Annali di Matematica Pura ed Applicata (1923 -) (2020) 199:1023–1072
https://doi.org/10.1007/s10231-019-00911-3

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# Asymptotic expansions in a general system of decaying functions for solutions of the Navier–Stokes equations

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Received: 16 January 2019 / Accepted: 13 September 2019 / Published online: 25 September 2019


**Abstract**

We study the long-time dynamics of the Navier–Stokes equations in the three-dimensional periodic domains with a body force decaying in time. We introduce appropriate systems of decaying functions and corresponding asymptotic expansions in those systems. We prove that if the force has a large-time asymptotic expansion in Gevrey–Sobolev spaces in such a general system, then any Leray–Hopf weak solution admits an asymptotic expansion of the same type. This expansion is uniquely determined by the force, and independent of the solutions. Various applications of the abstract results are provided which particularly include the previously obtained expansions for the solutions in case of power decay, as well as the new expansions in case of the logarithmic and iterated logarithmic decay.

**Keywords** Navier–Stokes equations · Long-time dynamics · Asymptotic expansions · Abstract expansions · Expansion theory

**Mathematics Subject Classification** Primary 35C20 (Asymptotic expansions); Secondary 35B40 (Asymptotic behavior of solutions) · 35Q30 (Partial differential equations: Navier-Stokes equations) · 37L05 (Infinite-dimensional dissipative dynamical systems: general theory) · 76D05 (Fluid mechanics: Navier-Stokes equations)

## Contents



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## 1 Introduction

We study the long-time behavior of viscous, incompressible fluid flows in space $\mathbb{R}^3$. First, we recall the Navier–Stokes equations (NSE) that describe the fluid dynamics.

Let $\mathbf{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$ denote the space and time variables, respectively. Let the (kinematic) viscosity be denoted by $\nu > 0$, the velocity vector field by $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, the pressure by $p(\mathbf{x}, t) \in \mathbb{R}$, and the body force by $\mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^3$. The NSE are

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p + \mathbf{f} \quad \text{on } \mathbb{R}^3 \times (0, \infty), \\ &\text{div } \mathbf{u} = 0 \quad \text{on } \mathbb{R}^3 \times (0, \infty). \end{aligned} \tag{1.1}$$

The initial condition is

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \tag{1.2}$$

where $\mathbf{u}^0(\mathbf{x})$ is a given divergence-free vector field.

We avoid the unbounded domains and the boundary conditions by considering only force $\mathbf{f}(\mathbf{x}, t)$ and solutions $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ that are $L$-periodic for some $L > 0$. Hereafter, a function $g(\mathbf{x})$ is said to be $L$-periodic if

$$g(\mathbf{x} + L\mathbf{e}_j) = g(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^3, \ j = 1, 2, 3,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$ and is said to have zero average over the domain $\Omega = (-L/2, L/2)^3$ if

$$\int_\Omega g(\mathbf{x})d\mathbf{x} = 0.$$

By using a particular Galilean transformation, see details in, e.g., [19], we can also assume, for all $t \geq 0$, that $\mathbf{f}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$, have zero averages over the domain $\Omega$. In light of the Leray–Helmholtz decomposition, and for the sake of convenience, we also assume that $\mathbf{f}(\mathbf{x}, t)$ is divergence-free for all $t \geq 0$.

By rescaling the variables $\mathbf{x}$ and $t$, we assume throughout, without loss of generality, that $L = 2\pi$ and $\nu = 1$.

Throughout the paper, we use the following notation

$$u(t) = \mathbf{u}(\cdot, t), \ f(t) = \mathbf{f}(\cdot, t), \ u^0 = \mathbf{u}^0(\cdot),$$

which are function-valued.

In the case of potential force, that is, $\mathbf{f}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t)$, for some scalar function $\phi$, Foias and Saut proved in [15] that any non-trivial, regular solution $u(t)$ in bounded or periodic domains admits an asymptotic expansion (as $t \to \infty$)

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-nt} \tag{1.3}$$

in Sobolev spaces $H^m(\Omega)^3$, for all $m \geq 0$. The interested reader is referred to [3,13] for early results on the solutions' asymptotic behavior, Foias and Saut [12–16] for associated normalization map and invariant nonlinear manifolds, Foias et al. [6–8] for the corresponding Poincaré–Dulac normal form, Foias et al. [4,5] for their applications to analysis of helicity, statistical solutions, and decaying turbulence. The recent paper [9] is a survey on the subject.

In case of periodic domains, it was then improved in [18] that the expansion (1.3) holds in Gevrey–Sobolev spaces $G_{\alpha,\sigma}$ for any $\alpha, \sigma > 0$, see definition (2.1) in Sect. 2, which have much stronger norms than those in $H^m(\Omega)^3$. When the force $f$ is not potential, the asymptotic expansion of Leray–Hopf weak solutions is established in [19] for an exponentially decaying force: if the force has an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} p_n(t)e^{-nt}, \tag{1.4}$$

then $u(t)$ has an asymptotic expansion of type (1.3).

The case of power-decaying forces is treated in [1]: if

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-\gamma_n}, \tag{1.5}$$

then all Leray–Hopf weak solutions $u(t)$ admit the same expansion

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-\mu_n}. \tag{1.6}$$

Above $\phi_n$'s and $\xi_n$'s belong to some Gevrey–Sobolev space $G_{\alpha,\sigma}$. The meanings of the expansions (1.3), (1.4), (1.5), (1.6) are specified precisely in the cited papers.

The current paper aims to develop the results in [1] to cover a very large class of forces. For example, we will prove that if

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n (\ln t)^{-\gamma_n}, \text{ or }, f(t) \sim \sum_{n=1}^{\infty} \phi_n (\ln(\ln t))^{-\gamma_n},$$

then

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n (\ln t)^{-\mu_n}, \text{ or, respectively, } u(t) \sim \sum_{n=1}^{\infty} \xi_n (\ln(\ln t))^{-\mu_n}.$$

In fact, we obtain a much more general result which is described very roughly here. Let $(\psi_n)_{n=1}^{\infty}$ be a sequence of time-decaying functions with $\psi_{n+1}(t)$ decays to zero, as $t \to \infty$, much faster than $\psi_n(t)$. The functions $\psi_n$'s are assumed to satisfy a certain set of conditions.

Suppose there exist $\alpha \geq 1/2$ and $\sigma \geq 0$ such that

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n \psi_n(t) \quad \text{in } G_{\alpha,\sigma}. \tag{1.7}$$

We will prove that any Leray–Hopf weak solution $u(t)$ will admit an expansion

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n \psi_n(t) \quad \text{in } G_{\alpha,\sigma}, \tag{1.8}$$

where $\xi_n$'s are explicitly determined by $\phi_n$'s. The meaning of the expansions (1.7) and (1.8) is more sophisticated than (1.3)–(1.6), thanks to their generality, and will be made clear later in the paper.

The paper is organized as follows. Section 2 reviews the functional setting for the NSE which is suitable for studying the solutions' dynamics in time. It also recalls basic inequalities for the Stokes operator and the bilinear form in the NSE. In Sect. 3, we establish the long-time estimates for solutions of both linearized NSE (Sect. 3.1) and the NSE (Sect. 3.2) with very general decaying forces. The results for the linearized NSE play a key role in improving the long-time estimates for the solutions of the NSE. Having their own merits, these estimates are also crucial to the proofs in Sects. 5 and 6. Section 4 introduces the definitions of systems of decaying functions in time and the asymptotic expansions in those systems. In Definition 4.3, we aim to balance between the generality, such as in Definition 4.1, and the technical requirements. Condition 4.4 is particularly emphasized on applications to ordinary and partial differential equations with quadratic or integral power nonlinearity. Condition 4.5 is focused on functions which are larger than the exponential ones. We state and prove elementary properties for these systems and expansions. In Sect. 5, we obtain in Theorem 5.4 the expansions in Gevrey–Sobolev spaces for all Leray–Hopf solutions of the NSE, when a continuum system of decaying functions is available as the expansions' basis. The result gives precise meanings to the above expansions (1.7) and (1.8). A version of finite sum asymptotic approximations is proved in Theorem 5.6. It is suitable for a force that has limited information about their long-time behavior. In Sect. 6, we study the situation when the discrete system of functions for expansions cannot be embedded directly into a continuum system as in Sect. 5. However, by using a continuum background system, we can still obtain in Theorem 6.3 the asymptotic expansions for solutions of the NSE. An asymptotic approximation result for the discrete system is similarly obtained in Theorem 6.4. Section 7 provides many applications of the abstract results in Sects. 5 and 6. They consist of the recovery of the power decay case previously established in [1], see Sect. 7.1, as well as the new logarithmic and iterated logarithmic decay cases, see Theorem 7.3 and Corollary 7.4. Examples 7.6 and 7.7 demonstrate some asymptotic expansions with trigonometric functions. More complicated expansions are presented in Propositions 7.8 and 7.11, particularly, the latter one is achieved by simply using of the background systems developed in Sect. 6. "Appendix A" contains some criteria for a convergent series of functions to have corresponding asymptotic expansions of the types specified in Sects. 4 and 6.

## 2 Functional setting and basic facts for the NSE

We recall the standard functional setting for the NSE, see e.g. [2,10,21,22], and some basic inequalities and estimates.

Let $L^2(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$, for integers $m \geq 0$, denote the standard Lebesgue and Sobolev spaces on $\Omega$. The standard inner product and norm in $L^2(\Omega)^3$ are denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. (We warn that this notation $|\cdot|$ also denotes the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{C}^n$, for any $n \in \mathbb{N}$, but its meaning will be clear based on the context.)

Let $\mathcal{V}$ be the set of all $2\pi$-periodic trigonometric polynomial vector fields which are divergence-free and have zero average over $\Omega$. Define

$$H, \text{ resp. } V = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^3, \text{ resp. } H^1(\Omega)^3.$$

Notice that each element of $H$ is divergence-free and has zero average over $\Omega$, and each element of $V$ is $2\pi$-periodic.

We use the following embeddings and identification

$$V \subset H = H' \subset V',$$

where each space is dense in the next one, and the embeddings are compact.

Let $\mathcal{P}$ denote the orthogonal (Leray) projection in $L^2(\Omega)^3$ onto $H$.

The Stokes operator $A$ is a bounded linear mapping from $V$ to its dual space $V'$ defined by

$$\langle A\mathbf{u}, \mathbf{v}\rangle_{V',V} = \langle\langle \mathbf{u}, \mathbf{v}\rangle\rangle \stackrel{\text{def}}{=} \sum_{j=1}^{3} \langle \frac{\partial \mathbf{u}}{\partial x_j}, \frac{\partial \mathbf{v}}{\partial x_j}\rangle \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

As an unbounded operator on $H$, the operator $A$ has the domain $\mathcal{D}(A) = V \cap H^2(\Omega)^3$, and, under the current consideration of periodicity conditions,

$$A\mathbf{u} = -\mathcal{P}\Delta\mathbf{u} = -\Delta\mathbf{u} \in H \quad \text{for all} \quad \mathbf{u} \in \mathcal{D}(A).$$

The spectrum of $A$ is known to be

$$\mathfrak{S}(A) = \{|\mathbf{k}|^2 : \mathbf{k} \in \mathbb{Z}^3, \mathbf{k} \neq \mathbf{0}\},$$

and each $\lambda \in \mathfrak{S}(A)$ is an eigenvalue. Note that $\mathfrak{S}(A) \subset \mathbb{N}$ and $1 \in \mathfrak{S}(A)$, hence, the additive semigroup generated by $\mathfrak{S}(A)$ is $\mathbb{N}$.

For $n \in \mathfrak{S}(A)$, we denote by $R_n$ the orthogonal projection in $H$ on the eigenspace of $A$ corresponding to $n$, and set

$$P_n = \sum_{j \in \mathfrak{S}(A), j \le n} R_j.$$

Note that each vector space $P_n H$ is finite dimensional.

For $\alpha, \sigma \in \mathbb{R}$ and $\mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} \widehat{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$, define

$$A^\alpha \mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|^{2\alpha} \widehat{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad e^{\sigma A^{1/2}}\mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} e^{\sigma|\mathbf{k}|} \widehat{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}},$$

and, hence,

$$A^\alpha e^{\sigma A^{1/2}}\mathbf{u} = e^{\sigma A^{1/2}} A^\alpha \mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|^{2\alpha} e^{\sigma|\mathbf{k}|} \widehat{\mathbf{u}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The Gevrey–Sobolev spaces are defined by

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) \stackrel{\text{def}}{=} \{\mathbf{u} \in H : |\mathbf{u}|_{\alpha,\sigma} \stackrel{\text{def}}{=} |A^\alpha e^{\sigma A^{1/2}}\mathbf{u}| < \infty\}, \tag{2.1}$$

and, in particular, when $\sigma = 0$, the domain of the fractional operator $A^\alpha$ is

$$\mathcal{D}(A^\alpha) = G_{\alpha,0} = \{\mathbf{u} \in H : |A^\alpha \mathbf{u}| = |\mathbf{u}|_{\alpha,0} < \infty\}.$$

Thanks to the zero-average condition, the norm $|A^{m/2}\mathbf{u}|$ is equivalent to $\|\mathbf{u}\|_{H^m(\Omega)^3}$ on the space $\mathcal{D}(A^{m/2})$ for $m = 0, 1, 2, \ldots$

Note that $\mathcal{D}(A^0) = H$, $\mathcal{D}(A^{1/2}) = V$, and $\|\mathbf{u}\| \stackrel{\text{def}}{=} |\nabla \mathbf{u}|$ is equal to $|A^{1/2}\mathbf{u}|$ for $\mathbf{u} \in V$. Also, the norms $|\cdot|_{\alpha,\sigma}$ are increasing in $\alpha, \sigma$, hence, the spaces $G_{\alpha,\sigma}$ are decreasing in $\alpha, \sigma$.

Regarding the nonlinear term in the NSE, a bounded linear map $B : V \times V \to V'$ is defined by

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{V',V} = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \stackrel{\text{def}}{=} \int_\Omega ((\mathbf{u} \cdot \nabla)\mathbf{v}) \cdot \mathbf{w}\, d\mathbf{x}, \quad \text{for all} \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in V.$$

In particular,

$$B(\mathbf{u}, \mathbf{v}) = \mathcal{P}((\mathbf{u} \cdot \nabla)\mathbf{v}), \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{D}(A).$$

The problems (1.1) and (1.2) can now be rewritten in the functional form as

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = f(t) \quad \text{in } V' \text{ on } (0, \infty), \tag{2.2}$$

$$u(0) = u^0 \in H. \tag{2.3}$$

(We refer the reader to the books [2,20–22] for more details.)

The next definition makes precise the meaning of weak solutions of (2.2).

**Definition 2.1** Let $f \in L^2_{\text{loc}}([0, \infty), H)$. A *Leray–Hopf weak solution* $u(t)$ of (2.2) is a mapping from $[0, \infty)$ to $H$ such that

$$u \in C([0, \infty), H_{\text{w}}) \cap L^2_{\text{loc}}([0, \infty), V), \quad u' \in L^{4/3}_{\text{loc}}([0, \infty), V'),$$

and satisfies

$$\frac{d}{dt}\langle u(t), v\rangle + \langle\langle u(t), v\rangle\rangle + b(u(t), u(t), v) = \langle f(t), v\rangle \tag{2.4}$$

in the distribution sense in $(0, \infty)$, for all $v \in V$, and the energy inequality

$$\frac{1}{2}|u(t)|^2 + \int_{t_0}^t \|u(\tau)\|^2 d\tau \le \frac{1}{2}|u(t_0)|^2 + \int_{t_0}^t \langle f(\tau), u(\tau)\rangle d\tau \tag{2.5}$$

holds for $t_0 = 0$ and almost all $t_0 \in (0, \infty)$, and all $t \ge t_0$. Here, $H_{\text{w}}$ denotes the topological vector space $H$ with the weak topology.

A *regular* solution is a Leray–Hopf weak solution that belongs to $C([0, \infty), V)$.

If $t \mapsto u(T + t)$ is a Leray–Hopf weak/regular solution, then we say $u$ is a Leray–Hopf weak/regular solution on $[T, \infty)$.

It is well known that a regular solution is unique among all Leray–Hopf weak solutions.

We denote by $\mathcal{T}$ the set of $t_0 \in [0, \infty)$ such that (2.5) holds for all $t \ge t_0$. Note that $[0, \infty)\setminus\mathcal{T}$ has zero measure.

If $u(t)$ is a Leray–Hopf weak solution and $t_0 \in \mathcal{T}$, then $u(t_0+t)$ is also a Leray–Hopf weak solution. Assume additionally that there exists a regular solution $w(t)$ with $w(0) = u(t_0)$. Then by the uniqueness of $w(t)$, we have $u(t_0 + t) = w(t)$ and, hence, $u(t)$ is a regular solution on $[t_0, \infty)$.

We assume throughout the paper that

**Assumption 2.2** The function $f$ belongs to $L^\infty_{\text{loc}}([0, \infty), H)$.

Under Assumption 2.2, for any $u^0 \in H$, there exists a Leray–Hopf weak solution $u(t)$ of (2.2) and (2.3), see e.g. [10]. The large-time behavior of $u(t)$ is the focus of our study. More specific conditions on $f$ will be imposed later.

We note that, thanks to Remark 1(e) of [11], the Leray–Hopf weak solutions in Definition 2.1 are the same as the weak solutions defined in [10, Chapter II, section 7]. Hence, according to inequality (A.39) in [10, Chap. II], we have for any such solution $u(t)$ that

$$|u(t)|^2 \le e^{-t}|u(0)|^2 + \int_0^t e^{-(t-\tau)}|f(\tau)|^2 \mathrm{d}\tau \quad \forall t > 0. \tag{2.6}$$

Below are some inequalities that will be needed in later estimates. First, for any $\sigma, \alpha > 0$, one has

$$\max_{x\ge 0}(x^\alpha e^{-\sigma x}) = d_0(\alpha,\sigma) \stackrel{\text{def}}{=} \left(\frac{\alpha}{e\sigma}\right)^\alpha. \tag{2.7}$$

Thanks to (2.7), one can verify, for all $\alpha, \sigma > 0$, that

$$\begin{aligned} |A^\alpha e^{-\sigma A} v| &\le d_0(\alpha,\sigma)|v| \quad \forall v \in H, \\ |A^\alpha e^{-\sigma A^{1/2}} v| &\le d_0(2\alpha,\sigma)|v| \quad \forall v \in H, \end{aligned} \tag{2.8}$$

and, consequently,

$$|A^\alpha v| = |(A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} v| \le d_0(2\alpha,\sigma)|e^{\sigma A^{1/2}} v| \quad \forall v \in G_{0,\sigma}. \tag{2.9}$$

For the bilinear mapping $B(u, v)$, it follows from its boundedness that there exists a constant $K_* > 0$ such that

$$\|B(u,v)\|_{V'} \le K_* \|u\|\,\|v\| \quad \forall u, v \in V. \tag{2.10}$$

The estimate of the Gevrey norms $|B(u, v)|_{0,\sigma}$ was initiated by Foias–Temam [17]. Here we recall an extended and convenient (though not sharp) version from [18, Lemma 2.1].

*There exists a constant $K > 1$ such that if $\sigma \ge 0$ and $\alpha \ge 1/2$, then*

$$|B(u,v)|_{\alpha,\sigma} \le K^\alpha |u|_{\alpha+1/2,\sigma}\, |v|_{\alpha+1/2,\sigma} \quad \forall u, v \in G_{\alpha+1/2,\sigma}. \tag{2.11}$$

**Notation** We make clear the meaning of the "big oh" and "small oh" notation. Let $f$ and $g$ be two non-negative functions defined on a neighborhood of infinity (in $\mathbb{R}$).

- We write $f(t) = \mathcal{O}(g(t))$ (implicitly means as $t \to \infty$) if there exist $T, C > 0$, such that $f(t) \le Cg(t)$ for all $t \ge T$.
- We say $f(t) \stackrel{\mathcal{O}}{=} g(t)$, if $f(t) = \mathcal{O}(g(t))$ and $g(t) = \mathcal{O}(f(t))$.
- We write $f(t) = o(g(t))$ (implicitly means as $t \to \infty$) if for any $\varepsilon > 0$, there exist $T_\varepsilon > 0$, such that $f(t) \le \varepsilon g(t)$ for all $t \ge T_\varepsilon$.

Let $u(t) = \mathcal{O}(f(t))$ and $v(t) = \mathcal{O}(g(t))$. Then clearly $(uv)(t) = \mathcal{O}(g(t)g(t))$, which we simply write as

$$\mathcal{O}(f(t))\mathcal{O}(g(t)) = \mathcal{O}(f(t)g(t)), \text{ and particularly, } f(t)\mathcal{O}(g(t)) = \mathcal{O}(f(t)g(t)). \tag{2.12}$$

If $f(t) = \mathcal{O}(g(t))$, then $(u + v)(t) = \mathcal{O}(g(t))$, which we write as

$$\mathcal{O}(f(t)) + \mathcal{O}(g(t)) = \mathcal{O}(g(t)). \tag{2.13}$$

Note that the identities in (2.12) and (2.13) are only for convenience and should be read from left to right.

## 3 Large-time estimates

This section contains long-time estimates for solutions of the linearized NSE and of the NSE with the force decaying in time.

First, we have a convenient integral estimate which will be used repeatedly later.

**Lemma 3.1** *Let $F$ be a continuous, decreasing function from $[0,\infty)$ to $[0,\infty)$. For any $\sigma>0$ and $\theta\in(0,1)$, one has*

$$\int_0^t e^{-\sigma(t-\tau)}F(\tau)\mathrm{d}\tau \le \frac{1}{\sigma}\Big(F(0)e^{-(1-\theta)\sigma t}+F(\theta t)\Big)\quad \forall t\ge 0. \tag{3.1}$$

***Proof*** We follow the proof of [1, Lemma 2.2]. We split

$$\int_0^t e^{-\sigma(t-\tau)}F(\tau)\mathrm{d}\tau = I_1+I_2,$$

where $I_1$ is the integral from 0 to $\theta t$, and $I_2$ is the integral from $\theta t$ to $t$. Using the monotonicity of $F$, we estimate

$$I_1\le F(0)\int_0^{\theta t} e^{-\sigma(t-\tau)}\mathrm{d}\tau \le F(0)\frac{e^{-(1-\theta)\sigma t}}{\sigma},$$
$$I_2\le F(\theta t)\int_{\theta t}^{t} e^{-\sigma(t-\tau)}\mathrm{d}\tau \le F(\theta t)\frac{1}{\sigma}.$$

Thus, we obtain (3.1). □

### 3.1 The linearized NSE

We establish an explicit estimate for solutions of the linearized NSE in terms of the decaying force.

