



Positivity for the clamped plate equation under high tension

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Abstract

In this article, we consider positivity issues for the clamped plate equation with high tension $\gamma > 0$. This equation is given by $\Delta^2 u - \gamma \Delta u = f$ under clamped boundary conditions. Here, we show that given a positive f , i.e. upwards pushing, we find a $\gamma_0 > 0$ such that for all $\gamma \geq \gamma_0$ the bending u is indeed positive. This γ_0 only depends on the domain and the ratio of the L^1 and L^∞ norm of f . In contrast to a recent result by Cassani and Tarsia, our approach is valid in all dimensions.

Keywords Bi-Laplace equation · Maximum principle · High tension

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1 Introduction

The Boggio-Hadamard conjecture states, that for a given convex, open, bounded set $\Omega \subseteq \mathbb{R}^n$, an $f \geq 0$ and outer normal ν of Ω , a solution u to

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

is nonnegative, i.e. $u \geq 0$ (cf. [26, 27]). Problem (1.1) models the bending u of a clamped plate Ω under a force f . Hence the problem can be restated as:

Does upward pushing yield upward bending?

The conjecture was substantiated by Boggio's explicit formula [6] (see also [15, Lemma 2.27] or [28]) for the Greens function of problem (1.1) on the unit disc, because this function is positive, and furthermore by Almansi's result to calculate the Greens function for certain domains by the Greens function of the unit ball (see [3]).

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Several other domains than the disc have been found on which such a positivity preserving property holds (see the references below). Hadamard himself claimed in [27] that such a property for all limaçons is true, which turned out to be wrong in general, though some of them still possess this property (see [15, Fig. 1.2]). Remarkable is that such limaçons are not convex. In [20], Grunau and Robert showed that positivity preserving is preserved under small regular perturbations of the domain in dimensions $n \geq 3$.

The conformal invariance of the problem was also successfully used to construct domains with a positivity preserving property by, e.g. Dall'Acqua and Sweers in [10] (see also the references therein for more information on such domains).

On the other hand, several counterexamples have been found by now. The first one was by Duffin on an infinite strip [12] and shortly after Garabedian [14] showed that on an elongated ellipse the Greens function changes sign. By now even for uniform forces, i.e. $f \equiv 1$, counterexamples have been found by Grunau and Sweers in [23] and [24]. We refer to [15, Sect. 1.2] for a comprehensive historical overview to this problem.

Instead of examining (1.1) for positivity, Cassani and Tarsia in [9] examined positivity issues for

$$\begin{cases} \Delta^2 u - \gamma \Delta u = f & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

with $\gamma > 0$ big enough. The basic motivation is that for γ big enough, the influence of Δu ($-\Delta u = f$ satisfies positivity preserving via the maximum principle) becomes stronger than that of $\Delta^2 u$. In more technical detail Cassani and Tarsia conjectured the existence of a $\gamma_0 = \gamma_0(f, \Omega) \geq 0$, such that $u \geq 0$ for all $\gamma \geq \gamma_0$ and provided a proof for dimensions $n = 2, 3$, smooth, bounded Ω and positive $f \in L^2(\Omega)$. In this article we provide a different approach, which is valid for all dimensions, see Theorem 1.1.

In dimension $n = 1$, this positivity preserving property is true for all $\gamma > 0$ independent of f . This was shown by Grunau in [19] Proposition 1.

The parameter γ is usually called tension, if it is positive. Several results concerning (1.2) have been achieved, which are usually concerned with vibrations of the plate, i.e. eigenvalue problems. Bickley gave some explicit calculations for the spectrum in the unit disc in [5] already in 1933. Hence the existing literature for these eigenvalue problems is quite vast and is still developing, see, e.g. [8, 4] and the references therein.

Other modifications for (1.1) concerning positivity issues are, e.g. changing the boundary conditions to so called Steklov conditions. This has been examined by, e.g. Bucur and Gazzola in [7].

Different elliptic differential operators of higher order, their respective fundamental solutions and their sign close to a singularity have also been examined by Grunau, Romani and Sweers in [22] in a more systematic approach to understand better the loss of positivity preserving.

Instead of (1.2), we examine the following boundary value problem for positivity preserving. This is obviously equivalent, but (1.3) yields the advantage, that the singularity of the equation is more prominent and hence yields easier access to necessary estimates.

$$\begin{cases} \varepsilon^2 \Delta^2 u_\varepsilon - \Delta u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

Here $\Omega \subset \subset \mathbb{R}^n$, $\partial\Omega \in C^4$, $f \in L^\infty(\Omega)$ and $\varepsilon > 0$, and the solution $u_\varepsilon \in W^{4,p}(\Omega)$ for all $1 < p < \infty$ (see, e.g. [15, Corollary 2.21] and the references therein for existence, regularity and uniqueness to (1.3)).

Theorem 1.1 *For connected $\Omega \subset \subset \mathbb{R}^n$, $\partial\Omega \in C^4$, $f \in L^\infty(\Omega)$, $\tau > 0$, $f \geq 0$ with*

$$\int f d\mathcal{L}^n \geq \tau \|f\|_{L^\infty(\Omega)} > 0, \quad (1.4)$$

there exists $\varepsilon_0 = \varepsilon_0(\Omega, \tau) > 0$ such that

$$u_\varepsilon > 0 \quad \text{in } \Omega \text{ for all } 0 < \varepsilon \leq \varepsilon_0. \quad (1.5)$$

Please note that we do not have any restrictions on the dimension, i.e. $n \in \mathbb{N}$ arbitrary. Furthermore, our method yields that ε_0 does not depend on f directly, only on τ .

The strategy of the proof is as follows: The limiting problem of (1.3) is

$$\begin{cases} -\Delta u = f \geq 0, \neq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which admits a maximum principle, and establishes positivity of u_ε on any $\Omega' \subset \subset \Omega$ for ε small.

We proceed by contradiction and assume Theorem 1.1 is false. Hence for every $\varepsilon > 0$, we find a nonnegative $f_\varepsilon \in L^\infty(\Omega)$ satisfying (1.4), such that u_ε is not positive in Ω .

Then, we examine a blow-up of our solutions u_ε , which is weighted by the supremum of the modulus of the Laplacian at the boundary, i.e. $\sup_{\partial\Omega} \varepsilon^2 |\Delta u_\varepsilon|$. After a careful analysis (see Sects. 2 and 3), we can show that this blow-up converges in a suitable sense to a solution of $\Delta^2 u - \Delta u = 0$ on the half-space with Dirichlet boundary conditions (see Sect. 4). With a uniqueness result shown in Appendix A, we explicitly calculate this limit and obtain positivity of the Laplacian of u_ε on the boundary for ε small. This is crucial, as in the presence of Dirichlet boundary conditions in (1.3) the Laplacian is the second normal derivative of u_ε on the boundary, and therefore positivity of the Laplacian on the boundary gives positivity of u_ε close to the boundary, see Sect. 4.

Similar strategies of examining a blow-up to the half space and using explicit formulas have been employed by Grunau and Robert in [20] and Grunau, Robert and Sweers in [21] to show lower bounds for the Greens function of a polyharmonic operator. This method was later refined by Pulst in his PhD-thesis [30] to also obtain such estimates, if non-constant lower-order terms are present. If variable coefficients in the principal part of the operator are given by a power of a second-order elliptic linear operator, such estimates were found by the same method by Grunau in [18].

Our blow-up strategy needs careful estimates for the singular problem (1.3). Estimates for these kinds of problems have a long history, see, e.g. [13, 17, 25] and [29]. We are not aware of any specific estimates, which would help in our specific situation. For this reason and for the sake of completeness, we derive them here.

2 Preliminary estimates

We proceed by contradiction and assume we find nonnegative $f_\epsilon \in L^\infty(\Omega)$ for $\epsilon \downarrow 0$ satisfying (1.4), such that the solution u_ϵ of (1.3) is sign-changing. As (1.3) is homogeneous of degree one, we may assume by scaling

$$0 \leq f_\epsilon \leq 1 \quad \text{and} \quad \int f_\epsilon d\mathcal{L}^n \geq \tau > 0. \tag{2.1}$$

By the Banach–Alaoglu Theorem, we get after passing to subsequence and relabelling $f_\epsilon \rightarrow f$ weakly* in $L^\infty(\Omega)$ with

$$0 \leq f \leq 1 \quad \text{and} \quad \int f d\mathcal{L}^n \geq \tau > 0. \tag{2.2}$$

In particular, we have $f \not\equiv 0$.

