Annali di Matematica Pura ed Applicata (1923 -) (2020) 199:709–735
https://doi.org/10.1007/s10231-019-00897-y

Stability of geometric flows on the circle

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Received: 10 October 2018 / Accepted: 17 August 2019 / Published online: 24 August 2019

Abstract
In this paper, we prove a general stability result for higher-order geometric flows on the circle, which basically states that if the initial condition is close to a round circle, the curve evolves smoothly and exponentially fast towards a circle (possibly not the one it started close to), and we improve on known convergence rates (which we believe are almost sharp). The polyharmonic flow is an instance of the flows to which our result can be applied. We will also present general families of flows for which our stability result applies.

Keywords Geometric flows in the circle · Stability · Convergence rate

Mathematics Subject Classification Primary 53C44 · 35K55 · 58J35

1 Introduction

The study of deformation of curves by using flow methods has become an important area of research in geometric analysis. In general, the purpose is to deform the curve as follows. We let

$$x : \mathbb{S}^1 \times [0, T) \longrightarrow \mathbb{R}^2$$

be a family of smooth convex embeddings of $\mathbb{S}^1$, the unit circle, into $\mathbb{R}^2$. and we assume that this family of embeddings satisfies an equation of the form

$$\frac{\partial x}{\partial t} = F\left(k, k_s, \ldots, k_s^{(n)}\right) N, \tag{1}$$

César Reyes wants to thank the Fondo de Investigaciones de la Facultad de Ciencias de la Universidad de los Andes for funding this project by means of “Convocatoria 2017-2 para la Financiación de proyectos de Investigación Categoría Estudiantes de Doctorado”, under the Project “Flujo de Ricci sobre el cilindro con frontera”.

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where k is the curvature of the embedding and N is the normal vector pointing outwards; the region bounded by $x(\cdot, t)$ and $k_s^{(m)}$ denotes the mth derivative of k with respect to the arclength parameter.

The dean of all these family of flows is the curve shortening flow, which occurs when in (1) we make $F = -k$. The curve shortening flow has been widely studied and is very well understood ([8]). The family of p-curve shortening flows is obtained by making $F = -\frac{1}{p}k^p$, and the family of polyharmonic flows (which gives an example of an application of our main result) is obtained with $F = (-1)^{p+1}k_s^{(2p)}$.

In most cases, as in the case of the curve shortening and p-curve shortening flow (see [1,8]) this flows does what is expected of them: they deform convex embedded curves into circles (after normalisation). However, some interesting behaviour occurs in the case of higher-order flow (see, for instance, [9,15]), and even proving that solutions starting close to a circle exist globally and converge to a circle can be challenging (see [7]).

Our purpose in this paper is to look at the structure of the evolution equation satisfied by the curvature k of a solution to a flow of form (1) (written in terms of a parameter θ, known as an indicatrix, which corresponds to the angle between the tangent to the curve at a given point with respect to a fixed direction on the plane), so that the following stability property can be deduced: if we start close to a circle, in a sense to be specified below, the curve evolves smoothly and exponentially fast towards a circle, although perhaps not the same circle the curve started close to.

With these considerations in mind, the semilinear parabolic equations we shall consider are of the following general form,

$$\begin{cases} \dfrac{\partial k}{\partial t} = (-1)^{p+1}k^M \dfrac{\partial^{2p} k}{\partial \theta^{2p}} + G\left(k, \dfrac{\partial k}{\partial \theta}, \ldots, \dfrac{\partial^{2p-1} k}{\partial \theta^{2p-1}}\right) & [0, 2\pi] \times (0, T), \\ k(\theta, 0) = \psi(\theta), \end{cases} \tag{2}$$

endowed with periodic boundary conditions, with $\psi > 0$. In our case, $G\left(z_0, \ldots, z_{2p-1}\right)$ is any polynomial in $2p$ variables for which the following conditions hold. Any term $z_0^{\alpha_0} z_1^{\alpha_1} \ldots z_{2p-1}^{\alpha_{2p-1}}$ of G satisfies

(I) either $\alpha_1 + \alpha_2 + \cdots + \alpha_{2p-1} > 1$,
(II) or in the case that $\alpha_1 + \alpha_2 + \cdots + \alpha_{2p-1} = 1$, there is a j such that $\alpha_{2j} = 1$.

Notice that these two requirements imply that for any constant k_0 we have that $F = 0$. Let us justify this claim. First of all, observe that the left-hand side of (2) is 0 if we replace k by a constant: the reason is that all the terms on the right-hand side of (2) contain a derivative of k. On the other hand, it is well known that given a flow (1), the curvature of the solution satisfies, when given in terms of θ, an equation

$$\frac{\partial k}{\partial t} = \frac{\partial^2 F}{\partial s^2} + k^2 F - k\left(\frac{\partial k}{\partial \theta}\right)^2,$$

where then we use $\dfrac{\partial}{\partial s} = k\dfrac{\partial}{\partial \theta}$, to write everything in terms of θ. Then the only way for (2) not to contain any terms of the form k^l (which is not allowed by conditions (I) and (II)) is that F does not have terms which are just powers of k, so every term of F involves a derivative of k with respect to s. Therefore, circles are fixed points of (1), whenever the equation satisfied by the curvature satisfies the two conditions required above. Observe that the equation satisfied by k in the case of the curve shortening flow does not satisfy the requirements we have made on the right-hand side of (2).

We shall also require something else from solutions to (2), but before revealing what our requirement is, and to put our results into context, let us first analyse the linearisation of the operator on the right-hand side of (2) around the solution $k \equiv 1$, which is given by

$$\mathcal{L}u = (-1)^{p+1}\frac{\partial^{2p}u}{\partial\theta^{2p}} + \sum_{j=0}^{2p-1}\frac{\partial G}{\partial z_j}(1,0,\ldots,0)\frac{\partial^j u}{\partial\theta^j}.$$

Hence, the eigenvalues of $\mathcal{L}$ of the eigenfunctions $e^{in\theta}$ with $n \in \mathbb{Z}$ can be computed as

$$\lambda_n = -n^{2p} + \sum_{j=0}^{p-1}(-1)^j\frac{\partial G}{\partial z_{2j}}(1,0,\ldots,0)n^{2j}.$$

Here increasing n does not imply any monotonicity of the eigenvalues; it is just a label to indicate that λ_n is the eigenvalue associated with the eigenfunction $e^{in\theta}$. In the case of the higher-order curvature flows we want to study, besides having $\lambda_0 = 0$ we also have that $\lambda_1 = 0$. It is because of this that it might not occur, in principle, that solutions with the initial data close to the constant function 1 will evolve towards a constant function.

On the other hand, solutions to (2), as long as k is the curvature function of a closed curve and remains positive, satisfy the identities

$$U_{\pm}(k(\theta,t)) := \int_0^{2\pi}\frac{e^{\pm i\theta}}{k(\theta,t)}\,\mathrm{d}\theta = 0. \tag{3}$$

These identities are the key to showing exponential convergence of the curvature function k towards a constant function if the initial data ψ are close to a constant function. They give us control over the Fourier coefficient of the wave numbers $n = \pm 1$. So we will require that solutions to (2) also satisfy (3). We shall call these flows *geometric flows*.

Henceforth, we will use the following seminorms

$$\|f\|_\beta = \max\left\{\sup_{n\neq 0}|n|^\beta\left|\mathrm{Re}\left\{\hat{f}(n)\right\}\right|, \sup_{n\neq 0}|n|^\beta\left|\mathrm{Im}\left\{\hat{f}(n)\right\}\right|\right\},$$

where $\hat{f}(n)$ represents the nth Fourier coefficient of f.