**Theorem 3.2** *Given $\alpha,\sigma\ge 0$, let $\xi\in G_{\alpha,\sigma}$, and $f$ be a function from $(0,\infty)$ to $G_{\alpha,\sigma}$ that satisfies*

$$|f(t)|_{\alpha,\sigma}\le MF(t)\quad \text{a.e. in } (0,\infty), \tag{3.2}$$

*where $M$ is a positive constant, and $F$ is a continuous, decreasing function from $[0,\infty)$ to $[0,\infty)$.*

*Let $w_0\in G_{\alpha,\sigma}$. Suppose $w\in C([0,\infty),H_{\mathrm{w}})\cap L^1_{\mathrm{loc}}([0,\infty),V)$, with $w'\in L^1_{\mathrm{loc}}([0,\infty),V')$, is a weak solution of*

$$w'=-Aw+\xi+f \text{ in } V' \text{ on } (0,\infty),\quad w(0)=w_0,$$

*i.e., it holds, for all $v\in V$, that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle w,v\rangle = -\langle\langle w,v\rangle\rangle+\langle \xi+f,v\rangle \text{ in the distribution sense on } (0,\infty).$$

*Then the following statements hold true.*

(i) *$w(t)\in G_{\alpha+1-\varepsilon,\sigma}$ for all $\varepsilon\in(0,1)$ and $t>0$.*
(ii) *For any numbers $a,a_0\in(0,1)$ with $a+a_0<1$ and any $\varepsilon\in(0,1)$, there exists a positive constant $C$ depending on $a_0$, $a$, $\varepsilon$, $M$, $F(0)$, $|\xi|_{\alpha,\sigma}$ and $|w_0|_{\alpha,\sigma}$ such that*

$$|w(t)-A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma}\le C\big(e^{-a_0t}+F(at)\big)\quad \forall t\ge 1. \tag{3.3}$$

(iii) *Assume, in addition, that*

- *There exist* $k_0 > 0$ *and* $D_1 > 0$ *such that*
$$e^{-k_0 t} \le D_1 F(t) \quad \forall t \ge 0, \text{ and} \tag{3.4}$$
- *For any* $a \in (0,1)$*, there exists* $D_2 = D_{2,a} > 0$ *such that*
$$F(at) \le D_2 F(t) \quad \forall t \ge 0. \tag{3.5}$$

*Then there exists* $C > 0$ *such that*
$$|w(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \le CF(t) \quad \forall t \ge 1. \tag{3.6}$$

***Proof*** (i) This regularity result is the same as [1, Lemma 2.4(i)] in which we set $f := \xi + f$.
(ii) First, we state and prove a more technical version of (3.3).
*For any* $\varepsilon, \delta, \theta, \theta' \in (0,1)$*, there exists* $C > 0$ *depending on* $\varepsilon, \delta, \theta, M, F(0), |\xi|_{\alpha,\sigma}$ *and* $|w_0|_{\alpha,\sigma}$ *such that*
$$|w(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \le C\big(e^{-(1-\theta')\theta\delta t} + F(\theta'\theta t)\big) \quad \forall t \ge 1. \tag{3.7}$$

*Proof of* (3.7) We follow the proof of [1, Lemma 2.3].

(a) Let $N \in \mathfrak{S}(A)$. We recall formula (2.19) of [1]:
$$P_N\big(w(t) - A^{-1}\xi\big) = e^{-tA}P_N w_0 - A^{-1}e^{-tA}P_N\xi + \int_0^t e^{-(t-\tau)A}P_N f(\tau)\mathrm{d}\tau \quad \forall t \ge 0. \tag{3.8}$$

(This formula was derived by using the equation of $P_N w$ and the variation of constant formula.)

Let $\varepsilon \in (0,1)$. Applying $A^{1-\varepsilon}$ to both sides of (3.8), and estimating the $|\cdot|_{\alpha,\sigma}$ norm of the resulting quantities, we obtain
$$\begin{aligned} |P_N(w(t) - A^{-1}\xi)|_{\alpha+1-\varepsilon,\sigma} &\le |A^{1-\varepsilon}e^{-tA}w_0|_{\alpha,\sigma} + |A^{-\varepsilon}e^{-tA}\xi|_{\alpha,\sigma} \\ &\quad + \int_0^t |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\mathrm{d}\tau. \end{aligned} \tag{3.9}$$

Let $\theta, \delta \in (0,1)$ and $t \ge 1$. We estimate each term on the right-hand side of (3.9) separately.

(b) Rewriting the first term on the right-hand side of (3.9) and applying (2.8) yield
$$\begin{aligned} |A^{1-\varepsilon}e^{-tA}w_0|_{\alpha,\sigma} &= |A^{1-\varepsilon}e^{-(1-\delta)tA}(e^{-\delta tA}w_0)|_{\alpha,\sigma} \le \left[\frac{1-\varepsilon}{e(1-\delta)t}\right]^{1-\varepsilon}|e^{-\delta tA}w_0|_{\alpha,\sigma} \\ &\le \left[\frac{1-\varepsilon}{e(1-\delta)}\right]^{1-\varepsilon}e^{-\delta t}|w_0|_{\alpha,\sigma}. \end{aligned}$$

The second term on the right-hand side of (3.9) can be easily estimated by
$$|A^{-\varepsilon}e^{-tA}\xi|_{\alpha,\sigma} \le |e^{-tA}\xi|_{\alpha,\sigma} \le e^{-t}|\xi|_{\alpha,\sigma}.$$

(c) Dealing with the last integral in (3.9), we split it into two integrals
$$\int_0^t |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\mathrm{d}\tau = I_1 + I_2,$$

where $I_1$ is the integral from 0 to $\theta t$, and $I_2$ from $\theta t$ to $t$.

For $I_1$, we have

$$I_1 = \int_0^{\theta t} \Big|e^{-(t-\tau)(1-\delta)A}\Big(e^{-(t-\tau)\delta A}A^{1-\varepsilon}f(\tau)\Big)\Big|_{\alpha,\sigma}\mathrm{d}\tau$$
$$\le \int_0^{\theta t} \Big|e^{-(1-\theta)t(1-\delta)A}A^{1-\varepsilon}\Big(e^{-(t-\tau)\delta A}f(\tau)\Big)\Big|_{\alpha,\sigma}\mathrm{d}\tau.$$

Utilizing (2.8) and then using hypothesis (3.2), we obtain

$$I_1 \le \int_0^{\theta t}\Big[\frac{1-\varepsilon}{e(1-\theta)(1-\delta)t}\Big]^{1-\varepsilon}|e^{-(t-\tau)\delta A}f(\tau)|_{\alpha,\sigma}\mathrm{d}\tau$$
$$\le \Big[\frac{1-\varepsilon}{e(1-\theta)(1-\delta)t}\Big]^{1-\varepsilon}\int_0^{\theta t} e^{-(t-\tau)\delta}MF(\tau)\mathrm{d}\tau.$$
$$\le M\Big[\frac{1-\varepsilon}{e(1-\theta)(1-\delta)}\Big]^{1-\varepsilon}\int_0^{\theta t} e^{-\delta(\theta t-\tau)}F(\tau)\mathrm{d}\tau.$$

Let $\theta' \in (0, 1)$. Then by Lemma 3.1,

$$I_1 \le M\Big[\frac{1-\varepsilon}{e(1-\theta)(1-\delta)}\Big]^{1-\varepsilon}\frac{1}{\delta}\Big(F(0)e^{-(1-\theta')\delta\theta t} + F(\theta'\theta t)\Big).$$

For $I_2$, we apply (2.8) and use (3.2) to have

$$I_2 = \int_{\theta t}^{t} |e^{-(t-\tau)\delta A}(e^{-(t-\tau)(1-\delta)A}A^{1-\varepsilon}f(\tau))|_{\alpha,\sigma}\mathrm{d}\tau$$
$$\le \int_{\theta t}^{t} e^{-(t-\tau)\delta}|e^{-(t-\tau)(1-\delta)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\mathrm{d}\tau$$
$$\le \int_{\theta t}^{t} e^{-\delta(t-\tau)}\Big[\frac{1-\varepsilon}{e(1-\delta)(t-\tau)}\Big]^{1-\varepsilon}|f(\tau)|_{\alpha,\sigma}\mathrm{d}\tau$$
$$\le \Big[\frac{1-\varepsilon}{e(1-\delta)}\Big]^{1-\varepsilon}MF(\theta t)\int_{\theta t}^{t}\frac{e^{-\delta(t-\tau)}}{(t-\tau)^{1-\varepsilon}}\mathrm{d}\tau.$$

We estimate the last integral by

$$\int_{\theta t}^{t}\frac{e^{-\delta(t-\tau)}}{(t-\tau)^{1-\varepsilon}}\mathrm{d}\tau = \int_0^{(1-\theta)t} z^{\varepsilon-1}e^{-\delta z}\mathrm{d}z \le \int_0^{1-\theta} z^{\varepsilon-1}\mathrm{d}z + (1-\theta)^{\varepsilon-1}\int_{1-\theta}^{(1-\theta)t} e^{-\delta z}\mathrm{d}z$$
$$\le \frac{(1-\theta)^{\varepsilon}}{\varepsilon} + \frac{(1-\theta)^{\varepsilon-1}}{\delta}e^{-\delta(1-\theta)}.$$

(d) Combining the above calculations, we obtain

$$|P_N\big(w(t) - A^{-1}\xi\big)|_{\alpha+1-\varepsilon,\sigma} \le C\big(e^{-(1-\theta')\theta\delta t} + F(\theta'\theta t)\big) \quad \forall t \ge 1, \tag{3.10}$$

with constant $C$ independent of $N$. By passing $N \to \infty$ in (3.10), and the fact $A^{-1}\xi \in G_{\alpha+1-\varepsilon,\sigma}$, we obtain $w(t) \in G_{\alpha+1-\varepsilon,\sigma}$ together with the estimate (3.7).

*Proof of* (3.3) Take $\theta \in (a + a_0, 1)$, and set $\delta = a_0/(\theta - a)$ and $\theta' = a/\theta$. Then $\theta - a > a_0 > 0$, which gives $\delta, \theta' \in (0, 1)$. It is also clear that $\theta\theta' = a$ and $(1 - \theta')\theta\delta = (\theta - a)\delta = a_0$. Therefore, with these values of $\theta, \theta', \delta$, inequality (3.3) follows (3.7).

(iii) Note, by the monotonicity of $F$, that the property (3.5) in fact holds true for all $a > 0$, with $D_{2,a} = 1$ for all $a \geq 1$. Then we have

$$\begin{aligned} e^{-a_0 t} + F(at) &= e^{-k_0 \cdot a_0 t/k_0} + F(at) \\ &\leq D_1 F(a_0 t/k_0) + F(at) \leq (D_1 D_{2,a_0/k_0} + D_{2,a}) F(t). \end{aligned} \tag{3.11}$$

Combining (3.11) with (3.3), we obtain inequality (3.6). The proof is complete. □

### 3.2 The NSE

This subsection aims at establishing the large-time estimates for weak solutions of the NSE. First, we obtain a result for small initial data and force.

**Theorem 3.3** *Let $F$ be a continuous, decreasing, non-negative function on $[0, \infty)$. Given $\alpha \geq 1/2$ and numbers $\theta_0, \theta \in (0, 1)$ such that $\theta_0 + \theta < 1$. Then there exist positive numbers $c_k = c_k(\alpha, \theta_0, \theta, F)$, for $k = 0, 1, 2, 3$, such that the following holds true. If*

$$|A^\alpha u^0| \leq c_0, \tag{3.12}$$

$$|f(t)|_{\alpha-1/2,\sigma} \leq c_1 F(t) \quad \text{a.e. in } (0, \infty) \text{ for some } \sigma \geq 0, \tag{3.13}$$

*then there exists a unique regular solution $u(t)$ of (2.2) and (2.3), which also belongs to $C([0, \infty), \mathcal{D}(A^\alpha))$ and satisfies, for all $t \geq 8\sigma(1-\theta)/(1-\theta-\theta_0)$,*

$$|u(t)|_{\alpha,\sigma} \leq c_2 (e^{-2\theta_0 t} + F^2(\theta t))^{1/2}, \tag{3.14}$$

$$\int_t^{t+1} |u(\tau)|^2_{\alpha+1/2,\sigma} d\tau \leq c_3^2 (e^{-2\theta_0 t} + F^2(\theta t)). \tag{3.15}$$

***Proof*** The proof follows [1, Theorem 3.1]. The calculations below are formal, but can be made rigorous by using the Galerkin approximations and then pass to the limit.

Let $\theta_* = \theta_0/(1-\theta) \in (\theta_0, 1)$ and denote $t_* = 8\sigma/(1-\theta_*) = 8\sigma(1-\theta)/(1-\theta-\theta_0)$.

(a) For $\sigma > 0$, let $\varphi$ be a $C^\infty$-function on $\mathbb{R}$ such that $\varphi((-\infty, 0]) = \{0\}$, $\varphi([t_*, \infty)) = \{\sigma\}$, and $0 < \varphi' < 2\sigma/t_*$ on $(0, t_*)$. We derive from (2.2) that

$$\begin{aligned} \frac{d}{dt}(A^\alpha e^{\varphi(t)A^{1/2}} u) &= \varphi'(t) A^{1/2} A^\alpha e^{\varphi(t)A^{1/2}} u + A^\alpha e^{\varphi(t)A^{1/2}} \frac{du}{dt} \\ &= \varphi'(t) A^{\alpha+1/2} e^{\varphi(t)A^{1/2}} u + A^\alpha e^{\varphi(t)A^{1/2}} (-Au - B(u,u) + f). \end{aligned} \tag{3.16}$$

By taking the inner product in $H$ of (3.16) with $A^\alpha e^{\varphi(t)A^{1/2}} u(t)$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u|^2_{\alpha,\varphi(t)} + |A^{1/2}u|^2_{\alpha,\varphi(t)} &= \varphi'(t)\langle A^{\alpha+1/2} e^{\varphi(t)A^{1/2}} u, A^\alpha e^{\varphi(t)A^{1/2}} u\rangle \\ &- \langle A^\alpha e^{\varphi(t)A^{1/2}} B(u,u), A^\alpha e^{\varphi(t)A^{1/2}} u\rangle + \langle A^{\alpha-1/2} e^{\varphi(t)A^{1/2}} f, A^{\alpha+1/2} e^{\varphi(t)A^{1/2}} u\rangle. \end{aligned}$$

Using the Cauchy–Schwarz inequality, and estimating the second term on the right-hand side by (2.11), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u|^2_{\alpha,\varphi(t)} + |A^{1/2}u|^2_{\alpha,\varphi(t)} &\leq \varphi'(t)|A^{1/2}u|^2_{\alpha,\varphi(t)} \\ &+ K^\alpha |A^{1/2}u|^2_{\alpha,\varphi(t)} |u|_{\alpha,\varphi(t)} + |f(t)|_{\alpha-1/2,\varphi(t)} |A^{1/2}u|_{\alpha,\varphi(t)}. \end{aligned} \tag{3.17}$$

Using the bound of $\varphi'(t)$ and applying Cauchy's inequality to the last term gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + |A^{1/2}u|^2_{\alpha,\varphi(t)} \le \frac{2\sigma}{t_*}|A^{1/2}u|^2_{\alpha,\varphi(t)}$$
$$+ K^\alpha |u|_{\alpha,\varphi(t)}|A^{1/2}u|^2_{\alpha,\varphi(t)} + \frac{2\sigma}{t_*}|A^{1/2}u|^2_{\alpha,\varphi(t)} + \frac{t_*}{8\sigma}|f(t)|^2_{\alpha-1/2,\varphi(t)},$$

which, together with the fact $\varphi(t) \le \sigma$, implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + \Big(1 - \frac{4\sigma}{t_*} - K^\alpha |u|_{\alpha,\varphi(t)}\Big)|A^{1/2}u|^2_{\alpha,\varphi(t)} \le \frac{t_*}{8\sigma}|f(t)|^2_{\alpha-1/2,\sigma}.$$

Thus,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^\alpha u|^2 + \Big(1 - \frac{1-\theta_*}{2} - K^\alpha |A^\alpha u|\Big)|A^{\alpha+1/2}u|^2 \le \frac{1}{1-\theta^*}|A^{\alpha-1/2}f|^2. \quad (3.18)$$

(b) For $\sigma = 0$, let $\varphi \equiv 0$ on $\mathbb{R}$. Then the first term in (3.17) vanishes. Applying Cauchy's inequality to the last term of (3.17):

$$|f(t)|_{\alpha-1/2,\varphi(t)}|A^{1/2}u|_{\alpha,\varphi(t)} \le \frac{1-\theta_*}{2}|A^{1/2}u|^2_{\alpha,\varphi(t)} + \frac{1}{1-\theta_*}|f(t)|^2_{\alpha-1/2,\varphi(t)},$$

we obtain the same inequality (3.18).

(c) In the calculations below, we use the following constants

$$c_* = c_*(\alpha, \theta_0, \theta, F) = \frac{1-\theta_*}{4K^\alpha}, \qquad \gamma = \gamma(F) = \frac{1}{F(0)+1} \in (0, 1],$$
$$c_0 = c_0(\alpha, \theta_0, \theta, F) = \gamma c_*, \qquad c_1 = c_1(\alpha, \theta_0, \theta, F) = \gamma^2 c_*(\theta_*(1-\theta_*))^{1/2},$$
$$c_2 = c_2(\alpha, \theta_0, \theta, F) = \sqrt{2}\gamma c_*, \qquad c_3 = c_3(\alpha, \theta_0, \theta, F) = (1+\theta_*^{-1})^{1/2}\gamma c_*.$$

At the initial time, we have

$$|u(0)|_{\alpha,\varphi(0)} = |A^\alpha u^0| < 2c_0 \le 2c_*.$$

Let $T \in (0, \infty)$. Assume that

$$|u(t)|_{\alpha,\varphi(t)} \le 2c_* \quad \forall t \in [0, T). \quad (3.19)$$

This and the definition of $c_*$ give

$$K^\alpha |u(t)|_{\alpha,\varphi(t)} \le 2c_* K^\alpha = (1-\theta_*)/2 \quad \forall t \in [0, T). \quad (3.20)$$

For $t \in (0, T)$, we have from (3.18) for both $\sigma > 0$ and $\sigma = 0$, and (3.20) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + 2\theta_*|A^{1/2}u|^2_{\alpha,\varphi(t)} \le 2(1-\theta_*)^{-1}|f(t)|^2_{\alpha-1/2,\sigma}. \quad (3.21)$$

Applying Gronwall's inequality in (3.21) and using (3.12), (3.13) yield, for all $t \in (0, T)$, that

$$|u(t)|^2_{\alpha,\varphi(t)} \le e^{-2\theta_* t}|u^0|^2_{\alpha,0} + 2(1-\theta_*)^{-1}\int_0^t e^{-2\theta_*(t-\tau)}|f(\tau)|^2_{\alpha-1/2,\sigma}\mathrm{d}\tau$$
$$\le e^{-2\theta_0 t}c_0^2 + 2(1-\theta_*)^{-1}c_1^2\int_0^t e^{-2\theta_*(t-\tau)}F^2(\tau)\mathrm{d}\tau.$$

For the last integral, applying (3.1) with $F := F^2, \sigma := 2\theta_*$ and noting that $(1-\theta)\theta_* = \theta_0$ give

$$\int_0^t e^{-2\theta_*(t-\tau)} F^2(\tau)\mathrm{d}\tau \le \frac{1}{2\theta_*}\Big(F^2(0)e^{-2\theta_0 t} + F^2(\theta t)\Big) \quad \forall t \ge 0.$$

Then we obtain

$$\begin{aligned}|u(t)|^2_{\alpha,\varphi(t)} &\le c_0^2 e^{-2\theta_0 t} + [\theta_*(1-\theta_*)]^{-1} c_1^2 \big(\gamma^{-2} e^{-2\theta_0 t} + F^2(\theta t)\big)\\ &\le 2c_*^2\gamma^2(e^{-2\theta_0 t} + F^2(\theta t)).\end{aligned}$$

This implies

$$|u(t)|_{\alpha,\varphi(t)} \le \sqrt{2} c_*\gamma(e^{-2\theta_0 t} + F^2(\theta t))^{1/2} \quad \forall t \in [0, T). \tag{3.22}$$

Letting $t \to T^-$ in (3.22) and using the monotonicity of $F$ give

$$\lim_{t\to T^-} |u(t)|_{\alpha,\varphi(t)} \le \sqrt{2} c_*\gamma(1 + F^2(0))^{1/2} < 2c_*. \tag{3.23}$$

By the standard contradiction argument, it follows (3.19) and (3.23) that the inequalities (3.19) and (3.22), in fact, hold true for $T = \infty$. Then, due to the fact $\varphi(t) = \sigma$ for all $t \ge t_*$, inequality (3.22) implies (3.14).

