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Positive solutions for semilinear elliptic systems with boundary measure data

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Abstract
In this paper, we study the Dirichlet problem of elliptic systems

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{g}(\mathbf{u}) & \text{in } \Omega, \\ \mathbf{u} = \varrho\boldsymbol{\mu} & \text{on } \partial\Omega, \end{cases}$$

where $\varrho \geq 0$, Ω is an open bounded C^2 domain in $\mathbb{R}^N$ with $N \geq 2$, and $\mathbf{u}$, $\mathbf{g}(\mathbf{u})$, $\boldsymbol{\mu}$ are non-negative vector-valued functions. We obtain the existence of weak positive solutions for the systems. In the special case $\mathbf{g}(\mathbf{u}) = |\mathbf{u}|^{p-1}\mathbf{u}$ with $p > 1$, we shall give a better description about the positive solutions including the priori estimate, regularity, existence and nonexistence.

Keywords Semilinear elliptic systems · Boundary measure data · Existence · Priori estimate

Mathematics Subject Classification Primary 35J57 · Secondary 35A01 · 35B09

1 Introduction

Let Ω be an open bounded C^2 domain in $\mathbb{R}^N$, $N \geq 2$. The Dirichlet problem with measure data

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$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = \varrho\mu & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

had been studied extensively in many literatures, where $p > 1$, $\varrho \geq 0$, $\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and u is a nonnegative function defined on Ω and μ is a Radon measure defined on $\partial\Omega$. The existence of the weak positive solution of problem (1.1) had been investigated by a lot of people. One of the first attempt in this direction was obtained by Bidaut-Véron-Vivier [7], showing the existence of a critical exponent $p^* := \frac{N+1}{N-1}$ for the solvability of this equation. More precisely, in the subcritical case $1 < p < p^*$, there exists a threshold value ϱ^* such that the problem admits a solution if and only if $\varrho \in [0, \varrho^*]$ (see [7, Theorem 1.3]). They also proved that in the case $p \geq p^*$, the problem (1.1) does not admit any solution with $\varrho > 0$, $\mu = \delta_z$ being a Dirac measure at z, $z \in \partial\Omega$. Then Bidaut-Véron-Yarur [8] considered again the problem with both interior and boundary measure data

$$\begin{cases} -\Delta u = u^p + \nu & \text{in } \Omega, \\ u = \varrho\mu & \text{on } \partial\Omega, \end{cases}$$

where ν is a nonegative Radon measure in Ω. They gave a complete description of the solutions in the subcritical case, and sufficient conditions for the existence in the supercritical case by establishing sharp estimates of Green kernel and Poisson kernel. For a more general case,

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = \varrho\mu & \text{on } \partial\Omega, \end{cases}$$

if g satisfies the so-called subcriticality condition, an existence result was recently obtained by Chen-Felmer-Véron [12] using the Schauder fixed point theorem, essentially based on estimates related to weighted Marcinkiewicz spaces. For the solvability of this problem, Bidaut-Véron-Hoang-Nguyen-Véron [4] provided new criteria, expressed in terms of appropriate capacities, where Ω is a bounded smooth domain or a half space. On the other hand, the positive solutions for the corresponding boundary value problem with the other sign also have been studied by Marcus-Véron in [25, 26].

Another topic about the priori estimate near the boundary for the solution of problem (1.1) also had been established. Bidaut-Véron-Vivier proved in [7] that if $p > 1$, μ is nonnegative, and u is a nonnegative solution of (1.1), then

$$\|u\|_{L^1(\Omega)} + \|u^p\|_{L^1(\Omega, \delta dx)} \leq C(N, p, \Omega)[1 + \varrho\mu(\partial\Omega)],$$

where, for any $x \in \Omega$,

$$\delta(x) := \mathrm{dist}(x, \partial\Omega).$$

Moreover, for $1 < p < \frac{N+1}{N-1}$,

$$\varrho\mathbb{P}[\mu] \leq u \leq C(N, p, \Omega, \mu(\partial\Omega))(\varrho\mathbb{P}[\mu] + \delta) \quad \text{in} \quad \Omega,$$

where $\mathbb{P}$ is the Poisson operator (see Sect. 2 for its definition). Furthermore, suppose that $0 \in \partial\Omega$, Bidaut-Véron-Ponce-Véron [5] proved that, for $1 < p < \frac{N+2}{N-2}$ and $\mu = \delta_0$, there exists a positive constant C, independent of u, such that

$$u(x) \leq C|x|^{-\frac{2}{p-1}} \quad \text{near } x = 0.$$

The theory of the singularity analysis also developed well. The asymptotic behavior of the positive singular solutions in $C^2(B_1^+) \cap C(\overline{B_1^+}\setminus\{0\})$ for

$$\begin{cases} -\Delta u = u^p & \text{in } B_1^+, \\ u = 0 & \text{on } \partial' B_1^+\setminus\{0\} \end{cases}$$

has been established by many works, where $B_1^+ := B_1 \cap \mathbb{R}^N_+$ and $\partial' B_1^+ := \overline{B_1^+} \cap \partial\mathbb{R}^N_+$. See Bidaut-Véron-Vivier [7] for $1 < p < \frac{N+1}{N-1}$, Bidaut-Véron-Ponce-Véron [5, 6] for $\frac{N+1}{N-1} \le p < \frac{N+2}{N-2}$, and Xiong [28] for $p = \frac{N+2}{N-2}$. On the other hand, a series of paper considered the internal isolated singularity; see [2, 10, 20, 22, 24] for the details.

Recently, the systems

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u}$$

also have been attracted a lot of attention, where $\mathbf{u}(x) = (u_1(x), \dots, u_m(x))$, $m \ge 1$, $\mathbf{u} : \Omega \to \mathbb{R}^m_{++}$, $\mathbb{R}^m_{++} := \{\mathbf{y} = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m : y_i \ge 0, i = 1, 2, \dots, m\}$. They naturally appear in the Hartree-Fock theory for Bose-Einstein double condensates, the fiber-optic theory, the langmuir waves theory for plasma physics, and in studying the behavior of deep water waves and freak waves in the ocean. A general reference in book focus on such systems and their role in physics is by Ablowitz-Prinari-Trubatch [1]. For the critical exponent $p = \frac{N+2}{N-2}$, Druet-Hebey-Vetóis [17] proved the Liouville theorem in $\mathbb{R}^N$ and Caju-do Ó-Silva Santos [11] obtained the qualitative properties of positive singular solutions for the nonlinear elliptic systems in $\mathbb{R}^N\setminus\{0\}$. For the subcritical exponent $1 < p < \frac{N+2}{N-2}$, Ghergu-Kim-Shahgholian [19] recently established that $\mathbf{u} = 0$ is the only nonnegative C^2 solution in $\mathbb{R}^N$. Furthermore, they also classified the solutions in the punctured space $\mathbb{R}^N\setminus\{0\}$, and derived the priori estimate and the asymptotic radial symmetry around the singularity. The behavior of the singular positive solutions for the semilinear elliptic systems with Dirichlet boundary value condition was analyzed in [23].

The aforementioned results are motivations of this paper, the goal of which is two-fold: (i) to study the existence of positive weak solutions for

$$\begin{cases} -\Delta \mathbf{u} = \mathbf{g}(\mathbf{u}) & \text{in } \Omega, \\ \mathbf{u} = \varrho\boldsymbol{\mu} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with boundary measures, where $\varrho \ge 0$, Ω is an open bounded C^2 domain in $\mathbb{R}^N$ with $N \ge 2$, $\mathbf{u} := (u_1, u_2, \dots, u_m)$, $\mathbf{g}(\mathbf{u}) = (g_1(\mathbf{u}), g_2(\mathbf{u}), \dots, g_m(\mathbf{u}))$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_m)$, $m \ge 1$, and $u_i : \Omega \to \mathbb{R}_+$, $g_i : \mathbb{R}^m_{++} \to \mathbb{R}_+$, μ_i is a positive Radon measure on $\partial\Omega$ with $\mu_i(\partial\Omega) < \infty$ for $i = 1, 2, \dots, m$; (ii) to study the systems

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u},$$

including the interior regularity of positive solutions, and then establish the priori estimate, the boundary regularity of positive solutions as well as the existence of the minimal weak positive solutions for the systems with boundary measure data.

Notice that all the solutions are understood in the usual weak sense, and we also include a definition here for completeness. Suppose $\mathbf{a} := (a_1, a_2, \dots, a_m) \in \mathbb{R}^m$ is a vector, then we say $\mathbf{a}$ has some properties means that every component of $\mathbf{a}$ has the same properties. For example, $\mathbf{a} \ge 0$ means that $a_i \ge 0$, $i = 1, 2, \dots, m$ and $\mathbf{u} \in (L^1(\Omega))^m$ means that every component of $\mathbf{u}$ belongs to $L^1(\Omega)$ in this article.

Definition 1.1 $\mathbf{u}$ is a weak solution of (1.2) in the sense that $\mathbf{u} \in (L^1(\Omega))^m$, $\mathbf{g}(\mathbf{u}) \in (L^1(\Omega, \delta dx))^m$, and

$$\int_\Omega \mathbf{u}(-\Delta \boldsymbol{\xi})dx = \int_\Omega \mathbf{g}(\mathbf{u})\xi dx - \boldsymbol{\varrho} \int_{\partial\Omega} \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{n}} d\boldsymbol{\mu}(x), \quad \forall \xi \in X_0^{1.1}(\overline{\Omega}),$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$ at point x, and $X_0^{1.1}(\overline{\Omega})$ is the space of function C^1 functions vanishing on $\partial\Omega$ with Lipschitz continuous gradient.

Our first main result is devoted to studying the existence of weak solution for problem (1.2).

Theorem 1.2 *Suppose that*, $1 < p < p^* := \frac{N+1}{N-1}$ *and* $\mathbf{g}$ *is nonnegative, continuous and*

$$|\mathbf{g}(\mathbf{s})| \le a|\mathbf{s}|^p + \epsilon, \qquad \forall \mathbf{s} \in \mathbb{R}^m_{++}, \tag{1.3}$$

for some $a, \epsilon > 0$, *then there exist positive constants* ϱ_0 *and* ϵ_0, *depending on a and* p, *such that, for* $\varrho \in (0, \varrho_0)$ *and* $\epsilon \in (0, \epsilon_0)$, *problem* (1.2) *admits a positive weak solution* $\mathbf{u}$ *which satisfies*

$$\mathbf{u} \ge \varrho \mathbb{P}[\boldsymbol{\mu}] \quad \text{in} \quad \Omega.$$

To prove Theorem 1.2, we translate (1.2) to an equivalent problem with zero boundary condition satisfied by $\mathbf{v} = \mathbf{u} - \varrho\mathbb{P}[\boldsymbol{\mu}]$. By the spirit of [12–15], owing to the estimates of Green kernel and Possion kernel, together with the Schauder fixed point theorem, we can construct a sequence of approximating solutions $\{\mathbf{v_n}\}$ for the new problem provided that $\varrho\mathbb{P}[\boldsymbol{\mu}]$ is small (see Lemma 2.4). Putting $\mathbf{u_n} := \mathbf{v_n} + \varrho\mathbb{P}[\boldsymbol{\mu}]$ and using the Vitali convergence theorem, we can finally show that the sequence $\{\mathbf{u_n}\}$ converges to a weak solution of (1.2). This result is consistent with the scalar case [12].

