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A remark on regularity of liquid crystal equations in critical Lorentz spaces

Xiangao Liu[1] · Yueli Liu[1] · Zixuan Liu[1]

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Abstract
The regularity for the 3-D nematic liquid crystal equations is considered in this paper, it is proved that the Leray–Hopf weak solutions (u, d) is in fact smooth, if the velocity field $u \in L^\infty(0,T;L_x^{3,\infty}(\mathbb{R}^3))$ satisfies some addition local small condition

$$r^{-3}\Big|\big\{x \in B_r(x_0) : |u(x,t_0)| > \varepsilon r^{-1}\big\}\Big| \le \varepsilon,$$

which is inspired by the papers [2, 35].

Keywords Weak -L^3 space · Backward uniqueness · Nematic liquid crystals equations

Mathematics Subject Classification 76A15 · 35B65 · 76D03

1 Introduction

The three-dimensional incompressible liquid crystals system are the following coupled equations

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = -\mathrm{div}(\nabla d \odot \nabla d), \\ \mathrm{div}\ u = 0, \\ d_t + u \cdot \nabla d - \Delta d = -f(d), \end{cases} \tag{1.1}$$

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✉ Xiangao Liu
xgliu@fudan.edu.cn

Yueli Liu
liuyuel04@126.com

Zixuan Liu
16110180007@fudan.edu.cn

[1] School of Mathematic Sciences, Fudan University, Shanghai, China

in the domain $Q_T \equiv \mathbb{R}^3 \times (0,T)$. Here, the unknowns $u=(u_1,u_2,u_3)$ is the velocity field, p is the scalar pressure and $d=(d_1,d_2,d_3)$ is the optical molecule direction after penalization, and $f(d)=\frac{1}{\sigma^2}(|d|^2-1)d$, $\nabla d \odot \nabla d$ is a symmetric tensor with its component $(\nabla d \odot \nabla d)_{ij}$ is given by $\partial_i d \cdot \partial_j d$. And the initial conditions are

$$u(x,0)=u_0(x), \quad \operatorname{div}(u_0)=0, \quad d(x,0)=d_0(x), \tag{1.2}$$

with $|d_0|=1$.

System (1.1) is the simplified system of the original Ericksen–Leslie system of variable length for the flow of liquid crystals, that is the Ginzburg–Landau energy $\int_\Omega(\frac{1}{2}|\nabla d|^2+\frac{(1-|d|^2)^2}{4\sigma^2})$. For this system, Lin and Liu [19–21] first proved a global existence of weak solutions under L^2 data and regularity result of the suitable weak solution under the C-K-N condition. Other results to liquid crystals equations refer to [8–11, 15, 16, 18, 22, 23, 28].

Let us now recall the notion of a suitable weak solution of liquid crystals equations.

Definition 1.1 ([20]) a triple $(u,\ d,\ p)$ is called a suitable weak solution of (1.1) in $\mathbb{R}^3\times(0,T)$ if the following conditions hold:

(1). the weak solution $(u,\ d)$ satisfies system (1.1) in the distribution sense;
(2). the solution $(u,\ d)$ satisfy the energy inequality, i.e.,

$$\begin{aligned}&\|u\|^2_{L^{2,\infty}(\mathbb{R}^3\times(0,T))}+\|\nabla d\|^2_{L^{2,\infty}(\mathbb{R}^3\times(0,T))}\\&\quad+\|\nabla^2 d\|^2_{L^2(\mathbb{R}^3\times(0,T)))}+\|\nabla u\|^2_{L^2(\mathbb{R}^3\times(0,T)))}\le c_0;\end{aligned}$$

(3). the press $p\in L^{\frac{3}{2}}_{loc}(\mathbb{R}^3\times(0,T))$;
(4). the triple $(u,\ d,\ p)$ satisfy the modified generalized local energy inequality, for a.e. $t\in(0,T)$ and for all $\phi\in C_0^\infty(\mathbb{R}^3\times(0,T))$ with $\phi\ge 0$,

$$\begin{aligned}&\int_{\mathbb{R}^3\times\{t\}}(|u|^2+|\nabla d|^2)\phi \mathrm{d}x+2\int_0^t\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla^2 d|^2)\phi\\&\quad\le\int_0^t\int_{\mathbb{R}^3}(|u|^2+|\nabla d|^2)(\phi_t+\Delta\phi)\\&\qquad+\int_0^t\int_{\mathbb{R}^3}(|u|^2+|\nabla d|^2+2p)(u\cdot\nabla\phi)\\&\qquad+2\int_0^t\int_{\mathbb{R}^3}((u\cdot\nabla)d\odot\nabla d)\cdot\nabla\phi-2\int_0^t\int_{\mathbb{R}^3}\nabla_x f(d)\cdot\nabla \mathrm{d}\phi.\end{aligned}$$

Denote for $z=(x,t)\in\mathbb{R}^3\times\mathbb{R}_+$ the standard notations

$$B_r(x)=\{y\in\mathbb{R}^3\,:\,|y-x|<r\},\quad Q_r(z)=B_r(x)\times(t-r^2,t),$$

to be the Euclidean ball and parabolic cylinder. For $z=(0,0)$, we simply write them as B_r and Q_r.

Now, we mention some relevant results on regularity. For 3D Navier–Stokes system, the study of partial regularity was originated by Scheffer in a series of papers [31, 32] and [33]. The notion of suitable weak solutions was introduced in a celebrated paper

[1] by Caffarelli, Kohn and Nirenberg. It is proved that, for any suitable weak solution (u, p), there is an open subset in which the velocity field u is Hölder continuous, and the complement of it has zero 1-D Hausdorff measure. Latter Lin [17] gave a simpler proof for the CKN theorem, Lin's method was used extensively by Seregin's papers for example in [3, 12]. These results are based on some regularity criterions when certain dimensionless quantities are small. For liquid crystal system (1.1), Lin and Liu[20] proved the following theorem.

Theorem(A). Let the triple (u, d, p) be a suitable weak solution to system (1.1). There exist a small constant $\epsilon_0 > 0$, such that if

$$\int_{Q_1} |u|^3 + |\nabla d|^3 + |p|^{\frac{3}{2}} < \epsilon_0,$$

then u and d are smooth in $\overline{Q_{\frac{1}{2}}}$. In particular, for any z_0 if

$$\frac{1}{r^2}\int_{Q_r(z_0)} |u|^3 + |\nabla d|^3 < \epsilon_0, \quad \text{for all } 0 < r \leq 1,$$

then z_0 is a regular point.

We can drop the pressure p by Wolf' method of local suitable weak solutions, the proof to see Sect. 5.

Proposition 1.2 *Let the triple* (u, d, p) *be a suitable weak solution to system* (1.1). *There exist a small constant* $\varepsilon_0 > 0$, *such that if*

$$\int_{Q_1} |u|^3 + |\nabla d|^3 < \varepsilon_0,$$

then u and d are smooth in $\overline{Q_{\frac{1}{2}}}$.

There is another type of regularity criterion called the Ladyzhenskaya–Prodi–Serrin condition (to see [13, 30, 36]). For system (1.1), [24] showed that if

$$||u||_{L^{p,q}(Q)} = \left(\int_{-1}^{0}\left(\int_B |u|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty,$$

where $\frac{3}{p} + \frac{2}{q} = 1$ and $(p, q) \neq (3, \infty)$, then the weak solution (u, d) is regular in $Q = B \times (-1, 0)$, Serrin's method [36] and Struwe's method [38] dealing with Navier–Stokes equations are applied. For the Borderline case where $(p, q) = (3, \infty)$, [28] showed that the weak solution (u, d) is smooth in $\mathbb{R}^3 \times (0, T]$ when $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ (also to see [25]). The $(3, \infty)$ case requires a technique utilizing the backward uniqueness of heat operator and unique continuation through spatial boundary, which was used to deal with the Navier–Stokes equations in [3]. For general Lorenz space $L^{3,q}(\mathbb{R}^3), 3 < q < \infty$ case, we prove in [26] if $u \in L^\infty(0, T; L^{3,q}(\mathbb{R}^3))$ then (u, d) is smooth. For N-S equations we refer to [29]. For $L^{3,\infty}(\mathbb{R}^3)$ case, Choe, Wolf and Yang [2] prove that if $u \in L^\infty(0,T;L^{3,\infty}(\mathbb{R}^3))$ and an additional local small condition

$$\frac{1}{r^3}|\{x \in B_r(x_0)|\, |u(\cdot,t_0)| > \frac{\epsilon}{r}\}| \le \epsilon, \tag{1.3}$$

then weak solution u of Navier–Stokes equations is smooth on $Q_{\epsilon r}(z_0)$. Also to see Seregin [35].

In this paper, we shall established the regularity of weak solutions of liquid crystals equations in Lorentz space $L^{3,\infty}$, the technicality is different from Navier–Stokes equations, the main technique is to deal with the new norms. Our main results can be stated as following.

Theorem 1.3 *Assume* (u, d, p) *is a weak solution of* (1.1) *with* $u \in L^\infty(0,T;L^{3,\infty}(\mathbb{R}^n))$. *There exists a positive constant* ε *small with following property. If* $z_0 = (x_0, t_0) \in Q_T$ *and* $R > 0$ *such that* $Q_R(z_0) \subset Q_T$ *and* (d, p) *satisfy*

$$C(\nabla d, R, z_0) + D(p, R, z_0) \le N,$$

for some $0 < r \le R/2$,

$$r^{-3}\Big|\big\{x \in B_r(x_0) \,:\, |u(x,t_0)| > \varepsilon r^{-1}\big\}\Big| \le \varepsilon.$$

Then (u, d) *is smooth in* $Q_{\varepsilon r}(z_0)$. *Here* $C(\nabla d, R, z_0)$ *and* $D(p, R, z_0)$ *are dimensionless quantities in Sect.* 2.

Theorem 1.4 *Let* (u, d) *be a suitable weak solution to the liquid crystal equations in* Q_T *with* $u \in L^\infty(0,T;L^{3,\infty}(\mathbb{R}^n))$. *Then there exist at most finite number* $\mathcal{N}$ *of singular points at any singular time* t.

