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Some results on surfaces with different mean curvatures in $\mathbb{R}^{N+1}$ and $\mathbb{L}^{N+1}$

Guowei Dai[1]

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Abstract
We discuss the following mean curvature equation

$$-a\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)+b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=\lambda f(x,u)$$

with 0-Dirichlet boundary condition on a bounded domain. We obtain the global gradient estimate of classical solutions. Furthermore, we investigate the existence and uniqueness of classical solution. By variational method, we also establish the multiplicity of strong solutions. Moreover, according to the behavior of f near 0, we obtain the global structure of positive solutions. Finally, we also investigate the symmetry of positive solutions when Ω is radially symmetric.

Keywords Bifurcation · A priori bounds · Mean curvature operator · Variational method · Positive solution · Uniqueness · Regularity · Symmetry

Mathematics Subject Classification 35B32 · 35B45 · 53A10 · 35J20 · 35B40 · 35B65

1 Introduction and main results

It is well known that a hypersurface in the Lorentz–Minkowski space $\mathbb{L}^{N+1}$ is said to be spacelike if its induced metric is a Riemannian one. Consider two different mean curvature functions on a spacelike hypersurface, the mean curvature function related to the metric induced by $\mathbb{R}^{N+1}$, that we will denote by H_R, and the one related to the metric inherited from $\mathbb{L}^{N+1}$, H_L. For $N=2$, Kobayashi [20] proved that the only surfaces satisfying $H_R=H_L=0$ are open pieces of a spacelike plane or of a helicoid in the region where the helicoid is spacelike. Further, Albujer and Caballero [1] obtained some

✉ Guowei Dai
daiguowei@dlut.edu.cn

[1] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, People's Republic of China

geometric properties of surfaces with $H_R = H_L$ and $N = 2$. In particular, when $\Omega \subset \mathbb{R}^2$, they showed that the following problem

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } \Omega, \\ |\nabla u| < 1 & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, where $\psi \in C(\partial\Omega)$.

A nature question is that what will happen if $H_R \not\equiv H_L$? Therefore, this paper is devoted to the following more general problem

$$\begin{cases} -a\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + b\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ with $N \geq 1$ is a bounded domain, a and b are two nonnegative constants with $a > b$, f denotes the difference of aNH_R and $2bNH_L$ and λ is a nonnegative parameter which indicates the strength of f. Here, we do not require $H_R \equiv H_L$. Thus, a solution of problem (1.1) may denote a hypersurface with different mean curvatures in $\mathbb{R}^{N+1}$ and $\mathbb{L}^{N+1}$. Let d denote the diameter of Ω. It is easy to verify that any spacelike solution u is uniformly bounded by $d/2$ and $\|u\|_\infty \leq \left(d\|\nabla u\|_\infty\right)/2$. It follows that the image of u lies in $\left[-d/2, d/2\right] = I_d$. Therefore, we assume that f is a real function defined on $\Omega \times I_d$. We always assume that $\operatorname{ess\,sup}_{\Omega\times I_d} |f(x,s)| \leq \Lambda < +\infty$ and

$$\frac{\sqrt{N}(d\lambda\Lambda + b)}{\sqrt{a^2 + (d\lambda\Lambda + b)^2}} < 1 \tag{1.2}$$

for any fixed λ. Clearly, this condition can be satisfied if $d\lambda\Lambda + b$ is sufficiently small, or a is sufficiently large.

If $a = 1$ and $b = 0$, the first equation of problem (1.1) is the well-known mean curvature equation in the Minkowski space. In this case, extensive research has been done regarding the existence, nonexistence, uniqueness, multiplicity and regularity of nontrivial solutions to problem (1.1), here we refer to [2–10, 13, 14, 19, 25] and the references therein.

The *aim* of this paper is to investigate the existence/nonexistence, regularity, symmetry, uniqueness and multiplicity of spacelike solutions for equation (1.1). In order to achieve main aim, we establish the following *global gradient estimate*.

Theorem 1.1 *Let $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ be any nontrivial spacelike solution of problem* (1.1) *with $\lambda = 1$. Then, there exists a positive constant $\theta = \theta(\Lambda, d, a, b)$ such that $\max_{\overline{\Omega}} |\nabla u| \leq 1 - \theta$.*

When $a = 1$ and $b = 0$, gradient estimates have been obtained in [2, Theorem 3.6]. Here, we do not require Ω being a C^2 domain. Based on Theorem 1.1, we can obtain the following *existence and uniqueness* of classical solution.

Theorem 1.2 *Assume that Ω has $C^{2,\alpha}$ boundary $\partial\Omega$ for some $\alpha \in (0,1)$. If $f \in C^{0,\alpha}\left(\overline{\Omega} \times I_d\right)$, problem* (1.1) *with $\lambda = 1$ has at least one spacelike solution $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$ such that*

$\max_{\overline{\Omega}} |\nabla u| \leq 1-\theta$ *for some positive constant* θ, *which only depends on* $\sup_{\overline{\Omega}\times I_d} |f(x,s)|$, d, *a and b. Moreover, the solution is unique if* $f(x, s)$ *is decreasing with respect to* s.

Now, the *natural question* is whether there exist multiple solutions of problem (1.1). We will use variational method to give a confirmed answer for this question. A function $u \in W^{2,p}(\Omega)$ for some $p > N$ with $\|\nabla u\|_\infty < 1$ and satisfying the problem (1.1) is called *strong* (spacelike) solution. Then, we have the following *multiplicity* of strong spacelike solutions.

Theorem 1.3 *Suppose that* $f : \Omega \times [0, d/2] \to \mathbb{R}$ *satisfies the Carathéodory conditions and the* L^∞*-growth condition*

$$f(x,s) \leq h(x) \text{ for a.e. } x \in \Omega, \forall s \in [0, d/2] \tag{1.3}$$

for some function $h \in L^\infty(\Omega)$. *Assume that* Ω *has* C^2 *boundary* $\partial\Omega$, $f(x,s) > 0$ *for a.e.* $x \in \Omega$ *and* $\forall s \in (0,R)$ *with any fixed* $R \in (0, d/2)$ *such that*

$$\lim_{s\to 0^+} \frac{f(x,s)}{s} = 0, \text{ uniformly with a.e. } x \in \Omega. \tag{1.4}$$

Then, there exists $\lambda_* > 0$ *such that problem* (1.1) *has at least two nontrivial nonnegative strong spacelike solutions for any* $\lambda > \lambda_*$.

If Ω has $C^{2,\alpha}$ boundary and $f \in C^{0,\alpha}\left(\overline{\Omega} \times [0, d/2]\right)$, applying Theorem 1.2, it is clear that solutions obtained in Theorem 1.3 are belonging to $C^{2,\alpha}\left(\overline{\Omega}\right)$. Finally, we will use bifurcation method to investigate the global structure of positive solutions set of problem (1.1). Let λ_1 be the first eigenvalue of

$$\begin{cases} -(a-b)\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Let

$$X = \left\{u \in C^1\left(\overline{\Omega}\right) \ : \ u = 0 \text{ on } \partial\Omega\right\}$$

with the norm $\|u\| := \|\nabla u\|_\infty$. Let $P = \{u \in X : u > 0 \text{ on } \Omega\}$. From now on, we add the point ∞ to our space $\mathbb{R} \times X$. Then, we have the following theorem, which is the last *main result* of this paper.

Theorem 1.4 *Assume that* Ω *has* $C^{2,\alpha}$ *boundary*, $f \in C^{0,\alpha}\left(\overline{\Omega} \times [0, d/2]\right)$ *such that* $f(x,s) > 0$ *for any* $x \in \overline{\Omega}$ *and* $s \in (0, d/2]$, *and there exists* $f_0 \in [0, +\infty]$ *such that*

$$\lim_{s\to 0^+} \frac{f(x,s)}{s} = f_0$$

uniformly for $x \in \Omega$. *Then,*

(a) *if* $f_0 = 1$, *there is an unbounded component* $\mathscr{C}$ *of the set of nontrivial solutions of problem* (1.1) *bifurcating from* $(\lambda_1, 0)$ *such that* $\mathscr{C} \subseteq (\mathbb{R} \times P) \cup \{(\lambda_1, 0)\}$, $(\lambda_1, +\infty) \subseteq \mathrm{pr}_{\mathbb{R}}(\mathscr{C})$, $\|u_\lambda\| < 1$ *and* $\lim_{\lambda\to+\infty} \|u_\lambda\| = 1$ *for* $(\lambda, u_\lambda) \in \mathscr{C} \setminus \{(\lambda_1, 0)\}$, *where* $\mathrm{pr}_{\mathbb{R}}(\mathscr{C})$ *denotes the projection of* $\mathscr{C}$ *on* $\mathbb{R}$,

(b) *if* $f_0 = +\infty$, *there is an unbounded component* $\mathscr{C}$ *of the set of nontrivial solutions of problem* (1.1) *emanating from* $(0, 0)$ *such that* $\mathscr{C} \subseteq (\mathbb{R} \times P) \cup \{(0, 0)\}$, *joins to* $(+\infty, 1)$ *and* $\|u_\lambda\| < 1$ *for* $(\lambda, u_\lambda) \in \mathscr{C} \setminus \{(0, 0)\}$,

(c) *if* $f_0 = 0$, *there is an unbounded component* $\mathscr{C}$ *of the set of nontrivial solutions of problem* (1.1) *such that* $\mathscr{C} \subseteq \mathbb{R} \times P$, *joins* $(+\infty, 1)$ *to* $(+\infty, 0)$ *and* $\|u_\lambda\| < 1$ *for any* $(\lambda, u_\lambda) \in \mathscr{C}$ *with* $\lambda < +\infty$.

