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Some local properties of subsolution and supersolutions for a doubly nonlinear nonlocal p-Laplace equation

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Abstract
We establish a local boundedness estimate for weak subsolutions to a doubly nonlinear parabolic fractional p-Laplace equation. Our argument relies on energy estimates and a parabolic nonlocal version of De Giorgi's method. Furthermore, by means of a new algebraic inequality, we show that positive weak supersolutions satisfy a reverse Hölder inequality. Finally, we also prove a logarithmic decay estimate for positive supersolutions.

Keywords Doubly nonlinear parabolic equation · Fractional p-Laplace equation · Energy estimates · De Giorgi's method

Mathematics Subject Classification 35K92 · 35B45 · 35R11

1 Introduction

This work studies the local behavior of subsolutions and supersolutions to the doubly nonlinear parabolic nonlocal problem

$$\partial_t(u^{p-1}) + \mathcal{L}u = 0 \text{ in } \Omega \times (0, T), \quad p > 2, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $T > 0$ and the operator $\mathcal{L}$ is defined by

$$\mathcal{L}u(x,t) = \text{P.V.} \int_{\mathbb{R}^n} |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))K(x,y,t)\,dy,$$

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and where P.V. stands for the principal value. We assume that K is a symmetric kernel with respect to x and y satisfying

$$\frac{\Lambda^{-1}}{|x-y|^{n+sp}} \leq K(x,y,t) \leq \frac{\Lambda}{|x-y|^{n+sp}}, \tag{1.2}$$

uniformly in $t \in (0,T)$ for some $\Lambda \geq 1$ and $s \in (0,1)$. If $K(x,y,t) = |x-y|^{-(n+sp)}$, then $\mathcal{L}$ becomes the fractional p-Laplace operator $(-\Delta_p)^s$, which further reduces to the fractional Laplacian $(-\Delta)^s$ for $p = 2$.

The partial differential equation in (1.1) constitutes a nonlocal counterpart of the doubly nonlinear equation,

$$\partial_t(u^{p-1}) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0. \tag{1.3}$$

We refer the reader to [1–7] and the references therein. To the best of our knowledge, there is no literature available concerning the corresponding nonlocal equation. This paper is a first step toward a regularity theory where we prove a local boundedness estimate for weak subsolutions to (1.1) when $p > 2$. To this end, we establish an energy estimate (Lemma 3.1) and apply De Giorgi's method to obtain our main result (Theorem 2.15). We also prove a reverse Hölder inequality for strictly positive weak supersolutions (Theorem 2.17) by means of a new algebraic inequality (Lemma 2.9) and a logarithmic decay estimate (Lemma 5.3). In particular, Lemma 2.9 generalizes an inequality due to Felsinger and Kassmann for $p = 2$, see Lemma 3.3 in [8]. Finally, we note that in the local case as for (1.3), such a reverse Hölder property as well as the logarithmic estimate constitute some of the key ingredients in the proof of weak Harnack inequality, see for instance [4]. To the best of our knowledge, weak Harnack inequality seems to be an open question in the non-local case for the doubly nonlinear equation (1.1) and therefore we believe that our results will be important in investigating such question along with further qualitative and quantitative properties of weak solutions to (1.1).

Fractional Laplace equations have been a topic of considerable attention recently. We refer to the survey [9] by Di Nezza, Palatucci and Valdinoci for an elementary introduction to the theory of the fractional Sobolev spaces and fractional Laplace equations. For globally nonnegative solutions of the elliptic fractional Laplace equation $(-\Delta)^s u = 0$, Landkof [10] obtained scale-invariant Harnack inequality, which fails for sign-changing solutions as shown by Kassman [11]. Indeed, an additional tail term appears in the Harnack estimate. Di Castro, Kuusi and Palatucci studied local boundedness and Hölder continuity results for the equation $(-\Delta_p)^s u = 0$ with $p > 1$ in [12]. They also obtained Harnack inequality with a tail dealing with sign-changing solutions in [13]. The nonhomogeneous case $(-\Delta_p)^s u = f$ has been settled for local and global boundedness along with a discussion of eigenvalue problem by Brasco and Parini [14]. Moreover in this case, Brasco, Lindgren and Schikorra established higher and optimal regularity results in [15]. See also [16, 17] and the references therein.

In the parabolic setting, for the fractional heat equation, $\partial_t u + (-\Delta)^s u = 0$, weak Harnack inequality has been established by Felsinger and Kassman in [8], see also [18, 19] for related results. Caffarelli, Chan and Vasseur established boundedness and Hölder continuity results in [20] for different type of kernels. For regularity results up to the boundary, see [21]. Bonforte, Sire and Vázquez established optimal existence and uniqueness results in [22], along with a scale-invariant Harnack inequality for globally positive solutions. For sign-changing solutions, Strömqvist proved Harnack inequality with a tail in [23], see [24] for a different approach.

In the nonlinear framework, we mention the work of Vázquez [25] where global boundedness results for the equation

$$\partial_t u + (-\Delta_p)^s u = 0$$

have been obtained. See also [26]. For such an equation, local boundedness result with a tail term has been investigated by Strömqvist in [27]. More recently, Hölder continuity results have been established for the same equation by Brasco, Lindgren and Strömqvist in [28]. In the doubly nonlinear case, Hynd and Lindgren [29] addressed the question of pointwise behavior of viscosity solutions for the following doubly nonlinear equation

$$|\partial_t u|^{p-2}\partial_t u + (-\Delta_p)^s u = 0.$$

See also [30, 31] for related results in the local case.

This paper is organized as follows: In Sect. 2, we introduce some basic notations, gather some preliminary results that are relevant to our work and then state our main results. In Sect. 3–5, we prove our main results. Finally, in Sect. 6, appendix, we give a proof of the algebraic inequality in Lemma 2.9 which is applied in the proof of Theorem 2.17.

2 Preliminaries and main results

We first present some facts about fractional Sobolev spaces. For more details we refer the reader to [9].

Definition 2.1 Let $1 < p < \infty$ and $0 < s < 1$ and assume that $\Omega \subset \mathbb{R}^n$ is an open and connected subset of $\mathbb{R}^n$. The fractional Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \, : \, \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}$$

and endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$

The fractional Sobolev space with zero boundary values is defined by

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^n) \, : \, u = 0 \text{ on } \mathbb{R}^n \setminus \Omega \right\}.$$

Both $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ are reflexive Banach spaces, see [9]. The parabolic Sobolev space $L^p(0,T;W^{s,p}(\Omega))$ is the set of measurable functions u on $\Omega \times (0,T)$, $T > 0$, such that

$$||u||_{L^p(0,T;W^{s,p}(\Omega))} = \left(\int_0^T ||u(\cdot,t)||^p_{W^{s,p}(\Omega)} \, dt \right)^{\frac{1}{p}} < \infty.$$

The spaces $W^{s,p}_{\text{loc}}(\Omega)$ and $L^p_{\text{loc}}(0,T;W^{s,p}_{\text{loc}}(\Omega))$ are defined analogously. Next we discuss Sobolev embedding theorems, see [9]. We write by C to denote a positive constant which may vary from line to line or even in the same line depending on the situation. If C depends on $r_1, r_2, \dots, r_k$, we write $C = C(r_1, r_2, \dots, r_k)$.

Theorem 2.2 *Let* $1 < p < \infty$ *and* $0 < s < 1$ *with* $sp < n$ *and* $\kappa^* = \frac{n}{n-sp}$. *For every* $u \in W^{s,p}(\mathbb{R}^n)$, *we have*

$$\|u\|^p_{L^{\kappa^* p}(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy,$$

for some positive constant $C = C(n, p, s)$. *If* Ω *is a bounded extension domain for* $W^{s,p}$ *and* $u \in W^{s,p}(\Omega)$, *then for any* $\kappa \in [1, \kappa^*]$,

$$\|u\|_{L^{\kappa p}(\Omega)} \leq C||u||_{W^{s,p}(\Omega)},$$

for some positive constant $C(n, p, s, \Omega)$. *If* $sp = n$, *then the second inequality hold for any* $\kappa \in [1, \infty)$ *and for* $sp > n$, *the second inequality holds for any* $\kappa \in [1, \infty]$ *respectively.*

The following Sobolev type inequality follows by arguing similarly as in the proof of [27, Lemma 2.1]. We give a brief sketch of the proof below. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ denotes the ball in $\mathbb{R}^n$ of radius r and center x_0. The barred integral sign denotes the corresponding integral average.

Lemma 2.3 *Let* $1 < p < \infty$ *and* $0 < s < 1$. *Assume that* $u \in W^{s,p}(B_r)$, *where* $B_r = B_r(x_0)$, *and let* $\kappa^* = \frac{n}{n-sp}$, *if* $sp < n$, *and* $\kappa^* = 2$, *if* $sp \geq n$. *There exists a constant* $C = C(n, p, s)$ *such that for every* $\kappa \in [1, \kappa^*]$, *we have*

$$\left(\fint_{B_r} |u(x)|^{\kappa p} \, dx \right)^{\frac{1}{\kappa}} \leq C r^{sp-n} \int_{B_r} \int_{B_r} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy + C \fint_{B_r} |u(x)|^p \, dx.$$

Proof Let $0 < s < 1$ and κ^* be as given by the hypothesis. Suppose $u \in W^{s,p}(B_1(0))$, then by choosing $\Omega = B_1(0)$ in Theorem 2.2, for every $\kappa \in [1, \kappa^*]$, we have

$$\left(\int_{B_1(0)} |u(x)|^{\kappa p} \, dx \right)^{\frac{1}{\kappa}} \leq C \left(\int_{B_1(0)} \int_{B_1(0)} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dxdy + \int_{B_1(0)} |u(x)|^p \, dx \right), \tag{2.1}$$

for some positive constant $C = C(n, p, s)$. Using change of variable in (2.1) the result follows. □

Next, we state and prove the parabolic Sobolev inequality, whose proof is similar to the proof of [27, Lemma 2.2].

Lemma 2.4 *Let* p, s *and* κ^* *be as in Lemma 2.3. Assume that* $u \in L^p(t_1, t_2; W^{s,p}(B_r))$. *There exists a constant* $C = C(n, p, s)$ *such that for every* $\kappa \in [1, \kappa^*]$, *we have*

$$\begin{aligned} \int_{t_1}^{t_2} \fint_{B_r} |u(x,t)|^{\kappa p} \, dx \, dt &\leq C \Big(r^{sp-n} \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \frac{|u(x,t) - u(y,t)|^p}{|x-y|^{n+sp}} \, dx \, dy \, dt \\ &+ \int_{t_1}^{t_2} \fint_{B_r} |u(x,t)|^p \, dx \, dt \Big) \cdot \\ &\left(\sup_{t_1 < t < t_2} \fint_{B_r} |u(x,t)|^{\frac{p\kappa^*(\kappa-1)}{\kappa^*-1}} \, dx \right)^{\frac{\kappa^*-1}{\kappa^*}}. \end{aligned}$$

Proof Let $0 < s < 1$ and κ^* be as given by the hypothesis. Using Hölder's inequality with exponents κ^* and $\frac{\kappa^*}{\kappa^*-1}$, for every $\kappa \in [1, \kappa^*]$, we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{B_r} |u(x,t)|^{\kappa p}\,dxdt \\
&= \int_{t_1}^{t_2}\int_{B_r} |u(x,t)|^{p}|u(x,t)|^{(\kappa-1)p}\,dxdt \\
&\leq \int_{t_1}^{t_2}\left(\int_{B_r} |u(x,t)|^{\kappa^* p}\,dx\right)^{\frac{1}{\kappa^*}}\left(\int_{B_r} |u(x,t)|^{\frac{p\kappa^*(\kappa-1)}{\kappa^*-1}}\,dx\right)^{\frac{\kappa^*-1}{\kappa^*}} dt \\
&= r^{n-\frac{n}{\kappa^*}}\int_{t_1}^{t_2}\left(\int_{B_r} |u(x,t)|^{\kappa^* p}\,dx\right)^{\frac{1}{\kappa^*}} dt\left(\sup_{t_1<t<t_2} \fint_{B_r} |u(x,t)|^{\frac{p\kappa^*(\kappa-1)}{\kappa^*-1}}\,dx\right)^{\frac{\kappa^*-1}{\kappa^*}}.
\end{aligned} \tag{2.2}$$

We now bound the following term in (2.2),

$$\int_{t_1}^{t_2}\left(\int_{B_r} |u(x,t)|^{\kappa^* p}\,dx\right)^{\frac{1}{\kappa^*}} dt,$$

using Lemma 2.3 and consequently we obtain,

$$\begin{aligned}
\int_{t_1}^{t_2}\fint_{B_r} |u(x,t)|^{\kappa p}\,dxdt \leq C\Bigg(& r^{sp-n}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+sp}}\,dxdydt \\
&+ \int_{t_1}^{t_2}\fint_{B_r} |u(x,t)|^{p}\,dxdt\Bigg) \\
&\cdot\left(\sup_{t_1<t<t_2}\fint_{B_r} |u(x,t)|^{\frac{p\kappa^*(\kappa-1)}{\kappa^*-1}}\,dx\right)^{\frac{\kappa^*-1}{\kappa^*}},
\end{aligned}$$

for some positive constant $C = C(n, p, s)$. This completes the proof. □

We now state the following weighted Poincaré inequality in fractional Sobolev spaces, see [32, Corollary 6].

Lemma 2.5 *Let $1 < p < \infty$, $0 < s_0 \leq s < 1$. Assume that $\phi(x) = \Phi(|x|)$ is a radially decreasing function on $B_1 = B_1(0)$. Then there exists a constant $C = C(p, n, s_0, \phi)$ such that for all $f \in L^p(B_1)$,*

$$\int_{B_1} |f(x) - f_{B_1}^{\phi}|^p\phi(x)\,dx \leq C(1-s)\int_{B_1}\int_{B_1}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\min\{\phi(x),\phi(y)\}\,dx\,dy,$$

where

$$f_{B_1}^{\phi} = \frac{\int_{B_1} f(x)\phi(x)\,dx}{\int_{B_1}\phi(x)\,dx}.$$

Using change of variables in Lemma 2.5, we obtain the following weighted Poincaré inequality which will be useful in establishing a logarithmic estimate for weak supersolutions (see Lemma 5.3).

Lemma 2.6 *Let $1 < p < \infty$, $0 < s < 1$ and $\psi(x) = \Psi(|x - x_0|)$ be a radially decreasing function on $B_r = B_r(x_0)$. Then there exists a constant $C = C(n,p,s)$ such that for every $f \in L^p(B_r)$,*

$$\int_{B_r} |f(x) - f_{B_r}^{\psi}|^p \psi(x)\,dx \le C r^{sp} \int_{B_r}\int_{B_r} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \min\{\psi(x),\psi(y)\}\,dx\,dy,$$

where

$$f_{B_r}^{\psi} = \frac{\int_{B_r} f(x)\psi(x)\,dx}{\int_{B_r}\psi(x)\,dx}.$$

We also need the following real analysis lemmas. For the proof of Lemma 2.7, see [33, Lemma 4.1].