The limiting problem of (1.3) is thought to be the second-order boundary-value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

We also consider

$$\begin{aligned} -\Delta u_{0,\epsilon} &= f_\epsilon && \text{in } \Omega, \\ u_{0,\epsilon} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.4}$$

These two problems admit some important estimates:

Proposition 2.1 *For u and $u_{0,\epsilon}$ in (2.3) resp. (2.4) we find a constant $c_0 = c_0(\Omega, \tau) > 0$, such that*

$$-\partial_\nu u, -\partial_\nu u_{0,\epsilon} \geq c_0 > 0 \quad \text{on } \partial\Omega \tag{2.5}$$

and

$$u, u_{0,\epsilon} \geq c_0 d(\cdot, \partial\Omega) \quad \text{on } \Omega \tag{2.6}$$

for $\epsilon > 0$ small enough.

Proof Both problems admit by standard elliptic theory, see [16] Theorem 9.15, unique solutions u respectively $u_{0,\epsilon} \in W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega)$ for all $1 < p < \infty$ and $1 - (n/p) > \alpha > 0$ with

$$\|u, u_{0,\epsilon}\|_{W^{2,p}(\Omega) \cap C^{1,\alpha}(\Omega)} \leq C(\Omega, p, \alpha) < \infty \quad \forall 1 < p < \infty, 0 < \alpha < 1. \tag{2.7}$$

In particular, the set of all $u, u_{0,\epsilon}$ for any f, f_ϵ with (2.2) and (2.1) is compact in $C^1(\bar{\Omega})$, and in particular

$$u_{0,\epsilon} \rightarrow u \quad \text{strongly in } C^{1,\alpha}(\Omega), \forall 0 < \alpha < 1, \tag{2.8}$$

and weakly in $W^{2,p}(\Omega)$ for all $1 < p < \infty$.

As $f, f_\epsilon \geq 0, f, f_\epsilon \not\equiv 0$ and Ω is connected, we get by the strong maximum principle, see [16] Theorem 8.19, that

$$u, u_{0,\epsilon} > 0 \text{ in } \Omega. \tag{2.9}$$

By Hopf’s maximum principle, see [16] Lemma 3.4, and by compactness of the $u, u_{0,\epsilon}$ in $C^1(\overline{\Omega})$ we find a constant $c_0 = c_0(\Omega, \tau) > 0$ such that (2.5) and (2.6) both hold for $\epsilon > 0$ small enough.

We put $v_\epsilon := u_\epsilon - u_{0,\epsilon} \in W^{2,p}(\Omega) \cap C^{1,\alpha}(\Omega)$ with $v_\epsilon = 0$ on $\partial\Omega$. Subtracting in (1.3), we see

$$\Delta(\epsilon^2 \Delta u_\epsilon - v_\epsilon) = 0 \quad \text{in } \Omega, \tag{2.10}$$

that is $\epsilon^2 \Delta u_\epsilon - v_\epsilon$ is harmonic in Ω .

Equation (2.3) is indeed the limiting problem of (1.3) in the sense of the following proposition.

Proposition 2.2 *For $u_\epsilon, u, v_\epsilon = u_\epsilon - u_{0,\epsilon}$ as in (1.3), (2.1), (2.4), we have*

$$\begin{aligned} u_\epsilon &\rightarrow u, v_\epsilon \rightarrow 0 \text{ strongly in } W_0^{1,2}(\Omega), \\ \epsilon \Delta u_\epsilon &\rightarrow 0 \text{ strongly in } L^2(\Omega). \end{aligned} \tag{2.11}$$

Proof Multiplying (2.10) by v_ϵ and integrating by parts, we get

$$\begin{aligned} 0 &= \int_\Omega \Delta(\epsilon^2 \Delta u_\epsilon - v_\epsilon) v_\epsilon \, d\mathcal{L}^n \\ &= \int_\Omega \epsilon^2 (\Delta^2 u_\epsilon) u_\epsilon \, d\mathcal{L}^n - \int_\Omega \epsilon^2 (\Delta^2 u_\epsilon) u_{0,\epsilon} \, d\mathcal{L}^n + \int_\Omega |\nabla v_\epsilon|^2 \, d\mathcal{L}^n. \end{aligned}$$

Replacing $\epsilon^2 \Delta^2 u_\epsilon$ by Δv_ϵ in the second term with (2.10), we continue

$$\begin{aligned} \int_\Omega \epsilon^2 |\Delta u_\epsilon|^2 \, d\mathcal{L}^n + \int_\Omega |\nabla v_\epsilon|^2 \, d\mathcal{L}^n &= \int_\Omega (\Delta v_\epsilon) u_{0,\epsilon} \, d\mathcal{L}^n \\ &= - \int_\Omega \nabla v_\epsilon \nabla u_{0,\epsilon} \, d\mathcal{L}^n \leq \| \nabla v_\epsilon \|_{L^2(\Omega)} \| \nabla u_{0,\epsilon} \|_{L^2(\Omega)}, \end{aligned} \tag{2.12}$$

in particular $\| \nabla v_\epsilon \|_{L^2(\Omega)} \leq \| \nabla u_{0,\epsilon} \|_{L^2(\Omega)} \leq C(\Omega)$ by (2.7), hence

$$\int_\Omega \epsilon^2 |\Delta u_\epsilon|^2 \, d\mathcal{L}^n + \int_\Omega |\nabla v_\epsilon|^2 \, d\mathcal{L}^n \leq \int_\Omega |\nabla u_{0,\epsilon}|^2 \, d\mathcal{L}^n \leq C(\Omega).$$

Passing to a subsequence, we get

$$\begin{aligned} v_\epsilon &\rightarrow v \text{ weakly in } W_0^{1,2}(\Omega), \\ \epsilon^2 \Delta u_\epsilon &\rightarrow 0 \text{ strongly in } L^2(\Omega). \end{aligned}$$

Multiplying (2.10) by some $\eta \in C_0^\infty(\Omega)$ and passing to a subsequence, we get

$$0 = \int_\Omega (\epsilon^2 \Delta^2 u_\epsilon - \Delta v_\epsilon) \eta \, d\mathcal{L}^n = \int_\Omega \epsilon^2 \Delta u_\epsilon \cdot \Delta \eta \, d\mathcal{L}^n + \int_\Omega \nabla v_\epsilon \nabla \eta \, d\mathcal{L}^n \rightarrow \int_\Omega \nabla v \nabla \eta \, d\mathcal{L}^n,$$

and $v \in W_0^{1,2}(\Omega)$ is harmonic in Ω , hence $v = 0$ and $u_\epsilon \rightarrow u$ weakly in $W_0^{1,2}(\Omega)$. Returning to (2.12), we improve now to

$$\int_{\Omega} \epsilon^2 |\Delta u_\epsilon|^2 d\mathcal{L}^n + \int_{\Omega} |\nabla v_\epsilon|^2 d\mathcal{L}^n = - \int_{\Omega} \nabla v_\epsilon \nabla u_{0,\epsilon} d\mathcal{L}^n \rightarrow 0,$$

which is (2.11). □

The following proposition shows that the Laplacian cannot be bounded throughout Ω .