(d) For $t \ge t_*$, by integrating (3.21) from $t$ to $t + 1$, and using estimates (3.14), (3.13), we obtain

$$\begin{aligned}\int_t^{t+1} |A^{1/2}u(\tau)|^2_{\alpha,\sigma}\mathrm{d}\tau &\le \frac{1}{2\theta_*}|u(t)|^2_{\alpha,\sigma} + [\theta_*(1-\theta_*)]^{-1}c_1^2 \int_t^{t+1} F^2(\tau)\mathrm{d}\tau\\ &\le c_*^2\gamma^2\theta_*^{-1}(e^{-2\theta_0 t} + F^2(\theta t)) + c_*^2\gamma^2 F^2(\theta t).\end{aligned}$$

Then inequality (3.15) follows. The proof is complete. □

**Theorem 3.4** *Let $F$ be a continuous, decreasing, non-negative function on $[0,\infty)$ that satisfies*

$$\lim_{t\to\infty} F(t) = 0. \tag{3.24}$$

*Suppose there exist $\sigma \ge 0$, $\alpha \ge 1/2$ such that*

$$|f(t)|_{\alpha,\sigma} = \mathcal{O}(F(t)). \tag{3.25}$$

*Let $u(t)$ be a Leray–Hopf weak solution of (2.2). Then there exists $\hat{T} > 0$ such that $u(t)$ is a regular solution of (2.2) on $[\hat{T},\infty)$, and for any $\varepsilon, \lambda \in (0,1)$, and $a_0, a, \theta_0, \theta \in (0,1)$ with $a_0 + a < 1$, $\theta_0 + \theta < 1$, there exists $C > 0$ such that*

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon,\sigma} \le C(e^{-a_0 t} + e^{-2\theta_0 a t} + F^{2\lambda}(\theta a t) + F(at)) \quad \forall t \ge 0. \tag{3.26}$$

*If, in addition, $F$ satisfies (3.4) and (3.5), then*

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon,\sigma} \le CF(t) \quad \forall t \ge 0. \tag{3.27}$$

***Proof*** By (3.25), there exist $T_1 > 0$ and $C_1 > 0$ such that

$$|f(t)|_{\alpha,\sigma} \le C_1 F(t) \quad t \ge T_1. \tag{3.28}$$

We claim the following fact which is weaker than the desired estimate (3.27).

**Claim** *For any* $\lambda \in (0,1)$, *and* $\theta, \theta_0 \in (0,1)$ *with* $\theta+\theta_0<1$, *there exists* $\hat T \ge T_1$ *such that* $u(t)$ *is a regular solution of* (2.2) *on* $[\hat T,\infty)$, *and one has for all* $t\ge 0$ *that*

$$|u(\hat T+t)|_{\alpha+1/2,\sigma} \le C(e^{-2\theta_0 t}+F^{2\lambda}(\theta t))^{1/2}, \tag{3.29}$$

*for some positive constant* $C$.

Accepting this Claim at the moment, we prove (3.27). Rewrite the NSE (2.2) as the linearized NSE:

$$u_t + Au = \tilde f(t) \stackrel{\text{def}}{=} -B(u(t),u(t)) + f(t). \tag{3.30}$$

Then from (3.29) and (2.11) we obtain for $t$ large,

$$|B(u(\hat T+t),u(\hat T+t))|_{\alpha,\sigma} \le K^\alpha |u(\hat T+t)|^2_{\alpha+1/2,\sigma} \le C_2(e^{-2\theta_0 t}+F^{2\lambda}(\theta t))$$

for some positive constant $C_2$. From this and (3.28), we have, for $t\ge 0$,

$$|\tilde f(\hat T+t)|_{\alpha,\sigma} \le C_2(e^{-2\theta_0 t}+F^{2\lambda}(\theta t)) + C_1 F(\hat T+t) \le C_3 \tilde F(t), \tag{3.31}$$

where $C_3 = C_1+C_2$ and $\tilde F(t) = e^{-2\theta_0 t} + F^{2\lambda}(\theta t) + F(t)$.

By (3.30) and (3.31), we apply part (iii) of Theorem 3.2 with $w(t) := u(\hat T+t)$, $f(t) := \tilde f(\hat T+t)$, $\xi=0$, $M=C_3$, $F(t) := \tilde F(t)$ to obtain from (3.3), with $t := t+1$, that

$$\begin{aligned}|u(\hat T+t+1)|_{\alpha+1-\varepsilon,\sigma} &\le C_4(e^{-a_0(t+1)}+\tilde F(a(t+1)))\\ &\le C_4(e^{-a_0 t}+e^{-2\theta_0 a t}+F^{2\lambda}(\theta a t)+F(at)).\end{aligned}$$

for all $t\ge 0$ and some constant $C_4>0$. By re-denoting $\hat T := \hat T+1$, we obtain (3.26).

Now, assume (3.4) and (3.5). Taking $\lambda=1/2$, $a_0=\theta_0\in(0,1/2)$ and $a=1/2$, we obtain from (3.26)

$$|u(\hat T+t)|_{\alpha+1-\varepsilon,\sigma} \le C(e^{-\theta_0 t}+F(\theta t/2)) \quad \forall t\ge 0. \tag{3.32}$$

Similarly to proving (3.11), we obtain inequality (3.27) from (3.32).

The rest of this proof is to prove the Claim.

(a) By Assumption 2.2, there exists $C_0>0$ such that

$$|f(t)| \le C_0, \quad \text{a.e. in } (0,T_1). \tag{3.33}$$

On the one hand, using (2.6), (3.28), (3.33) we have, for all $t\ge T_1$, that

$$\begin{aligned}|u(t)|^2 &\le e^{-t}|u_0|^2 + C_0^2\int_0^{T_1} e^{-(t-\tau)}\mathrm{d}\tau + C_1^2\int_{T_1}^t e^{-(t-\tau)}F^2(\tau)\mathrm{d}\tau\\ &\le e^{-t}|u_0|^2 + C_0^2 e^{-t}e^{T_1} + C_1^2\int_0^t e^{-(t-\tau)}F^2(\tau)\mathrm{d}\tau.\end{aligned}$$

To estimate the last integral, we apply inequality (3.1) with $\sigma := 1$, $\theta := 1/2$, $F := F^2$, hence, obtain

$$|u(t)|^2 \le e^{-t}(|u_0|^2 + C_0^2 e^{T_1}) + C_1^2(F^2(0)e^{-t/2}+F^2(t/2)) \quad \forall t\ge T_1. \tag{3.34}$$

On the other hand, we estimate in (2.5)

$$|\langle f(\tau),u(\tau)\rangle| \le \frac12|u(\tau)|^2 + \frac12|f(\tau)|^2 \le \frac12\|u(\tau)\|^2 + \frac12|f(\tau)|^2.$$

Hence, we obtain

$$|u(t)|^2 + \int_{t_0}^{t} \|u(\tau)\|^2 \mathrm{d}\tau \le |u(t_0)|^2 + \int_{t_0}^{t} |f(\tau)|^2 \,\mathrm{d}\tau \tag{3.35}$$

for all $t_0 \in \mathcal{T}$ and $t \ge t_0$.

Let $t_0 \in \mathcal{T} \cap [T_1, \infty)$. Setting $t = t_0 + 1$ in (3.35), using (3.34) to estimate $|u(t_0)|^2$, and (3.28) to estimate $|f(\tau)|$, we derive

$$\int_{t_0}^{t_0+1} \|u(\tau)\|^2 \mathrm{d}\tau \le e^{-t_0}(|u_0|^2 + C_0^2 e^{T_1}) + C_1^2(F^2(0)e^{-t_0/2} + F^2(t_0/2)) + C_1^2 F^2(t_0),$$

thus,

$$\int_{t_0}^{t_0+1} \|u(\tau)\|^2 \mathrm{d}\tau \le e^{-t_0/2}(|u_0|^2 + C_0^2 e^{T_1} + C_1^2 F^2(0)) + 2C_1^2 F^2(t_0/2). \tag{3.36}$$

Let $t \ge T_1$ be arbitrary now. There exists a sequence $\{t_n\}_{n=1}^{\infty}$ in $\mathcal{T} \cap (T_1, \infty)$ such that $\lim_{n\to\infty} t_n = t$. Then (3.36) holds for $t_0 = t_n$, and letting $n \to \infty$ gives

$$\int_{t}^{t+1} \|u(\tau)\|^2 \mathrm{d}\tau \le M_t \stackrel{\text{def}}{=} e^{-t/2}(|u_0|^2 + C_0^2 e^{T_1} + C_1^2 F^2(0)) + 2C_1^2 F^2(t/2) \quad \forall t \ge T_1. \tag{3.37}$$

Note that the quantity $M_t$ in (3.37) is decreasing in $t$, and goes to zero as $t$ tends to infinity.

(b) Consider $\sigma > 0$. Let $\lambda \in (0, 1)$. For $T > 0$, we write

$$F(t + T) = F^{1-\lambda}(t + T)F^{\lambda}(t + T) \le F(T)^{1-\lambda} F^{\lambda}(t). \tag{3.38}$$

Choose $T_2 > T_1$ such that

$$M_{T_2} < c_0(1/2, F^{\lambda})/2 \text{ and } F(T_2)^{1-\lambda} \le c_1(1/2, F^{\lambda})/C_1.$$

By applying inequality (3.37) to $t = T_2$, there exists $t_0 \in \mathcal{T} \cap (T_2, T_2 + 1)$ such that

$$|A^{1/2}u(t_0)| \le 2M_{t_0} \le 2M_{T_2} < c_0(1/2, F^{\lambda}).$$

Moreover, for $t \ge 0$, by (3.28) and (3.38),

$$|f(t_0 + t)|_{0,\sigma} \le C_1 F(t_0 + t) \le C_1 F(t_0)^{1-\lambda} F^{\lambda}(t) \le C_1 F(T_2)^{1-\lambda} F^{\lambda}(t) \le c_1(1/2, F^{\lambda}) F^{\lambda}(t). \tag{3.39}$$

Applying Theorem 3.3 to the unique regular solution $u(t) := u(t_0 + t)$, force $f(t) := f(t_0 + t)$ with parameters $\alpha = 1/2$ and $F(t) := F^{\lambda}(t)$, we obtain from (3.14) that

$$|u(t_0 + t)|_{1/2,\sigma} \le c_2(1/2, F^{\lambda})(e^{-\theta_0 t} + F^{\lambda}(\theta t)) \quad \forall t \ge t_*, \tag{3.40}$$

where $t_*$ is a non-negative number. Then by (2.9), we have for all $t \ge t_*$ that

$$|A^{\alpha+1/2}u(t_0 + t)| \le d_0(2\alpha + 1, \sigma)|e^{\sigma A^{1/2}} u(t_0 + t)| \le d_0(2\alpha + 1, \sigma)|u(t_0 + t)|_{1/2,\sigma},$$

and, hence, thanks to (3.40),

$$\lim_{t\to\infty} |A^{\alpha+1/2}u(t_0 + t)| = 0. \tag{3.41}$$

Using (3.41), and similar to (3.39) with the norm $|\cdot|_{\alpha,\sigma}$ replacing $|\cdot|_{0,\sigma}$, we deduce that there is $T \in \mathcal{T} \cap (t_0 + t_*, \infty)$ so that

$$|A^{\alpha+1/2}u(T)| \le c_0(\alpha + 1/2, F^\lambda), \tag{3.42}$$

$$|f(T+t)|_{\alpha,\sigma} \le c_1(\alpha + 1/2, F^\lambda)F^\lambda(t) \quad \forall t \ge 0. \tag{3.43}$$

(c) We will establish (3.42) and (3.43) when $\sigma = 0$. First, we observe the following: if $j \in \mathbb{N}$ such that $j \le 2\alpha + 1$ and

$$\lim_{t\to\infty} \int_t^{t+1} |A^{j/2}u(\tau)|^2 \mathrm{d}\tau = 0, \tag{3.44}$$

then

$$\lim_{t\to\infty} \int_t^{t+1} |A^{(j+1)/2}u(\tau)|^2 \mathrm{d}\tau = 0. \tag{3.45}$$

Indeed, since $(j-1)/2 \le \alpha$, and thanks to (3.25), we have

$$|A^{\frac{j-1}{2}} f(t)| = \mathcal{O}(F(t)). \tag{3.46}$$

By (3.44) and (3.46), we obtain, similar to (3.42) and (3.43) that there exists $T_3 \in \mathcal{T} \cap [T_1, \infty)$ so that

$$|A^{j/2}u(T_3)| \le c_0(j/2, F^\lambda),$$
$$|A^{j/2-1/2} f(T_3 + t)| \le c_1(j/2, F^\lambda)F^\lambda(t) \quad \forall t \ge 0.$$

Applying Theorem 3.3 to $u(t) := u(T_3+t)$, $f(t) := f(T_3+\cdot)$, $F(t) := F^\lambda(t)$, $\alpha := j/2$, $\sigma := 0$, we obtain from (3.15) that

$$\int_t^{t+1} |A^{(j+1)/2}u(\tau)|^2 \mathrm{d}\tau = \int_{t-T_3}^{t+1-T_3} |A^{(j+1)/2}u(T_3+\tau)|^2 \mathrm{d}\tau$$
$$= \mathcal{O}(e^{-2\theta_0(t-T_3)} + F^{2\lambda}(\theta(t-T_3))),$$

which proves (3.45), thanks to (3.24).

Now, let $m$ be a non-negative integer such that $2\alpha \le m < 2\alpha + 1$.

Note that $m \ge 1$, and, because of (3.37), condition (3.44) holds true for $j = 1$. Hence we obtain (3.45) with $j = 1$, which is (3.44) for $j = 2$. We apply the arguments recursively for $j = 1, 2, \ldots, m$, and obtain, when $j = m$, from (3.45) that

$$\lim_{t\to\infty} \int_t^{t+1} |A^{(m+1)/2}u(\tau)|^2 \mathrm{d}\tau = 0.$$

Since $\alpha \le m/2$, it follows that

$$\lim_{t\to\infty} \int_t^{t+1} |A^{\alpha+1/2}u(\tau)|^2 \mathrm{d}\tau = 0. \tag{3.47}$$

By (3.47), (3.28), (3.38) and (3.24), there exists $T \in \mathcal{T} \cap [T_1, \infty)$ so that (3.42) and (3.43) similarly hold true (for this case of $\sigma = 0$.)

(d) With $T \in \mathcal{T} \cap [T_1, \infty)$ in (b) and (c), we apply Theorem 3.3 to the unique regular solution $u(t) := u(T+t)$, $f(t) := f(T+t)$, $F(t) := F^\lambda(t)$, $\alpha := \alpha + 1/2$, and obtain that there is $t_* \geq 0$ such that, following (3.14) with $t := t + t_*$,

$$|u(T + t_* + t)|_{\alpha+1/2,\sigma} \leq c_2(\alpha + 1/2, \theta_0, \theta, F^\lambda)\Big(e^{-2\theta_0(t_*+t)} + F^{2\lambda}(\theta(t_* + t))\Big)^{1/2}$$
$$\leq C(e^{-2\theta_0 t} + F^{2\lambda}(\theta t))^{1/2}$$

for all $t \geq 0$. By setting $\hat{T} = T + t_*$, this estimate implies (3.29). The proof is complete. □

***Remark 3.5*** Because the constants $c_0$ and $c_1$ in (3.42), (3.43) can be small, we could not prove (3.29) for $\lambda = 1$ directly. Rather, we use (3.30) and the estimate (3.6) in Theorem 3.2 for the linearized NSE to improve (3.29) to (3.27).

## 4 General asymptotic expansions

Now, we introduce a very general definition of an asymptotic expansion in a normed space with respect to a system of time-decaying functions.

**Definition 4.1** Let $(\psi_n)_{n=1}^\infty$ be a sequence of non-negative functions defined on $[T_*, \infty)$ for some $T_* \in \mathbb{R}$ that satisfies the following two conditions:

(a) For each $n \in \mathbb{N}$,

$$\lim_{t\to\infty} \psi_n(t) = 0. \tag{4.1}$$

(b) For $n > m$,

$$\psi_n(t) = o(\psi_m(t)). \tag{4.2}$$

Let $(X, \|\cdot\|)$ be a normed space, and $g$ be a function from $[T_*, \infty)$ to $X$. We say $g$ has an asymptotic expansion (implicitly as $t \to \infty$)

$$g(t) \sim \sum_{n=1}^\infty \xi_n \psi_n(t) \text{ in } X, \tag{4.3}$$

where $\xi_n \in X$ for all $n \in \mathbb{N}$, if, for any $N \in \mathbb{N}$,

$$\left\| g(t) - \sum_{n=1}^N \xi_n \psi_n(t) \right\| = o(\psi_N(t)). \tag{4.4}$$

Obviously, if $g(t) = \sum_{n=1}^N \xi_n \psi_n(t)$ for some $N \in \mathbb{N}$, then $g(t) \sim \sum_{n=1}^\infty \xi_n \psi_n(t)$ where $\xi_n = 0$ for $n > N$. In case of the infinite sum, the convergent series

$$g(t) = \sum_{n=1}^\infty \xi_n \psi_n(t) \tag{4.5}$$

does not necessarily imply the expansion (4.3). We refer to "Appendix A" for some criteria for both (4.3) and (4.5) to hold, with the infinite sum not reduced to a finite one.

Note that the expansion (4.3) does not determine the function $g$. Indeed, if $h : [T_*, \infty) \to X$ is a function that satisfies $\|h(t)\| = o(\psi_n(t))$ for all $n \in \mathbb{N}$, then both $g$ and $g + h$ have the same expansion on the right-hand side of (4.3). The converse is considered in the next proposition.

**Proposition 4.2** *Let $(\psi_n)_{n=1}^{\infty}$, $(X, \|\cdot\|)$ and $g$ be as in Definition 4.1. Suppose, for each $n \in \mathbb{N}$, that the function $\psi_n$ is not identically zero on $[T, \infty)$ for all $T \geq T_*$. Then the asymptotic expansion (4.3), if exists, is unique.*

***Proof*** Suppose $g(t)$ has two expansions

$$g(t) \sim \sum_{n=1}^{\infty} \phi_n \psi_n(t) \text{ and } g(t) \sim \sum_{n=1}^{\infty} \xi_n \psi_n(t). \tag{4.6}$$

We will prove by induction that $\phi_n = \xi_n$ for all $n \in \mathbb{N}$.

One has from the triangle inequality and each expansion in (4.6) that

$$\|(\phi_1 - \xi_1)\psi_1(t)\| \leq \|\phi_1\psi_1(t) - g(t)\| + \|g(t) - \xi_1\psi_1(t)\| = o(\psi_1(t)). \tag{4.7}$$

Since $\psi_1$ is asymptotically non-trivial, then one can verify from (4.7) that $\phi_1 = \xi_1$.

Let $N \in \mathbb{N}$ and assume $\phi_n = \xi_n$ for $n = 1, 2, \ldots, N$. Then

$$\|(\phi_{N+1} - \xi_{N+1})\psi_{N+1}(t)\| = \left\| \sum_{n=1}^{N+1} (\phi_n - \xi_n)\psi_n(t) \right\|$$
$$\leq \left\| \sum_{n=1}^{N+1} \phi_n\psi_n(t) - g(t) \right\| + \left\| g(t) - \sum_{n=1}^{N+1} \xi_n\psi_n(t) \right\| = o(\psi_{N+1}(t)).$$

Hence, $\phi_{N+1} = \xi_{N+1}$. By the induction principle, $\phi_n = \xi_n$ for all $n \in \mathbb{N}$. □

Note that the rate of convergence, as $t \to \infty$, in (4.4), in fact, can be related to the next term $\psi_{N+1}(t)$. Indeed,

$$\left\| g(t) - \sum_{n=1}^{N} \xi_n\psi_n(t) \right\| \leq \left\| g(t) - \sum_{n=1}^{N+1} \xi_n\psi_n(t) \right\| + \psi_{N+1}(t)\|\xi_{N+1}\|.$$

Hence, we can replace equivalently (4.4) by

$$\left\| g(t) - \sum_{n=1}^{N} \xi_n\psi_n(t) \right\| = O(\psi_{N+1}(t)). \tag{4.8}$$

The equivalence of (4.4) and (4.8) is essentially due to the infinite sum in (4.3). If the sum is finite, this is no more the case. Moreover, for general $\psi_n$'s, the relation (4.2) is not informative enough to work with.

These prompt us to have the following more specific definition.

**Definition 4.3** Let $\Psi = (\psi_\lambda)_{\lambda>0}$ be a system of functions that satisfies the following two conditions.

(a) There exists $T_* \geq 0$ such that, for each $\lambda > 0$, $\psi_\lambda$ is a positive function defined on $[T_*, \infty)$, and

$$\lim_{t\to\infty} \psi_\lambda(t) = 0. \tag{4.9}$$

(b) For any $\lambda > \mu$, there exists $\eta > 0$ such that

$$\psi_\lambda(t) = \mathcal{O}(\psi_\mu(t)\psi_\eta(t)). \tag{4.10}$$

Let $(X, \|\cdot\|)$ be a real normed space, and $g$ be a function from $(0, \infty)$ to $X$. The function $g$ is said to have the asymptotic expansion

$$g(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t) \text{ in } X, \tag{4.11}$$

where $\xi_n \in X$ for all $n \in \mathbb{N}$, and $(\lambda_n)_{n=1}^{\infty}$ is a strictly increasing, divergent sequence of positive numbers, if it holds, for any $N \geq 1$, that there exists $\varepsilon > 0$ such that

$$\left\| g(t) - \sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t) \right\| = \mathcal{O}(\psi_{\lambda_N}(t)\psi_{\varepsilon}(t)). \tag{4.12}$$

We have the following remarks on Definition 4.3.

(a) If $\lambda > \mu$, it follows (4.10) and (4.9) that

$$\psi_{\lambda}(t) = o(\psi_{\mu}(t)). \tag{4.13}$$

(b) If a function $g$ has an expansion (4.11), then $g(t) \sim \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t)$ in $X$ in the sense of Definition 4.1.
(c) Thanks to (b) and Proposition 4.2, the $\xi_n$'s in (4.11) are unique. Similarly, following the proof of Proposition 4.2, we also have the uniqueness of $\xi_n$'s in (4.16).
(d) The main difference between Definition 4.1 and Definition 4.3 is the specific decaying rate $\psi_{\eta}(t)$ on the right-hand side of (4.10), in contrast with the non-specific one in (4.2). In the proofs, this crucially allows comparisons and estimates for different quantities.

We have the following special cases for the expansion (4.11).