Lemma 2.7 *Let $(Y_j)_{j=0}^{\infty}$ be a sequence of positive real numbers satisfying $Y_{j+1} \le c_0 b^j Y_j^{1+\beta}$, for some constants $c_0 > 1$, $b > 1$ and $\beta > 0$. If $Y_0 \le c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}}$, then $\lim_{j\to\infty} Y_j = 0$.*

The next inequality is as in [12, Lemma 3.1].

Lemma 2.8 *Let $p \ge 1$ and $\epsilon \in (0,1]$. Then for every $a, b \in \mathbb{R}^n$, we have*

$$|a|^p \le |b|^p + C(p)\epsilon|b|^p + \big(1 + C(p)\epsilon\big)\epsilon^{1-p}|a-b|^p,$$

where $C(p) = (p-1)\Gamma(\max\{1, p-2\})$ and Γ denotes the gamma function.

The following elementary inequality will play a crucial role in the proof of reverse Hö lder inequality for supersolutions as in Theorem 2.17. A proof for Lemma 2.9 is given in appendix. This generalizes an inequality of Felsinger and Kassmann [8] to the p-case.

Lemma 2.9 *Let $a, b > 0$, $\tau_1, \tau_2 \ge 0$. Then for any $p > 1$, there exists a constant $C = C(p) > 1$ large enough such that*

$$\begin{aligned} |b-a|^{p-2}(b-a)(\tau_1^p a^{-\epsilon} - \tau_2^p b^{-\epsilon}) &\ge \frac{\zeta(\epsilon)}{C(p)}\Big|\tau_2 b^{\frac{p-\epsilon-1}{p}} - \tau_1 a^{\frac{p-\epsilon-1}{p}}\Big|^p \\ &- \Big(\zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}}\Big)|\tau_2 - \tau_1|^p\big(b^{p-\epsilon-1} + a^{p-\epsilon-1}\big), \end{aligned} \tag{2.3}$$

where $0 < \epsilon < p-1$ and $\zeta(\epsilon) = \epsilon(\frac{p}{p-\epsilon-1})^p$. If $0 < p-\epsilon-1 < 1$, we may choose $\zeta(\epsilon) = \frac{\epsilon p^p}{p-\epsilon-1}$ in (2.3).

For $v, k > 0$, the auxiliary function defined by

$$\xi((v-k)_+) = \int_{k^{p-1}}^{v^{p-1}} \big(\eta^{\frac{1}{p-1}} - k\big)_+\,d\eta = (p-1)\int_k^v (\eta-k)_+\eta^{p-2}\,d\eta, \tag{2.4}$$

would be very useful to deduce the energy estimate below. Indeed, from [2, Lemma 2.2], we have the following result.

Lemma 2.10 *There exists a constant $\lambda = \lambda(p) > 0$ such that for all $v, k > 0$, we have*

$$\frac{1}{\lambda}(v+k)^{p-2}(v-k)_+^2 \le \xi((v-k)_+) \le \lambda(v+k)^{p-2}(v-k)_+^2.$$

For more applications of such functions in the doubly nonlinear context, we refer to [2, 3, 6].

For $t_0 \in (r^{sp}, T-r^{sp})$, we consider the space-time cylinders

$$U^-(r) = U^-(x_0, t_0, r) = B_r(x_0) \times (t_0 - r^{sp}, t_0)$$

and

$$U^+(r) = U^+(x_0, t_0, r) = B_r(x_0) \times (t_0, t_0 + r^{sp}).$$

We denote the positive and negative parts of u by

$$u_+(x,t) = \max\{u(x,t), 0\} \quad \text{and} \quad u_-(x,t) = \max\{-u(x,t), 0\},$$

respectively. For any $a, b \in \mathbb{R}$, we have $|a_+ - b_+| \le |a-b|$ which implies $u_+ \in W^{s,p}(\Omega)$ when $u \in W^{s,p}(\Omega)$. Analogously, we have $u_- \in W^{s,p}(\Omega)$. Throughout the paper, we denote by

$$\mathcal{A}(u(x,y,t)) = |u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t)) \quad \text{and} \quad d\mu = K(x,y,t)\,dx\,dy.$$

It is well known that a tail term appears in nonlocal problems. If u is a measurable function in $\mathbb{R}^n \times (0,T)$ and $x_0 \in \mathbb{R}^n$, $r > 0$, $0 < t_1 < t_2 < T$, the parabolic tail of u with respect to x_0, r, t_1 and t_2 is defined by

$$\mathrm{Tail}_\infty(u; x_0, r, t_1, t_2) = \left(r^{sp} \sup_{t_1 < t < t_2} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|u(x,t)|^{p-1}}{|x-x_0|^{n+sp}}\, dx \right)^{\frac{1}{p-1}}. \tag{2.5}$$

Next we define the notion of weak sub- and supersolution.

Definition 2.11 A function $u \in L^\infty(0,T; L^\infty(\mathbb{R}^n))$, with $u > 0$ in $\mathbb{R}^n \times (0,T)$, is a weak subsolution (or supersolution) of the equation (1.1) in $\Omega \times (0,T)$ if $u \in C_{\mathrm{loc}}(0,T; L^p_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0,T; W^{s,p}_{\mathrm{loc}}(\Omega))$ and for every $\Omega' \times (t_1, t_2) \Subset \Omega \times (0,T)$, and nonnegative test function $\phi \in W^{1,p}_{\mathrm{loc}}(0,T; L^p(\Omega')) \cap L^p_{\mathrm{loc}}(0,T; W^{s,p}_0(\Omega'))$, one has

$$\begin{aligned} &\int_{\Omega'} u(x,t_2)^{p-1}\phi(x,t_2)\,dx - \int_{\Omega'} u(x,t_1)^{p-1}\phi(x,t_1)\,dx \\ &- \int_{t_1}^{t_2}\int_{\Omega'} u(x,t)^{p-1}\partial_t\phi(x,t)\,dx\,dt \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{A}(u(x,y,t))(\phi(x,t)-\phi(y,t))\,d\mu\,dt \le 0 \quad (\text{or } \ge 0) \end{aligned}, \tag{2.6}$$

respectively.

Remark 2.12 The assumption $u \in L^\infty(0,T; L^\infty(\mathbb{R}^n))$ ensures that the last term in the left-hand side of (2.6) and $\mathrm{Tail}_\infty(u; x_0, r, t_1, t_2)$ defined by (2.5) are finite for every $x_0 \in \mathbb{R}^n$ and every $0 < t_1 < t_2 < T$.

Remark 2.13 Moreover, we would like to emphasize that the global boundedness assumption $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n))$ in Definition 2.11 can be replaced with the local boundedness assumption $u \in L^\infty_{\mathrm{loc}}(0,T;L^\infty_{\mathrm{loc}}(\mathbb{R}^n))$ together with the boundedness of $\mathrm{Tail}_\infty(u;x_0,r,t_1,t_2)$ defined by (2.5), for every $x_0 \in \mathbb{R}^n$ and every $0 < t_1 < t_2 < T$. Furthermore, the hypothesis $\phi_t \in L^p_{\mathrm{loc}}(0,T;L^p(\Omega'))$ in Definition 2.11 can be replaced with $\phi_t \in L^1_{\mathrm{loc}}(0,T;L^1(\Omega'))$.

Remark 2.14 To establish energy estimates for weak subsolutions or supersolutions of (1.1), we choose test functions ϕ that depend on the weak subsolution or supersolution itself and thus ϕ_t may not exist as a L^p function as opposed to what Definition 2.11 requires. This aspect can, however, be rectified by using the following mollification in time,

$$f_h(x,t) = \frac{1}{h}\int_0^t e^{\frac{s-t}{h}} f(x,s)\,ds, \tag{2.7}$$

combined with a limiting argument, i.e., by eventually letting $h \to 0$. See for instance the proof of Lemma 3.1. For more details on such a mollification, we refer to [2, 34].

2.1 Statement of the main results

Below, we state our main results. Our first main result is following local boundedness estimate for subsolutions.

Theorem 2.15 *Let $p > 2$, $x_0 \in \mathbb{R}^n$, $r > 0$ and $t_0 \in (r^{sp},T)$. Assume that $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n)) \cap C_{\mathrm{loc}}(0,T;L^p_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0,T;W^{s,p}_{\mathrm{loc}}(\Omega))$ is a weak subsolution of (1.1) in $\Omega\times(0,T)$ such that $U^-(r) = U^-(x_0,t_0,r) = B_r(x_0)\times(t_0-r^{sp},t_0) \Subset \Omega\times(0,T)$ with $u > 0$ in $\mathbb{R}^n\times(t_0-r^{ps},t_0)$. Then there exists a positive constant $C = C(n,p,s,\Lambda)$ such that for any $\delta \in (0,1)$, we have*

$$\sup_{(x,t)\in U^-(\frac{r}{2})} u(x,t) \le C\delta^{-\frac{(p-1)\kappa}{p(\kappa-1)}}\left(\fint_{U^-(r)} u(x,t)^p\,dx\,dt\right)^{\frac{1}{p}} + \delta\,\mathrm{Tail}_\infty\left(u;x_0,\frac{r}{2},t_0-r^{sp},t_0\right),$$

where $\kappa = \frac{n+sp}{n}$, if $sp < n$, and $\kappa = \frac{3}{2}$, if $sp \ge n$.

Remark 2.16 One should note that even when $\Omega = \mathbb{R}^n$, Theorem 2.15 will remain valid and the global boundedness assumption on $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n))$ can be replaced by the local boundedness assumption $u \in L^\infty_{\mathrm{loc}}(0,T;L^\infty_{\mathrm{loc}}(\mathbb{R}^n))$ together with the boundedness of $\mathrm{Tail}_\infty(u;x_0,r,t_1,t_2)$ defined by (2.5), for every $x_0 \in \mathbb{R}^n$ and every $0 < t_1 < t_2 < T$.

Our second main result constitutes the following reverse Hölder inequality for positive supersolutions.

Theorem 2.17 *Let $p > 2$, $x_0 \in \mathbb{R}^n$, $r > 0$ and $t_0 \in (0,T-r^{sp})$. Suppose that $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n)) \cap C_{\mathrm{loc}}(0,T;L^p_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0,T;W^{s,p}_{\mathrm{loc}}(\Omega))$ is a weak supersolution of (1.1) in $\Omega\times(0,T)$ such that $U^+(r) = U^+(x_0,t_0,r) = B_r(x_0)\times(t_0,t_0+r^{sp}) \Subset \Omega\times(0,T)$ with $u \ge \rho > 0$ in $\mathbb{R}^n\times(t_0,t_0+r^{ps})$. Then for any $\theta \in [\frac{1}{2},1)$ there exists positive constants $\mu = \mu(\kappa,p)$ and $C = C(n,p,q,s,\Lambda) \ge 1$ such that*

$$\left(\fint_{U^+(\theta r)} u(x,t)^q \, dx\, dt \right)^{\frac{1}{q}} \le \left(\frac{C}{(1-\theta)^\mu} \fint_{U^+(r)} u(x,t)^{\bar q} \, dx\, dt \right)^{\frac{1}{\bar q}}, \tag{2.8}$$

for all $0 < \bar q < q < q_0$ *where* $q_0 = \kappa(p-1)$ *with* $\kappa = \frac{n+sp}{n}$, *if* $sp < n$ *and* $\kappa = \frac{3}{2}$, *if* $sp \ge n$.

Remark 2.18 We would like to emphasize that the constant C in the reverse Hölder inequality (2.8) is independent of $\bar q$ as $\bar q \to 0$ and this is precisely where the algebraic lemma 2.9 plays a crucial role. It is well known that such a stable behavior of the constant C is needed in order to establish the Harnack inequality for local equations using the approach of Bombieri as in [35] (see also [4] for an adaptation of such an idea in the case of (1.3)). We therefore believe that such a reverse Hölder inequality will have similar future applications in the nonlocal case.

3 Energy estimate

To prove Theorem 2.15, we need the following Caccioppoli type estimate for subsolutions.

Lemma 3.1 *Let* $p > 2$, $x_0 \in \mathbb{R}^n$, $0 < \tau_1 < \tau_2$ *and* $l > 0$ *with* $B_r = B_r(x_0) \Subset \Omega$ *and* $0 < \tau_1 - l < \tau_2 < T$. *Assume that* $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n)) \cap C_{\mathrm{loc}}(0,T;L^p_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0,T;W^{s,p}_{\mathrm{loc}}(\Omega))$ *is a weak subsolution of* (1.1) *in* $\Omega \times (0,T)$ *with* $u > 0$ in $\mathbb{R}^n \times (\tau_1 - l, \tau_2)$. *Let* $k > 0$ *and denote* $w(x,t) = (u-k)_+(x,t)$. *Then there exists a positive constant* $C = C(n,p,s,\Lambda)$ *such that*

$$\begin{aligned}
&\int_{\tau_1-l}^{\tau_2}\int_{B_r}\int_{B_r} |w(x,t)\psi(x) - w(y,t)\psi(y)|^p \eta(t)^p \, d\mu\, dt \\
&\quad + C \sup_{\tau_1<t<\tau_2} \int_{B_r} w(x,t)^p \psi(x)^p \, dx \\
&\le \int_{\tau_1-l}^{\tau_2}\int_{B_r}\int_{B_r} |w(x,t)\psi(x) - w(y,t)\psi(y)|^p \eta(t)^p \, d\mu\, dt \\
&\quad + C \sup_{\tau_1<t<\tau_2} \int_{B_r} \xi(w)(x,t)\psi(x)^p \, dx \\
&\le C\Bigg(\int_{\tau_1-l}^{\tau_2}\int_{B_r}\int_{B_r} \max\{w(x,t), w(y,t)\}^p |\psi(x)-\psi(y)|^p \eta(t)^p \, d\mu\, dt \\
&\quad + \Bigg(\sup_{x\in \operatorname{supp}\psi,\, \tau_1-l<t<\tau_2} \int_{\mathbb{R}^n\setminus B_r} \frac{w(y,t)^{p-1}}{|x-y|^{n+sp}} \, dy \Bigg) \int_{\tau_1-l}^{\tau_2}\int_{B_r} w(x,t)\psi(x)^p \eta(t)^p \, dx\, dt \\
&\quad + \int_{\tau_1-l}^{\tau_2}\int_{B_r} \xi(w)\psi(x)^p \partial_t \eta(t)^p \, dx\, dt \Bigg),
\end{aligned}$$

for all nonnegative $\psi \in C_0^\infty(B_r)$ *and nonnegative* $\eta \in C^\infty(\mathbb{R})$ *such that* $\eta(t) = 0$ *for* $t \le \tau_1 - l$ *and* $\eta(t) = 1$ *for* $t \ge \tau_1$ *where* ξ *is as in* (2.4) *defined as follows,*

$$\xi(w) = \int_{k^{p-1}}^{u^{p-1}} \left(\eta^{\frac{1}{p-1}} - k\right)_+ d\eta = (p-1)\int_k^u (\eta - k)_+ \eta^{p-2} \, d\eta.$$

Proof Since $p > 2$, we observe that the first inequality, i.e.,

$$\int_{\tau_1-l}^{\tau_2}\int_{B_r}\int_{B_r}|w(x,t)\psi(x)-w(y,t)\psi(y)|^p\eta(t)^p\,d\mu\,dt+C\sup_{\tau_1<t<\tau_2}\int_{B_r}w(x,t)^p\psi(x)^p\,dx$$
$$\leq\int_{\tau_1-l}^{\tau_2}\int_{B_r}\int_{B_r}|w(x,t)\psi(x)-w(y,t)\psi(y)|^p\eta(t)^p\,d\mu\,dt$$
$$+C\sup_{\tau_1<t<\tau_2}\int_{B_r}\xi(w)(x,t)\psi(x)^p\,dx$$

follows directly from Lemma 2.10. Therefore, it is enough to prove the second inequality.

