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Quasilinear equations involving indefinite nonlinearities and exponential critical growth in $\mathbb{R}^N$

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Abstract

In this work, we establish the existence of nonzero solutions for a class of quasilinear elliptic equations involving indefinite nonlinearities with exponential critical growth of Trudinger–Moser type. Our proofs rely on variational arguments in a Orlicz–Sobolev space with a version of the Trudinger–Moser inequality.

Keywords Quasilinear equations · Exponential growth · Trudinger–Moser inequality · Indefinite nonlinearities

Mathematics Subject Classification 35A15 · 35J20 · 35J62

1 Introduction and main results

In this paper, we consider a class of quasilinear elliptic equations involving a sign-changing weight function and a nonlinearity with exponential critical growth. More precisely, we study the existence of nonzero solutions for the equation

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$$-\Delta_\Phi u + \Phi'(|u|)\frac{u}{|u|} = W(x)f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $\Delta_\Phi u = \operatorname{div}\big(\Phi'(|\nabla u|)\nabla u/|\nabla u|\big)$, $N \geq 2$, $W : \mathbb{R}^N \to \mathbb{R}$ is a continuous function changing sign, $f \in C(\mathbb{R})$ has exponential critical growth and $\Phi : \mathbb{R} \to \mathbb{R}_+$ is a function satisfying some appropriate conditions.

Elliptic problems with indefinite nonlinearities have been intensively studied in the last years. We would like to mention that existence of solutions for indefinite elliptic problems of the type

$$\begin{cases} -\Delta u = \lambda u + W(x)f(u) & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $\lambda > 0$, has already been established in various contexts when the dimension $N \geq 3$. If the domain Ω is a compact manifold of dimension $N \geq 3$, the critical exponent case $f(s) = |s|^{2^*-2}s$, where $2^* = 2N/(N-2)$, arises in the prescribed scalar curvature problem (see [36]). For manifolds carrying scalar flat metrics, sufficient conditions for the existence of positive solutions were given in [29]. Results for more general nonlinearities were obtained by Alama–Tarantello [5]. After that, many authors have studied indefinite semilinear elliptic problems when the nonlinear term $f(s)$ has polynomial growth (see [4, 6, 11, 20, 22, 23, 28, 40, 44] and references therein). Indefinite problems of type (1.2) involving critical growth in the Sobolev case were treated by various authors; see, for instance, [19, 33, 34].

We quote that there are few results involving indefinite nonlinearity with exponential critical growth. In the paper [41], the authors establish a version for dimension two of the main result in [5] (see also [40]) when the nonlinearity $f(s)$ has exponential critical growth. In [1], the authors consider an indefinite problem having exponential subcritical growth in all $\mathbb{R}^2$ with the nonlinearity being of the form $f(s) = \phi(s)e^s$ and $\phi(s)$ between two powers. We also emphasize that Alves et al. [8] studied the existence of solution for the problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{N-2}\nabla u\big) + |u|^{N-2}u = W(x)f(u) & \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is an exterior domain of $\mathbb{R}^N (N \geq 2)$ and $f(s)$ has exponential critical growth.

Motivated by the previous works, our main purpose here is to study Eq. (1.1) by considering the maximal growth on the nonlinear term $f(s)$ which allows us to treat the problem variationally in the Orlicz–Sobolev space $W^{1,\Phi}(\mathbb{R}^N)$. Furthermore, $W(x)$ is a weight function changing sign and having a thick zero set. In what follows, $\Phi : \mathbb{R} \to \mathbb{R}_+$ is a N-function of class C^1 fulfilling the conditions

(Φ_1) There exists $C > 0$ such that $t^N/C \leq \Phi(t) \leq Ct^N$ for all $t \in [0, 1/C)$;

(Φ_2) $\displaystyle\lim_{t\to+\infty} \frac{\Phi(t)}{t^N \log^\alpha t} = 1$, for some $\alpha \in [0, N-1)$;

and $W^{1,\Phi}(\mathbb{R}^N)$ is the Orlicz–Sobolev space that consists of functions in $L_\Phi(\mathbb{R}^N)$ (the Orlicz space associated with the N-function Φ) such that its weak derivatives exist and belong to $L_\Phi(\mathbb{R}^N)$. For the definition and information about N-functions, see Sect. 2. We regard $W^{1,\Phi}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{1,\Phi} = |\nabla u|_\Phi + |u|_\Phi,$$

where $|\cdot|_\Phi$ denotes the Luxemburg norm associated with $L_\Phi(\mathbb{R}^N)$. It is known that the Lorentz–Zygmund space $L^{N,N,\tau}(\mathbb{R}^N)$, $\tau = \alpha/N$, reproduces (up to equivalent norms) the Orlicz spaces $L_\Phi(\mathbb{R}^N)$ (see [10, 43]). Thus, $W^{1,\Phi}(\mathbb{R}^N)$ is equivalent to $W^1L^{N,N,\tau}(\mathbb{R}^N)$.

When $\alpha = 0$, $\Phi(t) = |t|^N$ satisfies conditions (Φ_1) and (Φ_2), and therefore, (1.1) becomes an elliptic equation involving the N-Laplacian operator, namely

$$-\Delta_N u + |u|^{N-2}u = \hat{g}(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.4}$$

where $\hat{g}(x,u) = W(x)f(u)/N$. Equations of type (1.4), with $\hat{g}(x,u)$ having definite sign and critical exponential growth with respect to the Trudinger–Moser inequality, have been intensively investigated by many authors; see, for example, [2, 15, 24, 26, 27, 39, 42].

Throughout this paper, we assume that $f(t)$ can behave like $\exp(b|t|^\gamma)$ as $t \to +\infty$; more precisely, we suppose the following growth condition on the nonlinearity $f(t)$:

(f_1) there exist constants $C > 0$ and $b > 0$ such that

$$f(t) \leq Ct^{N-1} + C\left[\exp(bt^\gamma) - S_{N,\alpha}(bt^\gamma)\right]$$

for all $t \in \mathbb{R}$, where $\gamma = N/(N-1-\alpha)$ and

$$S_{N,\alpha}(bt^\gamma) = \sum_{0 \leq j < \frac{N}{\gamma}} \frac{(bt^\gamma)^j}{j!}.$$

This growth on $f(t)$ is motivated by a version of the Trudinger–Moser inequality in the space $W^{1,\Phi}(\mathbb{R}^N)$ (see Lemma 3.3) which was proved in [18, 21].

Next, let us to obtain some properties of the function Φ. As a consequence of (Φ_1), it follows that

$$\liminf_{t\to 0^+} \frac{\Phi(2t)}{\Phi(t)} \geq \frac{2^N}{C^2}.$$

Moreover, by (Φ_2) we have

$$\lim_{t\to+\infty} \frac{\Phi(2t)}{\Phi(t)} = 2^N.$$

Thus, there exists $K_1 \geq 2^N$ such that $\Phi(2t) \leq K_1\Phi(t)$ for all $t \geq 0$; that is, Φ satisfies the Δ_2 -condition. Moreover, since Φ is convex, we reach $(K_1 - 1)\Phi(t) \geq \Phi(2t) - \Phi(t) \geq \Phi'(t)t$, and therefore,

$$c_\alpha := \sup_{t>0} \frac{\Phi'(t)t}{\Phi(t)} \leq K_1 - 1. \tag{1.5}$$

On the other hand, given any $t_0 > 0$, by the mean value theorem there exists $s \in (0, t_0)$ verifying $\Phi(t_0) = \Phi'(s)t_0$, and since $\Phi'(t)$ is nondecreasing for $t > 0$, it follows that $\Phi(t_0) \leq \Phi'(t_0)t_0$, and thus,

$$1 \leq m_\alpha := \inf_{t>0} \frac{\Phi'(t)t}{\Phi(t)}. \tag{1.6}$$

By using (Φ_1) and (Φ_2), we shall show that $m_\alpha > 1$ (see Proposition 2.7). Moreover, by deriving $\Phi(t)/t^{c_\alpha}$ we have that (1.5) implies that $\Phi(t)/t^{c_\alpha}$ is nonincreasing for $t > 0$. Similarly, from (1.6) we deduce that $\Phi(t)/t^{m_\alpha}$ is nondecreasing for $t > 0$. Hence,

$$\Phi(1)t^{c_\alpha} \leq \Phi(t) \leq \Phi(1)t^{m_\alpha}, \quad \text{for all} \quad t \in [0, 1].$$

Combining these inequalities with condition (Φ_1), for all $t \in [0, \min\{1, 1/C\})$ we reach

$$\Phi(1)t^{c_\alpha} \leq \Phi(t) \leq Ct^N \quad \text{and} \quad \frac{1}{C}t^N \leq \Phi(t) \leq \Phi(1)t^{m_\alpha},$$

which imply that $m_\alpha \leq N \leq c_\alpha$.