Notice also that

$$\frac{\partial G}{\partial z_{2l}}(1,0,\ldots,0) = \sum_{j_l} a_{j_l}$$

where (and this notation will be important in the statement of our main result)

$$a_{j_l} \quad \text{is the coefficient of the term} \quad k^{j_l}\frac{\partial^{2l}k}{\partial\theta^{2l}}.$$

We can now state our main theorem.

Theorem 1 *Consider Eq. (2) with the initial data $\psi > 0$, $U_{\pm}(\psi) = 0$, in the Sobolev space $H^{2p+3}(\mathbb{S}^1)$, a geometric flow. There exists a $\delta > 0$, which may also depend on the L^∞-norm of ψ, such that if*

$$\|\psi\|_{2p+1} \leq \delta\hat{\psi}(0), \tag{4}$$

and there exists a $c < 0$ such that for all integers $n \geq 2$

$$P_{n,5\delta} = -n^{2p}(1-5\delta)^M \hat{\psi}^M(0) + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\left(1+\operatorname{sgn}\left((-1)^l a_{j_l}\right)5\delta\right)^{j_l}\hat{\psi}^{j_l}(0) < c,$$

then the solution k of (2) with the initial data $\psi > 0$ exists for all time, remains positive, and converges exponentially fast to a constant that we shall denote by $\hat{k}(0,\infty)$. Moreover, if for every $0 < \epsilon < 5\delta$ we have the inequality

$$P_{2,\epsilon} \geq P_{n,\epsilon}$$

then we have the following estimate on the rate of convergence of the solution towards a constant: for any $\epsilon > 0$, there exists a $C_{l,\epsilon}$ such that

$$\left\| k(\theta,t) - \hat{k}(0,\infty)\right\|_{C^l(\mathbb{S}^1)} \leq C_{l,\epsilon} e^{P_{2,\epsilon} t}, \quad l = 0,1,2,\ldots.$$

A few comments are in place here. To verify that the assumption on $P_{n,5\delta}$ holds for $\delta > 0$ small, it is enough to verify that it holds for $\delta = 0$ (perhaps with a different negative c). Also, the regularity required on ψ is assumed in order to show that finite-dimensional approximations of the infinite-dimensional system of ODEs for the Fourier coefficients of solutions to (2) approximate well the solution of the full system, and in principle, it might be weakened. Theorem 1 implies that the solution to the original flow, via the support function (see the discussion in Sect. 1 in [16] or in Sect. 2.1 in [5]), when starting close enough to a circle in the sense of condition (4), will end up converging towards a circle, again, perhaps a different circle to the one it started close to.

As we said before, the main difficulty to prove Theorem 1 is the fact that the eigenvalue λ_1 might be equal to 0, and thus, we cannot expect smooth exponential convergence of the function $k(\cdot,t)$ towards a constant. The reader should compare our discussion with Theorem 1 (and the remark right after its statement) in Simon's celebrated paper [14]. Also, the fact that the (sub)set (of the set) of equilibria (we are interested in) has dimension 1, whereas the kernel of the linearisation of the right-hand side of (2) has dimension 3 shows that Theorem 1 does not follow from the general results presented in [13]. We deal with this issue using the approach presented in [6] for the p-curve shortening flow: that is, we make use of (3) to control the Fourier coefficients corresponding to the wave number $n = \pm 1$ in terms of the Fourier coefficients for the higher wave numbers; one difference with the family of flows considered in [6] to those considered in this paper is the fact that in that family of flows the average of the curvature blows up which in turn helps on sharpening the convergence rates, whereas in the flows considered in this paper the average of the curvature remains bounded, but also large enough, so that it can be shown that the convexity of the curve is preserved (something that is not obvious in the case of higher-order flows, as pointed out to us by the referee [3]; see Lemma 2 which shows how small $\delta > 0$ must be to guarantee that convexity is preserved). In any case, though we do not obtain sharp rates of convergence, our method allows us to give better convergence rates than the known ones, which are basically related to the expected rate $e^{\lambda_2 t}$.

Notice that the requirement in Theorem 1 on how close the initial condition must be to a constant (in this case its average) can be also recasted in terms of the norm of the Hölder spaces $C^{2p+3,\alpha}\left(\mathbb{S}^1\right)$ $(0 < \alpha < 1)$ or the Sobolev spaces $H^{2p+3}\left(\mathbb{S}^1\right)$.

As hinted above, we will be using the Fourier method to prove our results. To the best of our knowledge, this method was introduced by Palais [11] to show blow-up of certain nonlinear equations. Later, Mattingly and Sinai [10] used similar ideas to show regularity for the Navier–Stokes equations. The main idea of the method, as developed in [10,11], and as expressed by the authors of these beautiful works, is to trade a PDE by a system of ODEs, which approximate the infinite system of ODEs for the Fourier coefficients of the solutions to the PDE. Further, in [10], the authors introduced the following important idea: assuming certain growth condition on the Fourier coefficients of the initial data can be used, along with techniques borrowed from the theory of dynamical systems (in fact, the idea of the existence a trapping region), to demonstrate regularity results, in particular that under certain conditions the solutions to the Navier–Stokes equations become analytic in space instantaneously. Then in [4–6] these ideas were adapted to show stability of the flat profile of blow-up for certain equations with geometric flavour, the main difficulty being that, in the case of the equations coming from a geometric framework, the linearisation around an equilibrium of the operators depending on the spatial derivatives of the curvature might posses nonnegative eigenvalues (which is not the case in [10]). These ideas are borrowed once again in this paper, this time to show a proper stability result.

To show how our main theorem applies, we will first consider the case of the polyharmonic flows which corresponds to the case when $F = (-1)^m k_s^{(2m)}$ in (1). For polyharmonic flows, assuming that the initial curve encloses an area of π, it can be shown that if the flow converges, necessarily $\hat{k}(0, \infty) = 1$ (see Theorem 1 in [12]). Also, in this case, for the equation satisfied by the curvature function, with the notation employed in (2), we have that $p = m + 1$ Then we obtain the following corollary.

Corollary 1 *Consider Eq.* (2) *corresponding to the case when*

$$F = (-1)^m k_s^{(2m)},$$

i.e. when (2) *represents the evolution of the curvature in the case of the polyharmonic flows, with the initial condition* $\psi \in H^{2m+5}\left(\mathbb{S}^1\right)$*. There exists a* $\delta > 0$*, which may depend on the* L^∞ *norm of* ψ*, such that if the curvature of the initial curve satisfies*

$$\|\psi\|_{2m+3} \leq \delta \hat{\psi}(0), \tag{5}$$

and the area enclosed by it is equal to π*, then the solution of* (2) *with the initial data* ψ *exists for all time and converges smoothly and exponentially towards the constant function 1 (i.e. the curve is evolving towards a unit circle). Moreover, we have the following estimate on the rate of convergence of the solution towards a constant: for any* $\epsilon > 0$*, there exists a* $C_{l,\epsilon}$ *such that*

$$\|k(\theta, t) - 1\|_{C^l(\mathbb{S}^1)} \leq C_{l,\epsilon} e^{P_\epsilon t}, \quad l = 0, 1, 2, \ldots,$$

where P_ϵ *is given by*

$$-2^{2m+2}(1-\epsilon)^{2m+2} + (1+\epsilon)^{2m+2} 2^{2m}.$$

We must point out that the rates given in Corollary 1 improve on those obtained in [12,15]. Also, condition (5) is not sharp and we will discuss this issue in the very last section of this paper.