Proposition 2.3 *For u_ϵ, τ as in (1.3), (2.1), (2.2), we have for any $\Omega' \subset\subset \Omega$ that*

$$\liminf_{\epsilon \rightarrow 0} \int_{\Omega \setminus \Omega'} (\Delta u_\epsilon)_+ d\mathcal{L}^n \geq \tau > 0.$$

Proof We see for any $\eta \in C_0^\infty(\Omega)$ with the previous Proposition 2.2

$$\int (\Delta u_\epsilon) \eta d\mathcal{L}^n = \int u_\epsilon \Delta \eta d\mathcal{L}^n \rightarrow \int u \Delta \eta d\mathcal{L}^n = \int (\Delta u) \eta d\mathcal{L}^n = - \int f \eta d\mathcal{L}^n$$

and the homogeneous boundary conditions in (1.3) that

$$\begin{aligned} \int \Delta u_\epsilon (1 - \eta) d\mathcal{L}^n &= \int \Delta u_\epsilon d\mathcal{L}^n - \int (\Delta u_\epsilon) \eta d\mathcal{L}^n \\ &= \int_{\partial\Omega} \partial_\nu u_\epsilon d\text{area}_{\partial\Omega} - \int (\Delta u_\epsilon) \eta d\mathcal{L}^n \rightarrow \int f \eta d\mathcal{L}^n. \end{aligned}$$

Choosing $\eta \in C_0^\infty(\Omega)$ with $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in Ω' , we get

$$\int_{\Omega \setminus \Omega'} (\Delta u_\epsilon)_+ d\mathcal{L}^n \geq \int (\Delta u_\epsilon) (1 - \eta) d\mathcal{L}^n \rightarrow \int f \eta d\mathcal{L}^n.$$

Letting $\eta \nearrow \chi_{\Omega'}$, we get from (2.2) that

$$\int f \eta d\mathcal{L}^n \rightarrow \int f d\mathcal{L}^n \geq \tau > 0,$$

and the proposition follows.

3 The Laplacian on the boundary

In this section, we investigate the values of the Laplacian on the boundary and put

$$\ell_\epsilon^- := \min_{\partial\Omega} \epsilon^2 \Delta u_\epsilon \leq \max_{\partial\Omega} \epsilon^2 \Delta u_\epsilon =: \ell_\epsilon^+. \tag{3.1}$$

With subscripts \pm , we denote the positive, respectively, negative part, i.e.

$$\ell_{\epsilon,+}^\pm := \max(0, \ell_\epsilon^\pm), \ell_{\epsilon,-}^\pm := \max(0, -\ell_\epsilon^\pm). \tag{3.2}$$

Furthermore, we set

$$M_\epsilon := \|\epsilon^2 \Delta u_\epsilon\|_{L^\infty(\partial\Omega)} = \max(\ell_{\epsilon,+}^+, \ell_{\epsilon,-}^-). \tag{3.3}$$

The quantity M_ϵ will be crucial throughout the exposition. Our goal is to show that it has the same asymptotic as ϵ itself, i.e. we find constants $c_0, C > 0$ such that

$$c_0\epsilon \leq M_\epsilon \leq C\epsilon.$$

for $\epsilon > 0$ small. A first step in this direction is Proposition 3.3, which will later be improved to our desired result in Proposition 4.2 and (4.19).

With the maximum principle, we get the following estimates.

Proposition 3.1 *For $u_\epsilon, f_\epsilon, u_{0,\epsilon}, v_\epsilon = u_\epsilon - u_{0,\epsilon}, \ell_\epsilon^\pm$ as in (1.3), (2.1), (2.4), (3.1), we have*

$$\ell_{\epsilon}^- \leq \epsilon^2 \Delta u_\epsilon - v_\epsilon \leq \ell_{\epsilon,+}^+ \quad \text{in } \Omega, \tag{3.4}$$

$$-\ell_{\epsilon,+}^+ - \epsilon^2 \|f_\epsilon\|_{L^\infty(\Omega)} \leq v_\epsilon \leq \ell_{\epsilon,-}^- \quad \text{in } \Omega, \tag{3.5}$$

$$u_\epsilon > -\ell_{\epsilon,+}^+ \quad \text{in } \Omega. \tag{3.6}$$

Proof As $v_\epsilon = u_\epsilon - u_{0,\epsilon} = 0$ on $\partial\Omega$ by (1.3) and (2.4), we get (3.4) from (2.10).

Adding (2.4), we see

$$\epsilon^2 \Delta v_\epsilon - v_\epsilon = -\epsilon^2 \Delta u_{0,\epsilon} + \epsilon^2 \Delta u_\epsilon - v_\epsilon \leq \epsilon^2 f_\epsilon + \ell_{\epsilon,+}^+ \quad \text{in } \Omega,$$

in particular

$$\epsilon^2 \Delta v_\epsilon < 0 \quad \text{in } [v_\epsilon < -\epsilon^2 f_\epsilon - \ell_{\epsilon,+}^+] =: \Omega_0.$$

Since $v_\epsilon \geq -\epsilon^2 f_\epsilon - \ell_{\epsilon,+}^+$ on $\partial\Omega_0$, as $v_\epsilon = 0$ on $\partial\Omega$ by above, we get from the mean-value estimate for superharmonic functions or by Alexandroff’s maximum principle, as $v_\epsilon \in W^{2,n}(\Omega)$, see [16] Theorem 9.1, that $v_\epsilon \geq -\epsilon^2 f_\epsilon - \ell_{\epsilon,+}^+$ in Ω_0 , hence $\Omega_0 = \emptyset$, and the left estimate in (3.5) follows. The right estimate is obtained by symmetry observing that $f_\epsilon \geq 0$.

Next for $x \in \bar{\Omega}$ with $u_\epsilon(x) = \min_{\bar{\Omega}} u_\epsilon$ and assuming that this minimum is negative, we see $x \in \Omega$, as $u_\epsilon = 0$ on $\partial\Omega$ by (1.3), hence $\Delta u_\epsilon(x) \geq 0$, as $u_\epsilon \in W^{4,p}(\Omega) \hookrightarrow C^2(\Omega)$ for $2 - (n/p) > 0$. Then, we get with (3.4) and (2.9) that

$$-u_\epsilon(x) \leq \epsilon^2 \Delta u_\epsilon(x) - v_\epsilon(x) - u_{0,\epsilon}(x) < \ell_{\epsilon,+}^+,$$

which is (3.6). □

Using the fourth-order equation, we get estimates for the Laplacian.

Proposition 3.2 *For $u_\epsilon, f_\epsilon, u_{0,\epsilon}, v_\epsilon = u_\epsilon - u_{0,\epsilon}, \ell_\epsilon^\pm$ as in (1.3), (2.1), (2.4), (3.1), we have*

$$-\ell_{\epsilon,-}^- - \epsilon^2 \|f_\epsilon\|_{L^\infty(\Omega)} \leq \epsilon^2 \Delta u_\epsilon \leq \ell_{\epsilon,+}^+ \quad \text{in } \Omega. \tag{3.7}$$

Proof We have with (1.3) that

$$\epsilon^2 \Delta^2 u_\epsilon - \Delta u_\epsilon = f_\epsilon \quad \text{in } \Omega$$

and get

$$\epsilon^2 \Delta(\Delta u_\epsilon) < 0 \quad \text{in } [\Delta u_\epsilon < - \|f_{\epsilon,+}\|_{L^\infty(\Omega)} - \epsilon^{-2} \ell_{\epsilon,-}^-] =: \Omega_0.$$

Since $\Delta u_\epsilon \geq - \|f_{\epsilon,+}\|_{L^\infty(\Omega)} - \epsilon^{-2} \ell_{\epsilon,-}^-$ on $\partial\Omega_0$, as $\Delta u_\epsilon \geq \epsilon^{-2} \ell_{\epsilon,-}^- \geq -\epsilon^{-2} \ell_{\epsilon,-}^-$ on $\partial\Omega$ with (3.1), we get from the mean-value estimate for superharmonic functions or by Alexandroff’s maximum principle, as $\Delta u_\epsilon \in W^{2,n}(\Omega)$, see [16] Theorem 9.1, that $\Delta u_\epsilon \geq - \|f_{\epsilon,+}\|_{L^\infty(\Omega)} - \epsilon^{-2} \ell_{\epsilon,-}^-$ in Ω_0 , hence $\Omega_0 = \emptyset$, and the left estimate in (3.7) follows. The right estimate is obtained by symmetry observing that $f_\epsilon \geq 0$. \square

Here, we can give a preliminary asymptotic estimate for M_ϵ . Actually, we will improve this asymptotic later in Proposition 4.2 and (4.19). Anyway we present this estimate at this stage to get more compact bounds already now.