With respect to the term $W(x)$, we require the following conditions:

(W_1) $W : \mathbb{R}^N \to \mathbb{R}$ is a continuous function changing sign and $W \in L^r(\mathbb{R}^N)$ for some $r \in [N, +\infty)$;

(W_2) Ω^+ is a bounded set and $\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset$, where

$$\Omega^+ = \left\{x \in \mathbb{R}^N : W(x) > 0\right\} \quad \text{and} \quad \Omega^- = \left\{x \in \mathbb{R}^N : W(x) < 0\right\}.$$

Since $\overline{\Omega^+}$ is compact and $\overline{\Omega^+} \subset \mathbb{R}^N \backslash \overline{\Omega^-}$, there exists a function $\zeta \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \zeta(x) \leq 1$ for all $x \in \mathbb{R}^N$, $\zeta(x) = 1$ for $x \in \Omega^+$ and $\zeta(x) = 0$ for $x \in \Omega^-$. From now on, we set

$$K := \sup_{\mathbb{R}^N} |\nabla\zeta| > 0. \tag{1.7}$$

Besides the condition (f_1) on the nonlinearity $f(t)$, setting $F(t) = \int_0^t f(s)\mathrm{d}s$, we also consider the following assumptions:

(f_2) There exists $\sigma > (1 + K)c_\alpha$ such that

$$0 < \sigma F(t) \leq f(t)t, \quad \text{for all} \quad t \neq 0;$$

(f_3) There exist $\theta > m_\alpha$ and $\mu > 0$ such that $F(t) \geq \mu t^\theta$ for all $t \in [0, 1]$.

We observe that by (f_2) and deriving the quotient $F(t)/t^\sigma$ we deduce that $F(t)/t^\sigma$ is nondecreasing for $t > 0$. Thus, $F(t) \leq F(1)t^\sigma = C_1 t^\sigma$ for all $t \in [0, 1]$. Consequently, in view of (Φ_1), $F(t)/\Phi(t) \leq C_1 C t^{\sigma - N}$ for all $t \in (0, \min\{1, 1/C\})$, and therefore, we obtain

$$\lim_{t \to 0^+} \frac{F(t)}{\Phi(t)} = 0. \tag{1.8}$$

Assumption (f_3) is used to estimate the minimax level of the energy functional associated with (1.1). Note that we require this condition only for $t \in [0, 1]$.

We say that $u : \mathbb{R}^N \to \mathbb{R}$ is a weak solution of problem (1.1) if $u \in W^{1,\Phi}(\mathbb{R}^N)$ and it holds

$$\int_{\mathbb{R}^N} \left[\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla v + \Phi'(|u|)\frac{u}{|u|}v\right]\mathrm{d}x - \int_{\mathbb{R}^N} W(x)f(u)v\mathrm{d}x = 0, \quad \text{for all} \quad v \in W^{1,\Phi}(\mathbb{R}^N).$$

In order to state our main result, let us introduce some notations. Without loss of generality, we can assume that $0 \in \Omega^+$. Let $\delta > 0$ be such that $B_\delta := B_\delta(0) \subset\subset \Omega^+$. For $A \subset \mathbb{R}^N$ measurable, from now on we will denote its Lebesgue measure by $|A|$ and we introduce the number

$$\mu^* := \max\left\{\mu_1, \left[\frac{a_1|B_\delta|^{\frac{\theta-m_\alpha}{m_\alpha}}\left(\frac{\mu_1 m_\alpha}{\theta}\right)^{\frac{\theta}{\theta-m_\alpha}}}{\left(1-\frac{c_\alpha}{\sigma}-\frac{Kc_\alpha}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{(r'b)^{1/\gamma}}\right)}\right]^{\frac{\theta-m_\alpha}{m_\alpha}}\right\}, \tag{1.9}$$

where $\xi_0(t) = \min\{t^{m_\alpha}, t^{c_\alpha}\}$ for $t \geq 0$, $a_1 := \max\{W(x); x \in \overline{B_\delta}\} > 0$, $r' = r/(r-1)$ (r is given in (W_1)),

$$\mu_1 := \frac{2\Phi(1)|B_\delta|}{a_0|B_{\delta/2}|},$$

$a_0 := \min\{W(x); x \in \overline{B_\delta}\} > 0$ and $K_{N,\alpha} = B^{1/B}N\omega_{N-1}^{\gamma/N}$, $B = 1 - \alpha/(N-1)$ and ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$.

Now, we are ready to present our main result:

Theorem 1.1 *Assume that Φ is a N-function verifying $(\Phi_1)-(\Phi_2)$. Moreover, suppose that $(W_1)-(W_2)$, $(f_1)-(f_3)$ are satisfied with $\mu \geq \mu^*$ in (f_3). Then, problem (1.1) has a nonzero weak solution.*

The main features of this class of problems, considered in this paper, are that it is defined in the whole $\mathbb{R}^N$ and involves exponential critical growth (according to Lemma 3.3) and the operator inhomogeneous $\Delta_\Phi u = \mathrm{div}\big(\Phi'(|\nabla u|)\nabla u/|\nabla u|\big)$. We will show that the functional energy associated with (1.1) verifies the Palais–Smale compactness condition in certain energy levels. By applying the mountain pass theorem, we will establish the existence of nonzero solution for Eq. (1.1). Here, we improve and complement some previously cited works. As far as we know, there are no papers which deal with Eq. (1.1) in the Orlicz context, where the nonlinearities have exponential critical growth and changes sign. Besides, to prove the existence of nonzero solution, we do not assume the conditions (Φ_3), (1.7), (1.12), (1.13), (1.16) and (1.17) in [18] (see also similar assumptions in [16]). We also mention that we do not impose an specific hypothesis on $F(t)$ at the origin and we do not assume the condition

$$\text{there exist } M > 0,\ t_0 > 0 \text{ such that } F(t) \leq M|f(t)|, \quad \text{for all } |t| \geq t_0,$$

which is often used in semilinear and quasilinear problems involving exponential critical growth. In this direction, our paper improves and complements, for example, the works [16, 18, 24–27, 39, 46].

Example 1.2 Notice that, for $0 \leq \alpha < N-1$, the hypotheses of Theorem 1.1 are, for example, satisfied by $\Phi(t)$ and $f(t)$ given by:

(i) $\Phi(t) = |t|^N + N\int_0^{|t|} s^{N-1}\operatorname{arc\,sinh}^\alpha s\,\mathrm{d}s$;
(ii) $f(t) = F'(t)$ with $F(t) = \mu t^p \exp(bt^\gamma)$, where $\mu \geq \mu^*$, $p > c_\alpha$, $b > 0$ and $\gamma = N/(N-1-\alpha)$.

Notice that in this example we have $m_\alpha = N$ and $c_\alpha = N + \alpha$.

Remark 1.3 We emphasize that the approach used in this paper can be adapted with slight modifications to deal with a more general condition than (Φ_2), namely

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t^N \left(\prod_{j=1}^{l-1} \log_{[j]}^{\alpha}(t)\right) \log_{[l]}^{\alpha}(t)} = 1, \quad l \in \mathbb{N} \quad \text{and} \quad \alpha \in [0, N-1),$$

where $\log_{[k]}(t) = \log(\log_{[k-1]}(t))$ and $\log_{[1]}(t) = \log(t)$, which was considered in [16–18]. For the sake of simplicity, we prefer to treat only the case $l = 1$.

