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Strong unique continuation for second-order hyperbolic equations with time-independent coefficients

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Abstract
In this paper we prove that if u is a solution to second-order hyperbolic equation $\partial_t^2 u + a(x)\partial_t u - (\,\mathrm{div}_x\big(A(x)\nabla_x u\big) + b(x)\cdot\nabla_x u + c(x)u) = 0$ and u is flat on a segment $\{x_0\}\times(-T,T)$ (T finite), then u vanishes in a neighborhood of $\{x_0\}\times(-T,T)$. The novelty with respect to earlier papers on the subject is the nonvanishing damping coefficient $a(x)$ in the hyperbolic equation.

Keywords Unique continuation property · Stability estimates · Hyperbolic equations · Inverse problems

Mathematics Subject Classification 35R25 · 35L10 · 35B60 · 35R30

1 Introduction

In this paper we study strong unique continuation property (SUCP) for the equation

$$\partial_t^2 u + a(x)\partial_t u - L(u) = 0, \quad \text{in } B_{\rho_0}\times(-T,T), \tag{1.1}$$

where ρ_0, T are given positive numbers, B_{ρ_0} is the ball of $\mathbb{R}^n$, $n\geq 2$, of radius ρ_0 and center at 0, $a\in L^\infty(\mathbb{R}^n)$, L is the second-order elliptic operator

$$L(u) = \,\mathrm{div}_x\big(A(x)\nabla_x u\big) + b(x)\cdot\nabla_x u + c(x)u, \tag{1.2}$$

$b\in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $c\in L^\infty(\mathbb{R}^n)$ and $A(x)$ is a real-valued symmetric $n\times n$ matrix that satisfies a uniform ellipticity condition and entries of $A(x)$ are functions of Lipschitz class.

We say that Eq. (1.1) has the SUCP if there exists a neighborhood $\mathcal{U}$ of $\{0\}\times(-T,T)$ such that for every solution, u, to Eq. (1.1) we have

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$$\|u\|_{L^2(B_r\times(-T,T))} = \mathcal{O}(r^N),\ \forall N \in \mathbb{N},\ \text{as } r \to 0, \quad \Longrightarrow \quad u = 0,\ \text{in } \mathcal{U}. \tag{1.3}$$

Property (1.3) was proved (if the matrix A belongs to C^2), under the additional condition $T = +\infty$ and u is bounded, by Masuda in 1968, [25]. Later on, in 1978, Baouendi and Zachmanoglou, [5], proved the SUCP whenever the coefficients of equation (1.1) are analytic functions. In 1999, Lebeau, [23], proved the SUCP for solution to (1.1) when $a = b = c = 0$. The proof of [23] requires the symmetry of the differential operator, and there seems no obvious extension of the proof to the nonsymmetric case, in particular, to the case of damped wave equation $\partial_t^2 u + a(x)\partial_t u - \Delta u = 0$. We also refer to [29, 32] where the SUCP at the boundary and the quantitative estimate of unique continuation related to property was proved when $a = 0$.

The novelty of the present paper with respect to earlier papers, with finite T, is the nonvanishing damping coefficient $a(x)$ in the hyperbolic Eq. (1.1).

It is worth noting that SUCP and the related quantitative estimates, have been extensively studied and today well understood in the context of second-order elliptic and parabolic equation. Among the extensive literature on the subject here we mention, for the elliptic equations, [3, 4, 15, 19], and, for the parabolic equations, [2, 8, 20]. In the context of elliptic and parabolic equations, the quantitative estimates of unique continuation appear in the form of three sphere inequalities [21], doubling inequalities [13], or two-sphere one-cylinder inequality [9]. We refer to [1] and [31] for a more extensive literature concerning the elliptic context and the parabolic context respectively.

In the present paper we prove (Theorem 2.1) a quantitative estimate of unique continuation from which we derive (Corollary 2.2) property (1.3) for equation (1.1). The crucial step of the proof is Theorem 3.1, in such a Theorem 3.1 we exploit in a suitable way the simple and classical idea of converting a hyperbolic equation into an elliptic equation, see, for instance, [12, Ch.6]. Formally, such a classical idea consists in substitute, in (1.1), the variable t by iy. More precisely, the idea can be told as follows. Let us define the integral transform

$$v(x,y) = \int_{-T}^{T} u(x,t)\Phi(t+iy)\mathrm{d}t, \tag{1.4}$$

where the kernel Φ is a holomorphic function in variable $z = t + iy$. It is simple to check that v satisfies the elliptic equation

$$\partial_y^2 v + L(v) - ia(x)\partial_y v = F(x,y), \tag{1.5}$$

where F is an "error term" which depends on $u(\cdot, \pm T)$, $\partial_t u(\cdot, \pm T)$ and $\Phi(\pm T + iy)$.

The use of converting a hyperbolic equation into an elliptic equation, in the issue of weak unique continuation property (WUCP) for finite time T, can be tracked back to Robbiano in 1991, [26], see also [27]. By WUCP for (1.1) we mean: let R be a given positive number, there exists a neighborhood $\mathcal{V}$ of $\{0\} \times (-T, T)$ such that for every solution, u to equation (1.1) we have

$$u = 0,\ \text{in } B_R \times (-T,T) \Longrightarrow u = 0,\ \text{in } \mathcal{V}. \tag{1.6}$$

Subsequently, in [16, 28] and [30], the WUCP was studied in the general context of equation with partially analytic coefficients (not only of hyperbolic type) and the exact dependence domain $\mathcal{V}$ was determined, see also [6, 17, 22] for the related quantitative estimates. The above mentioned papers rely on the so-called Fourier–Bros–Iagolnitzer (FBI) transform that is an integral transform like (1.4) whose kernel is the Gaussian function $\Phi(z) = \sqrt{\mu/2\pi}e^{-\mu z^2}$, where μ is a large parameter. Although the FBI transform works very well to prove the WUCP for equation (1.1), it seems no obvious whether the FBI works well to tackle the SUCP.