It is worth pointing out that Theorem 1 allows us to show stability results for families of flows by only having to check the form of the function F in (1). This will be considered in Sects. 3.3 and 3.4 of this work.

This paper is organised as follows. In Sect. 2, we give a proof of our main result; in Sect. 3, we apply it to the polyharmonic flow. Then we give an example of another flow and propose families of flows for which stability (in the sense described above) holds (see Propositions 10 and 11); in Sect. 3.5, we will present a brief discussion on how condition (5) might be weakened in the case of the polyharmonic flow and a flow recently studied in [2]. We have complemented our exposition with an appendix where we show that finite-dimensional approximations of the ODE system for the Fourier coefficients of a solution to (2) approximate quite well the full infinite-dimensional system of ODEs for the Fourier coefficients equivalent to (2).

2 Proof of Theorem 1

2.1 Preliminaries

Given a function f, we shall denote by $\hat{f}$ its Fourier transform. For the purpose of the discussion below, we shall call the values $\hat{f}(q)$ the Fourier coefficients of f and the integer q will be called the *wave number* of the *Fourier coefficient*.

Applying the Fourier transform to the evolution equation (2), we obtain

$$\frac{\partial \hat{k}}{\partial t}(\xi, t) = (-1)^{p+1}\hat{k}^{*M}(\xi, t) * (i\xi)^{2p}\hat{k}(\xi, t) + \hat{G}(\xi, t).$$

If G has the form

$$G\left(k, \frac{\partial k}{\partial \theta}, \ldots, \frac{\partial^{2p-1}k}{\partial \theta^{2p-1}}\right) = \sum_{j_l \in \{0,1,\ldots,m\}} a_{j_0 j_1 \ldots j_{2p-1}} k^{j_0} \left[\frac{\partial k}{\partial \theta}\right]^{j_1} \cdots \left[\frac{\partial^{2p-1}k}{\partial \theta^{2p-1}}\right]^{j_{2p-1}},$$

then $\hat{G}$ is given by

$$\hat{G}(\xi, t) = \sum_{j_l \in \{0,1,\ldots,m\}} a_{j_0 j_1 \ldots j_{2p-1}} \hat{k}^{*j_0}(q_0) * \left[i\xi\hat{k}\right]^{*j_1}(q_1) * \cdots * \left[[i\xi]^{2p-1}\hat{k}\right]^{*j_{2p-1}}\left(q_{2p-1}\right),$$

where

$$\left[[i\xi]^l\hat{k}\right]^{*j_l}(q_l) = i^{lj_l}\sum_{q_l}^{\star}\left[q_{l,1}q_{l,2}\cdots q_{l,j_l}\right]^l \hat{k}(q_{l,1}, t)\hat{k}(q_{l,2}, t)\cdots\hat{k}(q_{l,j_l}, t),$$

and the notation $\sum_{q_l}^{\star}$ indicates that the sum is over all tuples of integers $\left(q_{l,1}, q_{l,2}, \ldots, q_{l,j_l}\right)$ whose sum is q_l.

Then the evolution of $\hat{k}(\xi, t)$ equation turns out to be

$$\begin{aligned}
&\partial_t\hat{k}(\xi, t)\\
&= -\sum_{q\in\mathbb{Z}} q^{2p}\hat{k}(q, t)\hat{k}^{*M}(\xi - q, t)\\
&\quad + \sum_{j_l \in \{0,1,\ldots,m\}} \sum_{\xi}^{\star} a_{j_0 j_1 \cdots j_{2p-1}} i^{j_1+\cdots+(2p-1)j_{2p-1}}\\
&\quad \times \left[q_{1,1}q_{1,2}\cdots q_{1,j_1}\right]\cdots\left[q_{2p-1,1}q_{2p-1,2}\cdots q_{2p-1,j_{2p-1}}\right]^{2p-1}\\
&\quad \left[\hat{k}(q_{0,1}, t)\hat{k}(q_{0,2}, t)\cdots\hat{k}(q_{0,j_0}, t)\right]\cdots\left[\hat{k}(q_{2p-1,1}, t)\hat{k}(q_{2p-1,2}, t)\cdots\hat{k}(q_{2p-1,j_{2p-1}}, t)\right]
\end{aligned}$$

$$= -\sum_{q\in\mathbb{Z}} q^{2p}\hat{k}(q,t)\hat{k}^{*M}(\xi-q,t)$$

$$+ \sum_J \sum_\xi^\star a_J i^{J\cdot(0,1,\ldots,2p-1)} Q_1 \cdots Q_{2p-1}^{2p-1} \hat{k}(Q_0,t)\cdots\hat{k}(Q_{2p-1},t)$$

$$= -\sum_{q\in\mathbb{Z}} q^{2p}\hat{k}(q,t)\hat{k}^{*M}(\xi-q,t) + \sum_J \sum_\xi^\star a_J i^{J\cdot R} Q^R \hat{k}(Q,t),$$

here $J = (j_0, j_1, \ldots, j_{2p-1})$, $R = (0, 1, \ldots, 2p-1)$, $Q = (Q_0, Q_1, \ldots, Q_{2p-1})$ and $Q_l = (q_{l,0}, q_{l,1}, \ldots, q_{l,j_l})$, and we adopt the notation

$$\hat{k}(Q,t) = \hat{k}(Q_0,t)\cdots\hat{k}(Q_{2p-1},t),$$

and

$$\hat{k}(Q_l) = \hat{k}(q_{l,0})\,\hat{k}(q_{l,1})\ldots\hat{k}(q_{l,j_l}).$$

Notice that the sum over J contains only a finite number of multi-indices. Also, given a multi-index $J = (j_1, \ldots, j_m)$ we write its size, as is customary, as

$$|J| = j_1 + \cdots + j_m.$$

We will rewrite the system above in a more convenient way. In order to do so, we define the sets

$$B_n = \left\{ Q = (q_{1,1}, \ldots, q_{1,j_1}, \ldots, q_{2p-1,1}, \ldots, q_{2p-1,j_{2p-1}}) : \sum_{l,k} q_{l,k} = n \right\}$$

$$A_n = \left\{ Q \in B_n : Q \text{ has at least two nonzero components} \right\}.$$

Using these sets, we can rewrite the system above as

$$\partial_t\hat{k}(n,t) = -n^{2p}\hat{k}^M(0,t)\hat{k}(n,t) + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{k}^{j_l}(0,t)\hat{k}(n,t)$$

$$- \sum_{q\in\mathcal{Z}-\{n\}} q^{2p}\hat{k}(q,t)\hat{k}^{*M}(n-q,t) + \sum_J\sum_{A_n} a_J i^{J\cdot R} Q^R\hat{k}(Q,t),$$

for $n \neq 0$, and

$$\partial_t\hat{k}(0,t) = -\sum_{q\in\mathcal{Z}} q^{2p}\hat{k}(q,t)\hat{k}^{*M}(-q,t) + \sum_J\sum_{A_0} a_J i^{J\cdot R} Q^R\hat{k}(Q,t),$$

for $n = 0$.