2 Notations and preliminaries

Let us recall the scaling property of (1.1). Denote

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad d_\lambda(x,t) = d(\lambda x, \lambda^2 t).$$

If (u, p, d) is a solution in $\mathbb{R}^3 \times (0, T)$, then obviously $(u_\lambda, p_\lambda, d_\lambda)$ is a solution to the following equations

$$\begin{cases} u_{\lambda t} - \Delta u_\lambda + u_\lambda \cdot \nabla u_\lambda + \nabla p_\lambda = -\nabla \cdot (\nabla d_\lambda \odot \nabla d_\lambda) & \text{in} \quad \mathbb{R}^3 \times (0, \lambda^2 T), \\ \nabla \cdot u_\lambda = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \lambda^2 T), \\ d_{\lambda t} - \Delta d_\lambda + u_\lambda \cdot \nabla d_\lambda = -\lambda^2 f(d_\lambda) & \text{in} \quad \mathbb{R}^3 \times (0, \lambda^2 T). \end{cases}$$

Thus, the scaling dimension of corresponding quantities are $\dim u = -1$, $\dim p = -2$, and $\dim d = 0$ (we assign x with dimension 1 and t with 2). There are some useful dimensionless quantities and we list them here, let $z_0 = (x_0, t_0)$,

$$A(u,r,z_0)=\sup_{t_0-r^2<t<t_0} r^{-1}\int_{B_r(x_0)}|u|^2\mathrm{d}x,\quad E(u,r,z_0)=r^{-1}\int_{Q_r(z_0)}|\nabla u|^2\mathrm{d}x\mathrm{d}t,$$
$$C(u,r,z_0)=r^{-\frac{16}{7}}\int_{t_0-r^2}^{t_0}\|u\|^4_{L^{\frac{14}{5}}(B_r(x_0))}\mathrm{d}t,\quad K(u,r,z_0)=r^{-3}\int_{Q_r(z_0)}|u|^2\mathrm{d}x\mathrm{d}t,$$
$$C_1(u,r,z_0)=r^{-2}\int_{Q_r(z_0)}|u|^3\mathrm{d}x\mathrm{d}t,\quad D_1(p,r,z_0)=r^{-2}\int_{Q_r(z_0)}|p|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t,$$
$$D(p,z_0,r)=r^{-\frac{16}{7}}\int_{t_0-r^2}^{t_0}\|p\|^2_{L^{\frac{7}{5}}(B_r(x_0))}\mathrm{d}t.$$

Similarly, we denote these notations for ∇d:

$$A(\nabla d,r,z_0)=\sup_{t_0-r^2<t<t_0} r^{-1}\int_{B_r(x_0)}|\nabla d|^2\mathrm{d}x,\quad E(\nabla d,r,z_0)=r^{-1}\int_{Q_r(z_0)}|\nabla^2 d|^2\mathrm{d}x\mathrm{d}t,$$
$$C(\nabla d,r,z_0)=r^{-\frac{16}{7}}\int_{t_0-r^2}^{t_0}\|\nabla d\|^4_{L^{\frac{14}{5}}(B_r(x_0))}\mathrm{d}t,\quad K(\nabla d,r,z_0)=r^{-3}\int_{Q_r(z_0)}|\nabla d|^2\mathrm{d}x\mathrm{d}t,$$
$$C_1(\nabla d,r,z_0)=r^{-2}\int_{Q_r(z_0)}|\nabla d|^3\mathrm{d}x\mathrm{d}t.$$

For simplicity when $z_0=(0,0)$, we write $A(u,r)=A(u,r,(0,0))$, and write $A(r)\equiv A(u,\nabla d,r)=A(u,r)+A(\nabla d,r)$, and the meaning of $E(r)$, $C(r)$, $K(r)$ are alike.

Next, we write down several facts about Lorentz spaces. We say a locally integrable function $f\in L^{p,q}(\Omega)$, if the quasi-norm below is bounded

$$\begin{aligned}||f||_{L^{p,q}(\Omega)}&=\left(p\int_0^\infty \alpha^q d_{f,\Omega}(\alpha)^{\frac{q}{p}}\frac{d\alpha}{\alpha}\right)^{\frac{1}{q}},\quad q<\infty,\\ ||f||_{L^{p,\infty}(\Omega)}&=\sup_{\alpha>0}\alpha d_{f,\Omega}(\alpha)^{\frac{1}{p}},\quad q=\infty,\end{aligned}\tag{2.1}$$

where

$$d_{f,\Omega}(\alpha)=|\{x\in\Omega:|f(x)|>\alpha\}|.$$

A basic fact for such spaces is

$$L^{p,q_1}\subset L^{p,p}=L^p\subset L^{p,q_2}\subset L^{p,\infty}=L^p_w,\tag{2.2}$$

where $0<q_1<p<q_2<\infty$, and L^p_w is the weak-L^p space. If $|\Omega|$ is finite then $L^{p,q}(\Omega)\subset L^r(\Omega)$ for all $0<q\leq\infty$ and $0<r<p$,

$$\|g\|_{L^r(\Omega)}\leq|\Omega|^{\frac{1}{r}-\frac{1}{p}}\|g\|_{L^{p,q}(\Omega)}.\tag{2.3}$$

Lemma 2.1 *Let (u,d,p) be a weak solution to the liquid crystal Eq. (1.1) in $Q=\Omega\times(a,b)$. Let $z_0=(x_0,t_0)$ and let $\rho>0$ be such that $Q_\rho(z_0)\subset Q$. For every $r<(0,\frac{\rho}{4}]$, we have*

$$D_1(p,r,z_0) \le c\left(\frac{\rho}{r}\right)^{\frac{3}{2}}\left[A(u,\rho,z_0)^{\frac{3}{4}}E(u,\rho,z_0)^{\frac{3}{4}} + A(\nabla d,\rho,z_0)^{\frac{3}{4}}E(\nabla d,\rho,z_0)^{\frac{3}{4}}\right] + c\frac{r}{\rho}D(p,z_0,\rho)^{\frac{3}{4}}, \tag{2.4}$$

$$D(p,r,z_0) \le c\left[\left(\frac{r}{\rho}\right)^2 D(p,\rho,z_0) + \left(\frac{\rho}{r}\right)^{\frac{16}{7}} C(u,\nabla d,\rho,z_0)\right]. \tag{2.5}$$

Proof Let $z_0 = (0,0)$, we decompose p so that

$$p = p_1 + p_2,$$

where p_1 satisfies in B_ρ for a.e. $t \in [-\rho^2, 0]$, in the weak sense,

$$\begin{cases} \Delta p_1 = -\mathrm{div}\,\mathrm{div}\,(u\otimes u - [u\otimes u]_{B_\rho})) - \mathrm{div}\,\mathrm{div}\,(\nabla d\otimes\nabla d - [\nabla d\otimes\nabla d]_{B_\rho}) \\ p_1|_{\partial B_\rho} = 0. \end{cases} \tag{2.6}$$

And p_2 is a harmonic function in B_ρ, i.e.,

$$\Delta p_2 = 0.$$

Regarding p_1, by theory of Laplace operator and Calderón–Zygmund theorem, we have

$$\begin{aligned} &\left(\int_{B_\rho} |p_1|^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \\ &\le c\left(\int_{B_\rho} |u\otimes u - [u\otimes u]_{B_\rho}|^{\frac{3}{2}} + |\nabla d\otimes\nabla d - [\nabla d\otimes\nabla d]_{B_\rho}|^{\frac{3}{2}})dx\right)^{\frac{2}{3}} \\ &\le c\int_{B_\rho} |\nabla u||u| + |\nabla^2 d||\nabla d| \\ &\le c\left[\|\nabla u\|_{L^2(B_\rho)}\|u\|_{L^2(B_\rho)} + \|\nabla^2 d\|_{L^2(B_\rho)}\|\nabla d\|_{L^2(B_\rho)}\right], \\ &\int_{Q_\rho} |p_1|^{\frac{3}{2}} \le c\rho^2\left[A^{\frac{3}{4}}(u,\rho)E^{\frac{3}{4}}(u,\rho) + A^{\frac{3}{4}}(\nabla d,\rho)E^{\frac{3}{4}}(\nabla d,\rho)\right]. \end{aligned}$$

For $x \in B_{\frac{\rho}{2}}$,

$$|p_2(x,t)| \le c -\!\!\!\!\!\!\int_{B_\rho} |p_2| \le c\left(-\!\!\!\!\!\!\int_{B_\rho} |p_2|^l\right)^{\frac{1}{l}}, \quad l > 1,$$

i.e., for $r \le \frac{\rho}{2}$,

$$\frac{1}{r^2}\int_{Q_r}|p_2|^{\frac{3}{2}} \le \frac{c\,r}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^{0}\|p_2\|^{3/2}_{L^{\frac{7}{5}}(B_\rho)}.$$

$$\begin{aligned} D_1(p,r) &\le \frac{1}{r^2}\int_{Q_\rho}|p_1|^{\frac{3}{2}} + \frac{1}{r^2}\int_{Q_r}|p_2|^{\frac{3}{2}} \\ &\le c(\frac{\rho}{r})^2\Big[A^{\frac{3}{4}}(u,\rho)E^{\frac{3}{4}}(u,\rho) + A^{\frac{3}{4}}(B,\rho)E^{\frac{3}{4}}(\nabla d,\rho)\Big] \\ &\quad + \frac{c\,r}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^{0}\left(\|p\|^{3/2}_{L^{\frac{7}{5}}(B_\rho)} + \|p_1\|^{3/2}_{L^{\frac{7}{5}}(B_\rho)}\right). \end{aligned}$$