Let $t_1=\tau_1-l$ and $t_2=\tau_2$ and for fixed $t_1<l_1<l_2<t_2$ and $\epsilon>0$ small enough, following [2] we define the function $\zeta_\epsilon\in W^{1,\infty}\big((t_1,t_2),[0,1]\big)$ by

$$\zeta_\epsilon(t):=\begin{cases}0 & \text{for } t_1\leq t\leq l_1-\epsilon,\\ 1+\frac{t-l_1}{\epsilon} & \text{for } l_1-\epsilon<t\leq l_1,\\ 1, & \text{for } l_1<t\leq l_2,\\ 1-\frac{t-l_2}{\epsilon}, & \text{for } l_2<t\leq l_2+\epsilon,\\ 0, & \text{for } l_2+\epsilon<t\leq t_2,\end{cases}$$

and we choose

$$\phi(x,t)=w(x,t)\psi(x)^p\zeta_\epsilon(t)\eta(t)^p$$

as a test function in (2.6). Recalling the definition of $(\cdot)_h$ from (2.7), we denote by

$$v_h^{p-1}=(u^{p-1})_h\quad\text{and}\quad\mathcal{V}(u(x,y,t))=\mathcal{A}(u(x,y,t))K(x,y,t).$$

Then following [2, 34], we observe that the subsolution u of (1.1) satisfies the following mollified inequality

$$\lim_{\epsilon\to0}\lim_{h\to0}(I_{h,\epsilon}+J_{h,\epsilon})\leq0,\tag{3.1}$$

where

$$I_{h,\epsilon}=\int_{t_1}^{t_2}\int_{B_r}\partial_t v_h^{p-1}\phi(x,t)\,dx\,dt=\int_{t_1}^{t_2}\int_{B_r}\partial_t v_h^{p-1}w(x,t)\psi(x)^p\zeta_\epsilon(t)\eta(t)^p\,dx\,dt,$$

and

$$J_{h,\epsilon}=\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(\mathcal{V}(u(x,y,t))\big)_h(\phi(x,t)-\phi(y,t))\,dx\,dy\,dt$$
$$=\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(\mathcal{V}(u(x,y,t))\big)_h\big(w(x,t)\psi(x)^p-w(y,t)\psi(y)^p\big)\zeta_\epsilon(t)\eta(t)^p\,dx\,dy\,dt.$$

Estimate of $I_{h,\epsilon}$: Proceeding similarly as in the proof of [2, Proposition 3.1], we have

$$\lim_{\epsilon\to0}\lim_{h\to0}I_{h,\epsilon}\geq\int_{B_r}\xi(w)(x,l_2)\psi(x)^p\eta(l_2)^p\,dx-\int_{B_r}\xi(w)(x,l_1)\psi(x)^p\eta(l_1)^p\,dx$$
$$-\int_{l_1}^{l_2}\int_{B_r}\xi(w)(x,t)\psi(x)^p\partial_t\eta(t)^p\,dx\,dt.\tag{3.2}$$

Estimate of $J_{h,\epsilon}$: First, we claim that $\lim_{h\to 0} J_{h,\epsilon} = J_\epsilon$, where

$$J_\epsilon = \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{V}(u(x,y,t))\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\zeta_\epsilon(t)\eta(t)^p\,dx\,dy\,dt.$$

Indeed, we can write

$$J_{h,\epsilon} - J_\epsilon = L_{h,\epsilon} + N_{h,\epsilon}, \tag{3.3}$$

where

$$L_{h,\epsilon} = \int_{t_1}^{t_2}\int_{B_r}\int_{B_r} \Big(\big(\mathcal{V}(u(x,y,t))\big)_h - \mathcal{V}(u(x,y,t))\Big)\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\zeta_\epsilon(t)\eta(t)^p\,dx\,dy\,dt,$$

and

$$N_{h,\epsilon} = 2\int_{t_1}^{t_2}\int_{B_r}\int_{\mathbb{R}^n\setminus B_r} \Big(\big(\mathcal{V}(u(x,y,t))\big)_h - \mathcal{V}(u(x,y,t))\Big)w(x,t)\psi(x)^p\zeta_\epsilon(t)\eta(t)^p\,dx\,dy\,dt.$$

Estimate of $L_{h,\epsilon}$: We can rewrite $L_{h,\epsilon}$ as

$$\begin{aligned} L_{h,\epsilon} = \int_{t_1}^{t_2}\int_{B_r}\int_{B_r} &\Big(\big(\mathcal{V}(u(x,y,t))\big)_h - \mathcal{V}(u(x,y,t))\Big) \\ &\frac{\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\zeta_\epsilon(t)\eta(t)^p}{|x-y|^{-\frac{(n+sp)}{p}}|x-y|^{\frac{n+sp}{p}}}\,dx\,dy\,dt, \end{aligned}$$

and using Hölder's inequality with exponents $p' = \frac{p}{p-1}$ and p, we obtain

$$\begin{aligned} L_{h,\epsilon} \le &\left(\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\Big|\Big(\big(\mathcal{V}(u(x,y,t))\big)_h - \mathcal{V}(u(x,y,t))\Big)|x-y|^{\frac{n+sp}{p}}\Big|^{p'}dx\,dy\,dt\right)^{\frac{1}{p'}} \cdot \\ &\left(\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\frac{\big|\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\zeta_\epsilon(t)\eta(t)^p\big|^p}{|x-y|^{n+sp}}\,dx\,dy\,dt\right)^{\frac{1}{p}}. \end{aligned} \tag{3.4}$$

Now using the property (1.2), we observe that

$$|x-y|^{\frac{n+sp}{p}}|\mathcal{V}(u(x,y,t)| \le \Lambda\frac{|u(x,t)-u(y,t)|^{p-1}}{|x-y|^{\frac{n+sp}{p'}}} \in L^{p'}((t_1,t_2)\times B_r\times B_r),$$

From [34, Lemma 2.9], we have

$$\Big(\big(\mathcal{V}(u(x,y,t))\big)_h - \mathcal{V}(u(x,y,t))\Big)|x-y|^{\frac{n+sp}{p}} \to 0 \text{ in } L^{p'}((t_1,t_2)\times B_r\times B_r),$$

and therefore from (3.4), it follows that $\lim_{h\to 0} L_{h,\epsilon} = 0$.

Estimate of $N_{h,\epsilon}$: We note that given the pointwise convergence of mollified functions together with the fact that $u \in L^\infty((t_1,t_2);L^\infty(\mathbb{R}^n))$, we can therefore apply the Lebesgue dominated convergence theorem to conclude that $\lim_{h\to 0} N_{h,\epsilon} = 0$.

Estimate of J_ϵ: We can rewrite $J_\epsilon = J_\epsilon^1 + J_\epsilon^2$, where

$$J_\epsilon^1 = \int_{t_1}^{t_2}\int_{B_r}\int_{B_r} \mathcal{A}(u(x,y,t))(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p)\zeta_\epsilon(t)\eta(t)^p\, d\mu\, dt,$$

and

$$J_\epsilon^2 = 2\int_{t_1}^{t_2}\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} \mathcal{A}(u(x,y,t))w(x,t)\psi(x)^p\zeta_\epsilon(t)\eta(t)^p\, d\mu\, dt.$$

Estimate of J_ϵ^1: To estimate the integral J_ϵ^1, we mainly adapt an idea from the proof of [12, Theorem 1.4]. By symmetry we may assume $u(x,t) \geq u(y,t)$. In this case, for every fixed t, we observe that

$$\begin{aligned}
&|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\eta(t)^p\\
&= (u(x,t)-u(y,t))^{p-1}\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\eta(t)^p\\
&= \begin{cases} (w(x,t)-w(y,t))^{p-1}\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\eta(t)^p, \text{ if } u(x,t), u(y,t) > k,\\ (u(x,t)-u(y,t))^{p-1}w(x,t)\psi(x)^p\eta(t)^p, \text{ if } u(x,t) > k,\ u(y,t)\leq k,\\ 0, \text{ otherwise.}\end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
&|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))\big(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p\big)\eta(t)^p\\
&\quad\geq |w(x,t)-w(y,t)|^{p-1}(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p)\eta(t)^p.
\end{aligned}$$

This implies,

$$J_\epsilon^1 \geq \int_{t_1}^{t_2}\int_{B_r}\int_{B_r}(w(x,t)-w(y,t))^{p-1}(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p)\zeta_\epsilon(t)\eta(t)^p\, d\mu\, dt.$$

Let us now consider the case when $w(x,t) > w(y,t)$ and $\psi(x)\leq\psi(y)$. By Lemma 2.8 we obtain

$$\psi(x)^p \geq (1-C(p)\epsilon)\psi(y)^p - (1+C(p)\epsilon)\epsilon^{1-p}|\psi(x)-\psi(y)|^p \tag{3.5}$$

for any $\epsilon\in(0,1]$ where $C(p) = (p-1)\Gamma(\max\{1,p-2\})$. Now by letting

$$\epsilon = \frac{1}{\max\{1, 2C(p)\}}\frac{w(x,t)-w(y,t)}{w(x,t)} \in (0,1],$$

we deduce from above that the following inequality holds for some positive constant $C = C(p)$,

$$\begin{aligned}
&(w(x,t)-w(y,t))^{p-1}w(x,t)\psi(x)^p \geq (w(x,t)-w(y,t))^{p-1}w(x,t)\max\{\psi(x),\psi(y)\}^p\\
&\qquad -\frac{1}{2}(w(x,t)-w(y,t))^p\max\{\psi(x),\psi(y)\}^p\\
&\quad - C\max\{w(x,t),w(y,t)\}^p|\psi(x)-\psi(y)|^p.
\end{aligned}$$

Note that over here, we used that under the assumption $\psi(x) \leq \psi(y)$, we have $\max\{\psi(x), \psi(y)\} = \psi(y)$. In the other cases $w(x,t) \geq w(y,t)$, $\psi(x) \geq \psi(y)$ or $w(x,t) = w(y,t)$, the above estimate is clear. Therefore, when $w(x,t) \geq w(y,t)$, we have

$$\begin{aligned}
&(w(x,t) - w(y,t))^{p-1}(w(x,t)\psi(x)^p - w(y,t)\psi(y)^p) \\
&\quad \geq (w(x,t) - w(y,t))^{p-1}(w(x,t)\max\{\psi(x), \psi(y)\}^p - w(y,t)\psi(y)^p) \\
&\quad - \frac{1}{2}(w(x,t) - w(y,t))^p \max\{\psi(x), \psi(y)\}^p \\
&\quad - C\max\{w(x,t), w(y,t)\}^p |\psi(x) - \psi(y)|^p \\
&\quad \geq \frac{1}{2}(w(x,t) - w(y,t))^p \max\{\psi(x), \psi(y)\}^p \\
&\quad - C\max\{w(x,t), w(y,t)\}^p |\psi(x) - \psi(y)|^p.
\end{aligned} \tag{3.6}$$

If $w(x,t) < w(y,t)$, we may interchange the roles of x and y above to obtain (3.6). We then observe that

$$\begin{aligned}
|w(x,t)\psi(x) - w(y,t)\psi(y)|^p &\leq 2^{p-1}|w(x,t) - w(y,t)|^p \max\{\psi(x), \psi(y)\}^p \\
&\quad + 2^{p-1}\max\{w(x,t), w(y,t)\}^p |\psi(x) - \psi(y)|^p.
\end{aligned} \tag{3.7}$$

Now (3.6) and (3.7) gives

$$\begin{aligned}
J_\epsilon^1 &\geq c\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} |w(x,t)\psi(x) - w(y,t)\psi(y)|^p \zeta_\epsilon(t)\eta(t)^p \, d\mu \, dt \\
&- C\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} \max\{w(x,t), w(y,t)\}^p |\psi(x) - \psi(y)|^p \zeta_\epsilon(t)\eta(t)^p \, d\mu \, dt,
\end{aligned} \tag{3.8}$$

for some positive constants $c = c(p), C = C(p)$.

Estimate of J_ϵ^2: To estimate J_ϵ^2, we observe that

$$\begin{aligned}
|u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))w(x,t) &\geq -(u(y,t) - u(x,t))^{p-1}w(x,t) \\
&\geq -(u(y,t) - k)_+^{p-1}w(x,t) \\
&\geq -w(y,t)^{p-1}w(x,t).
\end{aligned}$$

As a consequence, we obtain,

$$\begin{aligned}
J_\epsilon^2 &\geq -\int_{t_1}^{t_2}\int_{\mathbb{R}^n \setminus B_r}\int_{B_r} K(x,y,t)w(y,t)^{p-1}w(x,t)\psi(x)^p \zeta_\epsilon(t)\eta(t)^p \, dx \, dy \, dt \\
&\geq -\Lambda\Big(\sup_{t_1<t<t_2, x\in \operatorname{supp}\psi}\int_{\mathbb{R}^n \setminus B_r} \frac{w(y,t)^{p-1}}{|x-y|^{n+sp}} \, dy\Big)\int_{t_1}^{t_2}\int_{B_r} w(x,t)\psi(x)^p \zeta_\epsilon(t)\eta(t)^p \, dx \, dt.
\end{aligned} \tag{3.9}$$

Therefore from (3.8) and (3.9), we obtain for some positive constants $c = c(p)$ and $C = C(p)$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{h\to 0} J_{h,\epsilon} = \lim_{\epsilon\to 0} J_\epsilon = \lim_{\epsilon\to 0}(J^1_\epsilon + J^2_\epsilon)\\
&\geq c\int_{l_1}^{l_2}\int_{B_r}\int_{B_r} |w(x,t)\psi(x) - w(y,t)\psi(y)|^p \eta(t)^p \, d\mu \, dt\\
&\quad - C\int_{l_1}^{l_2}\int_{B_r}\int_{B_r} \max\{w(x,t), w(y,t)\}^p |\psi(x)-\psi(y)|^p \eta(t)^p \, d\mu \, dt\\
&\quad - \Lambda\Big(\sup_{t_1<t<t_2,\, x\in \operatorname{supp}\psi} \int_{\mathbb{R}^n\setminus B_r} \frac{w(y,t)^{p-1}}{|x-y|^{n+sp}}\, dy\Big) \int_{l_1}^{l_2}\int_{B_r} w(x,t)\psi(x)^p\eta(t)^p \, dx\, dt.
\end{aligned} \tag{3.10}$$