(i) Assume (4.11). If there exists $N \in \mathbb{N}$, such that

$$\xi_n = 0 \text{ for all } n > N, \tag{4.14}$$

then it holds for all $\lambda > 0$ that

$$\left\| g(t) - \sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t) \right\| = \mathcal{O}(\psi_{\lambda}(t)). \tag{4.15}$$

(ii) Assume there exist $N \in \mathbb{N}$, $\xi_n \in X$ for $1 \leq n \leq N$, and $\lambda_n$'s, for $1 \leq n \leq N$, are positive numbers, strictly increasing in $n$ such that (4.15) holds for all $\lambda > 0$. We extend $\xi_n \in X$ for $1 \leq n \leq N$ to a sequence $(\xi_n)_{n=1}^{\infty}$ with (4.14), and extend $\lambda_n$ for $1 \leq n \leq N$ to any sequence $(\lambda_n)_{n=1}^{\infty}$ that is a strictly increasing and divergent. Then one can verify that (4.11) holds true.
Therefore, we say in cases (i) and (ii) that the function $g$ has the asymptotic expansion

$$g(t) \overset{\Psi}{\sim} \sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t) \text{ in } X. \tag{4.16}$$

(iii) If $\xi_n = 0$ for all $n \in N$ in (4.11), then

$$\|g(t)\| = \mathcal{O}(\psi_{\lambda}(t)) \tag{4.17}$$

for all $\lambda > 0$.

(iv) Assume (4.17) holds for all $\lambda > 0$. Let $\xi_n = 0$ for all $n \in \mathbb{N}$. Let $(\lambda_n)_{n=1}^{\infty}$ be any strictly increasing, divergent sequence of positive numbers. Then we have (4.11).
Therefore, we say in cases (iii) and (iv) that the function $g$ has the asymptotic expansion

$$g(t) \overset{\Psi}{\sim} 0 \text{ in } X.$$

(v) For $N = 0$, we conveniently set the sum $\sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t)$ to be zero in (4.16), and see that the condition (4.15) is, in fact, (4.17). Thus the expression

$$g(t) \overset{\Psi}{\sim} \sum_{n=1}^{0} \xi_n \psi_{\lambda_n}(t) \quad \text{will mean} \quad g(t) \overset{\Psi}{\sim} 0.$$

(vi) If a function $g$ has an asymptotic expansion

$$g(t) \overset{\Psi}{\sim} \sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t) \text{ in } X,$$

for $N \in \mathbb{N} \cup \{0, \infty\}$, then by remark (c) above, this expansion is unique for $g$.

For solutions of ODEs or PDEs, the linear and nonlinear structures of the equations will impose more conditions on the system. We consider below the ones that are appropriate to our current study of the NSE.

**Condition 4.4** *The system* $\Psi = (\psi_\lambda)_{\lambda>0}$ *satisfies* (a) *and* (b) *in Definition* 4.3 *and the following.*

*(i) For any* $\lambda, \mu > 0$*, there exist* $\gamma > \max\{\lambda, \mu\}$ *and a nonzero constant* $d_{\lambda,\mu}$ *such that*

$$\psi_\lambda \psi_\mu = d_{\lambda,\mu} \psi_\gamma. \tag{4.18}$$

*(ii) For each* $\lambda > 0$*, the function* $\psi_\lambda$ *is continuous and differentiable on* $[T_*, \infty)$*, and its derivative* $\psi_\lambda'$ *has an expansion in the sense of Definition* 4.3

$$\psi_\lambda'(t) \overset{\Psi}{\sim} \sum_{k=1}^{N_\lambda} c_{\lambda,k} \psi_{\lambda^\vee(k)}(t) \text{ in } \mathbb{R}, \tag{4.19}$$

*where* $N_\lambda \in \mathbb{N} \cup \{0, \infty\}$*, all* $c_{\lambda,k}$ *are constants, all* $\lambda^\vee(k) > \lambda$*, and, for each* $\lambda > 0$*,* $\lambda^\vee(k)$*'s are strictly increasing in* $k$*.*

The following remarks on Condition 4.4 are in order.

(a) By (4.13), the numbers $\gamma$ and $d_{\lambda,\mu}$ in (4.18) are unique.
(b) By (4.18), we have the reverse of (4.10) in the following sense:

$$\psi_\mu(t)\psi_\eta(t) = \mathcal{O}(\psi_\lambda(t)) \text{ for some } \lambda > \mu. \tag{4.20}$$

(c) Thanks to (4.10) and (4.20), the condition (4.12) is equivalent to

$$\left\| g(t) - \sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t) \right\| = \mathcal{O}(\psi_\lambda(t)),$$

for some $\lambda > \lambda_N$.

(d) Expansion (4.19) of $\psi'_\lambda$, in all cases of $N_\lambda$, and property (4.10) imply that there exists $\eta > 0$ such that

$$|\psi'_\lambda(t)| = \mathcal{O}(\psi_\lambda(t)\psi_\eta(t)). \tag{4.21}$$

(e) If, instead of (4.19), $\psi'_\lambda = c_\lambda \psi_\lambda$ for all $\lambda$, then $\psi_\lambda$'s are exponential functions. This case was studied in [19]. For other examples of (4.19), see Sects. 7.3 and 7.4.

**Notation** Denote $\gamma$ in (4.18) by $\lambda \wedge \mu$, which is uniquely determined thanks to remark (a) above.

For the current study, we focus on decaying functions that are larger than the exponentially decaying ones; hence, we impose more specific conditions.

**Condition 4.5** *The system* $\Psi = (\psi_\lambda)_{\lambda>0}$ *satisfies* (a), (b) *of Definition* 4.3, *and the following.*

(i) *For each* $\lambda > 0$, *the function* $\psi_\lambda$ *is decreasing (in* $t$*).*
(ii) *If* $\lambda, \alpha > 0$, *then*

$$e^{-\alpha t} = o(\psi_\lambda(t)). \tag{4.22}$$

(iii) *For any number* $a \in (0, 1)$,

$$\psi_\lambda(at) = \mathcal{O}(\psi_\lambda(t)). \tag{4.23}$$

The followings are direct consequences of Condition 4.5.

(a) By (4.22), for any $\alpha, \lambda > 0$, there exists a positive constant $C_{\alpha,\lambda}$ such that

$$e^{-\alpha(t+T_*)} \le C_{\alpha,\lambda}\psi_\lambda(t + T_*) \quad \forall t \ge 0,$$

hence, by denoting $D_1(\lambda, \alpha) = e^{\alpha T_*} C_{\lambda,\alpha}$, we have

$$e^{-\alpha t} \le D_1(\lambda, \alpha)\psi_\lambda(t + T_*). \tag{4.24}$$

(b) For $a \in (0, 1)$, we have from (i) that $\psi_\lambda(t) = \mathcal{O}(\psi_\lambda(at))$. Thus, the condition (4.23), in fact, is equivalent to

$$\psi_\lambda(at) \stackrel{\mathcal{O}}{=} \psi_\lambda(t).$$

(c) Property (4.23) and the decrease of $\psi_\lambda(t)$ in $t$ imply, for $a \in (0, 1)$, that

$$\psi_\lambda(at) \le D_2(a, \lambda)\psi_\lambda(t) \quad \forall t \ge T_*/a,$$

where $D_2(a, \lambda)$ is a constant in $[1, \infty)$. Consequently, for $a \in (0, 1)$ and $t \ge 0$,

$$\psi_\lambda(at + T_*) = \psi_\lambda(a(t + T_*/a)) \le D_2(a, \lambda)\psi_\lambda(t + T_*/a) \le D_2(a, \lambda)\psi_\lambda(t + T_*). \tag{4.25}$$

Then by the decrease of $\psi_\lambda$ in $t$, we have

$$\psi_\lambda(at + T_*) \le D_2(a, \lambda)\psi_\lambda(t) \quad \forall t \ge T_*. \tag{4.26}$$

In particular, for any $T \ge 0$ and $t \ge 2(T_* + T)$, we have $t - T \ge t/2 + T_*$, then by (4.26),

$$\psi_\lambda(t - T) \le \psi_\lambda(t/2 + T_*) \le D_3(\lambda)\psi_\lambda(t), \text{ where } D_3(\lambda) = D_2(1/2, \lambda).$$

Combining this with the boundedness of $\psi_\lambda(t-T)/\psi_\lambda(t)$ for small $t \in [T_*+T, 2(T_*+T)]$, we obtain

$$\psi_\lambda(t-T) \le D_4(\lambda, T)\psi_\lambda(t) \quad \forall t \ge T_* + T,$$

which yields

$$\psi_\lambda(t) \le D_4(\lambda, T)\psi_\lambda(t+T) \quad \forall t \ge T_*,$$

for some positive constant $D_4(\lambda, T)$. Consequently, for any $T \in \mathbb{R}$,

$$\psi_\lambda(t) \stackrel{\mathcal{O}}{=} \psi_\lambda(t+T). \tag{4.27}$$

In applications to the NSE, suppose that the force $f$ has an expansion containing the terms $\psi_{\gamma_n}$'s for some numeric sequence $(\gamma_n)_{n=1}^\infty$. Then the operators in the NSE require that a solution $u(t)$, in case of having an expansion itself, may need many more terms in addition to $\psi_{\gamma_n}$'s. We describe below a general principle to find those other terms.

For any $x \in (0, \infty)$, define the set

$$G_x = \begin{cases} \emptyset, & \text{if } N_x = 0,\\ \{x^\vee(k) : 1 \le k \le N_x\}, & \text{if } N_x \in \mathbb{N},\\ \{x^\vee(k) : k \in \mathbb{N}\}, & \text{if } N_x = \infty. \end{cases}$$

A non-empty subset $S$ of $(0, \infty)$ is said to preserve the operation $\vee$ if

$$\forall x \in S : G_x \subset S. \tag{4.28}$$

Similarly, $S$ is said to preserve the operation $\wedge$ if

$$\forall x, y \in S : x \wedge y \in S. \tag{4.29}$$

**Lemma 4.6** *Let $S$ be any non-empty subset of* $(0, \infty)$.

(i) *There exists a smallest set $S_* \subset (0, \infty)$ that contains $S$, and preserves the operations $\vee$ and $\wedge$.*
(ii) *In fact, $S_* = S^*$, where $S^*$ is constructed explicitly in* (4.30) *below.*

***Proof*** (i) For any non-empty subset $M$ of $(0, \infty)$, we denote

$$M^\wedge = \{x \wedge y : x, y \in M\},$$
$$M^\vee = \bigcup_{x \in M} G_x.$$

(a) Let $S_0 = S$. We define recursively the sets $S_n$, for $n \in \mathbb{N}$, by

$$S_{2k+1} = S_{2k} \cup S_{2k}^\vee \text{ and } S_{2k+2} = S_{2k+1} \cup S_{2k+1}^\wedge \text{ for } k \ge 0.$$

Define

$$S^* = \bigcup_{n=0}^\infty S_n. \tag{4.30}$$

We obviously have

$$S_{2k} \subset S_{2k+1} \subset S_{2k+2} \subset S_{2k+3} \quad \forall k \ge 0. \tag{4.31}$$

It follows (4.31) that $(S_n)$, $(S_{2n})$ and $(S_{2n+1})$ are increasing sequences, and, hence,

$$S^* = \bigcup_{k=0}^{\infty} S_{2k} = \bigcup_{k=0}^{\infty} S_{2k+1}. \tag{4.32}$$

Clearly, $S = S_0 \subset S^*$. Next, we prove $S^*$ preserves the operations $\vee$ and $\wedge$.

Let $x \in S^*$. By (4.32), $x \in S_{2k}$ for some $k \geq 0$. Then obviously by definition $G_x \subset S_{2k}^{\vee} \subset S_{2k+1} \subset S^*$. Thus, $G_x \subset S^*$.

For $x, y \in S^*$, then by (4.32), $x \in S_{2k+1}$, $y \in S_{2m+1}$ for some $k, m \geq 0$. Assume $k \geq m$, then $y \in S_{2k+1}$ by (4.31). This implies $x \wedge y \in S_{2k+2} \subset S^*$.

(b) Let $\mathcal{C}$ be the collection of sets $M$ that contain $S$ and preserve the operations $\vee$ and $\wedge$. Because $S^* \in \mathcal{C}$, then the collection $\mathcal{C}$ is non-empty.

Let $S_*$ be the intersections of all the elements in $\mathcal{C}$. Then $S_* \subset S^*$. Let $M \in \mathcal{C}$, properties (4.28) and (4.29) for $S := M$ clearly imply

$$M^{\vee} \subset M \text{ and } M^{\wedge} \subset M. \tag{4.33}$$

Thus,

$$(S_*)^{\vee} \subset M^{\vee} \subset M \quad \text{and } (S_*)^{\wedge} \subset M^{\wedge} \subset M.$$

It follows that

$$(S_*)^{\vee} \subset \bigcap_{M\in C} M = S_* \quad \text{and } (S_*)^{\wedge} \subset \bigcap_{M\in C} M = S_*.$$

Therefore, $S_* \in \mathcal{C}$. By its definition, $S_*$ is the smallest set in $\mathcal{C}$.

(ii) We prove $S_* = S^*$. It suffices to show $S^* \subset S_*$.

Let $M$ be an arbitrary element in $\mathcal{C}$. We shall show that $S^* \subset M$. First, we see that $S_0 \subset M$. By (4.33),

$$S_1 = S_0 \cup S_0^{\vee} \subset M \cup M^{\vee} = M,$$

and then,

$$S_2 = S_1 \cup S_1^{\wedge} \subset M \cup M^{\wedge} = M.$$

By induction, we can prove similarly that $S_k \subset M$ for all $k$. Therefore $S^* = \bigcup_{k=0}^{\infty} S_k \subset M$. Then $S^* \subset \bigcap_{M\in\mathcal{C}} M = S_*$. This completes the proof of the lemma. □

**Notation** We denote the set $S_*$ in Lemma 4.6 by $\mathcal{G}_{\Psi}(S)$.

## 5 Asymptotic expansions in a continuum system

Let $\Psi = (\psi_\lambda)_{\lambda>0}$ be a system of functions that satisfies both Conditions 4.4 and 4.5.

**Assumption 5.1** Suppose there exist real numbers $\sigma \geq 0$, $\alpha \geq 1/2$, a strictly increasing, divergent sequence of positive numbers $(\gamma_n)_{n=1}^{\infty}$ and a sequence $(\tilde{\phi}_n)_{n=1}^{\infty}$ in $G_{\alpha,\sigma}$ such that, in the sense of Definition 4.3,

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \tilde{\phi}_n \psi_{\gamma_n}(t) \text{ in } G_{\alpha,\sigma}. \tag{5.1}$$

Note from (5.1) that $f(t)$ belongs to $G_{\alpha,\sigma}$ for all $t$ sufficiently large.

Let $u(t)$ be a Leray–Hopf weak solution of the NSE. We search for an asymptotic expansion of $u(t)$ in the form

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t). \tag{5.2}$$

Formally substituting expansion (5.2) into the NSE (2.2), we find that the indices $\lambda_n$'s naturally take values in the set

$$\mathcal{G}_\Psi(\{\gamma_n : n \in \mathbb{N}\}). \tag{5.3}$$

However, the expansion (5.2) only agrees with (4.11) in Definition 4.3 if the set in (5.3) does not have a finite cluster point. Therefore, we impose one more condition.

**Assumption 5.2** There exists a set $S_*$ that contains $\{\gamma_n : n \in \mathbb{N}\}$, preserves the operations $\vee$ and $\wedge$, and can be ordered so that

$$S_* = \{\lambda_n : n \in \mathbb{N}\}, \text{ where } \lambda_n\text{'s are strictly increasing to infinity.} \tag{5.4}$$

We usually choose $S_*$ in Assumption 5.2 to be (5.3), but this is not the only choice.

Under Assumption 5.2, we can show that the expansion (5.1) implies

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n \psi_{\lambda_n}(t) \quad \text{in } G_{\alpha,\sigma} \quad \text{as } t \to \infty, \tag{5.5}$$

where the sequence $(\phi_n)_{n=1}^{\infty}$ in $G_{\alpha,\sigma}$ is defined by $\phi_n = \tilde{\phi}_k$ if there exists $k \geq 1$ such that $\lambda_n = \gamma_k$, and $\phi_n = 0$ otherwise. Note in the former case that such an index $k$, when exists, is unique.

***Remark 5.3*** In case the set $S = \{\gamma_n : n \in \mathbb{N}\}$ itself preserves the operations $\vee$ and $\wedge$, then $S = \mathcal{G}_\Psi(S)$. Hence, Assumption 5.2 is met with $S_* = S$ and (5.5) holds with $\lambda_n = \gamma_n$, $\phi_n = \tilde{\phi}_n$, i.e., expansion (5.5) is just the original (5.1).

Our first main result on the expansion of the Leray–Hopf weak solutions is the following.

**Theorem 5.4** *Let Assumptions 5.1 and 5.2 hold true, and let $f$ have the asymptotic expansion (5.5). Then any Leray–Hopf weak solution $u(t)$ of (2.2) has the asymptotic expansion*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t) \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1), \tag{5.6}$$

*where $\xi_n$'s are defined recursively by*

$$\xi_1 = A^{-1}\phi_1, \tag{5.7}$$

$$\xi_n = A^{-1}\Big(\phi_n - \chi_n - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k\wedge\lambda_m=\lambda_n}} d_{\lambda_k,\lambda_m} B(\xi_k,\xi_m)\Big) \text{ for } n \geq 2, \tag{5.8}$$

*where*

$$\chi_n = \begin{cases} \sum_{\substack{(p,k)\in[1,n-1]\times\mathbb{N}:\\ \lambda_p^\vee(k)=\lambda_n}} c_{\lambda_p,k}\xi_p, & \text{if } \exists p \in [1,n-1], k \in \mathbb{N} : \lambda_p^\vee(k) = \lambda_n,\\ 0, & \text{otherwise.} \end{cases} \tag{5.9}$$

***Proof*** The proof is divided into parts A, B, …, steps 1,2, and substeps (a), (b), …

A. Notation. For $n \in \mathbb{N}$, denote

$$F_n(t) = \phi_n \psi_{\lambda_n}(t), \quad \bar{F}_n(t) = \sum_{j=1}^{n} F_j(t), \quad \text{and} \quad \tilde{F}_n(t) = f(t) - \bar{F}_n(t),$$

$$u_n(t) = \xi_n \psi_{\lambda_n}(t), \quad \bar{u}_n(t) = \sum_{j=1}^{n} u_j(t), \quad \text{and} \quad v_n = u(t) - \bar{u}_n(t).$$

According to the expansion (5.5) and Definition 4.3, we can assume that

$$|\tilde{F}_N(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_N}(t)\psi_{\delta_N}(t)), \tag{5.10}$$

for any $N \in \mathbb{N}$, with some $\delta_N > 0$.

B. We observe that

$$\xi_n \in G_{\alpha+1,\sigma} \quad \forall n \geq 1. \tag{5.11}$$

The proof of (5.11) is by induction and is the same as in [1, Lemma 4.2].

By (5.11), we have

$$|\bar{u}_n(t)|_{\alpha+1,\sigma} = \mathcal{O}(\psi_{\lambda_1}(t)) \quad \forall n \in \mathbb{N}. \tag{5.12}$$

C. As a preparation, we need to establish the large-time decay for $u(t)$ first. Letting $N = 1$ in (5.10) gives $|f(t) - \phi_1\psi_{\lambda_1}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_1}(t)\psi_{\delta_1}(t))$, which implies

$$|f(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_1}(t)) = \mathcal{O}(\psi_{\lambda_1}(t + T_*)).$$

The last relation is due to (4.27).

Let $F(t) = \psi_{\lambda_1}(t + T_*)$. Then $|f(t)|_{\alpha,\sigma} = \mathcal{O}(F(t))$, and, by (4.24) and (4.25), the function $F$ satisfies (3.4) and (3.5). We now apply Theorem 3.4 with $\varepsilon = 1/2$. Then there exists time $\hat{T} > 0$ and a constant $C > 0$ such that $u(t)$ is a regular solution of (2.2) on $[\hat{T}, \infty)$, and

$$|u(\hat{T} + t)|_{\alpha+1/2,\sigma} \leq C\psi_{\lambda_1}(t + T_*) \quad \forall t \geq 0. \tag{5.13}$$

It follows (2.11) and (5.13) that

$$|B(u(\hat{T} + t), u(\hat{T} + t))|_{\alpha,\sigma} \leq C|u(\hat{T} + t)|^2_{\alpha+1/2,\sigma} \leq C\psi^2_{\lambda_1}(t + T_*) \quad \forall t \geq 0. \tag{5.14}$$

D. It suffices to prove, for any $N \in \mathbb{N}$, that there exists a number $\varepsilon_N > 0$ such that

$$|v_N(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_N}(t)\psi_{\varepsilon_N}(t)). \tag{5.15}$$

We will prove (5.15) by induction in $N$. In calculations below, all differential equations hold in $V'$-valued distribution sense on $(T, \infty)$ for any $T > 0$, which is similar to (2.4). One can easily verify them by using (2.10), and the facts $u \in L^2_{\rm loc}([0,\infty), V)$ and $u' \in L^1_{\rm loc}([0,\infty), V')$ in Definition 2.1.

*Step 1: $N = 1$* Define $w_1(t) = \psi^{-1}_{\lambda_1}(t)u(t)$.