In the present paper to prove SUCP for (1.1) we use integral transform (1.4) with well-chosen family of *polynomial kernels*. More precisely, we define

$$v_k(x,y) = \int_{-T}^{T} u(x,t)\varphi_k(t+iy)dt, \quad \forall k \in \mathbb{N}, \tag{1.7}$$

where $\varphi_k(t+iy)$ is a polynomial with the following property:

(a) $\varphi_k(t+i0)$ is an approximation of Dirac's δ-function,
(b) $\left|\varphi_k(\pm T+iy)\right| \le C^k|y|^k$ for $k \in \mathbb{N}$ and $|y| \le 1$, where C is a constant.

In this way functions v_k turn out solutions to the elliptic equation

$$\partial_y^2 v_k - ia(x)\partial_y v_k + L(v_k) = F_k(x,y), \quad \text{in } B_1 \times \mathbb{R}, \tag{1.8}$$

where $|F_k| \le C^k|y|^k$, for $k \in \mathbb{N}$ and $|y| \le 1$. This behavior of F_k allows us to handle in a suitable way a Carleman estimate with singular weight for second-order elliptic operators, see Sect. 2.3, in such a way to get $u(x,0) = 0$ for $x \in B_\rho$, where $\rho \le \rho_0/C$. Similarly, we prove for every $t \in (-T,T)$, $u(\cdot,t) = 0$ in $B_{\rho(t)}$, where $\rho(t) = (1-tT^{-1})\rho$. So that we obtain (1.3) with $\mathcal{U} = \bigcup_{t\in(-T,T)}(B_{\rho(t)} \times \{t\})$. As a consequence of this result and using the WUCP, we have that $u = 0$ in the domain of dependence of $\mathcal{U}$.

The quantitative estimate of unique continuation that we prove in Theorem 2.1 can be read, roughly speaking, as a continuous dependence estimate of $u_{|\mathcal{U}}$ from $u_{|B_{r_0}\times(-T,T)}$, where r_0 is arbitrarily small. The sharp character of such a continuous dependence result is related to the logarithmic character of this estimate, that, at the light of counterexample of John [18], cannot be improved and to the fact that this quantitative estimate implies the SUCP property. The quantitative estimate of strong unique continuation (at the interior and at the boundary) was a crucial tool, see [33], to prove sharp stability estimate for inverse problems with unknown boundaries for wave equation $\partial_t^2 u - \operatorname{div}_x\left(A(x)\nabla_x u\right) = 0$.

Before concluding Introduction, we mention an open question (to the author knowledge). Such an open question concerns the SUCP, (1.3), for the second-order hyperbolic equation with coefficients that are analytic in variable t and smooth enough (but not analytic) in variables x. This is, for instance, the case of the equation

$$\partial_t^2 u + a(x,t)\partial_t u - \Delta_x u = 0,$$

where $a(x,t)$ is smooth enough w.r.t x and analytic w.r.t. t. Concerning this topic we mention [24] in which it is proved that if u satisfies the conditions: (a) $(\{0\}\times(-T,T)) \cap \operatorname{supp} u$ is compact and (b) $D^j u(x,t) = \mathcal{O}\left(e^{-k/|x|}\right)$, $j = 1,2$, for every k as $x \to 0$, $t \in (-T,T)$, then u vanishes in a neighborhood of $\{0\}\times(-T,T)$.

The plan of the paper is as follows: In Sect. 2 we state the main result of this paper, and in Sect. 3 we prove the main theorem.

2 The main results

2.1 Notation and definition

Let $n \in \mathbb{N}$, $n \geq 2$. For any $x \in \mathbb{R}^n$ we will denote $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ and $|x| = \left(\sum_{j=1}^n x_j^2\right)^{1/2}$. Given $r > 0$, we will denote by B_r, B'_r and $\widetilde{B}_r$ the ball of $\mathbb{R}^n$, $\mathbb{R}^{n-1}$ and $\mathbb{R}^{n+1}$ of radius r centered at 0, respectively. For any open set $\Omega \subset \mathbb{R}^n$ and any function (smooth enough) u we denote by $\nabla_x u = (\partial_{x_1} u, \cdots, \partial_{x_n} u)$ the gradient of u. Also, for the gradient of u we use the notation $D_x u$. If $j = 0, 1, 2$ we denote by $D_x^j u$ the set of the derivatives of u of order j, so $D_x^0 u = u$, $D_x^1 u = \nabla_x u$ and $D_x^2 u$ is the hessian matrix $\{\partial_{x_i x_j} u\}_{i,j=1}^n$. Similar notation is used whenever other variables occur and Ω is an open subset of $\mathbb{R}^{n-1}$ or a subset $\mathbb{R}^{n+1}$. By $H^\ell(\Omega)$, $\ell = 0, 1, 2$, we denote the usual Sobolev spaces of order ℓ (in particular, $H^0(\Omega) = L^2(\Omega)$), with the standard norm

$$\|v(x)\|_{H^\ell(\Omega)} = \left(\sum_{0 \leq j \leq \ell} \int_\Omega \left|D^j v(x)\right|^2 dx\right)^{1/2}.$$

For any interval $J \subset \mathbb{R}$ and Ω as above we denote

$$\mathcal{W}(J;\Omega) = \left\{u \in C^0\left(J; H^2(\Omega)\right) \,:\, \partial_t^\ell u \in C^0\left(J; H^{2-\ell}(\Omega)\right), \ell = 1, 2\right\}.$$

We shall use the letters $c, C, C_0, C_1, \cdots$ to denote constants. The value of the constants may change from line to line, but we shall specified their dependence everywhere they appear. Generally we will omit the dependence of various constants by n.