Now employing the estimates (3.2) and (3.10) into (3.1) and then first letting $l_1 \to t_1$ and then by $l_2 \to t_2$, we get

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} |w(x,t)\psi(x) - w(y,t)\psi(y)|^p \eta(t)^p \, d\mu \, dt\\
&\leq C\Big(\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} \max\{w(x,t), w(y,t)\}^p |\psi(x)-\psi(y)|^p \eta(t)^p \, d\mu \, dt\\
&\quad + \Big(\sup_{x\in \operatorname{supp}\psi,\, t_1<t<t_2} \int_{\mathbb{R}^n\setminus B_r} \frac{w(y,t)^{p-1}}{|x-y|^{n+sp}}\, dy\Big) \int_{t_1}^{t_2}\int_{B_r} w(x,t)\psi(x)^p\eta(t)^p \, dx\, dt\\
&\quad + \int_{t_1}^{t_2}\int_{B_r} \xi(w)\psi(x)^p \partial_t\eta(t)^p \, dx\, dt\Big).
\end{aligned} \tag{3.11}$$

Again using (3.2) and (3.10) and then first letting $l_1 \to t_1$ and then by choosing $l_2 \in (\tau_1, \tau_2)$ such that

$$\int_{B_r} \xi(w)(x,l_2)\psi(x)^p \, dx \geq \frac{1}{2}\sup_{\tau_1<t<\tau_2}\int_{B_r} \xi(w)(x,t)\psi(x)^p\, dx,$$

we observe that

$$\begin{aligned}
&\sup_{\tau_1<t<\tau_2}\int_{B_r} \xi(w)(x,t)\psi(x)^p\, dx\\
&\leq C\Big(\int_{t_1}^{t_2}\int_{B_r}\int_{B_r} \max\{w(x,t), w(y,t)\}^p |\psi(x)-\psi(y)|^p \eta(t)^p \, d\mu \, dt\\
&\quad + \Big(\sup_{x\in \operatorname{supp}\psi,\, t_1<t<t_2} \int_{\mathbb{R}^n\setminus B_r} \frac{w(y,t)^{p-1}}{|x-y|^{n+sp}}\, dy\Big) \int_{t_1}^{t_2}\int_{B_r} w(x,t)\psi(x)^p\eta(t)^p \, dx\, dt\\
&\quad + \int_{t_1}^{t_2}\int_{B_r} \xi(w)\psi(x)^p \partial_t\eta(t)^p \, dx\, dt\Big).
\end{aligned} \tag{3.12}$$

Now from (3.11) and (3.12), we get the required estimate. □

4 Proof of Theorem 2.15

Let $0 < s < 1$ and $\kappa = \frac{n+sp}{n}$, if $sp < n$ and $\kappa = \frac{3}{2}$ in the case when $sp \geq n$. For $j = 0, 1, 2, \ldots$, we denote by

$$r_j = \frac{1 + 2^{-j}}{2} r, \quad s_j = \frac{r_j + r_{j+1}}{2},$$

and

$$B_j = B_{r_j}(x_0), \quad \bar{B}_j = B_{s_j}(x_0), \quad \Gamma_j = (t_0 - r_j^{sp}, t_0), \quad \bar{\Gamma}_j = (t_0 - s_j^{sp}, t_0).$$

Moreover, for $\bar{k} > 0$ to be chosen later, we let

$$k_j = (1 - 2^{-j})\bar{k}, \quad \bar{k}_j = \frac{k_{j+1} + k_j}{2}, \quad w_j = (u - k_j)_+ \quad \text{and} \quad \bar{w}_j = (u - \bar{k}_j)_+.$$

We observe that since $\bar{k}_j > k_j$, $w_j \geq \bar{w}_j$, one has that the following inequality holds,

$$w_j^p \geq (\bar{k}_j - k_j)^{p-1} \bar{w}_j = (2^{-j-2}\bar{k})^{p-1} \bar{w}_j. \tag{4.1}$$

Indeed, (4.1) can be seen as follows. Suppose $u < \bar{k}_j$, then $\bar{w}_j = 0$ and thus (4.1) holds. Instead if $u \geq \bar{k}_j$, then one has that $\bar{k}_j - k_j \leq u - k_j \leq w_j$ and also by using $\bar{k}_j - k_j = 2^{-j-2}\bar{k}$ and $\bar{w}_j \leq w_j$, we obtain

$$(2^{-j-2}\bar{k})^{p-1} \bar{w}_j = (\bar{k}_j - k_j)^{p-1} \bar{w}_j \leq w_j^{p-1} \bar{w}_j \leq w_j^p,$$

which proves (4.1). Additionally, we choose $\psi_j \in C_0^\infty(B_j)$, $\eta_j \in C^\infty(\Gamma_j)$ such that $0 \leq \psi_j \leq 1$ in B_j, $\psi_j \equiv 1$ on B_{j+1}, $|\nabla \psi_j| < \frac{2^{j+3}}{r}$ in B_j and $0 \leq \eta_j \leq 1$ in Γ_j, and $\eta_j(t) = 1$ if $t \geq t_0 - r_{j+1}^{sp}$ with $\eta_j(t) = 0$ if $t \leq t_0 - s_j^{sp}$ and $|\partial_t \eta_j| \leq \frac{2^{jps}}{r^{ps}}$ in Γ_j. Let $\kappa = \frac{n+sp}{n}$ and $\kappa^* = \frac{n}{n-sp}$ if $sp < n$, and $\kappa = \frac{3}{2}$, $\kappa^* = 2$ if $sp \geq n$. Then noting that $\frac{p\kappa^*(\kappa-1)}{\kappa^*-1} = p$, by Lemma 2.4 we have for some positive constant $C = C(n,p,s)$ that the following inequality holds,

$$\begin{aligned}
&\int_{\Gamma_{j+1}} \fint_{B_{j+1}} \bar{w}_j^{p\kappa}\, dx\, dt \\
&\leq C r_{j+1}^{sp-n} \int_{\Gamma_{j+1}} \int_{B_{j+1}} \int_{B_{j+1}} \frac{|\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p}{|x-y|^{n+sp}}\, dx\, dy\, dt \cdot \left(\sup_{\Gamma_{j+1}} \fint_{B_{j+1}} \bar{w}_j^p\, dx \right)^{\kappa-1} \\
&\quad + C \int_{\Gamma_{j+1}} \fint_{B_{j+1}} \bar{w}_j^p\, dx\, dt \cdot \left(\sup_{\Gamma_{j+1}} \fint_{B_{j+1}} \bar{w}_j^p\, dx \right)^{\kappa-1} \\
&= C r_{j+1}^{sp-n} I_1 \cdot \left(\frac{I_2}{|B_{j+1}|} \right)^{\kappa-1} + C \int_{\Gamma_{j+1}} \fint_{B_{j+1}} |\bar{w}_j|^p\, dx\, dt \cdot \left(\frac{I_2}{|B_{j+1}|} \right)^{\kappa-1},
\end{aligned} \tag{4.2}$$

where

$$I_1 = \int_{\Gamma_{j+1}} \int_{B_{j+1}} \int_{B_{j+1}} \frac{|\bar{w}_j(x,t) - \bar{w}_j(y,t)|^p}{|x-y|^{n+sp}}\, dx\, dy\, dt \quad \text{and} \quad I_2 = \sup_{\Gamma_{j+1}} \int_{B_{j+1}} |\bar{w}_j|^p\, dx.$$

Let $U_j = B_j \times \Gamma_j$ and $\bar{U}_j = \bar{B}_j \times \bar{\Gamma}_j$. Since $r_{j+1} < r_j$, $s_j < r_j$, we have $\bar{B}_j \subset B_j$, $\bar{\Gamma}_j \subset \Gamma_j$, $B_{j+1} \subset B_j$ and $\Gamma_{j+1} \subset \Gamma_j$. To estimate I_1 and I_2 we apply Lemma 3.1 with $r = r_j$, $\tau_2 = t_0$,

$\tau_1 = t_0 - r_{j+1}^{sp}$, $l = s_j^{sp} - r_{j+1}^{sp}$ and $\phi_j(x,t) = \psi_j(x)\eta_j(t)$ with $\eta_j(t) = 0$ if $t \le \tau_1 - l$ and $\eta_j(t) = 1$ if $t \ge \tau_1$. Observing that $B_{j+1} \subset \bar{B}_j$ and $\Gamma_{j+1} \subset \bar{\Gamma}_j$, using Lemma 3.1, for some positive constant $C = C(n,p,s,\Lambda)$ we get

$$\begin{aligned} I_1 + C I_2 &\le \int_{\bar{\Gamma}_j}\int_{B_j}\int_{B_j} |\bar{w}_j(x,t)\psi_j(x) - \bar{w}_j(y,t)\psi_j(y)|^p \eta_j(t)^p \, d\mu \, dt \\ &\quad + C \sup_{\Gamma_{j+1}} \int_{B_j} \bar{w}_j(x,t)^p \psi_j(x)^p \, dx \\ &\le C(J_1 + J_2 + J_3), \end{aligned} \tag{4.3}$$

where

$$J_1 = \int_{\Gamma_j}\int_{B_j}\int_{B_j} \max\{\bar{w}_j(x,t)^p, \bar{w}_j(y,t)^p\} |\psi_j(x) - \psi_j(y)|^p \eta_j(t)^p \, d\mu \, dt,$$

$$J_2 = \sup_{t\in\Gamma_j, x\in \operatorname{supp}\psi_j} \int_{\mathbb{R}^n\setminus B_j} \frac{\bar{w}_j(y,t)^{p-1}}{|x-y|^{n+sp}} \, dy \int_{B_j} \bar{w}_j(x,t)\psi_j(x)^p \eta_j(t)^p \, dx,$$

and

$$J_3 = \int_{\Gamma_j}\int_{B_j} \xi(\bar{w}_j)(x,t)\psi_j(x)^p \partial_t \eta_j(t)^p \, dxdt.$$

Now we estimate each J_i, $i = 1,2,3$ separately.

Estimate of J_1: Using $r_j < r$ and $\bar{w}_j \le w_j$, we have

$$\begin{aligned} J_1 &= \int_{\Gamma_j}\int_{B_j}\int_{B_j} \max\{\bar{w}_j(x,t)^p, \bar{w}_j(y,t)^p\} |\psi_j(x) - \psi_j(y)|^p \eta_j(t)^p \, d\mu \, dt \\ &\le C(n,p,s,\Lambda)\Big(\sup_{x\in B_j} \int_{B_j} \frac{|\psi_j(x) - \psi_j(y)|^p}{|x-y|^{n+sp}} \, dy\Big) \int_{\Gamma_j}\int_{B_j} \bar{w}_j(x,t)^p \, dx \, dt \\ &\le C(n,p,s,\Lambda) \frac{2^{j(n+sp+p)}}{r_j^{sp}} \int_{\Gamma_j}\int_{B_j} w_j(x,t)^p \, dx \, dt. \end{aligned} \tag{4.4}$$

Estimate of J_2: Without loss of generality, we may assume $x_0 = 0$. Using the fact that $\bar{w}_j \le w_0$, under the assumptions on ψ_j, we have for $x \in \operatorname{supp}\psi_j$, and $y \in \mathbb{R}^n \setminus B_j$,

$$\frac{1}{|x-y|} = \frac{1}{|y|}\frac{|x-(x-y)|}{|x-y|} \le \frac{1}{|y|}\left(1 + 2^{j+3}\right) \le \frac{2^{j+4}}{|y|}.$$

This implies

$$\begin{aligned} J_2 &= \sup_{t\in\Gamma_j, x\in \operatorname{supp}\psi_j} \int_{\mathbb{R}^n\setminus B_j} \frac{\bar{w}_j(y,t)^{p-1}}{|x-y|^{n+sp}} \, dy \int_{\Gamma_j}\int_{B_j} \bar{w}_j(x,t)\psi_j(x)^p \eta_j(t)^p \, dx \, dt \\ &\le C \frac{2^{j(n+sp+p)}}{r^{sp}\bar{k}^{p-1}} \mathrm{Tail}_\infty^{p-1}(w_0; x_0, \frac{r}{2}, t_0 - r^{sp}, t_0) \int_{\Gamma_j}\int_{B_j} w_j(x,t)^p \, dx \, dt \\ &\le C \frac{2^{j(n+sp+p)}}{\delta^{p-1} r_j^{sp}} \int_{\Gamma_j}\int_{B_j} w_j(x,t)^p \, dx \, dt, \end{aligned} \tag{4.5}$$

where we also used the fact $\bar{w}_j \leq \left(\frac{2^{j+2}}{\bar{k}}\right)^{p-1} w_j^p$ from (4.1) and also that $\bar{k}$ would be chosen finally such that $\bar{k} \geq \delta\,\mathrm{Tail}_\infty(w_0;x_0,\frac{r}{2},t_0-r^{sp},t_0)$.

Estimate of J_3: To estimate J_3, we first note that by Lemma 2.10 and the fact that $p>2$ we have

$$\begin{aligned} J_3 &= \int_{\Gamma_j}\int_{B_j} \xi(\bar{w}_j)(x,t)\psi_j(x)^p\partial_t\eta_j(t)^p\,dx\,dt \\ &\leq C(p)\int_{\Gamma_j}\int_{B_j}(\bar{w}_j(x,t)+\bar{k}_j)^{p-2}\bar{w}_j(x,t)^2\psi_j(x)^p|\partial_t\eta_j(t)^p|\,dx\,dt \\ &= J_4+J_5, \end{aligned} \tag{4.6}$$

where

$$J_4 = \int_{(\Gamma_j\times B_j)\cap\{0<u-\bar{k}_j<\bar{k}_j\}}(\bar{w}_j(x,t)+\bar{k}_j)^{p-2}\bar{w}_j(x,t)^2\psi_j(x)^p|\partial_t\eta_j(t)^p|\,dx\,dt,$$

and

$$\begin{aligned} J_5 &= \int_{(\Gamma_j\times B_j)\cap\{\bar{w}_j\geq\bar{k}_j\}}(\bar{w}_j(x,t)+\bar{k}_j)^{p-2}\bar{w}_j(x,t)^2\psi_j(x)^p|\partial_t\eta_j(t)^p|\,dx\,dt \\ &\leq 2^{p-2}\int_{(\Gamma_j\times B_j)\cap\{\bar{w}_j\geq\bar{k}_j\}}\bar{w}_j(x,t)^p\psi_j(x)^p|\partial_t\eta_j(t)^p|\,dx\,dt \\ &\leq C(p,s)\frac{2^{j(n+sp+p)}}{r_j^{sp}}\int_{\Gamma_j}\int_{B_j}w_j(x,t)^p\,dx\,dt, \end{aligned} \tag{4.7}$$

where to deduce the estimate (4.7) we have again used the fact that $p>2$.