Figure 1 illustrates the global bifurcation branches of Theorem 1.4. From Theorem 1.2, we see that these positive solutions also belong to $C^{2,\alpha}\left(\overline{\Omega}\right)$. The rest of this paper is arranged as follows. In Sect. 2, we study the uniqueness of solution and present the proofs of Theorems 1,1–1.2. The proofs of Theorems 1.3–1.4 will be given in Sects. 3 and 4, respectively. Moreover, a result involving the nonexistence of positive solution is also given in Sect. 4. In the last Section, we show a result concerning the symmetry of positive solutions when Ω is the unit ball.

2 Proofs of Theorems 1.1–1.2

In this section, we always assume that $\lambda = 1$ and f satisfies the L^∞-growth condition (1.3) on $\Omega \times I_d$. Let $C^{0,1}(\Omega)$ denote the class of locally Lipschitz functions on Ω and

$$\mathscr{S} = \left\{ w \in C^{0,1}(\Omega) : w = 0 \text{ on } \partial\Omega \text{ and } |\nabla w| \le 1 \text{ a.e. in } \Omega \right\}.$$

Define the energy functional $I : \mathscr{S} \to \mathbb{R}$ as follows

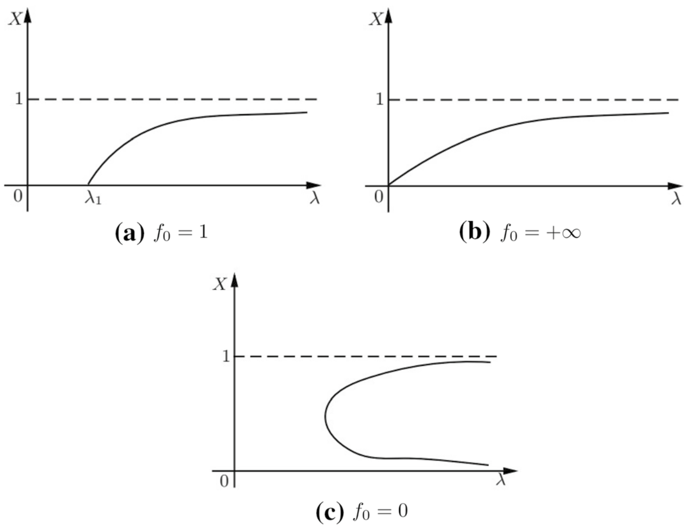

Fig. 1 Bifurcation diagrams of Theorem 1.4

$$I(u) = \int_\Omega \left(a + b - a\sqrt{1 - |\nabla u|^2} - b\sqrt{1 + |\nabla u|^2}\right) \mathrm{d}x - \int_\Omega \left(\int_0^u f(x, s)\, \mathrm{d}s\right) \mathrm{d}x.$$

It is obvious that I is uniformly bounded on $\mathscr{S}$. The equicontinuity of $\mathscr{S}$ gives a uniformly convergent minimizing sequence $u_n \rightrightarrows u \in \mathscr{S}$ as $n \to +\infty$. Consider the function $g(s) := a\sqrt{1 - s^2} + b\sqrt{1 + s^2}$ for $|s| < 1$. Then, we have that

$$g''(s) = \frac{b}{\left(1 + s^2\right)^{3/2}} - \frac{a}{\left(1 - s^2\right)^{3/2}} < 0.$$

Therefore, $a\sqrt{1 - p^2} + b\sqrt{1 + p^2}$ is concave with respect to $|p|$. Thus, it is not difficult to verify that $\int_\Omega \left(a + b - a\sqrt{1 - |\nabla u|^2} - b\sqrt{1 + |\nabla u|^2}\right) \mathrm{d}x$ is convex. Consequently, a semi-continuity theorem of [22, Theorem 1.8.1] shows that

$$I(u) \le \liminf_{n \to +\infty} I\left(u_n\right).$$

It follows that u is the *ground-state* (least energy) solution of problem (1.1). Moreover, we have the following *uniqueness*.

Lemma 2.1 *The ground-state solution of*

$$\begin{cases} -a\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

is unique for any $h(x) \in L^\infty(\Omega)$, which is denoted by $\Psi(h)$.

Proof Suppose that u, w are two ground-state solutions of problem (2.1). By the concavity of $a\sqrt{1 - p^2} + b\sqrt{1 + p^2}$, we have that

$$\begin{aligned} \int_\Omega \left(a\sqrt{1 - |\nabla u_t|^2} + b\sqrt{1 + |\nabla u_t|^2}\right) \mathrm{d}x \ge &(1 - t) \int_\Omega \left(a\sqrt{1 - |\nabla u|^2} + b\sqrt{1 + |\nabla u|^2}\right) \mathrm{d}x \\ &+ t \int_\Omega \left(a\sqrt{1 - |\nabla w|^2} + b\sqrt{1 + |\nabla w|^2}\right) \mathrm{d}x, \end{aligned}$$

where $u_t = u + t(w - u)$. This yields $I\left(u_t\right) \le (1 - t)I(u) + tI(w)$. Since u and w are both least energy solutions, we conclude that

$$\begin{aligned} &\int_\Omega \left(a\sqrt{1 - |\nabla u_t|^2} + b\sqrt{1 + |\nabla u_t|^2} + u_t h(x)\right) \mathrm{d}x \\ =&(1 - t) \int_\Omega \left(a\sqrt{1 - |\nabla u|^2} + b\sqrt{1 + |\nabla u|^2} + uh(x)\right) \mathrm{d}x \\ &+ t \int_\Omega \left(a\sqrt{1 - |\nabla w|^2} + b\sqrt{1 + |\nabla w|^2} + wh(x)\right) \mathrm{d}x. \end{aligned}$$

Thus, it is direct to check that

$$\int_\Omega \left(a\sqrt{1-|\nabla u_t|^2}+b\sqrt{1+|\nabla u_t|^2}\right)\mathrm{d}x = (1-t)\int_\Omega \left(a\sqrt{1-|\nabla u|^2}+b\sqrt{1+|\nabla u|^2}\right)\mathrm{d}x$$
$$+t\int_\Omega \left(a\sqrt{1-|\nabla w|^2}+b\sqrt{1+|\nabla w|^2}\right)\mathrm{d}x.$$

Then, the concavity of $a\sqrt{1-p^2}+b\sqrt{1+p^2}$ and $u=w$ on $\partial\Omega$ imply that $u=w$ in Ω. □

Furthermore, in view of Lemma 2.1, by an argument similar to that of [2, Lemma 1.2], we have the following comparison principle for the ground-state solutions.

Lemma 2.2 *Assume that u_i $(i=1,2)$ is the ground-state solution of problem (2.1) with $h_i\in L^\infty(\Omega)$ and $h_1(x)\le h_2(x)$ for a.e. $x\in\Omega$. Then, $u_1\le u_2$ in Ω.*

By Lemma 2.2, we can show the following comparison principle for strong spacelike solutions.