Proposition 3.3 For $u_\epsilon, M_\epsilon, \ell_\epsilon^\pm$ as in (1.3), (2.1), (3.1), (3.3), we have

$$\epsilon^{-2} M_\epsilon \geq \epsilon^{-2} \ell_{\epsilon,+}^+ \rightarrow \infty.$$

in particular $M_\epsilon \geq \ell_\epsilon^+ = \ell_{\epsilon,+}^+ > 0$ for ϵ small depending on Ω and τ .

Proof Combining Proposition 3.2 (3.7) and Proposition 2.3, we get for any $\Omega' \subset\subset \Omega$ that

$$\mathcal{L}^n(\Omega \setminus \Omega') \liminf_{\epsilon \rightarrow 0} \epsilon^{-2} \ell_{\epsilon,+}^+ \geq \liminf_{\epsilon \rightarrow 0} \int_{\Omega \setminus \Omega'} (\Delta u_\epsilon)_+ d\mathcal{L}^n \geq \tau > 0,$$

hence, as $\mathcal{L}^n(\Omega \setminus \Omega')$ can be made arbitrarily small, that

$$\epsilon^{-2} \ell_{\epsilon,+}^+ \rightarrow \infty,$$

which yields the assertion. \square

With the above asymptotic, we can already bound v_ϵ , and we can prove that u is positive on large parts of Ω .

Proposition 3.4 For $u_\epsilon, u_{0,\epsilon}, v_\epsilon = u_\epsilon - u_{0,\epsilon}, M_\epsilon$ as in (1.3), (2.1), (2.2), (2.4), (3.3), we have

$$\limsup_{\epsilon \rightarrow 0} \|M_\epsilon^{-1} v_\epsilon\|_{L^\infty(\Omega)} \leq 1.$$

Proof Combining Proposition 3.1 (3.5) and Proposition 3.3, we get observing (2.1) that

$$\limsup_{\epsilon \rightarrow 0} \|M_\epsilon^{-1} v_\epsilon\|_{L^\infty(\Omega)} \leq \limsup_{\epsilon \rightarrow 0} M_\epsilon^{-1} (M_\epsilon + \epsilon^2 \|f_\epsilon\|_{L^\infty(\Omega)}) \leq 1.$$

\square

Proposition 3.5 For $u_\epsilon, \ell_\epsilon^\pm$ as in (1.3), (2.1), (3.1), we have

$$u_\epsilon > 0 \quad \text{in } [d(\cdot, \partial\Omega) \geq C\ell_\epsilon^+] \cap \Omega \tag{3.8}$$

for some $C = C(\Omega, \tau) < \infty$ and ϵ small depending on Ω and τ .

Proof From (2.6), we see for $x \in \Omega$ with $c_0d(x, \partial\Omega) > \ell_{\epsilon,+}^+ + \epsilon^2 \|f_\epsilon\|_{L^\infty(\Omega)}$ by (3.5) that

$$u_\epsilon(x) = v_\epsilon(x) + u(x) \geq -\ell_{\epsilon,+}^+ - \epsilon^2 \|f_\epsilon\|_{L^\infty(\Omega)} + c_0d(x, \partial\Omega) > 0.$$

By Proposition 3.3 and (2.1) clearly

$$2\ell_\epsilon^+ \geq \ell_{\epsilon,+}^+ + \epsilon^2 \|f_\epsilon\|_{L^\infty(\Omega)}$$

for ϵ small, and (3.8) follows for $C = 2c_0^{-1} < \infty$. □

4 Blow-up

In this section, we consider a blow-up of our solutions u_ϵ by translating and rescaling with $x_{0,\epsilon} \in \mathbb{R}^n$. We will have to choose $x_{0,\epsilon}$ differently in different steps in the proof of Theorem 1.1. In, e.g. the proof of Claim 1 in the proof of Theorem 1.1, we choose $x_{0,\epsilon} \in \partial\Omega$ such that $\max_{x \in \partial\Omega} |\epsilon^2 \Delta u_\epsilon(x)|$ is attained, while in the proof of Claim 2, we choose $x_{0,\epsilon} \in \partial\Omega$ to attain ℓ_ϵ^- . This will not cause a problem, because the constants yielded by these claims do not depend on ϵ . We put

$$\tilde{u}_\epsilon(x) := u_\epsilon(x_{0,\epsilon} + \epsilon x) \quad \text{for } x \in \tilde{\Omega}_\epsilon := \epsilon^{-1}(\Omega - x_{0,\epsilon}). \tag{4.1}$$

Then

$$\nabla \tilde{u}_\epsilon = \epsilon \nabla u_\epsilon(x_{0,\epsilon} + \epsilon \cdot), \quad \Delta \tilde{u}_\epsilon = \epsilon^2 \Delta u_\epsilon(x_{0,\epsilon} + \epsilon \cdot) \tag{4.2}$$

and by (1.3) that

$$\begin{aligned} \Delta^2 \tilde{u}_\epsilon - \Delta \tilde{u}_\epsilon &= \epsilon^2 f_\epsilon(x_{0,\epsilon} + \epsilon \cdot) =: \tilde{f}_\epsilon && \text{on } \tilde{\Omega}_\epsilon, \\ \tilde{u}_\epsilon, \partial_{\nu_{\tilde{\Omega}_\epsilon}} \tilde{u}_\epsilon &= 0 && \text{on } \partial \tilde{\Omega}_\epsilon \end{aligned} \tag{4.3}$$

and with (2.1) and Proposition 3.3 that

$$\|\tilde{f}_\epsilon\|_{L^\infty(\tilde{\Omega}_\epsilon)} = \|f_\epsilon\|_{L^\infty(\Omega)} \epsilon^2 = o(1)M_\epsilon \quad \text{for } \epsilon \rightarrow 0, \tag{4.4}$$

where $o(1) \rightarrow 0$ depending on Ω and τ . We extend u_ϵ respectively \tilde{u}_ϵ by putting 0 outside Ω respectively outside $\tilde{\Omega}_\epsilon$. By the homogeneous boundary conditions in (1.3) and (4.3), we see $u_\epsilon, \tilde{u}_\epsilon \in W_{loc}^{2,2}(\mathbb{R}^n)$.

We also have to stretch \tilde{u}_ϵ to get a nontrivial limit, and it turns out that reaching bounded values of the Laplacian of \tilde{u}_ϵ on the boundary is the right measure for stretching.

Proposition 4.1 For $\tilde{u}_\epsilon, \tilde{f}_\epsilon, u_\epsilon, f_\epsilon, M_\epsilon$ as in (4.1), (1.3), (2.1), (3.3) and $x_{0,\epsilon} \in \partial\Omega$, we get for any subsequence with after rotating $\tilde{\Omega}_\epsilon$ such that

$$v_{\Omega}(x_{0,\varepsilon}) \rightarrow -e_n \tag{4.5}$$

after passing to a subsequence

$$\begin{aligned} M_\varepsilon^{-1}\tilde{u}_\varepsilon &\rightarrow \tilde{u}_\infty \quad \text{weakly in } W^{2,2}(B_R(0)) \text{ for all } R > 0, \\ M_\varepsilon^{-1}\tilde{u}_\varepsilon &\rightarrow \tilde{u}_\infty \quad \text{weakly in } W^{4,p}(B_R(0) \cap \tilde{\Omega}_\varepsilon) \text{ for all } R > 0, 1 < p < \infty \end{aligned} \tag{4.6}$$

as $\varepsilon \rightarrow 0$ after flattening the boundary of $\partial\tilde{\Omega}_\varepsilon$. Further

$$\begin{aligned} \Delta^2\tilde{u}_\infty - \Delta\tilde{u}_\infty &= 0 \quad \text{in } \mathbb{R}_+^n, \\ \tilde{u}_\infty, \partial_n\tilde{u}_\infty &= 0 \quad \text{on } \mathbb{R}^{n-1} \times \{0\}, \\ \tilde{u}_\infty &= 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}_+^n, \\ \tilde{u}_\infty &\geq -1, \quad |\Delta\tilde{u}_\infty| \leq 1 \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{4.7}$$