Next, we consider the systems

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} \quad \text{in} \quad \Omega. \tag{1.4}$$

We first give the following interior regularity result for the above problem.

Theorem 1.3 *Let* $1 < p < \frac{N}{N-1}$. *If* $\mathbf{u}$ *is a positive weak solution of* (1.4), *then* $\mathbf{u} \in (C^\infty(\Omega))^m$.

Notice that the precise estimates of the Green operator $\mathbb{G}[\cdot]$ in weighted Marcinkiewicz spaces will be used in our proof. Then the following we will learn this systems

$$\begin{cases} -\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} & \text{in} \ \ \Omega, \\ \mathbf{u} = \varrho\boldsymbol{\mu} & \text{on} \ \ \partial\Omega \end{cases} \tag{1.5}$$

with boundary measure data further. In fact, suppose that $0 \in \partial\Omega$, it is worth noting that in [23], the authors have provided one of the priori estimates about the solution of (1.5) for $1 < p < \frac{N+2}{N-2}$, $\mu = (\delta_0, \delta_0, \dots, \delta_0) \in \mathbb{R}^m$. Precisely speaking, there exists a positive constant C, independent of the solution $\mathbf{u}$, such that

$$|\mathbf{u}(x)| \le C|x|^{-\frac{2}{p-1}} \qquad \text{near } x = 0.$$

Then we shall study the priori estimates and boundary regularity for the solution of (1.5).

Theorem 1.4 *For* $p > 1$ *and any positive Radon measure* $\boldsymbol{\mu}$ *with* $|\boldsymbol{\mu}|(\partial\Omega) < \infty$, *if* $\mathbf{u}$ *is a positive weak solution of systems* (1.5), *then there exists a positive constant* C, *depending on* N, p, Ω, m *and* ϱ, *such that*

$$\|\mathbf{u}\|_{(L^1(\Omega))^m} + \||\mathbf{u}|^p\|_{L^1(\Omega,\delta dx)} \leq C[1 + |\boldsymbol{\mu}|(\partial\Omega)]. \tag{1.6}$$

In particular, for $1 < p < p^*$, *there exists a positive constant* $\widetilde{C}$ *depending on,* N, p, Ω, $\|\boldsymbol{\mu}\|_{\mathfrak{M}_+(\partial\Omega)}$, ϱ *and* m, *such that*

$$\varrho|\mathbb{P}[\boldsymbol{\mu}]| \leq |\mathbf{u}| \leq \widetilde{C}(\mathbb{P}[|\boldsymbol{\mu}|] + \delta) \quad \text{in} \quad \Omega. \tag{1.7}$$

Moreover, if $\boldsymbol{\mu} = 0$, *then* $\mathbf{u} \in (C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega}))^m$ *for any* $\alpha \in (0, 1)$.

The above results can be seen as a generalization of [7]. In the proof of this theorem, the first eigenvalue and corresponding positive eigenfunction (λ_1, ϕ_1) of $-\Delta$ with Dirichlet conditions on $\partial\Omega$ play an important role, and the fact that there exists a positive constant C such that $C^{-1}\delta(x) \leq \phi_1(x) \leq C\delta(x)$ for all $x \in \Omega$ (see [16]) will be used. As a consequence of Theorem 1.4, we have

Corollary 1.5 *Suppose that* $0 \in \partial\Omega$ *and* $1 < p < p^*$. *If* $\mathbf{u}$ *is a positive weak solution of*

$$\begin{cases} -\Delta\mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega\backslash\{0\}, \end{cases}$$

then either $\mathbf{u}$ *can be continuously extended at* 0, *or there exists a positive constant* C, *depending on* $\mathbf{u}$, *such that*

$$|\mathbf{u}(x)| = C|x|^{-N}\delta(x)[1 + o(1)] \quad \text{near} \quad x = 0.$$

The last part of this paper is about the existence of the minimal weak positive solutions for (1.5). We say $\underline{\mathbf{u}}$ is the minimal weak positive solution of (1.5) in the sense that if $\mathbf{u}$ is a weak positive solution of (1.5), then $\underline{\mathbf{u}} \leq \mathbf{u}$ in Ω.

Theorem 1.6 *Assume* $|\boldsymbol{\mu}|(\partial\Omega) = 1$.

Case 1) $1 < p < p^*$. *There exists a threshold value* $\varrho^* > 0$ *such that the problem* (1.5) *admits a minimal positive weak solution* $\underline{\mathbf{u}}_\varrho$ *for* $\varrho \in (0, \varrho^*]$. *Moreover,* $\{\underline{\mathbf{u}}_\varrho\}$ *is an increasing sequence which converges to the minimal solution* $\underline{\mathbf{u}}_{\varrho^*}$ *in* $(L^1(\Omega))^m$ *and in* $(L^p(\Omega, \delta dx))^m$ *as* $\varrho \to \varrho^*$. *If* $\varrho > \varrho^*$, *then the problem does not admit any positive weak solution.*

Case 2) $p \geq p^*$. *For any* $\varrho > 0$, *if there exists* $i \in \{1, 2, \dots, m\}$ *such that* $\mu_i = \delta_{z_i}$, *where* δ_{z_i} *is the Dirac measure concentrated at* z_i, $z_i \in \partial\Omega$, *then the problem* (1.5) *admits no positive weak solution.*

It is worth noting that any solution of (1.5) is naturally bounded from below by the Poisson operator $\varrho\mathbb{P}[\boldsymbol{\mu}]$ which also is a subsolution. Hence, in the proof of the existence of the minimal weak solutions, we only need to construct a supersolution. The result is an extension of Bidaut-Véron-Vivier [7].

The rest of the paper is organized as follows. Using the estimates about the Green and Poisson operators, we shall prove Theorem 1.2 in Sect. 2. In Sect. 3, we discuss the priori estimates, as well as the regularity properties of the weak solutions for a typical case. Section 4 deals with the existence of the minimal positive solution.

2 Proof of Theorem 1.2

In this section, we establish the existence of the positive weak solution for systems (1.2), that is, we prove Theorem 1.2. We begin with few notions and properties. For $\beta \in [0,1]$, $\mathfrak{M}(\Omega, \delta^\beta dx)$ denotes the space of Radon measures μ in Ω satisfying

$$\|\mu\|_{\mathfrak{M}(\Omega,\delta^\beta dx)} := \int_\Omega \delta^\beta d|\mu| < \infty,$$

where, for any $x \in \Omega$, $\delta(x) := \text{dist}(x, \partial\Omega)$. For simplicity, we denote $\mathfrak{M}(\Omega) := \mathfrak{M}(\Omega, \delta^0 dx)$. The associate positive cones are denoted by $\mathfrak{M}_+(\Omega, \delta^\beta dx)$ and $\mathfrak{M}_+(\Omega)$, respectively. Recall that, for almost every $y \in \Omega$ and $z \in \partial\Omega$, the Green kernel $G(\cdot, y)$ and Poisson kernel $P(\cdot, z)$ are the integral solutions of

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta P(\cdot, z) = 0 & \text{in } \Omega, \\ P(\cdot, z) = \delta_z & \text{on } \partial\Omega, \end{cases}$$

where δ_y and δ_z are Dirac measures at points $y \in \Omega$ and $z \in \partial\Omega$, respectively. The *Green operator* $\mathbb{G}[\cdot]$ and the *Poisson operator* $\mathbb{P}[\cdot]$ are, respectively, defined by setting

$$\mathbb{G}[\nu](x) := \int_\Omega G(x,y)d\nu(y), \quad \nu \in \mathfrak{M}(\Omega, \delta dx), \quad \text{and} \quad \mathbb{P}[\mu](x) := \int_{\partial\Omega} P(x,z)d\mu(z), \quad \mu \in \mathfrak{M}(\partial\Omega).$$

Then $\Phi = \mathbb{G}[\nu]$ and $\Psi = \mathbb{P}[\mu]$ are, respectively, the solution of problems

$$\begin{cases} -\Delta\Phi = \nu & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta\Psi = 0 & \text{in } \Omega, \\ \Psi = \mu & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Moreover, for any $\nu_1, \nu_2 \in \mathfrak{M}(\Omega, \delta dx)$ with $\nu_1 \le \nu_2$, by the weak maximum principle (see, for instance, [18, p. 344]), we have

$$\mathbb{G}[\nu_1] \le \mathbb{G}[\nu_2]; \tag{2.2}$$

see [7] for the details.

We recall the following property on Green operator, which can be found in [7, Remark 2.5].

Lemma 2.1 *Assume $N \ge 2$. The Green operator $\mathbb{G}$ is compact from $L^1(\Omega, \delta^\alpha dx)$ to $L^q(\Omega, \delta^\beta dx)$ for any $\alpha \in (0,1]$, $\beta \in (-N/(N+\alpha-1), N\alpha/(N-2))$ or $\alpha = \beta = 0$, and $q \in [1, (N+\beta)/(N+\alpha-2))$ and $q > -\beta$.*

Recall that, for any $k > 0$ and positive Borel measure ν on Ω, the *Marcinkiewicz space of exponent k* or *weak L^k space* $M^k(\Omega, d\nu)$ is defined to be the set of all $u \in L^1_{\text{loc}}(\Omega, d\nu)$ satisfying

$$\|u\|_{M^k(\Omega,d\nu)} := \inf\left\{ c\in[0,\infty] \,:\, \int_E |u|d\nu \le c[\nu(E)]^{\frac{k-1}{k}},\ \forall E\subset\Omega \text{ and } E \text{ is a Borel set} \right\} < \infty.$$

The following lemma shows the boundedness of the Poisson operator. For clarity, we provide the details.

Lemma 2.2 *Let* $N\ge 2$, $\gamma\ge -2$, *and* $\mu\in\mathfrak{M}_+(\partial\Omega)$. *There exists a positive constant* $C(N,\Omega,\gamma)$ *such that*

$$\|\mathbb{P}[\mu]\|_{M^{\frac{N+\gamma}{N-1}}(\Omega,\delta^\gamma dx)} \le C(N,\Omega,\gamma)\|\mu\|_{\mathfrak{M}(\partial\Omega)}. \tag{2.3}$$

Moreover, if $\gamma\ge 0$ *and* $1\le q<\frac{N+\gamma}{N-1}$, *then there exists a positive constant* $C(q,N,\Omega,\gamma)$ *such that*

$$\|\mathbb{P}[\mu]\|_{L^q(\Omega,\delta^\gamma dx)} \le C(q,N,\Omega,\gamma)\|\mu\|_{\mathfrak{M}(\partial\Omega)}.$$

Proof (2.3) was proved in [7, Theorem 2.5]. It is well known that, for any $1\le p<k<\infty$ and any $u\in M^k(\Omega,\delta^\gamma dx)$,

$$\left(\int_\Omega |u|^p\delta^\gamma dx\right)^{\frac{1}{p}} \le C(p,k)\|u\|_{M^k(\Omega,\delta^\gamma dx)}\left(\int_\Omega \delta^\gamma dx\right)^{\frac{1}{p}-\frac{1}{k}}; \tag{2.4}$$

see, for instance, [3, Lemma A.2]. Since Ω is bounded, it follows that $\int_\Omega \delta^\gamma dx<\infty$. A combination of this, (2.4), and (2.3) yields the assertion. □

Let $\mathbf{g} := (g_1, g_2, \dots, g_m)$ be a nonnegative and continuous vector-valued function satisfying (1.3). Consider a sequence $\{\mathbf{g_n}\}_{n\in\mathbb{N}}$ of C^1 nonnegative functions defined on $\mathbb{R}^m_{++}$ such that $\mathbf{g_n}(0)=\mathbf{g}(0)$ for every $n\in\mathbb{N}$ and, for every $i\in\{1,2,\dots,m\}$,

$$g_{n,i}\le g_{n+1,i}\le g_i,\quad \limsup_{s\in\mathbb{R}^m_{++}} g_{n,i}(s)=n,\quad \lim_{n\to\infty}\|g_{n,i}-g_i\|_{L^\infty_{\text{loc}}(\mathbb{R}^m_{++})}=0, \tag{2.5}$$

where $\mathbf{g_n} := (g_{n,1}, g_{n,2}, \dots, g_{n,m})$. Let $\varrho>0$ and $\boldsymbol{\mu}\in(\mathfrak{M}_+(\partial\Omega))^m$. Define the operators $\{\mathbf{T_n}\}_{n\in\mathbb{N}}$ by, for every $n\in\mathbb{N}$,

$$\mathbf{T_n}(\mathbf{v}) := \mathbb{G}\big[\mathbf{g_n}(\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}])\big],\qquad \mathbf{v}\in(L^1(\Omega))^m, \mathbf{v}\ge 0,$$

where $\mathbf{T_n}(\mathbf{v}) := \big(T_{n,1}(\mathbf{v}), T_{n,2}(\mathbf{v}),\dots,T_{n,m}(\mathbf{v})\big)$ and $T_{n,i}(\mathbf{v}) := \mathbb{G}\big[g_{n,i}(\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}])\big]$ for every i.