Lemma 2.3 *Assume that $\partial\Omega$ is C^1 and u_i $(i=1,2)$ is strong solution of problem (2.1) with $h_i\in L^\infty(\Omega)$ and $h_1(x)\le h_2(x)$ for a.e. $x\in\Omega$. Then, $u_1\le u_2$ in Ω.*

Proof By virtue of Lemma 2.2, it suffices to show that any strong solution u of problem (2.1) is also the ground-state solution. According to the concavity of $a\sqrt{1-p^2}+b\sqrt{1+p^2}$, for any $v\in\mathscr{S}$, we observe that

$$\int_\Omega \left(a\sqrt{1-|\nabla v|^2}+b\sqrt{1+|\nabla v|^2}\right)\mathrm{d}x-\int_\Omega \left(a\sqrt{1-|\nabla u|^2}+b\sqrt{1+|\nabla u|^2}\right)\mathrm{d}x$$
$$\le\int_\Omega \left(\frac{a\nabla u}{\sqrt{1-|\nabla u|^2}}-\frac{b\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\nabla(u-v)\,\mathrm{d}x.$$

Multiplying problem (2.1) by $u-v$ and integrating over Ω, we obtain that

$$\int_\Omega \left(b\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)-a\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)\right)(u-v)\,\mathrm{d}x=\int_\Omega h(x)(u-v)\,\mathrm{d}x.$$

Integrating by parts, we get

$$\int_\Omega \left(\frac{a\nabla u}{\sqrt{1-|\nabla u|^2}}-\frac{b\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\nabla(u-v)\,\mathrm{d}x=\int_\Omega h(x)(u-v)\,\mathrm{d}x.$$

Therefore, we conclude that

$$\int_\Omega \left(a\sqrt{1-|\nabla v|^2}+b\sqrt{1+|\nabla v|^2}\right)\mathrm{d}x-\int_\Omega \left(a\sqrt{1-|\nabla u|^2}+b\sqrt{1+|\nabla u|^2}\right)\mathrm{d}x$$
$$\leq \int_\Omega h(x)(u-v)\,\mathrm{d}x.$$

which ensures that $I(v)\geq I(u)$. Consequently, u is the ground-state solution of problem (2.1). □

Furthermore, the uniqueness for monotonous nonlinearity read as follows.

Proposition 2.1 *The ground-state solution of problem* (1.1) *in* $\mathscr{S}$ *is unique if* $f(x,s)$ *is decreasing with respect to* s.

Proof Let u, w be any two ground-state solutions of problem (1.1) in $\mathscr{S}$. It is seen from the Lemmas 2.1–2.2 that

$$0\leq (u-w)^2=\left(\Psi\left(\mathcal{N}_f(u)\right)-\Psi\left(\mathcal{N}_f(w)\right)\right)(u-w)\leq 0,$$

where $\mathcal{N}_f(u):=f(x,u(x))$ is the Nemytskii operator of $f(x,u)$. It follows that $u=w$ in Ω. □

Now, we present the proof of Theorem 1.1.

Proof of Theorem 1.1 We assume that $\max_{\overline{\Omega}}|\nabla u|=|\nabla u(x^*)|:=\gamma$. Since u is non-trivial, one has that $\gamma>0$. We choose two distinct points $x_0,y_0\in\partial\Omega$ such that $l_{x_0,y_0}:=\left\{x=tx_0+(1-t)y_0\,:\,t\in(0,1)\right\}$ contains in Ω and $x^*\in\bar{l}_{x_0,y_0}$. After rotation of the coordinates $\left(x_1,\dots,x_n\right)$, we may assume that

$$x_0^i=y_0^i \text{ for } i\neq 1,$$

where x_0^i and y_0^i denote the ith component of x_0 and y_0, respectively. For any $x\in l_{x_0,y_0}$, we obtain that

$$t=\frac{\left(x-y_0\right)\left(x_0-y_0\right)}{\left|x_0-y_0\right|^2}\in(0,1).$$

Set $u(x)=:w(t)$ and $z(t)=w(t)\left|x_0-y_0\right|^{-1}$ for any $x\in l_{x_0,y_0}$. Then, we have

$$u_i=w'(t)\frac{\partial t}{\partial x_i}=w'(t)\frac{x_0^i-y_0^i}{\left|x_0-y_0\right|^2}=z'(t)\frac{x_0^i-y_0^i}{\left|x_0-y_0\right|}$$

and

$$u_{ij}=w''(t)\frac{\left(x_0^i-y_0^i\right)\left(x_0^j-y_0^j\right)}{\left|x_0-y_0\right|^4}=z''(t)\frac{\left(x_0^i-y_0^i\right)\left(x_0^j-y_0^j\right)}{\left|x_0-y_0\right|^3},$$

where $u_i = \nabla_i u = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, $i, j \in \{1, \dots, N\}$. By some elementary calculations, we reach that

$$\begin{cases} -a\left(\dfrac{z'}{\sqrt{1-z'^2}}\right)' + b\left(\dfrac{z'}{\sqrt{1+z'^2}}\right)' = \left|x_0 - y_0\right| \widetilde{f}(t,z), \ t \in (0,1), \\ z(0) = z(1) = 0, \end{cases} \tag{2.2}$$

where $\widetilde{f}(t,z) = f\left(tx_0 + (1-t)y_0, \left|x_0 - y_0\right| z\right)$.

Obviously, there exists $\widehat{t} \in (0,1)$ such that $z'\left(\widehat{t}\right) = 0$. Integrating the first equation of problem (2.2) from $\widehat{t}$ to t, we observe that

$$b\frac{z'(t)}{\sqrt{1+|z'(t)|^2}} - a\frac{z'(t)}{\sqrt{1-|z'(t)|^2}} = \int_{\widehat{t}}^{t} \left|x_0 - y_0\right| \widetilde{f}(t,z)\, dt.$$

This gives that

$$\frac{a|z'(t)|}{\sqrt{1-|z'(t)|^2}} \leq d\Lambda + \frac{b|z'(t)|}{\sqrt{1+|z'(t)|^2}} \leq d\Lambda + b,$$

which implies that

$$\left|z'(t)\right| \leq \frac{d\Lambda + b}{\sqrt{a^2 + (d\Lambda + b)^2}} := \rho.$$

Since $a > 0$, we get $\rho < 1$. Thus, we conclude that

$$\left|\nabla_1 u(x^*)\right| \leq \rho.$$

Similarly, we show that

$$\left|\nabla_i u(x^*)\right| \leq \rho \text{ for any } i \in \{2, \dots, N\}.$$

It is seen from (1.2) that

$$\gamma = |\nabla u(x^*)| \leq \sqrt{N}\rho < 1.$$

Taking $\theta = 1 - \sqrt{N}\rho$, noting that ρ only depends on Λ, d, a and b, we get the desired conclusion. □

It is straightforward to see that the argument of Theorem 1.1 is more simple than the corresponding ones of [2, Corollary 3.4 and Theorem 3.5] even in the case of $b = 0$. To prove Theorem 1.2, we need the following lemma, which roughly says that the classical solution is also the ground-state solution of problem (1.1).

Lemma 2.4 *If $f(x, s)$ is decreasing with respect to s, any spacelike solution $u \in C^1\left(\overline{\Omega}\right) \cap C^2(\Omega)$ is also the ground-state solution of problem* (1.1).

Proof Combining Theorem 1.1 and an argument similar to that of Lemma 2.3, for any $v \in \mathscr{S}$, gives

$$\int_\Omega \left(a\sqrt{1-|\nabla v|^2}+b\sqrt{1+|\nabla v|^2}\right)\mathrm{d}x-\int_\Omega \left(a\sqrt{1-|\nabla u|^2}+b\sqrt{1+|\nabla u|^2}\right)\mathrm{d}x$$
$$\le \int_\Omega f(x,u)(u-v)\,\mathrm{d}x.$$

Since f is decreasing, we find that

$$\begin{aligned}\int_\Omega\left(\int_0^v f(x,s)\,\mathrm{d}s-\int_0^u f(x,s)\,\mathrm{d}s\right)\mathrm{d}x &= \int_\Omega\int_u^v f(x,s)\,\mathrm{d}s\,\mathrm{d}x\\ &= \int_{u\le v}\int_u^v f(x,s)\,\mathrm{d}s\,\mathrm{d}x+\int_{u>v}\int_u^v f(x,s)\,\mathrm{d}s\,\mathrm{d}x\\ &\le \int_\Omega f(x,u)(v-u)\,\mathrm{d}x.\end{aligned}$$

Thus, showing that

$$\int_\Omega \left(a\sqrt{1-|\nabla v|^2}+b\sqrt{1+|\nabla v|^2}\right)\mathrm{d}x-\int_\Omega \left(a\sqrt{1-|\nabla u|^2}+b\sqrt{1+|\nabla u|^2}\right)\mathrm{d}x$$
$$\le \int_\Omega\left(\int_0^u f(x,s)\,\mathrm{d}s-\int_0^v f(x,s)\,\mathrm{d}s\right)\mathrm{d}x,$$

which implies that $I(v)\ge I(u)$. This ensures that u is the ground-state solution of problem (1.1). □

According to Lemma 2.4 and reasoning as that of [2, Lemma 1.3], we can establish the following result, which will be used later.

