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# Helicoids and catenoids in $M \times \mathbb{R}$

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**Abstract**

Given an arbitrary $C^\infty$ Riemannian manifold $M^n$, we consider the problem of introducing and constructing minimal hypersurfaces in $M \times \mathbb{R}$ which have the same fundamental properties of the standard helicoids and catenoids of Euclidean space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$. Such hypersurfaces are defined by imposing conditions on their height functions and horizontal sections and then called *vertical helicoids* and *vertical catenoids*. We establish that vertical helicoids in $M \times \mathbb{R}$ have the same fundamental uniqueness properties of the helicoids in $\mathbb{R}^3$. We provide several examples of properly embedded vertical helicoids in the case where $M$ is one of the simply connected space forms. Vertical helicoids which are entire graphs of functions on $\mathrm{Nil}_3$ and $\mathrm{Sol}_3$ are also presented. We show that vertical helicoids of $M \times \mathbb{R}$ whose horizontal sections are totally geodesic in $M$ are locally given by a "twisting" of a fixed totally geodesic hypersurface of $M$. We give a local characterization of hypersurfaces of $M \times \mathbb{R}$ which have the gradient of their height functions as a principal direction. As a consequence, we prove that vertical catenoids exist in $M \times \mathbb{R}$ if and only if $M$ admits families of isoparametric hypersurfaces. If so, properly embedded vertical catenoids can be constructed through the solutions of a certain first-order linear differential equation. Finally, we give a complete classification of the hypersurfaces of $M \times \mathbb{R}$ whose angle function is constant.

**Keywords** Helicoid · Catenoid · Product space

**Mathematics Subject Classification** 53B25 (primary) · 53C24 · 53C42 (secondary)

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## 1 Introduction

In this paper, we address the problem of defining and constructing minimal hypersurfaces in $M \times \mathbb{R}$ with special properties, where $M^n$ is an arbitrary $C^\infty$ Riemannian manifold. We will focus our attention on those fundamental properties of the standard helicoids and catenoids of Euclidean space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, so that the corresponding minimal hypersurfaces of $M \times \mathbb{R}$ will be called *vertical helicoids* and *vertical catenoids*.

More specifically, these hypersurfaces will be introduced by imposing conditions on their horizontal sections (intersections with $M \times \{t\}$, $t \in \mathbb{R}$), and also on the trajectories of the gradient of their height functions (*height trajectories*, for short). Vertical helicoids, for instance, are defined as those hypersurfaces of $M \times \mathbb{R}$ whose horizontal sections are minimal hypersurfaces of $M \times \{t\}$, and whose height trajectories are asymptotic lines. Vertical catenoids, in turn, have nonzero constant mean curvature hypersurfaces as horizontal sections, and lines of curvature as height trajectories.

In this setting, we show that vertical helicoids of $M \times \mathbb{R}$ have all the classical uniqueness properties of the standard helicoids of $\mathbb{R}^3$. Namely, they are minimal hypersurfaces of $M \times \mathbb{R}$ and, as such, they are the only ones which are foliated by horizontal minimal hypersurfaces. They are also the only minimal local graphs of harmonic functions (defined on domains in $M$), and the only minimal non-totally geodesic hypersurfaces of $M \times \mathbb{R}$ whose spacelike pieces are maximal with respect to the standard Lorentzian product metric of $M \times \mathbb{R}$.

This last property extends the analogous classical result, set in Lorentzian space $\mathbb{L}^3$, established by O. Kobayashi [16]. In our approach, we briefly consider the class of hypersurfaces of $M \times \mathbb{R}$ whose mean curvatures with respect to both the Riemannian and Lorentzian metrics of $M \times \mathbb{R}$ coincide. We call them *mean isocurved*. These hypersurfaces have been studied by Albujer-Caballero [3] in the case where the ambient space is $\mathbb{L}^3$ (see [1] as well). Actually, during the preparation of this paper, we became acquainted with the recent works by Alarcón-Alias-Santos [2] and Albujer-Caballero [4] which have some overlapping with ours on this subject. Mean isocurved surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ have also been considered by Kim et al in [15].

Concerning examples of vertical helicoids in $M \times \mathbb{R}$, we show that they can be constructed by considering one-parameter groups of isometries of $M$ acting on suitable minimal hypersurfaces. When $M$ is one of the simply connected space forms, this method allows us to construct properly embedded minimal vertical helicoids in $M \times \mathbb{R}$ which are foliated by vertical translations of totally geodesic hypersurfaces of $M$. We also construct properly embedded vertical helicoids in $\mathbb{R}^n \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ which are foliated by vertical translations of helicoids of $\mathbb{R}^n$ and $\mathbb{H}^3$, respectively. In the same way, we construct vertical helicoids in $\mathbb{S}^3_\delta \times \mathbb{R}$, where $\mathbb{S}^3_\delta$ is a Berger sphere. Finally, we obtain a family of properly embedded minimal vertical helicoids in $\mathbb{S}^{2n+1} \times \mathbb{R}$ which are foliated by $2n$-dimensional Clifford tori, and also a corresponding family of vertical helicoids in $\mathbb{R}^{2n+2} \times \mathbb{R}$ (previously constructed by Choe and Hoppe [7]), whose horizontal sections are the cones of these tori in $\mathbb{R}^{2n+2}$.

Other examples of vertical helicoids that we give are graphs of harmonic and horizontally homothetic functions defined on domains of certain manifolds $M$, such as the Nil and Sol three-dimensional spaces (see Sect. 4.1). We remark that, all the vertical helicoids presented here, graphs or not, contain spacelike zero mean isocurved open sets.

We also give a local characterization of vertical helicoids of $M \times \mathbb{R}$ with totally geodesic horizontal sections and nonvanishing angle function by showing that each of its points has a neighborhood which can be expressed as a "twisting" of a totally geodesic hypersurface of $M$ (see Sect. 4.2 for more details).

Regarding vertical catenoids in $M \times \mathbb{R}$, their study naturally leads to the consideration of a broader class of hypersurfaces of $M \times \mathbb{R}$; those which have the gradient of their height functions as a principal direction. These hypersurfaces have been given a local characterization by R. Tojeiro [22] assuming that $M$ is one of the simply connected space forms. Here, we extend this result to general products $M \times \mathbb{R}$ and conclude that a necessary and sufficient condition for the existence of minimal or constant mean curvature (CMC) hypersurfaces in $M \times \mathbb{R}$ with this property (in particular, vertical catenoids) is that $M$ admits families of isoparametric hypersurfaces.

This extension of Tojeiro's result, in fact, provides a way of constructing such minimal and CMC hypersurfaces (as long as they are admissible) by solving a first-order linear differential equation. This can be performed, for instance, when $M$ is any of the simply connected space forms, a Damek–Ricci space or any of the simply connected 3-homogeneous manifolds with isometry group of dimension 4: $\mathbb{E}(k, \tau)$, $k - 4\tau^2 \neq 0$. This result will also be applied for constructing properly embedded vertical catenoids in $M \times \mathbb{R}$ when $M$ is a Hadamard manifold or the sphere $\mathbb{S}^n$. As a further application, we give a complete classification of hypersurfaces of $M \times \mathbb{R}$ whose angle function is constant.

The paper is organized as follows. In Sect. 2, we set some notation and formulae. In Sect. 3, we introduce mean isocurved hypersurfaces and establish some basic lemmas. We discuss on vertical helicoids in Sect. 4. In Sect. 5, we consider hypersurfaces of $M \times \mathbb{R}$ which have the gradient of their height functions as a principal direction. Finally, in Sect. 6, we discuss on vertical catenoids.

## 2 Preliminaries

Throughout this paper, $M$ will denote an arbitrary $n(\geq 2)$-dimensional $C^\infty$ orientable Riemannian manifold. For such an $M$, we will consider the product manifold $M \times \mathbb{R}$ with its standard differentiable structure. We will set

$$T(M \times \mathbb{R}) = TM \oplus T\mathbb{R}$$

for the tangent bundle of $M \times \mathbb{R}$, where $TM$ and $T\mathbb{R}$ denote the tangent bundles of $M$ and $\mathbb{R}$, respectively. We will endow $M \times \mathbb{R}$ with the Riemannian product metric:

$$\langle\, , \,\rangle = \langle\, , \,\rangle_M + dt^2.$$

We shall write $\pi_M$ and $\pi_{\mathbb{R}}$ for the projection of $M \times \mathbb{R}$ on its first and second factors, respectively, and $\partial_t$ for the gradient of $\pi_{\mathbb{R}}$ with respect to the Riemannian metric $\langle\, , \,\rangle$. We remark that $\partial_t$ is a parallel field on $M \times \mathbb{R}$.

Let $\Sigma$ be an orientable hypersurface of $M \times \mathbb{R}$. Given a unit normal field $N \in T\Sigma^\perp \subset T(M \times \mathbb{R})$, we will denote by $A$ the shape operator of $\Sigma$ relative to $N$, i.e.,

$$AX = -\overline{\nabla}_X N,$$

where $\overline{\nabla}$ stands for the Levi-Civita connection of $M \times \mathbb{R}$. The gradient of a differentiable function $\zeta$ on $\Sigma$ will be denoted by $\nabla\zeta$.

The *height function* $\xi$ and the *angle function* $\Theta$ of $\Sigma$ are defined as

$$\xi := \pi_{\mathbb{R}}|_{\Sigma} \quad \text{and} \quad \Theta := \langle N, \partial_t \rangle.$$

Regarding these functions, the following fundamental identities hold:

$$\nabla\xi = \partial_t - \Theta N \quad \text{and} \quad \nabla\Theta = -A\nabla\xi, \tag{1}$$

where the second one follows from the fact that $\partial_t$ is parallel in $M \times \mathbb{R}$. We point out that $\Theta \in [-1, 1]$, and that $x \in \Sigma$ is a critical point of the height function $\xi$ if and only if $\Theta^2(x) = 1$. If so, we say that $x$ is a *horizontal point* of $\Sigma$. Any field $X \in TM \subset T(M \times \mathbb{R})$ will be called *horizontal* as well.

## 3 Basic lemmas

Given a product manifold $M \times \mathbb{R}$, for each $t \in \mathbb{R}$, we will call the submanifold $M_t := M \times \{t\}$ a *horizontal section* of $M \times \mathbb{R}$. If $\Sigma$ intersects a horizontal section $M_t$ transversally, we call the set

$$\Sigma_t := M_t \cap \Sigma$$

a *horizontal section* of the hypersurface $\Sigma$.

Notice that, for all $t \in \mathbb{R}$, $M_t$ is isometric to $M$, and that any horizontal section $\Sigma_t$ is a hypersurface of $M_t$. In this setting, it is easily checked that

$$\eta := \phi(N - \Theta\partial_t), \quad \phi = -(1 - \Theta^2)^{-1/2}, \tag{2}$$

is a well defined unit normal field to $\Sigma_t$.

Now, denote the shape operator of $\Sigma_t$ with respect to $\eta$ by $A_\eta$, and set $H$ and $H_{\Sigma_t}$ for the (non-normalized) mean curvature functions of $\Sigma$ and $\Sigma_t$, respectively.

**Lemma 1** *Let $\Sigma_t$ be a horizontal section of a hypersurface $\Sigma$ of $M \times \mathbb{R}$. Then,*

$$\langle A_\eta X, Y \rangle = \phi\langle AX, Y \rangle \quad \forall X, Y \in T\Sigma_t.$$

*As a consequence, for $T = \nabla\xi/\|\nabla\xi\|$, the following equality holds along $\Sigma_t$:*

$$H_{\Sigma_t} = \phi(H - \langle AT, T \rangle). \tag{3}$$

***Proof*** We have that $M_t = M \times \{t\}$ is totally geodesic in $M \times \mathbb{R}$. Hence, its Riemannian connection coincides with the restriction of the Riemannian connection $\overline{\nabla}$ of $M \times \mathbb{R}$ to $TM_t \times TM_t$. Therefore, for all $X \in T\Sigma_t$, we have

$$A_\eta X = -\overline{\nabla}_X \eta = -\overline{\nabla}_X \phi(N - \Theta\partial_t) = -X(\phi)(N - \Theta\partial_t) + \phi(AX + X(\Theta)\partial_t).$$

Thus, for all $Y \in T\Sigma_t = TM_t \cap T\Sigma$,

$$\langle A_\eta X, Y \rangle = \phi\langle AX, Y \rangle.$$

Now, in a suitable neighborhood $U \subset \Sigma$ of an arbitrary point on $\Sigma_t$, consider an orthonormal frame $\{X_1 \; \ldots, X_{n-1}, T\}$ such that $X_1, \ldots X_{n-1}$ are all tangent to $\Sigma_t$. Then, on $U \cap \Sigma_t$, we have

$$H_{\Sigma_t} = \sum_{i=1}^{n-1} \langle A_\eta X_i, X_i \rangle = \phi \sum_{i=1}^{n-1} \langle AX_i, X_i \rangle = \phi(H - \langle AT, T \rangle),$$

which concludes the proof. □

### 3.1 Mean isocurved hypersurfaces

Let us consider in $M \times \mathbb{R}$ the *Lorentzian* product metric, which is defined as

$$\langle\, , \,\rangle_L := \langle\, , \,\rangle_M - dt^2.$$

This metric relates to the Riemannian metric $\langle\, , \,\rangle$ of $M \times \mathbb{R}$ through the identity

$$\langle X, Y \rangle_L = \langle X, Y \rangle - 2\langle X, \partial_t \rangle \langle Y, \partial_t \rangle, \tag{4}$$

which, as one can verify, is valid for all $X, Y \in T(M \times \mathbb{R})$.

Denote by $\Sigma_L := (\Sigma, \langle\, , \,\rangle_L)$ a hypersurface $\Sigma$ of $M \times \mathbb{R}$ with the induced Lorentzian metric of $M \times \mathbb{R}$. We say that $\Sigma$ is *spacelike* if $\Sigma_L$ is a Riemannian manifold, that is, the Lorentzian metric on $\Sigma$ is positive definite. It is easily checked that $\Sigma$ is spacelike if and only if $\langle Z, Z \rangle_L < 0$ for all nonzero local field $Z \in T\Sigma_L^\perp$. Also, any spacelike hypersurface of $M \times \mathbb{R}$ is necessarily orientable.

Assuming $\Sigma \subset M \times \mathbb{R}$ spacelike, choose a unit normal $N_L$ to $\Sigma_L$, that is,

$$\langle N_L, N_L \rangle_L = -1 \quad \text{and} \quad \langle X, N_L \rangle_L = 0 \ \forall X \in T\Sigma.$$

It is a well-known fact that the connections of $M \times \mathbb{R}$ with respect to the Riemannian and Lorentzian metrics coincide. So, keeping the notation of Sect. 2, we define the Lorentzian shape operator of $\Sigma_L$ with respect to $N_L$ as

$$A_L X := -\overline{\nabla}_X N_L. \tag{5}$$

Finally, the (non-normalized) Lorentzian mean curvature $H_L$ of $\Sigma_L$ is defined as

$$H_L := -\operatorname{trace} A_L.$$

**Definition 1** A spacelike hypersurface $\Sigma \subset M \times \mathbb{R}$ is said to be *mean isocurved* if its Riemannian and Lorentzian mean curvature functions, $H$ and $H_L$, coincide. When $H = H_L = 0$, we say that $\Sigma$ is *zero mean isocurved.*

Let us consider the following map

$$\Phi(X) = X - 2\langle X, \partial_t \rangle \partial_t, \ \ X \in T(M \times \mathbb{R}),$$

which is easily seen to be an involution, that is, $\Phi \circ \Phi$ is the identity map of $T(M \times \mathbb{R})$. Moreover, for all $X, Y \in T(M \times \mathbb{R})$, the following identities hold:

$$\langle \Phi(X), Y \rangle = \langle X, Y \rangle_L \quad \text{and} \quad \langle \Phi(X), Y \rangle_L = \langle X, Y \rangle. \tag{6}$$

Given an oriented hypersurface $\Sigma \subset M \times \mathbb{R}$ with unit normal $N$, it follows from the second relation in (6) that $\Phi(N)$ is a Lorentzian normal field on $\Sigma$. Indeed,

$$\langle \Phi(N), X\rangle_L = \langle N, X\rangle = 0 \;\; \forall X \in T\Sigma.$$

Moreover, considering also the equality (4), we have

$$\langle \Phi(N), \Phi(N)\rangle_L = \langle N, \Phi(N)\rangle = \langle N, N\rangle_L = 1 - 2\Theta^2,$$

from which we conclude that $\Sigma$ *is spacelike if and only if* $2\Theta^2 > 1$. If so, set

$$N_L := \mu\Phi(N), \quad \mu := \frac{-1}{\sqrt{2\Theta^2 - 1}} < 0,$$

and write $A_L$ for the shape operator of $\Sigma_L$ with respect to $N_L$.