Proof We get from Proposition 3.1 (3.6) that

$$\tilde{u}_\varepsilon \geq -M_\varepsilon \quad \text{in } \tilde{\Omega}_\varepsilon \tag{4.8}$$

and also outside $\tilde{\Omega}_\varepsilon$, as $\tilde{u}_\varepsilon = 0$ there. Next by Proposition 3.2 (3.7), Proposition 3.3, (2.1) and (4.2) that

$$\|\Delta\tilde{u}_\varepsilon\|_{L^\infty(\tilde{\Omega}_\varepsilon)} = \|\varepsilon^2\Delta u_\varepsilon\|_{L^\infty(\Omega)} \leq M_\varepsilon + \varepsilon^2\|f_\varepsilon\|_{L^\infty(\Omega)} \leq (1 + o(1))M_\varepsilon. \tag{4.9}$$

Then, $\tilde{u}_\varepsilon + M_\varepsilon \geq 0$ in \mathbb{R}^n , and we can apply the Harnack inequality, see [16] Theorems 8.17 and 8.18, and get observing that $\tilde{u}_\varepsilon(0) = 0$, as $x_{0,\varepsilon} \in \partial\Omega$, that

$$\sup_{B_R(0)}(\tilde{u}_\varepsilon + M_\varepsilon) \leq C_n(\tilde{u}_\varepsilon + M_\varepsilon)(0) + C_{n,R}\|\Delta(\tilde{u}_\varepsilon + M_\varepsilon)\|_{L^\infty(B_{2R}(0))} \leq C_{n,R}M_\varepsilon,$$

hence

$$\|\tilde{u}_\varepsilon\|_{L^\infty(B_R(0))} \leq C_{n,R}M_\varepsilon \tag{4.10}$$

for ε small depending on Ω and τ .

Next by Friedrichs's Theorem in the interior, see [16] Theorem 8.8, [16] Exercise 8.2, (4.9) and (4.10) that

$$\|\tilde{u}_\varepsilon\|_{W^{2,2}(B_R(0))} \leq C_{n,R}\left(\|\Delta\tilde{u}_\varepsilon\|_{L^2(B_{2R}(0))} + \|\tilde{u}_\varepsilon\|_{L^2(B_{2R}(0))}\right) \leq C_{n,R}M_\varepsilon \tag{4.11}$$

for ε small depending on Ω and τ , which yields the first convergence in (4.6) after passing to a subsequence.

Proceeding from (4.3), we get from fourth-order L^p -estimates, see [1, 2] Sect. 10, after flattening the boundary of $\partial\tilde{\Omega}_\varepsilon$, as $\partial\Omega \in C^4$, with (4.4) and (4.10) that

$$\begin{aligned} \|\tilde{u}_\varepsilon\|_{W^{4,p}(B_R(0) \cap \tilde{\Omega}_\varepsilon)} &\leq C_{n,R,p}\left(\|\tilde{f}_\varepsilon\|_{L^\infty(B_{2R}(0) \cap \tilde{\Omega}_\varepsilon)} + \|\tilde{u}_\varepsilon\|_{L^\infty(B_{2R}(0) \cap \tilde{\Omega}_\varepsilon)}\right) \\ &\leq C_{n,R,p}M_\varepsilon \quad \forall R > 0, 1 < p < \infty \end{aligned}$$

and ε small depending on Ω and τ . After passing to a subsequence, we obtain with (4.5) the second convergence in (4.6).

Finally (4.7) follows from (4.3), (4.4), (4.8), (4.9), when recalling that $\tilde{u}_\varepsilon = 0$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$.

Actually by fourth-order higher-order L^p -estimates, see [1, 2] Sect. 10, we get that the blow-up \tilde{u}_∞ is smooth on $\overline{\mathbb{R}_+^n}$.

Now we are able to give a lower bound for M_ϵ which improves the asymptotic in Proposition 3.3.

Proposition 4.2 *For u_ϵ, M_ϵ as in (1.3), (2.1), (3.3), we have*

$$M_\epsilon \geq c_0 \epsilon$$

for some $c_0 = c_0(\Omega, \tau) > 0$ and ϵ small depending on Ω and τ .

Proof We see for $x_\epsilon \in \Omega$ with $|x_\epsilon - x_{0,\epsilon}| = d(x_\epsilon, \partial\Omega) = \epsilon$ for ϵ small by (2.6), Proposition 3.4 and by the local boundedness of \tilde{u}_ϵ in Proposition 4.1, or more directly by (4.10), for $\tilde{x}_\epsilon := (x_\epsilon - x_{0,\epsilon})/\epsilon \in B_1(0)$ that

$$\begin{aligned} c_0 \epsilon &\leq u_{0,\epsilon}(x_\epsilon) = u_\epsilon(x_\epsilon) - v_\epsilon(x_\epsilon) = \tilde{u}_\epsilon(\tilde{x}_\epsilon) - v_\epsilon(x_\epsilon) \\ &\leq \|\tilde{u}_\epsilon\|_{L^\infty(B_1(0))} + \|v_\epsilon\|_{L^\infty(\Omega)} \leq (C_{n,1} + o(1))M_\epsilon, \end{aligned}$$

hence

$$M_\epsilon \geq c_0 \epsilon$$

for ϵ small depending on Ω and τ . □

The blow-up for $u_{0,\epsilon}$ is rather elementary by the strong convergence in $C^{1,\alpha}(\Omega)$ in (2.8). As in (4.1), we put

$$\tilde{u}_{0,\epsilon}(x) := u_{0,\epsilon}(x_{0,\epsilon} + \epsilon x) \quad \text{for } x \in \tilde{\Omega}_\epsilon := \epsilon^{-1}(\Omega - x_{0,\epsilon}). \tag{4.12}$$

Proposition 4.3 *For $\tilde{u}_{0,\epsilon}, u_\epsilon, u_{0,\epsilon}, f_\epsilon, M_\epsilon$ as in (4.12), (1.3), (2.1), (3.3), and with (4.5), we have after passing to a subsequence such that*

$$\beta \leftarrow |\nabla u_{0,\epsilon}(x_{0,\epsilon})| \epsilon / M_\epsilon \geq c_0 \epsilon / M_\epsilon \geq 0 \tag{4.13}$$

for some $c_0 = c_0(\Omega, \tau) > 0$ for the linear function $\tilde{u}_{0,\infty} : (y, t) \rightarrow \beta t$ that

$$M_\epsilon^{-1} \tilde{u}_{0,\epsilon} \rightarrow \tilde{u}_{0,\infty} \quad \text{uniformly in } B_R(0) \cap \tilde{\Omega}_\epsilon \text{ for all } R > 0$$

after flattening the boundary of $\partial\tilde{\Omega}_\epsilon$. Further

$$|\tilde{u}_\infty - \tilde{u}_{0,\infty}| \leq 1 \quad \text{in } \mathbb{R}_+^n, \tag{4.14}$$

in particular $\tilde{u}_\infty - \tilde{u}_{0,\infty} \in L^\infty(\mathbb{R}_+^n)$.

Proof From (2.7) and, as $u_{0,\epsilon} = 0$ on $\partial\Omega$ by (2.4), we get by Taylor’s expansion for any $x \in \mathbb{R}_+^n$ and any $0 < \alpha < 1$ that

$$M_\epsilon^{-1} \tilde{u}_{0,\epsilon}(x) = M_\epsilon^{-1} u_{0,\epsilon}(x_{0,\epsilon} + \epsilon x) = M_\epsilon^{-1} \epsilon \nabla u_{0,\epsilon}(x_{0,\epsilon}) \cdot x + M_\epsilon^{-1} O_\alpha(\epsilon^{1+\alpha} |x|^{1+\alpha}). \tag{4.15}$$

As $u_{0,\epsilon} = 0$ on $\partial\Omega$ by (2.4), we get with (2.8), (4.5) and $\partial\Omega \in C^4$ after passing to a subsequence with $x_{0,\epsilon} \rightarrow x_0 \in \partial\Omega$ that

$$\nabla u_{0,\varepsilon}(x_{0,\varepsilon}) = \partial_{v_\Omega} u_{0,\varepsilon}(x_{0,\varepsilon}) v_\Omega(x_{0,\varepsilon}) \rightarrow -\partial_{v_\Omega} u(x_0) e_n,$$

in particular with (2.5) that

$$|\nabla u_{0,\varepsilon}(x_{0,\varepsilon})| \rightarrow -\partial_{v_\Omega} u(x_0) > 0$$

and $\nabla u_{0,\varepsilon}(x_{0,\varepsilon})/|\nabla u_{0,\varepsilon}(x_{0,\varepsilon})| \rightarrow e_n$. By Proposition 4.2 we extract a subsequence, such that $\varepsilon/M_\varepsilon$ converges for $\varepsilon \downarrow 0$. Then, (2.5) yields

$$\beta := \leftarrow |\nabla u_{0,\varepsilon}(x_{0,\varepsilon})| \varepsilon / M_\varepsilon \geq c_0 \varepsilon / M_\varepsilon \geq 0,$$

which is (4.13). Furthermore, Proposition 4.2 yields

$$\varepsilon^{1+\alpha} / M_\varepsilon \rightarrow 0.$$

Together, we get

$$M_\varepsilon^{-1} \varepsilon \nabla u_{0,\varepsilon}(x_{0,\varepsilon}) \rightarrow \beta e_n,$$

and the proposed convergence follows from the Taylor expansion (4.15).

