/L$. Since

$$\chi(\mathsf{SO}^*(2n)\mathsf{U}(1)) \subset \mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$$

clearly holds, there is also an invariant qs-H structure on K/L. Let us consider the $\mathsf{Ad}(L)$-invariant complex structure $I_o : \mathfrak{m} \to \mathfrak{m}$ induced by $\chi|_{\mathsf{U}(1)}$, that is $I_o = \chi_*(U) = \mathrm{ad}(U)$ for some $U \in \mathfrak{u}(1)$. Let us also denote by $\langle\ ,\ \rangle_{\mathfrak{m}}$ the $\mathsf{Ad}(L)$-invariant symmetric pseudo-Hermitian metric (with respect to I_o) on $\mathfrak{m}$. Note that I_o corresponds to a K-invariant complex structure I on M which we may use to build a local admissible base of the invariant quaternionic structure Q on the origin $o = eL \in K/L$, induced by the $\mathsf{Sp}(1)$ action on $\mathfrak{m}$. Then, since $M = K/L$ is isotropy irreducible, by Schur's lemma we deduce that the 2-form ω defined by $\omega_o(\cdot\,,\cdot) := \langle I_o\cdot\,,\cdot\rangle_{\mathfrak{m}}$ is an $\mathsf{Ad}(L)$-invariant scalar 2-form with respect to Q. In terms of Proposition 2.11, this means that $\langle\ ,\ \rangle_{\mathfrak{m}} = g_{I_o}$.

Remark 6.1 Note that $\langle\ ,\ \rangle_{\mathfrak{m}}$ induces the unique (up to scale) K-invariant Einstein metric on $M = K/L$ of signature $(2n, 2n)$, which is actually a multiple of the Killing form of $\mathsf{SO}^*(2n+2)$ restricted to $\mathfrak{m}$.

Next by using the classification of the pseudo-Wolf spaces, i.e., quaternionic pseudo-Kähler symmetric spaces, given by Alekseevsky-Cortes in [3], we prove that in addition to $M = K/L$ there are a few more symmetric spaces with the same property.

Theorem 6.2 *The symmetric space* $\mathsf{SO}^*(2n+2)/\mathsf{SO}^*(2n)\mathsf{U}(1)$ *and the pseudo-Wolf spaces*

$$\mathsf{SU}(2+p,q)/(\mathsf{SU}(2)\mathsf{SU}(p,q)\mathsf{U}(1)), \qquad \mathsf{SL}(n+1,\mathbb{H})/(\mathsf{GL}(1,\mathbb{H})\mathsf{SL}(n,\mathbb{H}))$$

are the only (*up to covering*) *symmetric spaces* K/L *with* K *semisimple*, *admitting an invariant torsion-free quaternionic skew-Hermitian structure* (Q,ω). *In particular*, *the corresponding canonical connections on these symmetric spaces coincides with the associated minimal quaternionic skew-Hermitian connection* $\nabla^{Q,\omega}$.

Proof As it was shown in [22], the classification of invariant quaternionic structures on semisimple symmetric spaces K/L is divided according to the dimension of intersection of $\chi(L)$ with $\mathsf{Sp}(1)$, where $\chi : L \to \mathsf{Aut}(\mathfrak{m})$ is the isotropy representation and $\mathfrak{m}$ is the symmetric reductive complement. This dimension cannot be zero. Up to covering, the above symmetric space $\mathsf{SO}^*(2n+2)/\mathsf{SO}^*(2n)\mathsf{U}(1)$ is the only one in the classification with one-dimensional intersection, i.e., $\chi(L)\cap\mathsf{Sp}(1)=\mathsf{U}(1)$. This coincides with the center of L.