Lemma 2.5 *Suppose that there is a sequence* $\{u_k\}_1^\infty$ *in* $C^1\left(\overline{\Omega}\right)\cap C^2(\Omega)$ *of spacelike solutions of problem* (1.1) *with nonlinearities* h_k, h_k *is measurable on* Ω *and* $\sup_\Omega |h_k|\le\Lambda<+\infty$, *such that* $\{u_k\}$ *converges uniformly and* $\{h_k\}_1^\infty$ *converges weakly,*

$$u_k \rightrightarrows u \text{ in } C^0\left(\overline{\Omega}\right),$$
$$h_k \rightharpoonup h \text{ in } L^2(\Omega).$$

Then, u is weakly spacelike and is the ground-state solution of problem (1.1) *with nonlinearity h.*

We end this section by providing the proof of Theorem 1.2.

Proof of Theorem 1.2 For any $\sigma\in[0,1]$, we consider the following problem

$$\begin{cases} -a\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \sigma f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

By Theorem 1.1, for any spacelike solution $u \in C^{2,\alpha}\left(\overline{\Omega}\right)$ of problem (2.3), there exists a positive constant $\theta = \theta\left(\sup_{\overline{\Omega}\times I_d} |f(x,s)|, d, a, b\right)$ such that $\max_{\overline{\Omega}} |\nabla u| \le 1-\theta$.

By some calculations, we find that u satisfies

$$\begin{cases} -\sum_{i,j=1}^N \left(A\delta_{ij} + Bu_iu_j\right)u_{ij} = \sigma f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$A = \frac{a}{\sqrt{1-|\nabla u|^2}} - \frac{b}{\sqrt{1+|\nabla u|^2}},\ B = \frac{a}{\left(1-|\nabla u|^2\right)^{3/2}} + \frac{b}{\left(1+|\nabla u|^2\right)^{3/2}}.$$

It is easy to see that

$$A \ge a-b,\ B \ge a. \tag{2.4}$$

Next, we use these inequalities in (2.4) and the inductive approach to show that all the leading principal minors of matrix $\left(A\delta_{ij} + Bu_iu_j\right)$ are positive. It is clear that $A + Bu_1^2 \ge A \ge a-b$. For any $k \in \{1,\dots,N\}$, we may assume

$$\begin{vmatrix} A+Bu_1^2 & Bu_1u_2 & \cdots & Bu_1u_{k-1} \\ Bu_2u_1 & A+Bu_2^2 & \cdots & Bu_2u_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ Bu_{k-1}u_1 & Bu_{k-1}u_2 & \cdots & A+Bu_{k-1}^2 \end{vmatrix} := A_{k-1,k-1} \ge (a-b)^{k-1}.$$

Then, there must hold

$$\begin{aligned} A_{k,k} =& \begin{vmatrix} A+Bu_1^2 & Bu_1u_2 & \cdots & 0 \\ Bu_2u_1 & A+Bu_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Bu_ku_1 & Bu_ku_2 & \cdots & A \end{vmatrix} + \begin{vmatrix} A+Bu_1^2 & Bu_1u_2 & \cdots & Bu_1u_k \\ Bu_2u_1 & A+Bu_2^2 & \cdots & Bu_2u_k \\ \vdots & \vdots & \ddots & \vdots \\ Bu_ku_1 & Bu_ku_2 & \cdots & Bu_k^2 \end{vmatrix} \\ =& AA_{k-1,k-1} + A^{k-1}Bu_k^2 \ge (a-b)^k. \end{aligned}$$

Thus, the matrix $\left(A\delta_{ij} + Bu_iu_j\right)$ is positive definite. In particular, the eigenvalues μ_i of $\left(A\delta_{ij} + Bu_iu_j\right)$, $i = 1,\dots,N$, are positive. Since $\left(A\delta_{ij} + Bu_iu_j\right)$ is symmetrical, we have that

$$\sum_{i=1}^N \mu_i = NA + B|\nabla u|^2 = \frac{Na}{\sqrt{1-|\nabla u|^2}} - \frac{Nb}{\sqrt{1+|\nabla u|^2}} + \frac{a|\nabla u|^2}{\left(1-|\nabla u|^2\right)^{3/2}} + \frac{b|\nabla u|^2}{\left(1+|\nabla u|^2\right)^{3/2}}.$$

It follows that

$$\begin{aligned} a-b \le& \frac{a}{\left(1-|\nabla u|^2\right)^{3/2}} - \frac{b}{\left(1+|\nabla u|^2\right)^{3/2}} \le \sum_{i=1}^N \mu_i \le \frac{Na}{\left(1-|\nabla u|^2\right)^{3/2}} - \frac{Nb}{\left(1+|\nabla u|^2\right)^{3/2}} \\ \le& \frac{Na}{\left(1-(1-\theta)^2\right)^{3/2}}, \end{aligned}$$

which implies that

$$N \min_i \mu_i \le \sum_{i=1}^{N} \mu_i \le \frac{Na}{\left(1-(1-\theta)^2\right)^{3/2}}, \ a-b \le \sum_{i=1}^{N} \mu_i \le N \max_i \mu_i.$$

Therefore, we obtain

$$\frac{\max_i \mu_i}{\min_i \mu_i} \ge \frac{(a-b)\left(1-(1-\theta)^2\right)^{3/2}}{Na}.$$

Noting that

$$\left(\max_i \mu_i\right)^{N-1} \min_i \mu_i \ge \mu_1 \mu_2 \cdots \mu_N = \det\left(A\delta_{ij} + Bu_i u_j\right) \ge (a-b)^N,$$

gives

$$\min_i \mu_i \ge \frac{(a-b)^N}{\max_i \mu_i} \ge \frac{(a-b)^N}{\sum_{i=1}^N \mu_i} \ge \frac{(a-b)^N}{\frac{Na}{\left(1-(1-\theta)^2\right)^{3/2}}} = \frac{(a-b)^N Na}{\left(1-(1-\theta)^2\right)^{3/2}}.$$

It is seen that

$$\frac{\max_i \mu_i}{\min_i \mu_i} \le \frac{\sum_{i=1}^N \mu_i}{\min_i \mu_i} \le \frac{\frac{Na}{\left(1-(1-\theta)^2\right)^{3/2}}}{\frac{(a-b)^N Na}{\left(1-(1-\theta)^2\right)^{3/2}}} = \frac{1}{(a-b)^N}.$$

Therefore, problem (2.3) is uniformly elliptic. By Theorem 13.7 of [18], there exists a constant $C_0 = C_0\left(N, \sup_{\overline{\Omega}\times I_d} |f(x,s)|, \Omega, a, b\right)$ such that $\|u\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \le C_0$ for some $\alpha > 0$. According to Theorem 11.4 of [18], problem (1.1) is solvable in $C^{2,\alpha}\left(\overline{\Omega}\right)$. Finally, combining Proposition 2.1 and Lemma 2.4, we can derive the desired conclusions. □

3 Existence of solutions via variational method

In this section, we still assume that f satisfies the L^∞-growth condition (1.3) on $\Omega \times I_d$. Define

$$K_0 = \left\{u \in W^{1,\infty}(\Omega) : \|\nabla u\|_\infty \le 1, u = 0 \text{ on } \partial\Omega\right\}.$$

Let $\Phi : C\left(\overline{\Omega}\right) :\longrightarrow (-\infty, +\infty]$ be defined by

$$\Phi(u) = \begin{cases} \int_\Omega \left(a + b - a\sqrt{1-|\nabla u|^2} - b\sqrt{1+|\nabla u|^2}\right) \mathrm{d}x & \text{if } u \in K_0, \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, Φ is convex. By an argument similar to that of [5, Lemma 4], we can show that Φ is lower semicontinuous.

Obviously, the Nemytskii operator $\mathcal{N}_f$ is continuous and maps the bounded sets in $C\left(\overline{\Omega}\right)$ into the bounded sets in $L^1(\Omega)$. For any $u \in C\left(\overline{\Omega}\right)$, we can see that $\mathcal{N}_f(u) \in L^\infty(\Omega)$. Define the functional

$$\mathcal{H}(u) = -\int_\Omega \left(\int_0^u f(x,s)\,\mathrm{d}s \right) \mathrm{d}x$$

on $C\left(\overline{\Omega}\right)$. Clearly, $\mathcal{H}$ is C^1. Following the definition of [24], $u \in K_0$ is critical point of I if it satisfies the following variational inequality

$$\Phi(v) - \Phi(u) + \left\langle \mathcal{H}'(u), v - u \right\rangle \geq 0 \text{ for all } v \in K_0.$$

According to [24], I is said to satisfy the (PS)-condition if $\left\{u_n\right\}$ is a sequence containing in K_0 such that $I\left(u_n\right) \to c \in \mathbb{R}$ and

$$\Phi(v) - \Phi\left(u_n\right) + \left\langle \mathcal{H}'\left(u_n\right), v - u_n \right\rangle \geq -\varepsilon_n \left\|v - u_n\right\|_\infty, \ \forall v \in K_0,$$

where $\varepsilon_n \to 0^+$ as $n \to +\infty$, then $\left\{u_n\right\}$ possesses a convergent subsequence. We infer from Lemma 2 of [5] that I satisfies the (PS)-condition. To prove Theorem 1.3, we first prove the following result.