**Lemma 2** *Let $\Sigma$ be a spacelike hypersurface of $M \times \mathbb{R}$ with no horizontal points. With the above notation, the following identities hold:*

(i) $\langle A_L X, Y\rangle_L = \mu\langle AX, Y\rangle \;\; \forall X, Y \in T\Sigma.$
(ii) $H_L + \mu H = \mu(1 - \mu^2)\langle AT, T\rangle, \;\; T = \nabla\xi/\|\nabla\xi\|.$

***Proof*** Given $X, Y \in TM$, one has

$$\langle A_L X, Y\rangle_L = \langle \overline{\nabla}_X Y, N_L\rangle_L = \langle \overline{\nabla}_X Y, \mu\Phi(N)\rangle_L = \mu\langle \overline{\nabla}_X Y, N\rangle = \mu\langle AX, Y\rangle,$$

which proves (i).

Now, let us consider a point $x \in \Sigma$ and a basis $\mathfrak{B} = \{X_1, \dots, X_n\}$ of $T_x\Sigma$ which is orthonormal with respect to the Riemannian metric $\langle\, ,\, \rangle$. Since $x$ is non-horizontal, we can assume that $X_1, \dots, X_{n-1}$ are horizontal, i.e., tangent to $M$, and $X_n = T$. Hence, by (4), $\{X_1, \dots, X_{n-1}\}$ is orthonormal with respect to the Lorentzian metric $\langle\, ,\, \rangle_L$, and $\langle X_i, T\rangle_L = 0 \; \forall i = 1, \dots, n-1$.

Denote by $[a_{ij}]$ and $[\ell_{ij}]$, the matrices of the shape operators $A$ and $A_L$, respectively, with respect to the basis $\mathfrak{B}$. From (i), we have

$$\ell_{ij} = \langle A_L X_i, X_j\rangle_L = \mu\langle AX_i, X_j\rangle = \mu a_{ij} \;\; \forall i,j = 1, \dots, n-1. \tag{7}$$

Also, for any index $j = 1, \dots, n$, one has

$$\mu a_{nj} = \mu\langle AX_j, T\rangle = \langle A_L X_j, T\rangle_L = \sum_{i=1}^{n} \ell_{ij}\langle X_i, T\rangle_L = \ell_{nj}\langle T, T\rangle_L. \tag{8}$$

However, by (1) and (4),

$$\langle T, T\rangle_L = 1 - 2\langle T, \partial_t\rangle^2 = 2\Theta^2 - 1 = \frac{1}{\mu^2}\cdot$$

This, together with (8), yields

$$\ell_{nj} = \mu^3 a_{nj} \quad \forall j = 1, \dots, n. \tag{9}$$

Putting (7) and (9) together, we have

$$[\ell_{ij}] = \mu \begin{bmatrix} a_{11} & \cdots & \mu^2 a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & \mu^2 a_{in} \\ \vdots & & \vdots \\ \mu^2 a_{n1} & \cdots & \mu^2 a_{nn} \end{bmatrix},$$

which implies that

$$\operatorname{trace}[\ell_{ij}] = \mu(\operatorname{trace}[a_{ij}] + (\mu^2 - 1)a_{nn}). \tag{10}$$

Since we have $a_{nn} = \langle AT, T\rangle$, $H_L = -\operatorname{trace}[\ell_{ij}]$, and $H = \operatorname{trace}[a_{ij}]$, the identity (10) clearly implies (ii).

The following result extends [3, Theorem 4], set in Lorentzian space $\mathbb{L}^3$, to hypersurfaces in $M \times \mathbb{R}$.

**Corollary 1** *Let $\Sigma$ be a mean isocurved hypersurface of $M \times \mathbb{R}$. Then, its second fundamental form $\sigma$ is nowhere definite. Furthermore, $\sigma$ is semi-definite at $x \in \Sigma$ if and only if $\Sigma$ is totally geodesic at $x$.*

***Proof*** Let us denote by $C \subset \Sigma$ the set of critical points of the height function $\xi$ of $\Sigma$. Keeping the notation of the proof of the preceding lemma, and considering the equality (10), we have that $H = \mu(1-\mu)a_{nn}$ on $\Sigma - C$, for $H_L = H$. Thus,

$$\sum_{i=1}^{n-1} a_{ii} + (1 + \mu(\mu - 1))a_{nn} = 0. \tag{11}$$

However, $1 + \mu(\mu - 1) > 0$ and $a_{ii} = \langle AX_i, X_i\rangle = \sigma(X_i, X_i)$, $i = 1, \dots, n$. Hence, the equality (11) implies that, at a point $x$ in the closure of $\Sigma - C$ in $\Sigma$, $\sigma$ is neither definite nor semi-definite, unless, in the latter case, it vanishes.

## 4 Vertical helicoids in $M \times \mathbb{R}$.

Inspired by some fundamental properties of the standard helicoids of $\mathbb{R}^3$ (see Example 1 below), we introduce in this section the concept of vertical helicoid in $M \times \mathbb{R}$. We shall establish the uniqueness properties of these hypersurfaces and present a variety of examples, as we mentioned in the introduction. In addition, we will characterize the vertical helicoids which are graphs of functions on $M$ and give a local characterization of vertical helicoids $\Sigma$ whose horizontal sections $\Sigma_t$ are totally geodesic in $M_t$.

**Definition 2** Let $\Sigma$ be a hypersurface of $M \times \mathbb{R}$ with no horizontal points and nonconstant angle function. We say that $\Sigma$ is a *vertical helicoid* if it satisfies the following conditions:

- The horizontal sections $\Sigma_t \subset \Sigma$ are minimal hypersurfaces of $M \times \{t\}$.
- $\nabla\xi$ is an asymptotic direction of $\Sigma$, that is, $\langle A\nabla\xi, \nabla\xi\rangle = 0$ on $\Sigma$.

***Remark 1*** Considering the standard helicoids in $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, one could expect that a right extension of this concept to the context of products $M \times \mathbb{R}$ should ask for the horizontal sections to be totally geodesic, since the horizontal sections of the helicoids in $\mathbb{R}^3$ are straight lines. However, as our results and examples shall show, the appropriate condition to be imposed to the horizontal sections is, in fact, minimality, as in the above definition.

***Remark 2*** The identity $\nabla\Theta = -A\nabla\xi$ implies that $\nabla\xi$ is an asymptotic direction of $\Sigma$ if and only if the equality $\langle \nabla\Theta, \nabla\xi \rangle = 0$ holds on $\Sigma$. In this case, we have that $\Theta$ is constant along any trajectory $\gamma(s)$ of $\nabla\xi$. However, $\langle \nabla\xi, \partial_t \rangle = 1 - \Theta^2$, which gives that the tangent directions $\gamma'(s)$ make a constant angle with the vertical direction $\partial_t$. Therefore, considering the concept of helix in $\mathbb{R}^3$ as a curve which makes a constant angle with a given direction, we can extend it to curves in $M \times \mathbb{R}$ in an obvious way and conclude that the trajectories of $\nabla\xi$ on a vertical helicoid in $M \times \mathbb{R}$ are *vertical helices*.

In what follows, let $Q_c^n$ denote the simply connected $n$-space form of constant sectional curvature $c \in \{0, 1, -1\}$, that is, the Euclidean space $\mathbb{R}^n$ ($c = 0$), the $n$-sphere $\mathbb{S}^n$ ($c = 1$), or the hyperbolic space $\mathbb{H}^n$ ($c = -1$).

***Example 1*** (*Helicoids in* $Q_c^2 \times \mathbb{R}$) Consider the following parametrization of the standard vertical helicoid $\Sigma$ of $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ with *pitch* $a > 0$,

$$\Psi(x, y) = (x \cos y, x \sin y, ay), \ \ (x, y) \in \mathbb{R}^2.$$

As its Riemannian unit normal field, we can choose

$$N = \frac{1}{\sqrt{x^2 + a^2}}(a \sin y, -a \cos y, x),$$

which gives $\Theta = x/(x^2 + a^2)^{1/2}$.

Since $\Psi$ is an orthogonal parametrization and $\Theta$ depends only on $x$, we have that $\nabla\Theta$ is parallel to $\Psi_x = (\cos y, \sin y, 0)$. In particular,

$$\langle \nabla\Theta, \nabla\xi \rangle = \langle \nabla\Theta, \partial_t \rangle = 0.$$

Hence, $\nabla\xi$ is an asymptotic direction of $\Sigma$.

We also have that all horizontal sections of $\Sigma$ are straight lines. Therefore, $\Sigma$ is a vertical helicoid as in Definition 2. Moreover, from the equality

$$2\Theta^2 - 1 = \frac{x^2 - a^2}{x^2 + a^2},$$

we conclude that the open subset $\Sigma' = \{\Psi(x, y) \in \Sigma \,;\, |x| > a\}$ is spacelike and, as is well known, zero mean isocurved (see, e.g., [16]).

Considering the standard inclusions $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and $\mathbb{H}^2 \hookrightarrow \mathbb{L}^3$, we can apply an analogous reasoning to the parametrizations (see, e.g., [8, Section 4]):

$$\Psi_{\rm sph}(x, y) = (\cos x \cos y, \cos x \sin y, \sin x, ay) \in \mathbb{S}^2 \times \mathbb{R};$$

$$\Psi_{\rm hyp}(x, y) = (\sinh x \cos y, \sinh x \sin y, \cosh x, ay) \in \mathbb{H}^2 \times \mathbb{R};$$

and conclude that their images are vertical helicoids in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, respectively. They are both minimal surfaces containing open spacelike zero mean isocurved subsets, as verified in [15].

We prove now, as suggested by the above examples, that vertical helicoids in product spaces $M \times \mathbb{R}$ are minimal hypersurfaces. As such, except for some constant angle hypersurfaces, they are the only ones foliated by horizontal minimal hypersurfaces. Moreover, spacelike pieces of vertical helicoids (if any) are zero mean isocurved hypersurfaces in $M \times \mathbb{R}$, and they are unique with respect to this property as well.

**Theorem 1** *Let $\Sigma$ be a hypersurface of $M \times \mathbb{R}$ with no horizontal points and nonconstant angle function. Then, the following statements are equivalent:*

(i) $\Sigma$ *is a vertical helicoid.*
(ii) $\Sigma$ *and all the horizontal sections $\Sigma_t$ are minimal hypersurfaces.*

If, in addition, $\Sigma$ is spacelike, then both (i) and (ii) are equivalent to:

(iii) $\Sigma$ *is zero mean isocurved.*

***Proof*** (i) $\Rightarrow$ (ii): Since we are assuming that $\Sigma$ is a vertical helicoid, we have $H_{\Sigma_t} = 0$ for all horizontal sections $\Sigma_t \subset \Sigma$, and $\langle A\nabla\xi, \nabla\xi\rangle = 0$ on $\Sigma$. Thus, from the identity (3) in Lemma 1, $H = 0$, that is, $\Sigma$ is minimal.

(ii) $\Rightarrow$ (i): Now, we have $H = H_{\Sigma_t} = 0$ for any horizontal section $\Sigma_t \subset \Sigma$. In this case, (3) yields $\langle AT, T\rangle = 0$, which implies that $\nabla\xi$ is an asymptotic direction, that is, $\Sigma$ is a vertical helicoid.

(ii) $\Rightarrow$ (iii): We have $H = 0$ and, as above, $\langle AT, T\rangle = 0$. Hence, by Lemma 2-(iii), $H_L = 0$, i.e., $\Sigma$ is zero mean isocurved.

(iii) $\Rightarrow$ (ii): From $H = H_L = 0$ and Lemma 2-(ii), one has $\langle AT, T\rangle = 0$. This, together with identity (3), gives that the horizontal sections $\Sigma_t \subset \Sigma$ are minimal hypersurfaces of $M \times \{t\}$. Hence, $\Sigma$ is a vertical helicoid.

Vertical helicoids can be constructed by "twisting" minimal hypersurfaces, as shown in the following examples.

***Example 2*** (*Twisted planes in* $\mathbb{R}^3 \times \mathbb{R}$) Given $a, k > 0$, consider the map

$$\Psi(x, y, s) := \begin{bmatrix} \cos ks & -\sin ks & 0 & 0 \\ \sin ks & \cos ks & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ y \\ 0 \end{bmatrix} + a \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix}, \quad (x, y, s) \in \mathbb{R}^3,$$

which we call a *vertical twisting* of the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ in $\mathbb{R}^3 \times \mathbb{R}$. It is easily verified that $\Psi$ is a parametrization of a properly embedded hypersurface $\Sigma$ of $\mathbb{R}^3 \times \mathbb{R}$. Also, direct computations give that

$$N = \frac{(a \sin ks, -a \cos ks, 0, kx)}{\sqrt{a^2 + (kx)^2}}$$

is a unit normal field on $\Sigma$. In particular, $\Theta = kx/\sqrt{a^2+(kx)^2}$ depends only on $x$ and $\Theta^2 \neq 1$, that is, $\xi$ has no critical points on $\Sigma$. Also, the inverse matrix $[g^{ij}]$ of the first fundamental form of $\Sigma$ in this parametrization is

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a^2+(kx)^2} \end{bmatrix}.$$

Therefore,

$$\nabla\Theta = \frac{\partial\Theta}{\partial x}\frac{\partial\Psi}{\partial x} = \frac{\partial\Theta}{\partial x}(\cos ks, \sin ks, 0, 0) \Rightarrow \langle\nabla\Theta, \nabla\xi\rangle = \langle\nabla\Theta, \partial_t\rangle = 0.$$

Thus, $\Sigma$ is a (minimal) vertical helicoid, since its horizontal sections $\Sigma_t$ are planes of $\mathbb{R}^3 \times \{t\}$. Moreover, its angle function $\Theta$ satisfies

$$2\Theta^2 - 1 = \frac{(kx)^2 - a^2}{(kx)^2 + a^2},$$

which implies that the nonempty open subset $\Sigma'$ of $\Sigma$ given by

$$\Sigma' := \{\Psi(x, y, s) \in \Sigma\,;\, |x| > a/k\}$$

is spacelike. So, by Theorem 1, $\Sigma'$ is zero mean isocurved in $\mathbb{R}^3 \times \mathbb{R}$.

***Example 3*** (*Twisted helicoids in* $\mathbb{R}^n \times \mathbb{R}$) Let us consider now the map

$$\Psi(x, y, s) := \begin{bmatrix} \cos ks & -\sin ks & 0 & 0 \\ \sin ks & \cos ks & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x\cos y \\ x\sin y \\ y \\ 0 \end{bmatrix} + a\begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix}, \quad (x, y, s) \in \mathbb{R}^3,$$

where $a, k > 0$.

Clearly, $\Sigma = \Psi(\mathbb{R}^3)$ is a properly embedded hypersurface of $\mathbb{R}^3 \times \mathbb{R}$, which we call a *twisted helicoid*. A unit normal field to $\Sigma$ is

$$N = \frac{1}{\sqrt{a^2(1+x^2)+(kx)^2}}(a\sin(y+ks), -a\cos(y+ks), ax, kx),$$

so that $\Theta = kx/\sqrt{a^2(1+x^2)+(kx)^2}$. Again, we have $\Theta^2 \neq 1$ and

$$\nabla\Theta = g^{11}\frac{\partial\Theta}{\partial x}\frac{\partial\Psi}{\partial x} = g^{11}\frac{\partial\Theta}{\partial x}(\cos(y+ks), \sin(y+ks), 0, 0),$$

which yields $\langle\nabla\Theta, \nabla\xi\rangle = 0$.