The remaining cases in the classification have the intersection $\chi(L)\cap\mathsf{Sp}(1)=\mathsf{Sp}(1)$ and this induces an invariant quaternionic structure Q on K/L induced by the isotropy representation χ. This means that in this case, the invariant pseudo-Riemannian metric g induced by restriction of the Killing form to $\mathfrak{m}$ is a quaternionic pseudo-Kähler metric, and so K/L is a pseudo-Wolf space. The classification of pseudo-Wolf spaces was obtained in [3, Theorem 2]. On the other hand, it is well known that invariant symplectic structures on a simple symmetric space correspond to the center of the isotropy algebra. Then, the isotropy action of the center of the stabilizer provides an invariant complex or paracomplex structure $I \notin \Gamma(Q)$ on K/L and

$$\omega(\cdot\,,\cdot) := g(I\cdot\,,\cdot)\,,$$

is an invariant scalar 2-form with respect to Q. In the classification of the pseudo-Wolf spaces obtained in [3, Theorem 2], we see that

$$\mathsf{SU}(2+p,q)/(\mathsf{SU}(2)\mathsf{SU}(p,q)\mathsf{U}(1)), \qquad \mathsf{SL}(n+1,\mathbb{H})/(\mathsf{GL}(1,\mathbb{H})\mathsf{SL}(n,\mathbb{H}))$$

are the only pseudo-Wolf spaces for which the isotropy algebra contains a non-trivial center. This proves our first assertion. Recall finally that by the Ambrose–Singer theorem the canonical connection ∇^0 on K/L must preserve (Q,ω) and is in particular a minimal connection because it is torsion free, see [27]. Then, we obtain the identification $\nabla^0=\nabla^{Q,\omega}$ by uniqueness of the minimal connection, see also Proposition 4.6. □

6.2 Examples with special symplectic holonomy

Let us recall that on a symplectic manifold (M,ω) a symplectic connection ∇ is said to be of special symplectic holonomy if $\mathrm{Hol}(\nabla)$ is a proper subgroup of $\mathsf{Sp}(2n,\mathbb{R})$ that acts absolutely irreducibly on the tangent space, i.e., it acts irreducibly and does not preserve a complex structure. A special symplectic connection is a symplectic connection with special symplectic holonomy, and it is known that such connections may exist only in dimensions ≥ 4. The first special symplectic holonomies were constructed by Bryant [7], and by Chi, Merkulov and Schwachhöfer [16, 17]. Finally, these exotic holonomies were classified by Merkulov and Schwachhöfer and include the Lie group $\mathsf{SO}^*(2n)\mathsf{Sp}(1)$, see for example [30, Table 3] (note that in contrast to $\mathsf{SO}^*(2n)$, the Lie group $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$ is a real non-symmetric Berger subgroup, see [38, Tab. II]).

The construction providing such special symplectic holonomies has been described in [11]. Let us recall how this procedure works for a $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure (Q, ω). Let P be the connected subgroup of $\mathsf{SO}^*(2n+4)$ which stabilizes an isotropic (with respect to ω) quaternionic line in $\mathbb{H}^{n+1}$. Then, the homogeneous space $N = \mathsf{SO}^*(2n+4)/P$ admits an invariant contact structure (see [13, p. 298]), and we denote by $\mathscr{D}$ the corresponding contact distribution. In such terms, we obtain the following local construction:

Proposition 6.3 *Let $(Q, \omega, \nabla^{Q,\omega})$ be a smooth torsion-free $\mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$-structure with special symplectic holonomy, i.e., $T^{Q,\omega} = 0$ and $\mathsf{Hol}(\nabla^{Q,\omega}) = \mathsf{SO}^*(2n)\,\mathsf{Sp}(1)$. Then, $(Q, \omega, \nabla^{Q,\omega})$ is analytic, and locally equivalent to a symplectic reduction $\mathbb{T}\backslash U$ by a one-parameter subgroup $\mathbb{T} \subset \mathsf{SO}^*(2n+4)$ with Lie algebra $\mathfrak{t}$, such that the corresponding right-invariant vector fields are transversal to $\mathscr{D}$ everywhere on U. Here, $U \subset N$ is a sufficiently small open subset of N. In particular, the moduli space of such structures is n-dimensional, where n represents the quaternionic dimension of the symplectic reduction.*

Proof This result occurs as the restriction of [11, Corollary C] to the particular case of torsion-free almost quaternionic skew-Hermitian structures. □

Note that the manifold $N = \mathsf{SO}^*(2n+4)/P$ happens to be a flat parabolic geometry. The interplay between the parabolic geometry on N and the almost conformal symplectic geometry on the symplectic reduction was explored in detail by Čap and Salač in a series of papers, see [14, 15] for example. Indeed, they described a generalization of the above construction in the presence of torsion. This construction requires $\mathbb{T}$ to be a flow of a transversal infinitesimal automorphism of the parabolic contact structure, and yields structures which are almost conformally symplectic, rather than almost symplectic, and hence less relevant to our current situation.