Remark 1.4 Equations involving the operator Δ_Φ appear in several physical contexts, as observed in [30] (and references [6, 7, 8] therein). They are related to concrete examples from fluid mechanics and plasticity theory. Moreover, if Ω is a domain of $\mathbb{R}^N$, with $N = 2$ or $N = 3$, and

$$\Phi(t) = |t|^N + N \int_0^{|t|} s^{N-1} \text{arc}\sinh^{\alpha} s \, \mathrm{d}s, \quad t \in \mathbb{R}, \tag{1.10}$$

$\alpha \in (0, N-1)$, the slow steady-state motion of a fluid of Prandtl–Eyring type in Ω can be modeled by the following set of equations:

$$\begin{cases} \operatorname{div}\big(\Phi'(|Du|)Du/|Du|\big) + (\text{potential term}) = 0 \quad \text{in} \quad \Omega, \\ \operatorname{div} u = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $u : \Omega \to \mathbb{R}^N$ denotes the velocity field of a incompressible fluid and $Du := \frac{1}{2}(\nabla u + \nabla u^{\perp})$ is the symmetric gradient of u (for more details see [13] and [31]).

Remark 1.5 In the papers [3, 12, 35], the authors establish integrability estimates for N-Laplace equation with an external force. In short, they consider the problem

$$\begin{cases} -\Delta_N u = f(x) \quad & in \quad \Omega, \\ \quad u = 0 \quad & on \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and f belongs to $L^1(\Omega)$ or to a better space, and they show that if $u \in W_0^{1,N}(\Omega)$ is a weak solution (or an entropy solution) then u satisfies an integrability estimate of type

$$\int_\Omega \exp\left[\frac{(N\omega_N^{1/(N-1)} - \delta)|u(x)|}{\|f\|_1^{1/(N-1)}}\right] \mathrm{d}x \le \frac{N\omega_N^{1/(N-1)}}{\delta}|\Omega|, \quad \text{for all } \delta \in (0, N\omega_N^{1/(N-1)}),$$

where ω_N is the measure of the unit sphere in $\mathbb{R}^N$ and $|\Omega|$ is the measure of Ω. Since the function Φ satisfies condition $(\Phi_1) - (\Phi_2)$, we believe that the approach used in [3, 12, 35] seems to lead to similar results in the case of the problem

$$\begin{cases} -\Delta_\Phi u = f(x) \quad & \text{in} \quad \Omega, \\ \quad u = 0 \quad & \text{on} \quad \partial\Omega, \end{cases}$$

which can enable us to get regularity results to the solutions of (1.1).

This paper is organized as follows: In Sect. 2, we present some preliminary results about Orlicz spaces which are used in the work. In Sect. 3, we establish the variational framework for our problem and we obtain some embedding results involving our working space.

Section 4 shows that the energy functional has the geometric structure of the mountain pass theorem, and in Sect. 5, we prove that this functional satisfies the Palais–Smale condition in certain energy levels. Finally, in Sect. 6, we prove Theorem 1.1.

Throughout this paper, $W^{1,N}(\mathbb{R}^N)$ denotes the Sobolev space endowed with the norm

$$\|u\|_{1,N} = \left(\int_{\mathbb{R}^N} (|\nabla u|^N + |u|^N) \mathrm{d}x \right)^{1/N}, \quad u \in W^{1,N}(\mathbb{R}^N).$$

We use $|\cdot|_p$ to denote the norm of the Lebesgue space $L^p(\mathbb{R}^N)$, $1 \le p \le \infty$. We denote by B_R the ball centered at the origin with radius $R > 0$ and the symbols C, C_i, $i = 0, 1, 2, \dots$ will denote different (possibly) positive constants.

2 Preliminaries

In order to facilitate the understanding of the paper, in this section we present briefly some results about Orlicz spaces. For the proofs and more details, see, for instance, [7, 37, 45].

A function $A : \mathbb{R} \to [0, +\infty)$ is called N-function if it is convex, even, $A(t) = 0$ if and only if $t = 0$, $A(t)/t \to 0$ as $t \to 0$ and $A(t)/t \to +\infty$ as $t \to +\infty$. In particular, we have $A'(0) = 0$, and if A is differentiable, then $A'(t)$ is nondecreasing for $t \ge 0$, which implies that $A(t)$ is increasing for $t > 0$. For a N-function A and an open set $\Omega \subset \mathbb{R}^N$, the Orlicz class is the set defined by

$$K_{A,\mu}(\Omega) = \left\{ u : \Omega \to \mathbb{R}; \text{u is measurable and } \int_\Omega A(|u(x)|)\, d\mu < \infty \right\}.$$

The linear space $L_{A,\mu}(\Omega)$ generated by $K_{A,\mu}(\Omega)$ is called *Orlicz space*. If μ is the Lebesgue measure, then we denote $K_{A,\mu}(\Omega)$ and $L_{A,\mu}(\Omega)$ by $K_A(\Omega)$ and $L_A(\Omega)$, respectively. When A satisfies the Δ_2-condition, namely, there exists a constant $k > 0$ such that

$$A(2t) \le kA(t), \quad \text{for all } t \ge 0,$$

the Orlicz class $K_{A,\mu}(\Omega)$ is a linear space and hence equal to $L_{A,\mu}(\Omega)$. We consider the following norm (called of *Luxemburg's norm*) on $L_{A,\mu}(\Omega)$:

$$|u|_{A,\Omega} = \inf \left\{ \lambda > 0; \int_\Omega A\left(\frac{|u(x)|}{\lambda} \right) d\mu \le 1 \right\}.$$

It can be shown that $\left(L_{A,\mu}(\Omega), |\cdot|_{A,\Omega}\right)$ is a Banach space (see [45, Theorem 10, p. 67]). In the case $\Omega = \mathbb{R}^N$, we denote $|\cdot|_{A,\mathbb{R}^N}$ by $|\cdot|_A$. The complement N-function of A is defined by

$$\tilde{A}(t) = \sup_{s>0} \{ts - A(s)\}.$$

It is not difficult to verify that $\tilde{\tilde{A}} = A$. In the spaces $L_{A,\mu}(\Omega)$ and $L_{\tilde{A},\mu}(\Omega)$, an extension of the Hölder inequality holds, namely

$$\left| \int_\Omega u(x)v(x)\, d\mu \right| \le 2|u|_{A,\Omega}|v|_{\tilde{A},\Omega}, \quad \text{for all } u \in L_{A,\mu}(\Omega),\ v \in L_{\tilde{A},\mu}(\Omega). \tag{2.1}$$

As a consequence, for every $\tilde{u} \in L_{\tilde{A},\mu}(\Omega)$ there corresponds a continuous linear functional $f_{\tilde{u}} \in (L_{A,\mu}(\Omega))'$ given by $f_{\tilde{u}}(v) = \int_\Omega \tilde{u}(x)v(x)\, d\mu$, $v \in L_{A,\mu}(\Omega)$. Thus, we can define

$$\|\tilde{u}\|_{\tilde{A},\Omega} = \sup_{|v|_{A,\Omega}\le 1} \int_\Omega \tilde{u}(x)v(x)\,d\mu$$

and $\|\cdot\|_{\tilde{A},\Omega}$ is called the Orlicz norm on the space $L_{\tilde{A},\mu}(\Omega)$. Similarly, we can define the Orlicz norm $\|\cdot\|_{A,\Omega}$ on $L_{A,\mu}(\Omega)$. The norms $|\cdot|_{A,\Omega}$ and $\|\cdot\|_{A,\Omega}$ are equivalent and satisfy the inequalities

$$|u|_{A,\Omega} \le \|u\|_{A,\Omega} \le 2|u|_{A,\Omega}, \quad \text{for all } u \in L_A(\Omega).$$