Further we get with Proposition 3.4 that

$$|\tilde{u}_\infty - \tilde{u}_{0,\infty}| \leftarrow |M_\varepsilon^{-1} \tilde{u}_\varepsilon - M_\varepsilon^{-1} \tilde{u}_{0,\varepsilon}| \leq \limsup_{\varepsilon \rightarrow 0} \|M_\varepsilon^{-1} v_\varepsilon\|_{L^\infty(\Omega)} \leq 1,$$

which is (4.14).

By our investigation of half space solutions in Appendix A, Proposition A.3 applied to \tilde{u}_∞ and $\tilde{u}_{0,\infty}$ with Proposition 4.1 (4.7) and Proposition 4.3 (4.14) determines \tilde{u}_∞ uniquely as the one-dimensional solution and immediately yields the following Proposition.

Proposition 4.4 *For \tilde{u}_∞ as in Proposition 4.1 and β as in Proposition 4.3, we have*

$$\tilde{u}_\infty \equiv 0, \quad \text{if } \beta = 0, \tag{4.16}$$

$$\Delta \tilde{u}_\infty \equiv \beta \geq 0 \quad \text{in } \mathbb{R}^{n-1} \times \{0\} \tag{4.17}$$

and

$$\tilde{u}_\infty > 0 \quad \text{in } \mathbb{R}_+^n, \quad \text{if } \beta > 0. \tag{4.18}$$

□

Now we are able to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1 We consider $u_\varepsilon, f_\varepsilon, u_{0,\varepsilon}, \ell_\varepsilon^\pm, M_\varepsilon$ as in (1.3), (2.1), (2.4), (3.1), (3.3) and $\tilde{u}_\varepsilon, \tilde{f}_\varepsilon, \tilde{u}_{0,\varepsilon}$ as in (4.1), (4.12) with their blow-ups $\tilde{u}_\infty, \tilde{u}_{0,\infty}$ and β obtained in the Propositions 4.1 and 4.3. We prove various claims.

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon / M_\varepsilon > 0,$$

Claim 1

hence with Proposition 4.2 that

$$c_0 \leq M_\epsilon / \epsilon \leq C \tag{4.19}$$

for some $c_0 = c_0(\Omega, \tau) > 0, C = C(\Omega, \tau) < \infty$ and ϵ small depending on Ω and τ .

Proof If on contrary $\epsilon/M_\epsilon \rightarrow 0$ for a subsequence $\epsilon \rightarrow 0$, then we get from Proposition 4.3 and (2.7) that $\beta = 0$, hence with Proposition 4.4 (4.16) after passing to this subsequence we have that $\tilde{u}_\infty \equiv 0$.

On the other hand, choosing $x_{0,\epsilon} \in \partial\Omega$ in such a way that

$$|\epsilon^2 \Delta u_\epsilon(x_{0,\epsilon})| = \|\epsilon^2 \Delta u_\epsilon\|_{L^\infty(\partial\Omega)} = M_\epsilon,$$

we get from the convergence in Proposition 4.1 (4.6) that

$$1 = M_\epsilon^{-1} |\epsilon^2 \Delta u_\epsilon(x_{0,\epsilon})| = |M_\epsilon^{-1} \Delta \tilde{u}_\epsilon(0)| \rightarrow |\Delta \tilde{u}_\infty(0)|, \tag{4.20}$$

hence $\Delta \tilde{u}_\infty(0) \neq 0$ and $\tilde{u}_\infty \not\equiv 0$. This is a contradiction, and the claim follows.

Claim 2

$$\liminf_{\epsilon \rightarrow 0} M_\epsilon^{-1} \ell_\epsilon^- > 0, \tag{4.21}$$

$$M_\epsilon^{-1} \Delta \tilde{u}_\epsilon \geq c_1 > 0 \quad \text{on } \partial\tilde{\Omega}_\epsilon \tag{4.22}$$

for some $c_1 = c_1(\Omega, \tau) > 0$ and ϵ small depending on Ω and τ and

$$\tilde{u}_\infty > 0 \quad \text{in } \mathbb{R}_+^n. \tag{4.23}$$

Proof (4.19) implies with Proposition 4.3 (4.13) that

$$\beta \geq c_0 \liminf_{\epsilon \rightarrow 0} \epsilon/M_\epsilon > 0,$$

which immediately gives (4.23) by Proposition 4.4 (4.18).

Next we choose $x_{0,\epsilon} \in \partial\Omega$ in such a way that

$$\epsilon^2 \Delta u_\epsilon(x_{0,\epsilon}) = \min_{\partial\Omega} \epsilon^2 \Delta u_\epsilon = \ell_\epsilon^-$$

and get as in (4.20) from the convergence in Proposition 4.1 (4.6) and Proposition 4.4 (4.17) that

$$M_\epsilon^{-1} \ell_\epsilon^- = M_\epsilon^{-1} \epsilon^2 \Delta u_\epsilon(x_{0,\epsilon}) = M_\epsilon^{-1} \Delta \tilde{u}_\epsilon(0) \rightarrow \Delta \tilde{u}_\infty(0) = \beta > 0,$$

and (4.21) follows. Clearly (4.21) implies with (3.1) that

$$M_\epsilon^{-1} \inf_{\partial\Omega_\epsilon} \Delta \tilde{u}_\epsilon = M_\epsilon^{-1} \inf_{\partial\Omega} \epsilon^2 \Delta u_\epsilon = M_\epsilon^{-1} \ell_\epsilon^- \geq c_1 > 0$$

for some $c_1 > 0$ and ϵ small, which is (4.22).

Further (4.21) implies that $\ell_\epsilon^+ \geq \ell_\epsilon^- > 0$ and $\ell_{\epsilon,-}^- = 0$ for ϵ small, hence with (3.3) that

$$M_\epsilon = \max(\ell_{\epsilon,+}^+, \ell_{\epsilon,-}^-) = \ell_\epsilon^+ \tag{4.24}$$

for ϵ small depending on Ω and τ .

Claim 3

$$u_\epsilon > 0 \quad \text{in } [d(\cdot, \partial\Omega) \geq C\epsilon] \cap \Omega \tag{4.25}$$

for some $C = C(\Omega, \tau) < \infty$ and ϵ small depending on Ω and τ .

This follows directly from Proposition 3.5, (4.19) and (4.24).

Claim 4

$$u_\epsilon > 0 \quad \text{in } [0 < d(\cdot, \partial\Omega) \leq c_2\epsilon] \cap \Omega \tag{4.26}$$

for some $c_2 = c_2(\Omega, \tau) > 0$ and ϵ small depending on Ω and τ .