Appendix A: Adapted bases with coordinates in $\mathbb{H}^n$

A.1 Left quaternionic vector space $\mathbb{H}^n$

Observe that after a choice of an admissible hypercomplex basis H as in Lemma 2.3 we can identify $a + bj \in [\mathsf{EH}]$ with $a + bj \in \mathbb{H}^n$, where the latter is viewed as a left quaternionic vector space. Consequently:

Corollary A. 1 *Let $(h, H = \{J_1, J_2, J_3\})$ be a linear hypercomplex skew-Hermitian structure on a $4n$-dimensional real vector space V, or let $H = \{J_1, J_2, J_3\}$ be an admissible basis of the linear quaternionic skew-Hermitian structure h on V, and set $\omega := \mathbb{R}\mathrm{e}(h)$. Then, a skew-Hermitian basis of (h, H), in terms of Definition 2.24, provides an isomorphism $V \cong \mathbb{H}^n$, such that*

(1) $(a_1 + a_2J_1 + a_3J_2 + a_4J_3)x = (a_1 + a_2i + a_3j + a_4k)x$, for any $x \in \mathbb{H}^n$.
(2) $\omega(x, y) = \frac{1}{2}(x^tj\bar{y} - y^tj\bar{x})$ for all $x, y \in \mathbb{H}^n$, where $\bar{x}$ is the quaternionic conjugate.

(3) $g_{\mathsf{J}}(x,y) = \frac{1}{2}(-x^t j\bar{y}(\mu_1 i + \mu_2 j + \mu_3 k) - (\mu_1 i + \mu_2 j + \mu_3 k)y^t j\bar{x})$, for $\mathsf{J} = \mu_1 J_1 + \mu_2 J_2 + \mu_3 J_3 \in \mathrm{S}^2$.
(4) $h(x,y) = x^t j\bar{y}$.

Remark A. 2 The formula $h(x,y) = x^t j\bar{y}$ is the usual formula for a skew-Hermitian form on the left quaternionic space vector space $\mathbb{H}^n$, but be aware that some authors replace equivalently j by i.

A.2 Right quaternionic vector space $\mathbb{H}^n$

Left and right quaternionic vector spaces are related by conjugation. Therefore, after the choice of an admissible hypercomplex basis H, the element $\bar{a} - bj \in [\mathsf{EH}]$ can be identified with $a + bj$ in the right quaternionic vector space $\mathbb{H}^n$ (by Lemma 2.3). Thus, we get the following

Corollary A.3 *Let $(h, H = \{J_1, J_2, J_3\})$ be a linear hypercomplex skew-Hermitian structure on V, or let $H = \{J_1, J_2, J_3\}$ be an admissible basis of the linear quaternionic skew-Hermitian structure h, and let $\omega = \mathbb{R}\mathrm{e}(h)$. Then, a skew-Hermitian basis of (h, H), in terms of Definition 2.24, provides an isomorphism $V \cong \mathbb{H}^n$, such that*

(1) $(a_1 + a_2 J_1 + a_3 J_2 + a_4 J_3)x = x(a_1 - a_2 i - a_3 j - a_4 k)$, for any $x \in \mathbb{H}^n$.
(2) $\omega(x,y) = \frac{1}{2}(x^* jy - y^* jx)$ for all $x, y \in \mathbb{H}^n$, where x^* is conjugate transpose.
(3) $g_{\mathsf{J}}(x,y) = \frac{1}{2}(x^*(-\mu_2 + \mu_3 i - \mu_1 k)y + y^*(-\mu_2 - i\mu_3 + \mu_1 k)x)$, for $\mathsf{J} = \mu_1 J_1 + \mu_2 J_2 + \mu_3 J_3 \in \mathrm{S}^2$.
(4) $h(x,y) = x^* jy$.

Remark A. 4 The formula $h(x,y) = x^* jy$ is the usual formula for a skew-Hermitian form on the right quaternionic space vector space $\mathbb{H}^n$. However, observe that some authors replace equivalently j by i, see [25, p. 8].