We first establish the following technical lemma.

Lemma 2.3 *Let* $\varrho>0$, $\boldsymbol{\mu}\in(\mathfrak{M}_+(\partial\Omega))^m$, $1<p<p^* := \frac{N+1}{N-1}$, *and* $q\in(p,p^*)$. *Assume that* $\mathbf{g}$ *is a nonnegative function satisfying* (1.3) *for some* $a,\epsilon>0$. *Let* $\{\mathbf{T_n}\}_{n\in\mathbb{N}}$ *be as above. There exists a positive constant* $\widehat{\Lambda}$ *such that, for every* $\mathbf{v}\in(L^q(\Omega,\delta dx))^m$,

$$\|\mathbf{v}\|_{(L^q(\Omega,\delta dx))^m}\le\widehat{\Lambda} \Rightarrow \big\|\mathbf{T_n}(\mathbf{v})\big\|_{(L^q(\Omega,\delta dx))^m}\le\widehat{\Lambda}.$$

Proof Observe that, for every $i\in\{1,2,\dots,m\}$,

$$\mathbb{G}\big[g_{n,i}(\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}])\big]\le\mathbb{G}\big[|\mathbf{g_n}(\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}])|\big].$$

It follows that

$$\left\|\mathbf{T_n}(\mathbf{v})\right\|_{(L^q(\Omega,\delta dx))^m} \le m\left\|\mathbb{G}\left[|\mathbf{g_n}(\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}])|\right]\right\|_{(L^q(\Omega,\delta dx))^m}.$$

Using this inequality and Lemma 2.1 with $\alpha = \beta = 1$, we have

$$\left\|\mathbf{T_n}(\mathbf{v})\right\|_{(L^q(\Omega,\delta dx))^m} \le C(q,N,\Omega)m\|\mathbf{g_n}(\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}])\|_{(L^1(\Omega,\delta dx))^m},$$

which, together with (2.5) and (1.3), further implies that

$$\begin{aligned}\left\|\mathbf{T_n}(\mathbf{v})\right\|_{(L^q(\Omega,\delta dx))^m} &\le C_1 m\|a|\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}]|^p+\epsilon\|_{(L^1(\Omega,\delta dx))^m}\\ &\le C_1 m\left(2^{p-1}a\int_\Omega |\mathbf{v}|^p\delta dx + 2^{p-1}a\varrho^p\int_\Omega |\mathbb{P}[\boldsymbol{\mu}]|^p\delta dx+\epsilon\int_\Omega \delta dx\right),\end{aligned} \tag{2.6}$$

where $C_1 = C_1(q,N,\Omega)$, $a > 0$, $\epsilon > 0$ and $\varrho > 0$. Since Ω is bounded, it follows that $\int_\Omega \delta dx < \infty$. For any $\Lambda \in (0,\infty)$, if $\|\mathbf{v}\|_{(L^q(\Omega,\delta dx))^m} \le \Lambda$, then by the Hölder inequality, we have

$$\begin{aligned}\int_\Omega |\mathbf{v}|^p\delta dx &\le \left(\int_\Omega |\mathbf{v}|^q\delta dx\right)^{\frac{p}{q}}\left(\int_\Omega \delta dx\right)^{\frac{q-p}{q}}\\ &\le C(q,p,\Omega)\left(\int_\Omega |\mathbf{v}|^q\delta dx\right)^{\frac{p}{q}}\\ &= C(q,p,\Omega)\Lambda^p.\end{aligned}$$

Hence, using Lemma 2.2 with $\gamma = 1$ and (2.6), we then obtain

$$\left\|\mathbf{T_n}(\mathbf{v})\right\|_{(L^q(\Omega,\delta dx))^m} \le C(q,p,N,m,\Omega)(a\Lambda^p + a\varrho^p+\epsilon).$$

Since $p > 1$, there exist positive constants ϱ_0 and ϵ_0 such that, for any $\varrho \in (0,\varrho_0)$ and $\epsilon \in (0,\epsilon_0)$, the algebraic equation

$$C(q,p,N,m,\Omega)(a\Lambda^p + a\varrho^p+\epsilon) = \Lambda$$

admits a positive root $\widehat{\Lambda}$. This finishes the proof of Lemma 2.3. □

We remark that the key point in the proof of Theorem 1.2 is to derive the uniform bound in $(L^p(\Omega,\delta dx))^m$ for the solutions of systems

$$\begin{cases} -\Delta\mathbf{v} = \mathbf{g_n}(\mathbf{v}+\varrho\mathbb{P}[\boldsymbol{\mu}]) & \text{in } \Omega,\\ \mathbf{v} = 0 & \text{on } \partial\Omega,\end{cases} \tag{2.7}$$

where $\{\mathbf{g_n}\}$ is defined by (2.5). By [7, Corollary 2.2], we know that (2.7) admits a positive weak solution $\mathbf{v_n}$ if and only if $\mathbf{v_n}$ can be written in the form

$$\mathbf{v_n} = \mathbb{G}[\mathbf{g_n}(\mathbf{v_n}+\varrho\mathbb{P}[\boldsymbol{\mu}])]. \tag{2.8}$$

Hence, we shall prove the following lemma first by the Schauder fixed point theorem.

Lemma 2.4 *Assume that* $\mathbf{g}$ *is a nonnegative function satisfying* (1.3) *for some* a, $\epsilon > 0$, *and* $p \in (1,p^*)$. *Let* $\{\mathbf{g_n}\}$ *be a sequence of* C^1 *nonnegative functions defined on* $\mathbb{R}^m_{++}$ *satisfying* (2.5). *Then there exist* $\varrho_0, \epsilon_0 > 0$ *such that, for any* $\varrho \in (0,\varrho_0)$ *and* $\epsilon \in (0,\epsilon_0)$, *problem* (2.7) *admits a positive weak solution* $\mathbf{v_n}$ *which satisfies*

$$\|\mathbf{v_n}\|_{(L^p(\Omega,\delta dx))^m} \leq \overline{\lambda}$$

for some positive constant $\overline{\lambda}$ independent of n.

Proof Let $\widehat{\Lambda}$ be the positive constant as in Lemma 2.3, $q \in (p, p^*)$, and let

$$\mathcal{O} := \left\{ \mathbf{v} = (v_1, v_2, \ldots, v_m) \in (L^1(\Omega))^m : \mathbf{v} \geq 0 \text{ and } \|\mathbf{v}\|_{(L^q(\Omega,\delta dx))^m} \leq \widehat{\Lambda} \right\}.$$

Let $\{\mathbf{T_n}\}_{n\in\mathbb{N}}$ be as in Lemma 2.3. Ensure to use the Schauder fixed point theorem, we shall first prove that $\mathcal{O}$ is a convex, closed subset of $(L^1(\Omega))^m$ and $\mathbf{T_n}$ is well defined in $\mathcal{O}$. After that we show that, for each n, $\mathbf{T_n}$ is a continuous and compact map.

Clearly, from the Minkowski inequality, it follows that $\mathcal{O}$ is a convex set. Next, we show that $\mathcal{O}$ is a closed subset of $(L^1(\Omega))^m$. Indeed, let $\{\mathbf{v_j}\} \subset \mathcal{O}$ be a sequence converging to $\mathbf{v}$ in $(L^1(\Omega))^m$. Obviously, $\mathbf{v} \geq 0$. We can extract a subsequence, still denotes by $\{\mathbf{v_j}\}$, such that $\mathbf{v_j} \to \mathbf{v}$ a.e in Ω. By the Fatou Lemma, we then obtain

$$\|\mathbf{v}\|_{(L^p(\Omega,\delta dx))^m} \leq \liminf_{j\to\infty} \left\|\mathbf{v_j}\right\|_{(L^p(\Omega,\delta dx))^m} \leq \widehat{\Lambda}.$$

Consequently, $\mathbf{v} \in \mathcal{O}$ and therefore $\mathcal{O}$ is a closed subset of $(L^1(\Omega))^m$. By (2.5), we have, for every $n \in \mathbb{N}$, $\mathbf{g_n}$ is bounded and hence $\mathbf{g_n}(\mathbf{v} + \varrho\mathbb{P}[\boldsymbol{\mu}]) \in (L^1(\Omega))^m$. Combining this and Lemma 2.1, we then deduce that $\mathbf{T_n}(\mathbf{v}) \in (L^1(\Omega))^m$. By this and Lemma 2.3, we conclude that $\mathbf{T_n}$ is well defined on $\mathcal{O}$ and $\mathbf{T_n}(\mathcal{O}) \subset \mathcal{O}$.

The following we shall prove that $\mathbf{T_n}$ is a continuous map on $\mathcal{O}$. Suppose that $\{\mathbf{v_j}\} \subset \mathcal{O}$ and

$$\mathbf{v_j} \to \mathbf{v} \quad \text{in} \quad (L^1(\Omega))^m \quad \text{as} \quad j \to \infty.$$

For each fixed n, we have $\mathbf{g_n}(\mathbf{v_j} + \varrho\mathbb{P}[\boldsymbol{\mu}]) \to \mathbf{g_n}(\mathbf{v} + \varrho\mathbb{P}[\boldsymbol{\mu}])$ in $(L^1(\Omega))^m$ as $j \to \infty$ by the fact that $\mathbf{g_n} \in (C^1)^m$. Using Lemma 2.1 that $\mathbb{G} : L^1(\Omega) \to L^1(\Omega)$ is compact, we have

$$\mathbb{G}[\mathbf{g_n}(\mathbf{v_j} + \varrho\mathbb{P}[\boldsymbol{\mu}])] \to \mathbb{G}[\mathbf{g_n}(\mathbf{v} + \varrho\mathbb{P}[\boldsymbol{\mu}])] \quad \text{in} \quad (L^1(\Omega))^m \quad \text{as} \quad j \to \infty,$$

that is,

$$\mathbf{T_n}(\mathbf{v_j}) \to \mathbf{T_n}(\mathbf{v}) \quad \text{in} \quad (L^1(\Omega))^m \quad \text{as} \quad j \to \infty.$$

Hence, we prove that $\mathbf{T_n}$ is continuous.