Now,

$$\frac{r}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^{0}\|p_1\|^{3/2}_{L^{\frac{7}{5}}(B_\rho)} \le \frac{r}{\rho}D_1(p_1,\rho),$$

$$\frac{r}{\rho^{\frac{45}{14}}}\int_{-\rho^2}^{0}\|p\|^{3/2}_{L^{\frac{7}{5}}(B_\rho)} \le \frac{r}{\rho}D(p,\rho)^{\frac{3}{4}},$$

so that

$$D_1(p,r) \le c\left(\frac{\rho}{r}\right)^2\Big[A^{\frac{3}{4}}(u,\rho)E^{\frac{3}{4}}(u,\rho) + A^{\frac{3}{4}}(\nabla d,\rho)E^{\frac{3}{4}}(\nabla d,\rho)\Big] + c\frac{r}{\rho}D(p,\rho)^{\frac{3}{4}}.$$

On the other hand

$$\int_{B_r}|p_1|^{\frac{7}{5}} \le c\int_{B_r}|u|^{\frac{14}{5}} + |\nabla d|^{\frac{14}{5}},$$

we have

$$D(p_1,r) \le c\left(\frac{\rho}{r}\right)^{\frac{16}{7}}C(u,\nabla d,\rho).$$

For $x \in B_{\frac{\rho}{2}}$,

$$|p_2(x,t)| \le c\left(-\!\!\!\!\!\!\int_{B_\rho}|p_2|^{\frac{7}{5}}dx\right)^{\frac{5}{7}},$$

we have

$$\begin{aligned} D(p_2,r) = r^{-\frac{16}{7}}\int_{-r^2}^{0}\|p_2\|^2_{L^{\frac{7}{5}}(B_r)} &\le c\left(\frac{r}{\rho}\right)^2 D(p_2,\rho) \\ &\le c\left(\frac{r}{\rho}\right)^2\big[D(p,\rho)+D(p_1,\rho)\big], \end{aligned}$$

therefore,

$$
\begin{aligned}
D(p,r) &\le c\big[D(p_1,r)+D(p_2,r)\big]\\
&\le c\left(\frac{\rho}{r}\right)^{\frac{16}{7}} C(u,\nabla d,\rho)+c\,D(p_2,r)\\
&\le c\left[\left(\frac{r}{\rho}\right)^{2} D(p,\rho)+\left(\frac{\rho}{r}\right)^{\frac{16}{7}} C(u,\nabla d,\rho)\right].
\end{aligned}
$$

□

We use following analysis Lemma 2.2 which can be found in ([6], Lemma 6.1) to prove local estimate Lemma 2.3.

Lemma 2.2 *Let $I(s)$ be a bounded nonnegative function in the interval $[R_1,R_2]$. Suppose that for any $s,\ \rho\in[R_1,R_2]$ and $s<\rho$, the following yields*

$$I(s)\le\big[a_1(\rho-s)^{-\alpha}+a_2(\rho-s)^{-\beta}+a_3(\rho-s)^{-\gamma}+a_4\big]+\theta I(\rho),$$

with $\alpha>\beta>\gamma>0, a_i>0, i=1,2,3,4$ and $\theta\in[0,1)$. Then,

$$I(R_1)\le c(\alpha,\beta,\gamma)[a_1(R_2-R_1)^{-\alpha}+a_2(R_2-R_1)^{-\beta}+a_3(R_2-R_1)^{-\gamma}+a_4].$$

Lemma 2.3 *Let $(u,\ d,\ p)$ be a suitable weak solution to the liquid crystal Eq. (1.1) in $Q=\Omega\times(a,b)$. Assume that $z_0=(x_0,t_0)$ and $1\ge r>0$ with $Q_r(z_0)\subset Q$. Then the following holds:*

$$
\begin{aligned}
&A(u,\nabla d,r/2,z_0)+E(u,\nabla d,r/2,z_0)\\
&\quad\le c\Big[C(u,\nabla d,r,z_0)^{\frac12}+C(u,r,z_0)^{\frac12}C(\nabla d,r,z_0)^{\frac12}+C(u,r,z_0)\\
&\qquad+D(p,r,z_0)^{\frac{7}{10}}C(u,r,z_0)^{\frac{3}{20}}\Big].
\end{aligned}\tag{2.7}
$$

Proof Let $r/2\le s<\rho\le r<1$, and $Q_r\subset Q_1\equiv Q$. Choosing test function $\phi(x,t)=\eta_1(x)\eta_2(t)$ with $\eta_1\in C_0^\infty(B_\rho(x_0))$, $0\le\eta_1\le1$ in $\mathbb{R}^3$, $\eta_1\equiv1$ on $B_s(x_0)$, and $|\nabla^\alpha\eta_1|\le\frac{C}{(\rho-s)^{|\alpha|}}$, for all multi-index α, with $|\alpha|\le3$. And $\eta_2\in C_0^\infty(t_0-\rho^2,t_0+\rho^2)$, $0\le\eta_2\le1$ in $\mathbb{R}$, $\eta_2(t)\equiv1$ for $t\in[t_0-s^2,t_0+s^2]$, with $|\eta_2'(t)|\le\frac{C}{\rho^2-s^2}\le\frac{C}{r(\rho-s)}$. From the local energy inequality we have

$$\begin{aligned}
&\int_\Omega (|u|^2+|\nabla d|^2)\phi \mathrm{d}x + 2\int_a^t\int_\Omega (|\nabla u|^2+|\nabla^2 d|^2)\phi \mathrm{d}x\mathrm{d}s\\
&\le c\int_{t_0-\rho^2}^{t_0} \||u|^2+|\nabla d|^2\|_{W^{-1,2}(B_\rho(x_0))}\|\nabla(\phi_t+\Delta\phi)\|_{L^2(B_\rho(x_0))}\mathrm{d}t\\
&\quad + c\int_{t_0-\rho^2}^{t_0} \||u|^2\|_{W^{-1,2}(B_\rho(x_0))}\|\nabla(u\cdot\nabla\phi)\|_{L^2(B_\rho(x_0))}\mathrm{d}t\\
&\quad + c\int_{t_0-\rho^2}^{t_0} \||u||\nabla d|\|_{W^{-1,2}(B_\rho(x_0))}\||\nabla^2 d||\nabla\phi|+|\nabla d||\nabla^2\phi|\|_{L^2(B_\rho(x_0))}\mathrm{d}t\\
&\quad + c\int_{t_0-\rho^2}^{t_0}\int_{B_\rho} |p\, u\cdot\nabla\phi|\mathrm{d}x\mathrm{d}t + c\int_{t_0-\rho^2}^{t_0}\int_{B_\rho} |\nabla d|^2\phi\\
&= J_1+J_2+J_3+J_4+J_5.
\end{aligned} \tag{2.8}$$

Here, we rewrite the term

$$\int_{t_0-\rho^2}^{t_0}\int_{B_\rho} |\nabla d|^2\, u\cdot\nabla\phi \mathrm{d}x\mathrm{d}t = \int_{t_0-\rho^2}^{t_0}\int_{B_\rho} (u\otimes\nabla d) : (\nabla\phi\otimes\nabla d)\mathrm{d}x\mathrm{d}t.$$

Denote

$$\begin{aligned}
I(s) &= \sup_{t_0-s^2\le t\le t_0}\int_{B_s(x_0)} |\nabla d|^2\mathrm{d}x + \sup_{t_0-s^2\le t\le t_0}\int_{B_s(x_0)} |u|^2\mathrm{d}x\\
&\quad + \int_{t_0-s^2}^{t_0}\int_{B_s(x_0)} |\nabla^2 d|^2\mathrm{d}x\mathrm{d}t + \int_{t_0-s^2}^{t_0}\int_{B_s(x_0)} |\nabla u|^2\mathrm{d}x\mathrm{d}t\\
&= sA(\nabla d, s, z_0) + sA(u,s,z_0) + sE(\nabla d,s,z_0) + sE(u,s,z_0)\\
&= I_1(\nabla d, s) + I_1(u,s) + I_2(\nabla d,s) + I_2(u,s),
\end{aligned}$$

and

$$I(u,s) = I_1(u,s)+I_2(u,s), \quad I(\nabla d,s) = I_1(\nabla d,s)+I_2(\nabla d,s).$$

Estimate J_1, J_2, J_3, J_4 and J_5, respectively, as the following:

$$\begin{aligned}
J_1 &\le \frac{c\rho^{3/2}}{(\rho-s)^3}\int_{t_0-\rho^2}^{t_0} \||u|^2+|\nabla d|^2\|_{W^{-1,2}(B_\rho)}\mathrm{d}t\\
&\le \frac{c\rho^{\frac{5}{2}}}{(\rho-s)^3}\left[\int_{t_0-\rho^2}^{t_0} \||u|^2+|\nabla d|^2\|^2_{W^{-1,2}(B_\rho)}\mathrm{d}t\right]^{\frac{1}{2}},
\end{aligned}$$

by Young's inequality, we get

$$
\begin{aligned}
J_2 \leq & c \int_{t_0-\rho^2}^{t_0} \left[\| |u|^2 \|_{W^{-1,2}(B_\rho)} \left(\frac{\|\nabla u\|_{L^2(B_\rho)}}{\rho - s} + \frac{\|u\|_{L^2(B_\rho)}}{(\rho - s)^2} \right) \right] \mathrm{d}t \\
\leq & \frac{c}{\rho - s} \left[\int_{t_0-\rho^2}^{t_0} \| |u|^2 \|_{W^{-1,2}(B_\rho)}^2 \mathrm{d}t \right]^{\frac{1}{2}} I_2(v,\rho)^{\frac{1}{2}} \\
& + \frac{c\rho}{(\rho - s)^2} \left[\int_{t_0-\rho^2}^{t_0} \| |u|^2 \|_{W^{-1,2}(B_\rho)}^2 \mathrm{d}t \right]^{\frac{1}{2}} I_1(u,\rho)^{\frac{1}{2}} \\
\leq & \frac{1}{4} I(u,\rho) + \left[\frac{c}{(\rho - s)^2} + \frac{c\rho^2}{(\rho - s)^4} \right] \int_{t_0-\rho^2}^{t_0} \| |u|^2 \|_{W^{-1,2}(B_\rho)}^2 \mathrm{d}t,
\end{aligned}
$$

similarly, we have

$$
\begin{aligned}
J_3 \leq & c \int_{t_0-\rho^2}^{t_0} \left[\| |u| |\nabla d| \|_{W^{-1,2}(B_\rho)} \left(\frac{\|\nabla^2 d\|_{L^2(B_\rho)}}{\rho - s} + \frac{\|\nabla d\|_{L^2(B_\rho)}}{(\rho - s)^2} \right) \right] \mathrm{d}t \\
\leq & \frac{1}{4} I(\nabla d,\rho) + \left[\frac{c}{(\rho - s)^2} + \frac{c\rho^2}{(\rho - s)^4} \right] \int_{t_0-\rho^2}^{t_0} \| |u \nabla d| \|_{W^{-1,2}(B_\rho)}^2 \mathrm{d}t.
\end{aligned}
$$