Proposition 3.1 *Assume that Ω has C^2 boundary $\partial\Omega$ and $h \in L^\infty(\Omega)$. Then, problem (2.1) has a unique solution $u \in W^{2,p}(\Omega)$ for some $p > N$ and there exists a positive constant $\theta = \theta\left(\|h\|_\infty, d, a, b\right)$ such that $\max_{\overline{\Omega}} |\nabla u| \leq 1 - \theta$. Moreover, if $h \geq 0$ in Ω, then $u \geq 0$ in Ω and u cannot achieve a minimum in Ω unless it is the trivial solution.*

Proof We first assume that $h \in C^1\left(\overline{\Omega}\right)$. By Theorem 1.1, for any spacelike solution $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ of problem (2.1), there exists a positive constant $\theta = \theta\left(\|h\|_\infty, d, a, b\right)$ such that $\max_{\overline{\Omega}} |\nabla u| \leq 1 - \theta$. As that of Theorem 1.2, we find that u satisfies

$$\begin{cases} -\sum_{i,j=1}^N \left(A\delta_{ij} + Bu_i u_j\right) u_{ij} = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

From the proof of Theorem 1.2, we know that problem (2.1) is uniformly elliptic. By Theorem 13.7 of [18], there exists a constant $C_1 = C_1\left(N, \sup_{\overline{\Omega}\times I_d} |f(x,s)|, \Omega, a, b\right)$ such that $\|u\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leq C_1$ for some $\alpha > 0$.

Define

$$\mathcal{X} = \left\{ w \in C^{1,\alpha}\left(\overline{\Omega}\right) : \max_{\overline{\Omega}} |\nabla w| \leq 1 - \theta, \|w\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leq C_1 \right\}.$$

For any fixed $w \in \mathcal{X}$, consider the following problem

$$\begin{cases} -\sum_{i,j=1}^N a^{ij} v_{ij} = h & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$a^{ij} = A\delta_{ij} + Bw_i w_j.$$

We *claim* that problem (3.1) is strictly elliptic. It is straightforward to see that

$$\max_i \mu_i \le \sum_{i=1}^{N} \mu_i \le \frac{Na}{\left(1-(1-\theta)^2\right)^{3/2}}.$$

This together with some elementary calculations gives that

$$\left(\max_i \mu_i\right)^{N-1} \min_i \mu_i \ge \mu_1\mu_2 \cdots \mu_N = \det\left(a^{ij}\right) = A^N + A^{N-1}B \ge (a-b)^N + (a-b)^{N-1}a.$$

It follows that

$$\min_i \mu_i \ge \frac{\left((a-b)^N + (a-b)^{N-1}a\right)\left(1-(1-\theta)^2\right)^{\frac{3(N-1)}{2}}}{N^{N-1}a^{N-1}},$$

which verify our desired claim. Applying Theorem 9.15 of [18], problem (3.1) has a unique solution $v \in W^{2,p}(\Omega)$ with $1 < p < +\infty$, which is denoted by $\mathcal{L}(w)$. Then, by Theorem 9.11 of [18], there exists a constant $C_2 = C_2(N, \theta, p, a, b, \Omega)$ such that

$$\|v\|_{W^{2,p}(\Omega)} \le C_2\left(\|v\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}\right).$$

We infer from Theorem 9.1 of [18] that

$$\|v\|_\infty \le C_3\|h\|_{L^p(\Omega)}$$

for some constant C_3 depends on d, p, N, θ, a and b. Therefore, it is direct to check that

$$\|v\|_{W^{2,p}(\Omega)} \le C_4\|h\|_{L^p(\Omega)} \tag{3.2}$$

for some constant $C_4 = C_4(N, \theta, p, a, b, \Omega)$. Since $p > N$, we choose α small enough such that $W^{2,p}(\Omega)$ compactly imbedded into $C^{1,\alpha}\left(\overline{\Omega}\right)$. (3.2) shows that $\mathcal{L} : \mathcal{X} \longrightarrow C^{1,\alpha}\left(\overline{\Omega}\right)$ is completely continuous.

Clearly, u is a solution of problem (2.1) if and only if it is a fixed point of $\mathcal{L}$ in $\mathcal{X}$. By the Leray-Schauder continuation theorem (see [21, Theorem 4.4.3]) and the similar proof as that of [9, Lemma 2.2], we can show that $\mathcal{L}$ has a fixed point $u \in \mathcal{X}$. The uniqueness can be deduced from Lemma 2.3. The general case of $h \in L^\infty(\Omega)$ can be proved by approximation. If $h \ge 0$ in Ω, by Theorem 9.1 of [18], we know that $u \ge 0$ in Ω. We conclude from Theorem 9.6 of [18] that u cannot achieve a minimum in Ω unless it is a constant. □

If Ω has C^2 boundary $\partial\Omega$ and $h \in L^\infty(\Omega)$, by the proof of Lemma 2.3, we see that the solution obtained in Proposition 3.1 is the unique ground-state solution of problem (2.1) in K_0. Conversely, if u is a critical point of I, it is also ground-state solution of the following problem

$$\begin{cases} -a\mathrm{div}\left(\frac{\nabla w}{\sqrt{1-|\nabla w|^2}}\right) + b\mathrm{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) = \mathcal{N}_f(u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Further, if $\partial\Omega$ is C^2, by Proposition 3.1, u is a strong spacelike solution of problem (1.1) with $\lambda = 1$. The first existence result to problem (1.1) of this section is the following proposition.

Proposition 3.2 *Assume that Ω has C^2 boundary $\partial\Omega$. Then, problem (1.1) with $\lambda = 1$ has a strong spacelike solution, which is also the ground-state solution.*

Proof Since f satisfies the L^∞-growth condition, it is easy to verify that I is bounded from below on $C\left(\overline{\Omega}\right)$. In view of Theorem 1.7 of [24], we obtain a critical point $u_0 \in K_0$ of I such that

$$I\left(u_0\right) = \inf_{u\in C\left(\overline{\Omega}\right)} I(u).$$

By the above remark, u_0 is a strong spacelike solution of problem (1.1) with $\lambda = 1$ and it is also the ground-state solution. □

Obviously, the energy functional associated to problem (1.1) takes the form

$$I_\lambda(u) = \Phi(u) - \lambda \int_\Omega \left(\int_0^u f(x,s)\,\mathrm{d}s \right) \mathrm{d}x$$

on $C\left(\overline{\Omega}\right)$. Then, by virtue of Proposition 3.1 and the reasoning as that of [5, Theorem 2], we can show the following existence result.

Proposition 3.3 *Besides the condition of Proposition 3.2, we also assume that there exists $R \in (0, \delta)$ such that $f(x,s) > 0$ for a.e. $x \in \Omega$ and $\forall s \in (0, R)$. Then, there exists $\lambda_* > 0$ such that problem (1.1) has at least one nontrivial strong spacelike solution for any $\lambda > \lambda_*$ which is a minimizer of I_λ with negative energy.*

Now, on the basis of Proposition 3.3, we can give the proof of Theorem 1.3 via the Mountain Pass Theorem [24, Theorem 3.2].

Proof of Theorem 1.3 We first extend continuously f to the whole I_d by taking $f = 0$ on $\Omega \times [-d/2, 0]$, which is still denoted by f. For any fixed $\lambda > \lambda_*$, by Proposition 3.3, I_λ has a nontrivial minimizer $e_\lambda \in K_0$ such that $I_\lambda\left(e_\lambda\right) < 0$.