Last, we shall show that $\mathbf{T_n}$ is a compact map. Let $\{\mathbf{v_j}\} \subset \mathcal{O}$ be a bounded sequence in $(L^1(\Omega))^m$. For each fixed n, from $|\mathbf{g_n}| \leq |\mathbf{g}|$, and by the same argument as in (2.6), we deduce that

$$\left\|\mathbf{g_n}\left(\mathbf{v_j} + \varrho\mathbb{P}[\boldsymbol{\mu}]\right)\right\|_{(L^1(\Omega,\delta dx))^m} \leq \left\|\mathbf{g}\left(\mathbf{v_j} + \varrho\mathbb{P}[\boldsymbol{\mu}]\right)\right\|_{(L^1(\Omega,\delta dx))^m} \leq C\widehat{\Lambda},$$

where $\widehat{\Lambda}$ is the constant in the definition of $\mathcal{O}$. Thus, $\mathbf{g_n}(\mathbf{v_j} + \varrho\mathbb{P}[\boldsymbol{\mu}])$ is uniformly bounded in $(L^1(\Omega, \delta dx))^m$. Since the map $\mathbb{G} : L^1(\Omega, \delta dx) \to L^1(\Omega)$ is compact (see Lemma 2.1), it then follows that there exists a subsequence, still denoted by $\{\mathbf{T_n}(\mathbf{v_j})\}$, and a function $\mathbf{v} \in \mathcal{O}$ such that

$$\mathbf{T_n}(\mathbf{v_j}) \to \mathbf{T_n}(\mathbf{v}) \quad \text{in} \quad (L^1(\Omega))^m \quad \text{as} \quad j \to \infty.$$

Therefore, $\mathbf{T_n}$ is a compact map.

Using the Schauder fixed point theorem, there is a vector-valued function $\mathbf{v_n} \in \mathcal{O}$ such that $\mathbf{T_n}(\mathbf{v_n}) = \mathbf{v_n}$. Hence, $\mathbf{v_n}$ satisfies (2.8). Moreover, $\|\mathbf{v_n}\|_{(L^q(\Omega,\delta dx))^m} \le \widehat{\Lambda}$, where $\widehat{\Lambda}$ is a positive constant independent of n. By this and [7, Corollary 2.2], we conclude that $\mathbf{v_n}$ is a weak solution of (2.7), that is,

$$\int_\Omega \mathbf{v_n}(-\Delta)\boldsymbol{\xi} dx = \int_\Omega \mathbf{g_n}\big(\mathbf{v_n} + \varrho\mathbb{P}[\boldsymbol{\mu}]\big)\xi dx, \quad \forall \xi \in X_0^{1,1}(\overline{\Omega}). \tag{2.9}$$

Furthermore, since $p < q$ and $\|\mathbf{v_n}\|_{(L^q(\Omega,\delta dx))^m} \le \widehat{\Lambda}$, it follows that $\|\mathbf{v_n}\|_{(L^p(\Omega,\delta dx))^m} \le \overline{\lambda}$, where $\overline{\lambda} = C(p,q,\Omega)\widehat{\Lambda}$. This finishes the proof of Lemma 2.4. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 For each n, set $\mathbf{u_n} := \mathbf{v_n} + \varrho\mathbb{P}[\boldsymbol{\mu}]$, where $\mathbf{v_n}$ is the positive weak solution of (2.7) constructed in Lemma 2.4. Integrating by parts, we have

$$\int_\Omega \mathbb{P}[\boldsymbol{\mu}](-\Delta)\xi\, dx = -\int_{\partial\Omega} \frac{\partial \xi}{\partial \mathbf{n}} d\boldsymbol{\mu}(x).$$

Combining this and (2.9), we then obtain

$$\int_\Omega \mathbf{u_n}(-\Delta)\boldsymbol{\xi} dx = \int_\Omega \mathbf{g_n}(\mathbf{u_n})\boldsymbol{\xi} dx - \varrho \int_{\partial\Omega} \frac{\partial \xi}{\partial \mathbf{n}} d\boldsymbol{\mu}(x), \quad \forall \xi \in X_0^{1,1}(\overline{\Omega}). \tag{2.10}$$

Since $\mathbf{v_n} \in \mathcal{O}$, it follows that $\|\mathbf{v_n}\|_{(L^q(\Omega,\delta dx))^m} \le \widehat{\Lambda}$, which, together with the Hölder inequality, implies that $|\mathbf{v_n}|^p$ is uniformly bounded in $L^1(\Omega,\delta dx)$. From $|\mathbf{g_n}| \le |\mathbf{g}|$, and by the same argument as that used in (2.6), we have $\{\mathbf{g_n}(\mathbf{v_n} + \varrho\mathbb{P}[\boldsymbol{\mu}])\}_{n\in\mathbb{N}}$ is uniformly bounded in $(L^1(\Omega,\delta dx))^m$. By the fact that $\mathbb{G} : L^1(\Omega,\delta dx) \to L^1(\Omega)$ is compact, hence, up to a subsequence, $\{\mathbb{G}[\mathbf{g_n}(\mathbf{v_n} + \varrho\mathbb{P}[\boldsymbol{\mu}])]\}_{n\in\mathbb{N}}$ is convergent in $(L^1(\Omega))^m$. Since, for every $n \in \mathbb{N}$, $\mathbf{v_n}$ is a positive weak solution of (2.7), it then follows that $\{\mathbf{v_n}\}_{n\in\mathbb{N}}$ is convergent in $(L^1(\Omega))^m$. Therefore, there exists a positive function $\mathbf{v} \in (L^1(\Omega))^m$ such that $\mathbf{v_n} + \varrho\mathbb{P}[\boldsymbol{\mu}] \to \mathbf{v} + \varrho\mathbb{P}[\boldsymbol{\mu}]$ in $(L^1(\Omega))^m$ and *a.e.* in Ω as $n \to \infty$. Let

$$\mathbf{u} := \mathbf{v} + \varrho\mathbb{P}[\boldsymbol{\mu}].$$

We have $\mathbf{u_n} \to \mathbf{u}$ in $(L^1(\Omega))^m$ and *a.e.* in Ω as $n \to \infty$. Consequently, $\mathbf{g_n}(\mathbf{u_n}) \to \mathbf{g}(\mathbf{u})$ *a.e.* in Ω as $n \to \infty$.

We finally prove that $\mathbf{g_n}(\mathbf{u_n}) \to \mathbf{g}(\mathbf{u})$ in $(L^1_{\text{loc}}(\Omega,\delta dx))^m$ as $n \to \infty$. In fact, for any Borel set $E \subset \Omega$, using condition (1.3) we have

$$\begin{aligned}
\left\|\mathbf{g_n}\big(\mathbf{u_n}\big)\right\|_{(L^1(E,\delta dx))^m} &\le \left\|\mathbf{g}\big(\mathbf{u_n}\big)\right\|_{(L^1(E,\delta dx))^m} \\
&\le C\big\|a|\mathbf{u_n}|^p + \epsilon\big\|_{L^1(E,\delta dx)} \\
&\le C\left(a\int_E |\mathbf{u_n}|^p \delta dx + \epsilon \int_E \delta dx\right).
\end{aligned}$$

Since $|\mathbf{v_n}|^p$ is uniformly bounded in $L^1(\Omega,\delta dx)$, it follows from Lemma 2.2 that $|\mathbf{u_n}|^p$ is uniformly bounded in $L^1(\Omega,\delta dx)$. Hence, we obtain $\{\mathbf{g_n}(\mathbf{u_n})\}$ is uniformly integrable in $(L^1(\Omega,\delta dx))^m$. On another hand, since $\mathbf{g_n}(\mathbf{u_n}) \to \mathbf{g}(\mathbf{u})$ *a.e.* in Ω as $n \to \infty$, using the Vitali convergence theorem, we conclude that $\mathbf{g_n}(\mathbf{u_n}) \to \mathbf{g}(\mathbf{u})$ in $(L^1_{\mathrm{loc}}(\Omega, \delta dx))^m$ as $n \to \infty$. Pass the limit of (2.10) as $n \to \infty$ to derive that

$$\int_\Omega \mathbf{u}(-\Delta)\boldsymbol{\xi} dx = \int_\Omega \mathbf{g}(\mathbf{u})\boldsymbol{\xi} dx - \varrho \int_{\partial\Omega} \frac{\partial \xi}{\partial \mathbf{n}} d\boldsymbol{\mu}(x), \quad \forall \xi \in X_0^{1.1}(\overline{\Omega}).$$

A combination of this and Definition 1.1 yields that $\mathbf{u}$ is a weak solution of problem (1.2). Since $\boldsymbol{\nu}$ is positive, it follows that $\mathbf{u}$ is positive and $\mathbf{u} \geq \varrho\mathbb{P}[\boldsymbol{\mu}]$ in Ω. This finishes the proof of Theorem 1.2. □

3 Regularity and priori estimates

In this section, we prove Theorems 1.3 and 1.4 , and Corollary 1.5.

3.1 Interior regularity

Now we shall give an interior regularity result for the solution of problem (1.4). We begin with the following lemma, which can be found in [27, Corollary 2.8].

Lemma 3.1 *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $f \in L^1_{\mathrm{loc}}(\Omega)$ be such that*

$$\int_\Omega u(-\Delta\xi)dx = \int_\Omega f\xi dx, \quad \forall \xi \in C_0^\infty(\Omega).$$

Then, for any open subsets G and G' satisfying $G \subset \overline{G} \subset G' \subset \overline{G'} \subset \Omega$ and $\overline{G'}$ compact, and $1 \leq k < \frac{N}{N-1}$, there exists a positive constant $C(G, G', k)$ such that

$$\|u\|_{W^{1,k}(G)} \leq C(G, G', k)\left[\|u\|_{L^1(G')} + \|f\|_{L^1(G')}\right].$$

We now prove Theorem 1.3.

Proof of Theorem 1.3 Let $\mathbf{f} := |\mathbf{u}|^{p-1}\mathbf{u}$. By Definition 1.1, we know that every components of $\mathbf{u}$ and $\mathbf{f}$ satisfy the assumption of Lemma 3.1. By this and Lemma 3.1, we find that, for any $1 \leq k < \frac{N}{N-1}$, $\mathbf{u} \in (L^k_{\mathrm{loc}}(\Omega))^m$. Choose k such that $k > p$, then $|\mathbf{u}|^p \in L^{j_0}_{\mathrm{loc}}(\Omega)$ for some $j_0 > 1$. Since $\mathbf{u}$ is a positive weak solution of problem (1.4), it then follows that $\mathbf{u} \in (W^{2,j_0}_{\mathrm{loc}}(\Omega))^m$.