For the term J_4, using Hölder's inequality and Sobolev inequality, we have

$$
\begin{aligned}
J_4 \leq & c \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho(x_0))} \|u \nabla \phi\|_{L^{\frac{7}{2}}(B_\rho(x_0))} \\
\leq & c \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \|\nabla(u \nabla \phi)\|_{L^2(B_\rho)}^{\frac{4}{7}} \|u \nabla \phi\|_{L^{\frac{9}{4}}(B_\rho)}^{\frac{3}{7}} \\
\leq & c \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \left[\|\nabla u \nabla \phi)\|_{L^2(B_\rho)} + \|u \nabla^2 \phi\|_{L^2(B_\rho)} \right]^{\frac{4}{7}} \|u \nabla \phi\|_{L^{\frac{9}{4}}(B_\rho)}^{\frac{3}{7}} \\
\leq & \frac{c}{\rho - s} \|\nabla u\|_{L^2(Q_\rho(z_0))}^{\frac{4}{7}} \left[\int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)}^{\frac{7}{5}} \|u\|_{L^{\frac{9}{4}}(B_\rho)}^{\frac{3}{5}} \right]^{\frac{5}{7}} \\
& + \frac{c}{(\rho - s)^{\frac{11}{7}}} \sup_t \|u\|_{L^2(B_\rho)}^{\frac{4}{7}} \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \|u\|_{L^{\frac{9}{4}}(B_\rho)}^{\frac{3}{7}} \\
\leq & \frac{1}{4} I(u,\rho) + \frac{c}{(\rho - s)^{\frac{7}{5}}} \int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)}^{\frac{7}{5}} \|u\|_{L^{\frac{9}{4}}(B_\rho)}^{\frac{3}{5}} \\
& + \frac{c}{(\rho - s)^{\frac{11}{5}}} \left[\int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)} \|u\|_{L^{\frac{9}{4}}(B_\rho)}^{\frac{3}{7}} \right]^{\frac{7}{5}} \\
\leq & \frac{1}{4} I(u,\rho) + \frac{c\rho^{\frac{16}{35}}}{(\rho - s)^{\frac{7}{5}}} \left[\int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)}^2 \right]^{\frac{7}{10}} \left[\int_{t_0-\rho^2}^{t_0} \|u\|_{L^{\frac{14}{5}}(B_\rho)}^4 \right]^{\frac{3}{20}} \\
& + \frac{c\rho^{\frac{44}{35}}}{(\rho - s)^{\frac{11}{5}}} \left[\int_{t_0-\rho^2}^{t_0} \|p\|_{L^{\frac{7}{5}}(B_\rho)}^2 \right]^{\frac{7}{10}} \left[\int_{t_0-\rho^2}^{t_0} \|u\|_{L^{\frac{14}{5}}(B_\rho)}^4 \right]^{\frac{3}{20}}. \\
J_5 \leq & c\, \rho^{\frac{13}{7}} \left[\int_{t_0-\rho^2}^{t_0} \|\nabla d\|_{L^{\frac{14}{5}}(B_\rho)}^4 \right]^{1/2}.
\end{aligned}
$$

From (2.8), using estimates above with respect to J_1, J_2, J_3, J_4 and J_5 we get

$$
\begin{aligned}
I(s) \le \frac{1}{2} I(\rho) + \frac{c r^{\frac{5}{2}}}{(\rho-s)^3}\left[\int_{t_0-r^2}^{t_0} \||u|^2+|\nabla d|^2\|_{W^{-1,2}(B_r)}^2\right]^{\frac{1}{2}} \\
+ \left[\frac{c}{(\rho-s)^2}+\frac{c r^2}{(\rho-s)^4}\right]\int_{t_0-r^2}^{t_0}\left[\||u\,\nabla d|\|_{W^{-1,2}(B_r)}^2+\||u|^2\|_{W^{-1,2}(B_r)}^2\right] \\
+ \frac{c r^{\frac{16}{35}}}{(\rho-s)^{\frac{7}{5}}}\left[\int_{t_0-r^2}^{t_0}\|p\|_{L^{\frac{7}{5}}(B_r)}^2\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|u\|_{L^{\frac{14}{5}}(B_r)}^4\right]^{\frac{3}{20}} \\
+ \frac{c r^{\frac{44}{35}}}{(\rho-s)^{\frac{11}{5}}}\left[\int_{t_0-r^2}^{t_0}\|p\|_{L^{\frac{7}{5}}(B_r)}^2\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|u\|_{L^{\frac{14}{5}}(B_r)}^4\right]^{\frac{3}{20}} \\
+ c\,\rho^{\frac{13}{7}}\left[\int_{t_0-\rho^2}^{t_0}\|\nabla d\|_{L^{\frac{14}{5}}(B_\rho)}^4\right]^{1/2}.
\end{aligned}
\tag{2.9}
$$

By Lemma 2.2, we have

$$
\begin{aligned}
I(r/2) \le c r^{-\frac{1}{2}}\left[\int_{t_0-r^2}^{t_0}\||u|^2+|\nabla d|^2\|_{W^{-1,2}(B_r)}^2\right]^{\frac{1}{2}} \\
+ c r^{-2}\int_{t_0-r^2}^{t_0}\left(\||u\,\nabla d|\|_{W^{-1,2}(B_r)}^2+\||u|^2\|_{W^{-1,2}(B_r)}^2\right)\mathrm{d}t \\
+ c r^{-\frac{33}{35}}\left[\int_{t_0-r^2}^{t_0}\|p\|_{L^{\frac{7}{5}}(B_r)}^2\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|u\|_{L^{\frac{14}{5}}(B_r)}^4\right]^{\frac{3}{20}} \\
+ c\,r^{\frac{13}{7}}\left[\int_{t_0-r^2}^{t_0}\|\nabla d\|_{L^{\frac{14}{5}}(B_r)}^4\right]^{1/2}.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
A(u,\nabla d, r/2, z_0) + E(u,\nabla d, r/2, z_0) \\
\le c r^{-\frac{3}{2}}\left[\int_{t_0-r^2}^{t_0}\||u|^2+|\nabla d|^2\|_{W^{-1,2}(B_r)}^2\right]^{1/2} \\
+ c r^{-3}\int_{t_0-r^2}^{t_0}\left(\||u\,\nabla d|\|_{W^{-1,2}(B_r)}^2+\||u|^2\|_{W^{-1,2}(B_r)}^2\right) \\
+ c r^{-\frac{68}{35}}\left[\int_{t_0-r^2}^{t_0}\|p\|_{L^{\frac{7}{5}}(B_r)}^2\right]^{\frac{7}{10}}\left[\int_{t_0-r^2}^{t_0}\|u\|_{L^{\frac{14}{5}}(B_r)}^4\right]^{\frac{3}{20}} \\
+ c\,r^{\frac{6}{7}}\left[\int_{t_0-r^2}^{t_0}\|\nabla d\|_{L^{\frac{14}{5}}(B_r)}^4\right]^{1/2}.
\end{aligned}
\tag{2.10}
$$

For $f \in L^{7/5}(B_r(x_0))$ and $\varphi \in C_0^\infty(B_r(x_0))$, we have

$$\begin{aligned}\left|\int_{B_r(x_0)}\varphi f(x)dx\right| &\le c\int_{B_r(x_0)}\left[\int_{B_r(x_0)}\frac{|\nabla\varphi(y)|}{|x-y|^2}\mathrm{d}y\right]|f(x)|\mathrm{d}x\\ &= c\int_{B_r(x_0)}|\nabla\varphi(y)|\left[\int_{B_r(x_0)}\frac{|f(x)|}{|x-y|^2}\mathrm{d}x\right]\mathrm{d}y\\ &\le c\|\nabla\varphi\|_{L^2(B_r(x_0))}\|\mathbf{I}_1(\chi_{B_r(x_0)}|f|)\|_{L^2(B_r(x_0))}.\end{aligned}$$

where $\mathbf{I}_1$ is the first order Riesz's potential defined by

$$\mathbf{I}_1(\mu)(x)=c\int_{\mathbb{R}^3}\frac{d\mu(y)}{|x-y|^2},\quad x\in\mathbb{R}^3.$$

By using Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}\|f\|_{W^{-1,2}(B_r(x_0))} \le&\|\mathbf{I}_1(\chi_{B_r(x_0)}|f|)\|_{L^2(B_r(x_0))}\\ \le& c\|f\|_{L^{\frac{6}{5}}(B_r(x_0))}\le cr^{\frac{5}{14}}\|f\|_{L^{\frac{7}{5}}(B_r(x_0))}.\end{aligned}\tag{2.11}$$

Applying (2.11) with $f=|y|^2+|\nabla d|^2$, and $f=|u||\nabla d|$, we obtain

$$\begin{aligned}&r^{-3}\int_{t_0-r^2}^{t_0}\||u|^2+|\nabla d|^2\|^2_{W^{-1,2}(B_r)}\\ &\le r^{\frac{-16}{7}}\int_{t_0-r^2}^{t_0}\left(\|u\|^4_{L^{\frac{14}{5}}(B_r)}+\|\nabla d\|^4_{L^{\frac{14}{5}}(B_r)}\right)\\ &=C(u,\nabla d,r,z_0),\\ &r^{-3}\int_{t_0-r^2}^{t_0}\left(\||u\,\nabla d|\|^2_{W^{-1,2}(B_r)}+\||u|^2\|^2_{W^{-1,2}(B_r)}\right)\\ &\le\left[C(u,r,z_0)^{\frac{1}{2}}C(\nabla d,r,z_0)^{\frac{1}{2}}+C(u,r,z_0)\right].\end{aligned}$$

□

We need the bounded estimates for $C(\nabla d,r)$ and $D(p,r)$ with the help of the bounded of $C(u,r)$.