Since, by construction, the horizontal sections $\Sigma_t$ of $\Sigma$ are two-dimensional helicoids in $\mathbb{R}^3 \times \{t\}$, we conclude from the above that $\Sigma$ is a vertical helicoid in $\mathbb{R}^3 \times \mathbb{R}$. Moreover, if $k > a$, then the set

$$\Sigma' := \left\{\Psi(x, y, s) \in \Sigma\,;\, |x| > a/\sqrt{k^2 - a^2}\right\}$$

is easily seen to be spacelike and, so, zero mean isocurved.

Now, define the functions $f,\ g : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$f(x_2, \dots, x_{n-1}, s) = \cos(x_2 + x_3 + \cdots + x_{n-1} + s).$$
$$g(x_2, \dots, x_{n-1}, s) = \sin(x_2 + x_3 + \cdots + x_{n-1} + s).$$

Applying induction on $n$ and proceeding as above, one concludes that the map

$$\Psi(x_1, \dots, x_{n-1}, s) = (x_1 f(x_2, \dots, ks), x_1 g(x_2, \dots, ks), x_2, x_3, \dots, x_{n-1}, as)$$

parametrizes a properly embedded minimal vertical helicoid $\Sigma^n \subset \mathbb{R}^n \times \mathbb{R}$ whose horizontal sections are vertical helicoids in $\mathbb{R}^{n-1} \times \mathbb{R}$. Furthermore, for $k > a$, $\Sigma$ contains open spacelike zero mean isocurved subsets.

***Example 4*** (*Twisted Clifford torus in* $\mathbb{S}^3 \times \mathbb{R}$) Given $k > 0$, consider the immersion

$$\Psi : \mathbb{R}^3 \to \mathbb{S}^3 \times \mathbb{R} \subset \mathbb{R}^5$$

defined by the equality

$$\Psi(x, y, s) = (\cos(x + ks)\cos y, \sin(x + ks)\cos y, \cos x \sin y, \sin x \sin y, s).$$

Then, $\Sigma = \Psi(\mathbb{R}^3)$ is proper and embedded in $\mathbb{S}^3 \times \mathbb{R}$. A computation shows that

$$N = \frac{(\sin y \sin(x + ks), -\sin y \cos(x + ks), -\sin x \cos y, \cos x \cos y, k \cos y \sin y)}{\sqrt{1 + (k \cos y \sin y)^2}}$$

is a unit normal to $\Sigma$, which implies that its angle function is given by

$$\Theta = \frac{k \cos y \sin y}{\sqrt{1 + (k \cos y \sin y)^2}} = \frac{k \sin(2y)/2}{\sqrt{1 + k^2 \sin^2(2y)/4}} \cdot$$

Also, the matrix $[g_{ij}]$ of the first fundamental form of $\Sigma$ is

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & k\cos^2 y \\ 0 & 1 & 0 \\ k\cos^2 y & 0 & k^2\cos^2 y + 1 \end{bmatrix}.$$

In particular, for its inverse $[g^{ij}]$, we have that $g^{12} = g^{32} = 0$, since the corresponding cofactors of $[g_{ij}]$ clearly vanish. This, together with the fact that $\Theta$ depends only on $y$, gives that

$$\nabla\Theta = g^{22} \frac{\partial\Theta}{\partial y} \frac{\partial\Psi}{\partial y} \Rightarrow \langle \nabla\Theta, \nabla\xi \rangle = \langle \nabla\Theta, \partial_t \rangle = 0,$$

for $\partial\Psi/\partial y$ is a horizontal vector. Therefore, $\nabla\xi$ is an asymptotic direction of $\Sigma$. Observing that each horizontal section of $\Sigma$ is a Clifford torus, which is a compact embedded minimal hypersurface of $\mathbb{S}^3$, we conclude that $\Sigma$ is a properly embedded minimal vertical helicoid of $\mathbb{S}^3 \times \mathbb{R}$.

Finally, we have that the angle function of $\Sigma$ satisfies

$$2\Theta^2 - 1 = \frac{k^2 \sin^2(2y) - 4}{k^2 \sin^2(2y) + 4} \cdot$$

Hence, if we assume $k > 2$, we have that the open set

$$\Sigma' := \{\Psi(x,y,s) \in \Sigma\,;\, y > \arcsin(2/k)/2\} \subset \Sigma$$

is nonempty and zero mean isocurved in $\mathbb{S}^3 \times \mathbb{R}$.

***Example 5*** (*Twisted hyperbolic helicoid in* $\mathbb{H}^3 \times \mathbb{R}$) Consider the Lorentzian model of hyperbolic space $\mathbb{H}^3 \hookrightarrow \mathbb{L}^4 = (\mathbb{R}^4, ds^2)$, $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$. It is well known that the map

$$(x,y) \in \mathbb{R}^2 \mapsto (\sinh x \cos y, \sinh x \sin y, \cosh x \sinh y, \cosh x \cosh y) \in \mathbb{H}^3$$

parametrizes a properly embedded minimal surface which is called the *hyperbolic helicoid* of $\mathbb{H}^3$. Considering its twisting $\Psi : \mathbb{R}^3 \to \mathbb{H}^3 \times \mathbb{R}$ defined, for $k > 0$, by

$$\Psi(x,y,s) = (\sinh x \cos(y+ks), \sinh x \sin(y+ks), \cosh x \sinh y, \cosh x \cosh y, as),$$

we have that the hypersurface $\Sigma = \Psi(\mathbb{R}^3)$ is proper and embedded in $\mathbb{H}^3 \times \mathbb{R}$. A unit normal field for $\Sigma$ is given by

$$N = \lambda \begin{bmatrix} \cosh x \sin(y+ks) \\ -\cosh x \cos(y+ks) \\ \sinh x \cosh y \\ \sinh x \sinh y \\ k \sinh x \cosh x \end{bmatrix},$$

where $\lambda = (\cosh^2 x + \sinh^2 x + (k \cosh x \sinh x)^2)^{-1/2}$. Therefore, the angle function of $\Sigma$ is $\Theta = k\lambda \sinh x \cosh x$, which depends only on $x$.

Proceeding as before, one easily concludes that $\nabla\Theta$ is horizontal, i.e., that $\nabla\xi$ is an asymptotic direction of $\Sigma$. Hence, $\Sigma$ is a properly embedded minimal vertical helicoid in $\mathbb{H}^3 \times \mathbb{R}$ whose horizontal sections $\Sigma_t$ are hyperbolic helicoids of $\mathbb{H}^3 \times \{t\}$. Also, for sufficiently large $k$, $\Sigma$ contains open spacelike zero mean isocurved subsets.

***Example 6*** (*Twisted helicoid in* $\mathbb{S}^3_\delta \times \mathbb{R}$) Consider the product $\mathbb{S}^3_\delta \times \mathbb{R}$, where the first factor is a Berger sphere. It is well known that, given $\alpha \in \mathbb{R}$, the map

$$(s,\tau) \in \mathbb{R}^2 \mapsto (e^{i\alpha s}\cos(\tau), e^{is}\sin(\tau)) \in \mathbb{S}^3_\delta$$

is a parametrization of a minimal helicoid of $\mathbb{S}^3_\delta$ (see, for instance, [21]).

From this helicoid, using the same twisting method of the previous examples, we obtain a vertical helicoid in $\mathbb{S}^3_\delta \times \mathbb{R}$ that is given by

$$\Psi(s,\tau,u) = (e^{i(\alpha s+u)}\cos(\tau), e^{i(s+u)}\sin(\tau), au), \quad a \neq 0.$$

To see that $\Psi$ is indeed a vertical helicoid, it suffices to compute the angle function $\Theta$ and check that its gradient is horizontal. After a long but straightforward computation, $\Theta$ can be written as

$$\Theta = \frac{-\alpha\cos(\tau)\sin(\tau)}{\omega(\tau)},$$

where $\omega(\tau)$ is given by

$$\begin{aligned}\omega(\tau) = [&\cos^4(\tau)((1-\delta^2)\delta^2(\alpha+1)^2a^2-\alpha^2)\\ &+\cos^2(\tau)(\delta^2(\alpha+1)(\delta^2(\alpha+1)-2)a^2+\alpha^2)+\delta^2a^2]^{1/2}.\end{aligned}$$

From these expressions, and after some further computations, we get that $\nabla\Theta$ is horizontal. Also, for a convenient choice of the parameters $\alpha$, $a, \delta$, and of the range of $s$, $\tau$, and $u$, $\Psi$ is a spacelike immersion.

## 4.1 Vertical Helicoids as Graphs

Let $u$ be a differentiable (i.e., $C^\infty$) function defined on a domain $\Omega \subset M$. It is easily checked that

$$N = \frac{-\nabla u + \partial_t}{\sqrt{1+\|\nabla u\|^2}}, \tag{12}$$

is a unit normal to $\Sigma = \mathrm{graph}(u) \subset M \times \mathbb{R}$, where, by abuse of notation, we are writing $\nabla u$ instead of $\nabla u \circ \pi_M$. In particular,

$$\Theta = \frac{1}{\sqrt{1+\|\nabla u\|^2}} \tag{13}$$

is the angle function of $\Sigma$.

Denoting by div the divergence of fields on $M$, as is well known, $\Sigma = \mathrm{graph}(u)$ is a minimal hypersurface of $M \times \mathbb{R}$ if and only if $u$ satisfies the equation

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) = 0. \tag{14}$$

**Lemma 3** *Let $\Sigma$ be the graph of a differentiable function $u$ on a domain $\Omega \subset M$, and let $\Sigma_t$ be a horizontal section of $\Sigma$. Then, the following holds:*

(i) *$\Sigma$ is minimal in $M \times \mathbb{R}$ if and only if $u$ satisfies:*

$$\Delta u - \frac{\|\nabla u\|}{1+\|\nabla u\|^2}\langle \nabla u, \nabla\|\nabla u\|\rangle = 0. \tag{15}$$

(ii) *The mean curvature of $\Sigma_t$ is given by:*

$$H_{\Sigma_t} = \frac{\Delta u}{\|\nabla u\|} - \frac{\langle \nabla u, \nabla\|\nabla u\|\rangle}{\|\nabla u\|^2}. \tag{16}$$

***Proof*** Given a differentiable function $\varrho$ on $\Omega$, it is an elementary fact that

$$\mathrm{div}(\varrho\nabla u) = \varrho\Delta u + \langle \nabla\varrho, \nabla u\rangle. \tag{17}$$

Then, considering (14) and setting $\varrho = 1/\sqrt{1 + \|\nabla u\|^2}$, one easily concludes that the equations (14) and (15) are equivalent.

From (12), we have that $\eta = -\nabla u/\|\nabla u\|$ is a unit normal field to $\Sigma_t$ . Therefore, if we choose an orthonormal frame $\{X_1, \dots, X_{n-1}\}$ in $T\Sigma_t$ , we have

$$H_{\Sigma_t} = \sum_{i=1}^{n-1} -\left\langle \overline{\nabla}_{X_i}\eta, X_i \right\rangle = \operatorname{div} \frac{\nabla u}{\|\nabla u\|} \cdot$$

Now, equality (16) follows from (17) if we set $\varrho = 1/\|\nabla u\|$.

The identities in the above lemma suggest the consideration of horizontally homothetic functions, which we now introduce (cf. [18, 19]).

**Definition 3** We say that a smooth function $u$ on $\Omega \subset M$ is *horizontally homothetic* if the identity $\langle \nabla u, \nabla \|\nabla u\| \rangle = 0$ holds on $\Omega$.

Our next result establishes the uniqueness of vertical helicoids as minimal hypersurfaces which are local graphs of harmonic functions.

**Theorem 2** *Let* $\Sigma = \operatorname{graph}(u)$, *where* $u$ *is a smooth function defined on a domain* $\Omega \subset M$ *whose gradient never vanishes. Then, if the angle function of* $\Sigma$ *is nonconstant, the following are equivalent:*

(i) $\Sigma$ *is a vertical helicoid in* $M \times \mathbb{R}$.
(ii) $u$ *is harmonic and* $\Sigma$ *is minimal.*
(iii) $u$ *is harmonic and horizontally homothetic.*

***Proof*** Assume that $\Sigma$ is a vertical helicoid. Then, $H_{\Sigma_t} = 0$ for any horizontal section $\Sigma_t$ of $\Sigma$. Also, by Theorem 1, $\Sigma$ is minimal. So, by Lemma 3, $u$ satisfies equation (15). Combining it with (16), we have

$$\frac{\langle \nabla u, \nabla \|\nabla u\| \rangle}{\|\nabla u\|(1 + \|\nabla u\|^2)} = 0,$$

which yields $\langle \nabla u, \nabla \|\nabla u\| \rangle = 0$. This, together with (15), implies that $u$ is a harmonic function, that is, (i) $\Rightarrow$ (ii).

Let us suppose now that (ii) holds. Then, $u$ satisfies (15). Since $u$ is harmonic, it follows that $u$ is also horizontally homothetic. Now, we have from (16) that the horizontal sections of $\Sigma$ are minimal. Hence, from Theorem 1, $\Sigma$ is a vertical helicoid, which shows that (i) and (ii) are equivalent.

The equivalence between (ii) and (iii) follows directly from Lemma 3-(i).

We now make use of Theorem 2 to obtain vertical helicoids $\Sigma \subset M \times \mathbb{R}$ which contain spacelike pieces of zero mean isocurved hypersurfaces. Before that, let us remark that, by (13), the angle function $\Theta$ of $\Sigma = \operatorname{graph}(u)$ satisfies

$$2\Theta^2 - 1 = \frac{(1 - \|\nabla u\|^2)}{(1 + \|\nabla u\|^2)} \cdot$$

Therefore, $\Sigma = \text{graph}(u)$ is a spacelike hypersurface if and only if $\|\nabla u\| < 1$.

***Example 7*** Consider the set $\Omega$ of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ which satisfy $x_{n-1} > 0$ and define on it the function

$$u(x_1, \dots, x_n) = \sum_{i=1}^{n-2} a_i x_i + b \arctan(x_n/x_{n-1}).$$

From a direct computation, one concludes that $u$ is harmonic and horizontally homothetic. Thus, Theorem 2 gives that $\Sigma = \text{graph}(u)$ is a vertical helicoid. Moreover, the gradient of $u$ is

$$\nabla u(x_1, \dots, x_n) = \left( a_1, \dots, a_{n-2}, \frac{-bx_n}{x_{n-1}^2 + x_n^2}, \frac{bx_{n-1}}{x_{n-1}^2 + x_n^2} \right),$$

which implies that

$$\|\nabla u\|^2 = \sum_{i=1}^{n-2} a_i^2 + \frac{b^2}{x_{n-1}^2 + x_n^2} \cdot \tag{18}$$

Therefore, if we assume $a_1^2 + \cdots + a_{n-2}^2 < 1$ and consider the set $\Omega' \subset \Omega$ of points $(x_1, \dots, x_n) \in \Omega$ for which the right hand side of (18) is $< 1$, we have that $\Sigma' = \text{graph}(u|_{\Omega'})$ is spacelike and, in particular, zero mean isocurved in $\mathbb{R}^n \times \mathbb{R}$.

***Example 8*** (*Y-L Ou examples*) The following functions $u : M \to \mathbb{R}$, which were considered by Y-L Ou in [18, 19], are all harmonic and horizontally homothetic. Therefore, by Theorem 2, their graphs are complete embedded vertical helicoids in the corresponding product $M \times \mathbb{R}$.