To justify these expressions notice that the G may contain terms of the form (because of requirement (II))

$$a_{j_l} k^{j_l}\left(\frac{\partial^{2l}k}{\partial\theta^{2l}}\right), \quad l \geq 1,$$

and for such a term when taking the Fourier transform, which gives a convolution, which is a sum of terms of the form,

$$a_{j_l}\hat{k}(q_1,t)\ldots\hat{k}(q_{j_l},t)\, i^{2l} q_{j_l+1}\hat{k}(q_{j_l+1},t), \quad \sum_m q_m = n.$$

Now, if we look at such a term in the convolution which has exactly one nonzero wave number, i.e. at least a $\hat{k}\left(q_j, t\right)$ with $q_j \neq 0$, the only possibility for it to be nonzero and to contribute to the Fourier transform is for it to be of the form

$$(-1)^l n^{2l} a_{j_l} \hat{k}^{j_l}(0, t) \hat{k}(n, t).$$

The other typical term of G for J a fixed multi-index is of the form

$$a_J k^{j_0} \left(\frac{\partial k}{\partial \theta}\right)^{j_1} \cdots \left(\frac{\partial^{2p-1}}{\partial \theta^{2p-1}}\right)^{j_{2p-1}},$$

with at least two indices $j_k \geq 1$, $k \geq 1$. Therefore, if we look at a term of the convolution giving the Fourier transform of the expression right above, which is a product of Fourier coefficients of the Fourier transform of k and its derivatives, and has exactly one wave number with $q \neq 0$, by the pigeonhole principle at least one term coming from a derivative, say of order j, will contribute with a term of the form $\hat{k}(0, t)$. But this term will be accompanied by a 0^j corresponding to the order of the derivative, and hence, this term of the convolution will be 0. This shows our assertion.

In some of the arguments below, we will not be working with the full system given above but with a conveniently defined finite-dimensional approximation. To define this finite-dimensional approximation, we let $\mathcal{Z}$ be a finite subset of integers of the form $\{-a, -a+1, \ldots, 0, \ldots, a-1, a\}$. The finite-dimensional approximation is then $\hat{k}_{\mathcal{Z}}(q, t) = 0$ if $q \notin \mathcal{Z}$, and for $n \in \mathcal{Z}$

$$\begin{aligned}\partial_t \hat{k}_{\mathcal{Z}}(n, t) = & -n^{2p} \hat{k}_{\mathcal{Z}}^M(0, t) \hat{k}_{\mathcal{Z}}(n, t) + \sum_{l=1}^{p-1} \sum_{j_l} (-1)^l n^{2l} a_{j_l} \hat{k}_{\mathcal{Z}}^{j_l}(0, t) \hat{k}_{\mathcal{Z}}(n, t) \\ & - \sum_{q \in \mathcal{Z} - \{n\}} q^{2p} \hat{k}(q, t) \hat{k}_{\mathcal{Z}}^{*M}(n-q, t) + \sum_J \sum_{A_n \cap \mathcal{Z}^{|J|}} a_J i^{J \cdot R} Q^R \hat{k}_{\mathcal{Z}}(Q, t),\end{aligned}$$

for $n \neq 0$, and

$$\partial_t \hat{k}(0, t) = -\sum_{q \in \mathcal{Z}} q^{2p} \hat{k}_{\mathcal{Z}}(q, t) \hat{k}_{\mathcal{Z}}^{*M}(-q, t) + \sum_J \sum_{A_0 \cap \mathcal{Z}^{|J|}} a_J i^{J \cdot R} Q^R \hat{k}_{\mathcal{Z}}(Q, t),$$

for $n = 0$.

2.2 Important remark

We shall be using the following fact freely. The sums (here $l \geq 1$) satisfy

$$\sum^{\star} \frac{1}{q_1^2} \frac{1}{q_2^2} \cdots \frac{1}{q_l^2} \frac{1}{(n - q_l - q_{l-1} - \ldots q_1)^2} \leq \frac{C_l}{n^2},$$

and

$$\sum^{\star} \frac{1}{q_1} \frac{1}{q_2^2} \cdots \frac{1}{q_l^2} \frac{1}{(n - q_l - q_{l-1} - \ldots q_1)^2} \leq \frac{C_l}{n},$$

where the $\star$ means that the sum is over all the $q_1, q_2, \ldots, q_k$ such that $\left|q_j\right| \geq 1$ and $|n - q_k - \cdots - q_1| \geq 1$. (This can be proved by induction; see Lemma 3.1 in [5].)

2.3 Beginning the proof of Theorem 1: the solution remains close to the initial condition

The main idea of the proof of Theorem 1 is to show that the Fourier coefficients with nonzero wave numbers decay exponentially in time. As a warm up and to introduce the main tools used in proving this exponential decay, we shall show that if the initial data ψ are close to its average (in the sense of the hypotheses of Theorem 1), the solution to (2) with ψ as the initial data remains close to its average. To do this, we must show that the set of Fourier coefficients with wave numbers $q = \pm 1$ and $|q| \geq 2$ control each other and remain small with respect to the average of k (here we use the concept of a Trapping region), as long as the average of k remains large enough, and that k remains positive.

Then, in the following section, we will prove the exponential decay of the Fourier coefficients of the solution to (2): Lemma 7 guarantees that if we start close enough to $\hat{\psi}(0)$ (the average of the initial data), the Fourier coefficients of the solution decay exponentially in time, as long as the average of k remains large enough. That the average of k remains large enough (and does not blow up) is the purpose of Lemma 8. We resume all these findings in Proposition 9 and then proceed on to finish our proof of Theorem 1.

From now on, all the hypotheses of Theorem 1 on the structure of (2) are assumed.

To begin, as we need our curves to remain convex as long as the solution to (2) exists, we shall prove the following.

Lemma 2 *There exists a $\delta > 0$ such that if a solution k to (2) with the initial data $\psi > 0$ satisfies on an interval of time $[0, \tau]$ that*

$$(1 - 4\delta)\hat{\psi}(0) \leq \hat{k}(0, t) \leq (1 + 4\delta)\hat{\psi}(0)$$

and for all $q \neq 0$ that

$$|q|^{2p+1} \max\left\{\left|Im\left(\hat{k}(q, t)\right)\right|, \left|Re\left(\hat{k}(q, t)\right)\right|\right\} \leq \delta\hat{\psi}(0),$$

then $k(\cdot, t) > 0$ on $[0, \tau]$.

Proof Using the Fourier expansion of k, and the hypotheses of the lemma, we have that

$$k(0, t) \geq (1 - 4\delta)\hat{\psi}(0) - \sum_{q \neq 0} \frac{2\delta\hat{\psi}(0)}{|q|^{2p+1}}.$$

Thence, we will have that $k(\cdot, t) > 0$ as long as $\delta > 0$ is such that

$$(1 - 4\delta) - 2\delta > 0,$$

which happens to be true whenever $0 \leq \delta \leq 0.16$.

□

From now on, to proceed safely with our arguments we shall require that $\delta \leq 0.05$.

Lemma 3 *Given a smooth function ψ (which from now on we assume to be the curvature function of a simple smooth closed convex curve), there is a $\delta > 0$, which may depend on $\|\psi\|_\infty$, such that if ψ satisfies*

$$\|\psi\|_{2p+1} \leq \delta\hat{\psi}(0),$$

and if on $[0, \tau)$, the solution to (2) with the initial condition ψ satisfies

$$(1 - 4\delta)\hat{\psi}(0) \leq \hat{k}(0, t) \leq (1 + 4\delta)\hat{\psi}(0),$$

and

$$\left|\hat{k}(\pm 1, t)\right| \leq 2\delta\hat{\psi}(0).$$

Then,

$$|q|^{2p+1} \max\left\{\left|Re\left(\hat{k}(q,t)\right)\right|, \left|Im\left(\hat{k}(q,t)\right)\right|\right\} \leq \delta\hat{\psi}(0) \quad q \in \mathbb{Z} \setminus \{-1, 1\}$$

on the same time interval.