Lemma 2.4 *Suppose that* (u,d,p) *is a suitable weak solution in* $Q_1(z_0)=B_1(x_0)\times(t_0-1,t_0)$. *Let*

$$C(u,r,z_0)\le M\quad\text{for any }0<r\le 1,$$

for $M>0$. *Then for every* $0<r<1/4$, *we have the following estimates:*

$$\begin{aligned}&A(u,\nabla d,r/2,z_0)+E(u,\nabla d,r/2,z_0)+C(\nabla d,r/2,z_0)+D(p,r/2,z_0)\\ &\quad\le c(M,C(\nabla d,1/2,z_0),D(p,1/2,z_0));\\ &C_1(u,\nabla d,r,z_0)+D_1(p,r,z_0)\le c(M,D(p,1/2,z_0),C(\nabla d,1/2,z_0)).\end{aligned}$$

Proof Without loss of generality, we consider $z_0=(0,0)$. It is easy to see that

$$C(\nabla d,r/2)\le c\left[E(\nabla d,r)^{\frac{5}{7}}C(\nabla d,r)^{\frac{2}{7}}+C(\nabla d,r)^{\frac{9}{14}}\right].\tag{2.12}$$

Combining with (2.7) and (2.12) we obtain

$$C(\nabla d, r/2) \leq c(M)\left[1 + C(\nabla d, r)^{\frac{1}{2}} + D(p, r)^{\frac{7}{10}}\right]^{\frac{5}{7}} C(\nabla d, r)^{\frac{2}{7}} + c\, C(\nabla d, r)^{\frac{9}{14}}. \tag{2.13}$$

Let $r = \theta\rho$ with $\theta \leq \frac{1}{4}$. From (2.5) (2.13), we have

$$\begin{aligned} &C(\nabla d, r) + D(p, r)^{\frac{5}{6}} \\ &\quad \leq c(M)\theta^{-\frac{32}{49}}\left[1 + \theta^{-\frac{8}{7}} C(\nabla d, \rho)^{\frac{1}{2}} + \theta^{-\frac{8}{5}} D(p, \rho)^{\frac{7}{10}}\right]^{\frac{5}{7}} C(\nabla d, \rho)^{\frac{2}{7}} \\ &\quad + c\theta^{-\frac{72}{49}} C(\nabla d, \rho)^{\frac{9}{14}} + c\theta^{\frac{5}{3}} D(p, \rho)^{\frac{5}{6}} + c\theta^{-\frac{40}{21}} C(\nabla d, \rho)^{\frac{5}{6}} + c(M, \theta) \end{aligned}$$

Using Young's inequality, we have

$$C(\nabla d, r) + D(p, r)^{\frac{5}{6}} \leq \eta C(\nabla d, \rho) + (\eta + c\theta^{\frac{5}{3}}) D(p, \rho)^{\frac{5}{6}} + c(\theta, \eta, M)$$

Set $F(r) = C(\nabla d, r) + D(p, r)^{\frac{5}{6}}$. Choose $\eta > 0$ and $\theta > 0$ small enough, we have

$$F(r) \leq \frac{1}{2} F(\rho) + c.$$

By the standard iterating argument

$$C(\nabla d, r) + D(p, r)^{\frac{5}{6}} \leq c(M, D(p, 1/2), C(\nabla d, 1/2)), \quad r \in (0, \frac{1}{4}].$$

So that for $0 < r \leq 1/4$,

$$A(u, \nabla d, r) + E(u, \nabla d, r) + C(\nabla d, r) + D(p, r) \leq c(M, C(\nabla d, 1/2), D(p, 1/2)).$$

The estimates of $C_1(u, \nabla d, r)$ and $D_1(p, r)$ are immediate results. □

3 Proof of Theorem 1.3

We shall prove the following Proposition 3.1, Theorem 1.3 is an immediate result.

Proposition 3.1 *Let (u, d, p) a weak solution of (1.1) with*

$$\|u\|_{L^\infty(0,T;L^{3,\infty}(\mathbb{R}^3))} \leq M,$$

for $z_0 = (x_0, t_0)$ and $R > 0$ such that $Q_R(z_0) \subset Q_T$, (d, p) satisfy

$$C(\nabla d, R, z_0) + D(p, R, z_0) \leq N.$$

there exists a positive number $\varepsilon(M, N) < \frac{1}{4}$ such that if for some $0 < r \leq R/2$,

$$r^{-3}\left|\left\{x \in B_r(x_0) : |u(x, t_0)| > \varepsilon r^{-1}\right\}\right| \leq \varepsilon, \tag{3.1}$$

then there exists $\rho \in [2r\varepsilon, r]$ such that

$$\frac{1}{\rho^2}\int_{Q_\rho(z_0)}|u|^3+|\nabla d|^3<\varepsilon_0 \tag{3.2}$$

where ε_0 is the same number in Proposition 1.2.

Proof Let (u, d, p) be a weak solution of (1.1) with $u\in L^\infty(0,T;L^{3,\infty}(\mathbb{R}^3))$. We assume

$$\sup_{0<r<T}\|u(t)\|_{L^{3,\infty}(\mathbb{R}^3)}\le M. \tag{3.3}$$

Note that $|d|\le 1$, we have (see [28])

$$||\nabla d||_{L^4_x}\le c||\nabla d||^{\frac12}_{\dot B^{-1}_{\infty,\infty}}||\nabla d||^{\frac12}_{\dot H^1}\le c\|d\|^{1/2}_{L^\infty}||\nabla d||^{\frac12}_{\dot H^1}.$$

Also since the real interpolation $L^4=[L^{6,\infty},L^{3,\infty}]_{\frac12,4}$ holds, then

$$||u||_{L^4_x}\le c||u||^{\frac12}_{L^{6,\infty}_x}||u||^{\frac12}_{L^{3,\infty}_x}\le c||\nabla u||^{\frac12}_{L^2_x}M^{\frac12}.$$

From energy inequality and estimates above we get

$$\|(u,\nabla d)\|_{L^4(Q_T)}\le c(M,c_0), \tag{3.4}$$

which yields (u, d, p) is a local suitable weak solution of (1.1) and $u\in C([0,T];L^2(\mathbb{R}^3))$.

We use a contradiction argument for $z_0=(0,0)$ and $R=1$. Fixed $N, M>0$ if the assertion of the proposition were false, then there would exist $\epsilon_k\downarrow 0$, and suitable weak solutions (u_k,d_k,p_k) of (1.1) and $r_k\le 1/2$ such that

$$\|u_k\|_{L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))}\le M; \tag{3.5}$$

$$C(\nabla d_k,1)+D(p_k,1)\le N; \tag{3.6}$$

$$r_k^{-3}\left|\left\{x\in B_{r_k}(0)\,:\,|u_k(x,0)|>\epsilon_k r_k^{-1}\right\}\right|\le\epsilon_k; \tag{3.7}$$

and for all $\rho\in[2r_k\epsilon_k,r_k]$,

$$\frac{1}{\rho^2}\int_{Q_\rho(0)}|u_k|^3+|\nabla d_k|^3>\varepsilon_0/2. \tag{3.8}$$

Since for $0<r\le 1$

$$\begin{aligned}C(u_k,r)=&r^{-16/7}\int_{-r^2}^0|B_r|^{2/21}\|u_k\|^4_{L^{3,\infty}(B_r)}\\ \le& c\,\|u_k\|^4_{L^\infty(-1,0;L^{3,\infty}(B_1))}\le M^4,\end{aligned}$$

combining the estimate and (3.6) with Lemma 2.4, we get, for $0<r\le 1/2$,

$$A(u_k,\nabla d_k,r)+E(u_k,\nabla d_k,r)+C(\nabla d_k,r)+D(p_k,r)\le c(M,N).$$

Similarly, for any $z_0\in Q_{1/2}$ and $0<r\le 1/2$, we have

$$A(u_k,\nabla d_k,r,z_0)+E(u_k,\nabla d_k,r,z_0)+C(\nabla d_k,r,z_0)+D(p_k,r,z_0)\le c(M,N).$$

Define, for $(x,t)\in Q_{r_k^{-1}}$,

$$\begin{cases} U_k(x,t)=r_ku_k(r_kx,r_k^2t),\\ D_k(x,t)=d_k(r_kx,r_k^2t),\\ P_k(x,t)=r_k^2p_k(r_kx,r_k^2t). \end{cases} \tag{3.9}$$

Obviously, (U_k,D_k,P_k) are weak solutions to system (1.1) with the right side of $(1.1)_3$ replaced by $r_k^2f(D_k)$ in $Q_{r_k^{-1}}$. Now, for $a>0$, and $ar_k\le 1/2$,

$$\begin{aligned} \|U_k\|_{L^\infty(-r_k^{-1},0;L^{3,\infty}(B_{r_k^{-1}}))}&=\|u_k\|_{L^\infty(-1,0;L^{3,\infty}(B_1))}\le M,\\ C(\nabla D_k,a)+D(P_k,a)&=C(\nabla d_k,ar_k)+D(p_k,ar_k)\le N,\\ C(U_k,a)&=C(u_k,ar_k)\\ &\le (ar_k)^{-16/7}\int_{-(ar_k)^2}^0|B_{ar_k}|^{2/21}\|u_k\|^4_{L^{3,\infty}(B_{ar_k})}\\ &\le c\,\|u_k\|^4_{L^\infty(-1,0;L^{3,\infty}(B_1))}\le M^4. \end{aligned}$$

We have by Lemma 2.4 again

$$\begin{aligned} &A(U_k,\nabla D_k,a)+E(U_k,\nabla D_k,a)+D(P_k,a)\\ &\quad+C_1(U_k,\nabla D_k,a)+D_1(P_k,a)\le c(M,N). \end{aligned} \tag{3.10}$$