On the right quaternionic vector space $\mathbb{H}^n$, we can find a skew-Hermitian basis of the linear $\mathsf{hs\text{-}H}$ structure $(\mathbb{H}^n, h)$, as follows:

(I) We start with a quaternionic skew-Hermitian form h on $\mathbb{H}^n$.
(IIa) If $n = 1$, then by definition $h(x,y) = \bar{x}(h_1 i + h_2 j + h_3 k)y$ for some $h_1, h_2, h_3 \in \mathbb{R}$, and by non-degeneracy exists some $q \in \mathbb{H}$ such that $\bar{q}(h_1 i + h_2 j + h_3 k)q = j$. Thus, we are done.
(IIb) If $n > 1$, then we start by finding $e_1 \in \mathbb{H}^n$, such that $h(e_1 x, e_1 y) = \bar{x}jy$, $x, y \in \mathbb{H}$. Since h is non-degenerate, there are $f_1, f_2 \in \mathbb{H}^n$ such that $h(f_1, f_2) \neq 0$. Thus, for either $e = f_1$, $e = f_2$, or $e = kf_1 + jf_2$ it holds $h(e,e) \neq 0$ and the step (IIa) can be applied, i.e., $e_1 = qe$.
(III) If $n > 1$, then we complete e_1 to a quaternionic basis of $\mathbb{H}^n$. In this basis, h can be expressed by the following block matrix

$$\begin{pmatrix} j & -X^* \\ X & \square \end{pmatrix}$$

for $X \in \mathbb{H}^{n-1}$, and $\square$ is a quantity not important for us. Thus, changing the quaternionic basis by left multiplication by $\begin{pmatrix} 1 & -jX^* \\ 0 & \mathrm{id}_{\mathbb{H}^{n-1}} \end{pmatrix}$, we compute

$$\begin{pmatrix} 1 & -jX^* \\ 0 & \mathrm{id}_{\mathbb{H}^{n-1}} \end{pmatrix}^* \begin{pmatrix} j & -X^* \\ X & \square \end{pmatrix} \begin{pmatrix} 1 & -jX^* \\ 0 & \mathrm{id}_{\mathbb{H}^{n-1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Xj & \mathrm{id}_{\mathbb{H}^{n-1}} \end{pmatrix} \begin{pmatrix} j & 0 \\ X & \square \end{pmatrix} = \begin{pmatrix} j & 0 \\ 0 & \square \end{pmatrix}.$$

Consequently, we have constructed an orthogonal complement of e_1, and we may repeat the algorithm for restriction of h to the orthogonal complement. Then, we conclude by induction with respect to the dimension.

At this point the right basis looks very useful, however there is still the following task.

Lemma A. 5 *Let $(h, H = \{J_1, J_2, J_3\})$ be a linear hypercomplex skew-Hermitian structure on V. Then, there is no basis $e_1, \ldots, e_{2n}, f_1, \ldots, f_{2n}$ of V satisfying*

$$J_1(e_c) = -e_{c+n}, \quad J_2(e_c) = -f_c, \quad J_3(e_c) = -f_{c+n},$$

which is at the same time a symplectic basis with respect to the induced scalar 2-form $\omega = \mathbb{Re}(h)$.

Proof In a skew-Hermitian basis of (H, ω) (in terms of Definition 2.24), and after identifying $\bar{a} - bj \in [\mathsf{EH}]$ with $a + bj \in \mathbb{H}^n$, we compute

$$\omega(e_c, f_c) = 1, \quad \omega(e_{c+n}, f_{c+n}) = -1$$

for $c = 1, \ldots, n$, and all the other combinations (up to antisymmetry) are zero. The action of $\mathsf{GL}(n, \mathbb{H})$ on ω preserves the property $\omega(e_c, f_c) = -\omega(e_{c+n}, f_{c+n})$, and thus, there is no basis satisfying the given condition. $\square$

However, as we show below, this problem can be resolved exactly when the quaternionic dimension is *even*. This is because of the existence of another natural ordering of a basis of a right quaternionic vector space, which can be adapted to our purpose.

Lemma A. 6 *Let $H = \{J_1, J_2, J_3\}$ be a linear hypercomplex structure on V. Then, V admits a basis $e_1, \ldots, e_{4n}$ satisfying*

$$J_1(e_{4a-3}) = -e_{4a-2}, \quad J_2(e_{4a-3}) = -e_{4a-1}, \quad J_3(e_{4a-3}) = -e_{4a},$$

and such that the set of vectors $e_1, \ldots, e_{2n}, f_1 = e_{2n+1}, \ldots, f_{2n} = e_{4n}$ is a symplectic basis for a scalar 2-form ω on V, if and only if n is even, $n = 2m$.