2.2 Statements of the main results

Let $\rho_0 > 0$, T, $\lambda \in (0, 1]$, $\Lambda > 0$ and $\Lambda_1 > 0$ be given numbers. Let $A(x) = \left\{a^{ij}(x)\right\}_{i,j=1}^n$ be a real-valued symmetric $n \times n$ matrix whose entries are measurable functions, and they satisfy the following conditions

$$\lambda |\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \text{for every } x, \xi \in \mathbb{R}^n, \tag{2.1a}$$

$$\left|A(x_*) - A(x)\right| \leq \frac{\Lambda}{\rho_0} \left|x_* - x\right|, \quad \text{for every } x_*, x \in \mathbb{R}^n. \tag{2.1b}$$

Let $b \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and $a, c \in L^\infty(\mathbb{R}^n)$ satisfy

$$T|a(x)| + T^2 \rho_0^{-1} |b(x)| + T^2 |c(x)| \leq \Lambda_1, \quad \text{for almost every } x \in \mathbb{R}^n, \tag{2.2}$$

Let

$$L(u) = \operatorname{div}_x\left(A(x)\nabla_x u\right) + b(x) \cdot \nabla_x u + c(x)u. \tag{2.3}$$

Let $u \in \mathcal{W}\left([-T, T]; B_{\rho_0}\right)$ be a solution to

$$\partial_t^2 u + a(x)\partial_t u - L(u) = 0, \quad \text{a.e. in } B_{\rho_0} \times (-T, T). \tag{2.4}$$

Let ε and H be given positive numbers and let $r_0 \in (0, \rho_0]$. We assume

$$\rho_0^{-n} T^{-1} \int_{-T}^{T} \int_{B_{r_0}} |u(x,t)|^2 dxdt \le \varepsilon^2 \tag{2.5}$$

and

$$\max_{t\in[-T,T]} \left(\rho_0^{-n} \int_{B_{\rho_0}} |u(x,t)|^2 dx + \rho_0^{-n+1} \int_{B_{\rho_0}} \left|\partial_t u(x,t)\right|^2 dx \right) \le H^2. \tag{2.6}$$

Theorem 2.1 *Let $u \in \mathcal{W}\big([-T,T]; B_{\rho_0}\big)$ be a weak solution to (2.4) and let (2.1), (2.2), (2.5) and (2.6) be satisfied. For every $\alpha \in (0, 1/2)$ there exist constants $s_0 \in (0,1)$ and $C \ge 1$ depending on λ, Λ, Λ_1, α and $T\rho_0^{-1}$ only such that for every $t_0 \in (-T,T)$ and every $0 < r_0 \le \rho \le s_0\rho_0$ the following inequality holds true*

$$\rho_0^{-n} \int_{B_{\rho(t_0)}} \left|u(x,t_0)\right|^2 dx \le C \frac{\big(\log \rho_0/r_0)\big)^{\alpha} (H+\varepsilon)^2}{\big(\log\big(e + H\varepsilon^{-1}\big)\big)^{\alpha}}, \tag{2.7}$$

where

$$\rho(t_0) = (1 - |t_0|T^{-1})\rho.$$

The proof of Theorem 2.1 is given in Sect. 3.

The proof of the following corollary is standard (see, for instance, [32, Remark 2.2]), but we give it for the reader convenience.

Corollary 2.2 (Strong Unique Continuation Property) *Let $u \in \mathcal{W}\big([-T,T]; B_{\rho_0}\big)$ be a weak solution to (2.4). Assume that (2.1) and (2.2) be satisfied. We have that, if*

$$\left(\rho_0^{-n} T^{-1} \int_{-T}^{T} \int_{B_{r_0}} |u(x,t)|^2 dxdt \right)^{1/2} = O(r_0^N), \quad \forall N \in \mathbb{N}, \text{ as } r_0 \to 0,$$

then

$$u(\cdot,t) = 0, \text{ for } |x| + \frac{\rho_0|t|}{T} \le s_0\rho_0, \tag{2.8}$$

where s_0 is defined in Theorem 2.1.

Proof We consider the case $t = 0$; similarly, we could proceed for $t \ne 0$. If $\|u(\cdot,0)\|_{L^2\left(B_{s_0\rho_0}\right)} = 0$ there is nothing to proof, otherwise, if

$$\|u(\cdot,0)\|_{L^2\left(B_{s_0\rho_0}\right)} > 0 \tag{2.9}$$

we argue by contradiction. By (2.9) it is not restrictive to assume that

$$\max_{t\in[-T,T]} \left(\rho_0^{-n} \int_{B_{\rho_0}} |u(x,t)|^2 dx + \rho_0^{-n+1} \int_{B_{\rho_0}} \left|\partial_t u(x,t)\right|^2 dx \right) = 1. \tag{2.10}$$

Now we apply inequality (2.7) with $\varepsilon = C_N r_0^N$, $H = 1$ and passing to the limit as $r_0 \to 0$ we derive

$$\|u(\cdot,0)\|_{L^2\left(B_{s_0\rho_0}\right)} \leq CN^{-\alpha/2}, \quad \forall N \in \mathbb{N}, \tag{2.11}$$

by passing again to the limit as $N \to 0$ in (2.11) we get $\|u(\cdot,0)\|_{L^2\left(B_{s_0\rho_0}\right)} = 0$ that contradicts (2.9). □

2.3 Auxiliary result: Carleman estimate with singular weight

In order to prove Theorem 2.1, we need a Carleman estimate proved by several authors; here we recall [3, 15]. In order to control the dependence of the various constants, we use here a version of such a Carleman estimate proved, in the context of parabolic operator, in [11], see also [7, Sect. 8].