So that

$$\begin{aligned} \|U_k\|^4_{L^4(Q_a)}&\le c\int_{-a^2}^0\|U_k\|^2_{L^{3,\infty}(B_a)}\|U_k\|^2_{L^6(B_a)}\\ &\le c\int_{-a^2}^0\|U_k\|^2_{L^{3,\infty}(B_a)}\|\nabla U_k\|^2_{L^2(B_a)}+c\int_{-a^2}^0|B_a|\|U_k\|^2_{L^{3,\infty}(B_a)}\|U_k\|^2_{L^2(B_a)}\\ &\le c\,aM^2\big(A(U_k,a)+E(U_k,a)\big),\\ \|\nabla D_k\|^4_{L^4(Q_a)}&\le a\,c(M,N)\big(A(\nabla D_k,a)+E(\nabla D_k,a)\big). \end{aligned}$$

Thus, the L^p estimate holds for (U_k,D_k,P_k) in Q_a, for any $a>0$,

$$\begin{aligned} &\int_{Q_a}|U_k|^4+|\nabla D_k|^4+|\partial_tU_k|^{\frac43}+|\partial_t\nabla D_k|^{\frac43}+|\nabla^2U_k|^{\frac43}+|\nabla^2\nabla D_k|^{\frac43}+|\nabla P_k|^{\frac43}\\ &\quad\le c_2(a,M,N). \end{aligned} \tag{3.11}$$

By Aubin–Lion's lemma, there exists a triplet (v,e,q) such that

$$\begin{cases} U_k\to v, & \text{in } L^3(Q_a),\\ \nabla D_k\to\nabla e, & \text{in } L^3(Q_a),\\ P_k\rightharpoonup q, & \text{in } L^{\frac32}(Q_a), \end{cases} \tag{3.12}$$

and

$$U_k \to v, \quad \nabla D_k \to \nabla e \quad \text{in } C([-a^2,0];L^{\frac{4}{3}}(B_a)).$$

Using estimates above, the limit function (v, e, q) satisfy, in the sense of suitable weak solutions on $\mathbb{R}^3 \times (-\infty, 0)$,

$$\begin{cases} v_t - \Delta v + v \cdot \nabla v + \nabla q = -\nabla \cdot (\nabla e \odot \nabla e), \\ \nabla \cdot v = 0, \\ e_t - \Delta e + v \cdot \nabla e = 0. \end{cases} \tag{3.13}$$

From (3.7) and (3.8), we get

$$\left|\left\{x \in B(0) \,:\, |U_k(x,0)| > \epsilon_k\right\}\right| < \epsilon_k, \tag{3.14}$$

and for $\rho \in [2\epsilon_k, 1]$

$$\frac{1}{\rho^2}\int_{Q_\rho} |U_k|^3 + |\nabla D_k|^3 > \varepsilon_0/2. \tag{3.15}$$

Taking limit we get

$$v(\cdot, 0) = 0, \quad \text{in } B_1(0), \tag{3.16}$$

and for $\rho \in (0, 1]$

$$\frac{1}{\rho^2}\int_{Q_\rho} |v|^3 + |\nabla e|^3 \geq \varepsilon_0/2. \tag{3.17}$$

The crucial point here is a reduction to backward uniqueness for the heat operator with lower order terms as [3]. Set

$$v_k = \rho_k v(\rho_k x, \rho_k^2 t), \quad e_k = e(\rho_k x, \rho_k^2 t), \quad q_k = \rho_k^2 q(\rho_k x, \rho_k^2 t).$$

Then (v_k, e_k, q_k) satisfy (3.13), and similar to (3.10) for any $a > 0$

$$\begin{aligned} A(v_k, \nabla e_k, a) &+ E(v_k, \nabla e_k, a) + D(q_k, a) \\ &+ C_1(v_k, \nabla e_k, a) + D_1(q_k, a) \leq c(M, N). \end{aligned} \tag{3.18}$$

As before there exists a triplet $(\widetilde{v}, \widetilde{e}, \widetilde{q})$ is suitable weak solution of (3.13) such that

$$\begin{cases} v_k \to \widetilde{v}, & \text{in } L^3(Q_a), \\ \nabla e_k \to \nabla \widetilde{e}, & \text{in } L^3(Q_a), \\ q_k \rightharpoonup \widetilde{q}, & \text{in } L^{\frac{3}{2}}(Q_a), \end{cases} \tag{3.19}$$

and

$$v_k \to \widetilde{v}, \quad \nabla e_k \to \nabla \widetilde{e} \quad \text{in } C([-a^2,0];L^{\frac{4}{3}}(B_a)).$$

From (3.16) we get

$$\widetilde{v}(\cdot, 0) = 0, \quad \text{in } \mathbb{R}^3, \tag{3.20}$$

from (3.17) and take $\rho = \rho_k$

$$\int_{Q_1} |\widetilde{v}|^3 + |\nabla \widetilde{e}|^3 \geq \varepsilon_0/2. \tag{3.21}$$

On the other hand, for fixed z_0 and $0 < R \leq \frac{1}{2r_k}$, as (3.18)

$$\begin{aligned} &A(v_k, \nabla e_k, R, z_0) + E(v_k, \nabla e_k, R, z_0) + D(q_k, R, z_0) \\ &\quad + C_1(v_k, \nabla e_k, R, z_0) + D_1(q_k, R, z_0) \leq c(M, N). \end{aligned} \tag{3.22}$$

By the Fubini theorem, we have,

$$\begin{aligned} &\left| \left\{ (x,t) \in \mathbb{R}^3 \times (-T, 0) \,:\, |\widetilde{v}(x,t)| > \gamma \right\} \right| \\ &\quad = \int_{-T}^{0} d_{\widetilde{v}(t)}(\gamma) \mathrm{d}t \leq \gamma^{-3} M^3 T. \end{aligned}$$

Hence, for any $\eta > 0$, there exists a B_R such that

$$\left| \left\{ (x,t) \in (\mathbb{R}^3 \backslash B_R) \times (-T, 0) \,:\, |\widetilde{v}(x,t)| > \gamma \right\} \right| < \eta.$$

Let $Q_1(z_0) \subset (\mathbb{R}^3 \backslash B_R) \times (-T, 0]$, by (3.22), we have $A(\widetilde{v}, \nabla \widetilde{e}, \theta, z_0) + E(\widetilde{v}, \nabla \widetilde{e}, \theta, z_0) \leq C(M, N)$ for any $0 < \theta \leq 1$. Thus, by the interpolation inequality we have

$$\theta^{-5/3} \int_{Q_\theta(z_0)} |\widetilde{v}|^{10/3} + |\nabla \widetilde{e}|^{10/3} \mathrm{d}x\mathrm{d}t \leq c(M, N).$$

Thus

$$\begin{aligned} C_1(\widetilde{v}, 1, z_0) &\leq \gamma^3 |Q_1(z_0)| + \iint_{Q_1(z_0) \cap \{|\widetilde{v}| > \gamma\}} |\widetilde{v}|^3 dxdt \\ &\leq c\gamma^3 + \|\widetilde{v}\|_{L^{10/3}(Q_1(z_0))} \left| Q_1(z_0) \cap \left\{ |\widetilde{v}| > \gamma \right\} \right|^{1/10} \\ &\leq c(\gamma^3 + \eta^{\frac{1}{10}}). \end{aligned}$$

For any $\epsilon > 0$ we choose γ and η such that $C_1(\widetilde{v}, 1, z_0) < \epsilon$.

It is easy to see that, by [28] (to see Lemma 3.2),

$$\begin{aligned} K(\nabla \widetilde{e}, \theta, z_0) &\leq c\theta^{-3} C_1(\widetilde{v}, 1, z_0)^{\frac{2}{3}} \left[A(\nabla \widetilde{e}, 1, z_0) + E(\nabla \widetilde{e}, 1, z_0) \right] + \theta^2 K(\nabla \widetilde{e}, 1, z_0)) \\ &\leq c(M, N)(\theta^{-3} \epsilon^{2/3} + \theta^2) \end{aligned} \tag{3.23}$$

Utilizing (3.23) and Hölder's inequality we have

$$\begin{aligned} C_1(\nabla\widetilde{e},\theta,z_0) &\equiv \theta^{-2}\int_{Q_\theta(z_0)}|\nabla\widetilde{e}|^3 \\ &\le \left[\theta^{-\frac{5}{3}}\int_{Q_\theta(z_0)}|\nabla\widetilde{e}|^{\frac{10}{3}}\right]^{\frac{3}{4}}\left[\theta^{-3}\int_{Q_\theta(z_0)}|\nabla\widetilde{e}|^2\right]^{\frac{1}{4}} \\ &\le c(M,N)K(\nabla\widetilde{e},\theta;z_0)^{\frac{1}{4}} \\ &\le c(M,N)(\theta^{-3/4}\epsilon^{1/6}+\theta^{1/2}). \end{aligned} \tag{3.24}$$

Thus

$$C_1(\widetilde{v},\nabla\widetilde{e},\theta,z_0) \le \theta^{-2}\epsilon + c(M,N)(\theta^{-3/4}\epsilon^{1/6}+\theta^{1/2}).$$

First we take θ such that $c(M,N)\theta^{1/2} \le \varepsilon_0/2$, then take ϵ such that $\theta^{-2}\epsilon + c(M,N)\theta^{-3/4}\epsilon^{1/6} \le \varepsilon_0/2$, i.e.,

$$C_1(\widetilde{v},\nabla\widetilde{e},\theta,z_0) \le \varepsilon_0,$$

which implies that z_0 is a regular point by Proposition 1.2, therefor $(\widetilde{v},\widetilde{e},\widetilde{q})$ are smooth and their derivatives are bounded in $(\mathbb{R}^3 \setminus B_{2R}) \times (-T/2,0)$. Next, we show

$$\nabla\widetilde{e}(\cdot,0) = 0. \tag{3.25}$$

For any $B(y)$ and $\phi \in C_0^\infty(B(y))$, since e is Hölder continuous (to see following Lemma 3.2), we have

$$\begin{aligned} &\left|\int_{B(y)}\nabla\widetilde{e}(x,0)\phi dx\right| \\ &\le \int_{B(y)}|\nabla\widetilde{e}(x,0)-\nabla e_k(x,0)|\mathrm{d}x + \left|\int_{B(y)}\nabla e_k\phi\right| \\ &\le c\,\|\nabla\widetilde{e}(x,0)-\nabla e_k(x,0)\|_{L^{4/3}(B(y))} + c\,r_k^{-3}\int_{B_{r_k}(r_k y)}|e(x,0)-e(0,0)|\mathrm{d}x \\ &\le o(1) + c\,r_k^{-3}r_k^3 r_k^\alpha \\ &\le o(1). \end{aligned}$$