We define the Orlicz–Sobolev space $W^{1,A}(\Omega)$ as follows

$$W^{1,A}(\Omega) = \left\{u : \Omega \to \mathbb{R} : \text{ u is measurable and } u, |\nabla u| \in L_A(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{W^{1,A}(\Omega)} := |u|_{A,\Omega} + |\nabla u|_{A,\Omega},$$

where ∇u is the gradient of u, and we are using its Euclidean norm in $\mathbb{R}^N$. An important property is that if A and $\tilde{A}$ verify the Δ_2-condition, then the spaces $L_A(\Omega)$ and $W^{1,A}(\Omega)$ are reflexive, separable and

$$\left(L_A(\Omega), |\cdot|_{A,\Omega}\right)' = \left(L_{\tilde{A}}(\Omega), \|\cdot\|_{\tilde{A},\Omega}\right) \text{ and } \left(L_{\tilde{A}}(\Omega), |\cdot|_{\tilde{A},\Omega}\right)' = \left(L_A(\Omega), \|\cdot\|_{A,\Omega}\right).$$

Proposition 2.1 *Let A be a N-function of class C^1. Then $\tilde{A}$ is a function C^1 and verifies the conditions below:*

(i) $A(\tilde{A}'(t)) = \tilde{A}'(t)t - \tilde{A}(t), \quad \text{for all } t \ge 0;$
(ii) $A'(\tilde{A}'(t)) = t, \quad \text{for all } t \ge 0.$

Proof See Lemma A.2 and Lemma 2.5 in [32]. □

Definition 2.2 We say that a N-function A satisfies the ∇_2 condition (we denote by $A \in \nabla_2$), if there exists $\gamma > 1$ such that

$$A(t) \le \frac{1}{2\gamma}A(\gamma t), \quad \text{for all } t \ge 0.$$

Proposition 2.3 *Let A and $\tilde{A}$ be a pair of differentiable complementary N-functions. Then, the following assertions are equivalent:*

(a) $A \in \Delta_2$;
(b) *there exists* $\alpha \in (1,\infty)$ *such that* $A'(t)t/A(t) < \alpha$ *for all* $t \ge 0$;
(c) *there exists* $\beta \in (1,\infty)$ *such that* $\tilde{A}'(s)s/\tilde{A}(s) > \beta$ *for all* $s \ge 0$;
(d) $\tilde{A} \in \nabla_2$.

Proof See Theorem 3 (p. 22) in [45]. □

Next, we state two lemmas due to Fukagai et al. [32, Lemma 2.1, Lemma 2.5] and Lieberman [38, Lemma 1.1 (e)] which will be used in our arguments.

Lemma 2.4 *Suppose that A is a N-function and let $\tilde{A}$ be the complement N-function of A. Then,*

(a) $\tilde{A}(A'(t)) \le A(2t)$ *for all* $t \ge 0$;
(b) $\tilde{A}\left(\frac{A(t)}{t}\right) \le A(t)$ *for all* $t > 0$;
(c) $A'(t)s \le A'(t)t + A'(s)s$ *for all* $t, s \ge 0$.

Lemma 2.5 *Suppose that A is a differentiable N-function satisfying*

$$m \le \frac{A'(t)t}{A(t)} \le M, \quad \text{for all } t > 0, \tag{2.2}$$

for some $M \ge m > 0$. Defining, for $t \ge 0$, $\xi_0(t) = \min\{t^m, t^M\}$ and $\xi_1(t) = \max\{t^m, t^M\}$, one has

$$\xi_0(\rho)A(t) \le A(\rho t) \le \xi_1(\rho)A(t) \quad \text{for} \quad \rho, t \ge 0$$

and

$$\xi_0(|u|_{A,\Omega}) \le \int_\Omega A(|u(x)|)\, d\mu \le \xi_1(|u|_{A,\Omega}) \quad \text{for} \quad u \in L_{A,\mu}(\Omega).$$

Lemma 2.6 *Let Φ be a N-function satisfying conditions (Φ_1) and (Φ_2). Then, Φ satisfies the condition ∇_2.*

Proof In order to show that $\Phi \in \nabla_2$, we will verify that there exists a constant $C > 0$ such that $C\Phi(t)\rho^N \le \Phi(\rho t)$ for each $\rho > 1$ and $t \ge 0$. In fact, setting $M_0 := \inf_{t\in[0,1]} \Phi(t)/t^N$ and $M_1 := \sup_{t\in[0,1]} \Phi(t)/t^N$, by (Φ_1) one has $0 < M_0 \le M_1 < \infty$. Therefore, $M_0 t^N \rho^N \le \Phi(\rho t) \le M_1 \rho^N t^N$ if $\rho t \le 1$. Thus,

$$\Phi(\rho t) \ge \frac{M_0}{M_1}\rho^N \Phi(t) \quad \text{for} \quad \rho t \le 1. \tag{2.3}$$

On the other hand, setting

$$M_2 := \inf_{t\ge 1} \frac{\Phi(t)}{\log^\alpha(t+1)} \quad \text{and} \quad M_3 := \sup_{t\ge 1} \frac{\Phi(t)}{\log^\alpha(t+1)},$$

from condition (Φ_2) we have $0 < M_2 \le M_3 < \infty$ and

$$M_3 t^N \log^\alpha(t+1) \ge \Phi(t) \ge M_2 t^N \log^\alpha(t+1), \quad \text{for all } t \ge 1.$$

Therefore, if $\rho t > 1$ then

$$\Phi(\rho t) \ge M_2 t^N \rho^N \log^\alpha(\rho t + 1) \ge M_2 t^N \rho^N \log^\alpha(t+1) \ge \frac{M_2}{M_3}\rho^N \Phi(t). \tag{2.4}$$

Hence, taking $C = \min\left\{M_0/M_1, M_2/M_3\right\}$, by (2.3) and (2.4) we get $\Phi(\rho t) \geq C\rho^N\Phi(t)$ for each $\rho > 1$ and $t \geq 0$. Finally, taking $\gamma = \rho > (2/C)^{1/(N-1)}$ we obtain $\Phi(t) \leq \Phi(\gamma t)/(2\gamma)$ for $t \geq 0$, which shows that $\Phi \in \nabla_2$ and the proof is finished. □

Proposition 2.7 *If Φ is a N-function satisfying $(\Phi_1)-(\Phi_2)$, then $m_\alpha > 1$, where m_α was defined in* (1.6).

Proof The proof follows combining Proposition 2.3 and Lemma 2.6. □

Remark 2.8 As we saw in Introduction, our N-function Φ satisfies Δ_2-condition and assumption (2.2) with $m = m_\alpha$ and $M = c_\alpha$.Combining Proposition 2.3 with Lemma 2.6, $\tilde{\Phi}$ satisfies Δ_2-condition. Therefore, it can be shown that

$$u_n \to 0 \ \text{ in } \ L_{\Phi,\mu}(\mathbb{R}^N) \Longleftrightarrow \int_{\mathbb{R}^N} \Phi(|u_n|)\, d\mu \to 0$$

and (u_n) is bounded in $L_{\Phi,\mu}(\mathbb{R}^N)$ if and only if $(\int_{\mathbb{R}^N} \Phi(|u_n|)d\mu)$ is bounded. Moreover, as observed above $\left(L_{\Phi,\mu}(\mathbb{R}^N), |\cdot|_\Phi\right)$ is a separable and reflexive Banach space as well as $\left(W^{1,\Phi}(\mathbb{R}^N), \|\cdot\|\right)$.

3 Variational framework

The next lemma presents some embeddings which will be used in our arguments.

Lemma 3.1 *If (Φ_1) and (Φ_2) are satisfied, then the following embeddings are continuous:*

(a) $L_\Phi(\mathbb{R}^N) \hookrightarrow L^N(\mathbb{R}^N)$;
(b) $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow W^{1,N}(\mathbb{R}^N)$;
(c) $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ *for any* $r \in [N, \infty)$.

Proof We observe that by $(\Phi_1)-(\Phi_2)$ there exists $C_1 > 0$ such that $t^N \leq C_1\Phi(t)$ for all $t \geq 0$. Thus, if $u_n \to 0$ in $L_\Phi(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} |u_n|^N \mathrm{d}x \leq C_1 \int_{\mathbb{R}^N} \Phi(|u_n|)\mathrm{d}x \to 0$$

and items a) and b) are proved. The proof of item c) follows directly from b) and by the continuous embedding from $W^{1,N}(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ for any $r \in [N, \infty)$. □

Now, we prove a result of convergence, which will be crucial in the sequel.