(a) Equation for $w_1(t)$. We have

$$\begin{aligned} w_1'(t) &= \psi^{-1}_{\lambda_1}(t)u'(t) - \psi^{-2}_{\lambda_1}(t)\psi'_{\lambda_1}(t)u(t) \\ &= \psi^{-1}_{\lambda_1}(t)\big(-Au(t) - B(u,u) + \phi_1\psi_{\lambda_1}(t) + \tilde{F}_1(t)\big) - \psi^{-2}_{\lambda_1}(t)\psi'_{\lambda_1}(t)u(t). \end{aligned}$$

Thus,

$$w_1'(t) + Aw_1(t) = \phi_1 + H_1(t), \quad t > T_*, \tag{5.16}$$

where

$$H_1(t) = \psi_{\lambda_1}^{-1}[\tilde{F}_1(t) - B(u(t), u(t))] - \psi_{\lambda_1}'\psi_{\lambda_1}^{-2}u(t).$$

(b) Estimation of $|H_1(t)|_{\alpha,\sigma}$. By estimates (5.13), (5.14), and the relations in (4.27), we have

$$|u(t)|_{\alpha+1/2,\sigma} = \mathcal{O}(\psi_{\lambda_1}(t)). \tag{5.17}$$

$$|B(u(t), u(t)))|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_1}^2(t)). \tag{5.18}$$

By (5.10), (5.17), (5.18), and properties (4.21), (4.27), there exist $T_0 \ge T_*$, $\eta_1 > 0$, and $D_0 > 0$ such that for $t \ge 0$,

$$\begin{aligned} \psi_{\lambda_1}^{-1}(T_0+t)|\tilde{F}_1(T_0+t)|_{\alpha,\sigma} &\le D_0\psi_{\lambda_1}^{-1}(T_0+t)\psi_{\lambda_1}(T_0+t)\psi_{\delta_1}(T_0+t) \\ &\le D_0\psi_{\delta_1}(T_0+t), \\ \psi_{\lambda_1}^{-1}(T_0+t)|B(u(T_0+t), u(T_0+t))|_{\alpha,\sigma} &\le D_0\psi_{\lambda_1}^{-1}(T_0+t)\psi_{\lambda_1}^2(T_0+t) \\ &\le D_0\psi_{\lambda_1}(T_0+t), \end{aligned}$$

and

$$\begin{aligned} &\psi_{\lambda_1}'(T_0+t)\psi_{\lambda_1}^{-2}(T_0+t)|u(T_0+t)|_{\alpha+1/2,\sigma} \\ &\quad\le D_0\psi_{\lambda_1}(T_0+t)\psi_{\eta_1}(T_0+t)\psi_{\lambda_1}^{-2}(T_0+t)\psi_{\lambda_1}(T_0+t) \\ &\quad\le D_0\psi_{\eta_1}(T_0+t). \end{aligned}$$

Let $\varepsilon_1 = \min\{\delta_1, \eta_1, \lambda_1\}$. Then

$$|H_1(T_0+t)|_{\alpha,\sigma} \le 3D_0\psi_{\varepsilon_1}(T_0+t) \quad \forall t \ge 0.$$

(c) We apply Theorem 3.2(iii) to Eq. (5.16) in $G_{\alpha,\sigma}$ with $w(t) := w_1(T_0+t)$, $f(t) := H_1(T_0+t)$, $F(t) := \psi_{\varepsilon_1}(T_0+t)$ and $\xi = \phi_1$. We obtain from (3.6) that

$$|w_1(T_0+t) - A^{-1}\phi_1|_{\alpha+1-\rho,\sigma} = \mathcal{O}(\psi_{\varepsilon_1}(T_0+t))$$

for any $\rho \in (0, 1)$, which yields

$$|w_1(t) - A^{-1}\phi_1|_{\alpha+1-\rho,\sigma} = \mathcal{O}(\psi_{\varepsilon_1}(t)).$$

Multiplying this equation by $\psi_{\lambda_1}(t)$ gives

$$|u(t) - \xi_1\psi_{\lambda_1}(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(\psi_{\lambda_1}(t)\psi_{\varepsilon_1}(t)) \quad \forall \rho \in (0, 1).$$

This proves that (5.15) holds for $N = 1$.

*Step 2: Induction step* Let $N \ge 1$ be an integer and assume there exists $\varepsilon_N > 0$ such that

$$|v_N(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(\psi_{\lambda_N}(t)\psi_{\varepsilon_N}(t)) \quad \forall \rho \in (0, 1). \tag{5.19}$$

(a) We will find an equation for $v_N$ which is suitable to study its asymptotic behavior. First, we have the preliminary calculations.

*Rewriting $u'$.* By the NSE,

$$\begin{aligned}u' &= -Au - B(u,u) + f(t)\\ &= -Av_N - A\bar{u}_N - B(\bar{u}_N + v_N, \bar{u}_N + v_N) + \bar{F}_N + F_{N+1} + \tilde{F}_{N+1}\\ &= -Av_N - A\bar{u}_N + \bar{F}_N - B(\bar{u}_N, \bar{u}_N) + \phi_{N+1}\psi_{\lambda_{N+1}}(t) + h_{N+1,1},\end{aligned}$$

where

$$h_{N+1,1} = -B(\bar{u}_N, v_N) - B(v_N, \bar{u}_N) - B(v_N, v_N) + \tilde{F}_{N+1}.$$

On the one hand,

$$-A\bar{u}_N + \bar{F}_N = -\sum_{n=1}^{N} \psi_{\lambda_n}(t)\Big(A\xi_n - \phi_n\Big).$$

On the other hand,

$$\begin{aligned}B(\bar{u}_N, \bar{u}_N) &= \sum_{m,j=1}^{N} \psi_{\lambda_m}(t)\psi_{\lambda_j}(t)B(\xi_m, \xi_j)\\ &= \sum_{n=1}^{N} \psi_{\lambda_n}(t)\left(\sum_{\substack{1\le m,j\le N,\\ \lambda_m\wedge\lambda_j=\lambda_n}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j)\right)\\ &\quad + \psi_{\lambda_{N+1}}(t)\sum_{\substack{1\le m,j\le N,\\ \lambda_m\wedge\lambda_j=\lambda_{N+1}}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j) + h_{N+1,2},\end{aligned}$$

where

$$h_{N+1,2} = \sum_{\substack{1\le m,j\le N,\\ \lambda_m\wedge\lambda_j\ge\lambda_{N+2}}} \psi_{\lambda_m}(t)\psi_{\lambda_j}(t)B(\xi_m, \xi_j).$$

Then we obtain the equation

$$\begin{aligned}u' = -Av_N - \sum_{n=1}^{N} \psi_{\lambda_n}(t)\left(A\xi_n - \phi_n + \sum_{\substack{1\le m,j\le N,\\ \lambda_m\wedge\lambda_j=\lambda_n}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j)\right)\\ - \psi_{\lambda_{N+1}}(t)\left(\sum_{\substack{1\le m,j\le N,\\ \lambda_m\wedge\lambda_j=\lambda_{N+1}}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j) - \phi_{N+1}\right) - h_{N+1,2} + h_{N+1,1}.\end{aligned} \tag{5.20}$$

In calculations below to the end of this proof, $\varepsilon$ denotes a *generic* positive index used for function $\psi_\varepsilon(t)$.

Utilizing (5.10), (5.12) and (5.19), we estimate, with the use of the short-hand notation $\psi_\lambda = \psi_\lambda(t)$,

$$\begin{aligned}|h_{N+1,1}(t)|_{\alpha,\sigma} &= \mathcal{O}(\psi_{\lambda_1})\mathcal{O}(\psi_{\lambda_N}\psi_{\varepsilon_N}) + \mathcal{O}(\psi_{\lambda_N}\psi_{\varepsilon_N})\mathcal{O}(\psi_{\lambda_1}) \\ &\quad + \mathcal{O}(\psi_{\lambda_N}\psi_{\varepsilon_N})\mathcal{O}(\psi_{\lambda_N}\psi_{\varepsilon_N}) + \mathcal{O}(\psi_{\lambda_{N+1}}\psi_{\delta_{N+1}}) \\ &= \mathcal{O}(\psi_{\lambda_1}\psi_{\lambda_N}\psi_{\varepsilon_N}) + \mathcal{O}(\psi_{\lambda_{N+1}}\psi_{\delta_{N+1}}) \\ &= \mathcal{O}(\psi_{\lambda_1\wedge\lambda_N}\psi_{\varepsilon_N}) + \mathcal{O}(\psi_{\lambda_{N+1}}\psi_{\delta_{N+1}}).\end{aligned}$$

Since $\lambda_1 \wedge \lambda_N \ge \lambda_{N+1}$, we have

$$|h_{N+1,1}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_{N+1}}(t)\psi_\varepsilon(t)).$$

It is also clear that

$$\begin{aligned}|h_{N+1,2}(t)|_{\alpha,\sigma} &\le \sum_{\substack{1\le m,j\le N,\\ \lambda_m\wedge\lambda_j\ge\lambda_{N+2}}} |d_{\lambda_m,\lambda_j}|\psi_{\lambda_m\wedge\lambda_j}(t)|B(\xi_m,\xi_j)|_{\alpha,\sigma} \\ &= \mathcal{O}(\psi_{\lambda_{N+2}}(t)) = \mathcal{O}(\psi_{\lambda_{N+1}}(t)\psi_\varepsilon(t)).\end{aligned}$$

<u>*Rewriting* $\bar u_N'$.</u> We have

$$\bar u_N' = \sum_{p=1}^N \psi_{\lambda_p}'\xi_p = \sum_{p=1}^N \xi_p\left(\sum_{k=1}^{\widetilde N_p} c_{\lambda_p,k}\psi_{\lambda_p^\vee(k)} + \psi_{\lambda_p}' - \sum_{k=1}^{\widetilde N_p} c_{\lambda_p,k}\psi_{\lambda_p^\vee(k)}\right),$$

where $\widetilde N_p$ is the largest integer $k$ such that

$$\lambda_p^\vee(k) \le \lambda_{N+1}.$$

Then we obtain

$$\bar u_N' = \sum_{n=1}^N \psi_{\lambda_n}\chi_n + \psi_{\lambda_{N+1}}\chi_{N+1} + h_{N+1,3}, \tag{5.21}$$

where

$$h_{N+1,3} = \sum_{p=1}^N \xi_p\left(\psi_{\lambda_p}' - \sum_{k=1}^{\widetilde N_p} c_{\lambda_p,k}\psi_{\lambda_p^\vee(k)}\right).$$

Note from (4.19) and the definition of $\widetilde N_p$ that

$$|\psi_{\lambda_p}'(t) - \sum_{k=1}^{\widetilde N_p} c_{\lambda_p,k}\psi_{\lambda_p^\vee(k)}(t)| = \mathcal{O}(\psi_\lambda(t)),\quad \text{some } \lambda > \lambda_{N+1}.$$

Together with (4.10), we have

$$|h_{N+1,3}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_{N+1}}(t)\psi_\varepsilon(t)).$$

*Equation for $v_N$.* Combining (5.20) and (5.21) yields

$$\begin{aligned}
v_N' &= u' - \bar{u}_N' \\
&= -Av_N - \sum_{n=1}^{N} \psi_{\lambda_n}(t) \left( A\xi_n + \sum_{\substack{1\le m,j\le N,\\ \lambda_m \wedge \lambda_j = \lambda_n}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j) - \phi_n + \chi_n \right) \\
&\quad + \psi_{\lambda_{N+1}}(t) \left( - \sum_{\substack{1\le m,j\le N,\\ \lambda_m \wedge \lambda_j = \lambda_{N+1}}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j) + \phi_{N+1} - \chi_{N+1} \right) + h_{N+1,4}(t),
\end{aligned}$$

where

$$h_{N+1,4} = h_{N+1,1} - h_{N+1,2} - h_{N+1,3}.$$

Note, for $1 \le n \le N+1$, that

$$\sum_{\substack{1\le m,j\le N,\\ \lambda_m \wedge \lambda_j = \lambda_n}} B(\xi_m, \xi_j) = \sum_{\substack{1\le m,j\le n-1,\\ \lambda_m \wedge \lambda_j = \lambda_n}} B(\xi_m, \xi_j).$$

Therefore, one has, for $1 \le n \le N$,

$$A\xi_n + \sum_{\substack{1\le m,j\le N,\\ \lambda_m \wedge \lambda_j = \lambda_n}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j) - \phi_n + \chi_n = 0,$$

and

$$- \sum_{\substack{1\le m,j\le N,\\ \lambda_m \wedge \lambda_j = \lambda_{N+1}}} d_{\lambda_m,\lambda_j} B(\xi_m, \xi_j) + \phi_{N+1} - \chi_{N+1} = A\xi_{N+1}.$$

These yield

$$v_N' = -Av_N + \psi_{\lambda_{N+1}}(t) A\xi_{N+1} + h_{N+1,4}(t). \tag{5.22}$$

(b) Estimation of $v_N(t)$. In Eq. (5.22), we have

$$|h_{N+1,4}(t)|_{\alpha,\sigma} \le |h_{N+1,1}|_{\alpha,\sigma} + |h_{N+1,2}|_{\alpha,\sigma} + |h_{N+1,3}|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_{N+1}}(t)\psi_\varepsilon(t)). \tag{5.23}$$

One has from (5.23) that

$$|\psi_{\lambda_{N+1}}(t) A\xi_{N+1} + h_{N+1,4}(t)|_{\alpha,\sigma} \le \psi_{\lambda_{N+1}}(t) |A\xi_{N+1}|_{\alpha,\sigma} + |h_{N+1,4}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\lambda_{N+1}}(t)).$$

Similar to part (c) of Step 1, we apply Theorem 3.2(iii) to the linearized NSE (5.22) with $w(t) = v_N(T_0 + t)$, $\xi = 0$, $f(t) = \psi_{\lambda_{N+1}}(T_0 + t)A\xi_{N+1} + h_{N+1,4}(T_0 + t)$, and $F(t) = \psi_{\lambda_{N+1}}(T_0 + t)$, where $T_0 \ge T_*$ is an appropriate, sufficient large time. We have from (3.6), for any $\rho \in (0, 1)$, that

$$|v_N(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(\psi_{\lambda_{N+1}}(t)). \tag{5.24}$$

(c) We will improve the precision of decay in (5.24). Define $w_{N+1}(t) = \psi_{\lambda_{N+1}}(t)^{-1} v_N(t)$ for $t \ge T_*$. We have

$$w_{N+1}' = \psi_{\lambda_{N+1}}^{-1}(t) v_N' + \psi_{\lambda_{N+1}}'(t) \psi_{\lambda_{N+1}}^{-2}(t) v_N,$$

which, thanks to (5.22), yields

$$w'_{N+1} = -Aw_{N+1} + A\xi_{N+1} + H_{N+1}(t), \tag{5.25}$$

where $H_{N+1}(t) = \psi^{-1}_{\lambda_{N+1}}(t)h_{N+1,4}(t) + \psi'_{\lambda_{N+1}}(t)\psi^{-2}_{\lambda_{N+1}}(t)v_N(t)$.

We estimate $H_{N+1}(t)$ next. By (5.23),

$$|\psi^{-1}_{\lambda_{N+1}}(t)h_{N+1,4}(t)|_{\alpha,\sigma} = \psi^{-1}_{\lambda_{N+1}}(t)\mathcal{O}(\psi_{\lambda_{N+1}}(t)\psi_\varepsilon(t)) = \mathcal{O}(\psi_\varepsilon(t)).$$

For the second term, we use (4.21) and (5.24) to obtain

$$|\psi'_{\lambda_{N+1}}(t)\psi^{-2}_{\lambda_{N+1}}(t)v_N(t)|_{\alpha,\sigma} = \psi^{-2}_{\lambda_{N+1}}(t)\mathcal{O}(\psi_{\lambda_{N+1}}(t)\psi_\varepsilon(t))\mathcal{O}(\psi_{\lambda_{N+1}}(t)) = \mathcal{O}(\psi_\varepsilon(t)).$$

Hence, there exists $\varepsilon_{N+1} > 0$ such that

$$|H_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(\psi_{\varepsilon_{N+1}}(t)).$$

(d) Note from (5.11) that $A\xi_{N+1} \in G_{\alpha,\sigma} \subset G_{\alpha-\frac{1}{2},\sigma}$. Again, by applying Theorem 3.2(iii) to equation (5.25) with $w(t) := w_{N+1}(T_1 + t)$, $\xi := A\xi_{N+1}$, $f(t) := H_{N+1}(t + T_1)$, $F(t) := \psi_{\varepsilon_{N+1}}(t + T_1)$ for some $T_1 \geq T_*$ sufficiently large, we obtain from (3.6), for any $\rho \in (0, 1)$, that

$$|w_{N+1}(T_1 + t) - A^{-1}(A\xi_{N+1})|_{\alpha+1-\rho,\sigma} \leq C\psi_{\varepsilon_{N+1}}(t + T_1) \quad \forall t \geq 1.$$

Thus, $|w_{N+1}(t) - \xi_{N+1}|_{\alpha+1-\rho,\sigma} = \mathcal{O}(\psi_{\varepsilon_{N+1}}(t))$. Multiplying this equation by $\psi_{\lambda_{N+1}}(t)$ yields

$$|v_N(t) - \xi_{N+1}\psi_{\lambda_{N+1}}(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(\psi_{\lambda_{N+1}}(t)\psi_{\varepsilon_{N+1}}(t)).$$

Since the left-hand side of this equation is $|v_{N+1}(t)|_{\alpha+1-\rho,\sigma}$, it proves that the statement (5.15) holds true for $N := N + 1$.

*Conclusion* By the induction principle, we have (5.15) holds true for all $N \in \mathbb{N}$. Our proof is complete. □

In Theorem 5.4, both force $f$ and solution $u$ have infinite sum expansions which means that they can be approximated by infinitely many terms $\psi_\lambda$'s as $\lambda \to \infty$. The case of finite sum approximations can be treated similarly. We briefly discuss the idea and result here.

**Assumption 5.5** Suppose there exist numbers $\sigma \geq 0$, $\alpha \geq 1/2$, an integer $N_0 \geq 1$, strictly increasing, positive numbers $\gamma_n$ and functions $\tilde{\phi}_n \in G_{\alpha,\sigma}$ for $1 \leq n \leq N_0$ such that

$$\left| f(t) - \sum_{n=1}^{N_0} \tilde{\phi}_n \psi_{\gamma_n}(t) \right|_{\alpha,\sigma} = \mathcal{O}(\psi_\lambda(t)) \text{ for some } \lambda > \gamma_{N_0}. \tag{5.26}$$

Assume further that there exists a set $S_\infty$ that contains $\{\gamma_n : 1 \leq n \leq N_0\}$ and preserves the operations $\vee$ and $\wedge$, so that the set $S_* \stackrel{\text{def}}{=} S_\infty \cap [\gamma_1, \gamma_{N_0}]$ is finite.

In applications, we often choose $S_\infty = \mathcal{G}_\Psi(\{\gamma_n : 1 \leq n \leq N_0\})$, but it can be more general than this.

We rewrite $S_* = \{\lambda_n : 1 \leq n \leq N_*\}$ for some integer $N_* \geq N_0$, where $\lambda_n$'s are strictly increasing. Note that $\lambda_{N_*} = \gamma_{N_0}$. Then from (5.26) we have

$$\left| f(t) - \sum_{n=1}^{N_*} \phi_n \psi_{\lambda_n}(t) \right|_{\alpha,\sigma} = \mathcal{O}(\psi_\lambda(t)) \text{ for some } \lambda > \lambda_{N_*}, \tag{5.27}$$

where $\phi_n \in G_{\alpha,\sigma}$ for all $1 \leq n \leq N_*$.

**Theorem 5.6** *Let Assumption 5.5 hold true, and let $f$ have the asymptotic approximation (5.27). Let $\xi_n$ be defined by (5.7) and (5.8) for $1 \le n \le N_*$. For any Leray–Hopf weak solution $u(t)$ of (2.2), it holds that*

$$\left| u(t) - \sum_{n=1}^{N_*} \xi_n \psi_{\lambda_n}(t) \right|_{\alpha,\sigma} = \mathcal{O}(\psi_\lambda(t)) \text{ for some } \lambda > \lambda_{N_*}.$$

***Proof*** The proof of Theorem 5.6 is the same as that of Theorem 5.4 except that we only use *finite* induction to establish (5.15) for $1 \le N \le N_*$. □

## 6 Asymptotic expansions in a discrete system with a continuum background

In this section, we investigate the case that the system of functions $(\psi_n)_{n=1}^\infty$ in Definition 4.1 cannot be mapped directly to a system $(\psi_{\lambda_n})_{n=1}^\infty$ to be embedded into a continuum system $(\psi_\lambda)_{\lambda>0}$. Hence, Definition 4.3 will not apply. However, we consider below the case when each $\psi_n$ is of the same decaying order, when $t \to \infty$, as $\varphi_{\lambda_n}$ with the functions $\varphi_{\lambda_n}$'s being part of a continuum system $(\varphi_\lambda)_{\lambda>0}$.

**Definition 6.1** Let $\Psi = (\psi_n)_{n=1}^\infty$ be a sequence of positive functions defined on $[T_*, \infty)$ for some $T_* \in \mathbb{R}$, and $\Phi = (\varphi_\lambda)_{\lambda>0}$ be a continuum system as in Definition 4.3 such that there exists a strictly increasing, divergent sequence $(\lambda_n)_{n=1}^\infty$ of positive numbers such that

$$\psi_n(t) \overset{\mathcal{O}}{=} \varphi_{\lambda_n}(t) \quad \text{for all } n \in \mathbb{N}. \tag{6.1}$$

Let $(X, \|\cdot\|)$ be a normed space, and $g$ be a function from $(0, \infty)$ to $X$. We define the asymptotic expansions

$$g(t) \underset{\Phi}{\sim} \sum_{n=1}^\infty \xi_n \psi_n(t), \quad g(t) \underset{\Phi}{\sim} \sum_{n=1}^N \xi_n \psi_n(t) \text{ with } N \in \mathbb{N}, \quad g(t) \underset{\Phi}{\sim} 0 \text{ in } X, \tag{6.2}$$

in the same way as Definition 4.3 and the special cases (i)–(iv) below it, where we replace $\psi_{\lambda_n}$ with $\psi_n$, replace (4.12) with

$$\left\| g(t) - \sum_{n=1}^N \xi_n \psi_n(t) \right\| = \mathcal{O}(\psi_N(t)\varphi_\varepsilon(t)), \tag{6.3}$$

replace (4.15) with

$$\left\| g(t) - \sum_{n=1}^N \xi_n \psi_n(t) \right\| = \mathcal{O}(\varphi_\lambda(t)), \tag{6.4}$$

replace (4.17) with

$$\|g(t)\| = \mathcal{O}(\varphi_\lambda(t)). \tag{6.5}$$

We refer to $\Phi$ as a *background system* of $(\psi_n)_{n=1}^\infty$. We will write expansions in (6.2) as

$$g(t) \underset{\Phi}{\sim} \sum_{n=1}^N \xi_n \psi_n(t) \text{ in } X, \ \text{ for } N = \infty, \ N \in \mathbb{N}, \ \text{ and } N = 0, \ \text{ respectively.}$$

The following remarks on Definition 6.1 are in order.