If $j_0 > \frac{N}{2}$, then by the Sobolev embedding theorem and the Schauder estimates, we obtain $\mathbf{u} \in (C^{2,\alpha}_{\mathrm{loc}}(\Omega))^m$ for some $\alpha \in (0, 1)$. As a result, the interior regularity gives that $\mathbf{u} \in (C^\infty(\Omega))^m$. If $j_0 < \frac{N}{2}$, by the L^p regularity theory, we obtain $\mathbf{u} \in (W^{2,j_0}_{\mathrm{loc}}(\Omega))^m$. Hence, $|\mathbf{u}|^p \in L^{j_1}_{\mathrm{loc}}(\Omega)$ for $j_1 = \frac{Nj_0}{p(N-2j_0)}$ by the Sobolev embedding theorem and the fact that $p < \frac{N}{N-1}$ gives that $j_1 > j_0 > 1$. By induction, we obtain that $|\mathbf{u}|^p \in L^{j_l}_{\mathrm{loc}}(\Omega)$ for $j_l = \frac{Nj_{l-1}}{p(N-2j_{l-1})} > j_{l-1}$. By the same argument as before, we conclude that there exists a j_l such that $j_l > \frac{N}{2}$. If not, that is $j_l < \frac{N}{2}$ for any l, then $j_l \to \widetilde{j} = \frac{N(p-1)}{2p} < 1$, which is impossible. If $j_0 = \frac{N}{2}$, we may replace it by choosing $\widetilde{j}_0 < \frac{N}{2}$, and we can obtain the same conclusion by iterating the same steps as before. Hence, we complete the proof of Theorem 1.3. □

3.2 A priori estimate and boundary regularity

In this subsection, we study the priori estimate and regularity near the boundary. The following result addresses the norm estimates of a positive weak solution of problem (1.5).

Proposition 3.2 *Let $p > 1$ and $\mathbf{u}$ be a positive weak solution of problem* (1.5). *Then there exists a positive constant C, depending on N, m, p, Ω and ϱ, such that* (1.6) *holds true.*

Proof Let λ_1 be the first eigenvalue of the operator $-\Delta$ with Dirichlet conditions on $\partial\Omega$ and ϕ_1 be its corresponding positive eigenfunction. Moreover, there exists a positive constant $\widetilde{C}$ such that

$$\widetilde{C}^{-1}\delta \leq \phi_1 \leq \widetilde{C}\delta; \tag{3.1}$$

see [16, p. 62] for the details.

By Definition 1.1, we know that

$$\lambda_1 \int_\Omega \mathbf{u}\phi_1 dx = \int_\Omega \mathbf{u}\big(-\Delta\phi_1\big)dx = \int_\Omega |\mathbf{u}|^{p-1}\mathbf{u}\phi_1 dx - \varrho \int_{\partial\Omega} \frac{\partial\phi_1}{\partial\mathbf{n}} d\boldsymbol{\mu}. \tag{3.2}$$

For any $i \in \{1, 2, \dots, m\}$, using the Young inequality, we have

$$u_i \leq (2\lambda_1)^{-1}u_i^p + p^{\frac{-1}{p-1}}\frac{p-1}{p}(2\lambda_1)^{\frac{1}{p-1}} \leq (2\lambda_1)^{-1}u_i^p + (2\lambda_1)^{\frac{1}{p-1}}.$$

It then follows that

$$\begin{aligned}\int_\Omega u_i\phi_1 dx &\leq (2\lambda_1)^{-1}\int_\Omega u_i^p\phi_1 dx + (2\lambda_1)^{\frac{1}{p-1}}\int_\Omega \phi_1 dx\\ &\leq (2\lambda_1)^{-1}\int_\Omega |\mathbf{u}|^{p-1}u_i\phi_1 dx + (2\lambda_1)^{\frac{1}{p-1}}\int_\Omega \phi_1 dx,\end{aligned}$$

which implies that

$$2\lambda_1 \int_\Omega \mathbf{u}\phi_1 dx \leq \int_\Omega |\mathbf{u}|^{p-1}\mathbf{u}\phi_1 dx + (2\lambda_1)^{\frac{p}{p-1}}\int_\Omega \phi_1\mathbf{e_m} dx,$$

where $\mathbf{e_m} := (1, 1, \dots, 1) \in \mathbb{R}^m$. Using the above inequality and (3.2), we have

$$\int_\Omega |\mathbf{u}|^{p-1}\mathbf{u}\phi_1 dx \leq 2\varrho\int_{\partial\Omega}\frac{\partial\phi_1}{\partial\mathbf{n}}d\boldsymbol{\mu} + (2\lambda_1)^{\frac{p}{p-1}}\int_\Omega \phi_1\mathbf{e_m} dx,$$

and hence,

$$\int_\Omega |\mathbf{u}|^{p-1}\mathbf{u}\phi_1 dx \leq 2\varrho\int_{\partial\Omega}\left|\frac{\partial\phi_1}{\partial\mathbf{n}}\right|\mathbf{e_m} d\boldsymbol{\mu} + (2\lambda_1)^{\frac{p}{p-1}}\int_\Omega \phi_1\mathbf{e_m} dx.$$

Since $\mathbf{u}$ is nonnegative, it follows that

$$|\mathbf{u}| = \left(u_1^2 + u_2^2 + \cdots + u_m^2\right)^{1/2} \leq u_1 + u_2 + \cdots + u_m,$$

which, together with the above inequality and (3.1), implies that there exists a positive constant C, depending on m, p, Ω and ϱ, such that

$$\| |\mathbf{u}|^p \|_{L^1(\Omega,\delta dx)} \leq C\Big[1 + \|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}\Big]. \tag{3.3}$$

Since $\mathbf{u}$ is a positive weak solution of problem (1.5), it follows from [7, Corollary 2.2] that

$$\mathbf{u} = \mathbb{G}\big[|\mathbf{u}|^{p-1}\mathbf{u}\big] + \varrho\mathbb{P}[\boldsymbol{\mu}]. \tag{3.4}$$

By this and the triangle inequality, we obtain

$$\begin{aligned}\|\mathbf{u}\|_{(L^1(\Omega))^m} &= \Big\|\mathbb{G}\big[|\mathbf{u}|^{p-1}\mathbf{u}\big] + \varrho\mathbb{P}[\boldsymbol{\mu}]\Big\|_{(L^1(\Omega))^m} \\ &\leq m\Big\{\Big\|\mathbb{G}\big[|\mathbf{u}|^p\big]\Big\|_{L^1(\Omega)} + \varrho\|\mathbb{P}[|\boldsymbol{\mu}|]\|_{L^1(\Omega)}\Big\},\end{aligned}$$

Applying the above inequality, Lemma 2.1 with $\alpha = 1$ and $\beta = 0$, and Lemma 2.2 with $\gamma = 0$, we then deduce that there exists a positive constant C, depending on N, m, p, and Ω, such that

$$\|\mathbf{u}\|_{(L^1(\Omega))^m} \leq C\Big[\| |\mathbf{u}|^p \|_{L^1(\Omega,\delta dx)} + \varrho\|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}\Big].$$

Combining this and (3.3), we further deduce that

$$\|\mathbf{u}\|_{(L^1(\Omega))^m} \leq C\Big[1 + \|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}\Big].$$

A combination of this and (3.3) yields the desired assertion. □

The following lemmas give precise estimates of $\mathbb{G}[\nu]$ and $\mathbb{P}[\mu]$, which can be found in [7, Theorem 2.6] and [7, Theorem 1.1].

Lemma 3.3 *Assume that* $\nu \in \mathfrak{M}(\Omega, \delta^\alpha dx)$, $\alpha \in [0, 1]$. *If* $N \geq 3$, *then* $\mathbb{G}[\nu] \in M^{\frac{N+\beta}{N-2+\alpha}}(\Omega, \delta^\beta dx)$ *for any* $\beta \in \left(\frac{-N}{N-1+\alpha}, \frac{\alpha N}{N-2}\right)$ *and* $\alpha \neq 0$, *or for any* $\beta \in ((-N)/(N-1), 0]$ *and* $\alpha = 0$. *In any case, there exists a positive constant* $C(N, \Omega, \alpha, \beta)$ *such that*

$$\|\mathbb{G}[\nu]\|_{M^{\frac{N+\beta}{N-2+\alpha}}(\Omega,\delta^\beta dx)} \leq C(N, \Omega, \alpha, \beta)\|\nu\|_{\mathfrak{M}(\Omega,\delta^\beta dx)}.$$

If $N = 2$ *and* $\alpha \neq 0$, *then* $\mathbb{G}[\nu] \in M^{\frac{2+\beta-\varepsilon}{\alpha}}(\Omega, \delta^\beta dx)$ *for any* $\beta \in \left(\frac{-2}{1+\alpha}, \infty\right)$ *and* $\varepsilon > 0$ *small enough. If* $\alpha = 0$, *then* $\mathbb{G}[\nu] \in M^p(\Omega, \delta^\beta dx)$ *for any* $\beta \in (-2, 0]$ *and* $p \in (\max\{1, -\beta\}, \infty)$.

Lemma 3.4 *Assume that* $\mu \in \mathfrak{M}_+(\partial\Omega)$ *and* $1 < p < p^*$. *Then there exists a positive constant* $C(N, \Omega, p)$ *such that*

$$\mathbb{G}\big[\mathbb{P}^p[\mu]\big] \leq C(N, \Omega, p)\|\mu\|^{p-1}_{\mathfrak{M}_+(\partial\Omega)}\mathbb{P}[\mu].$$

Typically, if $\mu = \delta_0$ *and* $0 \in \partial\Omega$, *then, for any* $x \in \Omega$,

$$\mathbb{G}[\mathbb{P}^p[\delta_0]](x) \leq C(N, \Omega, p)P(x, 0)|x|^{N+1-p(N-1)}.$$

We turn to show the pointwise estimates of a positive weak solution of problem (1.5).

Proof of Theorem 1.4 In view of Proposition 3.2, it suffices to show (1.7). Since $\mathbf{u}$ is a positive weak solution of systems (1.5), it follows that the lower estimate of (1.7) holds true. Then the following we shall prove the upper estimate of (1.7). For the purpose, we divide it into two cases: $p < \frac{N}{N-1}$, and $\frac{N}{N-1} \le p < \frac{N+1}{N-1}$.