Proposition 3.2 *Under conditions (W_1) and $(\Phi_1)-(\Phi_2)$, if $u_n \rightharpoonup u$ in $W^{1,\Phi}(\mathbb{R}^N)$ then*

$$\lim_{n\to+\infty} \left(\int_{\mathbb{R}^N} |W(x)||u_n - u|^s \,\mathrm{d}x\right)^{1/s} = 0, \quad \text{for all } s \in [N, \infty).$$

Proof In view of (W_1) and by the Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |W(x)||u_n-u|^s\,\mathrm{d}x &= \int_{B_R} |W(x)||u_n-u|^s\,\mathrm{d}x + \int_{B_R^c} |W(x)||u_n-u|^s\,\mathrm{d}x\\ &\le |W|_{\infty,B_R}\int_{B_R}|u_n-u|^s\,\mathrm{d}x\\ &\quad+\left(\int_{B_R^c}|W(x)|^r\,\mathrm{d}x\right)^{1/r}\left(\int_{B_R^c}|u_n-u|^{sr'}\,\mathrm{d}x\right)^{1/r'}.\end{aligned} \tag{3.1}$$

On the other hand, since the embedding $W^{1,N}(\mathbb{R}^N)\hookrightarrow L^s(B_R)$ is compact and $W^{1,\Phi}(\mathbb{R}^N)\hookrightarrow L^{sr'}(\mathbb{R}^N)$ is continuous, given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\int_{B_R}|u_n-u|^s\,\mathrm{d}x<\frac{\varepsilon}{2|W|_{\infty,B_R}}\quad\text{for all}\ \ n>n_0 \tag{3.2}$$

and there exists $C>0$ verifying

$$\left(\int_{B_R^c}|u_n-u|^{sr'}\,\mathrm{d}x\right)^{\frac{1}{r'}}\le C,\quad\text{for all}\ \ n\in\mathbb{N}. \tag{3.3}$$

Moreover, by (W_1) there exists $R_0>0$ sufficiently large such that

$$\left(\int_{B_R^c}|W(x)|^r\,\mathrm{d}x\right)^{\frac{1}{r}}<\frac{\varepsilon}{2C},\quad\text{for all}\ \ R\ge R_0. \tag{3.4}$$

Hence, from (3.1)–(3.4),

$$\int_{\mathbb{R}^N}|W(x)||u_n-u|^s\,\mathrm{d}x<\varepsilon,\qquad\text{for all}\ \ n>n_0,$$

and the proof is complete. □

The next lemma presents a version of the Trudinger–Moser inequality for functions in $W^{1,\Phi}(\mathbb{R}^N)$, which was proved by Cerný in [18] (see also [21]). It is necessary to use variational methods to find solutions for problem (1.1) with nonlinearities $f(t)$ satisfying the condition growth (f_1).

Lemma 3.3 *If* $N\ge 2$, $\widetilde{K}>0$, $\alpha\in[0,N-1)$, Φ *is a N-function verifying* $(\Phi_1)-(\Phi_2)$ *and* $u\in W^{1,\Phi}(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N}\left[\exp(\widetilde{K}|u|^\gamma)-S_{N,\alpha}(\widetilde{K}|u|^\gamma)\right]\mathrm{d}x<\infty.$$

Furthermore, if $|\nabla u|_\Phi\le 1$, $|u|_\Phi\le M<\infty$ *and* $\widetilde{K}<K_{N,\alpha}$ *then there exists a constant* $C=C(N,\alpha,M,\Phi,\widetilde{K})>0$, *which depends only* N,α,M,Φ *and* $\widetilde{K}$ *such that*

$$\int_{\mathbb{R}^N}\left[\exp(\widetilde{K}|u|^\gamma)-S_{N,\alpha}(\widetilde{K}|u|^\gamma)\right]\mathrm{d}x\le C,$$

where $K_{N,\alpha} = B^{1/B} N \omega_{N-1}^{\gamma/N}$, $B = 1 - \alpha/(N-1)$ *and* ω_{N-1} *is the measure of the unit sphere in* $\mathbb{R}^N$.

To finalize this section, we get two technical lemmas that will be necessary to show the regularity of the energy functional associated with our problem.

Lemma 3.4 *For each* $p \geq 1$, *there exists* $C = C(p) > 0$ *such that*

$$\left[\exp(t) - S_{N,\alpha}(t)\right]^p \leq C\left[\exp(pt) - S_{N,\alpha}(pt)\right], \quad \text{for all } t \geq 0.$$

Proof It suffices to proof that the limits

$$\lim_{t\to 0} \frac{\left[\exp(t) - S_{N,\alpha}(t)\right]^p}{\left[\exp(pt) - S_{N,\alpha}(pt)\right]} \quad \text{and} \quad \lim_{t\to +\infty} \frac{\left[\exp(t) - S_{N,\alpha}(t)\right]^p}{\left[\exp(pt) - S_{N,\alpha}(pt)\right]}$$

are finite, which is a direct consequence of the L'Hospital rule. □

Lemma 3.5 *Let* (u_n) *be a sequence in* $W^{1,\Phi}(\mathbb{R}^N)$ *strongly convergent. Then there exist a subsequence* (u_{n_k}) *of* (u_n) *and* $v \in W^{1,\Phi}(\mathbb{R}^N)$ *such that* $u_{n_k}(x) \leq v(x)$ *almost everywhere in* $x \in \mathbb{R}^N$.

Proof The arguments used to show this lemma follows the same lines of the proof of Proposition 1 of [26] with slight modifications and we omit it. □

The energy functional associated with problem (1.1) is given by

$$J(u) = \int_{\mathbb{R}^N} [\Phi(|\nabla u|) + \Phi(|u|)]\, dx - \int_{\mathbb{R}^N} W(x)F(u)\, dx.$$

Notice that by (f_1), Lemmas 3.3, 3.5 and Proposition 3.2, J is well defined on $W^{1,\Phi}(\mathbb{R}^N)$, and moreover, by using standard computations (see [18, Proposition 4.1]), we can see that $J \in C^1(W^{1,\Phi}(\mathbb{R}^N), \mathbb{R})$ and its derivative is given by

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} \left[\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla v + \Phi'(|u|)\frac{u}{|u|}v\right] dx - \int_{\mathbb{R}^N} W(x)f(u)v dx$$

for $u, v \in W^{1,\Phi}(\mathbb{R}^N)$. Consequently, critical points of J are precisely the weak solutions of (1.1).

4 Mountain pass structure

In order to get Theorem 1.1, we shall use the mountain pass theorem due to Ambrosetti and Rabinowitz [9]:

Theorem 4.1 *Let X be a Banach space and* $J \in C^1(X;\mathbb{R})$ *with* $J(0) = 0$. *Suppose that there exist* $\rho, \tau > 0$ *and* $e \in X$, *with* $\|e\| > \rho$, *such that*

$$\inf_{\|u\|=\rho} J(u) \geq \tau \quad \text{and} \quad J(e) < 0. \tag{4.1}$$

Then, J possesses a Palais–Smale sequence at level c characterized as

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \geq \tau,$$

where $\Gamma = \{\gamma \in C([0,1];X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. *Moreover, if J satisfies the Palais–Smale condition at level c then J has a critical point* u_0 *such that* $J(u_0) = c$.

The number c is called *mountain pass level* or *minimax level* of the functional J.

In the sequel, we show that the functional J has the mountain pass geometry, condition (4.1). This is proved in the next lemmas.