Estimate of J_4: Now we estimate J_4 by adapting some ideas from [3]. Indeed, we denote by $A_j=(\Gamma_j\times B_j)\cap\{0<u-\bar{k}_j<\bar{k}_j\}$ and using binomial expansion we observe that,

$$\begin{aligned} J_4 &= \int_{A_j}(\bar{w}_j(x,t)+\bar{k}_j)^{p-2}\bar{w}_j(x,t)^2\psi_j(x)^p|\partial_t\eta_j(t)^p|\,dx\,dt \\ &= \sum_{d=0}^{\infty}\int_{A_j}\binom{p-2}{d}\bar{k}_j^p\left(\frac{\bar{w}_j(x,t)}{\bar{k}_j}\right)^{d+2}|\partial_t\eta_j(t)^p|\,dx\,dt \\ &= J_4^1+J_4^2, \end{aligned} \tag{4.8}$$

where

$$J_4^1 = \sum_{d=0}^{[p-2]}\int_{A_j}\binom{p-2}{d}\bar{k}_j^p\left(\frac{\bar{w}_j(x,t)}{\bar{k}_j}\right)^{d+2}|\partial_t\eta_j(t)^p|\,dx\,dt,$$

and

$$J_4^2 = \sum_{d=[p-2]+1}^{\infty}\int_{A_j}\binom{p-2}{d}\bar{k}_j^p\left(\frac{\bar{w}_j(x,t)}{\bar{k}_j}\right)^{d+2}|\partial_t\eta_j(t)^p|\,dx\,dt.$$

Estimate of J_4^1: Let us estimate J_4^1 as follows. Using Hölder's inequality, we obtain

$$J_4^1 \leq \sum_{d=0}^{[p-2]} \left| \binom{p-2}{d} \right| (\bar{k}_j)^{p-2-d} \Big(\int_{A_j} \bar{w}_j^p (\partial_t \eta_j^p)^{\frac{p}{d+2}} \, dx \, dt \Big)^{\frac{d+2}{p}} |A_j|^{1-\frac{d+2}{p}}.$$

Now, since $u > \bar{k}_j$ in A_j, we observe that

$$\int_{A_j} w_j^p \, dx \, dt \geq \Big(\bar{k}_j - k_j \Big)^p |A_j| = \Big(\frac{\bar{k}}{2^{j+2}} \Big)^p |A_j|.$$

Therefore, we obtain

$$|A_j| \leq \Big(\frac{2^{j+2}}{\bar{k}} \Big)^p \int_{A_j} w_j^p \, dx \, dt. \tag{4.9}$$

Now using (4.9) together with the fact $\bar{w}_j \leq w_j$, $\bar{k}_j < \bar{k}$, $r_j < r$ and also by using the bounds on $|\partial_t \eta_j|$, we get

$$\begin{aligned} J_4^1 &\leq C(p) \sum_{d=0}^{[p-2]} \left| \binom{p-2}{d} \right| 2^{jp} \Big(\int_{A_j} w_j^p |\partial_t \eta_j|^{\frac{p}{d+2}} \, dx \, dt \Big)^{\frac{d+2}{p}} \Big(\int_{A_j} w_j^p \, dx \, dt \Big)^{1-\frac{d+2}{p}} \\ &\leq C(p) \frac{2^{jp(s+1)}}{r_j^{sp}} \int_{A_j} w_j^p \, dx dt. \end{aligned} \tag{4.10}$$

Estimate of J_4^2: Now since $\bar{w}_j < \bar{k}_j$, therefore for all $d \geq [p-2]+1$, we have that $\bar{w}_j^{d-[p-2]-1} \leq \bar{k}_j^{d-[p-2]-1}$. Thus $\bar{k}_j^{p-2-d} \bar{w}_j^{d+2} \leq \bar{k}_j^{p-3-[p-2]} \bar{w}_j^{[p-2]+3}$ and consequently we obtain

$$J_4^2 \leq \sum_{d=[p-2]+1}^{\infty} \left| \binom{p-2}{d} \right| \int_{A_j} \bar{k}_j^{p-3-[p-2]} \bar{w}_j^{[p-2]+3} |\partial_t \eta_j^p| \, dx \, dt.$$

Finally by using $\bar{k}_j^{p-3-[p-2]} < \bar{w}_j^{p-3-[p-2]}$, we have

$$\begin{aligned} J_4^2 &\leq \sum_{d=[p-2]+1}^{\infty} \left| \binom{p-2}{d} \right| \int_{A_j} \bar{w}_j^p |\partial_t \eta_j^p| \, dx \, dt \\ &\leq C(p) \frac{2^{jsp}}{r_j^{ps}} \int_{A_j} \bar{w}_j^p \, dx \, dt \\ &\leq C(p) \frac{2^{jsp}}{r_j^{sp}} \int_{\Gamma_j} \int_{B_j} w_j^p \, dx \, dt, \end{aligned} \tag{4.11}$$

where we have also used the fact that the series $\sum_{d=0}^{\infty} | \binom{p-2}{d} |$ is convergent. Therefore, using (4.10) and (4.11) into (4.8), we obtain

$$J_4 \leq C(p) \frac{2^{jp(s+1)}}{r_j^{ps}} \int_{\Gamma_j} \int_{B_j} w_j^p \, dx \, dt. \tag{4.12}$$

Now using the estimates (4.7) and (4.12) in (4.6) we conclude

$$J_3 \leq C(p,s)\frac{2^{j(n+sp+p)}}{r_j^{ps}}\int_{\Gamma_j}\int_{B_j} w_j^p\,dx\,dt. \tag{4.13}$$

Then using $\bar{w}_j^{p\kappa} \geq (2^{-j-2}\bar{k})^{p(\kappa-1)} w_{j+1}^p$ in (4.2), we get

$$\begin{aligned} I &= (2^{-j-2}\bar{k})^{p(\kappa-1)} \fint_{\Gamma_{j+1}}\fint_{B_{j+1}} w_{j+1}^p\,dx\,dt \\ &\leq \frac{Cr_{j+1}^{ps-n}}{|\Gamma_{j+1}|} I_1 \cdot \Big(\frac{I_2}{|B_{j+1}|}\Big)^{\kappa-1} + C\fint_{\Gamma_{j+1}}\fint_{B_{j+1}} w_j^p\,dx\,dt \cdot \Big(\frac{I_2}{|B_{j+1}|}\Big)^{\kappa-1}. \end{aligned} \tag{4.14}$$

Plugging the estimates (4.4), (4.5) and (4.13) into (4.3), we have

$$I_1, I_2 \leq C(n,p,s,\Lambda)\frac{2^{j(n+sp+p)}}{\delta^{p-1} r_j^{sp}}\int_{\Gamma_j}\int_{B_j} w_j^p\,dx\,dt. \tag{4.15}$$

Then using (4.15) in (4.14), we get

$$I \leq C(n,p,s,\Lambda)\Big(\frac{2^{j(n+sp+p)}}{\delta^{p-1}}\fint_{\Gamma_j}\fint_{B_j} w_j^p\,dx\,dt\Big)^{\kappa}$$

We now let

$$A_j = \Big(\fint_{U_j} w_j^p\,dx\,dt\Big)^{\frac{1}{p}}.$$

Then we have

$$(2^{-j-2}\bar{k})^{p(\kappa-1)} A_{j+1}^p \leq C(n,p,s,\Lambda)\Big(\frac{2^{j(n+sp+p)}}{\delta^{p-1}} A_j^p\Big)^{\kappa}.$$

Then for some positive constant $C = C(n,p,s,\Lambda)$ we have

$$\frac{A_{j+1}}{\bar{k}} \leq \frac{C}{\bar{k}^{\kappa}} 2^{j(\kappa-1)}\Big(\frac{2^{j(n+sp+p)}}{\delta^{p-1}} A_j^p\Big)^{\frac{\kappa}{p}} = C\frac{2^{j\left(\kappa-1+(n+sp+p)(\frac{\kappa}{p})\right)}}{\delta^{(p-1)\frac{\kappa}{p}}}\Big(\frac{A_j}{\bar{k}}\Big)^{\kappa}.$$

Noting that $w_0 = u$, we now let

$$\bar{k} = \delta \mathrm{Tail}_\infty(u; x_0, \frac{r}{2}, t_0 - r^{ps}, t_0) + C^{\frac{1}{\kappa-1}} b^{\frac{1}{(\kappa-1)^2}} \delta^{-\frac{(p-1)\kappa}{p(\kappa-1)}}\Big(\fint_{U^-(r)} u^p\,dx\,dt\Big)^{\frac{1}{p}},$$

such that for

$$\beta = \kappa - 1, \quad c_0 = \frac{C}{\delta^{(p-1)\frac{\kappa}{p}}} > 1, \quad b = 2^{\kappa-1+(n+sp+p)(\frac{\kappa}{p})} > 1 \quad \text{and} \quad Y_j = \frac{A_j}{\bar{k}},$$

the hypothesis of Lemma 2.7 is satisfied and consequently we have that

$$\sup_{U^-(\frac{r}{2})} u \leq \bar{k},$$

which proves Theorem 2.15.

5 Some qualitative and quantitative properties of supersolutions

In this section, we prove some qualitative and quantitative properties of supersolutions which are strictly bounded away from zero. Throughout this section, by a global supersolution u in $\mathbb{R}^n \times (0,T)$, we refer to a bounded positive function u which satisfies the hypothesis of Definition 2.11 in $\Omega \times (0,T)$ where Ω is any bounded domain in $\mathbb{R}^n$.

The following lemma is the nonlocal analogue of Lemma 3.1 in [4] which states that the inverse of a supersolution is a subsolution.

Lemma 5.1 *Let $p > 2$ and $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n)) \cap C_{\text{loc}}(0,T;L^p_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0,T;W^{s,p}_{\text{loc}}(\Omega))$ be a supersolution of (1.1) in $\Omega \times (0,T)$ such that $u \geq \rho > 0$ in $\mathbb{R}^n \times (0,T)$, then u^{-1} is a subsolution of (1.1).*

Proof Let $v = u^{-1}$, $\Omega' \Subset \Omega$ and $\psi \in W^{1,p}_{\text{loc}}(0,T;L^p(\Omega')) \cap L^p_{\text{loc}}(0,T;W^{s,p}_0(\Omega'))$ be nonnegative. Since u is a weak supersolution of (1.1), by formally choosing $\phi(x,t) = u(x,t)^{2(1-p)}\psi(x,t)$ as a test function in (2.6) which can be justified by mollifying in time as in the proof of Lemma 3.1, we obtain for every $(t_1,t_2) \Subset (0,T)$,

$$0 \leq I_1 + I_2, \tag{5.1}$$

where

$$\begin{aligned} I_1 &= \int_{\Omega'} u(x,t_2)^{p-1}\phi(x,t_2)\,dx - \int_{\Omega'} u(x,t_1)^{p-1}\phi(x,t_1)\,dx \\ &\quad - \int_{t_1}^{t_2}\int_{\Omega'} u(x,t)^{p-1}\partial_t\phi(x,t)\,dx\,dt, \\ &= \int_{\Omega'} u(x,t_2)^{1-p}\psi(x,t_2)\,dx - \int_{\Omega'} u(x,t_1)^{1-p}\psi(x,t_1)\,dx - I_3, \end{aligned}$$

with

$$\begin{aligned} I_3 &= \int_{t_1}^{t_2}\int_{\Omega'} u(x,t)^{p-1}\big(u(x,t)^{2(1-p)}\partial_t\psi(x,t) \\ &\quad - 2(p-1)\psi(x,t)u(x,t)^{1-2p}\partial_t u(x,t)\big)\,dx\,dt \\ &= \int_{t_1}^{t_2}\int_{\Omega'} u(x,t)^{1-p}\partial_t\psi(x,t)\,dx\,dt - 2(p-1)\int_{t_1}^{t_2}\int_{\Omega'} \psi(x,t)u(x,t)^{-p}\partial_t u(x,t)\,dx\,dt \\ &= \int_{t_1}^{t_2}\int_{\Omega'} u(x,t)^{1-p}\partial_t\psi(x,t)\,dx\,dt - 2\int_{t_1}^{t_2}\int_{\Omega'} u(x,t)^{1-p}\partial_t\psi(x,t)\,dx\,dt + 2I_4, \end{aligned}$$

and

$$I_4 = \int_{\Omega'} u(x,t_2)^{1-p}\psi(x,t_2)\,dx - \int_{\Omega'} u(x,t_1)^{1-p}\psi(x,t_1)\,dx.$$

We thus obtain from above,

$$I_1 = -\Big(\int_{\Omega'} v(x,t_2)^{p-1}\psi(x,t_2)\,dx - \int_{\Omega'} v(x,t_1)^{p-1}\psi(x,t_1)\,dx \\ - \int_{t_1}^{t_2}\int_{\Omega'} v^{p-1}\partial_t\psi\,dx\,dt\Big).$$

Here

$$\begin{aligned} I_2 &= \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{A}(u(x,y,t))(\phi(x,t)-\phi(y,t))\,d\mu\,dt \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t)) \\ &\quad \cdot (u(x,t)^{2(1-p)}\psi(x,t) - u(y,t)^{2(1-p)}\psi(y,t))\,d\mu\,dt \\ &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{A}(v(x,y,t))\Big(\Big(\frac{v(x,t)}{v(y,t)}\Big)^{p-1}\psi(x,t) - \Big(\frac{v(y,t)}{v(x,t)}\Big)^{p-1}\psi(y,t)\Big)\,d\mu\,dt. \end{aligned}$$

Now we estimate I_2. Let us first consider the case when $v(x,t) \geq v(y,t)$. In this case, we have

$$\begin{aligned} &\mathcal{A}(v(x,y,t))\Big(\Big(\frac{v(x,t)}{v(y,t)}\Big)^{p-1}\psi(x,t) - \Big(\frac{v(y,t)}{v(x,t)}\Big)^{p-1}\psi(y,t)\Big) \\ &\quad \geq \mathcal{A}(v(x,y,t))\big(\psi(x,t)-\psi(y,t)\big). \end{aligned}$$

Likewise when $v(x,t) < v(y,t)$, we have

$$\begin{aligned} &\mathcal{A}(v(x,y,t))\Big(\Big(\frac{v(x,t)}{v(y,t)}\Big)^{p-1}\psi(x,t) - \Big(\frac{v(y,t)}{v(x,t)}\Big)^{p-1}\psi(y,t)\Big) \\ &= |v(y,t)-v(x,t)|^{p-2}(v(y,t)-v(x,t))\Big(\Big(\frac{v(y,t)}{v(x,t)}\Big)^{p-1}\psi(y,t) - \Big(\frac{v(x,t)}{v(y,t)}\Big)^{p-1}\psi(x,t)\Big) \\ &\geq \mathcal{A}(v(y,x,t))(\psi(y,t)-\psi(x,t)). \end{aligned}$$

Therefore in either case, we obtain

$$I_2 \leq -\int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{A}(v(x,y,t))(\psi(x,t)-\psi(y,t))\,d\mu\,dt.$$

By inserting the above estimates for I_1 and I_2 into (5.1), we get

$$\begin{aligned} &\int_{\Omega'} v(x,t_2)^{p-1}\psi(x,t_2)\,dx - \int_{\Omega'} v(x,t_1)^{p-1}\psi(x,t_1)\,dx \\ &\quad - \int_{t_1}^{t_2}\int_{\Omega'} v(x,t)^{p-1}\partial_t\psi(x,t)\,dx\,dt \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{A}(v(x,y,t))(\psi(x,t)-\psi(y,t))\,d\mu\,dt \leq 0. \end{aligned}$$

Hence $v = u^{-1}$ is a subsolution of (1.1). □

Now we prove an energy estimate for strictly positive supersolutions of (1.1) which is the key ingredient needed to deduce reverse Hölder inequality for strictly positive supersolutions.