To get the second critical point of I_λ, it is sufficient to show that there exist two positive constants α and $\rho < \|e_\lambda\|_\infty$ such that

$$I_\lambda(u) \geq \alpha \text{ for all } u \in K_0 \text{ with } \|u\|_\infty = \rho. \tag{3.3}$$

By some simple calculations, we can verify the following elementary inequality

$$a + b - a\sqrt{1 - s^2} - b\sqrt{1 + s^2} \geq \frac{(a-b)s^2}{2}. \tag{3.4}$$

For any $u \in K_0$, combining (3.4) and the Poncaré inequality gives

$$\Phi(u) \geq \frac{(a-b)\lambda_1}{2} \int_\Omega |u|^2 \,\mathrm{d}x.$$

Since (1.4) and $a > b$, there exists $\sigma > 0$ such that

$$f(x,s) \leq \frac{(a-b)\lambda_1}{2\lambda} |s| \text{ for a.e. } x \in \Omega \text{ and } \forall s \in [-\sigma, \sigma].$$

It follows that

$$\lambda \int_\Omega \left(\int_0^u f(x,s)\,\mathrm{d}s \right) \mathrm{d}x \leq \frac{(a-b)\lambda_1}{4} \int_\Omega |u|^2 \,\mathrm{d}x.$$

Thus,

$$I_\lambda(u) \geq \frac{(a-b)\lambda_1}{4} \int_\Omega |u|^2 \,\mathrm{d}x$$

for any $u \in K_0$ with $\|u\|_\infty \in [-\sigma, \sigma]$. Let $\rho \in \left(0, \min\left\{\sigma, \|e_\lambda\|_\infty\right\}\right)$, it follows from the proof of [5, Theorem 3] that

$$0 < \inf_{u \in K_0, \|u\|_\infty = \rho} \int_\Omega |u|^2 \,\mathrm{d}x := \gamma,$$

which implies (3.3) with $\alpha = \left(\lambda_1(a-b)\gamma\right)/4$.

Therefore, using the Mountain Pass Theorem, we obtain a critical point $u_\lambda \in C\left(\overline{\Omega}\right)$ of I_λ such that $\alpha \leq I_\lambda\left(u_\lambda\right) < +\infty$. This ensures that $u_\lambda \in K_0 \setminus \left\{e_\lambda\right\}$ is the nontrivial solution of problem (1.1). Finally, we show that e_λ and u_λ are nonnegative. Indeed, for any strong solution u, setting $u^- = \min\{0, u\}$, multiplying the first equation of problem (1.1) by u^- and integrating over Ω, we conclude that

$$\int_\Omega \left(\frac{a}{\sqrt{1-|\nabla u|^2}} - \frac{b}{\sqrt{1+|\nabla u|^2}} \right) |\nabla u^-|^2 \,\mathrm{d}x = 0.$$

It follows that $u^- \equiv 0$.

4 Bifurcation

For any $t \in (0, 1]$, consider the following problem

$$\begin{cases} -a\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-t|\nabla u|^2}}\right) + b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+t|\nabla u|^2}}\right) = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

for a given $g \in C^\alpha\left(\overline{\Omega}\right)$ with some $\alpha \in (0, 1)$. Let $v = \sqrt{t}u$, problem (4.1) is equivalent to

$$\begin{cases} -a\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+b\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)=\sqrt{t}g(x) & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

According to Theorem 1.2, problem (4.2) has a unique spacelike solution $v \in C^{2,\alpha}(\Omega)$ which is denoted by $\Psi(\sqrt{t}g)$. It follows that $u = \Psi(\sqrt{t}g)/\sqrt{t}$ is the unique solution of problem (4.1). We also consider the following problem

$$\begin{cases} -(a-b)\Delta u = g(x) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

By virtue of Theorem 8.34, Theorem 4.3 and Theorem 4.6 of [18], we know that problem (4.3) has a unique solution $u \in C^{1,\alpha}\left(\overline{\Omega}\right) \cap C^{2,\alpha}(\Omega)$, which is denoted by $\Phi(g)$. Clearly, $\Phi : C^{\alpha}\left(\overline{\Omega}\right) \longrightarrow C^{1,\alpha}\left(\overline{\Omega}\right)$ is continuous and linear. Therefore, $\Phi : C^{\alpha}\left(\overline{\Omega}\right) \longrightarrow X$ is completely continuous and linear. Define

$$G(t,g) = \begin{cases} \frac{\Psi(\sqrt{t}g)}{\sqrt{t}} & \text{if } t \in (0,1], \\ \Phi(g) & \text{if } t = 0. \end{cases}$$

We have the following compact result.

Lemma 4.1 *$G : [0,1] \times C^{\alpha}\left(\overline{\Omega}\right) \longrightarrow X$ is completely continuous.*

Proof For any $g_n, g \in C^{\alpha}\left(\overline{\Omega}\right)$ and $t_n, t \in [0,1]$ with $g_n \to g$ in $C^{\alpha}\left(\overline{\Omega}\right)$ and $t_n \to t$ in $[0, 1]$ as $n \to +\infty$, it is sufficient to show that $u_n := G(t_n, g_n) \to u := G(t, g)$ in X.

If $t > 0$, by Theorem 1.2, $u_n\sqrt{t_n} := v_n, u\sqrt{t} := v \in C^{2,\alpha}\left(\overline{\Omega}\right)$ and $\|v_n\| \le 1-\theta$ for any $n \in \mathbb{N}$. Theorem 13.7 of [18] gives a priori estimate for $\|v_n\|_{C^{1,\beta}(\overline{\Omega})}$ for some $\beta > 0$. Then, up to a subsequence, there exists $w \in C^1\left(\overline{\Omega}\right)$ such that $v_n \to w$ in $C^1\left(\overline{\Omega}\right)$ as $n \to +\infty$. We infer from Lemma 2.5 that w is the minimum point of

$$I(z) = \int_{\Omega} \left(a+b-a\sqrt{1-|\nabla z|^2}-b\sqrt{1+|\nabla z|^2}\right) \mathrm{d}x - \int_{\Omega} \sqrt{t}g(x)z \,\mathrm{d}x$$

in $\mathscr{S}$. Further, Lemma 2.1 implies that w is also the unique minimum point of I. From Lemma 2.4, we get that $w = v$. It follows that $u_n \to u$ in X as $n \to +\infty$.

If $t = 0$ and there exists a subsequence t_{n_i} of t_n such that $t_{n_i} = 0$, then $u_{n_i} = G(t_{n_i}, g_{n_i}) = \Phi(g_{n_i}) \to \Phi(g) = u$ in X as $i \to +\infty$. Next, we assume that $t = 0$ and $t_n > 0$ for any $n \in \mathbb{N}$. We conclude from Theorem 1.2 that problem (4.2) has only the trivial solution when $t = 0$. Reasoning as the above, we can show that $v_n \to 0$ in X as $n \to +\infty$. Noting that u_n satisfies

$$\begin{cases} -\sum_{i,j=1}^{N} a^{ij}u_{ij} = g_n(x) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where

$$a^{ij} = \delta_{ij}\left(\frac{a}{\sqrt{1-|\nabla v_n|^2}} - \frac{b}{\sqrt{1+|\nabla v_n|^2}}\right) + \left(\frac{a}{\left(1-|\nabla v_n|^2\right)^{\frac{3}{2}}} + \frac{b}{\left(1+|\nabla v_n|^2\right)^{\frac{3}{2}}}\right)\nabla_i v_n \nabla_j v_n.$$

In view of the proof of Proposition 3.1, we know that problem (4.4) is strictly elliptic. Then, Theorem 3.7 of [18] implies a priori estimate for $\|u_n\|_{C^0(\overline{\Omega})}$. Furthermore, by virtue of Theorem 6.6 of [18], we have that $\|u_n\|_{C^{2,\alpha}(\overline{\Omega})} \le C$ for some positive constant C independing on n. Thus, up to a subsequence, there exists $w \in C^2\left(\overline{\Omega}\right)$ such that $u_n \to w$ in $C^2\left(\overline{\Omega}\right)$ as $n \to +\infty$. Letting $n \to +\infty$ in (4.4), we obtain that

$$\begin{cases} -(a-b)\Delta w = g(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

which yields $w = \Phi(g) = G(0, g) = u$.

Anyway, there exists a subsequence u_{m_k} of u_n such that $u_{m_k} \to u$ in X as $k \to +\infty$. We *claim* that $u_n \to u$ in X. Otherwise, there exists a subsequence u_{n_k} of u_n and $\varepsilon_0 > 0$ such that $\left\|u_{n_k} - u\right\| \ge \varepsilon_0$ for any $k \in \mathbb{N}$. While, in view of the above arguments, we obtain that u_{n_k} contains a further subsequence $u_{n_{k_j}}$ such that $u_{n_{k_j}} \to u$ in X as $j \to +\infty$, which contradicts $\left\|u_{n_{k_j}} - u\right\| \ge \varepsilon_0$. Therefore, G is continuous. The *compactness* of G can be got by a similar way of [15, Lemma 2.3]. □

Consider the following problem

$$\begin{cases} -a\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

It is obvious that problem (4.5) is equivalent to the operator equation $u = \Psi(\lambda u) := \Psi_\lambda(u)$. Choosing δ small enough such that there is no eigenvalue of problem (1.5) in $\left(\lambda_1, \lambda_1 + \delta\right)$, we can obtain the following topological degree jumping result.