Lemma 4.2 *Assume* (f_1) *and* (f_2). *Then, there exist* $\rho, \beta > 0$ *such that*

$$J(u) \geq \beta, \quad \text{for all } \|u\| = \rho.$$

Proof From (1.8), given $\epsilon > 0$, there exists $\delta > 0$ verifying $F(t) \leq \epsilon\Phi(t)$ for all $|t| \leq \delta$. On the other hand, by using (f_1) and taking $p > c_\alpha$ we have

$$F(t) \leq C|t|^p[\exp(b|t|^\gamma) - S_{N,\alpha}(b|t|^\gamma)], \quad \text{for all } |t| > \delta,$$

for some $C = C(\delta, p) > 0$. Therefore,

$$F(t) \leq \varepsilon\Phi(t) + C|t|^p[\exp(b|t|^\gamma) - S_{N,\alpha}(b|t|^\gamma)], \quad \text{for all } t \geq 0. \tag{4.2}$$

From (W_2), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} W(x)F(u)\mathrm{d}x &\leq \int_{\Omega^+} W(x)F(u)\mathrm{d}x \\ &\leq \varepsilon C_1 \int_{\Omega^+} \Phi(u)\mathrm{d}x + C_2 \int_{\Omega^+} |u|^p[\exp(b|u|^\gamma) - S_{N,\alpha}(b|u|^\gamma)]\mathrm{d}x. \end{aligned} \tag{4.3}$$

Fixing $\varepsilon = 1/2C_1$, using Hölder inequality, Lemmas 3.1 and 3.4(c), we reach

$$\begin{aligned} J(u) \geq{}& \left(1 - \varepsilon C_1\right) \int_{\mathbb{R}^N} [\Phi(|\nabla u|) + \Phi(u)]\mathrm{d}x \\ &- C_2 \left\{ \int_{\mathbb{R}^N} [\exp(2b|u|^\gamma) - S_{N,\alpha}(2b|u|^\gamma)]\mathrm{d}x \right\}^{\frac{1}{2}} |u|_{2p}^p \\ \geq{}& \frac{1}{2} \int_{\mathbb{R}^N} [\Phi(|\nabla u|) + \Phi(|u|)]\mathrm{d}x \\ &- C_3 \left\{ \int_{\mathbb{R}^N} \left[\exp\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right) - S_{N,\alpha}\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right)\right]\mathrm{d}x \right\}^{\frac{1}{2}} \|u\|^p. \end{aligned}$$

Now, if $0 < \rho < 1$ is such that $2b\rho^\gamma < K_{N,\alpha}$ then for $\|u\| = \rho$ we obtain $2b|\nabla u|_\Phi^\gamma \leq 2b\rho^\gamma < K_{N,\alpha}$. Thus, by Lemma 3.3

$$\left\{ \int_{\mathbb{R}^N} \left[\exp\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right) - S_{N,\alpha}\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right)\right]\mathrm{d}x \right\}^{\frac{1}{2}} \leq C.$$

Consequently, from Lemma 2.5, for $\|u\| = \rho$ we get

$$J(u) \geq \frac{1}{2}\left(|\nabla u|_{\Phi}^{c_\alpha} + |u|_{\Phi}^{c_\alpha}\right) - C_4\rho^p \geq 2^{-(c_\alpha+1)}\rho^{c_\alpha} - C_4\rho^p.$$

Choosing $\rho > 0$ sufficiently small so that $2^{-(c_\alpha+1)}\rho^{c_\alpha} - C_4\rho^p =: \beta > 0$, we conclude $J(u) \geq \beta$ for all $\|u\| = \rho$. □

Lemma 4.3 *There exists $v_0 \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|v_0\| > \rho$ such that $J(v_0) < 0$.*

Proof By (f_2), there exist constants $C_1, C_2 > 0$ such that

$$F(t) \geq C_1 t^\sigma - C_2, \quad \text{for all} \quad t \geq 0.$$

Since $\Omega^+ \neq \emptyset$, taking $0 \leq \varphi \in C_0^\infty(\Omega^+)$ such that $\mathcal{K} = supp(\varphi)$, we have

$$\begin{aligned} J(t\varphi) &\leq \xi_1(t) \int_\Omega [\Phi(|\nabla\varphi|) + \Phi(\varphi)]\,\mathrm{d}x - C_1 t^\sigma \int_{\mathcal{K}} W(x)|\varphi|^\sigma\,\mathrm{d}x + C_2 \int_{\mathcal{K}} W(x)\,\mathrm{d}x \\ &= C_0 t^{c_\alpha} - C_3 t^\sigma + C_4, \quad \text{for all} \quad t > 1. \end{aligned}$$

Since $\sigma > c_\alpha$, it follows that $J(t\varphi) \to -\infty$ as $t \to +\infty$. Thus, taking $v_0 := t_0\varphi$, with t_0 large enough, the proof is finished. □

5 On Palais–Smale sequences

First, we recall that $(u_n) \subset W^{1,\Phi}(\mathbb{R}^N)$ is a Palais–Smale ($(PS)_c$ for short) sequence at level $c \in \mathbb{R}$ for the functional J if $J(u_n) \to c$ and $J'(u_n) \to 0$ in the dual space $[W^{1,\Phi}(\mathbb{R}^N)]'$. We say that J satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence has a convergent subsequence. In this section, our main objective is to prove the $(PS)_c$ condition for J with c in a convenient interval.

Lemma 5.1 *If $(u_n) \subset W^{1,\Phi}(\mathbb{R}^N)$ is a $(PS)_c$ sequence associated with J, then (u_n) is bounded in $W^{1,\Phi}(\mathbb{R}^N)$.*

Proof Take $\zeta \in C^\infty(\mathbb{R}^N)$ given in (1.7). By condition (1.5), (f_2) and Lemma 2.4, we have

$$\begin{aligned}
&J(u_n) - \frac{1}{\sigma}\langle J'(u_n), u_n\zeta\rangle \\
&= \int_{\mathbb{R}^N} \left[\Phi(|\nabla u_n|) + \Phi(|u_n|)\right] \mathrm{d}x - \int_{\mathbb{R}^N} W(x)F(u_n)\,\mathrm{d}x \\
&\quad - \frac{1}{\sigma}\int_{\mathbb{R}^N} \Phi'(|\nabla u_n|)|\nabla u_n|\zeta\,\mathrm{d}x - \frac{1}{\sigma}\int_{\mathbb{R}^N} \Phi'(|\nabla u_n|)\frac{\nabla u_n \nabla\zeta}{|\nabla u_n|}u_n\,\mathrm{d}x \\
&\quad - \frac{1}{\sigma}\int_{\mathbb{R}^N} \Phi'(|u_n|)|u_n|\zeta\,\mathrm{d}x + \frac{1}{\sigma}\int_{\mathbb{R}^N} W(x)f(u_n)u_n\zeta\,\mathrm{d}x \\
&\geq \left(1 - \frac{c_\alpha}{\sigma}\right)\int_{\mathbb{R}^N} \left[\Phi(|\nabla u_n|) + \Phi(|u_n|)\right] \mathrm{d}x - \int_{\Omega^+} W(x)F(u_n)\,\mathrm{d}x \\
&\quad - \frac{1}{\sigma}K\int_{\mathbb{R}^N} \Phi'(|\nabla u_n|)|u_n|\,\mathrm{d}x + \frac{1}{\sigma}\int_{\Omega^+} W(x)f(u_n)u_n\,\mathrm{d}x \\
&\geq \left(1 - \frac{c_\alpha}{\sigma}\right)\int_{\mathbb{R}^N} \left[\Phi(|\nabla u_n|) + \Phi(|u_n|)\right] \mathrm{d}x \\
&\quad - \frac{1}{\sigma}K\left(\int_{\mathbb{R}^N} \Phi'(|\nabla u_n|)|\nabla u_n|\,\mathrm{d}x + \int_{\mathbb{R}^N} \Phi'(|u_n|)|u_n|\,\mathrm{d}x\right).
\end{aligned} \tag{5.1}$$

Thus,

$$J(u_n) - \frac{1}{\sigma}\langle J'(u_n), u_n\zeta\rangle \geq \left(1 - \frac{c_\alpha}{\sigma} - \frac{Kc_\alpha}{\sigma}\right)\int_{\mathbb{R}^N} [\Phi(|\nabla u_n|) + \Phi(|u_n|)]\,\mathrm{d}x. \tag{5.2}$$

On the other hand,

$$J(u_n) - \frac{1}{\sigma}\langle J'(u_n), u_n\zeta\rangle \leq c + o_n(1) + o_n(1)\|u_n\zeta\| \leq c + o_n(1) + o_n(1)\|u_n\|. \tag{5.3}$$

Combining (5.2) and (5.3) and using Lemma 2.5, we obtain

$$c + o_n(1) + o_n(1)\|u_n\| \geq \left(1 - \frac{c_\alpha(1+K)}{\sigma}\right)[\xi_0(|\nabla u_n|_\Phi) + \xi_0(|u_n|_\Phi)]. \tag{5.4}$$

Now, we argue by contradiction. Suppose that, up to a subsequence, $\|u_n\| \to \infty$. We have three possibilities to consider:

(i) $|\nabla u_n|_\Phi \to \infty$ and $(|u_n|_\Phi)$ is bounded;
(ii) $(|\nabla u_n|_\Phi)$ is bounded and $|u_n|_\Phi \to \infty$;
(iii) $|\nabla u_n|_\Phi \to \infty$ and $|u_n|_\Phi \to \infty$.