First we introduce some notation. Let $\mathcal{P}$ be the elliptic operator

$$\mathcal{P}(w) := \partial_y^2 w + L(w) - ia(x)\partial_y w. \tag{2.12}$$

Denote

$$\varrho(x,y) = \left(A^{-1}(0)x \cdot x + y^2\right)^{1/2}, \tag{2.13}$$

$$\widetilde{B}_r^\varrho = \left\{(x,y) \in \mathbb{R}^{n+1} \,:\, \varrho(x,y) < r\right\}, \qquad r > 0. \tag{2.14}$$

Notice that

$$\widetilde{B}^\varrho_{\sqrt{\lambda}r} \subset \widetilde{B}_r \subset \widetilde{B}^\varrho_{r/\sqrt{\lambda}}, \qquad \forall r > 0. \tag{2.15}$$

Theorem 2.3 *Let $\mathcal{P}$ be the operator (2.12) and assume that (2.1) is satisfied. There exists constants $C_* > 1$ depending on λ, and Λ only and $\tau_0 > 1$ depending on λ, Λ and Λ_1 only such that, denoting*

$$\Psi(r) = r \exp\left(\int_0^r \frac{e^{-C_*\eta} - 1}{\eta} d\eta\right), \tag{2.16a}$$

$$\psi(x,y) = \Psi\left(\varrho(x,y)/2\sqrt{\lambda}\right), \tag{2.16b}$$

for every $\tau \geq \tau_0$ and $w \in C_0^\infty\left(\widetilde{B}^\varrho_{2\sqrt{\lambda}/C_*} \setminus \{0\}\right)$ we have

$$\int_{\mathbb{R}^{n+1}} \left(\tau\psi^{1-2\tau}\left|\nabla_{x,y} w\right|^2 + \tau^3\psi^{-1-2\tau}|w|^2\right) dxdy \leq C_* \int_{\mathbb{R}^{n+1}} \psi^{2-2\tau}|\mathcal{P}(w)|^2 dxdy. \tag{2.17}$$

Remark 2.4 We emphasize that

$$\Psi(r) \simeq r, \qquad \text{as } r \to 0.$$

Moreover, Ψ is an increasing and concave function and there exists $C > 1$ depending on λ, and Λ such that

$$C^{-1}r \le \Psi(r) \le r, \qquad \forall r \in (0,1]. \tag{2.18}$$

3 Proof of Theorem 2.1

The primary step to achieve Theorem 2.1 consists in proving the following

Theorem 3.1 *Let us assume $\rho_0 = 1$ and $T = 1$. Let $u \in \mathcal{W}\big([-1,1];B_1\big)$ be a weak solution to (2.4) and let (2.1), (2.2), (2.5) and (2.6) be satisfied. For every $\alpha \in (0,1/2)$ there exist constants $s_0 \in (0,1)$ and $C \ge 1$ depending on λ, Λ, Λ_1 and α only such that for every $0 < r_0 \le s \le s_0$ the following inequality holds true*

$$\int_{B_s} |u(x,0)|^2 dx \le C \frac{\left(\log 1/r_0\right)^{\alpha}(H+\varepsilon)^2}{\left(\log\left(e + H\varepsilon^{-1}\right)\right)^{\alpha}}. \tag{3.1}$$

In order to prove Theorem 3.1, we define

$$v_k(x,y) = \int_{-1}^{1} u(x,t)\varphi_k(t+iy)dt, \quad \forall k \in \mathbb{N}, \tag{3.2}$$

where

$$\varphi_k(z) = \mu_k\left(1-z^2\right)^k, \quad z = t+iy \in \mathbb{C}, \tag{3.3}$$

and

$$\mu_k = \left(\int_{-1}^{1}\left(1-t^2\right)^k dt\right)^{-1}, \tag{3.4}$$

so that we have

$$\int_{-1}^{1} \varphi_k(t)dt = 1, \quad \forall k \in \mathbb{N}. \tag{3.5}$$

It is easy to check that

$$\mu_k \simeq \sqrt{\frac{k}{\pi}}, \qquad \text{as } k \to \infty. \tag{3.6}$$

We need some simple lemmas to state the properties of functions v_k.

Lemma 3.2 *We have*

$$\left\|v_k(\cdot,0)-u(\cdot,0)\right\|_{L^2(B_1)} \le c\frac{\log k}{\sqrt{k}}, \quad \forall k \in \mathbb{N}, \tag{3.7}$$

where c depends on n only.

Proof By (3.2) and (3.5) we have

$$v_k(x,0)-u(x,0) = \int_{-1}^{1} (u(x,t)-u(x,0))\varphi_k(t)dt, \quad \forall x \in B_1;$$

hence, by Schwarz inequality and integrating over B_1, we have

$$\int_{B_1} \left|v_k(x,0)-u(x,0)\right|^2 dx \le \int_{B_1} dx \int_{-1}^{1} |u(x,t)-u(x,0)|^2 \varphi_k(t)dt =$$
$$= \int_{[-\gamma,\gamma]} \varphi_k(t)dt \int_{B_1} |u(x,t)-u(x,0)|^2 dx + \int_{[-1,1]\setminus[-\gamma,\gamma]} \varphi_k(t)dt \int_{B_1} |u(x,t)-u(x,0)|^2 dx, \tag{3.8}$$

where $\gamma \in (0,1)$ is a number that we will choose. Now we have

$$\varphi_k(t) \le \mu_k (1-\gamma^2)^{k/2}, \quad \forall t \in [-1,1] \setminus [-\gamma,\gamma]$$

and, by (2.6),

$$\int_{B_1} |u(x,t)-u(x,0)|^2 dx \le t^2 H^2.$$

Hence, by (2.6), (3.6) and (3.8), we have

$$\left\|v_k(\cdot,0)-u(\cdot,0)\right\|_{L^2(B_1)} \le cH\left(\gamma + k^{1/4}(1-\gamma^2)^{k/2}\right), \quad \text{for every } \gamma \in (0,1), \tag{3.9}$$

where c depends on n only. Now, we choose $\gamma = k^{-1/2}\log k$ and we get (3.7). □

Lemma 3.3 *Let u be a solution to* (2.4), *and let* (2.1) *and* (2.2) *be satisfied, then* $v_k \in H^2\big(B_1 \times (-1,1)\big)$ *is a solution to the equation*

$$\partial_y^2 v_k - ia(x)\partial_y v_k + L(v_k) = F_k(x,y), \quad \text{in } B_1 \times \mathbb{R}, \tag{3.10}$$

where $F_k \in L^\infty(-1,1;L^2(B_1))$ *and it satisfies*

$$\left\|F_k(\cdot,y)\right\|_{L^2(B_1)} \le CHk\mu_k|\sqrt{5}y|^{k-1}, \quad \forall y \in [-1,1], \tag{3.11}$$

C depending on Λ_1 *only.*

In addition, v_k *satisfies the following properties*

$$\sup_{y\in[-1,1]} \left\|v_k(\cdot,y)\right\|_{L^2(B_1)} \le 2^k \mu_k H, \tag{3.12}$$

$$\int_{\widetilde{B}_{r_0/2}} \left(\left| v_k \right|^2 + r_0^2 \left| \nabla_{x,y} v_k \right|^2 \right) dxdy \leq C \left(r_0 4^k k \varepsilon^2 + H^2 k^3 \left(\sqrt{5} r_0 \right)^{2(k+2)} \right), \qquad (3.13)$$

where C depends on λ and Λ_1 only.