Proof We will show that for any $\rho > 0$ and any finite-dimensional approximation $k_{\mathcal{Z}}$ with $\mathcal{Z}$ large enough, and $t \in [0, \tau)$ the following holds

$$|q|^{2p+1} \max\left\{\left|\mathrm{Re}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right|, \left|\mathrm{Im}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right|\right\} \leq (1+\rho)\delta\hat{\psi}(0).$$

Being $\rho > 0$ arbitrary, we obtain the result for the full solution k. So let $\rho > 0$ and $\omega > 0$ be arbitrary. First we choose $\mathcal{Z}_0$ so that for any $\mathcal{Z} \supset \mathcal{Z}_0$ on $[0, \tau - \omega]$ we have that

$$|k_{\mathcal{Z}}(\pm 1, t)| \leq (2+\rho)\delta\hat{\psi}(0)$$

(this is, of course, justified by Theorem A1 in appendix), and

$$(1 - 4\delta - \rho)\hat{\psi}(0) \leq \hat{k}_{\mathcal{Z}}(0, t) \leq (1 + 4\delta + \rho)\hat{\psi}(0).$$

Now, fix $\mathcal{Z}$ as above, and consider the set (which will be our **trapping region**)

$$\Omega_{\mathcal{Z}} = \left\{w \in \mathbb{C}^{\mathcal{Z}} : (1+\rho)\delta\hat{\psi}(0) \geq |q|^{2p+1} \max\left\{|\mathrm{Re}\,(w(q))|, |\mathrm{Im}\,(w(q))|\right\}, \quad |q| \geq 2\right\}.$$

We shall argue that when the trajectory $\hat{k}_{\mathcal{Z}}$ hits the boundary of $\Omega_{\mathcal{Z}}$ at time $\tau' \in [0, \tau - \omega]$ then its velocity vector $\partial_t \hat{k}_{\mathcal{Z}}$ points towards the interior of $\Omega_{\mathcal{Z}}$ (i.e. the trajectory $\hat{k}_{\mathcal{Z}}$ is trapped in $\Omega_{\mathcal{Z}}$), which in turn shows that the inequality that defines $\Omega_{\mathcal{Z}}$ holds beyond time τ'. Notice that $\tau' > 0$ due to the fact that the initial condition belongs to the interior of the trapping region. Hence, since the set of times for which $\hat{k}_{\mathcal{Z}}$ belongs to $\Omega_{\mathcal{Z}}$ is clearly nonempty and closed by continuity, we would then have that it is also open, and by connectedness that for all times in $[0, \tau)$, $\hat{k}_{\mathcal{Z}}$ belongs to $\Omega_{\mathcal{Z}}$.

In order to proceed, assume that

$$|n|^{2p+1} \max\left\{\left|\mathrm{Re}\left(\hat{k}_{\mathcal{Z}}(n,\tau')\right)\right|, \left|\mathrm{Im}\left(\hat{k}_{\mathcal{Z}}(n,\tau')\right)\right|\right\} = (1+\rho)\delta\hat{\psi}(0)$$

for a given n occurs for a first time at time $\tau' > 0$, τ' as described above. Let us consider the real and imaginary parts of the evolution equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{k}_{\mathcal{Z}}(n,t) &= P_n \hat{k}_{\mathcal{Z}}(n,t) \\ &\quad - \sum_{q \in \mathcal{Z}-\{n\}} q^{2p}\hat{k}_{\mathcal{Z}}(q,t)\hat{k}_{\mathcal{Z}}^{*M}(n-q,t) + \sum_{J}\sum_{A_n \cap \mathcal{Z}^{|J|}} a_J i^{J\cdot R} Q^R \hat{k}_{\mathcal{Z}}(Q,t),\end{aligned}$$

where

$$P_n = -n^{2p}\hat{k}_{\mathcal{Z}}^{M}(0,t) + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{k}_{\mathcal{Z}}^{j_l}(0,t).$$

Do keep in mind that P_n is real valued. So what we must show in order to prove that $\Omega_{\mathcal{Z}}$ is truly a trapping region is that in the differential equation right above, the real and imaginary parts of the term containing P_n have the opposite sign of the real and imaginary parts of $\hat{k}_{\mathcal{Z}}$, and that it also dominates in size, for $\delta > 0$ small enough, the other terms in the equation when $\hat{k}_{\mathcal{Z}}$ hits the boundary of $\Omega_{\mathcal{Z}}$.

To begin, first, we examine the terms different from $P_n\hat{k}(n,t)$ in the last equation. We begin with

$$\left| -\sum_{q\in\mathcal{Z}-\{n\}} q^{2p}\hat{k}_{\mathcal{Z}}(q,t)\hat{k}_{\mathcal{Z}}^{*M}(n-q,t)\right|.$$

The condition $q \neq n$ in the sum implies that in the convolution term $\hat{k}_{\mathcal{Z}}^{*M}(n-q,t)$ there is at least one term $\hat{k}_{\mathcal{Z}}(q_j,t)$ with $q_j \neq 0$. (If all the terms in this convolution term are $\hat{k}_{\mathcal{Z}}(0,t)$, then necessarily $q = n$.) Hence, in any term of the sum there are at least two factors of order δ, since necessarily $q \neq 0$. The fact that

$$|q|^{2p+1}\left|\hat{k}_{\mathcal{Z}}(q,t)\right| \leq \sqrt{2}\delta(1+\rho)\hat{\psi}(0)$$

(where we have used the fact that for a complex number z, $|z| \leq \sqrt{2}\max\{|\mathrm{Re}(z)|, |\mathrm{Im}(z)|\}$) together with $\hat{k}_{\mathcal{Z}}(0,t) \leq (1+4\delta+\rho)\hat{\psi}(0)$ for $t \in [0,\tau']$ implies that the sum can be bounded independently of $\mathcal{Z}$ (recall the "important remark"). Hence, this term is of order $O\left(\delta^2/n\right)$ and the implicit constant in this bound may depend on powers of $\hat{\psi}(0)$.

Let us be a little more explicit regarding our estimates. In the sum being considered, every term in each of the products, which are parts of the sum, can be bounded as follows. First, by the discussion above, there are at least two Fourier coefficients with nonzero wave number, and each can be bounded above by (the extra $2+\rho$ is to include the Fourier coefficients $\hat{k}(\pm 1,t)$ in the bound)

$$\frac{\sqrt{2}(2+\rho)\delta(1+\rho)\hat{\psi}(0)}{|q|^{2p+1}}.$$

Therefore, taking into account that given a term in the sum among its $M+1$ factors there might be exactly d of them, with $2 \leq d \leq M+1$, Fourier coefficients with nonzero wave number, and that

$$\hat{k}_{\mathcal{Z}}(0,t) \leq (1+4\delta+\rho)\hat{\psi}(0),$$

the sum can bounded above by (here q_1 plays the role of q)

$$\left(\sqrt{2}\right)^{M+1}(2+\rho)^{M+1}\delta^2(1+4\delta+\rho)^{M+1}(1+\rho)^{M+1}\hat{\psi}(0)^{M+1}\sum_{k=2}^{M+1}\binom{M+1}{d}$$
$$\times \sum{}^{\star} \frac{1}{|q_1|}\frac{1}{|q_2|^{2p+1}}\cdots\frac{1}{|q_d|^{2p+1}}\frac{1}{|n-q_1-\cdots-q_d|^{2p+1}},$$

where $\sum^\star$ indicates that the sum is over all tuples $(q_1,\ldots,q_d)$ of integers such that $q_j \neq 0$ and $n-q_1-\cdots-q_d \neq 0$. Then using the important remark gives our claim.