(i) $M = \mathbb{H}^n = (\mathbb{R}^n_+, x_n^{-2} g_{\text{Euc}}),\ \ u(x_1, \dots, x_n) = ax_i,\ \ 1 \le i \le n-1.$
(ii) $M = (\mathbb{R}^3, g_{\text{Nil}}),\ g_{\text{Nil}} = dx^2 + dy^2 + (dz - xdy)^2,\ \ u(x, y, z) = a(z - xy/2).$
(iii) $M = (\mathbb{R}^3, g_{\text{Sol}}),\ g_{\text{Sol}} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,\ \ u(x, y, z) = az.$

We remark that, in contrast with (i), in (ii) and (iii) the horizontal sections of $\Sigma = \text{graph}(u)$ are non-totally geodesic. Also, in all cases, for certain suitable values of the parameter $a$, $\Sigma$ has nonempty spacelike zero mean isocurved open sets.

### 4.2 Construction and local characterization of vertical helicoids

In this section, we generalize the method for constructing vertical helicoids in $M \times \mathbb{R}$ which we applied in Examples 2–6. We also give a local characterization of vertical helicoids whose horizontal sections are totally geodesic.

Let $I \ni 0$ be an open interval in $\mathbb{R}$ and let

$$\Gamma_s : M \to M, \ s \in I,$$

be a one-parameter group of isometries of $M$ such that $\Gamma_0$ is the identity map. Choose a hypersurface $\Sigma_0^{n-1} \subset M^n$, define $\Sigma_s^{n-1} \subset M^n$ by

$$\Sigma_s = \Gamma_s(\Sigma_0), \ s \in I,$$

and let $\eta$ and $\eta_s = \Gamma_{s_*}\eta$ be unit normal fields on $\Sigma_0$ and $\Sigma_s$, respectively.

**Definition 4** Given a constant $a > 0$, we call the hypersurface

$$\Sigma := \{(\Gamma_s(p), as) \in M \times \mathbb{R}\,; p \in \Sigma_0, \ s \in I\} \subset M \times \mathbb{R} \tag{19}$$

the *a-pitched twisting* of $\Sigma_0$ *determined by* $\{\Gamma_s\,; s \in I\} \subset \mathrm{Isom}\,(M)$.

Given $p \in \Sigma_0$, denote by $\alpha_p$ the orbit of $p$ in $M$ under the action of $\Gamma_s$, that is,

$$\alpha_p(s) := \Gamma_s(p) \in \Sigma_s, \ s \in I. \tag{20}$$

Finally, define the *ν-function* of $\Sigma$ as

$$\nu(\alpha_p(s), as) := \langle \alpha_p'(s), \eta_s(\alpha_p(s))\rangle, \ \ (\alpha_p(s), as) \in \Sigma. \tag{21}$$

**Lemma 4** *Given $a > 0$, let $\Sigma \subset M \times \mathbb{R}$ be the a-pitched twisting of a hypersurface $\Sigma_0 \subset M$ determined by a one-parameter group $\{\Gamma_s\,; s \in I\} \subset \mathrm{Isom}(M)$. Then, $\nabla\xi$ never vanishes on $\Sigma$, and the following assertions hold*:

(i) *$\nabla\xi$ is an asymptotic direction on $\Sigma$ if and only if the gradient $\nabla\nu$ of the ν-function of $\Sigma$ is a horizontal field.*
(ii) *The open set $\Sigma' = \{x \in \Sigma\,; |\nu(x)| > a\} \subset \Sigma$ is spacelike (if nonempty).*

In particular, if $\nabla\nu$ is horizontal and the horizontal sections $\Sigma_t \subset \Sigma$ are minimal, then $\Sigma$ is a vertical helicoid in $M \times \mathbb{R}$, and $\Sigma'$ is zero mean isocurved.

***Proof*** Given a point $x = (\alpha_p(s), as) \in \Sigma$, we have that

$$T_x\Sigma = T_{\alpha_p(s)}\Sigma_s \oplus \mathrm{Span}\{\partial_s\}, \ \ \partial_s = \alpha_p'(s) + a\partial_t. \tag{22}$$

Hence, a unit normal field $N$ for $\Sigma$ in $T(M \times \mathbb{R})$ can be defined as

$$N(x) := \frac{-a\eta_s(\alpha_p(s)) + \nu(x)\partial_t}{\sqrt{a^2 + \nu^2(x)}}, \ \ x = (\alpha_p(s), as) \in \Sigma\,.$$

In particular, the angle function of $\Sigma$ at $x$ is given by

$$\Theta(x) = \frac{\nu(x)}{\sqrt{a^2 + \nu^2(x)}}\cdot \tag{23}$$

Hence, $\Theta^2 \neq 1$, which implies that $\nabla\xi$ never vanishes on $\Sigma$. Equality (23) also gives that $\nabla\Theta(x)$ is a multiple of $\nabla\nu(x)$. So, $\nabla\xi$ is an asymptotic direction of $\Sigma$ if and only if $\langle \nabla\nu(x), \partial_t\rangle = 0$ for all $x \in \Sigma$, which proves (i).

Now, a direct computation yields

$$2\Theta^2 - 1 = \frac{\nu^2 - a^2}{\nu^2 + a^2},$$

which implies that $\Sigma'$ is spacelike, as stated in (ii).

Let $\Sigma$ be as in the above lemma. Given $x = (\alpha_p(s), as) \in \Sigma$, considering the decomposition (22) of $T_x\Sigma$, we have that any vector $X \in T_x\Sigma$ can be written as

$$X = X^s + \lambda\partial_s, \quad X^s \in T_{\alpha_p(s)}\Sigma_s, \quad \lambda \in \mathbb{R}. \tag{24}$$

Since $X^s$ is horizontal, taking the inner product with $\partial_t$ on both sides of (24), one gets $\lambda = \langle X, \partial_t\rangle / a$. Thus, for $X = \nabla\nu(x)$, setting $X^s = \nabla^s\nu(x)$, one has

$$\nabla\nu(x) = \nabla^s\nu(x) + \frac{\langle \nabla\nu(x), \partial_t\rangle}{a}\partial_s \quad \forall x \in \Sigma. \tag{25}$$

**Lemma 5** *Let $\Sigma$ be as in Lemma 4. Assume that its $\nu$-function is independent of $s$, i.e., $\langle \nabla\nu, \partial_s\rangle = 0$ on $\Sigma$. Then, $\nabla\nu$ is a horizontal field on $\Sigma$ if and only if*

$$\langle \nabla^s\nu(x), \alpha_p'(s)\rangle = 0 \quad \forall x = (\alpha_p(s), as) \in \Sigma. \tag{26}$$

***Proof*** We have that $0 = \langle \nabla\nu, \partial_s\rangle = \langle \nabla\nu, \alpha_p' + a\partial_t\rangle$. This, together with (25), gives

$$\langle \nabla\nu(x), \partial_t\rangle = -\frac{1}{a}\langle \nabla\nu(x), \alpha_p'(s)\rangle = -\frac{1}{a}\left(\langle \nabla^s\nu(x), \alpha_p'(s)\rangle + \frac{\langle \nabla\nu(x), \partial_t\rangle}{a}\|\alpha_p'(s)\|^2\right).$$

Hence, $\langle \nabla\nu(x), \partial_t\rangle = 0$ if and only if $\langle \nabla^s\nu(x), \alpha_p'(s)\rangle = 0$.

Recall that the *cone* over a given hypersurface $\Sigma_0^{n-1}$ of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the hypersurface $\widehat{\Sigma}_0$ of $\mathbb{R}^{n+1}$ which is defined as

$$\widehat{\Sigma}_0 := \{rp \in \mathbb{R}^{n+1}\,; r \in (0, +\infty),\, p \in \Sigma_0\}.$$

It is an elementary fact that the unit normal $\hat\eta$ of the cone $\widehat{\Sigma}_0$ at $rp$ is the parallel transport of the unit normal $\eta$ at $p \in \Sigma_0$ along the radial line of $\mathbb{R}^{n+1}$ through $p$ and $rp$. So, they can be identified as vectors of $\mathbb{R}^{n+1}$, that is,

$$\eta(p) = \hat\eta(rp) \quad \forall p \in \Sigma_0,\; rp \in \widehat{\Sigma}_0. \tag{27}$$

Also, $\Sigma_0$ is minimal in $\mathbb{S}^n$ if and only if $\widehat{\Sigma}_0$ is minimal in $\mathbb{R}^{n+1}$.

**Lemma 6** *Assume that $\Sigma_0$ is a hypersurface of $\mathbb{S}^n$ and let $\widehat{\Sigma}_0$ be the cone of $\mathbb{R}^{n+1}$ over $\Sigma_0$. Assume further that $\{\Gamma_s\,; s \in I\}$ is a one-parameter subgroup of the orthogonal group $O(n+1) = \mathrm{Isom}(\mathbb{S}^n)$. Given $a > 0$, denote by $\Sigma \subset \mathbb{S}^n \times \mathbb{R}$ (respect. $\widehat{\Sigma} \subset \mathbb{R}^{n+1} \times \mathbb{R}$) the $a$-pitched twisting of $\Sigma_0$ in $\mathbb{S}^n \times \mathbb{R}$ (respect. $\mathbb{R}^{n+1} \times \mathbb{R}$) determined by $\{\Gamma_s\,; s \in I\}$, that is,*

- $\Sigma := \{(\Gamma_s(p), as) \in \mathbb{S}^n \times \mathbb{R}\,; p \in \Sigma_0,\ s \in I\} \subset \mathbb{S}^n \times \mathbb{R}$.
- $\widehat{\Sigma} := \{(\Gamma_s(rp), as) \in \mathbb{R}^{n+1} \times \mathbb{R}\,; rp \in \widehat{\Sigma}_0,\ s \in I\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$.

Under these conditions, $\Sigma$ is a vertical helicoid in $\mathbb{S}^n \times \mathbb{R}$ if and only if $\widehat{\Sigma}$ is a vertical helicoid in $\mathbb{R}^{n+1} \times \mathbb{R}$. Moreover, open spacelike subsets occur in $\widehat{\Sigma}$ if they occur in $\Sigma$.

***Proof*** Set $x = (\Gamma_s(p), as) \in \Sigma$ and $\hat{x} = (\Gamma_s(rp), as) \in \widehat{\Sigma}$. Let $\alpha_p$ and $\nu$ be as in (20) and (21) and denote the corresponding objects for $\widehat{\Sigma}$ by $\hat{\alpha}_{rp}$ and $\hat{\nu}$, that is,

$$\hat{\alpha}_{rp}(s) = \Gamma_s(rp) \quad \text{and} \quad \hat{\nu}(\hat{x}) = \langle \hat{\eta}(rp), \hat{\alpha}'_{rp}(s)\rangle.$$

Since $\Gamma_s$ is linear, we have that $\hat{\alpha}_{rp}(s) = \Gamma_s(rp) = r\Gamma_s(p) = r\alpha_p(s)$. Therefore, considering (27), we conclude that

$$\hat{\nu}(\hat{x}) = r\nu(x). \tag{28}$$

Therefore, denoting by $\partial_r \in T_{rp}\widehat{\Sigma}_0$ the gradient of the radial function $rp \in \widehat{\Sigma}_0 \mapsto r \in \mathbb{R}$ on $\widehat{\Sigma}_0$ , it follows from (28) that

$$\widehat{\nabla}\hat{\nu}(\hat{x}) = r\nabla\nu(x) + \nu(x)\partial_r\,,$$

where $\widehat{\nabla}$ denotes the gradient on $\widehat{\Sigma}$.

Since $\partial_r$ is horizontal, it follows from this last equality that $\widehat{\nabla}\hat{\nu}(\hat{x})$ is horizontal if and only if $\nabla\nu(x)$ is horizontal. Therefore, by Lemma 4, $\widehat{\nabla}\hat{\xi}$ is an asymptotic direction on $\widehat{\Sigma}$ if and only if $\nabla\xi$ is an asymptotic direction on $\Sigma$. In addition, any horizontal section $\widehat{\Sigma}_t \subset \mathbb{R}^{n+1} \times \{t\}$ is clearly the cone of $\Sigma_t \subset \mathbb{S}^n \times \{t\}$ in $\mathbb{R}^{n+1} \times \{t\}$. In particular, $\Sigma_t$ is minimal in $\mathbb{S}^n \times \{t\}$ if and only if $\widehat{\Sigma}_t$ is minimal in $\mathbb{R}^{n+1} \times \{t\}$. Thus, $\Sigma$ is a vertical helicoid in $\mathbb{S}^n \times \mathbb{R}$ if and only if $\widehat{\Sigma}$ is a vertical helicoid in $\mathbb{R}^{n+1} \times \mathbb{R}$.

From Lemma 4, $\Sigma' = \{x \in \Sigma\,; |\nu(x)| > a\} \subset \Sigma$ is the spacelike part of $\Sigma$, which we assume to be nonempty. Thus, by (28), the set

$$\widehat{\Sigma}' = \{\hat{x} \in \widehat{\Sigma}\,; |\hat{\nu}(\hat{x})| > a\} \subset \widehat{\Sigma}$$

is nonempty. Then, by Lemma 4, it is spacelike.

Now, by means of Lemmas 4–6, we construct properly embedded vertical helicoids in $Q^n_c \times \mathbb{R}$ whose horizontal sections project on totally geodesic hypersurfaces of $Q^n_c$. First, we handle the Euclidean case $c = 0$. For that, consider the matrices

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad e^{(ks)J} = \begin{bmatrix} \cos(ks) & -\sin(ks) \\ \sin(ks) & \cos(ks) \end{bmatrix},\ s \in \mathbb{R},$$

and, for $k > 0$, define $\Gamma_s = \Gamma_s(k)$ as the following $n \times n$ block diagonal matrix:

- $$\Gamma_s = \begin{bmatrix} e^{(ks)J} & & & & \\ & e^{(ks)J} & & & \\ & & \ddots & & \\ & & & e^{(ks)J} & \\ & & & & e^{(ks)J} \end{bmatrix} (n \text{ even}).$$

- $\Gamma_s = \begin{bmatrix} e^{(ks)J} & & & & \\ & e^{(ks)J} & & & \\ & & \ddots & & \\ & & & e^{(ks)J} & \\ & & & & 1 \end{bmatrix}$ ($n$ odd).

We have that $\mathscr{G} := \{\Gamma_s\,;\, s \in \mathbb{R}\}$ is a one-parameter group of isometries of $\mathbb{R}^n$. So, given $a > 0$, we can choose a totally geodesic hyperplane $\Sigma_0^{n-1} \subset \mathbb{R}^n$ through the origin $\mathbf{0} \in \mathbb{R}^n$ and consider the $a$-pitched twisting $\Sigma = \Sigma(a, k)$ determined by $\mathscr{G}$. In this setting, since $J$ and $e^{(ks)J}$ commute, we have that

$$\frac{d}{ds}e^{(ks)J} = kJe^{(ks)J} = ke^{(ks)J}J.$$

Hence, for any $(\Gamma_s(p), as) \in \Sigma$,

$$\alpha_p'(s) := \frac{d}{ds}\Gamma_s(p) = k\Gamma_s \boldsymbol{J}p,$$

where

- $\boldsymbol{J} = \begin{bmatrix} J & & & & \\ & J & & & \\ & & \ddots & & \\ & & & J & \\ & & & & J \end{bmatrix}$ ($n$ even).
- $\boldsymbol{J} = \begin{bmatrix} J & & & & \\ & J & & & \\ & & \ddots & & \\ & & & J & \\ & & & & 0 \end{bmatrix}$ ($n$ odd).

Thus,

$$\nu(\alpha_p(s), as) = \langle \alpha_p'(s), \eta_s(\alpha_p(s))\rangle = k\langle \Gamma_s\boldsymbol{J}p, \Gamma_s\eta(p)\rangle = k\langle \boldsymbol{J}p, \eta(p)\rangle, \tag{29}$$

i.e., $\nu$ is nonconstant and independent of $s$. Also, the orbits $\alpha_p(s) = \Gamma_s(p)$, $p \in \mathbb{R}^n$, lie on geodesic spheres of $\mathbb{R}^n$ centered at the origin $\mathbf{0}$. Thus, since the hypersurfaces $\Gamma_s(\Sigma_0) \subset \mathbb{R}^n$ all intersect these spheres orthogonally, we have, in particular, that (26) holds. So, by Lemma 5, $\nabla\nu$ is horizontal on $\Sigma$.