The backward uniqueness theorem of parabolic equations [3], we conclude

$$\widetilde{e}(x,t) = 0, \quad \text{in } \mathbb{R}^3\backslash\bar{B}_{2R}(0)\times(-T/2,0].$$

Using unique continuation theorem of parabolic equation in the bounded domain again [3], we conclude that

$$\widetilde{e}(x,t) = 0 \quad \text{in } \mathbb{R}^3\times(-T/2,0).$$

Thus, $\widetilde{v}$ satisfies Navier–Stokes equations in $\mathbb{R}^3\times(-T/2,0)$

$$\begin{cases} \partial_t\widetilde{v} - \Delta\widetilde{v} + \widetilde{v}\cdot\nabla\widetilde{v} + \nabla\widetilde{q} = 0, \\ \mathrm{div}\widetilde{v} = 0. \end{cases}$$

Using (3.20) and backward uniqueness of heat operator again [3], we get

$$\widetilde{v}(x,t) = 0 \quad \text{in } \mathbb{R}^3 \times (-T/2, 0),$$

which is a contradiction with (3.21). □

We need following lemma.

Lemma 3.2 *For $v \in L^\infty(0,T;L^{3,\infty}(\mathbb{R}^3))$, if e satisfies in $\mathbb{R}^3 \times (0,T)$*

$$\partial_t e - \Delta e = -v \cdot \nabla e.$$

Then e is Hölder continuous.

Proof From [34], if $v \in L^\infty(0,T;BMO^{-1}(\mathbb{R}^3))$, then e is Hölder continuous. We only prove the following inclusion relationship for $3 < p < \infty$,

$$L^3_w(\mathbb{R}^3) \subset \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3) \subset BMO^{-1}(\mathbb{R}^3),$$

where $L^3_w(\mathbb{R}^3) = L^{3,\infty}(\mathbb{R}^3)$ is the weak Lebesgue space, and $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ is a homogeneous Besov space.

The first inclusion is obtained through Sobolev embedding and real interpolation. To be specific, we write weak space L^3_w a real interpolation

$$L^3_w(\mathbb{R}^3) = (L^2(\mathbb{R}^3), L^p(\mathbb{R}^3))_{\theta,\infty},$$

where $\theta = \frac{p}{3(p-2)}$. Notice we have the following two embedding relations

$$L^2(\mathbb{R}^3) = F^0_{2,2}(\mathbb{R}^3) = B^0_{2,2}(\mathbb{R}^3) \subset B^{\frac{3}{p}-\frac{3}{2}}_{p,2}(\mathbb{R}^3),$$
$$L^p(\mathbb{R}^3) = F^0_{p,2}(\mathbb{R}^3) \subset B^0_{p,p}(\mathbb{R}^3),$$

where we have used Littlewood–Paley Theorem to characterize L^r by the Triebel–Lizorkin space $F^0_{r,2}$ (for $1 < r < \infty$). Thus the identity map is bounded:

$$id : L^2(\mathbb{R}^3) \to B^{\frac{3}{p}-\frac{3}{2}}_{p,2}(\mathbb{R}^3),$$
$$id : L^p(\mathbb{R}^3) \to B^0_{p,p}(\mathbb{R}^3).$$

By real interpolation, and $(1-\theta)\cdot(\frac{3}{p}-\frac{3}{2}) + \theta \cdot 0 = -1 + \frac{3}{p}$, this leads to

$$B^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3) = (B^{\frac{3}{p}-\frac{3}{2}}_{p,2}(\mathbb{R}^3), B^0_{p,p}(\mathbb{R}^3))_{\theta,\infty},$$

thus the identity map

$$id : L^3_w(\mathbb{R}^3) \to B^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3) \subset \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$$

is also bounded. The second inclusion is obtained by using the heat kernel characterization of the corresponding spaces, denote $s = -1 + \frac{3}{p}$,

$$||f||_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)} = \sup_{t>0} ||t^{-\frac{s}{2}} e^{t\Delta} f||_{L^p(\mathbb{R}^3)},$$

$$||f||_{BMO^{-1}(\mathbb{R}^3)} = \sup_{x\in\mathbb{R}^3, R>0} \left[\frac{1}{|B_R(x)|} \int_{B_R(x)} \int_0^{R^2} |e^{t\Delta} f|^2 \mathrm{d}y\mathrm{d}t \right]^{\frac{1}{2}}.$$

Direct calculation yields:

$$\begin{aligned} ||f||_{BMO^{-1}(\mathbb{R}^3)} &= \sup_{x\in\mathbb{R}^3, R>0} \left[\frac{1}{|B_R(x)|} \int_0^{R^2} \int_{B_R(x)} |e^{t\Delta} f|^2 \mathrm{d}y\mathrm{d} \right]^{\frac{1}{2}} \\ &\leq \sup_{x\in\mathbb{R}^3, R>0} \left[|B_R(x)|^{-\frac{2}{p}} \int_0^{R^2} (\int_{B_R(x)} |e^{t\Delta} f|^p dy)^{\frac{2}{p}} \mathrm{d}t \right]^{\frac{1}{2}} \\ &\leq \sup_{x\in\mathbb{R}^3, R>0} \left[|B_R(x)|^{-\frac{2}{p}} \int_0^{R^2} ||e^{t\Delta} f||^2_{L^p(\mathbb{R}^3)} \mathrm{d}t \right]^{\frac{1}{2}} \\ &\leq \sup_{x\in\mathbb{R}^3, R>0} \left[|B_R(x)|^{-\frac{2}{p}} \int_0^{R^2} t^s ||f||^2_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)} \mathrm{d}t \right]^{\frac{1}{2}} \\ &\leq c||f||_{\dot{B}^s_{p,\infty}(\mathbb{R}^3)}. \end{aligned}$$

The proof is thus finished. □

4 Proof of Theorem 1.4

Define for any $r > 0$ such that $Q_r(z_0) \subset Q_T$,

$$C_2(u, r, z_0) = r^{-1} \int_{Q_r(z_0)} |u|^4, \quad C_2(\nabla d, r, z_0) = r^{-1} \int_{Q_r(z_0)} |\nabla d|^4.$$

By interpolation inequality we have

$$C(\nabla d, R, z_0) \leq A(\nabla d, R, z_0)^{6/7} C_2(\nabla d, R, z_0)^{4/7} \leq c(c_0, R),$$

and

$$C_2(u, R, z_0) \leq c\, M^2 \left[A(u, R, z_0) + E(u, R, z_0) \right],$$

where $\|u\|_{L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))} \leq M$. On the other hand, by Calderón-Zygmund theorem,

$$\|p(t)\|^s_{L^s(\mathbb{R}^3)} \leq c\, \|(u, \nabla d)(t)\|^{2s}_{L^{2s}(\mathbb{R}^3)} \quad \text{for } 1 < s < \infty,$$

and

$$(u, \nabla d) \in L^4(0, T; L^3(\mathbb{R}^3)),$$

which implies

$$p \in L^2(0,T;L^{3/2}(\mathbb{R}^3)),$$

we have

$$D(p,R,z_0) \le c(c_0,R).$$

Since for any $0 < r \le R$,

$$C(u,r,z_0) \le c\,M^4,$$

by Lemma 2.4, we have for any $0 < r \le R/2$ and $z_0 \in \Omega \times (0,T), \Omega \subset\subset \mathbb{R}^3$,

$$\begin{aligned} &A(u,\nabla d,r,z_0) + E(u,\nabla d,r,z_0) + C_1(u,\nabla d,r,z_0) + C_2(u,\nabla d,r,z_0) \\ &\quad + C(\nabla d,r,z_0) + D(p,r,z_0) + D_1(p,r,z_0) \\ &\le c(M,R,c_0) \equiv N. \end{aligned} \tag{4.1}$$

The number $\varepsilon(M,N)$ of Proposition 3.1 can be determined.

Let S be a singular points set of $(u,\ d)$ at $\left\{(x,T)\,:\,x\in\mathbb{R}^3\right\}$. Assume that it contains more than $M^3\varepsilon^{-4}$ elements. Letting $P=[M^3\varepsilon^{-4}]+1$, we can find P different singular points $\left\{(x_k,T)\,:\,k=1,2,\dots,P\right\}$ of the set S. We can choose $R_0 \le R$ such that $B_{R_0}(x_k)\cap B_{R_0}(x_l)=\emptyset, k\ne l$, and bounded domain Ω such that $\cup_{k=1}^{P} B_{R_0}(x_k)\subset\Omega$. According to Proposition 3.1, for all $r\in(0,R_0/2]$, it holds true

$$\varepsilon \le \frac{1}{r^3}\left|\left\{x\in B_r(x_k)\,:\,|u(x,T)|>\frac{\varepsilon}{r}\right\}\right| \tag{4.2}$$

for all $k=1,2,\dots,P$. In particular, taking $r=r_0=R_0/2$, we have

$$\begin{aligned} P\varepsilon &\le \sum_{k=1}^{P}\frac{1}{r_0^3}\left|\left\{x\in B_{r_0}(x_k)\,:\,|u(x,T)|>\frac{\varepsilon}{r_0}\right\}\right| \\ &\le \frac{1}{r_0^3}\left|\left\{x\in \cup_{k=1}^{P}B_{r_0}(x_k)\,:\,|u(x,T)|>\frac{\varepsilon}{r_0}\right\}\right| \\ &\le \frac{1}{r_0^3}\left|\left\{x\in\Omega\,:\,|u(x,T)|>\frac{\varepsilon}{r_0}\right\}\right| \\ &\le \varepsilon^{-3}\|u(\cdot,T)\|^3_{L^{3,\infty}(\Omega)} \\ &\le \varepsilon^{-3}M^3, \end{aligned}$$

i.e., $P\le M^3\varepsilon^{-4}<P$, which is a contradiction. □

5 Appendix: Proof of Proposition 1.2

According to the L^p theorem of Stokes system in [5], if $\mathbf{f}\in W^{-1,q}(\Omega;\mathbb{R}^3)$, $1<q<\infty$, and Ω is a C^1 bounded domain, the following Stokes equations

$$\begin{cases} -\Delta v + \nabla p = \mathbf{f}, \\ \operatorname{div} v = 0, \quad \int_\Omega p = 0, \\ v|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

there exists exactly one solution $(v, p) \in W^{1,q}(\Omega) \times L^q(\Omega)$, and

$$\|\nabla v\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} \le c(q)\|\mathbf{f}\|_{W^{-1,q}(\Omega)}. \tag{5.2}$$

Wolf's the local pressure projection $\mathcal{P}_q$ tell us,

$$\mathcal{P}_q : W^{-1,q}(\Omega) \to W^{-1,q}(\Omega), \quad \mathcal{P}_q(\mathbf{f}) = p.$$

As in [39], we have following Lemma.