Now we look at the term

$$\left|\sum_J \sum_{A_n\cap\mathcal{Z}^{|J|}} a_J i^{J\cdot R} Q^R \hat{k}_{\mathcal{Z}}(Q,t)\right|.$$

In this case, by the definition of the set $A_n \cap \mathcal{Z}^{|J|}$, in the convolution $\hat{k}_{\mathcal{Z}}(Q,t)$ there are at least two terms $\hat{k}_{\mathcal{Z}}\left(q_{j_1},t\right)$ and $\hat{k}_{\mathcal{Z}}\left(q_{j_2},t\right)$ with $q_{j_1}, q_{j_2} \neq 0$. Also, the fact that in the expression Q^R the powers of the individual terms are less than or equal to $2p-1$ together with that on the interval $[0,\tau']$, $|q|^{2p+1}\left|\hat{k}_{\mathcal{Z}}(q,t)\right| \leq \sqrt{2}(1+\rho)\delta\hat{\psi}(0)$, and $\hat{k}_{\mathcal{Z}}(0,t) \leq (1+4\delta+\rho)\hat{\psi}(0)$ imply that the sums can be bounded independently of $\mathcal{Z}$, and hence, by our previous observation, this sum is $O\left(\delta^2/n\right)$ and the implicit constant may depend on powers of $\hat{\psi}(0)$. To be more precise, fixing a multi-index J, the term

$$\sum_{A_n\cap\mathcal{Z}^{|J|}} a_J i^{J\cdot R} Q^R \hat{k}_{\mathcal{Z}}(Q,t)$$

can be bounded as follows. Since the Fourier coefficient $\hat{k}_{\mathcal{Z}}(q,t)$, $q\neq 0$ can be bounded by

$$2\sqrt{2}(1+2\rho)\delta\hat{\psi}(0)/\left|q\right|^{2p+1},$$

and if this term comes from a derivative, it will contribute to a term of the convolution which can be bounded above as follows

$$|q|^m \max\left\{\left|\mathrm{Re}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right|, \left|\mathrm{Im}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right|\right\} \leq (2+\rho)\,(1+\rho)\,\delta\hat{\psi}(0)/|q|^2,$$

this taking into account that $m \leq 2p-1$. Since there are at least two terms for which $q\neq 0$, and each term in the sum, which is the product of $|J|$, can have exactly d, $2\leq d\leq |J|$, Fourier coefficients with nonzero wave number we obtain a bound

$$a_J\left(2\sqrt{2}\right)^{|J|}(1+2\rho)^{|J|}(2+\rho)^{|J|}\delta^2(1+4\delta+\rho)^{|J|}\hat{\psi}(0)^{|J|}$$
$$\times\sum_{d=2}^{|J|}\binom{|J|}{d}\sum^{\star}\frac{1}{|q_1|^2}\cdots\frac{1}{|q_d|^2},\frac{1}{|n-q_1-\cdots-q_d|^2}$$

where again the $\sum^{\star}$ indicates that the sum is over all tuples $(q_1,\ldots,q_d)$ of integers such that $q_j\neq 0$ and $n-q_1-\cdots-q_d\neq 0$. The fact that we are only adding over a finite number of multi-indices J allows us to conclude our claim.

By the hypotheses of Theorem 1 (more precisely that referring to $P_{n,5\delta}$), which are in place here, it is not difficult to see that there is a $c<0$ such that for all n, $P_n\leq c<0$, and for n large and ρ small, $P_{n,4\delta+\rho}\sim n^{2p}$ and that $\hat{k}_{\mathcal{Z}}(n,t)$ is of order δ/n^{2p+1}, at time τ'. Hence, by taking $\delta>0$ small enough, when the trajectory $\hat{k}_{\mathcal{Z}}$ hits the boundary of $\Omega_{\mathcal{Z}}$, the real and imaginary parts of $\partial_t\hat{k}_{\mathcal{Z}}(n,\tau')$ are dominated by the real and imaginary parts, respectively, of the term $P_n\hat{k}_{\mathcal{Z}}(n,\tau')$, which is of order δ/n, and this implies that the trajectory $\hat{k}_{\mathcal{Z}}(n,t)$ enters the interior of $\Omega_{\mathcal{Z}}$ at τ'. This shows that if the trajectory $\hat{k}_{\mathcal{Z}}$ hits the boundary of $\Omega_{\mathcal{Z}}$, it will then go to its interior; from this, we conclude that on $[0,\tau)$ the estimate claimed at the beginning of the argument holds for the full solution k.

□

To extend the control to the Fourier coefficients $\hat{k}(\pm 1,t)$, we use the following lemma.

Lemma 4 *There exists a $\delta>0$ such that if*

$$|q|^{2p+1}\max\left\{\left|Im\left(\hat{k}(q,t)\right)\right|, \left|Re\left(\hat{k}(q,t)\right)\right|\right\}\leq\delta\hat{\psi}(0)$$

for all $|q|\geq 2$, $\left|\hat{k}(\pm 1,t)\right|\leq 2\delta\hat{\psi}(0)$, and $(1-4\delta)\hat{\psi}(0)\leq\hat{k}(0,t)\leq(1+4\delta)\hat{\psi}(0)$, then

$$\max\left\{\left|Re\left(\hat{k}(\pm 1,t)\right)\right|, \left|Im\left(\hat{k}(\pm 1,t)\right)\right|\right\}\leq\delta^{\frac{3}{2}}\hat{\psi}(0).$$

Proof The proof of this lemma is the same as the proof of Lemma 3 in [6] (even though its statement is different), and the key fact used in the proof is identity (3). For completeness sake, we reproduce the proof given in [6] here.

Since we have that $\delta\hat{\psi}(0) \geq \left|\hat{\psi}(\pm 1)\right|$, we can choose a $\tau' \in [0, \tau]$ such that $\left|\hat{k}(\pm 1, t)\right| \leq 2\delta\hat{k}(0, t)$ on $[0, \tau']$. We have the following identity

$$\frac{1}{k(\theta, t)} = \frac{1}{\hat{k}(0, t)} \frac{1}{1 + \sum_{q \neq 0} \frac{\hat{k}(q, t)}{\hat{k}(0, t)} e^{iq\theta}} = \frac{1}{\hat{k}(0, t)} \frac{1}{1+z} = \frac{1}{\hat{k}(0, t)} \sum_{n=0}^{\infty} (-1)^n z^n,$$

where $z = z(\theta, t) = \sum_{q \neq 0} \frac{\hat{k}(q, t)}{\hat{k}(0, t)} e^{iq\theta}$. It can be easily seen that for the Fourier modes of z we have:

$$\hat{z}(p, t) = \begin{cases} 0 & \text{if } p = 0, \\ \dfrac{\hat{k}(p, t)}{\hat{k}(0, t)} & \text{otherwise.} \end{cases}$$

Taking Fourier transform of the previous expression yields

$$\widehat{\left(\frac{1}{k}\right)}(-1, t) = \frac{1}{\hat{k}(0, t)} \left(-\hat{z}(-1, t) + \sum_{m=2}^{\infty} (-1)^m \sum_{q_1 + \cdots + q_m = -1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t) \right).$$