Proof The fact that v_k belongs to $H^2\left(B_1 \times (-1,1)\right)$ is an immediate consequence of differentiation under the integral sign. Actually we have

$$\partial_y^m D_x^j v_k(x,y) = \int_{-1}^{1} D_x^j u(x,t) \partial_y^m \left(\varphi_k(t+iy) \right) dt, \qquad \text{for } j,m = 0,1,2; \qquad (3.14)$$

hence, by Schwarz inequality and taking into account that $u \in \mathcal{W}\left([-1,1];B_1\right)$, we have $v_k \in H^2\left(B_1 \times (-1,1)\right)$.

Now we prove (3.10).

By integration by parts and taking into account that

$$\partial_t \varphi_k(t+iy) = \frac{1}{i} \partial_y \varphi_k(t+iy),$$

,we have

$$\begin{aligned}\int_{-1}^{1} \partial_t u(x,t) \varphi_k(t+iy) dt &= \int_{-1}^{1} \partial_t u(x,t) \varphi_k(t+iy) dt \\ &= \left(u(x,t)\varphi_k(t+iy)\right)\Big|_{t=-1}^{t=1} - \frac{1}{i} \int_{-1}^{1} u(x,t) \partial_y \varphi_k(t+iy) dt \\ &= \left(u(x,t)\varphi_k(t+iy)\right)\Big|_{t=-1}^{t=1} + i\partial_y v_k(x,y).\end{aligned}$$

Hence, we have

$$-i\partial_y v_k(x,y) = -\int_{-1}^{1} \partial_t u(x,t) \varphi_k(t+iy) dt + \left(u(x,t)\varphi_k(t+iy)\right)\Big|_{t=-1}^{t=1}. \qquad (3.15)$$

Similarly, we have

$$\begin{aligned}\partial_y^2 v_k(x,y) = \partial_y^2 v_k(x,y) = -\int_{-1}^{1} \partial_t^2 u(x,t) \varphi_k(t+iy) dt + \\ + \left(\partial_t u(x,t)\varphi_k(t+iy)\right)\Big|_{t=-1}^{t=1} - \left(u(x,t)\varphi_k'(t+iy)\right)\Big|_{t=-1}^{t=1}.\end{aligned} \qquad (3.16)$$

Now, by (2.4), (3.14), (3.15) and (3.16) we have

$$\begin{aligned}&\partial_y^2 v_k - ia(x)\partial_y v_k + L(v_k) = \\ &= -\int_{-1}^{1} \left\{ \partial_t^2 u(x,t) + a(x)\partial_t u(x,t) - L(u)(x,t) \right\} \varphi_k(t+iy) dt + F_k(x,y) = F_k(x,y),\end{aligned} \qquad (3.17)$$

where

$$\begin{aligned}F_k(x,y) = \varphi_k(1+iy)\left(a(x)u(x,1) + \partial_t u(x,1)\right) - \varphi_k'(1+iy)u(x,1) \\ - \left[\varphi_k(-1+iy)\left(a(x)u(x,-1) + \partial_t u(x,-1)\right) - \varphi_k'(-1+iy)u(x,-1)\right].\end{aligned} \qquad (3.18)$$

and (3.10) is proved.

Now we prove (3.11).

It is easy to check that for every $k \in \mathbb{N}$ we have

$$\left|\varphi_k(\pm 1 + iy)\right| = \mu_k\left(4y^2 + y^4\right)^{k/2}, \quad \forall y \in \mathbb{R}, \tag{3.19a}$$

$$\left|\varphi_k'(\pm 1 + iy)\right| = 2k\mu_k\left(1 + y^2\right)^{1/2}\left(4y^2 + y^4\right)^{(k-1)/2}, \quad \forall y \in \mathbb{R}. \tag{3.19b}$$

In addition, since

$$\left|\varphi_k(t + iy)\right| = \mu_k\left[t^4 - 2t^2(1 - y^2) + (1 + y^2)^2\right]^{\frac{k}{2}},$$

we have

$$\left|\varphi_k(t + iy)\right| \le 2^k\mu_k, \quad \forall (t, y) \in [-1, 1] \times [-1, 1]. \tag{3.20}$$

By (2.2), (2.6), (3.19a) and (3.19b) we have (3.11).

By Schwarz inequality and (3.20) we have, for any $R \in (0, 1]$,

$$\sup_{y\in[-1,1]} \left\|v_k(\cdot, y)\right\|_{L^2(B_R)} \le 2^k\mu_k\left(\int_{-1}^{1}\int_{B_R} u^2(x, t)dxdt\right)^{1/2}; \tag{3.21}$$

hence, for $R = 1$, taking into account (2.6), we obtain (3.12).

Finally, let us prove (3.13). For this purpose, we firstly observe that applying (3.21) for $R = r_0$ and taking into account (2.5), we have

$$\sup_{y\in[-1,1]} \left\|v_k(\cdot, y)\right\|_{L^2(B_{r_0})} \le C2^k\mu_k\varepsilon. \tag{3.22}$$

Afterward, since v_k is solution to elliptic equation (3.10), the following Caccioppoli inequality, [10, 14], holds

$$\int_{\widetilde{B}_{r_0/2}} \left|\nabla_{x,y}v_k\right|^2 dxdy \le Cr_0^{-2}\int_{\widetilde{B}_{r_0}} \left(\left|v_k\right|^2 + r_0^4\left|F_k\right|^2\right)dxdy, \tag{3.23}$$

where C depends on λ only. Finally, by (3.6), (3.11), (3.22) and (3.23) we get (3.13). □

Proof of Theorem 3.1 Set

$$r_1 = \frac{\sqrt{\lambda}r_0}{8}.$$

By (3.13) we have

$$\int_{\widetilde{B}^o_{4r_1}} \left(\left|v_k\right|^2 + r_1^2\left|\nabla_{x,y}v_k\right|^2\right)dxdy \le Cr_1\sigma_k(\varepsilon, r_1), \tag{3.24}$$

where C depends on λ and Λ_1 only and

$$\sigma_k(\varepsilon, r_1) = 4^k k\varepsilon^2 + H^2k^3\left(C_1r_1\right)^{2k+2}, \tag{3.25}$$

where $C_1 = 2\sqrt{5}\lambda^{-1/2}$.