Now consider first $p < \frac{N}{N-1}$. By (3.3), we have $|\mathbf{u}|^p \in L^1(\Omega, \delta dx)$. Combining this and Lemma 3.3 with $\alpha = 1$ and $\beta = 1$, we then deduce that $\mathbb{G}[|\mathbf{u}|^{p-1}\mathbf{u}] \in (M^{\frac{N}{N-1}}(\Omega))^m$ when $N \ge 3$, and $\mathbb{G}[|\mathbf{u}|^{p-1}\mathbf{u}] \in (M^{2-\varepsilon}(\Omega))^m$ when $N = 2$. Since $\boldsymbol{\mu} \in (\mathfrak{M}_+(\partial\Omega))^m$, it follows from Lemma 2.2 that $\mathbb{P}[\boldsymbol{\mu}] \in (M^{\frac{N}{N-1}}(\Omega))^m$ when $N \ge 3$, and $\mathbb{P}[\boldsymbol{\mu}] \in (M^{2-\varepsilon}(\Omega))^m$ when $N = 2$. Therefore, by (3.4), we conclude that, if $N \ge 3$, $\mathbf{u} \in (M^{\frac{N}{N-1}}(\Omega))^m$, and if $N = 2$, $\mathbf{u} \in (M^{2-\varepsilon}(\Omega))^m$ for ε small enough. Using this, (2.4), and $p < \frac{N}{N-1}$, we have $|\mathbf{u}|^p \in L^{k_0}(\Omega)$ for some $1 < k_0 < \frac{N}{p(N-1)}$. For any $n \ge 1$, let

$$v_0 := |\mathbf{u}| \qquad \text{and} \qquad v_n := \mathbb{G}\left[v_{n-1}^p\right].$$

By (2.1), we have

$$\begin{cases} -\Delta v_n = v_{n-1}^p & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Since $\mathbf{u}$ is a positive weak solution of the system (1.5), it follows from (3.4) that

$$|\mathbf{u}| \le m(v_1 + \varrho\mathbb{P}[|\boldsymbol{\mu}|]), \tag{3.6}$$

and hence

$$|\mathbf{u}|^p \le 2^{p-1}m^p\left(v_1^p + \varrho^p\mathbb{P}^p[|\boldsymbol{\mu}|]\right). \tag{3.7}$$

Now we will show that, there exist an integer $n_0 \ge 1$, and constants C_2 and $\widetilde{C_2}$, depending on $N, p, \Omega, \|\boldsymbol{\mu}\|_{\mathfrak{M}_+(\partial\Omega)}, \varrho$ and m, such that $v_{n_0} \in C^0(\overline{\Omega})$ and

$$v_{n_0} \le C_2\left(\mathbb{P}[|\boldsymbol{\mu}|] + v_{n_0+1}\right) \qquad \text{and} \qquad |\mathbf{u}| \le \widetilde{C_2}\left(\mathbb{P}[|\boldsymbol{\mu}|] + v_{n_0+1}\right) \quad \text{in} \quad \Omega. \tag{3.8}$$

Since $v_0^p \in L^{k_0}(\Omega)$ for some $1 < k_0 < \frac{N}{p(N-1)}$, it follows that $v_1 \in W^{2,k_0}(\Omega)$. Clearly, if $N = 2$, then $k_0 > N/2$. It follows from the Sobolev inequality (see, for instance, [18, p. 284]) that $v_1 \in C^0(\overline{\Omega})$. A combination of (3.6) and (3.7) yields (3.8).

If $N \ge 3$, then $k_0 < N/2$. By this and the Sobolev inequality, we deduce that

$$v_1^p \in L^{k_1}(\Omega), \qquad \text{where} \quad k_1 := \frac{Nk_0}{p(N-2k_0)}. \tag{3.9}$$

Moreover, we have $k_1 > \frac{(N-1)k_0}{N-2k_0}k_0 > k_0$. If $k_1 < N/2$, by (2.1), we know that

$$\begin{cases} -\Delta v_1 = |\mathbf{u}|^p & \text{in } \Omega, \\ -\Delta\mathbb{G}[\varrho^p\mathbb{P}^p[|\boldsymbol{\mu}|] + v_1^p] = \varrho^p\mathbb{P}^p[|\boldsymbol{\mu}|] + v_1^p & \text{in } \Omega, \\ v_1 = 0 = \mathbb{G}[\mathbb{P}^p[|\boldsymbol{\mu}|] + v_1^p] & \text{on } \partial\Omega. \end{cases}$$

Combining this, (3.7), and the weak maximum principle (see, for instance, [18, p. 344]), we then deduce that

$$v_1 \le 2^{p-1}m^p\left(v_2 + \varrho^p\mathbb{G}\left[\mathbb{P}^p[|\boldsymbol{\mu}|]\right]\right), \tag{3.10}$$

where

$$v_2 := \mathbb{G}\left[v_1^p\right] \in L^{k_2 p}(\Omega) \qquad \text{and} \qquad k_2 := \frac{Nk_1}{p(N-2k_1)} > \frac{Nk_0}{p(N-2k_0)} = k_1.$$

Since $|\boldsymbol{\mu}| \in \mathfrak{M}_+(\partial\Omega)$, it follows from Lemma 3.4 that there exists a positive constant C, depending on N, p, and Ω, such that $\mathbb{G}[\mathbb{P}^p[|\boldsymbol{\mu}|]] \le C\|\boldsymbol{\mu}\|^{p-1}_{(\mathfrak{M}_+(\partial\Omega))^m}\mathbb{P}[|\boldsymbol{\mu}|]$. Combining this, (3.6), and (3.10), we then obtain

$$v_1 \le C_1\left(\mathbb{P}[|\boldsymbol{\mu}|] + v_2\right) \qquad \text{and} \qquad |\mathbf{u}| \le \widetilde{C_1}\left(\mathbb{P}[|\boldsymbol{\mu}|] + v_2\right),$$

where C_1 and $\widetilde{C_1}$ are positive constants which depend on N, p, Ω, $\|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}$, ϱ and m. By induction, for any $n \ge 2$, we can obtain (3.8) holds true. Moreover, $v_n^p \in L^{k_n}(\Omega)$ with

$$k_n = \frac{Nk_{n-1}}{p(N-2k_{n-1})} > k_{n-1}.$$

Now we claim that there exists $n_0 \ge 2$ such that $k_{n_0-1} > \frac{N}{2}$. If not, that is $k_n < \frac{N}{2}$ for all $n \ge 2$, then together with $p < p^* < \frac{N}{N-2}$, we have

$$k_n \to \frac{N(p-1)}{2p} < 1 \quad \text{as} \quad n \to \infty.$$

It is a contradiction with $\{k_n\}_{n\in\mathbb{N}}$ is an increasing sequence and $k_0 > 1$. This proves the above claim. Therefore, for $N \ge 3$, we can find a integer $n_0 \ge 2$ such that $v_{n_0} \in C^0(\overline{\Omega})$ and (3.8) holds true.

By (3.5), we obtain v_{n_0+1} is of Hölder continuous with exponent 1 and hence, there exists a positive constant C_3 such that

$$v_{n_0+1}(x) \le C_3\delta(x) \quad \text{in} \quad \Omega.$$

Combining this and (3.8), we conclude that there exists a constant $\widetilde{C}$, depending on N, p, Ω, $\|\boldsymbol{\mu}\|_{\mathfrak{M}_+(\partial\Omega)}$, and m such that

$$|\mathbf{u}| \le \widetilde{C}(\mathbb{P}[|\boldsymbol{\mu}|] + \delta(x)).$$

This proves (1.7) when $p < \frac{N}{N-1}$. Moreover, if $\boldsymbol{\mu} = 0$, then by (1.7), we obtain, for any $i \in \{1,2,\dots,m\}$, $|u_i| \le \widetilde{C}\delta(x)$. It follows from [21, p. 140] that $u_i \in C^{0,1}(\overline{\Omega})$. By this and Schauder estimates, we conclude that, for any $i \in \{1,2,\dots,m\}$, $u_i \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. Hence, $\mathbf{u} \in (C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega}))^m$. This proves Theorem 1.4 when $p < \frac{N}{N-1}$.

Next, we consider the case $\frac{N}{N-1} \le p < \frac{N+1}{N-1}$. Let $K \ge 2$ be some fixed integer such that

$$\frac{1}{K} < N + 1 - (N-1)p. \tag{3.11}$$

For any $n \in [0,K]$, let $\beta_n := 1 - \frac{n}{K} \in [0,1]$. By Lemma 3.3, if $N \ge 3$, then $\mathbf{u} \in (M^{\frac{N+1}{N-1}}(\Omega,\delta dx))^m$, and if $N = 2$, for ε small, then $\mathbf{u} \in (M^{2-\varepsilon}(\Omega,\delta dx))^m$. Let $\{v_n\}_{n\ge 0}$ be the same as in the first case. By (2.4) and $\beta_0 = 1$, we have

$$v_0^p \in L^{r_0}\left(\Omega, \delta^{\beta_0}dx\right),$$

where $1 < r_0 < \frac{N+\beta_0}{(N-1)p}$. Define $v_1 := \mathbb{G}[v_0^p]$ in $L^1(\Omega)$. By Lemma 3.3 and (2.4), then for $N \ge 3$ and $\varepsilon > 0$ small enough,

$$v_1 \in L^{\frac{N+\beta}{N-2+\beta_0}-\varepsilon}\left(\Omega, \delta^{\beta} dx\right),$$

for any $\beta \in \left(-1, \frac{N}{N-2}\right)$, and for $N = 2$. Since $\beta_1 = 1 - \frac{1}{K} \in (0,1)$, it follows from (3.11) that $N + \beta_1 - (N-2+\beta_0)p = N + 1 - (N-1)p - \frac{1}{K} > 0$ and

$$v_1^p \in L^{r_1}\left(\Omega, \delta^{\beta_1} dx\right),$$

where $1 < r_1 < \frac{N+\beta_1}{(N-2+\beta_0)p}$. For any $n \le K$, assume by induction that $v_{n-1} := \mathbb{G}[v_{n-2}^p]$ in $L^1(\Omega)$, and that

$$v_{n-1}^p \in L^{r_{n-1}}\left(\Omega, \delta^{\beta_{n-1}} dx\right),$$

where $1 < r_{n-1} < \frac{N+\beta_{n-1}}{(N-2+\beta_{n-2})p}$. Then we can define $v_n := \mathbb{G}[v_{n-1}^p]$ in $L^1(\Omega)$. By Lemma 3.3 and (2.4), we obtain

$$v_n \in L^{\frac{N+\beta}{N-2+\beta_{n-1}}-\varepsilon}\left(\Omega, \rho^{\beta} dx\right),$$

for any $\beta \in \left(-1, \frac{\beta_{n-1}N}{N-2}\right)$. Taking $\beta = \beta_n \ge 0$ and by (3.11), we have $N + \beta_n - (N-2+\beta_{n-1})p > \frac{(n-1)(p-1)}{K} > 0$, hence

$$v_n^p \in L^{r_n}\left(\Omega, \delta^{\beta_n} dx\right),$$

where $1 < r_n < \frac{N+\beta_n}{(N-2+\beta_{n-1})p}$. Now in case $n = K$, we have $\beta_K = 0$. This proves that $v_K^p \in L^{r_K}(\Omega)$, with $r_K > 1$ and we are reduced to (3.9) of the first case. By this and the same argument as that used in the first case, we conclude that there exists an integer n_0 such that $v_{n_0+K} \in C^0(\overline{\Omega})$. Until now, we finish the proof of Theorem 1.4. □

At last, we give a simple proof of Corollary 1.5.

Proof of Corollary 1.5 Since its trace is necessarily of the form $\mathbf{u} = \varrho\delta_0$ for some $\varrho \ge 0$, $\varrho \in \mathbb{R}^m$. To obtain this corollary, we just need to consider a typical case of problem (1.5) with $\mathbf{u} = \varrho\delta_0$ on $\partial\Omega$ for some $\varrho \ge 0$. If $\varrho = 0$, then by Theorem 1.4, we find that $\mathbf{u}$ is continuous at 0. If not, then by Lemma 3.4, we obtain a more precise estimate, that is, there exists a positive constant C, depending on N, p, Ω, and ϱ, such that

$$v_1 \le 2^{p-1} m^p \left\{ v_2 + |\varrho|^p \mathbb{G}\left[\mathbb{P}^p[\delta_0]\right] \right\} \le C_1 \left[v_2 + P(x,0)|x|^{N+1-p(N-1)} \right].$$

By induction, we then obtain, for any $n \in \mathbb{N}$,

$$\begin{aligned} v_n &\le C_1 \left[v_{n+1} + P(x,0)|x|^{N+1-p(N-1)} \right], \\ |\mathbf{u}| &\le C\left[P(x,0) + v_{n+1} + P(x,0)|x|^{N+1-p(N-1)} \right]. \end{aligned}$$

By [7, (2.8)], we know that there exists a constant $C = C(N, \Omega)$ such that, for any $(x,z) \in \Omega \times \partial\Omega$,

$$C^{-1}\delta(x)|x-z|^{-N} \le P(x,z) \le C\delta(x)|x-z|^{-N} \le C|x-z|^{1-N}. \tag{3.12}$$

From this and (1.7), it follows that

$$C|x|^{-N}\delta(x) \le |\mathbf{u}| \le C\left[|x|^{-N}\delta(x) + \delta(x) + |x|^{-N}\delta(x)|x|^{N+1-p(N-1)}\right].$$

Combining this inequality and $p < \frac{N+1}{N-1}$, we conclude that

$$|\mathbf{u}| = C|x|^{-N}\delta(x)[1 + o(1)] \quad \text{as} \quad x \to 0.$$

This finishes the proof of Corollary 1.5. □

4 Proof of Theorem 1.6

In order to show the existence of the minimal solution for the system (1.5), we first give a sufficient conditions for the existence of the minimal solution by Proposition 4.1. As a result, we can prove Proposition 4.2 and obtain that there exists a minimal solution for (1.5). At last, with the help of Proposition 4.2 we shall prove Theorem 1.6.