By the homogeneous boundary conditions in (1.3), we have $\Delta u_\epsilon(x_{0,\epsilon}) = \partial_{vv}u_\epsilon(x_{0,\epsilon})$, hence with (4.22), Proposition 4.1 (4.6) and the embedding $W^{4,p} \hookrightarrow C^3$ for $1 - (n/p) > 0$ that

$$\tilde{u}_\epsilon(-tv_{\tilde{\Omega}_\epsilon}(0)) \geq \frac{1}{2}t^2\Delta\tilde{u}_\epsilon(0) - C_nM_\epsilon t^3 \geq \frac{1}{2}t^2M_\epsilon(c_1 - 2C_n t) > 0$$

for $0 < t < c_1/(2C_n) < 1$ and ϵ small. As $x_{0,\epsilon} \in \partial\Omega$ can be chosen arbitrarily, we get for $c_2 = c_1/(4C_n)$ that

$$\tilde{u}_\epsilon > 0 \quad \text{in } [0 < d(\cdot, \partial\tilde{\Omega}_\epsilon) \leq c_2] \cap \tilde{\Omega}_\epsilon,$$

and the claim follows by rescaling in (4.1).

Claim 5

$$u_\epsilon > 0 \quad \text{in } [c_2\epsilon \leq d(\cdot, \partial\Omega) \leq C\epsilon] \cap \Omega \tag{4.27}$$

for ϵ small depending on Ω and τ .

Here for any $x_\epsilon \in \Omega$ with $c_2\epsilon \leq d(x_\epsilon, \partial\Omega) \leq C\epsilon$ for ϵ small, we select $x_{0,\epsilon} \in \partial\Omega$ with $d(x_\epsilon, \partial\Omega) = |x_\epsilon - x_{0,\epsilon}|$ and get for $\tilde{x}_\epsilon := \epsilon^{-1}(x_\epsilon - x_{0,\epsilon}) \in \tilde{\Omega}_\epsilon$ that

$$c_2 \leq d(\tilde{x}_\epsilon, \partial\tilde{\Omega}_\epsilon) = |\tilde{x}_\epsilon| \leq C.$$

Passing to subsequence, we get $\tilde{x}_\epsilon \rightarrow \tilde{x}$ with $\tilde{x} \in \overline{\mathbb{R}_+^n}$ and

$$d(\tilde{x}, \mathbb{R}^{n-1} \times \{0\}) \geq c_2 > 0,$$

hence $\tilde{x} \in \mathbb{R}_+^n$. Then by Proposition 4.1 (4.6) and (4.23) that

$$M_\epsilon^{-1}u_\epsilon(x_\epsilon) = M_\epsilon^{-1}\tilde{u}_\epsilon(\tilde{x}_\epsilon) \rightarrow \tilde{u}_\infty(\tilde{x}) > 0,$$

and we conclude that $u_\epsilon(x_\epsilon) > 0$ for ϵ small, and the claim follows.

Combining (4.25), (4.26) and (4.27), we get

$$u_\epsilon > 0 \quad \text{in } \Omega$$

for ϵ small depending on Ω and τ , which proves Theorem 1.1.

Appendix

Uniqueness of a bi-Laplace equation on the half space

In this section, we show the following uniqueness theorem.

Theorem A.1 *Let $v : \overline{\mathbb{R}_+^n} \rightarrow \mathbb{R}$ be a smooth solution to*

$$\begin{cases} \Delta^2 v - \Delta v = 0 \text{ on } \mathbb{R}_+^n, \\ v = \partial_n v = 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}, \\ v \in L^\infty(\mathbb{R}_+^n). \end{cases}$$

Then, $v = 0$.

The proof is based on an energy type estimate and on the one-dimensional case. We start with the one-dimensional case and show the following lemma:

Lemma A.1 *Let $v \in C_{loc}^4([0, \infty[)$ satisfy*

$$\begin{aligned} \frac{d^4 v}{dt^4} - \frac{d^2 v}{dt^2} &= 0 \text{ on }]0, \infty[, \\ v(0) = \frac{dv}{dt}(0) &= 0, \\ v &\in L^\infty(]0, \infty[). \end{aligned}$$

Then, $v = 0$.

Proof Since the differential equation is ordinary and linear, the solution space of the equation itself is of dimension 4. By inserting the following functions, we see that they constitute a basis of the solution space

$$t \mapsto \cosh(t), \sinh(t), 1, t.$$

The following functions are therefore a basis of the solution space including the initial conditions at $t = 0$:

$$v_1(t) = \cosh(t) - 1, \quad v_2(t) = \sinh(t) - t.$$

Hence v has to be of the form

$$v(t) = Av_1(t) + Bv_2(t) = A(\cosh(t) - 1) + B(\sinh(t) - t).$$

Inserting the exponential function for \cosh and \sinh yields

$$\begin{aligned} v(t) &= A \left(\frac{e^t + e^{-t}}{2} - 1 \right) + B \left(\frac{e^t - e^{-t}}{2} - t \right) \\ &= \frac{1}{2}(A + B)e^t + \frac{1}{2}(A - B)e^{-t} - A - Bt. \end{aligned}$$

Since the solution is bounded, we have $A = -B$. Again the boundedness then yields $B = 0$, to rule out linear growth. Hence $v = 0$.

For our next step, we introduce a bit of notation for half spaces

$$B_R^+ := B_R(0) \cap \mathbb{R}_+^n \subseteq \mathbb{R}^n, \quad R > 0.$$

Next we show an energy type estimate.

Lemma A.2 *Let v satisfy*

$$\begin{cases} \Delta^2 v - \Delta v = 0 \text{ on } \mathbb{R}_+^n, \\ v = \partial_n v = 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}. \end{cases}$$

Then, there exists a constant $C = C(n) > 0$, such that for all $R > 1$ we have

$$\int_{B_R^+} (|\Delta v|^2 + |\nabla v|^2) \, d\mathcal{L}^n \leq \frac{C}{R^2} \int_{B_{2R}^+} (|\nabla v|^2 + v^2) \, d\mathcal{L}^n.$$

Proof Let $\eta \in C_0^\infty(B_{2R}(0))$ with $0 \leq \eta \leq 1$, $\eta = 1$ on $B_R(0)$ and for any $k \geq 1$

$$|D^k \eta| \leq \frac{C_k}{R^k} \chi_{B_{2R}(0) \setminus B_R(0)}, \tag{A.1}$$

i.e. η is a cut-off function for the ball $B_R(0)$. Then, $v\eta^4$ and its first derivative are zero on ∂B_{2R}^+ . Therefore, partial integration and the differential equation itself yield

$$\begin{aligned} 0 &= \int_{B_{2R}^+} v\eta^4 (\Delta v - \Delta^2 v) \, d\mathcal{L}^n = - \int_{B_{2R}^+} \Delta(v\eta^4) \Delta v \, d\mathcal{L}^n - \int_{B_{2R}^+} \nabla(v\eta^4) \nabla v \, d\mathcal{L}^n \\ &= - \int_{B_{2R}^+} (|\Delta v|^2 \eta^4 + |\nabla v|^2 \eta^4) \, d\mathcal{L}^n \\ &\quad - \int_{B_{2R}^+} 2(\nabla v \nabla(\eta^4) \Delta v + v \Delta(\eta^4) \Delta v + v \nabla v \nabla(\eta^4)) \, d\mathcal{L}^n \\ &= - \int_{B_{2R}^+} (|\Delta v|^2 \eta^4 + |\nabla v|^2 \eta^4) \, d\mathcal{L}^n \\ &\quad - \int_{B_{2R}^+} (8 \nabla v \nabla \eta \eta^3 \Delta v + 4v \Delta v \Delta \eta \eta^3 + 12v \Delta v |\nabla \eta|^2 \eta^2 + 4v \nabla v \nabla \eta \eta^3) \, d\mathcal{L}^n. \end{aligned}$$

By the previous identity and Young’s inequality with an $\varepsilon > 0$ we get

$$\begin{aligned} &\int_{B_{2R}^+} (|\Delta v|^2 \eta^4 + |\nabla v|^2 \eta^4) \, d\mathcal{L}^n \\ &\leq \int_{B_{2R}^+} (\varepsilon |\Delta v|^2 \eta^4 + C_\varepsilon |\nabla v|^2 |\nabla \eta|^2 \eta^2 \\ &\quad + \varepsilon |\Delta v|^2 \eta^4 + C_\varepsilon v^2 |\Delta \eta|^2 \eta^2 \\ &\quad + \varepsilon |\Delta v|^2 \eta^4 + C_\varepsilon v^2 |\nabla \eta|^4 \\ &\quad + \varepsilon |\nabla v|^2 \eta^4 + C_\varepsilon v^2 |\nabla \eta|^2 \eta^2) \, d\mathcal{L}^n. \end{aligned}$$