Now we apply Theorem 2.3.

Denote

$$\psi_0(r) := \Psi(r/2\sqrt{\lambda}) \quad , \text{for every } r > 0$$

and

$$R = \frac{\sqrt{\lambda}}{2C_*}.$$

Let us define

$$\zeta(x,y) = h(\psi(x,y)).$$

where h belongs to $C_0^2\big(0, \psi_0(2R)\big)$ and satisfies

$$\begin{aligned} &0 \le h \le 1, \\ &h(r) = 1, \quad \forall r \in \big[\psi_0\big(2r_1\big), \psi_0(R)\big], \\ &h(r) = 0, \quad \forall r \in \big[0, \psi_0\big(r_1\big)\big] \cup \big[\psi_0(3R/2), \psi_0(2R)\big], \\ &r_1\big|h'(r)\big| + r_1^2\big|h''(r)\big| \le c, \quad \forall r \in \big[\psi_0\big(r_1\big), \psi_0\big(2r_1\big)\big], \\ &\big|h'(r)\big| + \big|h''(r)\big| \le c, \quad \forall r \in \big[\psi_0(R), \psi_0(3R/2)\big], \end{aligned}$$

where c depends on λ and Λ only. Notice that if $2r_1 \le \varrho(x,y) \le R$, then $\zeta(x,y) = 1$ and if $\varrho(x,y) \ge 2R$ or $\varrho(x,y) \le r_1$, then $\zeta(x,y) = 0$.

By density, we can apply (2.17) to the function $w = \zeta v_k$ and we have, for every $\tau \ge \tau_0$,

$$\int_{\widetilde{B}^{\varrho}_{2R}} \left(\tau\psi^{1-2\tau}\big|\nabla_{x,y}\big(\zeta v_k\big)\big|^2 + \tau^3\psi^{-1-2\tau}\big|\zeta v_k\big|^2\right)dxdy \le C\big(I_1 + I_2 + I_2\big), \tag{3.27}$$

where C depends on λ, Λ and Λ_1 only and

$$I_1 = \int_{\widetilde{B}^{\varrho}_{2R}} \psi^{2-2\tau}\big|F_k\big|^2\zeta^2 dxdy, \tag{3.28a}$$

$$I_2 = \int_{\widetilde{B}^{\varrho}_{2R}} \psi^{2-2\tau}|\mathcal{P}(\zeta)|^2\big|v_k\big|^2 dxdy, \tag{3.28b}$$

$$I_3 = \int_{\widetilde{B}^{\varrho}_{2R}} \psi^{2-2\tau}\big|\nabla_{x,y}v_k\big|^2\big|\nabla_{x,y}\zeta\big|^2 dxdy. \tag{3.28c}$$

□

Estimate of I_1.
By (2.18) we have

$$C_2^{-1} \le \big(|x|^2 + y^2\big)^{-1/2}\psi(x,y) \le C_2, \quad \forall (x,y) \in \widetilde{B}_1, \tag{3.29}$$

where $C_2 > 2$ depends on λ and Λ only.

By (3.11), (3.28a) and (3.29) we have

$$
\begin{aligned}
I_1 &= \int_{\widetilde{B}^\rho_{2R}} \psi^{2-2\tau} |F_k|^2 \zeta^2 dxdy \leq C_2^{2(\tau-1)} \int_{\widetilde{B}^\rho_{2R}} \left(|x|^2+y^2\right)^{1-\tau} |F_k|^2 dxdy \leq \\
&\leq CH^2 k^3 5^k C_2^{2(\tau-1)} \int_{\widetilde{B}^\rho_{2R}} \left(|x|^2+y^2\right)^{k-\tau} dxdy,
\end{aligned}
\tag{3.30}
$$

where C depends on λ and Λ only.

Now let k and τ satisfy

$$
k \geq \tau \geq \tau_0. \tag{3.31}
$$

By (3.30) and (3.31) we have

$$
I_1 \leq CH^2 k^3 5^k C_2^{2(k-1)}. \tag{3.32}
$$

Estimate of I_2.
By (3.12), (3.24) and (3.28b) we have

$$
\begin{aligned}
I_2 &\leq Cr_1^{-4} \int_{\widetilde{B}^\rho_{2r_1}\setminus \widetilde{B}^\rho_{r_1}} \psi^{2-2\tau} |v_k|^2 dxdy + C \int_{\widetilde{B}^\rho_{3R/2}\setminus \widetilde{B}^\rho_{R}} \psi^{2-2\tau} |v_k|^2 dxdy \leq \\
&\leq C\left(r_1^{-3} \psi_0^{2-2\tau}(r_1)\sigma_k(\varepsilon, r_1) + H^2 k 4^k \psi_0^{2-2\tau}(R)\right);
\end{aligned}
$$

hence, by (3.29) we have

$$
I_2 \leq C\left(\psi_0^{-1-2\tau}(r_1)\sigma_k(\varepsilon, r_1) + H^2 k 4^k \psi_0^{1-2\tau}(R)\right), \tag{3.33}
$$

where C depends on λ and Λ only.
Estimate of I_3.
By (3.28c) we have

$$
\begin{aligned}
I_3 &\leq Cr_1^{-2} \psi_0^{2-2\tau}(r_1) \int_{\widetilde{B}^\rho_{2r_1}\setminus \widetilde{B}^\rho_{r_1}} \left|\nabla_{x,y} v_k\right|^2 dxdy \\
&+ C\psi_0^{2-2\tau}(R) \int_{\widetilde{B}^\rho_{3R/2}\setminus \widetilde{B}^\rho_{R}} \left|\nabla_{x,y} v_k\right|^2 dxdy.
\end{aligned}
\tag{3.34}
$$