Now, Lemma 4 applies and gives that $\Sigma$ is a properly embedded vertical helicoid in $\mathbb{R}^n \times \mathbb{R}$, since its horizontal sections are minimal. In addition, equality (29) and the second part of Lemma 4 imply that, for a sufficiently large $k$, $\Sigma$ contains open spacelike zero mean isocurved subsets.

The above method can be easily adapted for constructing properly embedded vertical helicoids in $\mathbb{H}^n \times \mathbb{R}$. Indeed, one has just to consider the standard isometric immersion of $\mathbb{H}^n$ into the Lorentz space $\mathbb{L}^{n+1}$ and then define the isometries $\Gamma_s$ as

- $\Gamma_s = \begin{bmatrix} e^{(ks)J} & & & & & \\ & e^{(ks)J} & & & & \\ & & \ddots & & & \\ & & & e^{(ks)J} & & \\ & & & & e^{(ks)J} & \\ & & & & & 1 \end{bmatrix}$ ($n$ even).
- $\Gamma_s = \begin{bmatrix} e^{(ks)J} & & & & & \\ & e^{(ks)J} & & & & \\ & & \ddots & & & \\ & & & e^{(ks)J} & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$ ($n$ odd).

The rest of the argument is the same as in the Euclidean case.

For the spherical case $c = 1$, we consider the standard isometric immersion of $\mathbb{S}^n$ into $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, and then define $\Sigma_0$ as the totally geodesic sphere $\mathbb{S}^n \cap \widehat{\Sigma}_0$, where $\widehat{\Sigma}_0$ is an arbitrary totally geodesic hyperplane of $\mathbb{R}^{n+1}$ through the origin $\mathbf{0}$. For $a, k > 0$, the $a$-twisting of $\widehat{\Sigma}_0$ determined by $\Gamma_s(k) \in \text{Isom}(\mathbb{R}^{n+1})$, as described above, is a vertical helicoid in $\mathbb{R}^{n+1} \times \mathbb{R}$. Since $\widehat{\Sigma}_0 - \{\mathbf{0}\}$ is the cone of $\mathbb{R}^{n+1}$ over $\Sigma_0$, Lemma 6 gives that the corresponding $a$-twisting of $\Sigma_0$ is a properly embedded minimal vertical helicoid in $\mathbb{S}^n \times \mathbb{R}$.

We summarize these considerations in the following

**Theorem 3** *There exists a two-parameter family $\{\Sigma(a,k)\,;a,k > 0\}$ of properly embedded vertical helicoids in $Q_c^n \times \mathbb{R}$ whose horizontal sections are vertical translations of totally geodesic hypersurfaces of $Q_c^n$. Such a $\Sigma(a,k)$ is an a-pitched twisting of a totally geodesic hypersurface $\Sigma_0 \subset Q_c^n$ determined by a suitable one-parameter subgroup $\mathscr{G} = \{\Gamma_s = \Gamma_s(k)\,;s \in \mathbb{R}\}$ of $\text{Isom}(Q_c^n)$. Furthermore, for any fixed $a > 0$, the parameter k can be chosen in such a way that $\Sigma(a,k)$ contains open spacelike zero mean isocurved subsets.*

Our next result shows that any vertical helicoid in $M \times \mathbb{R}$ with nonvanishing angle function and totally geodesic horizontal sections is locally a twisting. In particular, Theorem 3 admits a local converse.

**Theorem 4** *Let $\Sigma \subset M \times \mathbb{R}$ be a vertical helicoid with non-vanishing angle function. Assume that each horizontal section $\Sigma_t \subset \Sigma$ is totally geodesic in $M \times \{t\}$. Then, given $x_0 \in \Sigma$, there exists a connected open set $\Sigma' \ni x_0$ of $\Sigma$, a totally geodesic hypersurface $\mathfrak{L}_0 \subset \pi_M(\Sigma') \subset M$, and a one-parameter group of isometries*

$$\Gamma_t \,:\, \pi_M(\Sigma') \to \Gamma_t(\pi_M(\Sigma')) \subset M\,, \quad t \in (-\epsilon, \epsilon),$$

*such that $\Sigma'$ is the 1-pitched twisting of $\mathfrak{L}_0$ determined by $\{\Gamma_t\,; t \in (-\epsilon, \epsilon)\}$, that is,*

$$\Sigma' = \{(\Gamma_t(p), t) \in \Sigma\,; p \in \mathfrak{L}_0\,,\ t \in (-\epsilon, \epsilon)\}.$$

***Proof*** Let $\varphi_t$ be the flow of the field $Z = \nabla\xi/\|\nabla\xi\|^2$ on $\Sigma$, i.e.,

$$\frac{d\varphi_t}{dt}(x) = Z(\varphi_t(x))\ \forall x \in \Sigma.$$

Considering that

$$\frac{d}{dt}\xi(\varphi_t(x)) = \left\langle \nabla\xi(\varphi_t(x)), \frac{d\varphi_t(x)}{dt} \right\rangle = 1,$$

we have $\xi(\varphi_t(x)) = t + \xi(x)$. In particular, $\varphi_t$ takes a horizontal section $\Sigma_s$ to $\Sigma_{s+t}$.

Since we are assuming $\Theta \neq 0$, we have that $\Sigma$ is locally a vertical graph. So, there exists a connected open set $\Sigma' \ni x_0$ of $\Sigma$ satisfying $\Sigma' = \text{graph}(u)$, where $u$ is a differentiable function defined on the domain $\Omega = \pi_M(\Sigma') \subset M$.

After a vertical translation, we can assume $\Sigma' \cap (M \times \{0\})$ nonempty and $\pi_{\mathbb{R}}(\Sigma') = (-2\epsilon, 2\epsilon)$ for some $\epsilon > 0$. In this setting, define the field $Z_0 \in T(\Omega)$ as

$$Z_0(\pi_M(x)) := \pi_{M_*}Z(x), \quad x \in \Sigma',$$

and let $\Gamma_t$ be the its flow on $\Omega$, that is,

$$\Gamma_t(\pi_M(x)) := \pi_M\varphi_t(x), \quad x \in \Sigma'.$$

Writing $\mathfrak{L}_t := u^{-1}(t)$, $t \in (-2\epsilon, 2\epsilon)$, one has $\Gamma_t(\mathfrak{L}_s) = \mathfrak{L}_{s+t}$ for $|s+t| < 2\epsilon$. (Here, we are identifying $M \times \{0\}$ with $M$.) Moreover, it follows from (12) that $\pi_{M_*}\nabla\xi$ is parallel to $\nabla u$, which implies that $Z_0$ is orthogonal to all level sets $\mathfrak{L}_t$, $t \in (-2\epsilon, 2\epsilon)$.

Noticing that the family $\{\mathfrak{L}_t, t \in (-2\epsilon, 2\epsilon)\}$ defines a totally geodesic foliation of $\Omega \subset M$, we conclude from [23, Corollary 6.6] that, for $t, s \in (-\epsilon, \epsilon)$, the restriction of $\Gamma_t$ to $\mathfrak{L}_s$ is an isometry over its image $\Gamma_t(\mathfrak{L}_s) = \mathfrak{L}_{s+t}$. Also, since $\Sigma$ is a vertical helicoid, we have that $\|\nabla\xi\|$, and so $\|Z\|$, is constant along the curves $t \mapsto \varphi_t(x)$, $x \in \Sigma'$ (see Remark 2). In addition, $Z_0 = Z - \langle Z, \partial_t\rangle\partial_t = Z - \partial_t$, and $\Gamma_{t_*} \circ Z_0 = Z_0 \circ \Gamma_t$. Thus, for any $p = \pi_M(x)$, $x \in \Sigma'$, we have

$$\|\Gamma_{t_*}Z_0(p)\|^2 = \|Z_0(\Gamma_t(p))\|^2 = \|Z(\varphi_t(x))\|^2 - 1 = \|Z(x)\|^2 - 1 = \|Z_0(p)\|^2.$$

It follows from the above considerations that, defining $\Omega_\epsilon \subset \Omega$ as the union of all level sets $\mathfrak{L}_t$ with $t \in (-\epsilon, \epsilon)$, any map $p \in \Omega_\epsilon \mapsto \Gamma_t(p)$, $t \in (-\epsilon, \epsilon)$, is an isometry from $\Omega_\epsilon$ to $\Gamma_t(\Omega_\epsilon) \subset \Omega$. Therefore, if we set, by abuse of notation, $\Sigma' = \pi_M^{-1}(\Omega_\epsilon) \cap \Sigma'$, and $\Omega = \Omega_\epsilon$, we have that

$$\Sigma' = \{(\Gamma_t(p), t) \in \Sigma\,; p \in \mathfrak{L}_0, t \in (-\epsilon, \epsilon)\},$$

as we wished to prove.

Since one-dimensional minimal submanifolds are totally geodesic, Theorem 4 has the following consequence.

**Corollary 2** *Any two-dimensional vertical helicoid $\Sigma^2 \subset M^2 \times \mathbb{R}$ with nonvanishing angle function is given, locally, by a twisting of a geodesic of M.*

As a further application of Lemma 4, we now generalize the construction made in Example 4. Namely, we will obtain a family of properly embedded vertical helicoids in the product $\mathbb{S}^{2n+1} \times \mathbb{R}$ by twisting $2n$-dimensional Clifford tori.

We will adopt the following notation. The identity matrix of order $n+1$ will be denoted by Id. We will write $\boldsymbol{J}$, now, for the $(2n+2) \times (2n+2)$ block matrix

$$\boldsymbol{J} := \begin{bmatrix} 0 & -\,\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}.$$

Then, setting $C(t) = (\cos t)\mathrm{Id}$, and $S(t) = (\sin t)\mathrm{Id}$, the following identity holds:

$$e^{t\boldsymbol{J}} = \begin{bmatrix} C(t) & -\,S(t) \\ S(t) & C(t) \end{bmatrix}.$$

In particular, the derivative of the map $t \in \mathbb{R} \mapsto e^{t\boldsymbol{J}} \in O(2n+2)$ is

$$\frac{d}{dt}e^{t\boldsymbol{J}} = \boldsymbol{J}e^{t\boldsymbol{J}}.$$

**Theorem 5** *Let* $\Sigma_0 = \mathbb{S}^n(1/\sqrt{2}) \times \mathbb{S}^n(1/\sqrt{2})$ *be the minimal Clifford torus of the sphere* $\mathbb{S}^{2n+1}$. *Then, for any* $a, k > 0$, *the a-pitched twisting*

$$\Sigma = \Sigma(a,k) := \{(e^{(ks)\boldsymbol{J}}p, as)\,;p \in \Sigma_0,\ s \in \mathbb{R}\} \subset \mathbb{S}^{2n+1} \times \mathbb{R}$$

*is a properly embedded vertical helicoid in* $\mathbb{S}^{2n+1} \times \mathbb{R}$. *Furthermore, for any fixed* $a > 0$, *the parameter k can be chosen in such a way that* $\Sigma(a,k)$ *contains open spacelike zero mean isocurved subsets.*

***Proof*** Consider the standard immersion of $\mathbb{S}^{2n+1} \times \mathbb{R}$ into $\mathbb{R}^{2n+2} \times \mathbb{R}$ and define the following local parametrization of $\Sigma$:

$$\Psi(x_1,\dots,x_n,y_1,\dots,y_n,s) = \left(\frac{1}{\sqrt{2}}\Gamma_s((\varphi(x_1,\dots,x_n),\psi(y_1,\dots,y_n)),as\right),$$

where $\Gamma_s = e^{(ks)\boldsymbol{J}}$ and $\varphi, \psi : \mathbb{R}^n \to \mathbb{S}^n$ are conformal parametrizations of $\mathbb{S}^n$.

Setting $\varphi_i = \partial\varphi/\partial x_i$ and $\psi_i = \partial\psi/\partial y_i$, we have that

$$\frac{\partial\Psi}{\partial x_i} = \frac{1}{\sqrt{2}}(\Gamma_s(\varphi_i,0),0) \quad \text{and} \quad \frac{\partial\Psi}{\partial y_i} = \frac{1}{\sqrt{2}}(\Gamma_s(0,\psi_i),0),\ \ 1 \le i \le n.$$

In particular, $\eta_s = \Gamma_s\eta$ is a unit normal field on $\Sigma_s = \Gamma_s\Sigma_0 \subset \mathbb{S}^{2n+1}$, where

$$\eta = \frac{1}{\sqrt{2}}(\varphi,-\psi).$$

Writing $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$, we have that the orbit of a point $p = \frac{1}{\sqrt{2}}(\varphi(x),\psi(y)) \in \Sigma_0$ under the action of $\Gamma_s$ is

$$\alpha_p(s) = \Gamma_s(p) = \frac{1}{\sqrt{2}}\Gamma_s(\varphi(x),\psi(y)).$$

From $\frac{d\Gamma_s}{ds} = k\boldsymbol{J}e^{(ks)\boldsymbol{J}} = k\boldsymbol{J}\Gamma_s = k\Gamma_s\boldsymbol{J}$, one has

$$\alpha_p'(s) = \frac{d}{ds}\Gamma_s(p) = k\Gamma_s\boldsymbol{J}p = \frac{k}{\sqrt{2}}\Gamma_s(-\psi(y),\varphi(x)). \tag{30}$$

Thus, with the notation of Lemma 4,

$$\nu(\alpha_p(s), as) = \langle \alpha_p'(s), \eta_s(\alpha_p(s)\rangle = k\langle \Gamma_s \boldsymbol{J}p, \Gamma_s \eta(p)\rangle = k\langle \boldsymbol{J}p, \eta(p)\rangle = -k\langle \varphi(x), \psi(y)\rangle,$$

so that $\nu$ is independent of $s$. Now, for $i = 1, \dots, n$, define

$$a_i := \frac{\partial \nu}{\partial x_i} = -k\langle \varphi_i, \psi\rangle \quad \text{and} \quad b_i := \frac{\partial \nu}{\partial y_i} = -k\langle \varphi, \psi_i\rangle,$$

and notice that

- $\|\psi\|^2 = \sum_{i=1}^{n} \frac{\langle \psi, \varphi_i\rangle^2}{\|\varphi_i\|^2} + \frac{\langle \psi, \varphi\rangle^2}{\|\varphi\|^2} = \frac{1}{k^2}\sum_{i=1}^{n} \frac{a_i^2}{\|\varphi_i\|^2} + \frac{\langle \psi, \varphi\rangle^2}{\|\varphi\|^2}\,.$
- $\|\varphi\|^2 = \sum_{i=1}^{n} \frac{\langle \psi_i, \varphi\rangle^2}{\|\psi_i\|^2} + \frac{\langle \psi, \varphi\rangle^2}{\|\psi\|^2} = \frac{1}{k^2}\sum_{i=1}^{n} \frac{b_i^2}{\|\psi_i\|^2} + \frac{\langle \psi, \varphi\rangle^2}{\|\psi\|^2}\,.$

Hence, setting $\lambda = \langle \varphi_i, \varphi_i\rangle$ and $\mu = \langle \psi_i, \psi_i\rangle$, $i = 1, \dots, n$, (recall that $\varphi$ and $\psi$ are both conformal), we have that

$$\sum_{i=1}^{n} a_i^2 = \frac{\lambda k^2}{2} - 2\lambda k^2\langle \psi, \varphi\rangle^2 \quad \text{and} \quad \sum_{i=1}^{n} b_i^2 = \frac{\mu k^2}{2} - 2\mu k^2\langle \psi, \varphi\rangle^2, \tag{31}$$

for $\|\varphi\|^2 = \|\psi\|^2 = 1/2$.