Proof Let us consider $e_1, \ldots, e_{4n}$ as the reordering of a symplectic basis adapted to a hypercomplex structure H, after identifying $\bar{a} - bj \in [\mathsf{EH}]$ with $a + bj \in \mathbb{H}^n$, so that the conditions on the action of H are satisfied. Then, the formula $\omega(x, y) = \frac{1}{2}(x^* jy - y^* jx)$ is still *not* the standard symplectic form in this basis. For $n = 1$, it is immediate to show that the transformation $q^* jq$ never induces the standard symplectic form. But if $n = 2$, then we can directly compute that the following matrix

$$\mathscr{C} := \begin{pmatrix} -\frac{1}{2}k & i \\ -\frac{1}{2}j & -1 \end{pmatrix}$$

satisfies

$$h(\mathscr{C}x,\mathscr{C}y) = x^*\begin{pmatrix} \frac{1}{2}k & \frac{1}{2}j \\ -i & -1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}i & -k \\ \frac{1}{2} & j \end{pmatrix} y = x^*\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y\,.$$

This provides the standard expression for ω in our new basis, by setting $\omega = \mathbb{Re}(h)$. This procedure can be successfully generalized to the general case $n = 2m$, via the matrix

$$\mathscr{C}_{2m} := \begin{pmatrix} -\frac{1}{2}k\,\mathrm{id}_{\mathbb{H}^m} & i\,\mathrm{id}_{\mathbb{H}^m} \\ -\frac{1}{2}j\,\mathrm{id}_{\mathbb{H}^m} & -\,\mathrm{id}_{\mathbb{H}^m} \end{pmatrix}.$$

In particular, note that $\mathscr{C}_2 = \mathscr{C}$. However, for $n = 2m + 1$ this is not possible. Indeed, we can restrict $\mathscr{C}_{2m}$ to the first $4m$ vectors of the above basis, and for the remaining vectors in the basis, we get j in the diagonal of the matrix corresponding to ω. Hence, for $n = 2m + 1$ this task cannot be resolved, in line with the case $n = 1$. □

It is reasonable to provide a special name for the basis constructed in Lemma A. 6, as we do below.

Definition A.7 Let $e_1, \dots, e_{4m}, f_1, \dots, f_{4m}$ be a symplectic basis on a symplectic vector space (V, ω) of real dimension $8m$. Then, we say that it forms a quaternionic Darboux basis for the linear hypercomplex structure $H = \{J_1, J_2, J_3\}$ on V, if the following holds:

$$\begin{aligned} J_1(e_{4a-3}) &= -e_{4a-2}\,, & J_1(f_{4a-3}) &= -f_{4a-2}\,, \\ J_2(e_{4a-3}) &= -e_{4a-1}\,, & J_2(f_{4a-3}) &= -f_{4a-1}\,, \\ J_3(e_{4a-3}) &= -e_{4a}\,, & J_3(f_{4a-3}) &= -f_{4a}\,, \end{aligned}$$

for any $a = 1, \dots, m$.

Note that this definition coincides with the definition of an admissible basis to a linear hypercomplex structure, given in [4], but be aware that their V is a left quaternionic vector space.

Example A. 8 For $n = 2$, assume that e_1, e_2 are nonzero vectors in $\mathbb{H}^2$ for which the quaternionic lines $e_1 \cdot \mathbb{H}$ and $e_2 \cdot \mathbb{H}$ do not coincide. If H is the linear hypercomplex structure induced by right multiplication via $-i, -j, -k$, then a quaternionic Darboux basis is given by

$$\{e_1, e_1 i, e_1 j, e_1 k, e_2, e_2 i, e_2 j, e_2 k\}\,.$$

Let us now consider the skew-Hermitian basis $\mathfrak{B} = \{e_1, e_2, ie_1, ie_2, je_1, je_2, ke_1, ke_2\}$ of $\mathbb{H}^2$ (viewed as a left quaternionic vector space) described in Example 2.25. Then, we can multiply the transition matrix $\mathscr{C}$ in the proof of Lemma A. 6 with the matrix corresponding to the linear quaternionic conjugation. This composition maps the coordinates $(a_1, \dots, a_8)$ in the quaternionic Darboux basis to