Lemma 5.2 *Let $p > 2$, $x_0 \in \mathbb{R}^n$, $r > 0$ and $\alpha \in (0, p-1)$ with $B_r = B_r(x_0) \Subset \Omega$ and $(\tau_1, \tau_2 + l) \Subset (0, T)$. Suppose that $u \in L^\infty(0,T;L^\infty(\mathbb{R}^n)) \cap C_{\mathrm{loc}}(0,T;L^p_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0,T;W^{s,p}_{\mathrm{loc}}(\Omega))$ is a weak supersolution of (1.1) in $\Omega \times (0,T)$ with $u \geq \rho > 0$ in $\mathbb{R}^n \times (\tau_1, \tau_2 + l)$. Then there exists positive constants $C = C(n,p,s,\Lambda)$ and $c = c(p)$ large enough such that*

$$
\begin{aligned}
&\frac{p-1}{\alpha} \sup_{\tau_1 < t < \tau_2} \int_{B_r} \psi(x)^p u(x,t)^\alpha \, dx \\
&\quad + \frac{\zeta(\epsilon)}{c(p)} \int_{\tau_1}^{\tau_2+l} \int_{B_r} \int_{B_r} \left|\psi(x)u(x,t)^{\frac{\alpha}{p}} - \psi(y)u(y,t)^{\frac{\alpha}{p}}\right|^p \eta(t)\, d\mu\, dt \\
&\leq \left(\zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}}\right) \int_{\tau_1}^{\tau_2+l} \int_{B_r} \int_{B_r} |\psi(x) - \psi(y)|^p (u(x,t)^\alpha + u(y,t)^\alpha)\eta(t)\, d\mu\, dt \\
&\quad + C(\Lambda) \sup_{x \in \operatorname{supp} \psi} \int_{\mathbb{R}^n \setminus B_r} \frac{dy}{|x-y|^{n+sp}} \int_{\tau_1}^{\tau_2+l} \int_{B_r} u(x,t)^\alpha \psi(x)^p \eta(t)\, dx\, dt \\
&\quad + \frac{p-1}{\alpha} \int_{\tau_1}^{\tau_2+l} \int_{B_r} u(x,t)^\alpha \psi(x)^p |\partial_t \eta(t)|\, dx\, dt,
\end{aligned}
$$

where $\epsilon = p - \alpha - 1$ and $\zeta(\epsilon) = \frac{\epsilon p^p}{\alpha^p}$, if $\alpha \geq 1$ and $\zeta(\epsilon) = \frac{\epsilon p^p}{\alpha}$ if $\alpha \in (0,1)$. Moreover, $\psi \in C_0^\infty(B_r)$ is taken to be nonnegative and $\eta \in C^\infty(\mathbb{R})$ is also nonnegative such that $\eta(t) = 1$ if $\tau_1 \leq t \leq \tau_2$ and $\eta(t) = 0$ if $t \geq \tau_2 + l$.

Proof Let $t_1 \in (\tau_1, \tau_2)$ and $t_2 = \tau_2 + l$. We consider $\eta \in C^\infty(t_1, t_2)$ such that $\eta(t_2) = 0$ and $\eta(t) = 1$ for all $t \leq t_1$. Let $\epsilon \in (0, p-1)$ and $\alpha = p - \epsilon - 1$. Then since u is a strictly positive weak supersolution of (1.1), choosing $\phi(x,t) = u(x,t)^{-\epsilon}\psi(x)^p\eta(t)$ as a test function in (2.6) (which is again justified by mollifying in time), we obtain

$$
0 \leq I_1 + I_2 + 2I_3, \tag{5.2}
$$

where

$$
\begin{aligned}
I_1 &= \int_{t_1}^{t_2} \int_{B_r} \frac{\partial}{\partial t}(u^{p-1})\phi(x,t)\, dx\, dt, \\
I_2 &= \int_{t_1}^{t_2} \int_{B_r} \int_{B_r} \mathcal{A}(u(x,y,t))(u(x,t)^{-\epsilon}\psi(x)^p - u(y,t)^{-\epsilon}\psi(y)^p)\eta(t)\, d\mu\, dt,
\end{aligned}
$$

and

$$
I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} \mathcal{A}(u(x,y,t))u(x,t)^{-\epsilon}\psi(x,t)^p \eta(t)\, d\mu\, dt.
$$

We observe that for any $x \in B_r$ and $y \in \mathbb{R}^n \setminus B_r$, we have that the integrand in I_3 is nonnegative precisely in the set where $u(x,t) \geq u(y,t)$. In view of this, we observe that I_3 can be estimated from above in the following way,

$$
\begin{aligned}
I_3 &= \int_{t_1}^{t_2}\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} |u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))u(x,t)^{-\epsilon}\psi(x)^p\eta(t)\,d\mu\,dt \\
&\le \int_{t_1}^{t_2}\int_{\mathbb{R}^n\setminus B_r}\int_{B_r} u(x,t)^{p-\epsilon-1}\psi(x)^p\eta(t)\,d\mu\,dt \\
&\le C(\Lambda)\sup_{x\in \operatorname{supp}\psi}\int_{\mathbb{R}^n\setminus B_r}\frac{dy}{|x-y|^{n+sp}}\int_{t_1}^{t_2}\int_{B_r} u(x,t)^{p-\epsilon-1}\psi(x)^p\eta(t)\,dx\,dt.
\end{aligned}
\tag{5.3}
$$

Then we note that I_2 can be estimated using Lemma 2.9 as follows,

$$
\begin{aligned}
I_2 \le& -\frac{\zeta(\epsilon)}{C(p)}\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}\left|\psi(x)u(x,t)^{\frac{\alpha}{p}}-\psi(y)u(y,t)^{\frac{\alpha}{p}}\right|^p\eta(t)\,d\mu\,dt \\
&+\left(\zeta(\epsilon)+1+\frac{1}{\epsilon^{p-1}}\right)\int_{t_1}^{t_2}\int_{B_r}\int_{B_r}|\psi(x)-\psi(y)|^p(u(x,t)^\alpha+u(y,t)^\alpha)\eta(t)\,d\mu\,dt.
\end{aligned}
\tag{5.4}
$$

For I_1 we have

$$
\begin{aligned}
I_1 =& -\frac{p-1}{p-\epsilon-1}\int_{B_r} u^{p-\epsilon-1}(x,t_1)\psi(x)^p\,dx \\
&-\frac{p-1}{p-\epsilon-1}\int_{t_1}^{t_2}\int_{B_r} u(x,t)^{p-\epsilon-1}\psi(x)^p\partial_t\eta(t)\,dx\,dt.
\end{aligned}
\tag{5.5}
$$

Now using (5.3), (5.4), (5.5) into (5.2) and letting $t_1\to\tau_1$, we obtain

$$
\begin{aligned}
&\frac{\zeta(\epsilon)}{C(p)}\int_{\tau_1}^{\tau_2+l}\int_{B_r}\int_{B_r}\left|\psi(x)u(x,t)^{\frac{\alpha}{p}}-\psi(y)u(y,t)^{\frac{\alpha}{p}}\right|^p\eta(t)\,d\mu\,dt \\
&\le \left(\zeta(\epsilon)+1+\frac{1}{\epsilon^{p-1}}\right)\int_{\tau_1}^{\tau_2+l}\int_{B_r}\int_{B_r}|\psi(x)-\psi(y)|^p(u(x,t)^\alpha+u(y,t)^\alpha)\eta(t)\,d\mu\,dt \\
&\quad + C(\Lambda)\sup_{x\in \operatorname{supp}\psi}\int_{\mathbb{R}^n\setminus B_r}\frac{dy}{|x-y|^{n+sp}}\int_{\tau_1}^{\tau_2+l}\int_{B_r} u(x,t)^\alpha\psi(x)^p\eta(t)\,dx\,dt \\
&\quad + \frac{p-1}{\alpha}\int_{\tau_1}^{\tau_2+l}\int_{B_r} u(x,t)^\alpha\psi(x)^p|\partial_t\eta(t)|\,dx\,dt.
\end{aligned}
\tag{5.6}
$$

We then choose t_1 such that

$$
\int_{B_r} u(x,t_1)^{p-\epsilon-1}\psi(x)^p\,dx \ge \frac{1}{2}\sup_{\tau_1<t<\tau_2}\int_{B_r} u(x,t)^{p-\epsilon-1}\psi(x)^p\,dx. \tag{5.7}
$$

Again using (5.3), (5.4), (5.5) and (5.7), we get

$$
\begin{aligned}
&\frac{p-1}{\alpha} \sup_{\tau_1<t<\tau_2} \int_{B_r} \psi(x)^p u(x,t)^\alpha \, dx \\
&\leq \left(\zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}}\right) \int_{\tau_1}^{\tau_2+l} \int_{B_r} \int_{B_r} |\psi(x)-\psi(y)|^p (u(x,t)^\alpha + u(y,t)^\alpha) \eta(t) \, d\mu \, dt \\
&\quad + C(\Lambda) \sup_{x\in \operatorname{supp}\psi} \int_{\mathbb{R}^n\setminus B_r} \frac{dy}{|x-y|^{n+sp}} \int_{\tau_1}^{\tau_2+l} \int_{B_r} u(x,t)^\alpha \psi(x)^p \eta(t) \, dx \, dt \\
&\quad + \frac{p-1}{\alpha} \int_{\tau_1}^{\tau_2+l} \int_{B_r} u(x,t)^\alpha \psi(x)^p |\partial_t \eta(t)| \, dx \, dt.
\end{aligned} \tag{5.8}
$$

Therefore from (5.6) and (5.8) we get the required estimate. □

Following the energy estimate, we now proceed with the proof of the reverse Hölder inequality for strictly positive supersolutions as in Theorem 2.17.

Proof of Theorem 2.17 Let $0 < s < 1$ and $\kappa = \frac{n+sp}{n}$ if $sp < n$ and $\kappa = \frac{3}{2}$ if $sp \geq n$. Let us denote by

$$
r_0 = r, \quad r_j = \left(1 - (1-\theta)\frac{1-2^{-j}}{(1-2^{-m})}\right) r, \quad \delta_j = 2^{-j} r, \quad j = 1, 2, \dots, m
$$

and $U_j = B_j \times \Gamma_j = B_{r_j}(x_0) \times (t_0, t_0 + r_j^{sp})$. We shall fix m later. Now we choose nonnegative test functions $\psi_j \in C_0^\infty(B_j)$ such that $0 \leq \psi_j \leq 1$ in B_j, $\psi_j \equiv 1$ in B_{j+1}, $|\nabla \psi_j| \leq \frac{2^{j+3}}{(1-\theta)r}$ and $\operatorname{dist}(\operatorname{supp}\psi_j, \mathbb{R}^n \setminus B_j) \geq \frac{\delta_j(1-\theta)}{2}$. Moreover, we choose $\eta_j \in C^\infty(\Gamma_j)$ such that $0 \leq \eta_j \leq 1$ in Γ_j, $\eta_j \equiv 1$ in Γ_{j+1}, and $|\partial_t \eta_j| \leq \frac{2^{sp(j+3)}}{(1-\theta)r^{sp}}$, $\eta_j(t) = 0$ if $t \geq t_0 + r_j^{sp}$. Let $\alpha = p - \epsilon - 1$ where $\epsilon \in (0, p-1)$. Then $\alpha \in (0, p-1)$. Denote by $v = u^{\frac{\alpha}{p}}$. Let $r = r_j$, $\tau_1 = t_0$, $\tau_2 = t_0 + r_{j+1}^{sp}$, $l = r_j^{sp} - r_{j+1}^{sp}$. Let $\kappa = \frac{n+sp}{n}$ and $\kappa^* = \frac{n}{n-sp}$ if $sp < n$, and $\kappa = \frac{3}{2}$, $\kappa^* = 2$ if $sp \geq n$. Then noting that $\frac{p\kappa^*(\kappa-1)}{\kappa^*-1} = p$, by the Sobolev embedding theorem (Lemma 2.4), we obtain for some positive constant $C = C(n,p,s)$ that the following inequality holds,

$$
\int_{\Gamma_{j+1}} \fint_{B_{j+1}} v^{p\kappa} \, dx \, dt \leq C\left(r_{j+1}^{sp-n} I_1 + \int_{\Gamma_{j+1}} \fint_{B_{j+1}} v^p \, dxdt\right) \cdot \left(\frac{I_2}{|B_{j+1}|}\right)^{\kappa-1}, \tag{5.9}
$$

where

$$
I_1 = \int_{\Gamma_{j+1}} \int_{B_{j+1}} \int_{B_{j+1}} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{n+sp}} \, dx \, dy \, dt,
$$

and

$$
I_2 = \sup_{\Gamma_{j+1}} \int_{B_{j+1}} v^p \, dx.
$$

Using the fact that $\psi_j \equiv 1$ on B_{j+1} and also that $\eta_j \equiv 1$ on Γ_{j+1}, we obtain using Lemma 5.2 that the following holds,

$$
I_1, I_2 \leq C(J_1 + J_2 + J_3), \tag{5.10}
$$

for some positive constant C which is independent of α as long as α is away from $p-1$, where

$$\begin{aligned} J_1 &= \int_{\Gamma_j}\int_{B_j}\int_{B_j}(v(x,t)^p + v(y,t)^p)\frac{|\psi_j(x)-\psi_j(y)|^p}{|x-y|^{n+sp}}\eta_j(t)\,dx\,dy\,dt \\ &\le C\frac{2^{j(n+sp+p)}}{(1-\theta)^p r_j^{sp}}\int_{\Gamma_j}\int_{B_j} v(x,t)^p\,dx\,dt, \end{aligned} \tag{5.11}$$

since $r_j < r$,

$$\begin{aligned} J_2 &= \sup_{x\in\operatorname{supp}\psi_j}\int_{\mathbb{R}^n\setminus B_j}\frac{dy}{|x-y|^{n+sp}}\int_{\Gamma_j}\int_{B_j} v(x,t)^p\psi_j(x)^p\eta_j(t)\,dx\,dt \\ &\le C\frac{2^{j(n+sp+p)}}{r_j^{sp}}\int_{\Gamma_j}\int_{B_j} v(x,t)^p\,dx\,dt, \end{aligned} \tag{5.12}$$

and

$$J_3 = \int_{\Gamma_j}\int_{B_j}\psi_j(x)^p v(x,t)^p|\partial_t\eta_j(t)|\,dx\,dt \le C\frac{2^{ps(j+3)}}{(1-\theta)r_j^{sp}}\int_{\Gamma_j}\int_{B_j} v(x,t)^p\,dx\,dt, \tag{5.13}$$

again since $r_j < r$.