From (30), we have that $\partial\Psi/\partial s = \left(\frac{k}{\sqrt{2}}\Gamma_s(-\psi, \varphi), a\right)$. So,

$$\left\langle \frac{\partial \Psi}{\partial x_i}, \frac{\partial \Psi}{\partial s}\right\rangle = -\frac{k}{2}\langle \varphi_i, \psi\rangle = \frac{a_i}{2} \quad \text{and} \quad \left\langle \frac{\partial \Psi}{\partial y_i}, \frac{\partial \Psi}{\partial s}\right\rangle = \frac{k}{2}\langle \varphi, \psi_i\rangle = -\frac{b_i}{2}\,,$$

from which we conclude that the $[g_{ij}]$ matrix of $\Sigma$ with respect to $\Psi$ is

$$[g_{ij}] = \frac{1}{2}\begin{bmatrix} \lambda & & & & & & a_1 \\ & \ddots & & & & & \vdots \\ & & \lambda & & & & a_n \\ & & & \mu & & & -b_1 \\ & & & & \ddots & & \vdots \\ & & & & & \mu & -b_n \\ a_1 & \cdots & a_n & -b_1 & \cdots & -b_n & k^2 + 2a^2 \end{bmatrix},$$

where the non-dotted missing entries are all zero.

Computing the cofactors of the first $2n$ entries of the last line of $[g_{ij}]$, we conclude that the first $2n$ entries of the last line of $[g^{ij}] = [g_{ij}]^{-1}$ are

$$-\frac{\lambda^{n-1}\mu^n}{2^{2n}\mathcal{D}}a_1\,, \dots, -\frac{\lambda^{n-1}\mu^n}{2^{2n}\mathcal{D}}a_n\,, \frac{\lambda^n\mu^{n-1}}{2^{2n}\mathcal{D}}b_1\,, \dots, \frac{\lambda^n\mu^{n-1}}{2^{2n}\mathcal{D}}b_n\,, \tag{32}$$

where $\mathcal{D} = \det[g_{ij}]$. Since the coordinates of $\nabla\nu$ with respect to the frame

$$\mathfrak{B} := \left\{ \frac{\partial\Psi}{\partial x_1}, \dots, \frac{\partial\Psi}{\partial x_n}, \frac{\partial\Psi}{\partial y_1}, \dots, \frac{\partial\Psi}{\partial y_n}, \frac{\partial\Psi}{\partial s}\right\} \subset T\Sigma$$

are the entries of the column matrix

$$[g^{ij}]\begin{bmatrix}\frac{\partial \nu}{\partial x_1}\\ \vdots\\ \frac{\partial \nu}{\partial x_n}\\ \frac{\partial \nu}{\partial y_1}\\ \vdots\\ \frac{\partial \nu}{\partial y_n}\\ \frac{\partial \nu}{\partial s}\end{bmatrix} = [g^{ij}]\begin{bmatrix}a_1\\ \vdots\\ a_n\\ b_1\\ \vdots\\ b_n\\ 0\end{bmatrix},$$

it follows from (31) and (32) that the last coordinate of $\nabla\nu$ with respect to $\mathfrak{B}$ is

$$\frac{1}{2^{2n}\mathcal{D}}\left(-\lambda^{n-1}\mu^n\sum_{i=1}^{n}a_i^2+\lambda^n\mu^{n-1}\sum_{i=1}^{n}b_i^2\right)=\frac{k^2}{2^{2n}\mathcal{D}}(-\lambda^n\mu^n+\lambda^n\mu^n)=0,$$

so that $\nabla\nu$ is a horizontal field on $\Sigma$.

Finally, we observe that

$$|\nu(\alpha_p(s), as)| = k|\langle\varphi(x), \psi(y)\rangle| \;\forall(\alpha_p(s), as)\in\Sigma.$$

Thus, given $a > 0$, for a sufficiently large $k > 0$, the open set of points of $\Sigma$ on which $|\nu| > a$ is nonempty. The result, then, follows from Lemma 4.

From the above theorem and Lemma 6, we have:

**Corollary 3** *Let $\widehat{\Sigma}_0\subset\mathbb{R}^{2n+2}$ be the cone over the Clifford torus $\Sigma_0$ of $\mathbb{S}^{2n+1}$. Then, for any $a, k > 0$, the $a$-pitched twisting*

$$\widehat{\Sigma}(a,k) := \{(e^{(ks)\boldsymbol{J}}p, as)\,;p\in\widehat{\Sigma}_0,\ s\in\mathbb{R}\}\subset\mathbb{R}^{2n+2}\times\mathbb{R} \tag{33}$$

*is an embedded vertical helicoid of $\mathbb{R}^{2n+2}\times\mathbb{R}$. Furthermore, for any fixed $a > 0$, the parameter $k$ can be chosen in such a way that $\Sigma(a,k)$ contains open spacelike zero mean isocurved subsets.*

It should be mentioned that, through a method different from ours, Choe and Hoppe [7] showed that the twisted cones in the above corollary are minimal hypersurfaces of $\mathbb{R}^{2n+3}$. (We are grateful to Alma Albujer for let us know about this work.) A distinguished property of these $a$-twisted cones is that, for sufficiently large $a > 0$, they constitute nodal sets of the solutions of the Allen-Cahn differential equation (see [9]).

## 5 Hypersurfaces with a canonical direction

With the aim of introducing and studying vertical catenoids in $M\times\mathbb{R}$, we proceed now to the characterization of hypersurfaces of $M\times\mathbb{R}$ which have $\nabla\xi$ as a principal direction. Our approach will be based on the work of Tojeiro [22], who considered the case where $M$ is a constant sectional curvature space form $Q_c^n$.

We start with an arbitrary isometric immersion

$$f : \Sigma_0^{n-1} \to M^n,$$

with $\Sigma_0$ orientable, assuming that there is a neighborhood $\mathscr{U}$ of $\Sigma_0$ in $T\Sigma_0^\perp$ without focal points of $f$, that is, the restriction of the normal exponential map $\exp^\perp_{\Sigma_0} : T\Sigma_0^\perp \to M$ to $\mathscr{U}$ is a diffeomorphism onto its image. In other words, denoting by $\eta$ the unit normal field of $f$, we are assuming that there is an open interval $I \ni 0$ such that, for all $p \in \Sigma_0$, the curve

$$\gamma_p(s) = \exp_M(f(p), s\eta(p)),\ s \in I,$$

is a well defined geodesic of $M$ without conjugate points. In particular, for all $s \in I$,

$$\begin{array}{rccc} f_s : & \Sigma_0 & \to & M \\ & p & \mapsto & \gamma_p(s) \end{array}$$

is an immersion of $\Sigma_0$ into $M$, which is said to be *parallel* to $f$. Observe that, given $p \in \Sigma_0$, the tangent space $f_{s_*}(T_p\Sigma_0)$ of $f_s$ at $p$ is the parallel transport of $f_*(T_p\Sigma_0)$ along $\gamma_p$ from 0 to $s$. Also, with the induced metric, the unit normal $\eta_s$ of $f_s$ at $p$ is $\eta_s(p) = \gamma_p'(s)$.

Now, define in $M \times \mathbb{R}$ the hypersurface

$$\Sigma := \{(f_s(p), a(s)) \in M \times \mathbb{R}\,; p \in \Sigma_0,\ s \in I\}, \tag{34}$$

where $a : I \to a(I) \subset \mathbb{R}$ is an increasing diffeomorphism, i.e., $a' > 0$. We call $\Sigma$ an $(f_s, a)$ *-graph* of $M \times \mathbb{R}$.

For any point $x = (f_s(p), a(s)) \in \Sigma$, one has

$$T_x\Sigma = f_{s_*}(T_p\Sigma_0) \oplus \text{Span}\,\{\partial_s\},\quad \partial_s = \eta_s + a'(s)\partial_t.$$

A unit normal to $\Sigma$ is

$$N = \frac{-a'}{\sqrt{1+(a')^2}}\eta_s + \frac{1}{\sqrt{1+(a')^2}}\partial_t\,.$$

In particular, its angle function is

$$\Theta = \frac{1}{\sqrt{1+(a')^2}}\,\cdot \tag{35}$$

**Theorem 6** *If* $\Sigma$ *is an* $(f_s, a)$*-graph in* $M \times \mathbb{R}$, *the following holds*:

(i) $\Theta$ *and* $\nabla\xi$ *never vanish on* $\Sigma$.
(ii) $\nabla\xi$ *is a principal direction of* $\Sigma$.
(iii) $\Theta$ *and the principal curvature of* $\Sigma$ *in the direction* $\nabla\xi$ *are constant along the horizontal sections* $\Sigma_t$ *of* $\Sigma$.

Conversely, if $\Sigma \subset M \times \mathbb{R}$ is a hypersurface with nonvanishing angle function which has $\nabla\xi$ as a principal direction, then $\Sigma$ is locally an $(f_s, a)$-graph.

***Proof*** Assume that $\Sigma$ is an $(f_s, a)$-graph of $M \times \mathbb{R}$. Then, by (35), $\Theta \neq 0$ and $\Theta^2 \neq 1$. In particular, $\nabla\xi$ never vanishes on $\Sigma$.

Since, for any $p \in \Sigma_0$, $\gamma_p$ is a geodesic of $M$ (and so of $M \times \mathbb{R}$), and $\eta_s = \gamma_p'(s)$, we have $\overline{\nabla}_{\partial_s}\eta_s = 0$. Then, noticing that $N = \Theta(-a'\eta_s + \partial_t)$, one has

$$\overline{\nabla}_{\partial_s}N = \overline{\nabla}_{\partial_s}\Theta(-a'\eta_s + \partial_t) = \frac{\Theta'}{\Theta}N - \Theta(a''\eta_s + a'\overline{\nabla}_{\partial_s}\eta_s) = \frac{\Theta'}{\Theta}N - \Theta a''\eta_s\,.$$

Hence, for all $X \in \{\partial_s\}^\perp \cap T\Sigma$, we have that $\langle\overline{\nabla}_{\partial_s}N, X\rangle = 0$, which implies that $\partial_s$ is a principal direction of $\Sigma$. In addition, one has

$$\langle A\partial_s, \partial_s\rangle = -\langle\overline{\nabla}_{\partial_s}N, \partial_s\rangle = a''\Theta.$$

So, the corresponding eigenvalue of $A$ is

$$\lambda := a''\Theta^3 = \frac{a''}{\sqrt{(1+(a')^2)^3}}$$

(for $\|\partial_s\|^2 = 1 + (a')^2 = 1/\Theta^2$), which gives that $\lambda$ is a function of $s$ alone, and so it is constant along the horizontal sections of $\Sigma$. By (35), the same is true for $\Theta$.

Finally, observing that $\nabla\xi = \partial_t - \Theta N = a'\Theta^2\partial_s$, we conclude that $\nabla\xi$ is also a principal direction of $\Sigma$ with principal curvature $\lambda = a''\Theta^3$, i.e.,

$$A\nabla\xi = (a''\Theta^3)\nabla\xi. \tag{36}$$

This proves the first part of the theorem.

Conversely, let us suppose that $\Sigma \subset M \times \mathbb{R}$ is a hypersurface which has $\nabla\xi$ as a principal direction and whose angle function $\Theta$ never vanishes. Then, $\Sigma$ is (locally) a graph of a differentiable function $u$ defined on a domain $\Omega \subset M$. (By abuse of notation, we keep denoting this local graph by $\Sigma$.)

As we have seen in Section 4.1, in this setting,

$$\Theta = \frac{1}{\sqrt{1+\|\nabla u\|^2}}\,, \tag{37}$$

where, as before, we are writing $\nabla u$ instead of $\nabla u \circ \pi_M$. Notice that, since we are assuming that $\nabla\xi$ is a principal direction, we have $\nabla\xi \neq 0$. In particular, $\Theta^2 \neq 1$, so that $\|\nabla u\|$ never vanishes.

Considering the flow $\varphi_t$ of $\nabla\xi/\|\nabla\xi\|^2$ on $\Sigma$, and possibly restricting the domain $\Omega$, we can assume that the horizontal sections $\Sigma_t \subset \Sigma$ are all connected and homeomorphic to a certain Riemannian manifold $\Sigma_0$. In other words, there exists an open interval $I_0 \ni 0$ such that the map $G : \Sigma_0 \times I_0 \to \Sigma \subset M \times \mathbb{R}$ given by

$$G(p,t) = \varphi_t(p)$$

is a well defined immersion satisfying $G(\Sigma_0 \times \{t\}) = \Sigma_t$.

Define the map $f_t : \Sigma_0 \to M$ by

$$f_t = \pi_M G(\cdot, t), \ \ t \in I_0,$$

and observe that each $f_t$ is an immersion whose image $f_t(\Sigma_0)$ is a level set of $u$. In particular, $\nabla u$ is orthogonal to $f_t$ with respect to the induced metric. Furthermore, since $\nabla\xi$ is a principal direction and $\nabla\Theta = -A\nabla\xi$, we have that $\Theta$ is constant along the horizontal sections $\Sigma_t$ (so, the same is true for $\|\nabla\xi\|$, since $\|\nabla\xi\|^2 + \Theta^2 = 1$). This, together with (37), gives that, for each $t\in I_0$, $\|\nabla u\|$ is constant on the level set $f_t(\Sigma_0)$. Consequently, the (normalized) trajectories of $\nabla u$ are geodesics of $M$ (see [22, Lemma 1]).

For a fixed $p\in\Sigma_0$, let us denote by $\varphi_t'(p)$ the velocity vector of the trajectory $t\in I_0\mapsto\varphi_t(p)\in\Sigma$ at $t$, that is,

$$\varphi_t'(p)=\frac{\nabla\xi}{\|\nabla\xi\|^2}(\varphi_t(p)).$$

In particular, the curve $\gamma_p(t):=\pi_M\circ\varphi_t(p)$ is tangent to $\nabla u$ and, by the above considerations, is a geodesic of $M$ (when reparametrized by arclength). Also, from

$$\|\varphi_t'(p)\|=\frac{1}{\|\nabla\xi(\varphi_t(p))\|}\quad\text{and}\quad\langle\varphi_t'(p),\partial_t\rangle=1,$$

we have $\gamma_p'=\varphi_t'(p)-\langle\varphi_t'(p),\partial_t\rangle\partial_t=\varphi_t'(p)-\partial_t$, which yields

$$\|\gamma_p'\|=\frac{\sqrt{1-\|\nabla\xi\|^2}}{\|\nabla\xi\|}. \tag{38}$$

Let $s=L_p(t)\in I\subset\mathbb{R}$ be the arclength parameter of $\gamma_p$ from an arbitrary point $t_0\in I_0$. Since $\|\nabla\xi\|$ is a function of $t$ alone, it follows from (38) that the same is true for $L_p(t)$. Hence, the function $a=L_p^{-1}:I\to I_0$ depends only on $s$ and satisfies $a'>0$. Writing, by abuse of notation, $\gamma_p=\gamma_p\circ a$, and $f_s=f_{a(s)}$, one has that each $\gamma_p$ is an arclength geodesic of $M$, so that the immersions $f_s$ are parallel and $\Sigma$ is the corresponding $(f_s,a)$-graph. This finishes the proof.

We get from Theorem 6 the following result, which classifies the hypersurfaces of $M\times\mathbb{R}$ whose angle function is constant. For $M=Q_c^n$, this was done in [17, 22].

**Corollary 4** *Let $\Sigma$ be a connected hypersurface of $M\times\mathbb{R}$. Then, if the angle function $\Theta$ of $\Sigma$ is constant, one of the following holds*:

(i) *$\Sigma$ is an open set of $M\times\{t\}$, $t\in\mathbb{R}$.*
(ii) *$\Sigma$ is an open set of a vertical cylinder over a hypersurface of $M$.*
(iii) *$\Sigma$ is locally an $(f_s,a)$-graph with $a'$ constant.*

Conversely, if one of these possibilities occurs, then $\Theta$ is constant.

***Proof*** Suppose that $\Theta$ is constant on $\Sigma$. Clearly, (i) occurs if $\Theta^2=1$, and (ii) occurs if $\Theta=0$. Otherwise, $\nabla\xi\neq 0$ and $\Theta\neq 0$. Since, $A\nabla\xi=-\nabla\Theta=0$, it follows that $\nabla\xi$ is a principal direction of $\Sigma$. Hence, by Theorem 6, $\Sigma$ is locally an $(f_s,a)$-graph and, by (35), $a'$ is constant.

The converse is immediate in cases (i) and (ii). The case (iii) follows directly from equality (35).