If item (i) occurs, then there exists $n_1 \in \mathbb{N}$ such that $|\nabla u_n|_\Phi > 1$ for all $n > n_1$. Thus, by the definition of ξ_0 and inequality (5.4) we get

$$c + o_n(1) + o_n(1)|\nabla u_n|_\Phi \geq \left(1 - \frac{c_\alpha(1+K)}{\sigma}\right)|\nabla u_n|_\Phi^{m_\alpha}, \quad \text{for all } n > n_1. \tag{5.5}$$

Dividing this estimate by $|\nabla u_n|_\Phi^{m_\alpha}$, we get a contradiction doing $n \to \infty$. Thus, (i) does not happen. Similarly, we can show that items (ii) and (iii) do not happen as well. Therefore, (u_n) should be bounded in $W^{1,\Phi}(\mathbb{R}^N)$ and the proof is finalized. □

Corollary 5.2 *If $(u_n) \subset W^{1,\Phi}(\mathbb{R}^N)$ is a $(PS)_c$ sequence for J, then*

$$\xi_0(|\nabla u_n|_\Phi) \le \left(1 - \frac{c_\alpha(1+K)}{\sigma}\right)^{-1} c + o_n(1).$$

Proof This estimate is a direct consequence of (5.4). □

Before to show that the functional J satisfies the Palais–Smale condition in a convenient interval, we shall need of the following convergence result:

Lemma 5.3 *Let (u_n) be a Palais–Smale sequence for the functional J at any level $c \in \mathbb{R}$ such that*

$$c < \left(1 - \frac{c_\alpha(1+K)}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{(r'b)^{1/\gamma}}\right).$$

Then, there exists $u \in W^{1,\Phi}(\mathbb{R}^N)$ verifying

$$\int_{\mathbb{R}^N} W(x)f(u_n)(u_n - u)\,\mathrm{d}x \to 0.$$

Proof By Corollary 5.2, we have

$$\xi_0(|\nabla u_n|_\Phi) \le \left(1 - \frac{c_\alpha(1+K)}{\sigma}\right)^{-1} c + o_n(1) < \xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{(r'b)^{1/\gamma}}\right) + o_n(1).$$

Since $\xi_0(t)$ is increasing for $t \ge 0$, we can obtain $n_1 \in \mathbb{N}$ such that

$$|\nabla u_n|_\Phi \le \frac{K_{N,\alpha}^{1/\gamma}}{(r'b)^{1/\gamma}} - \delta, \quad \text{for all } n > n_1,$$

for some $\delta > 0$ sufficiently small. Choosing still $s > 1$ close to 1, we obtain

$$r'sb|\nabla u_n|_\Phi^\gamma \le K_{N,\alpha} - \delta_1, \quad \text{for all } n > n_1, \tag{5.6}$$

for some appropriate $\delta_1 > 0$. Now, by Lemma 5.1, there exists $u \in W^{1,\Phi}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,\Phi}(\mathbb{R}^N)$. Next, setting $Q_n := [\exp(b|u_n|^\gamma) - S_{N,\alpha}(b|u_n|^\gamma)]$, by assumption (f_1), Hölder inequality and Lemma 3.4, it follows that

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N} W(x)f(u_n)(u_n-u)\,\mathrm{d}x\Big| \\
&\quad\le C\int_{\mathbb{R}^N}|W(x)||u_n|^{N-1}|u_n-u|\,\mathrm{d}x + C\int_{\mathbb{R}^N}|W(x)||u_n-u||Q_n|\,\mathrm{d}x \\
&\quad\le C\Big(\int_{\mathbb{R}^N}|W(x)||u_n|^N\mathrm{d}x\Big)^{\frac{N-1}{N}}\Big(\int_{\mathbb{R}^N}|W(x)||u_n-u|^N\,\mathrm{d}x\Big)^{\frac{1}{N}} \\
&\qquad + C\Big(\int_{\mathbb{R}^N}|W(x)||u_n-u|^{\frac{s}{s-1}}\,\mathrm{d}x\Big)^{\frac{s-1}{s}}\Big(\int_{\mathbb{R}^N}|W(x)||Q_n|^s\Big)^{\frac{1}{s}} \\
&\quad\le C|W|_r^{\frac{N-1}{N}}|u_n|_{r'N}^{N-1}\Big(\int_{\mathbb{R}^N}|W(x)||u_n-u|^N\,\mathrm{d}x\Big)^{\frac{1}{N}} + \\
&\qquad + C\Big(\int_{\mathbb{R}^N}|W(x)||u_n-u|^{\frac{s}{s-1}}\,\mathrm{d}x\Big)^{\frac{s-1}{s}}|W|_r^{\frac{1}{s}}|Q_n|_{r's}.
\end{aligned}
$$

By Proposition 3.2, we have

$$\int_{\mathbb{R}^N}|W(x)||u_n-u|^N\,\mathrm{d}x\to 0 \quad\text{and}\quad \int_{\mathbb{R}^N}|W(x)||u_n-u|^{\frac{s}{s-1}}\,\mathrm{d}x\to 0.$$

Hence, to finalize the proof, just to justify that

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^N}[\exp(r'sb|u_n|^\gamma)-S_{N,\alpha}(r'sb|u_n|^\gamma)]\mathrm{d}x<\infty.$$

Indeed, we can write this integral as

$$\int_{\mathbb{R}^N}\left[\exp\left(r'sb|\nabla u_n|_\Phi^\gamma\left(\frac{|u_n|}{|\nabla u_n|_\Phi}\right)^\gamma\right)-S_{N,\alpha}\left(r'sb|\nabla u_n|_\Phi^\gamma\left(\frac{|u_n|}{|\nabla u_n|_\Phi}\right)^\gamma\right)\right]\mathrm{d}x,$$

and by (5.6), we have $r'sb|\nabla u_n|_\Phi^\gamma<K_{N,\alpha}-\delta_1<K_{N,\alpha}$, for $n>n_1$. Therefore, invoking Lemma 3.3 we conclude that the above supreme is finite and the proof is complete. □

Lemma 5.4 *The functional J satisfies the $(PS)_c$ condition for all*

$$c<\left(1-\frac{c_\alpha(1+K)}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{(r'b)^{1/\gamma}}\right). \tag{5.7}$$

Proof Let (u_n) be in $W^{1,\Phi}(\mathbb{R}^N)$ such that $J(u_n)\to c$ and $J'(u_n)\to 0$ in $[W^{1,\Phi}(\mathbb{R}^N)]'$ with c satisfying (5.7). By Lemma 5.1, (u_n) is bounded in $W^{1,\Phi}(\mathbb{R}^N)$, and therefore, up to a subsequence, $u_n\rightharpoonup u$ in $W^{1,\Phi}(\mathbb{R}^N)$. Since the functional $I(u):=\int_{\mathbb{R}^N}[\Phi(|\nabla u|)+\Phi(|u|)]\,\mathrm{d}x$ is convex, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^N} [\Phi(|\nabla u|) + \Phi(|u|)]\,\mathrm{d}x - \int_{\mathbb{R}^N} [\Phi(|\nabla u_n|) + \Phi(|u_n|)]\,\mathrm{d}x \\
&\quad \geq \int_{\mathbb{R}^N} \Phi'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|}(\nabla u - \nabla u_n)\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} \Phi'(|u_n|)\frac{u_n}{|u_n|}(u - u_n)\,\mathrm{d}x \\
&\quad = \langle J'(u_n), u - u_n\rangle + \int_{\mathbb{R}^N} W(x)f(u_n)(u - u_n)\,\mathrm{d}x.
\end{aligned} \tag{5.8}
$$