$$(-a_6 + \frac{1}{2}a_4, -a_5 + \frac{1}{2}a_3, -a_5 - \frac{1}{2}a_3, a_6 + \frac{1}{2}a_4, a_8 + \frac{1}{2}a_2, a_7 + \frac{1}{2}a_1, -a_7 + \frac{1}{2}a_1, a_8 - \frac{1}{2}a_2)$$

in the skew-Hermitian basis $\mathfrak{B} = \{e_1, e_2, ie_1, ie_2, je_1, je_2, ke_1, ke_2\}$. Note that the indicated composition is a linear isomorphism between the linear hs-H structures provided by these bases.

Let us finally emphasize the following nice application of quaternionic Darboux bases in the theory of parabolic geometries.

Proposition A. 9 (1) *In a quaternionic Darboux basis,* $\mathfrak{so}^*(4n)$ *carries a* $|1|$*-grading*

$$\mathfrak{so}^*(4n)_{-1} \oplus \mathfrak{so}^*(4n)_0 \oplus \mathfrak{so}^*(4n)_1$$

represented by the following matrix

$$\begin{pmatrix} A & C \\ B & -A^* \end{pmatrix},$$

for $A, B, C \in \mathfrak{gl}(n, \mathbb{H})$, $A \in \mathfrak{so}^*(4n)_0$ $B^* = B \in \mathfrak{so}^*(4n)_{-1}$, $C^* = C \in \mathfrak{so}^*(4n)_1$. *This depth* 1*-gradation corresponds to the last node in the corresponding Satake diagram.*

(2) *In the basis from the proof of Lemma A. 6,* $\mathfrak{so}^*(4n+2)$ *carries a* $|2|$*-grading*

$$\mathfrak{so}^*(4n)_{-2} \oplus \mathfrak{so}^*(4n)_{-1} \oplus \mathfrak{so}^*(4n)_0 \oplus \mathfrak{so}^*(4n)_1 \oplus \mathfrak{so}^*(4n)_2$$

represented by the following matrix

$$\begin{pmatrix} A & Y & C \\ X & uj & jY^* \\ B & -X^*j & -A^* \end{pmatrix},$$

for $A, B, C \in \mathfrak{gl}(n, \mathbb{H})$, $A \in \mathfrak{so}^*(4n)_0, B^* = B \in \mathfrak{so}^*(4n)_{-2}, C^* = C \in \mathfrak{so}^*(4n)_2$, $X, Y \in \mathbb{H}^n$, $X \in \mathfrak{so}^*(4n)_{-1}$, $Y \in \mathfrak{so}^*(4n)_1$ *and* $u \in \mathbb{R} \subset \mathfrak{so}^*(4n)_0$. *This depth* 2*-gradation corresponds to the last two nodes in the corresponding Satake diagram.*

(3) *There is a diagonal Cartan subalgebra of* $\mathfrak{so}^*(4n)$ *and* $\mathfrak{so}^*(4n+2)$ *in both of these matrix representations.*

Proof We can use the transition matrix $\mathscr{C}_{2n}$ posed in the proof of Lemma A. 6 on our representation of $\mathfrak{so}^*(4n)$ and $\mathfrak{so}^*(4n+2)$ and obtain the claimed matrices. Also, it is a simple observation that the claimed decompositions are $|1|$-gradings and $|2|$-gradings, respectively, and that there is a diagonal Cartan subalgebra. □

We should mention that the $|2|$-grading described above differs from the unique contact $|2|$-grading corresponding to the second node of the Satake diagram associated to $\mathfrak{so}^*(2n)$, see [13]. In particular, a decomposition of the matrix given in Proposition A. 9 for the $\mathfrak{so}^*(4n)$-case into blocks of size 1, $(n-1)$, 1 and $(n-1)$, provides the contact $|2|$-grading of $\mathfrak{so}^*(4n)$. Similarly, a decomposition of the matrix given in the second part of Proposition A. 9 into blocks of size 1, n, 1 and $(n-1)$, determines the contact $|2|$-grading of $\mathfrak{so}^*(4n+2)$.

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