An important class of hypersurfaces of $Q_c^n\times\mathbb{R}$ having $\nabla\xi$ as a principal direction are the *rotational hypersurfaces*, which are those obtained by the rotation of a plane curve about an axis $\{o\}\times\mathbb{R}$, $o\in Q_c^n$. Clearly, any horizontal section $\Sigma_t$ of a rotational hypersurface $\Sigma\subset Q_c^n\times\mathbb{R}$ is contained in a geodesic sphere with center at $(o,t)\in Q_c^n\times\mathbb{R}$.

Considering this property, we introduce the following notion of rotational hypersurface in $M \times \mathbb{R}$.

**Definition 5** A hypersurface $\Sigma \subset M \times \mathbb{R}$ is called *rotational*, if there exists a fixed point $o \in M$ such that any horizontal section $\Sigma_t$ is contained in a geodesic sphere with center at $(o, t) \in M^n \times \mathbb{R}$. If so, we call $\{o\} \times \mathbb{R}$ the *axis* of $\Sigma$.

***Remark 3*** Let $\Sigma \subset M \times \mathbb{R}$ be a rotational hypersurface with no horizontal points and non-vanishing $\Theta$. Since concentric geodesic spheres constitute a parallel family $\{f_s\}$ of hypersurfaces of $M$, under these hypotheses, $\Sigma$ is locally an $(f_s, a)$-graph. Hence, by Theorem 6, $\nabla\xi$ *is a principal direction of any such rotational* $\Sigma$.

We introduce now a special type of family of parallel hypersurfaces which will play a fundamental role in the sequel.

**Definition 6** We call a family of parallel hypersurfaces $f_s : \Sigma_0 \to M$, $s \in I$, *isoparametric* if $f_s$ has constant mean curvature $H_s$ (depending on $s$) for all $s \in I$. If so, each hypersurface $f_s$ is also called *isoparametric*.

***Example 9*** It is well known that any totally umbilical hypersurface of $Q^n_c$ is isoparametric (see, e.g., [10]).

***Example 10*** There are certain Hadamard–Einstein manifolds, known as *Damek–Ricci spaces*, which have many families of isoparametric hypersurfaces, including its geodesic spheres. More specifically, geodesic spheres (of any radius) in symmetric Damek–Ricci spaces are isoparametric with constant principal curvatures, whereas geodesic spheres (of small radius) in non-symmetric Damek–Ricci spaces are isoparametric with nonconstant principal curvatures. The symmetric Damek–Ricci spaces are completely classified. They are the hyperbolic space $\mathbb{H}^n$, the complex hyperbolic space $\mathbb{C}\mathbb{H}^n$, the quaternionic hyperbolic space, and the octonionic hyperbolic plane (see [10, Section 6] and the references therein for an account of Damek–Ricci spaces).

***Example 11*** Let $\mathbb{E}(k, \tau)$, $k - 4\tau^2 \neq 0$, be one of the simply connected 3-homogeneous manifolds with isometry group of dimension 4: The products $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ $(\tau = 0)$, the Heisenberg space $\mathrm{Nil}_3$ $(k = 0, \tau \neq 0)$, the Berger spheres $(k > 0, \tau \neq 0)$, or the universal cover of the special linear group $\mathrm{SL}_2(\mathbb{R})$ $(k < 0, \tau \neq 0)$. In [11], the authors classified all isoparametric hypersurfaces of these spaces, showing, in particular, that none of them is spherical.

In our next result, we show that there exist minimal or constant mean curvature $(f_s, a)$ -graphs in $M \times \mathbb{R}$ if and only if $M$ has isoparametric hypersurfaces.

**Theorem 7** *Let $\Sigma \subset M \times \mathbb{R}$ be an $(f_s, a)$-graph, $s \in I \subset \mathbb{R}$, such that $f_s$ is isoparametric with constant mean curvature $H_s$. Assume that, for a given constant $H \in \mathbb{R}$, the diffeomorphism $a : I \to a(I) \subset \mathbb{R}$ is defined by the equality*

$$a(s) = \int_{s_0}^{s} \frac{\varrho(u)}{\sqrt{1-\varrho(u)^2}}du, \ \ s_0 \in I, \tag{39}$$

*where* $y = \varrho(s)$ *is a solution of the linear differential equation of first-order*

$$y' = H_s y + H \tag{40}$$

*satisfying* $0 < \varrho(s) < 1$. *Under these conditions*, $\Sigma$ *has constant mean curvature* $H$. *Conversely*, *if* $\Sigma$ *has constant mean curvature* $H$, *then* $f_s$ *is isoparametric and the function* $a(s)$ *is necessarily given by* (39) *with* $\varrho = a'\Theta$.

***Proof*** Let us denote the mean curvature of $\Sigma$ by $H_\Sigma$. By equalities (3) and (36), we get $H_s = \phi(H_\Sigma - \lambda)$, where

$$\phi = -(1-\Theta^2)^{-1/2} = -(a'\Theta)^{-1} \quad \text{and} \quad \lambda = a''\Theta^3.$$

So, we have $H_\Sigma = -(a'\Theta)H_s + a''\Theta^3$. However, by (35), one has $(a'\Theta)' = a''\Theta^3$. Therefore, if we set $\zeta = a'\Theta$, we get

$$\zeta' = H_s\zeta + H_\Sigma \ \ \forall s \in I. \tag{41}$$

A direct computation gives that $0 < \zeta^2 = (a')^2/(1+(a')^2) < 1$, and also that

$$a' = \frac{\zeta}{\Theta} = \frac{\zeta}{\sqrt{1-\zeta^2}} \cdot \tag{42}$$

Thus, if $f_s$ is isoparametric and the function $a(s)$ is defined by (39) (with $\varrho$ satisfying (40)), it follows by (42) that $\zeta = \varrho$. Then, comparing (40) and (41), we conclude that $\Sigma$ has constant mean curvature $H$.

Conversely, if $\Sigma$ has constant mean curvature $H_\Sigma = H \in \mathbb{R}$, it follows from (41) that $f_s$ is isoparametric and, by (42), that $a(s)$ is given by equality (39) with $\varrho = \zeta = a'\Theta$.

## 6 Vertical catenoids in $M \times \mathbb{R}$.

In this section, we introduce the minimal hypersurfaces of $M \times \mathbb{R}$ which resemble the standard catenoids of $\mathbb{R}^3$ with respect to some of its fundamental properties. The definition is as follows.

**Definition 7** We say that a hypersurface $\Sigma$ of $M \times \mathbb{R}$ with no horizontal points and nonvanishing and nonconstant angle function is a *vertical catenoid* if the following conditions are satisfied:

(i) $\nabla\xi$ is a principal direction of $\Sigma$ with principal curvature $\lambda \neq 0$.
(ii) Any horizontal section $\Sigma_t \subset \Sigma$ has nonzero constant mean curvature (i.e., depending only on $t$) given by

$$H_{\Sigma_t} = \frac{\lambda}{\sqrt{1-\Theta^2}} \cdot \tag{43}$$

Regarding condition (ii) in the above definition, notice that, from Theorem 6, for any $\Sigma$ satisfying condition (i), the functions $\lambda$ e $\Theta$ depend only on $t$, so that $\lambda/\sqrt{1-\Theta^2}$ is constant along the horizontal sections $\Sigma_t$. It should also be noticed that a vertical catenoid, as defined, is not necessarily rotational (see Definition 5). At the end of this section, we construct non-rotational properly embedded vertical catenoids in $M \times \mathbb{R}$, where $M$ is a Hadamard manifold (see Theorems 10 and 11).

The result below establishes the minimality of catenoids as hypersurfaces of $M \times \mathbb{R}$, and also the uniqueness of rotational vertical catenoids as minimal rotational hypersurfaces of $M \times \mathbb{R}$. The latter is a well-known property of the standard catenoids of Euclidean space $\mathbb{R}^3$.

**Proposition 1** *The following assertions on a hypersurface $\Sigma \subset M \times \mathbb{R}$ with no horizontal points, no minimal horizontal sections, and nonconstant and non-vanishing angle function hold:*

(i) *If $\Sigma$ is a vertical catenoid, then $\Sigma$ is minimal.*
(ii) *If $\Sigma$ is rotational and minimal, then $\Sigma$ is a vertical catenoid.*

***Proof*** Suppose that $\Sigma$ is a vertical catenoid. Then, any horizontal section $\Sigma_t$ satisfies (43). Thus, by (3), we have

$$H = -\sqrt{1-\Theta^2}H_{\Sigma_t} + \langle AT, T\rangle = -\sqrt{1-\Theta^2}\frac{\lambda}{\sqrt{1-\Theta^2}} + \lambda = 0,$$

which proves (i).

Regarding (ii), if $\Sigma$ is rotational, then $\nabla\xi$ is a principal direction of $\Sigma$, so that $\Sigma$ is, locally, an $(f_s, \phi)$-graph (see Remark 3). In particular, the eigenvalue $\lambda$ associated with $\nabla\xi$ and the angle function $\Theta$ of $\Sigma$ depends only on $t$. If, in addition, $\Sigma$ is minimal, again by identity (3), we have that the mean curvature of any horizontal section $\Sigma_t$ satisfies (43) with $\lambda \neq 0$, since we are assuming $H_{\Sigma_t} \neq 0$. Hence, $\Sigma$ is a vertical catenoid.

It follows from Theorems 6 and 7 that, as long as $M$ contains isoparametric hypersurfaces, there exist vertical catenoids in $M \times \mathbb{R}$ which are $(f_s, a)$-graphs. This applies, for instance, to all manifolds $M$ described in Examples 9–11. In what follows, we use this fact to construct properly embedded vertical catenoids by "gluing" pieces of such graphs.

First, recall that $M$ is said to be a *Hadamard manifold* if it is complete, simply connected and has non-positive sectional curvature. Any Hadamard manifold $M^n$ is diffeomorphic to $\mathbb{R}^n$ through the exponential map, so that, for a given point $o \in M$, and $r > 0$, the geodesic sphere $S_r(o)$ with center at $o$ and radius $r$ is well defined. We will write $B_r(o)$ for the geodesic ball of $M$ with center at $o \in M$ and radius $r > 0$, and $\overline{B_r(o)}$ for its closure in $M$.

**Theorem 8** *Let $M^n$ be a Hadamard manifold whose geodesic spheres are all isoparametric. Then, there exists a one-parameter family of properly embedded rotational catenoids in the product $M \times \mathbb{R}$ which are all homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$ and symmetric with respect to the horizontal section $M \times \{0\} \subset M \times \mathbb{R}$.*

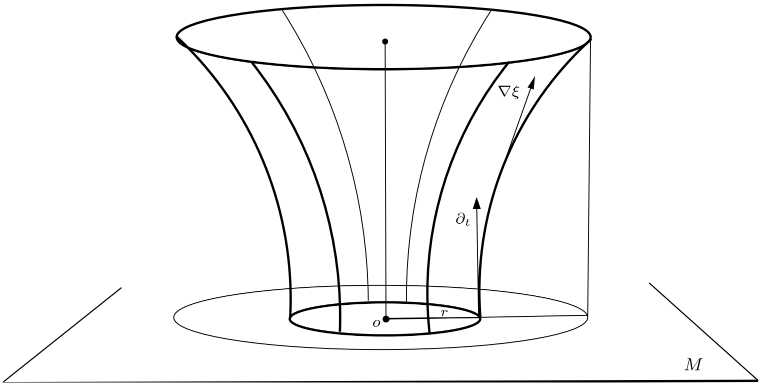


**Fig. 1** The rotational half-catenoid $\Sigma'_r$

***Proof*** Fix $o \in M$ and choose $r > 0$. For each $s \in (r, +\infty)$, let

$$f_s : \mathbb{S}^{n-1} \to M^n \simeq M^n \times \{0\} \subset M \times \mathbb{R}$$

be the geodesic sphere of $M$ with center at $o \in M$ and radius $s > r$. Since $M$ is a Hadamard manifold, each immersion $f_s$ is convex and non-totally geodesic. Hence, taking the "outward" unit normal $\eta_s$ of $f_s$, we have that the (constant) mean curvature $H_s$ of $f_s$ is negative. In particular, setting

$$\varrho(s) := \exp\left(\int_r^s H_u du\right), \ s \in (r, +\infty),$$

we have that $\varrho$ is a solution of $y' = H_s y$ which satisfies $0 < \varrho(s) < 1$ for all $s > r$.

Now, with the purpose of applying Theorem 7, we define the function

$$a(s) := \int_r^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du, \ \ s \in (r, +\infty).$$

The integral on the right is improper, for $\varrho(s) \to 1$ as $s \to r$. So, we have to prove that $a$ is well defined, i.e., that this integral is convergent. For that, notice that $\varrho'(s) \to H_r < 0$ as $s \to r$. In particular, there exist $\delta, C > 0$ such that

$$\varrho'(s) < -C \ \forall s \in (r, r + \delta).$$

This, and the fact that $\varrho$ is decreasing and satisfies $0 < \varrho(s) < 1$ for $s > r$, gives

$$\begin{aligned}\int_r^{r+\delta} \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du = \int_r^{r+\delta} \frac{\varrho'(u)\varrho(u)du}{\varrho'(u)\sqrt{1 - \varrho^2(u)}} &\leq \frac{1}{C} \int_{\varrho(r+\delta)}^{\varrho(r)} \frac{d\varrho}{\sqrt{1 - \varrho^2}} \\ &= \frac{1}{C}(\arcsin(\varrho(r)) - \arcsin(\varrho(r + \delta))) \leq \frac{\pi}{2C},\end{aligned}$$

which implies that the function $a$ is well defined, and that $a(s) \to 0$ as $s \to r$. From this and Theorems 6 and 7, we conclude that the $(f_s, a)$-graph, which we denote by $\Sigma'_r$, is a rotational vertical catenoid.

Furthermore, $\Sigma'_r$ is clearly a graph over $M - \overline{B_r(o)}$ contained in $M \times \mathbb{R}_+$ and with boundary $\partial\Sigma'_r = S_r(o)$. In addition,

$$a'(s) = \frac{\varrho(s)}{\sqrt{1 - \varrho^2(s)}} \to +\infty \quad \text{as} \quad s \to r,$$

which, together with (35), gives that $\Theta(s) \to 0$ as $s \to r$. Hence, the tangent spaces of $\Sigma'_r$ along any trajectory of $-\nabla\xi$ on $\Sigma'_r$ converge to a vertical space (i.e., parallel to $\partial_t$) at a point on $\partial\Sigma'_r = S_r(o)$ (see Fig. 1).

Now, let $\Sigma''_r \subset M \times \mathbb{R}$ be the reflection of $\Sigma'_r$ with respect to $M \times \{0\}$. Then, $\Sigma''_r$ is also a rotational catenoid in $M \times \mathbb{R}$ with boundary $\partial\Sigma''_r = S_r(o)$, which implies that it can be "glued" together with $\Sigma'_r$ along $S_r(o)$, that is, we can define

$$\Sigma_r := \text{closure}\,(\Sigma'_r) \cup \text{closure}\,(\Sigma''_r).$$

Since the tangent spaces of $\Sigma'_r$ and $\Sigma''_r$ are vertical along $S_r(o)$, we have that the tangent spaces of $\Sigma_r$ along $S_r(o)$ are well defined, so that $\Sigma_r$ is a differentiable manifold. Let us see that $\Sigma_r$ is, in fact, of class $C^\infty$. Indeed, being a geodesic sphere, $S_r(o)$ is a $C^\infty$ manifold. Also, the trajectories of $\nabla\xi$ on $\Sigma_r$ are geodesics (see [22, Lemma 1])—so, they are $C^\infty$ as well—and any of them intersects $S_r(o)$ transversally. These facts imply that $\Sigma_r$ is $C^\infty$.

Therefore, $\Sigma_r$ is a $C^\infty$ properly embedded rotational catenoid in $M \times \mathbb{R}$ which is clearly homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$ and symmetric with respect to $M \times \{0\}$. □

The above theorem and the considerations of Example 10 give the following result.