Proposition 4.1 *Assume that* $1 < p < p^*$. *If there exists a nonnegative function* $\mathbf{U} \in (L^p(\Omega, \delta dx))^m$ *such that* $\mathbf{U} \ge \mathbb{G}[|\mathbf{U}|^{p-1}\mathbf{U}] + \varrho\mathbb{P}[\boldsymbol{\mu}]$, *then the problem* (1.5) *admits a minimal positive weak solution* $\underline{\mathbf{u}}$ *satisfying*

$$\varrho\mathbb{P}[\boldsymbol{\mu}] \le \underline{\mathbf{u}} \le \mathbf{U}.$$

Proof Put $\mathbf{u}_0 := (0, 0, \dots, 0) \in \mathbb{R}^m$ and

$$\mathbf{u_n} := \mathbb{G}\left[|\mathbf{u_{n-1}}|^{p-1}\mathbf{u_{n-1}}\right] + \varrho\mathbb{P}[\boldsymbol{\mu}], \quad n \ge 1. \tag{4.1}$$

Clearly, $\mathbf{u}_0 \le \mathbf{U}$. It then follows from (2.2) that

$$\mathbf{u_1} = \mathbb{G}\left[|\mathbf{u_0}|^{p-1}\mathbf{u_0}\right] + \varrho\mathbb{P}[\boldsymbol{\mu}] \le \mathbb{G}\left[|\mathbf{U}|^{p-1}\mathbf{U}\right] + \varrho\mathbb{P}[\boldsymbol{\mu}] \le \mathbf{U}.$$

By induction, we can show that $\mathbf{u_n} \le \mathbf{U}$ for any $n \ge 1$. It is easy to deduce from (2.2) that $\{\mathbf{u_n}\}_{n\in\mathbb{N}}$ is an increasing sequence. Hence, there exists a positive vextor-valued function $\underline{\mathbf{u}} \in (L^p(\Omega, \delta dx))^m$ such that $\mathbf{u_n} \to \underline{\mathbf{u}}$ almost everywhere as $n \to \infty$. Moreover, by the dominated convergence theorem, we find that $|\mathbf{u_n}|^{p-1}\mathbf{u_n} \to |\underline{\mathbf{u}}|^{p-1}\underline{\mathbf{u}}$ in $(L^1(\Omega, \delta dx))^m$ as $n \to \infty$. Therefore, by Lemma 2.1, we have

$$\mathbb{G}[|\mathbf{u_n}|^{p-1}\mathbf{u_n}] \to \mathbb{G}[|\underline{\mathbf{u}}|^{p-1}\underline{\mathbf{u}}] \qquad \text{in} \quad (L^1(\Omega))^m$$

and $a.e.$ in Ω as $n \to \infty$. Letting $n \to \infty$ in (4.1), we then deduce that

$$\underline{\mathbf{u}} = \mathbb{G}\left[|\underline{\mathbf{u}}|^{p-1}\underline{\mathbf{u}}\right] + \varrho\mathbb{P}[\boldsymbol{\mu}].$$

By [7, Corollary 2.2], we know that $\underline{\mathbf{u}}$ is a weak solution of (1.5).

Now, we claim that $\underline{\mathbf{u}}$ is the minimal weak solution of (1.5). Indeed, for any positive weak solution $\mathbf{u}$ of (1.5), we have by (3.4) that

$$\mathbf{u} = \mathbb{G}[|\mathbf{u}|^{p-1}\mathbf{u}] + \varrho\mathbb{P}[\boldsymbol{\mu}] \ge \mathbf{u_0},$$

which, together with (2.2), implies that

$$\mathbf{u} \ge \mathbb{G}\left[|\mathbf{u_0}|^{p-1}\mathbf{u_0}\right] + \varrho\mathbb{P}[\boldsymbol{\mu}] = \mathbf{u_1}.$$

By induction, we know that $\mathbf{u} \geq \mathbf{u_n}$ for all $n \geq 1$. Hence $\mathbf{u} \geq \underline{\mathbf{u}}$. This proves the above claim and hence finishes the proof of Proposition 4.1. □

Using the above proposition, we establish the existence of minimal positive weak solution of systems (1.5).

Proposition 4.2 *Assume that* $1 < p < p^*$. *Then there exists a positive constant* $\widetilde{\varrho}$ *such that, for any* $\varrho \in (0, \widetilde{\varrho})$, *systems* (1.5) *admit a minimal positive weak solution* $\underline{\mathbf{u}}$.

Proof To use Proposition 4.1, we first construct a supersolution. For any $\theta > 0$, define

$$\mathbf{U} := \theta\varrho^p \mathbb{G}\big[|\mathbb{P}[\boldsymbol{\mu}]|^p \mathbf{e_m}\big] + \varrho\mathbb{P}[\boldsymbol{\mu}],$$

where $\mathbf{e_m} := (1, 1, \ldots, 1) \in \mathbb{R}^m$. By Lemma 3.4, we have

$$|\mathbf{U}| \leq C(N, \Omega, p)\|\boldsymbol{\mu}\|^{p-1}_{\mathfrak{M}_+(\partial\Omega)} m\theta\varrho^p |\mathbb{P}[\boldsymbol{\mu}]| + \varrho|\mathbb{P}[\boldsymbol{\mu}]|.$$

It follows that,

$$|\mathbf{U}|^p \leq |\mathbb{P}[\boldsymbol{\mu}]|^p (Cm\theta\varrho^p + \varrho)^p,$$

where C is a positive constant depends on N, Ω, p, and $\|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}$. If

$$(Cm\theta\varrho^p + \varrho)^p \leq \theta\varrho^p, \tag{4.2}$$

then $|\mathbf{U}|^{p-1}\mathbf{U} \leq |\mathbf{U}|^p \mathbf{e_m} \leq \theta\varrho^p |\mathbb{P}[\boldsymbol{\mu}]|^p \mathbf{e_m}$. Combining this, the definition of $\mathbf{U}$, and (2.2), we obtain

$$\mathbb{G}\big[|\mathbf{U}|^{p-1}\mathbf{U}\big] + \varrho\mathbb{P}[\boldsymbol{\mu}] \leq \mathbb{G}\big[\theta\varrho^p |\mathbb{P}[\boldsymbol{\mu}]|^p \mathbf{e_m}\big] + \varrho\mathbb{P}[\boldsymbol{\mu}] = \mathbf{U}.$$

By this and Proposition 4.1, we obtain the desired assertion. Now, we just need to deal with (4.2). Observe that (4.2) holds true if and only if

$$\big(Cm\theta\varrho^{p-1} + 1\big)^p \leq \theta.$$

Set

$$\varrho_0 := \left(\frac{1}{Cmp}\right)^{\frac{1}{p-1}} \left(\frac{p-1}{p}\right),$$

then if $\varrho \leq \varrho_0$, we have $(Cm\theta_0\varrho^{p-1} + 1)^p \leq \theta_0$, where $\theta_0 := \left(\frac{p}{p-1}\right)^p$. Hence, we have chosen θ_0 and ϱ_0 such that (4.2) satisfied. Using Proposition 4.1, there exists a minimal solution $\underline{\mathbf{u}}$ satisfying

$$\varrho\mathbb{P}[\boldsymbol{\mu}] \leq \underline{\mathbf{u}} \leq \mathbf{U}.$$

This finishes the proof of Proposition 4.2. □

To prove Theorem 1.6, we also need the following technical lemma, which was first proved in [9, Lemma 5.3].

Lemma 4.3 *Assume that $v_1, v_2 \in C^2(\Omega)$, and $v_2 > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 and concave function. Then*

$$-\Delta\left[v_2 f\left(\frac{v_1}{v_2}\right)\right] \geq f'\left(\frac{v_1}{v_2}\right)(-\Delta v_1) + \left[f\left(\frac{v_1}{v_2}\right) - \frac{v_1}{v_2} f'\left(\frac{v_1}{v_2}\right)\right](-\Delta v_2).$$

In order to prove the first part of Theorem 1.6, we shall define a set $\mathcal{A}$ as follows:

$$\mathcal{A} := \{\varrho > 0 : \text{ systems (1.5) admits a positive weak solution }\}.$$

By Proposition 4.2, we know that (1.5) admits a positive solution for ϱ small enough. Therefore, $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$ is well-defined. For the case $1 < p < p^*$, our aim is to show that $\varrho^* = \sup \mathcal{A}$. On the other hand, by the method of contradiction, we can complete the other case $p \geq p^*$.

Proof of Theorem 1.6 Case 1) $1 < p < p^*$. To prove

$$\varrho^* = \sup \mathcal{A},$$

we only need to prove the following 4 claims.

Claim 1: ϱ^* is finite.