Now choosing ε small enough, we can absorb the terms with ε as a prefactor into the left hand side

$$\begin{aligned} & \frac{1}{2} \int_{B_{2R}^+} (|\Delta v|^2 \eta^4 + |\nabla v|^2 \eta^4) \, d\mathcal{L}^n \\ & \leq C \int_{B_{2R}^+} (|\nabla v|^2 |\nabla \eta|^2 \eta^2 + v^2 |\Delta \eta|^2 \eta^2 + v^2 |\nabla \eta|^4 + v^2 |\nabla \eta|^2 \eta^2) \, d\mathcal{L}^n. \end{aligned}$$

Together with $0 \leq \eta \leq 1$, the estimates on the derivatives of the cut-off function (A.1) yield

$$\int_{B_R^+} (|\Delta v|^2 + |\nabla v|^2) \, d\mathcal{L}^n \leq C \int_{B_{2R}^+} \left(|\nabla v|^2 \frac{1}{R^2} + v^2 \frac{1}{R^4} + v^2 \frac{1}{R^4} + v^2 \frac{1}{R^2} \right) \, d\mathcal{L}^n.$$

Since we have chosen $R > 1$, we obtain

$$\int_{B_R^+} (|\Delta v|^2 + |\nabla v|^2) \, d\mathcal{L}^n \leq \frac{C}{R^2} \int_{B_{2R}^+} (|\nabla v|^2 + v^2) \, d\mathcal{L}^n.$$

Now we can show our main result Theorem A.1, by iterating Lemma A.2:

Proof of Theorem A.1 Since v is bounded and the differential equation is linear and elliptic, we can employ Schauder-type estimates (see [1, Thm. 6.2] and [11, Sect. 4]) to obtain

$$\|v\|_{C^k(\overline{\mathbb{R}_+^n})} \leq C_k \|v\|_{L^\infty(\overline{\mathbb{R}_+^n})} < \infty. \tag{A.2}$$

Let $x = (y, t) \in \overline{\mathbb{R}_+^n}$, such that $y \in \mathbb{R}^{n-1}$ and $t \geq 0$. By $\partial_y^k v$, we denote any partial derivative of v of order k only after horizontal directions, i.e. indices in $\{1, \dots, n-1\}$. Then, for every $k \in \mathbb{N}$ the function $\partial_y^k v : \overline{\mathbb{R}_+^n} \rightarrow \mathbb{R}$ still solves the differential equation, satisfies the Dirichlet boundary conditions and by (A.2) is again bounded. Now we can iteratively apply Lemma A.2 for $k \geq \ell \in \mathbb{N}$ and obtain

$$\begin{aligned} & \int_{B_R^+} |\partial_y^k v|^2 \, d\mathcal{L}^n \leq \int_{B_R^+} |\nabla \partial_y^{k-1} v|^2 \, d\mathcal{L}^n \\ & \leq \frac{C}{R^2} \int_{B_{2R}^+} (|\nabla \partial_y^{k-1} v|^2 + |\partial_y^{k-1} v|^2) \, d\mathcal{L}^n \\ & \leq \frac{C}{R^2} \int_{B_{2R}^+} (|\nabla \partial_y^{k-1} v|^2 + |\nabla \partial_y^{k-2} v|^2) \, d\mathcal{L}^n \\ & \leq \frac{C}{R^4} \int_{B_{4R}^+} (|\nabla \partial_y^{k-1} v|^2 + |\partial_y^{k-1} v|^2 + |\nabla \partial_y^{k-2} v|^2 + |\partial_y^{k-2} v|^2) \, d\mathcal{L}^n \\ & \leq \dots \leq \frac{C_{\ell,k}}{R^{2\ell}} \sum_{j=1}^{\ell} \int_{B_{2^j R}^+} (|\nabla \partial_y^{k-j} v|^2 + |\partial_y^{k-j} v|^2) \, d\mathcal{L}^n. \end{aligned}$$

By choosing $k = \ell$, (A.2) yields

$$\int_{B_R^+} |\partial_y^k v|^2 \, d\mathcal{L}^n \leq \frac{C_k}{R^{2k}} \mathcal{L}^n(B_{2^k R}^+) \leq C_k R^{n-2k}.$$

If $k > 2n$, this yields for $R \rightarrow \infty$:

$$\int_{\mathbb{R}_+^n} |\partial_y^k v|^2 \, d\mathcal{L}^n = 0.$$

Hence $\partial_y^k v = 0$. This implies

$$\partial_{y_i} \partial_y^{k-1} v = 0, \text{ for } i = 1, \dots, i - 1.$$

Therefore $\partial_y^{k-1} v(y, t)$ is independent of $y \in \mathbb{R}^{n-1}$ and $t \mapsto \partial_y^{k-1} v(y, t)$ satisfies the assumptions of Lemma A.1. Hence, we also have

$$\partial_y^{k-1} v = 0.$$

Especially we have

$$\partial_{y_i} \partial_y^{k-2} v = 0.$$

for all $i = 1, \dots, n - 1$. Hence again $\partial_y^{k-2} v(y, t)$ is independent of y and therefore again satisfies the assumptions of Lemma A.1. Iterating this final process yields

$$v = 0,$$

which is the desired conclusion. □

Remark: The proof above works, because both Δ^2 and $-\Delta$ are monotone operators which are added correctly. If we would destroy this monotonicity by, e.g. examining $\Delta^2 + \Delta$, Theorem A.1 is not true anymore. For example, a bounded nontrivial solution to $\Delta^2 v + \Delta v = 0$, is $(y, t) \mapsto 1 - \cos t$. □

Proposition A.3 *Let $u : \overline{\mathbb{R}_+^n} \rightarrow \mathbb{R}$ be a smooth solution of the fourth-order boundary-value problem*

$$\begin{aligned} \Delta^2 u - \Delta u &= 0 && \text{in } \mathbb{R}_+^n, \\ u, \partial_n u &= 0 && \text{on } \mathbb{R}^{n-1} \times \{0\}. \end{aligned} \tag{A.3}$$

Furthermore, let u for some $\beta \in \mathbb{R}$ and the corresponding linear function $u_0 : (y, t) \mapsto \beta t$ satisfy

$$u - u_0 \in L^\infty(\mathbb{R}_+^n).$$

Then, u is one dimensional, that is

$$u(y, t) = \beta(e^{-t} - 1 + t) \quad \text{for } y \in \mathbb{R}^{n-1}, t \geq 0, \tag{A.4}$$

in particular

$$\Delta u \equiv \beta \quad \text{in } \mathbb{R}^{n-1} \times \{0\} \tag{A.5}$$

and

$$u > 0 \quad \text{in } \mathbb{R}_+^n, \quad \text{if } \beta > 0. \tag{A.6}$$

Proof We put

$$v(y, t) := u(y, t) - \beta(e^{-t} - 1 + t) \quad \text{for } y \in \mathbb{R}^{n-1}, t \geq 0,$$

and see $v \in C_{loc}^\infty(\overline{\mathbb{R}_+^n})$ and

$$\begin{aligned} \Delta^2 v - \Delta v &= 0 && \text{in } \mathbb{R}_+^n, \\ v, \partial_n v &= 0 && \text{on } \mathbb{R}^{n-1} \times \{0\}, \\ v &= u - u_0 - \beta(\exp(\cdot) - 1) \in L^\infty(\mathbb{R}_+^n). \end{aligned}$$

Then, the uniqueness in Proposition A.1 gives $v \equiv 0$, which is (A.4), and by direct calculation

$$\Delta u(y, 0) = \partial_{tt}(\beta(e^{-t} - 1 + t))|_{t=0} = \beta \quad \text{for } y \in \mathbb{R}^{n-1},$$

which is (A.5). For $\beta > 0$, we get by the strict convexity of the exponential function

$$u(y, t) = \beta(e^{-t} - 1 + t) > 0 \quad \text{for } y \in \mathbb{R}^{n-1}, t > 0,$$

which is (A.6).

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