Lemma 4.2 *For any* $r \in (0, 1)$, *we have*

$$\deg\left(I - \Psi_\lambda, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in \left(0, \lambda_1\right), \\ -1 & \text{if } \lambda \in \left(\lambda_1, \lambda_1 + \delta\right), \end{cases}$$

where $B_r(0) = \{w \in X : \|w\| < r\}$.

Proof We *claim* that the Leray-Schauder degree $\deg\left(I - G(t, \lambda\cdot), B_r(0), 0\right)$ is well defined for any $\lambda \in \left(0, \lambda_1 + \delta\right) \setminus \left\{\lambda_1\right\}$ and $t \in [0, 1]$. The claim is obvious for $t = 0$. Thus, it is enough to show that $u = G(t, \lambda u)$ has no solution with $\|u\| = r$ for r sufficiently small and any $t \in (0, 1]$. Otherwise, there exists a sequence $\left\{u_n\right\}$ such that $u_n = \Psi_\lambda\left(\sqrt{t}u_n\right)/\sqrt{t}$ and $\left\|u_n\right\| \to 0$ as $n \to +\infty$. Let $w_n := u_n/\left\|u_n\right\|$, by an argument similar to that of Lemma 4.1, we can show that $w_n \to w$ as $n \to +\infty$ and w verifies problem (1.5) with $\|w\| = 1$. Hence, λ is an eigenvalue of problem (1.5), which is a contradiction. Combining the invariance of the degree under homotopies and Lemma 4.1 gives

$$\deg\big(I-\Psi_\lambda, B_r(0), 0\big) = \deg\big(I-G(1,\lambda\cdot), B_r(0), 0\big)$$
$$= \deg\big(I-G(0,\lambda\cdot), B_r(0), 0\big) = \deg\big(I-\lambda\Phi, B_r(0), 0\big).$$

Noting that Theorem 8.10 of [16] shows

$$\deg\big(I-\lambda\Phi, B_r(0), 0\big) = \begin{cases} 1 & \text{if } \lambda \in \big(0,\lambda_1\big), \\ -1 & \text{if } \lambda \in \big(\lambda_1, \lambda_1+\delta\big), \end{cases}$$

which implies the desired conclusion. □

Now, we present the proof of Theorem 1.4.

Proof of Theorem 1.4 (a) Let $\xi : \Omega \times [0, 2/d] \to \mathbb{R}$ be such that

$$f(x,s) = s + \xi(x,s)$$

with

$$\lim_{s\to 0^+} \frac{\xi(x,s)}{s} = 0$$

uniformly for $x \in \Omega$. Then, problem (1.1) is equivalent to

$$\begin{cases} -a\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) + b\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \lambda u + \lambda\xi(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Define

$$F(\lambda,u) = \lambda u + \lambda\xi(x,u) + a\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) - b\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

for any $(\lambda, u) \in \mathbb{R} \times X$. By some elementary calculations, we obtain that

$$F_u(\lambda,0)v = \lim_{t\to 0} \frac{F(\lambda, tv)}{t} = \lambda v + (a-b)\Delta v.$$

It follows that if $(\mu, 0)$ is a bifurcation point of problem (4.6), μ must be an eigenvalue of problem (1.5).

Consider the following problem

$$\begin{cases} -a\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) + b\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \lambda u + \lambda s\xi(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

for any $s \in [0,1]$. Clearly, problem (4.7) is equivalent to

$$u = \Psi(\lambda u + \lambda s\xi(x,u)) := F_\lambda(s,u).$$

According to Lemma 4.1, $F_\lambda : [0,1] \times X \longrightarrow X$ is completely continuous. Let

$$\widetilde{\xi}(x,w) = \max_{0\le s\le w} |\xi(x,s)| \text{ for any } x \in \Omega.$$

Then, $\widetilde{\xi}$ is nondecreasing with respect to w and

$$\lim_{w\to 0^+} \frac{\widetilde{\xi}(x,w)}{w} = 0. \tag{4.8}$$

It follows from (4.8) that

$$\left|\frac{\xi(x,u)}{\|u\|}\right| \le \frac{\widetilde{\xi}(x,u)}{\|u\|} \le \frac{\widetilde{\xi}\left(x,\|u\|_\infty\right)}{\|u\|} \le \frac{d}{2}\frac{\widetilde{\xi}\left(x,\frac{d\|u\|}{2}\right)}{\frac{d\|u\|}{2}} \to 0 \text{ as } \|u\| \to 0 \tag{4.9}$$

uniformly in $x \in \Omega$.

Noting (4.9) and an argument similar to that of Lemma 4.2, we may obtain that the Leray-Schauder degree $\deg\left(I - F_\lambda(s,\cdot), B_r(0), 0\right)$ is well defined for $\lambda \in \left(0, \lambda_1 + \delta\right) \setminus \left\{\lambda_1\right\}$. By the invariance of the degree under homotopies, we find that

$$\begin{aligned}\deg\left(I - H_\lambda, B_r(0), 0\right) &= \deg\left(I - F_\lambda(1,\cdot), B_r(0), 0\right) = \deg\left(I - F_\lambda(0,\cdot), B_r(0), 0\right)\\ &= \deg\left(I - \Psi_\lambda, B_r(0), 0\right),\end{aligned}$$

where $H_\lambda := F_\lambda(1,\cdot)$. It is seen from Lemma 4.2 that

$$\deg\left(I - H_\lambda, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in \left(0,\lambda_1\right), \\ -1 & \text{if } \lambda \in \left(\lambda_1, \lambda_1 + \delta\right). \end{cases}$$

By Theorem 4.12 of [23] (or Proposition 2.1 of [15]), there exists a continuum $\mathscr{C}$ of nontrivial solution of problem (1.1) bifurcating from $\left(\lambda_1, 0\right)$ which is either unbounded or $\mathscr{C} \cap \left(\mathbb{R} \setminus \left\{\lambda_1\right\} \times \{0\}\right) \neq \emptyset$. Since $u \equiv 0$ is the only solution of problem (1.1) for $\lambda = 0$ and 0 is not an eigenvalue of problem (1.5), $\mathscr{C} \cap (\{0\} \times X) = \emptyset$. Using Proposition 3.1, we have $u \ge 0$ in Ω for any $(\lambda, u) \in \mathscr{C}$.

We *claim* that $\mathscr{C} \cap \left(\mathbb{R} \setminus \left\{\lambda_1\right\} \times \{0\}\right) = \emptyset$. Otherwise, there exists a nontrivial solution sequence $\left(\lambda_n, u_n\right) \in \mathscr{C} \setminus \left\{\left(\lambda_1, 0\right)\right\}$ such that $\lambda_n \to \mu$ and $u_n \to 0$ as $n \to +\infty$. Set $w_n := u_n / \left\|u_n\right\|$, in view of (4.9) and reasoning as that of Lemma 4.1, we can show that $w_n \to w$ as $n \to +\infty$ and w verifies problem (1.5) with $\|w\| = 1$. It follows that $\mu = \lambda_1$, which is a contradiction. Therefore, $\mathscr{C}$ is unbounded. Moreover, using Proposition 3.1, we know that $u > 0$ in Ω for any $(\lambda, u) \in \mathscr{C} \setminus \left\{\left(\lambda_1, 0\right)\right\}$. We see from Theorem 1.2 that $\|u\| < 1$ for any fixed $(\lambda, u) \in \mathscr{C}$, which implies that the projection of $\mathscr{C}$ on $\mathbb{R}_+ := [0, +\infty)$ is unbounded.