Therefore, using (5.11), (5.12) and (5.13) into (5.10) we obtain

$$I_1, I_2 \le C\frac{2^{j(n+sp+p)}}{(1-\theta)^p r_j^{sp}}\int_{\Gamma_j}\int_{B_j} v(x,t)^p\,dx\,dt, \tag{5.14}$$

for some positive constant C independent of α as long as α is away from $p-1$, but may depend on n, p, s, Λ.

Using the estimate (5.14) and the fact that $r_{j+1} < r_j < 2r_{j+1}$ for every j, we obtain from (5.9), since $v = u^{\frac{\alpha}{p}}$ that

$$⨍_{U_{j+1}} u^{\kappa\alpha}\,dx\,dt \le C\Big(\frac{2^{j(n+sp+p)}}{(1-\theta)^p}⨍_{U_j} u^\alpha\,dx\,dt\Big)^\kappa, \tag{5.15}$$

for some positive constant C independent of α (given that our choice of α will be away from $p-1$) but may depend on n, p, s, Λ. Now we use Moser iteration technique into (5.15). Let us fix $q, \bar{q}$ such that $0 < \bar{q} < q < q_0 = \kappa(p-1)$ and m such that $\bar{q}\kappa^{m-1} \le q \le \bar{q}\kappa^m$. Let $t_0 = \frac{q}{\kappa^m}$, then $t_0 \le \bar{q}$. Denote by $t_j = \kappa^j t_0$ for $j = 0, 1, \cdots, m$. Then observing that $r_m = \theta r$ and $r_0 = r$, we get $U_m = U^+(\theta r)$ and $U_0 = U^+(r)$. Hence by (5.15), we obtain

$$\begin{aligned} \Big(⨍_{U^+(\theta r)} u^q\,dx\,dt\Big)^{\frac{1}{q}} &= \Big(⨍_{U_m} u^q\,dx\,dt\Big)^{\frac{1}{q}} \\ &\le \Big(\frac{C2^{\frac{(n+sp+p)m}{p}}}{(1-\theta)}\Big)^{\frac{p}{t_{m-1}}}\Big(⨍_{U_{m-1}} u^{t_{m-1}}\,dx\,dt\Big)^{\frac{1}{t_{m-1}}} \\ &\le \Big(\frac{C_{\text{prod}}(m)}{(1-\theta)^{m^*}}⨍_{U^+(r)} u^{t_0}\,dx\,dt\Big)^{\frac{1}{t_0}}, \end{aligned}$$

where

$$C_{\text{prod}}(m) = C^{m^*} \prod_{j=0}^{m-1} \left(2^{\frac{n+sp+p}{p}(j+1)}\right)^{p\kappa^{-j}},$$

and

$$m^* = p \sum_{j=0}^{m-1} \kappa^{-j} = \frac{p\kappa}{\kappa - 1}(1 - \kappa^{-m}).$$

It can be easily seen that $C_{\text{prod}}(m)$ is a positive constant uniformly bounded on m, where C is independent of $\bar{q}$ but depends on q due to the singularity of the constants involved in the energy inequality in Lemma 5.2 at $\epsilon = 0$. Finally using Hölder's inequality, we obtain

$$\left(\fint_{U^+(\theta r)} u^q \,dx\,dt \right)^{\frac{1}{q}} \leq \left(\frac{C}{(1-\theta)^{m^*}} \right)^{\frac{1}{t_0}} \left(\fint_{U^+(r)} u^{\bar{q}} \,dx\,dt \right)^{\frac{1}{\bar{q}}}.$$

Now, since $\bar{q}\kappa^{m-1} \leq t_0\kappa^m$, we have $t_0 \geq \frac{\bar{q}}{\kappa}$. As a consequence the required estimate follows with $\mu = \frac{p\kappa^2}{\kappa - 1}$.

In closing, we prove the following logarithmic estimate for strictly positive supersolutions which constitutes the nonlocal analogue of Lemma 6.1 in [4] and also constitutes one of the key ingredients in the proof of weak Harnack in the local case.

Lemma 5.3 *Let $p > 2$, $x_0 \in \mathbb{R}^n$, $r > 0$ and $t_0 \in (r^{ps}, T - r^{ps})$ with $B_{\frac{3r}{2}} = B_{\frac{3r}{2}}(x_0) \Subset \Omega$ and $(t_0 - r^{sp}, t_0 + r^{sp}) \Subset (0, T)$. Suppose that $u \in L^\infty(0, T; L^\infty(\mathbb{R}^n)) \cap C_{\text{loc}}(0, T; L^p_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0, T; W^{s,p}_{\text{loc}}(\Omega))$ is a weak supersolution of (1.1) in $\Omega \times (0, T)$ with $u \geq \rho > 0$ in $\mathbb{R}^n \times (t_0 - r^{sp}, t_0 + r^{sp})$. Then for every $\lambda > 0$, there exists a positive constant $C = C(n, p, s, \Lambda)$ such that*

$$\left|\{(x,t) \in U^+(x_0, t_0, r) : \log u(x,t) < -\lambda - b\}\right| \leq \frac{Cr^{n+sp}}{\lambda^{p-1}} \tag{5.16}$$

and

$$\left|\{(x,t) \in U^-(x_0, t_0, r) : \log u(x,t) > \lambda - b\}\right| \leq \frac{Cr^{n+sp}}{\lambda^{p-1}} \tag{5.17}$$

where

$$b = b(u(\cdot, t_0)) = -\frac{\int_{B_{\frac{3r}{2}}(x_0)} \log u(x, t_0)\psi(x)^p \,dx}{\int_{B_{\frac{3r}{2}}(x_0)} \psi(x)^p \,dx}.$$

Proof Following Lemma 6.1 in [4], we only prove (5.16) because the proof of (5.17) is analogous. Without loss of generality, we may assume $x_0 = 0$. Let $\psi \in C_0^\infty(B_{\frac{3r}{2}})$ be a non-negative radially decreasing function such that $0 \leq \psi \leq 1$ in $B_{\frac{3r}{2}}$, $\psi \equiv 1$ in B_r, $|\nabla\psi| \leq \frac{C}{r}$ in $B_{\frac{3r}{2}}$. Since u is a strictly positive supersolution of (1.1), choosing $\phi(x,t) = \psi(x)^p u(x,t)^{1-p}$ as a test function in (2.6), we get

$$I_1 + I_2 + 2I_3 \geq 0, \tag{5.18}$$

where for any $t_0 - r^{sp} \leq t_1 < t_2 \leq t_0 + r^{sp}$, we have

$$I_1 = \int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \frac{\partial}{\partial t}(u(x,t)^{p-1})\phi(x,t)\,dx\,dt = (p-1)\int_{B_{\frac{3r}{2}}} \log\, u(x,t)\psi(x)^p\,dx\Big|_{t=t_1}^{t_2},$$
$$I_2 = \int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} \mathcal{A}(u(x,y,t))(\phi(x,t)-\phi(y,t))\,d\mu\,dt, \tag{5.19}$$

and

$$I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} \mathcal{A}(u(x,y,t))\phi(x,t)\,d\mu\,dt.$$

Following the arguments in the proof of [12, Lemma 1.3], we obtain for some positive constant $C = C(p)$,

$$\begin{aligned} I_2 &= \int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} \mathcal{A}(u(x,y,t))(\phi(x,t)-\phi(y,t))\,d\mu\,dt \\ &\le -\frac{1}{C}\int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} K(x,y,t)|\log\, u(x,t) - \log\, u(y,t)|^p\psi(y)^p\,dx\,dy\,dt \\ &\quad + C\int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} K(x,y,t)|\psi(x)-\psi(y)|^p\,dx\,dy\,dt \\ &\le -\frac{1}{C}\int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} K(x,y,t)|\log\, u(x,t) - \log\, u(y,t)|^p\psi(y)^p\,dx\,dy\,dt \\ &\quad + C(t_2-t_1)r^{n-sp}, \end{aligned} \tag{5.20}$$

where the last inequality is obtained using the properties of ψ. Again following the proof of [12, Lemma 1.3], we get that

$$I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^n \setminus B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} \mathcal{A}(u(x,y,t))\phi(x,t)\,d\mu\,dt \le C(t_2-t_1)r^{n-sp}. \tag{5.21}$$

Therefore using the estimates (5.19), (5.20) and (5.21) into (5.18), we obtain

$$\begin{aligned} &\frac{1}{C}\int_{t_1}^{t_2} \int_{B_{\frac{3r}{2}}} \int_{B_{\frac{3r}{2}}} K(x,y,t)|\log\, u(x,t) - \log\, u(y,t)|^p\psi(y)^p\,dx\,dy\,dt \\ &\qquad -(p-1)\int_{B_{\frac{3r}{2}}} \log\, u(x,t)\psi(x)^p\,dx\Big|_{t=t_1}^{t_2} \le C(t_2-t_1)r^{n-sp}. \end{aligned} \tag{5.22}$$

Let $v(x,t) = -\log\, u(x,t)$ and

$$V(t) = \frac{\int_{B_{\frac{3r}{2}}} v(x,t)\psi(x)^p\,dx}{\int_{B_{\frac{3r}{2}}} \psi(x)^p\,dx}.$$

Since $0 \le \psi \le 1$ in $B_{\frac{3r}{2}}$ and $\psi \equiv 1$ in B_r, therefore we have that $\int_{B_{\frac{3r}{2}}} \psi(x)^p\,dx \approx r^n$. Hence dividing by $\int_{B_{\frac{3r}{2}}} \psi(x)^p\,dx$ on both sides of (5.22), we obtain using the weighted Poincaré inequality in Lemma 2.6 that the following holds,

$$V(t_2) - V(t_1) + \frac{r^{-sp}}{c(p-1)} \int_{t_1}^{t_2} \fint_{B_r} |v(x,t) - V(t)|^p\,dx\,dt \le \frac{cr^{-sp}}{p-1}(t_2 - t_1).$$

Let $A_1 = C(p-1)$, $A_2 = \frac{C}{p-1}$,

$$\bar{w}(x,t) = v(x,t) - A_2 r^{-sp}(t - t_1) \quad \text{and} \quad \bar{W}(t) = V(t) - A_2 r^{-sp}(t - t_1).$$

Therefore $v(x,t) - V(t) = \bar{w}(x,t) - \bar{W}(t)$. Hence, we get

$$\bar{W}(t_2) - \bar{W}(t_1) + \frac{1}{A_1 r^{n+sp}} \int_{t_1}^{t_2} \int_{B_r} |\bar{w}(x,t) - \bar{W}(t)|^p\,dx\,dt \le 0. \tag{5.23}$$

Therefore, $\bar{W}(t)$ is a monotone decreasing function in $t_0 - r^{sp} \le t_1 < t_2 \le t_0 + r^{sp}$. Hence, $\bar{W}(t)$ is differentiable almost everywhere with respect to t. Dividing by $t_2 - t_1$ on both sides of (5.23), we obtain after letting $t_2 \to t_1$,

$$\bar{W}'(t) + \frac{1}{A_1 r^{n+sp}} \int_{B_r} \left|\bar{w}(x,t) - \bar{W}(t)\right|^p dx \le 0. \tag{5.24}$$

Let $t_1 = t_0$, then $\bar{W}(t_0) = V(t_0)$ and we denote by $b(u(\cdot, t_0)) = \bar{W}(t_0)$. Let

$$\Omega_t^+(\lambda) = \left\{x \in B_r : \bar{w}(x,t) > b + \lambda\right\}.$$

Then for any $x \in \Omega_t^+(\lambda)$ and $t \ge t_0$, since $\bar{W}(t) \le \bar{W}(t_0) = b$, we have

$$\bar{w}(t,x) - \bar{W}(t) \ge b + \lambda - \bar{W}(t) \ge b + \lambda - \bar{W}(t_0) = \lambda > 0.$$

Hence from (5.24), we have

$$\bar{W}'(t) + \frac{|\Omega_t^+(\lambda)|}{A_1 r^{n+sp}} \left(b + \lambda - \bar{W}(t)\right)^p \le 0.$$

Therefore, we have

$$|\Omega_t^+(\lambda)| \le -\frac{A_1 r^{n+sp}}{p-1} \partial_t \left(b + \lambda - \bar{W}(t)\right)^{1-p}.$$

Integrating over t_0 to $t_0 + r^{sp}$, we obtain

$$\begin{aligned} &\left|\{(x,t) \in B_r \times (t_0, t_0 + r^{sp}) : \bar{w}(x,t) > b + \lambda\}\right| \\ &\le -\frac{A_1 r^{n+sp}}{p-1} \int_{t_0}^{t_0 + r^{sp}} \partial_t \left(b + \lambda - \bar{W}(t)\right)^{1-p} dt, \end{aligned}$$

which gives

$$\left|\{(x,t) \in B_r \times (t_0, t_0 + r^{sp}) : \log u(x,t) + A_2 r^{-sp}(t - t_0) < -\lambda - b\}\right| \le \frac{A_1}{p-1} \frac{r^{n+sp}}{\lambda^{p-1}}. \tag{5.25}$$

Finally, we note that

$$\left|\{(x,t)\in B_r\times(t_0,t_0+r^{sp}) : \log u(x,t) < -\lambda - b\}\right| \le A+B, \tag{5.26}$$

where

$$A = \left|\{(x,t)\in B_r\times(t_0,t_0+r^{sp}) : \log u(x,t) + A_2 r^{-sp}(t-t_0) < -\tfrac{\lambda}{2} - b\}\right| \le \frac{Cr^{n+sp}}{\lambda^{p-1}},$$

which follows from (5.25) and

$$B = \left|\{(x,t)\in B_r\times(t_0,t_0+r^{sp}) : A_2 r^{-sp}(t-t_0) > \tfrac{\lambda}{2}\}\right| \le \left(1-\frac{\lambda}{2A_2}\right) r^{n+sp}.$$

If $\frac{\lambda}{2A_2} < 1$, then

$$B \le \left(1-\frac{\lambda}{2A_2}\right) r^{n+sp} < r^{n+sp} < \left(\frac{2A_2}{\lambda}\right)^{p-1} r^{n+sp}.$$

If $\frac{\lambda}{2A_2} \ge 1$, then $B=0$. Hence in either case we have

$$B \le \frac{Cr^{n+sp}}{\lambda^{p-1}}.$$

Inserting the above estimates of A and B into (5.26), we obtain

$$\left|\{(x,t)\in B_r\times(t_0,t_0+r^{sp}) : \log u(x,t) < -\lambda - b\}\right| \le \frac{Cr^{n+sp}}{\lambda^{p-1}},$$

for some positive constant $C = C(n,p,s,\Lambda)$, which proves (5.16). The proof of (5.17) is analogous. □

Appendix

In this section, we prove Lemma 2.9. To this end, we establish the following auxiliary lemmas. Throughout this section, we assume $p > 1$.