(a) It follows (6.1) immediately that property (4.1) holds true. Moreover, for any $n > m$, there exists $\eta > 0$ such that

$$\psi_n(t) = \mathcal{O}(\psi_m(t)\varphi_\eta(t)). \tag{6.6}$$

where $\eta$ is a number such that $\varphi_{\lambda_n}(t) = \mathcal{O}(\varphi_{\lambda_m}(t)\varphi_\eta(t))$, thanks to property (4.10) for the system $\Phi$. Thus, property (4.2) is also true. Therefore, Definition 4.1 for the asymptotic expansions $g(t) \sim \sum_{n=1}^{\infty} \xi_n \psi_n(t)$ in $X$ still applies.
Obviously, if $g(t) \underset{\Phi}{\sim} \sum_{n=1}^{N} \xi_n \psi_n(t)$ then $g(t) \sim \sum_{n=1}^{N} \xi_n \psi_n(t)$ in the sense of Definition 4.1. Then, thanks to Proposition 4.2, the uniqueness of the latter expansion implies the uniqueness of the former one.

(b) We can *equivalently* replace $\mathcal{O}(\psi_N(t)\varphi_\varepsilon(t))$ in (6.3) with $\mathcal{O}(\varphi_{\lambda_N}(t)\varphi_\varepsilon(t))$, or $\mathcal{O}(\varphi_\lambda(t))$ for some $\lambda > \lambda_N$.

(c) For a given sequence $(\psi_n)_{n=1}^{\infty}$, there may be different background systems. However, the asymptotic expansion of a function $g$ as defined in (6.2), thanks to remark (a), is unique disregarding the choice of the background system $\Phi$.

(d) Let $\Phi = (\varphi_\lambda)_{\lambda>0}$ and $\Theta = (\vartheta_\lambda)_{\lambda>0}$ be two systems as in Definition 4.3. If there exists a strictly increasing bijection $\mu$ from $(0,\infty)$ to $(0,\infty)$ such that $\varphi_\lambda(t) \overset{\mathcal{O}}{=} \vartheta_{\mu(\lambda)}(t)$ for all $\lambda > 0$, then

$$g(t) \underset{\Phi}{\sim} \sum_{n=1}^{N} \xi_n \psi_n(t) \quad \text{if and only if} \quad g(t) \underset{\Theta}{\sim} \sum_{n=1}^{N} \xi_n \psi_n(t).$$

(e) Let $\Psi = (\psi_\lambda)_{\lambda>0}$ and $\Phi = (\varphi_\lambda)_{\lambda>0}$ satisfy (a) and (b) of Definition 4.3. Suppose there exists a strictly increasing bijection $\mu$ from $(0,\infty)$ to $(0,\infty)$ such that

$$\psi_\lambda(t) \overset{\mathcal{O}}{=} \varphi_{\mu(\lambda)}(t) \text{ for all } \lambda > 0. \tag{6.7}$$

Let $X$, $(\xi_n)_{n=1}^{\infty}$ and $(\lambda_n)_{n=1}^{\infty}$ be as in Definition 4.3. Set $\tilde{\psi}_n = \psi_{\lambda_n}$ for all $n \in \mathbb{N}$. If $g$ is a function from $(0,\infty)$ to $X$, then

$$g(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t) \text{ if and only if } g(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \xi_n \tilde{\psi}_n(t). \tag{6.8}$$

For simplicity, we will write the last expansion as

$$g(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t). \tag{6.9}$$

While the functions $\psi_n$'s, with discrete index $n$, are the actual functions presented in the expansions in (6.2), the functions $\varphi_\lambda$, with the continuum index $\lambda$, provide specific rates in comparison (6.6) and remainder estimates (6.5), (6.4), (6.3). The fact that $\lambda$ has the range $(0,\infty)$ gives $\varphi_\lambda$ the flexibility in many comparisons and estimates, while the structure of the expansions is maintained by $\psi_n$'s. Note also that $\psi_n$'s are not required to be decreasing anymore.

**Assumption 6.2** For the rest of this section, we assume that $\Psi = (\psi_n)_{n=1}^{\infty}$ and $\Phi = (\varphi_\lambda)_{\lambda>0}$ are a pair of systems as in Definition 6.1 that further satisfy

(i) For any $m, n \in \mathbb{N}$, there exist a natural number $k > \max\{m, n\}$ and a nonzero constant $d_{m,n}$ such that

$$\psi_m \psi_n = d_{m,n} \psi_k. \tag{6.10}$$

(ii) For each $n \in \mathbb{N}$, $\psi_n$ is continuous and differentiable on $[T_*, \infty)$, and $\psi_n'$ has an expansion in the sense of Definition 6.1

$$\psi_n'(t) \underset{\Phi}{\sim} \sum_{k=n+1}^{N_n} c_{n,k}\psi_k(t) \text{ in } \mathbb{R},$$

where $N_n \in \mathbb{N} \cup \{0, \infty\}$, all $c_{n,k}$ are constants.

(iii) The system $\Phi = (\varphi_\lambda)_{\lambda>0}$ satisfies Condition 4.5.

**Notation** We denote the unique number $k$ in (6.10) by $m \wedge n$.

We obtain the asymptotic expansions of type (6.2) for the NSE.

**Theorem 6.3** *Suppose there exist $\alpha \geq 1/2$, $\sigma \geq 0$, and $\phi_n \in G_{\alpha,\sigma}$ for all $n \in \mathbb{N}$ such that*

$$f(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \phi_n \psi_n(t) \textit{ in } G_{\alpha,\sigma}.$$

*Then any Leray–Hopf weak solution $u(t)$ of (2.2) has the asymptotic expansion*

$$u(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_n(t) \quad \textit{in } G_{\alpha+1-\rho,\sigma} \textit{ for all } \rho \in (0,1), \tag{6.11}$$

*where*

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\left(\phi_n - \chi_n - \sum_{\substack{1\leq k,m\leq n-1,\\ k\wedge m=n}} d_{k,m} B(\xi_k, \xi_m)\right) \textit{ for } n \geq 2, \tag{6.12}$$

*with $\chi_n = \sum_{p=1}^{n-1} c_{p,n}\xi_p$.*

***Proof*** We follow the proof of Theorem 5.4 and make the following replacements:

- $\psi_{\lambda_n}$ is replaced with $\psi_n$ for all $n \in \mathbb{N}$, and
- $\psi_\sharp$ is replaced with $\varphi_\sharp$ whenever the subscript symbol $\sharp$ is $\delta_1, \varepsilon, \varepsilon_1, \eta_1, \varepsilon_N, \varepsilon_{N+1}, \delta_{N+1}$.

It results in the expansion (6.11) as desired. □

For finite sum asymptotic approximations in a discrete system, we obtain the following counter part of Theorem 5.6.

**Theorem 6.4** *Suppose there exist numbers $\sigma \geq 0$, $\alpha \geq 1/2$, $N_* \in \mathbb{N}$, and functions $\phi_n \in G_{\alpha,\sigma}$ for $1 \leq n \leq N_*$ such that*

$$\left| f(t) - \sum_{n=1}^{N_*} \phi_n \psi_n(t) \right|_{\alpha,\sigma} = \mathcal{O}(\varphi_\lambda(t)) \textit{ for some } \lambda > \lambda_{N_*}.$$

*Let $\xi_n$ be defined by (6.12) for $1 \leq n \leq N_*$. Then any Leray–Hopf weak solution $u(t)$ satisfies*

$$\left| u(t) - \sum_{n=1}^{N_*} \xi_n \psi_n(t) \right|_{\alpha,\sigma} = \mathcal{O}(\varphi_\lambda(t)) \textit{ for some } \lambda > \lambda_{N_*}.$$

***Proof*** The proof of Theorem 6.4 is the same as that of Theorem 5.6 with the use of replacements in the proof of Theorem 6.3. □

## 7 Applications

We will apply results in Sects. 5 and 6 to obtain specific expansions for solutions of the NSE corresponding to different types of forces. We focus on the infinite expansions, hence, show only applications of Theorems 5.4 and 6.3. Their counterparts using the finite asymptotic approximations in Theorems 5.6 and 6.4 can be similarly obtained. However, they will not be presented here, for the sake of avoiding repetitions and keeping the paper concise.

First, we discuss a very frequently used type of systems of functions for long-time asymptotic expansions.

**Definition 7.1** A *P-system* is a system $\Psi = (\psi_\lambda)_{\lambda>0}$, with $\psi_\lambda = \varphi^\lambda$, where $\varphi$ is a positive function defined on $[T_*, \infty)$ for some $T_* \ge 0$, and $\varphi(t) \to 0$ as $t \to \infty$.

**Property (P)** *Clearly, a P-system* $\Psi$ *satisfies (a) and (b) in Definition* 4.3 *with* $\eta = \lambda - \mu$, *and (i) in Condition* 4.4 *with* $d_{\lambda,\mu} = 1$ *and* $\gamma = \lambda \wedge \mu = \lambda + \mu$.

In this case, a set $S \subset (0, \infty)$ preserves the operation $\wedge$, see (4.29), if and only if it *preserves the addition*, i.e., $x + y \in S$ whenever $x, y \in S$.

In Sects. 7.1–7.4, let $\sigma \ge 0$ and $\alpha \ge 1/2$ be given numbers, $(\gamma_n)_{n=1}^\infty$ be a strictly increasing, divergent sequence of positive numbers, and $(\tilde{\phi}_n)_{n=1}^\infty$ be a sequence in $G_{\alpha,\sigma}$.

### 7.1 The system of power-decaying functions

We quickly demonstrate how to apply Theorem 5.4 to recover one of the main theorems in [1] on the expansions in the system of power-decaying functions.

Let $\Psi = (t^{-\lambda})_{\lambda>0}$ which is a P-system.

(i) By Property (P), $\Psi$ satisfies (a) and (b) of Definition 4.3.
(ii) By Property (P), $\Psi$ satisfies (i) of Condition 4.4. In addition, it satisfies (ii) of Condition 4.4 with

$$N_\lambda = 1, \quad c_{\lambda,1} = -\lambda, \quad \lambda^\vee(1) = \lambda + 1 \text{ for all } \lambda > 0. \tag{7.1}$$

Thus, $\Psi$ meets Condition 4.4.
(iii) Elementary calculations show $\Psi$ meets Condition 4.5.

Therefore, $\Psi$ satisfies the conditions set from the beginning of Sect. 5.

Note from (7.1) that a set $S \subset (0, \infty)$ preserves the operation $\vee$, see (4.28), if and only if it *preserves the increments by* 1, i.e., $x + 1 \in S$ whenever $x \in S$.

We assume the force has an expansion

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \tilde{\phi}_n t^{-\gamma_n} \quad \text{in } G_{\alpha,\sigma}. \tag{7.2}$$

Let

$$S_* = \left\{ \sum_{j=1}^p \gamma_{n_j} + k : \ p, n_1, n_2, \ldots, n_p \in \mathbb{N}, \ k \in \mathbb{N} \cup \{0\} \right\}. \tag{7.3}$$

Clearly, the set $S_*$ in (7.3) satisfies Assumption 5.2. We assume (5.4), and rewrite (7.2) as

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \phi_n t^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

where the sequence $(\phi_n)_{n=1}^\infty \subset G_{\alpha,\sigma}$ is defined as in (5.5). Then Theorem 5.4 implies that any Leray–Hopf weak solution $u(t)$ of the NSE (2.2) has the asymptotic expansion

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n t^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

where

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\left(\phi_n + \chi_n - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k,\xi_m)\right) \quad \text{for } n \ge 2,$$

with $\chi_n = \lambda_p \xi_p$ if there exists an integer $p \in [1, n-1]$ such that $\lambda_p + 1 = \lambda_n$, and $\chi_n = 0$ otherwise.

We have recovered Theorem 4.3 in [1] as a consequence of Theorem 5.4.

### 7.2 Systems of iterated logarithmic, decaying functions

We consider the case when the force decays as logarithmic or iterated logarithmic functions.

For $k, m \in \mathbb{N}$, let

$$L_k(t) = \underbrace{\ln(\ln(\cdots \ln(t)))}_{k\text{-times}} \quad \text{and} \quad \mathcal{L}_m(t) = (L_1(t), L_2(t), \ldots, L_m(t)).$$

Let $Q_0 : \mathbb{R}^m \to \mathbb{R}$ be a polynomial in $m$ variables:

$$Q_0(z) = \sum_{\alpha} c_\alpha z^\alpha \text{ for } z \in \mathbb{R}^m, \tag{7.4}$$

where the sum is taken over finitely many multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, and $c_\alpha$'s are (real) constants. We use the lexicographic order for the multi-indices in (7.4).

We assume that $Q_0(z)$ has positive degree and positive leading coefficient. Denote by $\alpha_* = (\alpha_{*1}, \alpha_{*2}, \ldots, \alpha_{*m})$ the largest multi-index (with the lexicographic order) in (7.4) for which $c_{\alpha_*} \neq 0$. Then we have $|\alpha_*| \ge 1$ and $c_{\alpha_*} > 0$.

Let $Q_1$ be a polynomial in one variable of positive degree with positive leading coefficient. Denote the degree of $Q_1$ by $d \ge 1$, and the leading coefficient by $a_d > 0$.

Given a number $\beta > 0$, we define

$$\omega(t) = (Q_0 \circ \mathcal{L}_m \circ Q_1)(t^\beta)) \text{ with } t \in \mathbb{R}. \tag{7.5}$$

One can see that there exists $T_* > 0$ such that $\omega$ is a positive function defined on $[T_*, \infty)$ and $\omega(t) \to \infty$ as $t \to \infty$.

Let $\psi_\lambda(t) = \omega(t)^{-\lambda}$ for $\lambda > 0$, and let $\Psi$ be the P-system $(\psi_\lambda)_{\lambda>0}$.

**Lemma 7.2** *If* $\lambda > 0$, *then*

$$\lim_{t\to\infty} \psi_\lambda(t)\mathcal{L}_m(t)^{\lambda\alpha_*} = (c_{\alpha_*}(\beta d)^{\alpha_{*1}})^{-\lambda}. \tag{7.6}$$

***Proof*** First, if $k < j$, then $L_j(t) = o(L_k(t))$. With the lexicographic order, we have

$$\lim_{t\to\infty} \frac{(Q_0 \circ \mathcal{L}_m)(t)}{c_{\alpha_*}\mathcal{L}_m(t)^{\alpha_*}} = 1, \text{ which implies } \lim_{t\to\infty} \frac{\omega(t)}{\mathcal{L}_m(Q_1(t^\beta))^{\alpha_*}} = c_{\alpha_*} > 0.$$

Moreover,

$$\lim_{t\to\infty} \frac{\mathcal{L}_m(Q_1(t^\beta))^{\alpha_*}}{\mathcal{L}_m(a_d t^{\beta d})^{\alpha_*}} = 1.$$

By the properties of the logarithmic function, one has, for any $a, r > 0$, that

$$\lim_{t\to\infty} \frac{L_k(at^r)}{L_k(t)} = \begin{cases} r, & \text{for } k = 1, \\ 1, & \text{for } k > 1. \end{cases} \tag{7.7}$$

Combining these gives

$$\lim_{t\to\infty} \frac{\omega(t)}{\mathcal{L}_m(t)^{\alpha_*}}$$
$$= \lim_{t\to\infty} \frac{\omega(t)}{\mathcal{L}_m(Q_1(t^\beta))^{\alpha_*}} \cdot \lim_{t\to\infty} \frac{\mathcal{L}_m(Q_1(t^\beta))^{\alpha_*}}{\mathcal{L}_m(a_d t^{\beta d})^{\alpha_*}} \cdot \lim_{t\to\infty} \frac{\mathcal{L}_m(a_d t^{\beta d})^{\alpha_*}}{\mathcal{L}_m(t)^{\alpha_*}} = c_{\alpha_*}(\beta d)^{\alpha_{*1}}.$$

Thus, (7.6) follows. □

As a consequence of (7.6), we have

$$\psi_\lambda(t) \overset{\mathcal{O}}{=} \mathcal{L}_m(t)^{-\lambda\alpha_*}. \tag{7.8}$$

In particular, if $\alpha_* = p_0 e_k$ for some $p_0 \in \mathbb{N}$, where $e_k$ is the $k$-th unit vector of the canonical basis of $\mathbb{R}^m$, then

$$\psi_\lambda(t) \overset{\mathcal{O}}{=} L_k(t)^{-p_0\lambda}. \tag{7.9}$$

We verify Conditions 4.4 and 4.5 for the P-system $\Psi$.

*Verification of Condition* 4.4 Because of Property (P) for $\Psi$, we only need to check (ii) of Condition 4.4. By Chain Rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{L}_m(t)^\alpha) = \sum_{k=1}^m \alpha_k \mathcal{L}_m(t)^{\alpha-e_k} \frac{\mathrm{d}}{\mathrm{d}t} L_k(t) = \sum_{k=1}^m \alpha_k \mathcal{L}_m(t)^{\alpha-e_k} \frac{1}{t\prod_{p=1}^{k-1} L_p(t)}$$
$$= \frac{1}{t}\sum_{k=1}^m \alpha_k \mathcal{L}_m(t)^{\alpha-e_1-e_2-\cdots e_k}.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{L}_m(Q_1(t^\beta))^\alpha) = \frac{\beta t^{\beta-1} Q_1'(t^\beta)}{Q_1(t^\beta)} \sum_{k=1}^m \alpha_k \mathcal{L}_m(Q_1(t^\beta))^{\alpha-e_1-e_2-\cdots-e_k}.$$

We estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{L}_m(Q_1(t^\beta))^\alpha)\right| = \mathcal{O}(t^{-1}\mathcal{L}_m(Q_1(t^\beta))^{\alpha_*}) = \mathcal{O}(t^{-1}\mathcal{L}_m(t))^{\alpha_*}).$$

Hence,

$$|\omega'(t)| = \mathcal{O}(t^{-1}\mathcal{L}_m(t))^{\alpha_*}).$$

Since $\psi_\lambda'(t) = -\lambda\omega(t)^{-\lambda-1}\omega'(t)$, we combine these with Lemma 7.2 to obtain

$$|\psi_\lambda'(t)| = \mathcal{O}\left(\frac{\mathcal{L}_m(t)^{\alpha_*}}{t\cdot\mathcal{L}_m(t)^{(1+\lambda)\alpha_*}}\right) = \mathcal{O}(t^{-1}).$$

This implies

$$|\psi_\lambda'(t)| = \mathcal{O}\left(\mathcal{L}_m(t)^{-\mu\alpha_*}\right) = \mathcal{O}(\psi_\mu(t)), \quad \forall \mu > 0.$$

Therefore, by definition,

$$\psi_\lambda'(t) \overset{\Psi}{\sim} 0 \text{ for all } \lambda > 0. \tag{7.10}$$

Thus, $\Psi$ satisfies (ii) of Condition 4.4 with $N_\lambda = 0$ for all $\lambda > 0$.

*Verification of Condition* 4.5 Thanks to (7.6) and (7.7), the requirements (ii) and (iii) are met. For (i), using the above calculations we find

$$\omega'(t) = \frac{\beta t^{\beta-1} Q_1'(t^\beta)}{Q_1(t^\beta)} \sum_{\alpha=(\alpha_1,\alpha_2,\dots,\alpha_m)} c_\alpha \sum_{k=1}^m \alpha_k \mathcal{L}_m(Q_1(t^\beta))^{\alpha-e_1-e_2-\cdots e_k}.$$

Let $\gamma_*$ be the largest multi-index among $\alpha - e_1 - e_2 - \cdots - e_k$ with nonzero $c_\alpha\alpha_k$.

Then $\gamma_* = \alpha_* - e_1 - e_2 - \cdots - e_k$ where $k$ is the smallest index for which the component $\alpha_{*k} \geq 1$. Hence the corresponding coefficient is $c_{\alpha_*}\alpha_{*k} > 0$. Note also that $Q_1'(t^\beta) > 0$ for large $t$. We conclude, for sufficiently large $t$, that $\omega'(t) > 0$, and hence $\psi_\lambda'(t) < 0$ .

Now, we assume the force has the following expansion

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \tilde{\phi}_n \omega(t)^{-\gamma_n} = \sum_{n=1}^\infty \tilde{\phi}_n \psi_{\gamma_n}(t) \text{ in } G_{\alpha,\sigma}.$$

Let $S_* = \{\sum_{j=1}^p \gamma_{n_j} : p, n_1, n_2, \dots, n_p \in \mathbb{N}\}$. Then, again, this set $S_*$ satisfies Assumption 5.2; hence, we can assume (5.4) and the expansion (5.5), which reads as

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \phi_n \omega(t)^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma}, \tag{7.11}$$

for a sequence $(\phi_n)_{n=1}^\infty$ in $G_{\alpha,\sigma}$.

**Theorem 7.3** *Let $\omega$ be defined by* (7.5) *and assume the expansion* (7.11).

(i) *Then any Leray–Hopf weak solution $u(t)$ of the NSE* (2.2) *has the asymptotic expansion*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \xi_n \omega(t)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1), \tag{7.12}$$

*where*

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\left(\phi_n - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k,\xi_m)\right) \quad \text{for } n \geq 2. \tag{7.13}$$

(ii) *By defining $\varphi_\lambda(t) = \mathcal{L}_m(t)^{-\lambda\alpha_*}$ and $\Phi = (\varphi_\lambda)_{\lambda>0}$, we can* equivalently *replace* $\overset{\Psi}{\sim}$ *with* $\underset{\Phi}{\sim}$ *in* (7.11) *and* (7.12)*, in the sense of* (6.9).
*In particular, if $\alpha_*$ is co-linear with the $k$-th unit vector $e_k$ of the canonical basis of $\mathbb{R}^m$, then this replacement still holds true for $\Phi = (L_k(t)^{-\lambda})_{\lambda>0}$.*

***Proof*** (i) Applying Theorem 5.4 while noting that $\chi_n = 0$ in (5.9) for all $n \in \mathbb{N}$ due to the fact (7.10), we deduce (7.12) from (5.6).