According to Lemma 5.3, we know that $\int_{\mathbb{R}^N} W(x)f(u_n)(u - u_n)\,\mathrm{d}x \to 0$. Thus, by (5.8) one has

$$
\int_{\mathbb{R}^N} [\Phi(|\nabla u|) + \Phi(|u|)]\,\mathrm{d}x \geq \int_{\mathbb{R}^N} [\Phi(|\nabla u_n|) + \Phi(|u_n|)]\,\mathrm{d}x + o_n(1),
$$

and consequently,

$$
\limsup_{n\to\infty} \int_{\mathbb{R}^N} [\Phi(|\nabla u_n|) + \Phi(|u_n|)]\,\mathrm{d}x \leq \int_{\mathbb{R}^N} [\Phi(|\nabla u|) + \Phi(|u|)]\,\mathrm{d}x. \tag{5.9}
$$

Since $I_1(u) := \int_{\mathbb{R}^N} \Phi(|\nabla u|)\,\mathrm{d}x$ is a sequentially weakly lower semicontinuous functional on $W^{1,\Phi}(\mathbb{R}^N)$, the weak convergence $u_n \rightharpoonup u$ in $W^{1,\Phi}(\mathbb{R}^N)$ implies that

$$
\int_{\mathbb{R}^N} \Phi(|\nabla u|)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n|)\,\mathrm{d}x. \tag{5.10}
$$

The same reason shows that

$$
\int_{\mathbb{R}^N} \Phi(|u|)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \Phi(|u_n|)\,\mathrm{d}x. \tag{5.11}
$$

By virtue of (5.9), we must have the equality in (5.10) and (5.11). Hence, up to subsequences,

$$
\int_{\mathbb{R}^N} \Phi(|\nabla u_n|)\,\mathrm{d}x \to \int_{\mathbb{R}^N} \Phi(|\nabla u|)\,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^N} \Phi(|u_n|)\,\mathrm{d}x \to \int_{\mathbb{R}^N} \Phi(|u|)\,\mathrm{d}x.
$$

Now, arguing as in [18, Lemma 6.2] we can see that, up to a subsequence, $\nabla u_n \to \nabla u$ almost everywhere in $\mathbb{R}^N$. Using a version of the Brezis–Lieb lemma (see [14, Theorem 2]), we conclude

$$
\int_{\mathbb{R}^N} \Phi(|\nabla u_n - \nabla u|)\,\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \Phi(|u_n - u|)\,\mathrm{d}x \to 0,
$$

and according to Remark 2.8, it follows that $\|u_n - u\| = |\nabla u_n - \nabla u|_\Phi + |u_n - u|_\Phi \to 0$ and the proof is finalized. □

6 Proof of Theorem 1.1

In order to apply Theorem 4.1 to find a nonzero critical point for J, we need to estimate the minimax level c of J, where

$$c^* := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} J(\gamma(t)) \quad \text{and} \quad \Gamma := \left\{\gamma \in C([0,1];W^{1,\Phi}(\mathbb{R}^N));\gamma(0)=0 \text{ and } J(\gamma(1))<0\right\}.$$

Before to state our next result, we need to fix some notations. We recall that $B_\delta := B_\delta(0) \subset\subset \Omega^+$ for some $\delta > 0$. We are going to consider a function $\varphi_0 \in C_0^\infty(\Omega^+)$ given by $\varphi_0(x) = 1$ if $|x| \le \delta/2$, $\varphi_0(x) = 0$ if $|x| \ge \delta$, $0 \le \varphi_0(x) \le 1$ for all $x \in \Omega^+$ and $|\nabla\varphi_0(x)| \le 1$ for all $x \in \Omega^+$. Recalling that

$$\mu_1 := \frac{2\Phi(1)|B_\delta|}{a_0|B_{\delta/2}|},$$

where $a_0 := \min\{W(x) : x \in \overline{B}_\delta\}$, by (f_3) we infer that if $\mu \ge \mu_1$ then

$$\begin{aligned} J(\varphi_0) &\le \int_{B_\delta} [\Phi(|\nabla\varphi_0|) + \Phi(|\varphi_0|)]\,\mathrm{d}x - \mu_1 \int_{B_\delta} W(x)|\varphi_0|^\theta\,\mathrm{d}x \\ &< 2\Phi(1)|B_\delta| - \mu_1 a_0 |B_{\delta/2}| = 0. \end{aligned}$$

In particular,

$$\int_{B_\delta} [\Phi(|\nabla\varphi_0|) + \Phi(|\varphi_0|)]\,\mathrm{d}x < \mu_1 \int_{B_\delta} W(x)|\varphi_0|^\theta\,\mathrm{d}x. \tag{6.1}$$

Lemma 6.1 (Minimax Estimate). *If condition (f_3) holds with $\mu \ge \mu^*$, where the number μ^* was defined in* (1.9), *then*

$$c* < \left(1 - \frac{c_\alpha(1+K)}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{(r'b)^{1/\gamma}}\right).$$

Proof By definition of c^*, (f_3) and (6.1), one has

$$\begin{aligned} c^* &\le \max_{t\in[0,1]} J(t\varphi_0) \\ &\le \max_{t\in[0,1]}\left[t^{m_\alpha}\int_{B_\delta}[\Phi(|\nabla\varphi_0|) + \Phi(|\varphi_0|)]\,\mathrm{d}x - \mu t^\theta \int_{B_\delta} W(x)|\varphi_0|^\theta\,\mathrm{d}x\right] \\ &\le \max_{t\in[0,1]}\left[t^{m_\alpha}\mu_1\int_{B_\delta} W(x)|\varphi_0|^\theta\,\mathrm{d}x - \mu t^\theta \int_{B_\delta} W(x)|\varphi_0|^\theta\,\mathrm{d}x\right] \\ &\le \max_{t\ge 0}\left[\mu_1 t^{m_\alpha} - \mu t^\theta\right] a_1 \int_{B_\delta} |\varphi_0|^\theta\,\mathrm{d}x. \end{aligned}$$

A straightforward calculation shows that

$$\max_{t\ge 0}\left[\mu_1 t^{m_\alpha} - \mu t^\theta\right] = \frac{1}{\mu^{\frac{m_\alpha}{\theta-m_\alpha}}}\,\frac{\theta-m_\alpha}{m_\alpha}\left(\frac{\mu_1 m_\alpha}{\theta}\right)^{\frac{\theta}{\theta-m_\alpha}},$$

and therefore,

$$c^* < \frac{a_1|B_\delta|}{\mu^{\frac{m_\alpha}{\theta - m_\alpha}}} \frac{\theta - m_\alpha}{m_\alpha} \left(\frac{\mu_1 m_\alpha}{\theta} \right)^{\frac{\theta}{\theta - m_\alpha}}.$$

Thus, by using that $\mu \geq \mu^*$, we reach the estimate

$$c^* < \left(1 - \frac{c_\alpha(1+K)}{\sigma}\right) \xi_0 \left(\frac{K_{N,\alpha}^{1/\gamma}}{(r'b)^{1/\gamma}} \right).$$

□

Finalizing the proof of Theorem 1.1: According to Lemmas 5.4 and 6.1, J satisfies $(PS)_c$ condition. Moreover, since J has the mountain pass geometry, it follows by invoking mountain pass theorem that there exists a nonzero critical $u \in W^{1,\Phi}(\mathbb{R}^N)$ for J such that $J(u) = c$ and the proof is finalized.

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