Lemma 5.1 *Let* (u, d) *be a weak solution of* (1.1), *then for every* C^2 *bounded sub-domain* Ω, *and any* $\phi \in C_0^\infty(\Omega \times (0, T))$, *there holds*

$$\begin{aligned} &-\int_0^T \int_\Omega (u + \nabla p_h) \cdot \phi_t - \int_0^T \int_\Omega (u \otimes u + \nabla d \odot \nabla d + p_1 \mathbf{I}) : \nabla \phi \\ &+ \int_0^T \int_\Omega (\nabla u - p_2 \mathbf{I}) : \nabla \phi = 0, \end{aligned} \tag{5.3}$$

i.e., set $v_\Omega = v := u + \nabla p_h$,

$$\partial_t v + \operatorname{div}(u \otimes u) + \nabla p_1 + \nabla p_2 = \Delta v - \nabla \cdot (\nabla d \odot \nabla d), \tag{5.4}$$

where $\mathbf{I}$ *is identity matrix, and*

$$\begin{cases} p_h = -\mathcal{P}_2(u), \\ p_1 = -\mathcal{P}_{3/2}(u \otimes u + \nabla d \odot \nabla d), \\ p_2 = \mathcal{P}_2(\Delta u). \end{cases}$$

In addition, following estimates hold for a.e. $t \in (0, T)$

$$\begin{cases} \|\nabla p_h(t)\|_{L^m(\Omega)} \le c\,\|u(t)\|_{L^m(\Omega)}, \quad 1 < m \le 6, \\ \|p_1(t)\|_{L^{3/2}(\Omega)} \le c\,\|u \otimes u + \nabla d \odot \nabla d\|_{L^{3/2}(\Omega)}, \\ \|p_2(t)\|_{L^2(\Omega)} \le c\,\|\nabla u(t)\|_{L^2(\Omega)}. \end{cases} \tag{5.5}$$

Here $c > 0$ *depends on the geometry of* Ω *and in* $(5.5)_1$ *on* m *only. In particular, if* Ω *is the ball* $B_R(x_0)$ *then* c *in* $(5.5)_1$ *depends only on* m, *while in* $(5.5)_2$ *and* $(5.5)_3$ c *is an absolute constant.*

Hence, we have local energy inequality, for $\varphi \in C_0^\infty(\Omega \times (0, T))$

$$
\begin{aligned}
&\int_\Omega (|v(t)|^2 + |\nabla d(t)|^2)\varphi + 2\int_0^t \int_\Omega (|\nabla v|^2 + |\nabla^2 d|^2)\varphi \\
&\leq \int_0^t \int_\Omega (|v|^2 + |\nabla d|^2)(\varphi_t + \Delta\varphi) + \int_0^t \int_\Omega (|u|^2 u + |\nabla d|^2 v)\cdot\nabla\varphi \\
&+ \int_0^t \int_\Omega 2(p_1 + p_2)v\cdot\nabla\varphi + 2\int_0^t \int_\Omega u^i u^j \partial_i(\partial_j p_h \varphi) \\
&+ 2\int_0^t \int_\Omega (u\cdot\nabla d)\cdot(\nabla d\nabla\varphi) \\
&+ \int_0^t \int_\Omega 2\nabla^2 p_h : (\nabla d \odot \nabla d)\varphi - |\nabla d|^2 \nabla p_h \cdot \nabla\varphi \\
&- 2\int_0^t \int_\Omega \nabla_x f(d)\nabla d\,\varphi.
\end{aligned}
\tag{5.6}
$$

Note that the suitable weak solution of (1.1) satisfies the local energy inequality (5.6).

From local energy inequality we can get the Caccioppoli-type estimates

$$
\begin{aligned}
&\|W\|^2_{L^{10/3}(Q_{R/2})} + \|\nabla W\|^2_{L^2(Q_{R/2})} \\
&\quad \leq c\,R^{-1/3}\|W\|^2_{L^3(Q_R)} + c\,R^{-1}\|W\|^3_{L^3(Q_R)}.
\end{aligned}
\tag{5.7}
$$

Here, $W = (u, \nabla d)$, and

$$
\begin{aligned}
\|W\|^2_{L^k(Q_r)} &= \|u\|^2_{L^k(Q_r)} + \|\nabla d\|^2_{L^k(Q_r)}, \\
\|\nabla W\|^2_{L^2(Q_r)} &= \|\nabla u\|^2_{L^2(Q_r)} + \|\nabla^2 d\|^2_{L^2(Q_r)}.
\end{aligned}
$$

Obviously,

$$
C_1(W, r, z_0) = C_1(u, \nabla d, r, z_0), \quad E(W, r, z_0) = E(u, \nabla d, r, z_0).
$$

Proposition 1.2 is an immediate result of following lemma and **Theorem (A)**.

Lemma 5.2 *Suppose that* (u, d) *is a local suitable weak solution of* (1.1). *Then there exist universal constants* $\varepsilon^* > 0$ *and* $\theta \in (0, \frac{1}{4}]$ *with following property. For any* $\varepsilon \in (0, \varepsilon^*]$ *if*

$$
C_1(W, 1, z_0) \leq \varepsilon,
$$

then

$$
C_1(W, \theta, z_0) \leq \varepsilon.
$$

Proof We prove by contradiction. Let $\theta \in (0, \frac{1}{4}]$ be a constant to be specified later. Suppose there exist a decreasing sequence $\{\varepsilon_n\}$ converging to 0, and a sequence of pairs of local suitable weak solutions (u_n, d_n, p_n) such that

$$
C_1(W_n, 1, z_0) = \varepsilon_n^3, \tag{5.8}
$$

and

$$
C_1(W_n, \theta, z_0) > \varepsilon_n^3. \tag{5.9}
$$

Define $(v_n, e_n, q_n) = (\frac{u_n}{\varepsilon_n}, \frac{d_n}{\varepsilon_n}, \frac{p_n}{\varepsilon_n})$, then they satisfy

$$\partial_t v_n + \varepsilon_n v_n \cdot \nabla v_n + \nabla q_n = \Delta v_n - \varepsilon_n \mathrm{div}\,(\nabla e_n \odot \nabla e_n), \quad \mathrm{div} v_n = 0,$$
$$\partial_t e_n + \varepsilon_n v_n \cdot \nabla e_n - \Delta e_n = -\sigma^{-2}(|d_n|^2 - 1))e_n.$$

Write $w_n = (v_n, \nabla e_n)$, then

$$C_1(w_n, 1, z_0) = 1, \tag{5.10}$$

and

$$C_1(w_n, \theta, z_0) > 1. \tag{5.11}$$

Using the Caccioppoli estimate (5.7) we conclude

$$\|w_n\|_{L^{10/3}(Q_{1/2}(z_0))} + \|\nabla w_n\|_{L^2(Q_{1/2}(z_0))} \le c, \tag{5.12}$$

which implies

$$\|w_n \cdot \nabla w_n\|_{L^{5/4}(Q_{1/2}(z_0))} \le \|\nabla w_n\|_{L^2(Q_{1/2}(z_0))} \|w_n\|_{L^{10/3}(Q_{1/2}(z_0))} \le c.$$

The coercive estimate for the Stokes system (see, for instance, [27]) with a suitable cutoff function implies

$$\int_{Q_{\frac{1}{3}}} |\partial_t w_n|^{\frac{5}{4}} + |\nabla^2 w_n|^{\frac{5}{4}} + |\nabla q_n|^{\frac{5}{4}} + |w_n|^{\frac{5}{4}} \le c,$$

where the constant c is independent of n. Thanks to the compact embedding theorem and (5.12), there exist $w \in L^3(Q_{1/3}(z_0))$ and $q \in L^{\frac{5}{4}}(Q_{1/3}(z_0))$ such that

$$w_n \to w = (v, \nabla e) \quad \text{in } L^3(Q_{1/3}(z_0)),$$
$$q_n \rightharpoonup q \quad \text{in } L^{\frac{5}{4}}(Q_{1/3}(z_0)).$$

Thus, (v, e, q) satisfy

$$\partial_t v - \Delta v + \nabla q = 0, \quad \mathrm{div}\, v = 0,$$
$$\partial_t e - \Delta e = \sigma^{-2} e.$$

Moreover

$$\|w\|_{L^3(Q_{1/3}(z_0))} + \|q\|_{L^{\frac{5}{4}}(Q_{3/4}(z_0))} \le c.$$

By the classical estimate of the Stokes system [37], we get

$$\sup_{Q_{1/3}(z_0)} |w| \le c,$$

which implies that for $0 < \theta \le 1/3$

$$C_1(w, \theta, z_0) \le c\,\theta^3.$$

This contradicts (5.11), if we choose θ sufficiently small. The lemma is proved. ☐

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