**Corollary 5** *Let $M$ be a symmetric Damek–Ricci space. Then, there exists a one-parameter family of properly embedded rotational catenoids in $M \times \mathbb{R}$ which are all homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$ and symmetric with respect to $M \times \{0\}$.*

Assume $M = \mathbb{R}^n$ and let $\Sigma_r$ be a rotational catenoid as in Theorem 8. When $n = 2$, $\Sigma_r$ is a standard catenoid of $\mathbb{R}^3$ obtained by rotating a catenary about a fixed axis. For the half catenoid $\Sigma'_r$ in $\mathbb{R}^n \times \mathbb{R}$, one has

$$a(s) = \int_r^s \frac{r^{n-1}}{\sqrt{u^{2n-2} - r^{2n-2}}}\,du.$$

It is easily checked that this function is bounded for $n \geq 3$. So, in this case, for any $r > 0$, the rotational catenoid $\Sigma_r$ is contained in a "slab" determined by two horizontal sections. For $n = 2$, we have

$$a(s) = r \log\left(\frac{s + \sqrt{s^2 - r^2}}{r}\right), \ \ s > r,$$

which is clearly an unbounded function.

In $\mathbb{H}^n \times \mathbb{R}$, the height function of any $\Sigma_r$ is *uniformly bounded*. More precisely, given $n \geq 2$, for any $r > 0$, $\Sigma_r$ is contained in a slab of width $\pi/(n-1)$. Indeed, in this setting, the mean curvature of $f_s$ is $H_s = (1-n)\coth s$, which gives, for $s \in (r, +\infty)$,

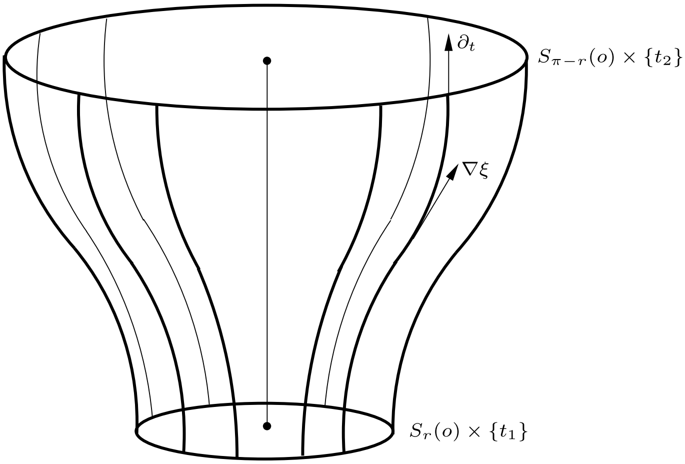


**Fig. 2** The "block" $\Sigma_r'$ of the rotational catenoid $\Sigma_r$

$$\varrho(s) = \exp\left(\int_r^s H_u du\right) = \exp\left((1-n)\int_r^s \coth u du\right) = \left(\frac{\sinh r}{\sinh s}\right)^{n-1}.$$

Thus, the function $a$ which defines $\Sigma_r'$ is

$$a(s) = \int_r^s \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}}du = \sinh^{n-1}(r)\int_r^s (\sinh^{2n-2}(u) - \sinh^{2n-2}(r))^{-1/2}du.$$

Applying, in the last integral, the change of variables $v = \sinh u/\sinh r$, we get

$$a(s) = \sinh r \int_1^{\frac{\sinh s}{\sinh r}} (v^{2n-2}-1)^{-1/2}(1+(\sinh^2 r)v^2)^{-1/2}dv.$$

However, $(1+(\sinh^2 r)v^2)^{-1/2} < ((\sinh r)v)^{-1}$, which implies that

$$a(s) \le \int_1^{\frac{\sinh s}{\sinh r}} \frac{dv}{v\sqrt{v^{2n-2}-1}} = \frac{1}{n-1}\arctan\sqrt{v^{2n-2}-1}\Big|_1^{\frac{\sinh s}{\sinh r}} \le \frac{\pi}{2(n-1)}.$$

***Remark 4*** In [5], the authors constructed the rotational catenoids $\Sigma_r$ in $\mathbb{H}^n \times \mathbb{R}$ by rotating suitable curves about an axis. They also obtained the bound $\pi/2(n-1)$ for the height of the half catenoids $\Sigma_r'$.

Next, we show that $\mathbb{S}^n \times \mathbb{R}$ admits a one-parameter family of rotational catenoids as well.

**Theorem 9** *There exists a one-parameter family* $\left\{\sum_{\mathrm{r}}; 0 < r < \pi/2\right\}$ *of properly embedded Delaunay-type rotational catenoids in* $\mathbb{S}^n \times \mathbb{R}$, *that is*, *each* $\Sigma_r$ *is periodic*, *homeomorphic to* $\mathbb{S}^{n-1} \times \mathbb{R}$ *and has unduloids as the trajectories of the gradient of its height function.*

***Proof*** Let $f_s : \mathbb{S}^{n-1} \to \mathbb{S}^n$, $s \in (0, \pi)$, be a family of concentric geodesic spheres of $\mathbb{S}^n$ with center at $o \in \mathbb{S}^n$ and outward normal orientation, that is, the mean curvature of $f_s$ is $H_s = -(n-1)\cot(s)$. Given $r \in (0, \pi/2)$, consider the function

$$\varrho_r(s) = \left(\frac{\sin r}{\sin s}\right)^{n-1}, \quad s \in [r, \pi - r],$$

which can be verified to be a solution of $y' = H_s y$ satisfying $0 < \varrho_r|_{(r,\pi-r)} < 1$.

Now, let us define the function

$$a_r(s) = \int_r^s \frac{\varrho_r(u)}{\sqrt{1 - \varrho_r^2(u)}} du, \quad s \in (r, \pi - r).$$

Since $\varrho_r'(r) = H_r \neq 0$ and $\varrho_r'(\pi - r) = H_{\pi-r} \neq 0$, we can proceed as in the proof of Theorem 8 to conclude that $a_r$ is well defined and bounded. In particular, $t_1 = a_r(r)$ and $t_2 = a_r(\pi - r)$ are well defined.

It follows from the above that $\Sigma_r'$ is homeomorphic to $\mathbb{S}^{n-1} \times (r, \pi - r)$ and has boundary $\partial\Sigma_r' = S_r(o) \times \{t_1\} \cup S_{\pi-r}(o) \times \{t_2\}$ (Fig. 2). Also, the tangent spaces of $\Sigma_r'$ are vertical along its boundary $\partial\Sigma_r'$, for $\varrho_r(r) = \varrho_r(\pi - r) = 1$. Therefore, from successive reflections of $\Sigma_r'$ with respect to suitable horizontal sections of $\mathbb{S}^n \times \mathbb{R}$, we obtain a periodic properly embedded rotational catenoid $\Sigma_r$ homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$.

***Remark 5*** The above Delaunay-type catenoids were also obtained in [20].

Given a Hadamard manifold $M$, recall that the *Busemann function* $\mathfrak{b}_\gamma(p)$ of $M$ corresponding to an arclength geodesic $\gamma : (-\infty, +\infty) \to M$ is defined as

$$\mathfrak{b}_\gamma(p) := \lim_{s\to+\infty} (\mathrm{dist}_M(p, \gamma(s)) - s), \quad p \in M.$$

The level sets $\mathscr{H}_s := \mathfrak{b}_\gamma^{-1}(s)$ of a Busemann function $\mathfrak{b}_\gamma$ are called *horospheres* of $M$. In this setting, as is well known, $\{\mathscr{H}_s\,; s \in (-\infty, +\infty)\}$ is a parallel family which foliates $M$. Furthermore, any geodesic of $M$ which is asymptotic to $\gamma$ (i.e., with the same point on the asymptotic boundary $M(\infty)$ of $M$) is orthogonal to each horosphere $\mathscr{H}_s$. We also remark that horospheres are submanifolds of class (at least) $C^2$ (see, e.g., [14, Proposition 3.1]).

In hyperbolic space $\mathbb{H}^n$, any horosphere is totally umbilical with constant principal curvatures equal to 1. Also, as shown in [6, Proposition-(vi), pg. 88], except for hyperbolic space,[1] any Damek–Ricci space contains a family $\{\mathscr{H}_s\,; s \in (-\infty, +\infty)\}$ of parallel horospheres such that the principal curvatures of each $\mathscr{H}_s$ are 1/2 and 1, both with constant multiplicities.

Let us see now that, when $M$ is a Hadamard manifold whose horospheres are properly embedded and isoparametric with the same mean curvature, as in the above examples, one can construct properly embedded vertical catenoids in $M \times \mathbb{R}$ with special properties.

**Theorem 10** *Let* $\{\mathscr{H}_s\,; s \in (-\infty, +\infty)\}$ *be a parallel family of properly embedded horospheres of constant mean curvature* $H_0 > 0$ *in a Hadamard manifold* $M$*. Then, there exists a properly embedded vertical catenoid* $\Sigma$ *in* $M \times \mathbb{R}$ *of class at least* $C^2$ *which is*

[1] In [6], hyperbolic space is not considered a Damek–Ricci space.

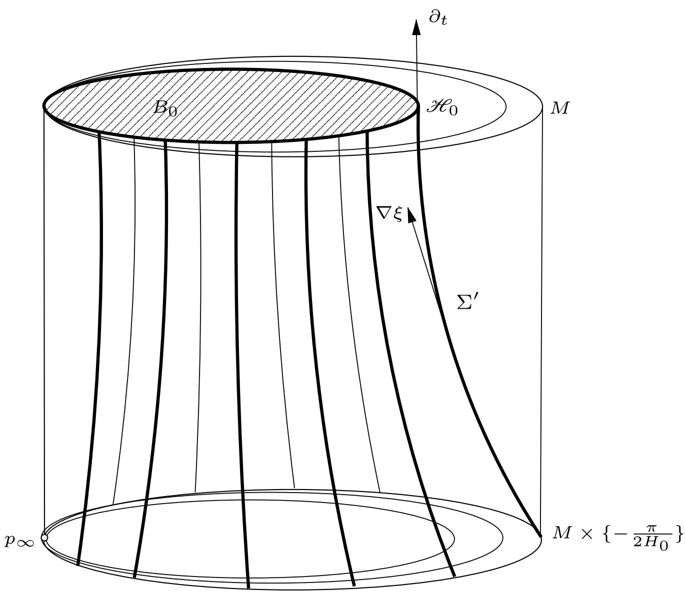


**Fig. 3** The half-catenoid $\Sigma'$ foliated by horospheres

*homeomorphic to* $\mathbb{R}^n$. *Furthermore*, $\Sigma$ *is foliated by horospheres*, *is symmetric with respect to* $M \times \{0\}$, *and is asymptotic to both* $M \times \{-\frac{\pi}{2H_0}\}$ *and* $M \times \{\frac{\pi}{2H_0}\}$.

***Proof*** For each $s \in (-\infty, \infty)$, consider the isometric immersion $f_s : \mathbb{R}^{n-1} \to M^n$ such that $f_s(\mathbb{R}^{n-1}) = \mathscr{H}_s$. Define the function

$$\varrho(s) := e^{H_0 s}, \quad s \in (-\infty, 0],$$

and notice that $\varrho$ is a solution of $y' = H_0 y$ satisfying

$$0 < \varrho(s) < 1 = \varrho(0) \quad \forall s \in (-\infty, 0).$$

Thus, by Theorem 7, defining

$$a(s) := \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du = \frac{1}{H_0}(\arcsin(e^{H_0 s}) - \pi/2),$$

one has that the $(f_s, a)$-graph $\Sigma'$ is a minimal hypersurface of $M \times \mathbb{R}$. In addition,

$$\lim_{s \to -\infty} a(s) = -\frac{\pi}{2H_0} \cdot$$

Hence, denoting by $B_0$ the mean convex side of $\mathscr{H}_0$, and identifying $M \times \{0\}$ with $M$, it follows that $\Sigma'$ is a minimal graph over $M - B_0$ which has boundary $\partial\Sigma' = \mathscr{H}_0$ and is asymptotic to $M \times \{-\frac{\pi}{2H_0}\}$ (see Fig. 3). In particular, $\Sigma'$ is homeomorphic to $\mathbb{R}^n$.

We also have that $\varrho(0) = 1$. So, as in the previous theorems, any trajectory of $\nabla\xi$ on $\Sigma'$ meets $\partial\Sigma'$ orthogonally. Therefore, setting $\Sigma''$ for the reflection of $\Sigma'$ with respect to

$M \times \{0\}$, and defining $\Sigma := \text{closure}(\Sigma') \cup \text{closure}(\Sigma'')$, we can argue just as before and conclude that $\Sigma$ is a properly embedded $C^2$-differentiable (for horospheres are, at least, $C^2$ differentiable) vertical catenoid of $M \times \mathbb{R}$ which has all the stated properties.

In our next result, we consider more general isoparametric foliations of Hadamard manifolds.

**Theorem 11** *Let $\mathscr{F} := \{f_s : \Sigma_0 \to M, s \in (-\infty, +\infty)\}$ be an isoparametric family of hypersurfaces in a Hadamard manifold $M^n$. Assume that:*

(i) *For all $s \in (-\infty, +\infty)$, $f_s$ is a $C^k$ $(k \geq 2)$ proper embedding with positive mean curvature $H_s$.*
(ii) *$\mathscr{F}$ foliates $M$, i.e., $M = \bigcup f_s(\Sigma_0)$, $s \in (-\infty, +\infty)$.*

Then, there exists a properly embedded $C^k$ catenoid $\Sigma$ in $M \times \mathbb{R}$ which is homeomorphic to $\Sigma_0 \times \mathbb{R}$. Furthermore, $\Sigma$ is foliated by (vertical translations of) the leaves of $\mathscr{F}$ and is symmetric with respect to $M \times \{0\}$.

***Proof*** Since $H_s > 0$ for all $s \in (-\infty, +\infty)$, we have that the function

$$\varrho(s) := \exp\left(\int_0^s H_u du\right), \quad s \in (-\infty, 0],$$

which is a solution of $y' = H_s y$, satisfies:

$$0 < \varrho(s) < 1 = \varrho(0) \quad \forall s \in (-\infty, 0).$$

In addition, $\varrho'(0) = H_0 > 0$. From this, as in the preceding proofs, we get that

$$a(s) := \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du, \quad s \in (-\infty, 0),$$

is a well defined function, i.e., this improper integral is convergent. So, the $(f_s, a)$-graph $\Sigma'$ is a minimal graph over $M - B_0$ whose $\nabla\xi$-trajectories meet $\partial\Sigma' = \mathfrak{L}_0 \times \{0\}$ orthogonally. Here, $B_0 \subset M$ is the mean convex side of $\mathfrak{L}_0$. In particular, $\Sigma'$ is homeomorphic to $\Sigma_0 \times \mathbb{R}$. Now, by reflecting $\Sigma'$ with respect to $M \times \{0\}$, as we did before, we obtain the desired vertical catenoid of $M \times \mathbb{R}$.

We conclude from the above proof that, under the conditions of Theorem 11, the result is still valid if we assume that $H_s > 0$ on an interval $(-\infty, c]$, $c \in \mathbb{R}$. In $\mathbb{H}^n$, this is the case of the well known family of equidistant hypersurfaces from a fixed totally geodesic hyperplane of $\mathbb{H}^n$. Also, each leaf of such a family is $C^\infty$ and homeomorphic to $\mathbb{R}^{n-1}$. So, we have the following final result, which was obtained in [8, 12], and [13] for the particular case $n = 2$.

**Corollary 6** *Let $\mathscr{F} := \{f_s : \mathbb{R}^{n-1} \to \mathbb{H}^n,\ s \in (-\infty, +\infty)\}$ be a family of parallel equidistant hypersurfaces in $\mathbb{H}^n$. Then, there exists a properly embedded $C^\infty$ vertical catenoid in $\mathbb{H}^n \times \mathbb{R}$ which is homeomorphic to $\mathbb{R}^n$. Moreover, $\Sigma$ is symmetric with respect to $\mathbb{H}^n \times \{0\}$ and is foliated by (vertical translations of) the leaves of $\mathscr{F}$.*

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