Finally, we show the *asymptotic behavior* of u_λ as $\lambda \to +\infty$ for $\left(\lambda, u_\lambda\right) \in \mathscr{C} \setminus \left\{\left(\lambda_1, 0\right)\right\}$. Otherwise, there exist a constant $\delta > 0$ and $\left(\lambda_n, u_n\right) \in \mathscr{C} \setminus \left\{\left(\lambda_1, 0\right)\right\}$ with $\lambda_n \to +\infty$ as $n \to +\infty$ such that $\left\|u_n\right\|^2 \le 1 - \delta^2$ for any $n \in \mathbb{N}$. Our assumptions on f imply that there exists a positive constant $\rho > 0$ such that

$$\frac{f(x, u_n(x))}{u_n(x)} \ge \rho$$

for any $x \in \Omega$ and $n \in \mathbb{N}$. Let φ_1 be a positive eigenfunction associated to λ_1. Multiplying the first equation of problem (1.1) by φ_1, we obtain after integrations by parts that

$$
\begin{aligned}
\frac{\lambda_1 a}{\delta(a-b)}\int_\Omega u_n\varphi_1\,\mathrm{d}x =&\frac{a}{\delta}\int_\Omega \nabla u_n\nabla\varphi_1\,\mathrm{d}x\\
&\geq \int_\Omega\left(\frac{a}{\sqrt{1-\left|\nabla u_n\right|^2}}-\frac{b}{\sqrt{1+\left|\nabla u_n\right|^2}}\right)\nabla u_n\nabla\varphi_1\,\mathrm{d}x\\
&=\lambda_n\int_\Omega\frac{f\left(x,u_n\right)}{u_n}u_n\varphi_1\,\mathrm{d}x\geq\lambda_n\rho\int_\Omega u_n\varphi_1\,\mathrm{d}x.
\end{aligned}
$$

This yields $\lambda_n \leq \lambda_1 a/(\delta\rho(a-b))$, which contradicts the fact of $\lambda_n \to +\infty$.

(b) For any $n \in \mathbb{N}$, define

$$
f^n(x,s)=\begin{cases} ns, & s\in\left[0,\frac{1}{n}\right],\\ \left(f\left(x,\frac{2}{n}\right)-1\right)ns+2-f\left(x,\frac{2}{n}\right), & s\in\left(\frac{1}{n},\frac{2}{n}\right),\\ f(x,s), & s\in\left[\frac{2}{n},+\infty\right).\end{cases}
$$

It is direct to see that $\lim_{n\to+\infty}f^n(x,s)=f(x,s)$ and $f_0^n=n$. Consider the following approximation problem

$$
\begin{cases} -a\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)+b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=\lambda f^n(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{4.10}
$$

By (a), there exists a sequence unbounded continua $\mathscr{C}_n$ of the set of nontrivial solutions of problem (4.10) emanating from $\left(\lambda_1/n,0\right)$ and joining to $(+\infty,1)$ such that

$$
\mathscr{C}_n\subseteq\left(\left(\mathbb{R}_+\times P\right)\cup\left\{\left(\lambda_1/n,0\right)\right\}\right).
$$

Taking $z^*=(0,0)$, then there must hold $z^*\in\liminf_{n\to+\infty}\mathscr{C}_n$. The compactness of Ψ implies that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right)\cap\mathbb{B}_R$ is pre-compact, where $\mathbb{B}_R=\{z\in\mathbb{R}\times X:\|z\|<R\}$ for any $R>0$. By Theorem 2.1 of [11], $\mathscr{C}=\limsup_{n\to+\infty}\mathscr{C}_n$ is unbounded and connected such that $z^*\in\mathscr{C}$ and $(+\infty,1)\in\mathscr{C}$. From the definition of superior limit (see [26]) and the continuity of Ψ, it is not difficult to see that u is a solution of problem (1.1) for any $(\lambda,u)\in\mathscr{C}$. Obviously, u is nonnegative for any $(\lambda,u)\in\mathscr{C}$. By the definition of inferior limit (see [26]), we can derive that $\mathscr{C}\cap((0,+\infty)\times\{0\})=\emptyset$. Therefore, by virtue of Proposition 3.1, we have that $u>0$ in Ω for any $(\lambda,u)\in\mathscr{C}\setminus\{(0,0)\}$.

(c) For any $n\in\mathbb{N}$, define

$$
f_n(x,s)=\begin{cases} \frac{1}{n}s, & s\in\left[0,\frac{1}{n}\right],\\ \left(f\left(x,\frac{2}{n}\right)-\frac{1}{n^2}\right)ns+2\frac{1}{n^2}-f\left(x,\frac{2}{n}\right), & s\in\left(\frac{1}{n},\frac{2}{n}\right),\\ f(x,s), & s\in\left[\frac{2}{n},+\infty\right).\end{cases}
$$

Consider the following problem

$$\begin{cases} -a\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) + b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f_n(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By modifying the argument given in [14, Theorem 1.3] with obvious changes, we can verify the desired conclusion. □

Finally, we show a result is concerned with the nonexistence of positive solution.

Theorem 4.1 *Assume that there exists a positive constant ϱ such that*

$$\frac{f(x,s)}{s} \le \varrho$$

for any $s \in (0, d/2]$ and a.e. $x \in \Omega$. Then, there exists $\varrho_* > 0$ such that problem (1.1) has no any positive classical solution for $\lambda \in (0, \varrho_*)$.

Proof Suppose that u is a positive classical solution of problem (1.1) with some $\lambda > 0$. Multiplying the first equation of problem (1.1) by u, in view of Theorem 1.1, we obtain after integrations by parts that

$$\begin{aligned}(a-b)\int_\Omega |\nabla u|^2\,\mathrm{d}x &\le \int_\Omega \left(\frac{(a-b)|\nabla u|^2}{\sqrt{1-|\nabla u|^2}} + \frac{b|\nabla u|^2}{\sqrt{1-|\nabla u|^2}} - \frac{b|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\right)\mathrm{d}x \\ &= \int_\Omega \left(\frac{a|\nabla u|^2}{\sqrt{1-|\nabla u|^2}} - \frac{b|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\right)\mathrm{d}x = \lambda\int_\Omega \frac{f(x,u)}{u}u^2\,\mathrm{d}x \\ &\le \lambda\varrho\int_\Omega u^2\,\mathrm{d}x \le \frac{\lambda\varrho(a-b)}{\lambda_1}\int_\Omega |\nabla u|^2\,\mathrm{d}x,\end{aligned}$$

which follows that $\lambda \ge \lambda_1/\varrho := \varrho_*$. □

5 Radial symmetry of solutions

Our purpose of this section is to provide sufficient conditions on the prescription function to ensure that any eventual positive solution of problem (1.1) must be radially symmetric when Ω is the unit ball B. More precisely, we shall show the following result.

Theorem 5.1 *Assume that $f(x, s)$ is continuous on $\overline{B} \times I_d$, has continuous first derivative with respect to s, is radially symmetric, decreasing on $(0, 1)$ with respect to the first variable and satisfies $f(x, 0) \ge 0$ on ∂B. Then, any positive solution $u \in C^2\left(\overline{B}\right)$ of problem (1.1) is radially symmetric and monotone decreasing about the origin.*

Proof For convenience, we assume $\lambda = 1$. Let u be any positive solution of problem (1.1). We infer from Theorem 1.1 that $|\nabla u| \le \theta < 1$ on $\overline{B}$ for some positive constant θ. Define the truncated function as follows

$$\varphi(t) = \begin{cases} \frac{1}{\sqrt{1-t}} & \text{if } t \in \left[0, \theta^2\right], \\ \alpha(t) & \text{if } t \in \left(\theta^2, 1\right), \\ c & \text{if } t \geq 1, \end{cases}$$

where the function α and the constant c are such that $\varphi \in C^1\left(\mathbb{R}_+\right)$ is increasing. We observe that both φ and φ' are bounded on $\mathbb{R}_+$.

Further, consider the following problem

$$\begin{cases} -a\mathrm{div}\left(\varphi\left(|\nabla u|^2\right)\nabla u\right) + b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(x,u) & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Set

$$F\left(\nabla u, D^2 u\right) := -a\mathrm{div}\left(\varphi\left(|\nabla u|^2\right)\nabla u\right) + b\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

It is straightforward to see that

$$F\left(x, \nabla u, D^2 u\right) = -\sum_{i,j=1}^{N} a^{ij}(x) u_{ij},$$

where

$$a^{ij}(x) = \left(a\varphi\left(|\nabla u|^2\right) - \frac{b}{\sqrt{1+|\nabla u|^2}}\right)\delta_{ij} + \left(2a\varphi'\left(|\nabla u|^2\right) + \frac{b}{\left(1+|\nabla u|^2\right)^{3/2}}\right)u_i u_j.$$

We can easily see that

$$a\varphi\left(|\nabla u|^2\right) - \frac{b}{\sqrt{1+|\nabla u|^2}} \geq a - b,\ \ 2a\varphi'\left(|\nabla u|^2\right) + \frac{b}{\left(1+|\nabla u|^2\right)^{3/2}} > 0.$$

Combining the arguments of Theorem 1.2 and Proposition 3.1, we conclude that F is strictly elliptic. Consequently, the desired conclusions can be deduced from Corollary 2.1 of [17].

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