(ii) Let $\varphi_\lambda(t) = \mathcal{L}_m(t)^{-\lambda\alpha_*}$ and $\Phi = (\varphi_\lambda)_{\lambda>0}$. Note by (7.8) that $\psi_\lambda(t) \overset{\mathcal{O}}{=} \varphi_\lambda(t)$ which implies (6.7) with $\mu(\lambda) = \lambda$. Then the replacement is valid thanks to (6.8) and (6.9) in remark (e) after Definition 6.1.
Now, consider the case when $\alpha_* = p_0 e_k$ for some $p_0 \in \mathbb{N}$. Let $\varphi_\lambda(t) = L_k(t)^{-\lambda}$ and, again, $\Phi = (\varphi_\lambda)_{\lambda>0}$. By (7.9), we have

$$\psi_\lambda(t) \overset{\mathcal{O}}{=} \varphi_{p_0\lambda}(t) = \varphi_{\mu(\lambda)}(t), \quad \text{where } \mu(\lambda) = p_0\lambda.$$

Hence, the replacement is valid again by the same (6.8) and (6.9). □

**Corollary 7.4** *Given $m \in \mathbb{N}$, define $\Psi = (L_m(t)^{-\lambda})_{\lambda>0}$. Suppose $(\lambda_n)_{n=1}^\infty$ is a strictly increasing, divergent sequence of positive numbers such that the set $\{\lambda_n : n \in \mathbb{N}\}$ preserves the addition. If*

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \phi_n L_m(t)^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

*then any Leray–Hopf weak solution $u(t)$ of the NSE (2.2) admits the same asymptotic expansion*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \xi_n L_m(t)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

*where $\xi_n$'s are defined by (7.13).*

***Proof*** We choose $Q_0(z_1, z_2, \ldots, z_m) = z_m$, $Q_1(t) = t$ and $\beta = 1$. Notice, in this case, that (7.5) becomes $\omega(t) = L_m(t)$. Then the result in this corollary follows Theorem 7.3(i) and Remark 5.3. □

***Example 7.5*** Thanks to Theorem 7.3, we can have asymptotic expansions of many different types. We illustrate it with just two here. Let $(\lambda_n)_{n=1}^\infty$ be a strictly increasing, divergent sequence of positive numbers that preserves the addition. By Remark 5.3, we can use, from the beginning, expansion (7.11) for the force. We will also use the replacements indicated in Theorem 7.3(ii).

(a) When $m = 5$, $Q_0(z_1, z_2, \ldots, z_5) = 3z_1^2 z_3 - 2z_2 z_5^4$, $Q_1(t) = t$, and $\beta = 1$, if

$$f(t) \underset{\Phi}{\sim} \sum_{n=1}^\infty \phi_n \big[3(\ln t)^2 L_3(t) - 2L_2(t) L_5(t)^4\big]^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

where $\Phi = \big((\ln t)^{-2\lambda} L_3(t)^{-\lambda}\big)_{\lambda>0}$, then

$$u(t) \underset{\Phi}{\sim} \sum_{n=1}^\infty \xi_n \big[3(\ln t)^2 L_3(t) - 2L_2(t) L_5(t)^4\big]^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1).$$

(b) When $m = 7$, $Q_0(z_1, z_2, \ldots, z_7) = 4z_2 - z_7^5 + 3$, $Q_1(t) = t^3 - 3t + 1$, and $\beta = 1/2$, if

$$f(t) \underset{\Phi}{\sim} \sum_{n=1}^\infty \phi_n \big[4L_2(t^{3/2} - 3t^{1/2} + 1) - L_7(t^{3/2} - 3t^{1/2} + 1)^5 + 3\big]^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

where $\Phi = (L_2(t)^{-\lambda})_{\lambda>0}$, then

$$u(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \xi_n \big[4L_2(t^{3/2} - 3t^{1/2} + 1) - L_7(t^{3/2} - 3t^{1/2} + 1)^5 + 3\big]^{-\lambda_n}$$

in $G_{\alpha+1-\rho,\sigma}$ for all $\rho \in (0, 1)$.

***Example 7.6*** Given $m \in \mathbb{N}$. Consider the system

$$\Psi = (\psi_\lambda)_{\lambda>0} = \left(\left[\sin(L_m^{-1}(t))\right]^\lambda\right)_{\lambda>0}.$$

Clearly, $\sin(L_m^{-1}(t))$ is a positive function defined on $[T_*, \infty)$ for some $T_* > 0$ and $\sin(L_m^{-1}(t)) \to 0$ as $t \to \infty$. Thus $\Psi$ is a P-system.

Since $\sin^\lambda(x)$ is increasing in $x$ on $(0, \pi/2)$, it implies that $\left[\sin(L_m^{-1}(t))\right]^\lambda$ is decreasing on $[T_1, \infty)$ for some sufficiently large number $T_1$. Noting that

$$\frac{2x}{\pi} \le \sin(x) \le x, \quad \text{for } 0 \le x \le \pi/2,$$

and $0 < L_m^{-1}(t) \le 1$ for large $t$, one has $\sin(L_m^{-1}(t)) \overset{\mathcal{O}}{=} L_m^{-1}(t)$.

This and Lemma 7.2 yield

$$\left[\sin(L_m^{-1}(at))\right]^\lambda \overset{\mathcal{O}}{=} \left[L_m^{-1}(at)\right]^\lambda \overset{\mathcal{O}}{=} \left[L_m^{-1}(t)\right]^\lambda \overset{\mathcal{O}}{=} \left[\sin(L_m^{-1}(t))\right]^\lambda.$$

Clearly,

$$\lim_{t\to\infty} \frac{e^{-\alpha t}}{\psi_\lambda(t)} = \lim_{t\to\infty} e^{-\alpha t} L_m(t)^\lambda = 0.$$

Therefore, $\Psi$ satisfies Condition 4.5. We write

$$\psi_\lambda'(t) = -\lambda \left(\sin(L_m^{-1}(t))\right)^{\lambda-1} \cos(L_m^{-1}(t)) \frac{1}{t\left(\Pi_{p=1}^{m-1} L_p(t)\right) L_m(t)^2},$$

and estimate

$$|\psi_\lambda'(t)| = \mathcal{O}\left(t^{-1}(\sin(L_m^{-1}(t))^{\lambda-1}\right) = \mathcal{O}\left(t^{-1} L_m(t)^{-\lambda+1}\right) = \mathcal{O}\left(L_m(t)^{-\mu}\right) = \mathcal{O}(\psi_\mu(t))$$

for all $\mu > 0$. Then $\Psi$ satisfies (ii) of Condition 4.4 with $N_\lambda = 0$ for all $\lambda > 0$, and thus, all parts of Condition 4.4 due to Property (P). Let $(\lambda_n)_{n=1}^\infty$ be as in Example 7.5. By Theorem 5.4, if

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n \left[\sin(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \left[\sin(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0, 1),$$

with $\xi_n$'s being (7.13).

***Example 7.7*** Given $m \in \mathbb{N}$. Consider the system

$$\Psi = (\psi_\lambda)_{\lambda>0} = \left(\left[\tan(L_m^{-1}(t))\right]^\lambda\right)_{\lambda>0}.$$

Similar to Example 7.6, using the fact that

$$x \le \tan(x) \le 2x, \quad \text{for } 0 \le x \le 1/2,$$

one can verify that $\Psi$ satisfies Condition 4.5 and Condition 4.4 with $N_\lambda = 0$ for all $\lambda > 0$.

Again, let $(\lambda_n)_{n=1}^\infty$ be as in Example 7.5. We obtain that if

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \phi_n \left[\tan(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \xi_n \left[\tan(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

with $\xi_n$'s from (7.13).

### 7.3 A system with infinite expansions for the derivatives

In the previous two subsections, the expansion (4.19) is zero or a finite sum. In this subsection, we demonstrate the case when each expansion (4.19) is an infinite sum.

Consider a particular P-system $\Psi = (\psi_\lambda)_{\lambda>0}$ with $\psi_\lambda = (\sqrt{t}+1)^{-\lambda}$. Let $\lambda > 0$. We see, for any $t > 0$, that

$$\begin{aligned}\psi_\lambda'(t) &= -\lambda(\sqrt{t}+1)^{-\lambda-1}\frac{1}{2}\frac{1}{\sqrt{t}} = -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-1}\frac{1}{\sqrt{t}+1}\cdot\frac{1}{1-\frac{1}{\sqrt{t}+1}}\\ &= -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-1}\sum_{k=1}^\infty \frac{1}{(\sqrt{t}+1)^k} = \sum_{k=1}^\infty -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-k-1}.\end{aligned}$$

Applying Lemma A.1 to $X = \mathbb{R}$, $\varphi(t) = (\sqrt{t}+1)^{-1}$, $\lambda_n = \lambda+n+1$, $M = 2$, $\xi_n = -\lambda/2$, $c_0 = \lambda/2$, $\kappa = 1$, we deduce that the derivative $\psi_\lambda'(t)$, in fact, has the expansion

$$\psi_\lambda'(t) \overset{\Psi}{\sim} \sum_{k=1}^\infty -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-k-1}.$$

Thus, we have expansion (4.19) with

$$N_\lambda = \infty, \text{ and } \quad c_{\lambda,k} = -\lambda/2, \quad \lambda^\vee(k) = \lambda + 1 + k \text{ for all } k \in \mathbb{N}.$$

With this, Property (P), and some elementary estimates, we can verify that Conditions 4.4 and 4.5 are met.

We assume the force has the expansion

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \tilde{\phi}_n \psi_{\gamma_n}(t) = \sum_{n=1}^\infty \tilde{\phi}_n (\sqrt{t}+1)^{-\gamma_n} \text{ in } G_{\alpha,\sigma}.$$

Define

$$S_* = \Big\{(\sum_{j=1}^{p} \gamma_{n_j}) + k : p, n_1, n_2, \ldots, n_p \in \mathbb{N},\ k = 0 \text{ or } k \in \mathbb{N} \cap [2, \infty)\Big\}.$$

Then the set $S_*$ satisfies Assumption 5.2, and we can assume (5.4) and the expansion

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n (\sqrt{t} + 1)^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma}, \tag{7.14}$$

for a sequence $(\phi_n)_{n=1}^{\infty}$ in $G_{\alpha,\sigma}$. We apply Theorem 5.4 and obtain the following.

**Proposition 7.8** *Assume* (7.14)*. Then any Leray–Hopf weak solution $u(t)$ of the NSE* (2.2) *admits the asymptotic expansion*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n (\sqrt{t} + 1)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0, 1),$$

*where*

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\left(\phi_n + \frac{1}{2}\sum_{p \in \mathcal{Z}_n} \lambda_p \xi_p - \sum_{\substack{1 \le k,m \le n-1,\\ \lambda_k + \lambda_m = \lambda_n}} B(\xi_k, \xi_m)\right) \quad \text{for } n \ge 2,$$

*with* $\mathcal{Z}_n = \{p \in \mathbb{N} \cap [1, n-1] : \exists k \in \mathbb{N}, \lambda_p + 1 + k = \lambda_n\}$.

### 7.4 Expansions using a background system

In this subsection, we present a scenario for which the use of the background systems in Sect. 6 is essential. To motivate our more general force $f$ later, we consider a simple case first. Let $\gamma \in (0, 1)$, $\beta_0 > 0$, and

$$f(t) = \frac{\phi}{(t^\gamma + 1)^{\beta_0}} \quad \text{with } \phi \in G_{\alpha,\sigma}.$$

We expect a solution $u(t)$ of the NSE (2.2) to have an asymptotic expansion containing at least $(t^\gamma + 1)^{-\beta}$ for some $\beta > 0$. (Here, the structure of $f(t)$ is maintained without being converted to a different form such as (7.2).) The derivative term $u_t$ in the NSE will contain $\frac{\mathrm{d}}{\mathrm{d}t}(t^\gamma + 1)^{-\beta}$, which is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(t^\gamma + 1)^{-\beta} &= \frac{-\gamma\beta}{(t^\gamma + 1)^{\beta+1} t^{1-\gamma}} = \frac{-\gamma\beta}{(t^\gamma + 1)^{\beta+1}(t^{1-\gamma} + 1)(1 - \frac{1}{t^{1-\gamma}+1})}\\ &= \frac{-\gamma\beta}{(t^\gamma + 1)^{\beta+1}} \sum_{k=1}^{\infty} \frac{1}{(t^{1-\gamma} + 1)^k},\end{aligned}$$

thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^\gamma + 1)^{-\beta} = \sum_{k=1}^{\infty} \frac{-\gamma\beta}{(t^\gamma + 1)^{\beta+1}(t^{1-\gamma} + 1)^k}. \tag{7.15}$$

Thanks to the term $Au$ in the NSE, (7.15) in turn suggests that a possible asymptotic expansion of $u(t)$ may have to include infinitely many terms $(t^\gamma + 1)^{-\lambda}(t^{1-\gamma} + 1)^{-\mu}$.

Because of this, we now consider a function $\psi(t) = (t^\gamma + 1)^{-\lambda}(t^{1-\gamma} + 1)^{-\mu}$. Taking the derivative by the product rule gives

$$\psi'(t) = \Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(t^\gamma+1)^\lambda}\Big)\frac{1}{(t^{1-\gamma}+1)^\mu} + \frac{1}{(t^\gamma+1)^\lambda}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(t^{1-\gamma}+1)^\mu}\Big).$$

Using (7.15) with $\beta := \lambda$ for the first derivative, and with $\gamma := 1-\gamma$, $\beta := \mu$ for second derivative, we obtain

$$\psi'(t) = \sum_{k=1}^\infty \frac{-\gamma\lambda}{(t^\gamma+1)^{\lambda+1}(t^{1-\gamma}+1)^{\mu+k}} + \sum_{k=1}^\infty \frac{-(1-\gamma)\mu}{(t^{1-\gamma}+1)^{\mu+1}(t^\gamma+1)^{\lambda+k}}. \tag{7.16}$$

Observe that the sums in (7.16) involve the functions of *the same form* as $\psi$, but with *different* powers. Also, the equality can be converted, under proper conditions, to an asymptotic expansion with the background system $\Phi = (t^{-\lambda})_{\lambda>0}$.

*Fixing a background system* Let us fix the P-system $\Phi = (\varphi_\lambda)_{\lambda>0}$, where $\varphi_\lambda(t) = t^{-\lambda}$. By (i)–(iii) in Sect. 7.1, we see that $\Phi$ satisfies Condition (iii) of Assumption 6.2.

From the above observation, we consider a force $f$ having the following general expansion

$$f(t) \underset{\Phi}{\sim} \sum_{n=1}^\infty \tilde\phi_n \tilde\psi_n(t) = \sum_{n=1}^\infty \frac{\tilde\phi_n}{(t^\gamma+1)^{\tilde\alpha_n}(t^{1-\gamma}+1)^{\tilde\beta_n}} \quad \text{in } G_{\alpha,\sigma}, \tag{7.17}$$

where $\tilde\psi_n(t) = (t^\gamma+1)^{-\tilde\alpha_n}(t^{1-\gamma}+1)^{-\tilde\beta_n}$, $\gamma$ is a constant in the interval $(0, 1)$, $(\tilde\alpha_n)_{n=1}^\infty$ and $(\tilde\beta_n)_{n=1}^\infty$ are sequences of non-negative numbers such that

$$\tilde\lambda_n \overset{\text{def}}{=} \gamma\tilde\alpha_n + (1-\gamma)\tilde\beta_n \text{ is positive, strictly increasing (in } n\text{) to infinity.} \tag{7.18}$$

Note that the expansion in (7.17) is understood in the sense of Definition 6.1 with $\tilde\Psi = (\tilde\psi_n)_{n=1}^\infty$ replacing $\Psi$, and $\tilde\lambda_n$ replacing $\lambda_n$.

A simple example of (7.17) is a finite sum $f(t) = \sum_{n=1}^N \tilde\phi_n\tilde\psi_n(t)$ for some $N \in \mathbb{N}$. For more complicated cases of infinite sums, see Corollary A.3.

**Assumption 7.9** The number $\gamma$ is irrational, while numbers $\tilde\alpha_n$ and $\tilde\beta_n$ are rational for all $n \in \mathbb{N}$.

Define the sets

$$\begin{aligned} S_* = \Bigg\{\gamma\Bigg(\sum_{j=1}^p \tilde\alpha_{n_j} + k\Bigg) + (1-\gamma)\Bigg(\sum_{j=1}^p \tilde\beta_{n_j} + \ell\Bigg) :& \\ p, n_1, n_2, \ldots, n_p \in \mathbb{N},\ (k,\ell) \in \mathbb{N}^2 \cup (0,0)\Bigg\}&, \end{aligned} \tag{7.19}$$

$$E_1 = \Bigg\{\sum_{j=1}^p \tilde\alpha_{n_j} + k : p, n_1, n_2, \ldots, n_p \in \mathbb{N},\ k \in \mathbb{N}\cup\{0\}\Bigg\},$$

$$E_2 = \Bigg\{\sum_{j=1}^q \tilde\beta_{k_j} + \ell : q, k_1, k_2, \ldots, k_q \in \mathbb{N},\ \ell \in \mathbb{N}\cup\{0\}\Bigg\}.$$

Note that

$$\{\tilde\lambda_n : n \in \mathbb{N}\} \subset S_* \subset E^* \overset{\text{def}}{=} \{\gamma\alpha + (1-\gamma)\beta : \alpha \in E_1,\ \beta \in E_2\}. \tag{7.20}$$

We see that $S_*$ and $E^*$ preserve the addition.

**Lemma 7.10** *For each $\mu \in E^*$, there exists a unique pair $(\alpha, \beta) \in E_1 \times E_2$ such that*

$$\mu = \gamma\alpha + (1-\gamma)\beta. \tag{7.21}$$

***Proof*** The existence of the decomposition (7.21) comes directly from (7.20). We prove the uniqueness now. Let $\mu \in E^*$ and suppose there are $(\alpha, \beta), (\alpha', \beta') \in E_1 \times E_2$ such that

$$\mu = \gamma\alpha + (1-\gamma)\beta = \gamma\alpha' + (1-\gamma)\beta'.$$

Then $\gamma(\alpha - \alpha') = -(1-\gamma)(\beta - \beta')$. Note from this relation that $\alpha = \alpha'$ if and only if $\beta = \beta'$.

Consider the case $\alpha \neq \alpha'$ and $\beta \neq \beta'$. Then

$$\frac{\gamma}{1-\gamma} = -\frac{\beta - \beta'}{\alpha - \alpha'}. \tag{7.22}$$

We have $\alpha - \alpha' = \sum_{j=1}^{p} \pm\tilde{\alpha}_{n_j} + k \neq 0$, and $\beta - \beta' = \sum_{j=1}^{q} \pm\tilde{\beta}_{k_j} + \ell \neq 0$, for some $k, \ell \in \mathbb{Z}$, and some particular $+/-$ signs in the two sums.

Since $\gamma$ is irrational, so is $\gamma/(1-\gamma)$, while the right-hand side of (7.22) is rational, which yields a contradiction. Therefore, we can only have $\alpha = \alpha'$ and $\beta = \beta'$. □

We rewrite $S_*$ as

$$S_* = \left\{ \sum_{j=1}^{p} \tilde{\lambda}_{n_j} + \gamma k + (1-\gamma)\ell : p, n_1, n_2, \ldots, n_p \in \mathbb{N},\ (k, \ell) \in \mathbb{N}^2 \cup (0,0) \right\}. \tag{7.23}$$

Since $\tilde{\lambda}_n \to \infty$, it follows (7.23) that we can order $S_*$ to be a sequence $(\lambda_n)_{n=1}^{\infty}$ as in (5.4).

Note that $\lambda_n \to \infty$, hence consequently, $\alpha_n + \beta_n \to \infty$.

*The discrete system for expansions* Let $\Psi = (\psi_n)_{n=1}^{\infty}$, where

$$\psi_n(t) = (t^{\gamma} + 1)^{-\alpha_n}(t^{1-\gamma} + 1)^{-\beta_n},$$

with $(\alpha_n, \beta_n)$, thanks to Lemma 7.10, being the unique pair in $E_1 \times E_2$ such that

$$\lambda_n = \gamma\alpha_n + (1-\gamma)\beta_n. \tag{7.24}$$

Clearly, $\psi_n(t) \overset{\mathcal{O}}{=} t^{-\lambda_n} = \varphi_{\lambda_n}(t)$. Hence, $\Psi$ and $\Phi$ satisfy condition (6.1) in Definition 6.1.

We still need to verify the remaining Conditions (i) and (ii) of Assumption 6.2.

*Verification of Condition* (i) For $m, n \in \mathbb{N}$, we have

$$\psi_m(t) = (t^{\gamma} + 1)^{-\alpha_m}(t^{1-\gamma} + 1)^{-\beta_m}, \quad \lambda_m = \gamma\alpha_m + (1-\gamma)\beta_m \in S_*, \tag{7.25}$$

$$\psi_n(t) = (t^{\gamma} + 1)^{-\alpha_n}(t^{1-\gamma} + 1)^{-\beta_n}, \quad \lambda_n = \gamma\alpha_n + (1-\gamma)\beta_n \in S_*, \tag{7.26}$$

where $\alpha_m, \alpha_n \in E_1$ and $\beta_m, \beta_n \in E_2$. Then

$$(\psi_m\psi_n)(t) = (t^{\gamma} + 1)^{-\alpha_m - \alpha_m}(t^{1-\gamma} + 1)^{-\beta_n - \beta_n}.$$

Since $S_*$ preserves the addition we have $\lambda_m + \lambda_n \in S_*$, hence there exists $k$ such that

$$\lambda_k = \lambda_m + \lambda_n. \tag{7.27}$$

By (7.27), (7.25) and (7.26), we have $\lambda_k = \gamma(\alpha_m + \alpha_n) + (1-\gamma)(\beta_m + \beta_n)$. Because $\alpha_m + \alpha_n \in E_1$, $\beta_m + \beta_n \in E_2$, and by the uniqueness of the decomposition of $\lambda_k$, we deduce $\alpha_m + \alpha_n = \alpha_k$ and $\beta_n + \beta_m = \beta_k$. Therefore,

$$(\psi_m \psi_n)(t) = (t^\gamma + 1)^{-\alpha_k}(t^{1-\gamma} + 1)^{-\beta_k} = \psi_k(t).$$

This proves that (6.10) of Assumption 6.2 holds true with $d_{m,n} = 1$ and $k = m \wedge n \in \mathbb{N}$ satisfying (7.27).

Thus, Condition (i) of Assumption 6.2 is met.