Now in order to estimate from above the right-hand side of (3.34), we use the Caccioppoli inequality, (3.11), (3.12) and (3.24) and we get

$$
\begin{aligned}
I_3 &\leq Cr_1^{-2} \psi_0^{2-2\tau}(r_1) \left(r_1^{-2} \int_{\widetilde{B}^\rho_{4r_1}\setminus \widetilde{B}^\rho_{r_1/2}} |v_k|^2 dxdy + r_1^2 \int_{\widetilde{B}^\rho_{4r_1}\setminus \widetilde{B}^\rho_{r_1/2}} |F_k|^2 dxdy \right) \\
&+ C\psi_0^{2-2\tau}(R) \left(R^{-2} \int_{\widetilde{B}^\rho_{2R}\setminus \widetilde{B}^\rho_{R/2}} |v_k|^2 dxdy + R^2 \int_{\widetilde{B}^\rho_{2R}\setminus \widetilde{B}^\rho_{R/2}} |F_k|^2 dxdy \right) \leq \\
&\leq C\sigma_k(\varepsilon, r_1) \psi_0^{-1-2\tau}(r_1) + CH^2 5^k k^3 \psi_0^{1-2\tau}(R) := \widetilde{I}_3,
\end{aligned}
\tag{3.35}
$$

where C depends on λ, Λ and Λ_1 only.

Let $r_1 \leq \frac{R}{2}$ and let s be such that $\frac{2r_1}{\sqrt{\lambda}} \leq s \leq \frac{R}{\sqrt{\lambda}}$. Denote

$$\widetilde{s} = \sqrt{\lambda} s.$$

By estimating from below trivially the left-hand side of (3.27) and taking into account (3.35), we get

$$\psi_0^{-1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{\widetilde{s}} \setminus \widetilde{B}^\varrho_{2r_1}} |v_k|^2 + \psi_0^{1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{\widetilde{s}} \setminus \widetilde{B}^\varrho_{2r_1}} \left|\nabla_{x,y} v_k\right|^2 \leq C\left(I_1 + I_2 + \widetilde{I}_3\right), \tag{3.36}$$

where C depends on λ, Λ and Λ_1 only.

Now, by (2.15), (3.24) and into account that $\psi_0(\widetilde{s}) \geq \psi_0(r_1)$ we have

$$\begin{aligned} &\psi_0^{-1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{2r_1}} |v_k|^2 + \psi_0^{1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{2r_1}} \left|\nabla_{x,y} v_k\right|^2 dxdy \\ &\leq C\psi_0^{-1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{2r_1}} \left(|v_k|^2 + r_1^2\left|\nabla_{x,y} v_k\right|^2\right) dxdy \leq C r_1 \sigma_k(\varepsilon, r_1) \psi_0^{-1-2\tau}(r_1). \end{aligned} \tag{3.37}$$

Now let us add at both the sides of (3.36) the quantity

$$\psi_0^{-1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{2r_1}} |v_k|^2 + \psi_0^{1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{2r_1}} \left|\nabla_{x,y} v_k\right|^2 dxdy$$

and by (3.37) we have

$$\psi_0^{-1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{\widetilde{s}}} |v_k|^2 + \psi_0^{1-2\tau}(\widetilde{s}) \int_{\widetilde{B}^\varrho_{\widetilde{s}}} \left|\nabla_{x,y} v_k\right|^2 \leq C\left(I_1 + I_2 + \widetilde{I}_3\right), \tag{3.38}$$

where C depends on λ, Λ and Λ_1 only. Moreover, by (3.32), (3.33) and (3.35) we have

$$I_1 + I_2 + \widetilde{I}_3 \leq C\sigma_k(\varepsilon, r_1)\psi_0^{-1-2\tau}(r_1) + CH^2 k^3 5^k C_2^{2k} \psi_0^{1-2\tau}(R). \tag{3.39}$$

Now by (3.29), (3.32), (3.33), (3.35) and (3.39) we have that if (3.31) is satisfied, then

$$\int_{\widetilde{B}_{\lambda s}} |v_k|^2 + s^2 \int_{\widetilde{B}_{\lambda s}} \left|\nabla_{x,y} v_k\right|^2 \leq C\omega_{k,\tau} \quad , \tag{3.40}$$

where C depends on λ, Λ and Λ_1 only and

$$\omega_{k,\tau}(\varepsilon, r_1) = \sigma_k(\varepsilon, r_1)\left(\frac{\psi_0(\widetilde{s})}{\psi_0(r_1)}\right)^{1+2\tau} + H^2 k^3 5^k C_2^{2k} \left(\frac{\psi_0(\widetilde{s})}{\psi_0(R)}\right)^{1+2\tau}. \tag{3.41}$$

By a standard trace inequality, we have

$$s\int_{B_{\lambda s/2}} |v_k(\cdot, 0)|^2 \leq C\omega_{k,\tau}(\varepsilon, r_1) \tag{3.42}$$

and Lemma (3.2) implies

$$s\int_{B_{\lambda s/2}} |u(\cdot, 0)|^2 \leq C\left(\frac{\log k}{\sqrt{k}} + \omega_{k,\tau}(\varepsilon, r_1)\right), \tag{3.43}$$

where C depend on λ, Λ and Λ_1 only.