Since k is the curvature of a convex curve, we have $U_{\pm}(k) = 0$; so,

$$\widehat{\left(\frac{1}{k}\right)}(-1, t) = \int_0^{2\pi} \frac{e^{-(-1)i\theta}}{k(\theta, t)} \, d\theta = U(k) = 0;$$

then,

$$\left|\hat{z}(-1, t)\right| = \left| \sum_{m=2}^{\infty} (-1)^m \sum_{q_1 + \cdots + q_m = -1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t) \right|.$$

In order to estimate the sum on the right-hand side, first notice that

$$\sum_{q_1 + \cdots + q_m = -1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t) = \sum_{j=0}^{m-1} \binom{m}{j} \hat{z}(1, t)^j \sum_{l=0}^{m-j-1} \binom{m-j}{l} \hat{z}(-1, t)^j \sum_{q_{j+l+1} + \cdots + q_m = -1-j+l} \hat{z}(q_{j+l+1}, t) \cdots \hat{z}(q_m, t).$$

Now we proceed to estimate $S_m = \sum_{q_1 + \cdots + q_m = -1} \hat{z}(q_1, t) \cdots \hat{z}(q_m, t)$. We shall use the fact that

$$|q|^{2p+1} |z(q, t)| \leq \delta/(1 - 4\delta) \leq 1$$

for $\delta > 0$ small enough:

$$|S_m| \le \sum_{j=0}^{m-1} \binom{m}{j} \left|\hat{z}(1,t)\right|^j \sum_{l=0}^{m-j-1} \binom{m-j}{l} \left|\hat{z}(-1,t)\right|^l \sum_{\substack{q_{j+l+1}+\cdots+q_m \\ =-1-j+l}} \delta^{m-j-l} \frac{1}{q_{j+l+1}^2} \cdots \frac{1}{q_m^2}$$

$$\le \sum_{j=0}^{m} \binom{m}{j} \left|\hat{z}(1,t)\right|^j \sum_{l=0}^{m-j} \binom{m-j}{l} \left|\hat{z}(-1,t)\right|^l \delta^{m-j-l} L^{m-j-l}$$

$$= \left(\left|\hat{z}(1,t)\right| + \left|\hat{z}(-1,t)\right| + \delta L\right)^m,$$

where L is a constant independent δ (actually one can take $L = \pi^2/3$).

As we have that $\left|\hat{z}(\pm 1,t)\right| \le 2\delta$, we get

$$|S_m| \le \left(\left|\hat{z}(1,t)\right| + \left|\hat{z}(-1,t)\right| + \delta L\right)^m \le \delta^m (4+L)^m.$$

Therefore,

$$\left|\hat{z}(-1,t)\right| \le \frac{\delta^2(4+L)^2}{1-\delta(4+L)} \le \frac{\delta^{\frac{3}{2}}}{1+4\delta},$$

for $\delta > 0$ small enough, and the conclusion of the lemma follows.

□

We resume what we have proved in this section in the following.

Proposition 5 *Given a function $\psi > 0$, the curvature function of a convex closed curve, there is a $\delta > 0$ (which may depend on $\|\psi\|_\infty$) such that if*

$$\|\psi\|_{2p+1} \le \delta\hat{\psi}(0),$$

and on $[0,\tau)$, a solution to (2) with the initial data ψ satisfies

$$(1-4\delta)\hat{\psi}(0) \le \hat{k}(0,t) \le (1+4\delta)\hat{\psi}(0).$$

Then

$$|q|^{2p+1} \max\left\{\left|Im\left(\hat{k}(q,t)\right)\right|, \left|Re\left(\hat{k}(q,t)\right)\right|\right\} \le \delta\hat{\psi}(0), \quad q \in \mathbb{Z}\setminus\{0\},$$

on the same time interval.

2.4 Exponential decay of the Fourier coefficients: finishing the proof of Theorem 1

Our purpose now is to show exponential decay for the Fourier coefficients with wave number $q \ne 0$.

Lemma 6 *Assume that on $[0,\tau]$, $\tau \ge 0$, we have estimates*

$$|q|^{2p+1} \max\left\{\left|Im\left(\hat{k}(q,t)\right)\right|, \left|Re\left(\hat{k}(q,t)\right)\right|\right\} \le \delta e^{-\gamma t}\hat{\psi}(0)$$

for $|q| \ge 2$, and $(1-4\delta)\hat{\psi}(0) \le \hat{k}(0,t) \le (1+4\delta)\hat{\psi}(0)$. Then there exists $\eta > 0$ so that the same estimate holds on $[0,\tau+\eta)$.

Proof As in the proof of Lemma 3, we will show that for any $\rho > 0$ small there exists $\eta > 0$ independent of ρ so that for any finite-dimensional approximation $k_{\mathcal{Z}}$, with $\mathcal{Z}$ and $t \in [0, \tau + \eta)$ the following holds

$$|q|^{2p+1} \max \left\{ \left| \mathrm{Im}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right|, \left|\mathrm{Re}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right| \right\} \leq (1+2\rho)\delta\hat{\psi}(0)e^{-\gamma t}.$$

Being $\rho > 0$ arbitrary, we obtain the result for the full solution k.

First of all, by Proposition 5 the fact that k is actually smooth on $(0, T)$ by parabolic regularity, where $(0, T)$ is the maximum interval of existence of k, and Theorem A1 in Appendix, there is an $\eta > 0$ so that for all $\mathcal{Z}$ large enough,

$$\left|\hat{k}_{\mathcal{Z}}(0,t)\right| \leq 2\delta\hat{\psi}(0)e^{-\gamma t},$$

and

$$(1-5\delta)\hat{\psi}(0) \leq \hat{k}_{\mathcal{Z}}(0,t) \leq (1+5\delta)\hat{\psi}(0),$$

for $t \in [0, \tau + \eta)$ (so $\eta > 0$ is independent of ρ). Also, by Theorem A1 in appendix, the choice of $\mathcal{Z}$ can be made such that for all $q \in \mathcal{Z}$ on $[0, \tau]$ we have

$$|q|^{2p+1} \max \left\{ \left| \mathrm{Im}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right|, \left|\mathrm{Re}\left(\hat{k}_{\mathcal{Z}}(q,t)\right)\right| \right\} \leq (1+\rho)\delta\hat{\psi}(0)e^{-\gamma t}. \tag{6}$$