*Verification of Condition* (ii) Using (7.16), we have, for $n \in \mathbb{N}$, $t > 0$,

$$\begin{aligned}\psi_n'(t) &= \sum_{k=1}^{\infty} \frac{-\gamma\alpha_n}{(t^\gamma + 1)^{\alpha_n+1}(t^{1-\gamma} + 1)^{\beta_n+k}} + \sum_{k=1}^{\infty} \frac{-(1-\gamma)\beta_n}{(t^\gamma + 1)^{\alpha_n+k}(t^{(1-\gamma} + 1)^{\beta_n+1}} \\ &= g_{n,1}(t) + g_{n,2}(t),\end{aligned}$$

where

$$\begin{aligned}g_{n,1}(t) &= \sum_{k=1}^{\infty} \frac{-\gamma\alpha_n}{(t^\gamma + 1)^{\alpha_n+1}(t^{1-\gamma} + 1)^{\beta_n+k}}, \\ g_{n,2}(t) &= \sum_{k=1}^{\infty} \frac{-(1-\gamma)\beta_n}{(t^\gamma + 1)^{\alpha_n+k}(t^{(1-\gamma} + 1)^{\beta_n+1}}.\end{aligned}$$

Let $n \in \mathbb{N}$ be fixed momentarily. We apply Lemma A.2 to

$$\begin{aligned}&\varphi(t) = t^{-1},\ \bar{\psi}_k(t) = (t^\gamma + 1)^{-\alpha_n-1}(t^{1-\gamma} + 1)^{-\beta_n-k},\ T_* = 1, \\ &X = \mathbb{R},\ \bar{\xi}_k = -\gamma\alpha_n,\ \bar{\nu}_k = \gamma(\alpha_n + 1) + (1-\gamma)(\beta_n + k).\end{aligned}$$

Note, for $t \geq 1$, that

$$2^{-\alpha_n-\beta_n-k-1} t^{-\bar{\nu}_k} = (2t^\gamma)^{-\alpha_n-1}(2t^{1-\gamma})^{-\beta_n-k} \leq \bar{\psi}_k(t) \leq t^{-\bar{\nu}_k},$$

which yields

$$D_k^{-1}\varphi(t)^{\bar{\nu}_k} \leq \bar{\psi}_k(t) \leq D_k \varphi(t)^{\bar{\nu}_k},$$

where $D_k = 2^{\alpha_n+\beta_n+k+1}$. Taking $c_0 = \gamma\alpha_n$, $\kappa = 1$, and $M = 3^{1/(1-\gamma)}$, we have

$$|\xi_k| \leq c_0 \kappa^{\bar{\nu}_k}, \quad \sum_{k=1}^{\infty} D_k M^{-\bar{\nu}_k} < \infty.$$

Since $(\bar{\nu}_k)_{k=1}^\infty$ is already strictly increasing, it is its own strictly increasing re-arrangement. Then, by Lemma A.2,

$$g_{n,1}(t) \underset{\Phi}{\sim} \sum_{k=1}^{\infty} \bar{\xi}_k \bar{\psi}_k(t) = \sum_{k=1}^{\infty} \frac{-\gamma\alpha_n}{(t^\gamma + 1)^{\alpha_n+1}(t^{1-\gamma} + 1)^{\beta_n+k}}. \tag{7.28}$$

Now, with $\hat{\xi}_k = -(1-\gamma)\beta_n$ replacing $\xi_k$, $\hat{\psi}_k(t) = (t^\gamma + 1)^{-\alpha_n-k}(t^{1-\gamma} + 1)^{-\beta_n-1}$ replacing $\bar{\psi}_k(t)$, and $\hat{\nu}_k = \gamma(\alpha_n + k) + (1-\gamma)(\beta_n + 1)$ replacing $\bar{\nu}_k$, we similarly obtain

$$g_{n,2}(t) \underset{\Phi}{\sim} \sum_{k=1}^{\infty} \hat{\xi}_k \hat{\psi}_k(t) = \sum_{k=1}^{\infty} \frac{-(1-\gamma)\beta_n}{(t^\gamma + 1)^{\alpha_n+k}(t^{1-\gamma} + 1)^{\beta_n+1}}. \tag{7.29}$$

Since $\bar{\nu}_k$ and $\hat{\nu}_k$ belong to $S_*$, all functions $\bar{\psi}_k(t)$ in (7.28), and $\hat{\psi}_k(t)$ in (7.29) belong to the collection $\{\psi_n : n \in \mathbb{N}\}$. Then we can rewrite

$$g_{n,i}(t) \underset{\Phi}{\sim} \sum_{k=1}^{\infty} \tilde{c}_{n,i,k}\psi_k(t), \quad \text{for } i = 1, 2,$$

for some constants $\tilde{c}_{n,i,k}$. Therefore,

$$\psi_n'(t) = g_{n,1}(t) + g_{n,2}(t) \underset{\Phi}{\sim} -\sum_{k=1}^{\infty} c_{n,k}\psi_k(t), \tag{7.30}$$

where

$$c_{n,k} = -(\tilde{c}_{n,1,k} + \tilde{c}_{n,2,k}) = \gamma \sum_j \alpha_j + (1-\gamma)\sum_\ell \beta_\ell, \tag{7.31}$$

with $\alpha_j + 1 = \alpha_k$, $\beta_j + p = \beta_k$, and $\alpha_\ell + q = \alpha_k$, $\beta_\ell + 1 = \beta_k$ for some $p, q \in \mathbb{N} \cup \{0\}$.

Note that these pairs $(j, p)$ and $(\ell, q)$ are only finitely many. Indeed, since

$$\alpha_j + \beta_j + p + 1 = \alpha_k + \beta_k \text{ and } \alpha_j + \beta_j \to \infty,$$

we have, for each fixed $k$, there are only finitely many $j$ and $p$. The same arguments apply to $(\ell, q)$. Therefore, the sums in (7.31) are only finite ones.

With (7.30), Condition (ii) of Assumption 6.2 is met.

*Conclusion on Assumption* 6.2 We have checked that the systems $\Psi$ and $\Phi$ satisfy Assumption 6.2.

We now return to the expansion (7.17) for the force. Since $\tilde{\lambda}_n \in S_*$, we have that each function $(t^\gamma + 1)^{-\tilde{\alpha}_n}(t^{1-\gamma} + 1)^{-\tilde{\beta}_n}$ in the sum in (7.17) belongs to $\{\psi_n : n \in \mathbb{N}\}$. Hence, we can rewrite (7.17) as

$$f(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \phi_n\psi_n(t) = \sum_{n=1}^{\infty} \frac{\phi_n}{(t^\gamma + 1)^{\alpha_n}(t^{1-\gamma} + 1)^{\beta_n}} \quad \text{in } G_{\alpha,\sigma}, \tag{7.32}$$

for some sequence $(\phi_n)_{n=1}^{\infty}$ in $G_{\alpha,\sigma}$.

By applying Theorem 6.3, we obtain the following result.

**Proposition 7.11** *Assume* (7.32). *Then any Leray–Hopf weak solution $u(t)$ of the NSE* (2.2) *admits the asymptotic expansion*

$$u(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \xi_n\psi_n(t) = \sum_{n=1}^{\infty} \frac{\xi_n}{(t^\gamma + 1)^{\alpha_n}(t^{1-\gamma} + 1)^{\beta_n}} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

*where*

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\left(\phi_n + \sum_{p=1}^{n-1} c_{p,n}\xi_p - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k, \xi_m)\right) \quad \text{for } n \ge 2,$$

*with $c_{p,n}$ being defined in* (7.31).

***Remark 7.12*** This is a counterpart of Remark 5.3, but applied to the expansion (7.17).

Assume $(\tilde{\alpha}_n)_{n=1}^{\infty}$ and $(\tilde{\beta}_n)_{n=1}^{\infty}$ are sequences of non-negative numbers such that

(a) Property (7.18) holds true,
(b) For each $n \in \mathbb{N}$, the right-hand side of (7.18) is the unique decomposition among $\gamma\tilde{\alpha}_k + (1-\gamma)\tilde{\beta}_j$ for $k, j \in \mathbb{N}$,
(c) Each set $E_{*,1} = \{\tilde{\alpha}_n : n \in \mathbb{N}\}$, $E_{*,2} = \{\tilde{\beta}_n : n \in \mathbb{N}\}$ preserves the addition, and the increments by 1,
(d) The set $E_* \stackrel{\text{def}}{=} \{\tilde{\lambda}_n : n \in \mathbb{N}\}$ is equal to $\bar{E} \stackrel{\text{def}}{=} \{\gamma\tilde{\alpha}_k + (1-\gamma)\tilde{\beta}_j : k, j \in \mathbb{N}\}$.

Define $E = \Big\{\gamma(\tilde{\alpha}_n + k) + (1-\gamma)(\tilde{\beta}_n + j) : n \in \mathbb{N},\ k, j \in \mathbb{N} \cup \{0\}\Big\}$.

By the preservation of $E_{*,1}$ and $E_{*,2}$ in (c), we have $\tilde{\alpha}_n + k \in E_{*,1}$ and $\tilde{\beta}_n + k \in E_{*,2}$. Then $E_* \subset E \subset \bar{E} = E_*$, which implies $E_* = E$.

Let $S_*$ be defined by (7.19). We have $E_* \subset S_* \subset E = E_*$. Hence, $S_* = E_*$, which, by (7.18), is already ordered by $(\tilde{\lambda}_n)_{n=1}^\infty$. By this and (5.4), (7.24), we have $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$, $\beta_n = \tilde{\beta}_n$, and $\psi_n = \tilde{\psi}_n$. Therefore, the expansion (7.32) is the original (7.17).

# A Appendix

One way to generate an infinite expansion of the type (4.3) is to start with a function as a convergent series in (4.5). We give a criterion for such a conversion.

**Lemma A.1** *Let $\varphi$ be a positive function defined on $[T_*, \infty)$ for some $T_* \geq 0$, and $\varphi(t) \to 0$ as $t \to \infty$.*

*Let $(\lambda_n)_{n=1}^\infty$ be a strictly increasing, divergent sequence of positive numbers and there exists a number $M > 1$ such that*

$$\sum_{n=1}^{\infty} M^{-\lambda_n} < \infty. \tag{A.1}$$

*Let $(X, \|\cdot\|)$ be a Banach space and let $(\xi_n)_{n=1}^\infty \subset X$ satisfy*

$$\|\xi_n\| \leq c_0 \kappa^{\lambda_n} \quad \forall n \in \mathbb{N}, \tag{A.2}$$

*for some positive constants $c_0$ and $\kappa$.*

*Then the series $\sum_{n=1}^\infty \xi_n \varphi(t)^{\lambda_n}$ converges absolutely and uniformly to a function $g(t)$ on $[T_0, \infty)$ for some $T_0 \geq T_*$, and $g$ has the expansion*

$$g(t) \stackrel{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \varphi(t)^{\lambda_n} \text{ in } X, \text{ where } \Psi = (\varphi^\lambda)_{\lambda>0}. \tag{A.3}$$

***Proof*** Since $\varphi(t) \to 0$ as $t \to \infty$, there exists $T_0 \geq T_*$ such that for all $t \geq T_0$,

$$\varphi(t) \leq \frac{1}{M\kappa}. \tag{A.4}$$

Combining (A.2) and (A.4) yields, for all $n \in \mathbb{N}$,

$$\sup_{[T_0,\infty)} \|\xi_n \varphi(t)^{\lambda_n}\| \leq c_0 M^{-\lambda_n}.$$

This and (A.1) imply that $\sum_{n=1}^\infty \xi_n \varphi(t)^{\lambda_n}$ converges absolutely and uniformly on $[T_0, \infty)$, with $g(t) = \sum_{n=1}^\infty \xi_n \varphi(t)^{\lambda_n}$ being its limit function.

It remains to prove the expansion (A.3). We note, by the convergence of the series $\sum_{n=1}^\infty \xi_n\varphi(T_0)^{\lambda_n}$, that

$$\sup_{n\in\mathbb{N}} \|\xi_n\|\varphi(T_0)^{\lambda_n} = c_1 < \infty,$$

which implies

$$\|\xi_n\| \le \frac{c_1}{\varphi(T_0)^{\lambda_n}} \quad \forall n \in \mathbb{N}. \tag{A.5}$$

Again, since $\varphi(t) \to 0$ as $t \to \infty$, there exists $T_1 \ge T_0$ such that for all $t \ge T_1$,

$$\frac{\varphi(t)}{\varphi(T_0)} \le \frac{1}{M}. \tag{A.6}$$

Using (A.5) and (A.6), we estimate, for all $t \ge T_1$,

$$\begin{aligned}\left\| g(t) - \sum_{n=1}^N \xi_n\varphi(t)^{\lambda_n} \right\| &= \left\| \sum_{n=N+1}^\infty \xi_n\varphi(t)^{\lambda_n} \right\| \\ &\le c_1 \sum_{n=N+1}^\infty \frac{\varphi(t)^{\lambda_n}}{\varphi(T_0)^{\lambda_n}} = c_1 \frac{\varphi(t)^{\lambda_{N+1}}}{\varphi(T_0)^{\lambda_{N+1}}} \sum_{n=N+1}^\infty \left(\frac{\varphi(t)}{\varphi(T_0)}\right)^{\lambda_n-\lambda_{N+1}} \\ &\le c_1 \frac{\varphi(t)^{\lambda_{N+1}}}{\varphi(T_0)^{\lambda_{N+1}}} \sum_{n=N+1}^\infty M^{-\lambda_n+\lambda_{N+1}} \le C_N\varphi(t)^{\lambda_{N+1}},\end{aligned}$$

where $C_N = c_1(M/\varphi(T_0))^{\lambda_{N+1}} \sum_{n=1}^\infty M^{-\lambda_n} < \infty$. Therefore, we obtain the expansion (A.3), according to Definition 4.3 with $\psi_\lambda = \varphi^\lambda$. □

We extend Lemma A.1 to cover the expansions with a background system such as those in Sect. 6.

**Lemma A.2** *Let $\varphi(t)$ and $\psi_n(t)$, for $n \in \mathbb{N}$, be positive functions defined on $[T_*, \infty)$ for some $T_* \ge 0$ that tend to zero as $t \to \infty$. Assume, for each $n \in \mathbb{N}$, there exist a numbers $D_n \ge 1$ such that*

$$D_n^{-1}\varphi(t)^{\lambda_n} \le \psi_n(t) \le D_n\varphi(t)^{\lambda_n} \quad \forall t \ge T_*, \tag{A.7}$$

*where $(\lambda_n)_{n=1}^\infty$ is a sequence of positive numbers and $\lambda_n \to \infty$ as $n \to \infty$. Assume further that there exists $M > 0$ such that*

$$\sum_{n=1}^\infty D_n M^{-\lambda_n} < \infty. \tag{A.8}$$

*Let $(X, \|\cdot\|)$ be a Banach space, and $(\xi_n)_{n=1}^\infty$ be a sequence in $X$ such that (A.2) holds.*

(i) *Then the series $\sum_{n=1}^\infty \xi_n\psi_n(t)$ converges absolutely and uniformly on $[T_0, \infty)$ for some $T_0 \ge T_*$. Define*

$$f(t) = \sum_{n=1}^\infty \xi_n\psi_n(t) \in X \quad \forall t \ge T_0. \tag{A.9}$$

(ii) *Assume the mapping $n \mapsto \lambda_n$ is one-to-one. Let $(\mu_n)_{n=1}^\infty$ be the strictly increasing rearrangement of $(\lambda_n)_{n=1}^\infty$. Define $\psi_n^* = \psi_k$ and $\xi_n^* = \xi_k$ with $\mu_n = \lambda_k$. Let $\Phi = (\varphi(t)^\lambda)_{\lambda>0}$.*

*Then*

$$f(t) \underset{\Phi}{\sim} \sum_{n=1}^{\infty} \xi_n^* \psi_n^*(t). \tag{A.10}$$

***Proof*** (i) There is $T_0 \geq T_*$ such that $\varphi(t) \leq 1/\kappa M$ for all $t \geq T_0$. Then for all $t \geq T_0$, by (A.2), (A.7), and (A.8),

$$\sum_{n=1}^{\infty} \|\xi_n\| \psi_n(t) \leq \sum_{n=1}^{\infty} \|\xi_n\| D_n \varphi(t)^{\lambda_n} \leq \sum_{n=1}^{\infty} c_0 D_n M^{-\lambda_n} < \infty.$$

Therefore, we obtain the absolute and uniform convergence on $[T_0, \infty)$.

(ii) Let $D_n^* = D_k$ with $\mu_n = \lambda_k$. After the re-arrangement we still have

$$\sum_{n=1}^{\infty} D_n^* M^{-\mu_n} = \sum_{k=1}^{\infty} D_k M^{-\lambda_k} < \infty, \tag{A.11}$$

and, because of the absolute convergence,

$$f(t) = \sum_{n=1}^{\infty} \xi_n^* \psi_n^*(t) \text{ for } t \geq T_0. \tag{A.12}$$

For each $n \in \mathbb{N}$, let $k \in \mathbb{N}$ such that $\mu_n = \lambda_k$, then we have

$$(D_n^*)^{-1} \varphi^{\mu_n} = D_k^{-1} \varphi^{\lambda_k} \leq \psi_n^* = \psi_k \leq D_k \varphi^{\lambda_k} = D_n^* \varphi^{\mu_n}.$$

The convergence of $f(T_0)$ in (A.12) implies that there exists $c_1 > 0$ such that

$$\|\xi_n^*\| \leq c_1 \psi_n^*(T_0)^{-1} \leq c_1 D_n^* \varphi(T_0)^{-\mu_n} \quad \forall n \in \mathbb{N}.$$

Let $T_1 \geq T_0$ such that $\varphi(t)/\varphi(T_0) \leq 1/M^2$ for all $t \geq T_1$. Then, for $t \geq T_1$,

$$\begin{aligned}
\left\| f(t) - \sum_{n=1}^{N} \xi_n^* \psi_n^*(t) \right\| &= \left\| \sum_{n=N+1}^{\infty} \xi_n^* \psi_n^*(t) \right\| \leq \sum_{n=N+1}^{\infty} c_1 D_n^* \varphi(T_0)^{-\mu_n} \cdot D_n^* \varphi(t)^{\mu_n} \\
&\leq c_1 (\varphi(t)/\varphi(T_0))^{\mu_{N+1}} \sum_{n=N+1}^{\infty} (D_n^*)^2 (\varphi(t)/\varphi(T_0))^{\mu_n - \mu_{N+1}} \\
&\leq c_1 (\varphi(t)/\varphi(T_0))^{\mu_{N+1}} \sum_{n=N+1}^{\infty} (D_n^*)^2 M^{-2\mu_n + 2\mu_{N+1}} \\
&\leq c_1 (M^2 \varphi(t)/\varphi(T_0))^{\mu_{N+1}} \left( \sum_{n=N+1}^{\infty} D_n^* M^{-\mu_n} \right)^2 \leq C \varphi(t)^{\mu_{N+1}},
\end{aligned}$$

where thanks to (A.11), $C$ is a positive number. Since $\mu_{N+1} > \mu_N$, we obtain (A.10). □

We emphasize that the sum in (A.9) must be re-arranged to have a meaningful expansion as in (A.10). In particular cases, Lemma A.2 is used to obtain expansions when $\psi_n$ is generated by two functions with two different sequences of powers.

**Corollary A.3** *Suppose $\zeta(t)$, $\vartheta(t)$ and $\varphi(t)$ are three positive functions defined on $[T_*, \infty)$ for some $T_* \geq 0$ that tend to zero as $t \to \infty$, and there exist numbers $D \geq 1$, $s_1, s_2 > 0$ such that*

$$D^{-1} \varphi(t)^{s_1} \leq \zeta(t) \leq D \varphi(t)^{s_1}, \quad D^{-1} \varphi(t)^{s_2} \leq \vartheta(t) \leq D \varphi(t)^{s_2} \quad \forall t \geq T_*. \tag{A.13}$$

*Let* $(\alpha_n)_{n=1}^\infty$ *and* $(\beta_n)_{n=1}^\infty$ *be two sequences of non-negative numbers such that* $\alpha_n + \beta_n \to \infty$.

*Define* $\lambda_n = s_1\alpha_n + s_2\beta_n$ *for* $n \in \mathbb{N}$. *Let* $(X, \|\cdot\|)$ *be a Banach space, and* $(\xi_n)_{n=1}^\infty$ *be a sequence in* $X$. *Assume* (A.1) *and* (A.2).

(i) *Then the series* $\sum_{n=1}^\infty \xi_n \zeta(t)^{\alpha_n}\vartheta(t)^{\beta_n}$ *converges absolutely and uniformly on* $[T_0, \infty)$ *for some* $T_0 \geq T_*$. *Define*

$$f(t) = \sum_{n=1}^\infty \xi_n \zeta(t)^{\alpha_n}\vartheta(t)^{\beta_n} \quad \forall t \geq T_0.$$

(ii) *Suppose the mapping* $n \mapsto \lambda_n$ *is one-to-one. Let* $(\mu_n)_{n=1}^\infty$ *be the strictly increasing re-arrangement of* $(\lambda_n)_{n=1}^\infty$. *Define* $\psi_n^* = \zeta^{\alpha_k}\vartheta^{\beta_k}$ *and* $\xi_n^* = \xi_k$ *with* $\mu_n = \lambda_k$.

*Let* $\Phi = (\varphi(t)^{-\lambda})_{\lambda>0}$. *Then*

$$f(t) \underset{\Phi}{\sim} \sum_{n=1}^\infty \xi_n^* \psi_n^*(t).$$

***Proof*** Let $\psi_n(t) = \zeta(t)^{\alpha_n}\vartheta(t)^{\beta_n}$ and $D_n = D^{\alpha_n+\beta_n}$. Thanks to (A.13), we have

$$D_n^{-1}\varphi^{\lambda_n} = D^{-(\alpha_n+\beta_n)}\varphi^{\lambda_n} \leq \psi_n \leq D^{\alpha_n+\beta_n}\varphi^{\lambda_n} = D_n\varphi^{\lambda_n}.$$

Note that $\lambda_n \to \infty$. Denote $s = \max\{1/s_1, 1/s_2\}$. Then

$$\sum_{n=1}^\infty D_n(MD^s)^{-\lambda_n} = \sum_{n=1}^\infty M^{-\lambda_n} D^{(1-s_1 s)\alpha_n + (1-s_2 s)\beta_n} \leq \sum_{n=1}^\infty M^{-\lambda_n} < \infty.$$

Hence, (A.8) holds true with $M := MD^s$. Applying Lemma A.2, we obtain the desired statements in (i) and (ii). □

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