Now, we choose $k=\tau$ in (3.43) and using trivial inequality we have that for any $0<\alpha<\frac{1}{2}$ there exist constants $C_3>1$ and k_0 depending on λ, Λ, Λ_1 and α only such that for every $k\geq k_0$ we have

$$s\int_{B_{\lambda s/2}}|u(\cdot,0)|^2\leq C_3H_1^2\left[\left(C_3sr_1^{-1}\right)^{2k+1}\varepsilon_1^2+\left(C_3s\right)^{2k+1}+k^{-\alpha}\right], \tag{3.44}$$

where

$$H_1:=H+e\varepsilon \quad \text{and} \quad \varepsilon_1:=\frac{\varepsilon}{H+e\varepsilon}.$$

Let us denote

$$\overline{s}=\frac{1}{2C_3}$$

and put $s=\overline{s}$, by (3.44) we have trivially

$$\begin{aligned}&\overline{s}\int_{B_{\lambda\overline{s}/2}}|u(\cdot,0)|^2\leq C_3H_1^2\left[\left(2r_1\right)^{-(2k+1)}\varepsilon_1^2+2^{-(2k+1)}+k^{-\alpha}\right],\\&k_*=\min\left\{p\in\mathbb{Z}:p\geq\frac{\log\varepsilon_1}{2\log r_1}\right\}.\end{aligned} \tag{3.45}$$

If $k_*\geq k_0$, then we choose $k=k_*$ and by (3.45) we have

$$\overline{s}\int_{B_{\lambda\overline{s}/2}}|u(\cdot,0)|^2\leq 2C_3H_1^2\left(\varepsilon_1^{2\theta_0}+\left(\frac{2\log(1/r_1)}{\log(1/\varepsilon_1)}\right)^{\alpha}\right), \tag{3.46}$$

where

$$\theta_0=\frac{\log 2}{2\log(1/r_1)}. \tag{3.47}$$

Otherwise, if $k_*<k_0$, then $\frac{\log\varepsilon_1}{2\log r_1}<k_0$, hence

$$\theta_0\log(1/\varepsilon_1)=\frac{\log\varepsilon_1}{2\log r_1}\log 2<k_0\log 2.$$

This implies

$$2^{-2k_0}\varepsilon_1^{2\theta_0}\geq 1,$$

that, in turns, taking into account (2.6), gives trivially

$$\int_{B_{\lambda\overline{s}/2}}|u(\cdot,0)|^2\leq H^2\leq 4^{k_0}\varepsilon_1^{2\theta_0}H^2\leq 4^{k_0}(H+e\varepsilon)^{2(1-\theta_0)}\varepsilon^{2\theta_0}. \tag{3.48}$$

Finally, by (3.46) and (3.48) we obtain (3.1), with $s_0=2\lambda^{-1}\overline{s}$. □

Conclusion of the proof of Theorem 2.1.

Let $t_0\in(-T,T)$. It is not restrictive to assume $t_0\geq 0$. Denote

$$\rho(t_0) = \left(1 - T^{-1}t_0\right)\rho_0, \quad T(t_0) = \left(1 - T^{-1}t_0\right)T$$

and

$$U(y,\eta) = u\left(\rho(t_0)y, \eta T(t_0) + t_0\right), \qquad \text{for } (y,\eta) \in B_1 \times (-1,1).$$

It is easy to check that $\widetilde{u}$ is a solution to

$$\partial^2_\eta U + \widetilde{a}(y)\partial^2_\eta U - \mathcal{L}U = 0, \qquad \text{for } (y,\eta) \in B_1 \times (-1,1),$$

where

$$\begin{aligned}
&\mathcal{L}U = \operatorname{div}_y\left(\widetilde{A}(y)\nabla_y U\right) + \widetilde{b}(y) \cdot \nabla_y U + \widetilde{c}(y)U, \\
&\widetilde{A}(y) = \left(T\rho_0^{-1}\right)^2 A\left(\rho(t_0)y\right), \quad \widetilde{a}(y) = (T - t_0)a\left(\rho(t_0)y\right), \\
&\widetilde{b}(y) = \left(T(T - t_0)\rho_0^{-1}\right)b\left(\rho(t_0)y\right), \quad \widetilde{c}(y) = \left(T - t_0\right)^2 c\left(\rho(t_0)y\right).
\end{aligned}$$

By (2.1a) and (2.1b) we have, respectively,

$$\begin{aligned}
&\lambda_0|\xi|^2 \le \widetilde{A}(y)\xi\cdot\xi \le \lambda_0^{-1}|\xi|^2, \qquad \text{for every } x, \xi \in \mathbb{R}^n, \\
&\left|\widetilde{A}(y_*) - A(y)\right| \le \frac{\Lambda_0}{\rho_0}\left|y_* - y\right|, \qquad \text{for every } y_*, y \in \mathbb{R}^n,
\end{aligned}$$

where

$$\lambda_0 = \lambda \min\{\left(T\rho_0^{-1}\right)^2, \left(T\rho_0^{-1}\right)^{-2}\}, \text{ and } \Lambda_0 = T^2\rho_0^{-1}\Lambda.$$

By (2.2) we have

$$\left|\widetilde{a}(y)\right| + \left|\widetilde{b}(y)\right| + \left|\widetilde{c}(y)\right| \le \Lambda_1, \qquad \text{for almost every } y \in \mathbb{R}^n.$$

In addition, by (2.5), (2.6) we have, respectively,

$$\int_{-1}^{1}\int_{B_{r_0\rho_0^{-1}}} |U(y,\eta)|^2 dy d\eta \le \varepsilon^2\left(1 - t_0T^{-1}\right)^{-n}$$

and

$$\max_{\eta\in[-1,1]}\left(\int_{B_1}|U(y,\eta)|^2 dy + \int_{B_1}\left|\partial_\eta U(y,\eta)\right|^2 dy\right) \le H^2\left(1 - t_0T^{-1}\right)^{-n}.$$

Now we apply Theorem 3.1. Denoting $s = \rho\rho_0^{-1}$ we have $0 < r_0\rho_0^{-1} < s \le s_0$; therefore,

$$\int_{B_s}|U(y,0)|^2 dy \le \frac{C}{\left(1 - t_0T^{-1}\right)^n}\frac{\left(\log(\rho_0/r_0)\right)^\alpha (H + e\varepsilon)^2}{\left(\log\left(e + H\varepsilon^{-1}\right)\right)^\alpha}.$$

Finally, come back to the variables x and t we get (2.7). □

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