By Theorem 1.2, we know that systems (1.5) admit a positive weak solution $\mathbf{u}$ for ϱ small enough. For any fixed $i \in \{1, 2, \dots, m\}$, let $v_{1,i} := u_i$ and $v_{2,i} := \varrho\mathbb{P}[\mu_i]$. Then, we have $-\Delta(u_i) = |\mathbf{u}|^{p-1} u_i$ in Ω. Let

$$f(s) := \begin{cases} \dfrac{1 - s^{1-p}}{p-1}, & \text{if } s \geq 1, \\ s - 1, & \text{if } s < 1. \end{cases}$$

Since $\Delta(\varrho\mathbb{P}[\mu_i]) = 0$ (see (2.1)) and $v_{1,i} \geq v_{2,i} > 0$, it follows from Lemma 4.3 that

$$\varrho^p \mathbb{P}^p[\mu_i] \leq \left(\frac{u_i}{\varrho\mathbb{P}[\mu_i]}\right)^{-p} |\mathbf{u}|^{p-1} u_i \leq -\Delta\left(\varrho\mathbb{P}[\mu_i] f\left(\frac{u_i}{\varrho\mathbb{P}[\mu_i]}\right)\right).$$

Using the above inequality and the fact that $-\Delta\mathbb{G}[\mathbb{P}^p[\mu_i]] = \mathbb{P}^p[\mu_i]$ in Ω (see (2.1)), we then deduce that, for any $i \in \{1, 2, \dots, m\}$,

$$-\Delta\left(\varrho\mathbb{P}[\mu_i] f\left(\frac{u_i}{\varrho\mathbb{P}[\mu_i]}\right) - \varrho^p \mathbb{G}\left[\mathbb{P}^p[\mu_i]\right]\right) \geq 0 \quad \text{in} \quad \Omega. \tag{4.3}$$

Since $u_i = \varrho\mathbb{P}[\mu_i]$ on $\partial\Omega$, it follows that $f(\frac{u_i}{\varrho\mathbb{P}[\mu_i]}) = f(1) = 0$ on $\partial\Omega$, which, together with the fact that $\mathbb{G}[\mathbb{P}^p[\mu_i]] = 0$ on $\partial\Omega$ (see (2.1)), further implies that

$$\varrho\mathbb{P}[\mu_i] f\left(\frac{u_i}{\varrho\mathbb{P}[\mu_i]}\right) - \varrho^p \mathbb{G}\left[\mathbb{P}^p[\mu_i]\right] = 0 \quad \text{on} \quad \partial\Omega.$$

From this, (4.3), and the weak maximum principle, it then follows that

$$\varrho^p \mathbb{G}\left[\mathbb{P}^p[\mu_i]\right] \leq \varrho\mathbb{P}[\mu_i] f\left(\frac{u_i}{\varrho\mathbb{P}[\mu_i]}\right) \quad \text{in} \quad \Omega.$$

Combining this, the fact that $u_i \geq \varrho\mathbb{P}[\mu_i]$, and $f \leq \frac{1}{p-1}$ on $[1, \infty)$, we conclude that

$$\varrho^p \mathbb{G}\big[\mathbb{P}^p[\mu_i]\big] \le \frac{\varrho}{p-1}\mathbb{P}[\mu_i] \quad \text{in} \quad \Omega.$$

Therefore, for any $i \in \{1, 2, \ldots, m\}$, $(\boldsymbol{\varrho}^*)^{p-1}\mathbb{G}\big[\mathbb{P}^p[\mu_i]\big] \le \frac{1}{p-1}\mathbb{P}[\mu_i]$. Since $|\boldsymbol{\mu}|(\partial\Omega) = 1$, it follows that ϱ^* is finite. This proves the first claim.

Claim 2: $\mathcal{A}$ is an interval.

To show this claim, it is enough to prove that if $\varrho' \in \mathcal{A}$ and $\varrho' < \varrho^*$, then $\tau \in \mathcal{A}$ for any $\tau \in (0, \varrho')$. By the definition of $\mathcal{A}$, we find that there exists a positive weak solution $\mathbf{u}_{\varrho'}$ of problem (1.5) with $\varrho = \varrho'$, and $\mathbf{u}_{\varrho'} \ge \varrho'\mathbb{P}[\boldsymbol{\mu}] \ge \tau\mathbb{P}[\boldsymbol{\mu}]$. It is easy to see that $\mathbf{u}_{\varrho'}$ is a supsolution of (1.5) with $\varrho = \tau$. By this and Proposition 4.1, we know that (1.5) with $\varrho = \tau$ admits a minimal positive weak solution $\underline{\mathbf{u}}_\tau < \mathbf{u}_{\varrho'}$. This implies that $\tau \in \mathcal{A}$ and hence proves the second claim.

Claim 3: $\varrho^* \in \mathcal{A}$.

To prove Claim 3, we only need to show that problem (1.5) admits a positive weak solution for $\varrho = \varrho^*$. Let $\{\varrho_n\} \in \mathcal{A}$ be a nondecreasing sequence converging to ϱ^*. For each n, let $\underline{\mathbf{u}}_n$ be the positive minimal weak solution of (1.5) with $\varrho = \varrho_n$. By Definition 1.1, we obtain, for any $\xi \in X_0^{1,1}(\overline{\Omega})$,

$$\int_\Omega \underline{\mathbf{u}}_n(-\Delta)\boldsymbol{\xi}dx = \int_\Omega \left|\underline{\mathbf{u}}_n\right|^{p-1}\underline{\mathbf{u}}_n\xi dx - \varrho_n\int_{\partial\Omega}\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{n}}d\mu. \tag{4.4}$$

It follows from Theorem 1.4 that the sequence $\{\underline{\mathbf{u}}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $(L^1(\Omega))^m$ and in $(L^p(\Omega, \delta dx))^m$. By the formulation

$$\underline{\mathbf{u}}_n = \mathbb{G}[|\underline{\mathbf{u}}_n|^{p-1}\underline{\mathbf{u}}_n] + \varrho_n\mathbb{P}[\boldsymbol{\mu}], \tag{4.5}$$

and the Green operator $\mathbb{G} : L^1(\Omega, \delta dx) \to L^1(\Omega)$ is compact (see Lemma 2.1), we derive that there exist a function $\underline{\mathbf{u}}_{\varrho^*}$ and a subsequence, still denoted by the same notation, such that $\underline{\mathbf{u}}_n$ converges to $\underline{\mathbf{u}}_{\varrho^*}$ in $(L^1(\Omega))^m$ and a.e in Ω as $n \to \infty$.

For $q \in (p, p^*)$, by (4.5), Lemma 2.1 with $\alpha = 1$ and $\beta = 1$, and Lemma 2.2 with $\gamma = 1$, we deduce that, for any $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\underline{\mathbf{u}}_n\right\|_{(L^q(\Omega,\delta dx))^m} &\le \left\|\mathbb{G}\left[|\underline{\mathbf{u}}_n|^{p-1}\underline{\mathbf{u}}_n\right]\right\|_{(L^q(\Omega,\delta dx))^m} + \varrho_n\|\mathbb{P}[|\boldsymbol{\mu}|]\|_{L^q(\Omega,\delta dx)} \\ &\le C(N, q, \Omega)\left[\left\||\underline{\mathbf{u}}_n|^{p-1}\underline{\mathbf{u}}_n\right\|_{(L^1(\Omega,\delta dx))^m} + \varrho_n\|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}\right],\end{aligned}$$

It follows from Theorem 1.4 that, for any $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\underline{\mathbf{u}}_n\right\|_{(L^q(\Omega,\delta dx))^m} &\le C(N, q, \Omega)\left[\left\|\left|\underline{\mathbf{u}}_n\right|^p\right\|_{L^1(\Omega,\delta dx)} + \varrho^*\|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}\right] \\ &\le C(N, q, \Omega)(1 + \varrho^*)\left[1 + \|\boldsymbol{\mu}\|_{(\mathfrak{M}_+(\partial\Omega))^m}\right].\end{aligned}$$

Thus, $\{\underline{\mathbf{u}}_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $(L^q(\Omega, \delta dx))^m$. Furthermore, we infer that $\{|\underline{\mathbf{u}}_n|^{p-1}\underline{\mathbf{u}}_n\}_{n\in\mathbb{N}}$ is uniformly integrable in $(L^1(\Omega, \delta dx))^m$ by the Hölder inequality. Using the Vitali convergence theorem, we further deduce that $|\underline{\mathbf{u}}_n|^{p-1}\underline{\mathbf{u}}_n \to |\underline{\mathbf{u}}_{\varrho^*}|^{p-1}\underline{\mathbf{u}}_{\varrho^*}$ in $(L^1(\Omega, \delta dx))^m$ as $n \to \infty$. Therefore, letting $n \to \infty$ in (4.4), we have

$$\int_\Omega \underline{\mathbf{u}}_{\varrho^*}(-\Delta)\boldsymbol{\xi}dx = \int_\Omega |\underline{\mathbf{u}}_{\varrho^*}|^{p-1}\underline{\mathbf{u}}_{\varrho^*}\xi dx - \boldsymbol{\varrho}^*\int_{\partial\Omega}\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{n}}d\boldsymbol{\mu}, \quad \forall\xi \in X_0^{1,1}(\overline{\Omega}).$$

This means $\underline{\mathbf{u}}_{\varrho^*}$ is a positive weak solution of (1.5) with $\varrho = \varrho^*$. Thus, $\varrho^* \in \mathcal{A}$. This finishes the proof of the claim.

Claim 4: The function $\underline{\mathbf{u}}_{\varrho^*}$ is the positive minimal weak solution of (1.5) for $\varrho = \varrho^*$.

Let $\mathbf{u}$ be any positive weak solution of (1.5) for $\varrho = \varrho^*$. Then $\mathbf{u}$ is a supsolution of (1.5) when $\varrho = \varrho_n$. It follows from Proposition 4.1 that $\mathbf{u} \geq \underline{\mathbf{u}}_n$, where $\underline{\mathbf{u}}_n$ is defined in the Claim 3. Therefore, $\mathbf{u} \geq \underline{\mathbf{u}}_{\varrho^*}$. By this, we conclude that $\underline{\mathbf{u}}_{\varrho^*}$ is the positive minimal weak solution of (1.5) for $\varrho = \varrho^*$. This proves the desired assertion.

For any $\varrho_1, \varrho_2 \in (0, \varrho^*]$ with $\varrho_1 \leq \varrho_2$, $\underline{\mathbf{u}}_{\varrho_2}$ is a supsolution of systems (1.5) when $\varrho = \varrho_1$. From this and Proposition 4.1, it follows that $\{\underline{\mathbf{u}}_\varrho\}_{\varrho \in (0,\varrho^*]}$ is an increasing sequence. Moreover, by the same argument as that used in the proof of Claim 3, we conclude that $\{\underline{\mathbf{u}}_\varrho\}_{\varrho \in (0,\varrho^*]}$ converges to $\underline{\mathbf{u}}_{\varrho^*}$ in $(L^1(\Omega))^m$ and in $(L^p(\Omega, \delta\, dx))^m$. Furthermore, if $\varrho > \varrho^*$, by the definition of $\mathcal{A}$, we find that systems (1.5) admit no positive weak solution. Hence, we finish the proof of Case 1).

Case 2) $p \geq p^*$. Suppose by contradiction that for some $\varrho > 0$ and $z_i \in \partial\Omega$, $i \in \{1, 2, \dots, m\}$, the system with $\mu_i = \delta_{z_i}$ admits a positive weak solution $\mathbf{u}$. Then from Theorem 1.4, we obtain that $\mathbf{u} \in (L^p(\Omega, \delta dx))^m$ and $\mathbf{u} \geq \varrho\mathbb{P}[\boldsymbol{\mu}]$. Combining this and (3.12), we then deduce that

$$\begin{aligned}\int_\Omega |u_i|^p \delta dx &\geq \varrho^p \int_\Omega \left|\mathbb{P}(\delta_{z_i})\right|^p \delta dx \\ &\geq C \int_\Omega |x - z_i|^{-Np} \delta^{p+1} dx \\ &\geq C \int_{\{x\in\Omega :\, |x-z_i| \leq 2\delta(x)\}} |x - z_i|^{-Np} \delta^{p+1} dx.\end{aligned}$$

Choose $r > 0$ such that

$$\mathcal{Q} := \left\{x \in \Omega : |x - z_i| \leq r, |x - z_i| \leq 2\delta(x)\right\} \subset \{x \in \Omega : |x - z_i| \leq 2\delta(x)\}.$$

Then

$$\int_\Omega |u_i|^p \delta dx \geq C \int_{\mathcal{Q}} |x - z_i|^{p+1-Np} dx.$$

Since $p \geq \frac{N+1}{N-1}$, it follows that the integral on the right hand-side of the above inequality is divergent. This implies that $\mathbf{u} \notin (L^p(\Omega, \delta dx))^m$. Thus, we obtain a contradiction. Therefore, we complete the proof of Case 2) and hence Theorem 1.6. □

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