Lemma 6.1 *Let $f, g \in C^1([a,b])$. Then*

$$\frac{f(b)-f(a)}{b-a} + \left|\frac{g(b)-g(a)}{b-a}\right|^p \le \max_{t\in[a,b]} \left[f'(t) + |g'(t)|^p\right].$$

Proof Suppose the result does not hold, then by contradiction, we get

$$\frac{f(b)-f(a)}{b-a} + \left|\frac{g(b)-g(a)}{b-a}\right|^p > f'(t) + |g'(t)|^p,$$

for all $t \in [a,b]$. Integrating over a to b, we obtain

$$\left|\frac{g(b)-g(a)}{b-a}\right|^p > \frac{1}{b-a}\int_a^b |g'(t)|^p\,dt,$$

which contradicts Jensen's inequality. □

Lemma 6.2 *Let* $a, b > 0$, $0 < \epsilon < p - 1$. *Then we have*

$$|b-a|^{p-2}(b-a)(a^{-\epsilon} - b^{-\epsilon}) \geq \zeta(\epsilon)\left|b^{\frac{p-\epsilon-1}{p}} - a^{\frac{p-\epsilon-1}{p}}\right|^p,$$

where $\zeta(\epsilon) = \frac{p^p\epsilon}{(p-\epsilon-1)^p}$. *Moreover, if* $0 < p - \epsilon - 1 < 1$, *then we may choose* $\zeta(\epsilon) = \frac{p^p\epsilon}{p-\epsilon-1}$.

Proof Let $0 < \epsilon < p - 1$ and $\zeta(\epsilon) = \frac{p^p\epsilon}{(p-\epsilon-1)^p}$. Let $f(t) = \frac{t^{-\epsilon}}{\zeta(\epsilon)}$ and $g(t) = t^{\frac{p-\epsilon-1}{p}}$. By Lemma 6.1, we have

$$\frac{1}{\zeta(\epsilon)}\frac{b^{-\epsilon} - a^{-\epsilon}}{b-a} + \left|\frac{b^{\frac{p-\epsilon-1}{p}} - a^{\frac{p-\epsilon-1}{p}}}{b-a}\right|^p \leq 0.$$

If $b \geq a$, multiplying by $(b-a)^p$, we obtain

$$(b-a)^{p-1}(a^{-\epsilon} - b^{-\epsilon}) \geq \zeta(\epsilon)\left|b^{\frac{p-\epsilon-1}{p}} - a^{\frac{p-\epsilon-1}{p}}\right|^p. \tag{6.1}$$

If $b < a$, interchanging a and b, the Lemma follows. If $0 < p - \epsilon - 1 < 1$, then we have $0 < (p-\epsilon-1)^p < p - \epsilon - 1$, therefore $\zeta(\epsilon) \geq \frac{p^p\epsilon}{p-\epsilon-1}$ and (6.1) implies

$$(b-a)^{p-1}(a^{-\epsilon} - b^{-\epsilon}) \geq \frac{p^p\epsilon}{p-\epsilon-1}\left|b^{\frac{p-\epsilon-1}{p}} - a^{\frac{p-\epsilon-1}{p}}\right|^p.$$

Hence the claim follows with $\zeta(\epsilon) = \frac{p^p\epsilon}{p-\epsilon-1}$ when $0 < p - \epsilon - 1 < 1$. □

Proof of Lemma 2.9

We denote the left-hand and right-hand sides of (2.3) by L.H.S and R.H.S, respectively. Let $\zeta_1(\epsilon) = \frac{\zeta(\epsilon)}{c(p)}$ and $\zeta_2(\epsilon) = \zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}}$. Then $\zeta_1(\epsilon) - \zeta_2(\epsilon) < -1$ since $C(p) > 1$ (to be finally chosen appropriately).

Case 1. If $\tau_1 = \tau_2 = 0$, then (2.3) holds trivially.

Case 2. If $\tau_1 > 0$ and $\tau_2 = 0$. In this case, we note that if $b > a$, then

$$\text{L.H.S} = |b-a|^{p-2}(b-a)(\tau_1^p a^{-\epsilon} - \tau_2^p b^{-\epsilon}) = (b-a)^{p-1}\tau_1^p a^{-\epsilon}$$

and

$$\begin{aligned}\text{R.H.S} &= \zeta_1(\epsilon)\tau_1^p a^{p-\epsilon-1} - \zeta_2(\epsilon)\tau_1^p(b^{p-\epsilon-1} + a^{p-\epsilon-1}) \\ &= (\zeta_1(\epsilon) - \zeta_2(\epsilon))\tau_1^p a^{p-\epsilon-1} - \zeta_2(\epsilon)\tau_1^p b^{p-\epsilon-1}.\end{aligned}$$

Now L.H.S is positive and since $\zeta_1(\epsilon) - \zeta_2(\epsilon) < 0$ and $\zeta_2(\epsilon) > 0$, the R.H.S is negative. Therefore, we have L.H.S $\geq$ R.H.S. On the other hand if $b \leq a$, then

$$\text{L.H.S} = -(a-b)^{p-1}\tau_1^p a^{-\epsilon} \geq -\tau_1^p a^{p-\epsilon-1},$$

and since $\zeta_1(\epsilon) - \zeta_2(\epsilon) < -1$ and $\zeta_2(\epsilon) > 0$, we have

$$\text{R.H.S} = (\zeta_1(\epsilon) - \zeta_2(\epsilon))\tau_1^p a^{p-\epsilon-1} - \zeta_2(\epsilon)\tau_1^p b^{p-\epsilon-1} < -\tau_1^p a^{p-\epsilon-1} \leq \text{L.H.S.}$$

Case 3. If $\tau_1 = 0$ and $\tau_2 > 0$. Then we have

$$\text{L.H.S} = -|b-a|^{p-2}(b-a)\tau_2^p b^{-\epsilon},$$

and

$$\text{R.H.S} = (\zeta_1(\epsilon) - \zeta_2(\epsilon))\tau_2^p b^{p-\epsilon-1} - \zeta_2(\epsilon)\tau_2^p a^{p-\epsilon-1}.$$

If $b > a$, then

$$\text{L.H.S} = -(b-a)^{p-1}\tau_2^p b^{-\epsilon} \geq -\tau_2^p b^{p-\epsilon-1},$$

and since $\zeta_1(\epsilon) - \zeta_2(\epsilon) < -1$ and $\zeta_2(\epsilon) > 0$, we have

$$\begin{aligned}\text{R.H.S} &= (\zeta_1(\epsilon) - \zeta_2(\epsilon))\tau_2^p b^{p-\epsilon-1} - \zeta_2(\epsilon)\tau_2^p a^{p-\epsilon-1}\\ &< -\tau_2^p b^{p-\epsilon-1} \leq \text{L.H.S.}\end{aligned}$$

If $b \leq a$, then the L.H.S is nonnegative and the R.H.S is negative. Therefore, we have L.H.S $\geq$ R.H.S.

Case 4. Let both $\tau_1, \tau_2 > 0$. By symmetry, we may assume that $b \geq a$. Let $t = \frac{b}{a} \geq 1$, $s = \frac{\tau_2}{\tau_1} > 0$ and $\lambda = s^p t^{-\epsilon}$. It can be easily seen that the inequality (2.3) is equivalent to the following inequality

$$\zeta_1(\epsilon)|st^{\frac{p-\epsilon-1}{p}} - 1|^p \leq (t-1)^{p-1}(1-\lambda) + \zeta_2(\epsilon)|s-1|^p(t^{p-\epsilon-1}+1). \tag{6.2}$$

We first estimate the following term.

$$\begin{aligned}|st^{\frac{p-\epsilon-1}{p}} - 1|^p &= |st^{\frac{p-\epsilon-1}{p}} - t^{\frac{p-\epsilon-1}{p}} + t^{\frac{p-\epsilon-1}{p}} - 1|^p\\ &= |(s-1)t^{\frac{p-\epsilon-1}{p}} + (t^{\frac{p-\epsilon-1}{p}} - 1)|^p\\ &\leq 2^{p-1}|s-1|^p t^{p-\epsilon-1} + 2^{p-1}|t^{\frac{p-\epsilon-1}{p}} - 1|^p\\ &= A + B,\end{aligned}$$

where

$$A = 2^{p-1}|s-1|^p t^{p-\epsilon-1} \text{ and } B = 2^{p-1}|t^{\frac{p-\epsilon-1}{p}} - 1|^p.$$

By Lemma 6.2, we have

$$B \leq \frac{2^{p-1}(t-1)^{p-1}(1-t^{-\epsilon})}{\zeta(\epsilon)}.$$

As a consequence, we obtain

$$|st^{\frac{p-\epsilon-1}{p}} - 1|^p \leq 2^{p-1}|s-1|^p t^{p-\epsilon-1} + \frac{2^{p-1}(t-1)^{p-1}(1-t^{-\epsilon})}{\zeta(\epsilon)}.$$

We observe that

$$
\begin{aligned}
1 - t^{-\epsilon} &= 1 - \lambda + \lambda - t^{-\epsilon} = 1 - \lambda + (s^p - 1)t^{-\epsilon} \\
&= 1 - \lambda + |s - 1|^p t^{-\epsilon} + (s^p - 1 - |s - 1|^p)t^{-\epsilon}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
|st^{\frac{p-\epsilon-1}{p}} - 1|^p &\le 2^{p-1}\big(1 + \frac{1}{\zeta(\epsilon)}\big)|s - 1|^p t^{p-\epsilon-1} \\
&+ \frac{2^{p-1}}{\zeta(\epsilon)}(t - 1)^{p-1}(1 - \lambda) + \frac{2^{p-1}}{\zeta(\epsilon)}(t - 1)^{p-1}(s^p - 1 - |s - 1|^p)t^{-\epsilon}.
\end{aligned}
\tag{6.3}
$$

Next we estimate the term $T = \frac{2^{p-1}}{\zeta(\epsilon)}(t - 1)^{p-1}(s^p - 1 - |s - 1|^p)t^{-\epsilon}$ for different values of t and s.

Case (a). If $t > 1$ and $s \ge 2$. Then using the fact that $s \ge 2$, it can be easily seen that there exists constant $C(p)$ large enough such that $s^p - 1 - (s - 1)^p \le C(p)(s - 1)^p$. Therefore, we get

$$
T \le \frac{C(p)}{\zeta(\epsilon)}|s - 1|^p t^{p-\epsilon-1}. \tag{6.4}
$$

By inserting (6.4) into (6.3), we get

$$
|st^{\frac{p-\epsilon-1}{p}} - 1|^p \le C(p)\big(1 + \frac{1}{\zeta(\epsilon)}\big)|s - 1|^p t^{p-\epsilon-1} + \frac{C(p)}{\zeta(\epsilon)}(t - 1)^{p-1}(1 - \lambda). \tag{6.5}
$$

Case (b). If $t = 1$ or $0 < s \le 1$. Then $T \le 0$. Hence, we get the estimate in (6.5).

Case (c). If $t > 1$, $s \in (1, 2)$. Let $r \ge p$ be the nearest integer to p. Again it follows that there exists a positive constant $C(p)$ large enough such that $s^p - 1 - |s - 1|^p \le C(p)|s - 1|$. We have further subcases.

Case (i). If

$$
t - 1 < \frac{r2^{r-1}}{\epsilon} t(s - 1).
$$

Note that we can choose $C(p)$ large enough such that $r2^{r-1} \le C(p)$. Hence, we have

$$
T \le \frac{C(p)}{\epsilon^{p-1}\zeta(\epsilon)} t^{p-\epsilon-1}|s - 1|^p. \tag{6.6}
$$

By inserting (6.6) into (6.3), we get

$$
|st^{\frac{p-\epsilon-1}{p}} - 1|^p \le C(p)\Big(1 + \frac{1}{\zeta(\epsilon)}\Big(1 + \frac{1}{\epsilon^{p-1}}\Big)\Big)|s - 1|^p t^{p-\epsilon-1} + \frac{C(p)}{\zeta(\epsilon)}(t - 1)^{p-1}(1 - \lambda). \tag{6.7}
$$

Case (ii). If

$$
t - 1 \ge \frac{r2^{r-1}}{\epsilon} t(s - 1).
$$

Since r is an integer, we observe that

$$
s^r + s - 2 = (s - 1)(s^{r-1} + s^{r-2} + \cdots + s + 2).
$$

By the mean value theorem there exists $\eta \in (1, t)$ such that $t^\epsilon - 1 = \epsilon \eta^{\epsilon-1}(t-1)$ and so $\epsilon = \frac{t^\epsilon - 1}{\eta^{\epsilon-1}(t-1)}$. Now, we have

$$\begin{aligned}\frac{s^r + s - 2}{t-1} &= \frac{s-1}{t-1}(s^{r-1} + s^{r-2} \cdots + s + 2)\\ &\leq \frac{\epsilon}{r2^{r-1}t}(s^{r-1} + s^{r-2} + \cdots + s + 2)\\ &\leq \frac{\epsilon}{t} \leq \frac{t^\epsilon - 1}{t\eta^{\epsilon-1}(t-1)},\end{aligned}$$

which gives $t\eta^{\epsilon-1}(s^r + s - 2) \leq t^\epsilon - 1$.

Now, the fact $\epsilon > 0$ and $1 < \eta < t$ gives $t\eta^{\epsilon-1} > \eta^\epsilon > 1$. Therefore since $r \geq p$ and $s > 1$, we get $s^p + s - 2 < s^r + s - 2 < t\eta^{\epsilon-1}(s^r + s - 2) \leq t^\epsilon - 1$. Hence, we have $s - 1 \leq t^\epsilon - s^p = t^\epsilon(1-\lambda)$. Thus

$$T \leq \frac{C(p)}{\zeta(\epsilon)}(t-1)^{p-1}(1-\lambda). \tag{6.8}$$

Using (6.8) into (6.3) we get

$$|st^{\frac{p-\epsilon-1}{p}} - 1|^p \leq C(p)\big(1 + \frac{1}{\zeta(\epsilon)}\big)|s-1|^p t^{p-\epsilon-1} + \frac{C(p)}{\zeta(\epsilon)}(t-1)^{p-1}(1-\lambda). \tag{6.9}$$

Finally from the estimates (6.5), (6.7) and (6.9), we obtain

$$\begin{aligned}|st^{\frac{p-\epsilon-1}{p}} - 1|^p \leq C(p)\Big(1 + \frac{1}{\zeta(\epsilon)}(1 + \frac{1}{\epsilon^{p-1}})\Big)|s-1|^p(t^{p-\epsilon-1} + 1)\\ + \frac{C(p)}{\zeta(\epsilon)}(t-1)^{p-1}(1-\lambda).\end{aligned} \tag{6.10}$$

Multiplying $\frac{\zeta(\epsilon)}{C(p)}$ on both sides of (6.10), we obtain

$$\frac{\zeta(\epsilon)}{C(p)}|st^{\frac{p-\epsilon-1}{p}} - 1|^p \leq \Big(\zeta(\epsilon) + 1 + \frac{1}{\epsilon^{p-1}}\Big)|s-1|^p(t^{p-\epsilon-1} + 1) + (t-1)^{p-1}(1-\lambda),$$

which corresponds to the inequality (6.2). The lemma thus follows.

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