Having fixed $\mathcal{Z}$, we let $\hat{v}(n,t) = \exp(\gamma|n|t)\hat{k}_{\mathcal{Z}}(n,t)$, then, as for the finite-dimensional approximation that we are considering we have that

$$\begin{aligned}\frac{d}{dt}\hat{k}_{\mathcal{Z}}(n,t) &= -n^{2p}\hat{k}^M_{\mathcal{Z}}(0,t)\hat{k}_{\mathcal{Z}}(n,t) + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{k}^{j_l}_{\mathcal{Z}}(0,t)\hat{k}_{\mathcal{Z}}(n,t)\\ &\quad - \sum_{q\in\mathcal{Z}-\{n\}} q^{2p}\hat{k}_{\mathcal{Z}}(q,t)\hat{k}^{*M}_{\mathcal{Z}}(n-q,t) + \sum_J \sum_{A_n\cap\mathcal{Z}^{|J|}} a_J i^{J\cdot R} Q^R \hat{k}_{\mathcal{Z}}(Q,t),\end{aligned}$$

we obtain the following equation for $v(n,t)$, $n \in \mathcal{Z}$

$$\begin{aligned}\partial_t\hat{v}(n,t) &= \gamma|n|\hat{v}(n,t)\\ &\quad + \exp(\gamma|n|t)\Bigg[-n^{2p}\hat{k}^M(0,t)\hat{k}(n,t) + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{k}^{j_l}(0,t)\hat{k}(n,t)\\ &\quad - \sum_{q\in\mathcal{Z}-\{n\}} q^{2p}\hat{k}(q,t)\hat{k}^{*M}(n-q,t) + \sum_J\sum_{A_n\cap\mathcal{Z}^{|J|}} a_J i^{J\cdot R}Q^R\hat{k}(Q,t)\Bigg]\\ &= \gamma|n|\hat{v}(n,t) - n^{2p}\hat{v}^M(0,t)\hat{v}(n,t) + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l}a_{j_l}\hat{v}^{j_l}(0,t)\hat{v}(n,t)\\ &\quad - \sum_{q\in\mathcal{Z}-\{n\}}\frac{\exp(\gamma|n|t)}{\exp(\gamma[|n-q|+|q|]t)} q^{2p}\hat{v}(q,t)\hat{v}^{*M}(n-q,t)\\ &\quad + \sum_J\sum_{A_n\cap\mathcal{Z}^{|J|}}\frac{\exp(\gamma|n|t)}{\exp(\gamma|Q|t)} a_J i^{J\cdot R}Q^R\hat{v}(Q,t).\end{aligned}$$

Consider the set

$$\Omega_{\mathcal{Z}} = \left\{ w \in \mathbb{C}^{\mathcal{Z}} : (1+2\rho)\delta\hat{\psi}(0) \geq |q|^{2p+1} \max\left\{ |\mathrm{Re}\,(w(q))|, |\mathrm{Im}\,(w(q))| \right\}, \quad |q| \geq 2 \right\}.$$

We shall argue that when $\hat{v}$ hits the boundary of $\Omega_{\mathcal{Z}}$ at time t then $\partial_t \hat{v}$ points towards the interior of $\Omega_{\mathcal{Z}}$. To proceed, let n be a witness of this; that is to say, let $n \in \mathcal{Z}$ be such that

$$|n|^{2p+1} \max\left\{ \left|\mathrm{Re}\left(\hat{v}(n,t)\right)\right|, \left|\mathrm{Im}\left(\hat{v}(n,t)\right)\right| \right\} = (1+2\rho)\delta\hat{\psi}(0).$$

Notice that as at time τ the estimate (6) holds, necessarily $t \in (\tau, \tau+\eta)$.

To begin our argumentation, we shall show that the term

$$-n^{2p}\hat{v}^M(0,t)\hat{v}(n,t) + \gamma|n| + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{v}^{j_l}(0,t)\hat{v}(n,t) \tag{7}$$

dominates over the other terms in the expression on the left-hand side of the equation given.

Notice that by the inequalities

$$(1-5\delta)\hat{\psi}(0) \leq \hat{k}_{\mathcal{Z}}(0,t) \leq (1+5\delta)\hat{\psi}(0),$$

and the assumption on $P_{n,5\delta}$ in Theorem 1 (recall that $\hat{v}(0,t) = \hat{k}_{\mathcal{Z}}(0,t)$), for a well chosen γ, and $\rho > 0$ small enough,

$$-n^{2p}\hat{v}^M(0,t) + \gamma|n| + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{v}^{j_l}(0,t) \leq c < 0,$$

and $\hat{v}(n,t) = O\left(\delta/n^{2p+1}\right)$, so (7) is of order δ/n and its real and imaginary parts have the opposite sign to the real and imaginary parts of $\hat{v}(n,t)$: This fact, if (7) were the only term we had to deal with, would imply that $\partial_t\hat{v}$ points towards the interior of $\Omega_{\mathcal{Z}}$. However, this is not the case and we must examine other terms and show that they are smaller than (7) in absolute value when δ is small.

Let us begin by the term

$$\sum_{q\in\mathcal{Z}-\{n\}} \left| \frac{\exp(\gamma|n|t)}{\exp(\gamma[|n-q|+|q|]t)} q^{2p}\hat{v}(q,t)\hat{v}^{*M}(n-q,t) \right|.$$

Observe that

$$0 \leq \frac{\exp(\gamma|n|t)}{\exp(\gamma[|n-q|+|q|]t)} \leq 1,$$

and hence, we will just have to bound

$$\sum_{q\in\mathcal{Z}-\{n\}} \left| q^{2p}\hat{v}(q,t)\hat{v}^{*M}(n-q,t) \right|.$$

As we argued before, the condition $q \neq n$ in the sum implies that in the convolution term there is at least one term $\hat{k}(q_j,t)$ with $q_j \neq 0$. (If all the terms in this convolution term are $\hat{v}(0,t)$, then necessarily $q = n$.) Hence, in any term of the sum there are at least two factors of order δ. The fact that $|q|^{2p+1}\left|\hat{v}(q,t)\right| < 2\delta\hat{v}(0,t)$ and $\hat{v}(0,t) = \hat{v}(0,t) \leq (1+5\delta)\hat{\psi}(0)$ imply that the sums can be bounded above independently of $\mathcal{Z}$, and that the sum being considered is of order δ^2/n.

On the other hand, the term

$$\sum_J \sum_{A_n \cap \mathcal{Z}^{|J|}} \left| \frac{\exp(\gamma|n|t)}{\exp(\gamma|Q|t)} a_J i^{J\cdot R} Q^R \hat{v}(Q,t) \right|$$

is relatively easy to deal with, because by the definition of $A_n \cap \mathcal{Z}^{|J|}$ in any of the terms in the sum which, as the reader knows, are basically products of $\hat{v}(q,t)$, there are at least two factors for which $q \neq 0$, which amounts to the fact that the whole sum is of order δ^2/n.

Then, we can conclude that for $\delta > 0$ small enough $\partial_t \hat{v}$ points towards the interior of $\Omega_{\mathcal{Z}}$ whenever $\hat{v}$ belongs to its boundary. Hence,

$$|n|^{2p+1} \max \left\{ \left|\mathrm{Re}\left(\hat{v}(n,t)\right)\right|, \left|\mathrm{Im}\left(\hat{v}(n,t)\right)\right| \right\}$$

decreases, and therefore, for times a bit after τ'

$$\max \left\{ \left|\mathrm{Re}\left(\hat{k}_{\mathcal{Z}}(n,t)\right)\right|, \left|\mathrm{Im}\left(\hat{k}_{\mathcal{Z}}(n,t)\right)\right| \right\} < \frac{(1+2\rho)\delta\hat{\psi}(0)e^{-\gamma|n|t}}{n^{2p+1}},$$

and as anticipated, we get the claimed estimate for the full solution, at least for wave numbers q with $|q| \geq 2$. We now use Lemma 4, to obtain the estimate for the Fourier coefficients corresponding to the wave numbers ± 1.

□

Lemma 7 *Given a function ψ, there is a $\delta > 0$, which may depend on $\|\psi\|_\infty$, such that if the initial datum ψ satisfies the following inequality*

$$\|\psi\|_{2p+1} \leq \delta\hat{\psi}(0),$$

and such that if for all $t \in [0,\tau]$

$$(1-4\delta)\hat{\psi}(0) \leq \hat{k}(0,t) \leq (1+4\delta)\hat{\psi}(0),$$

and $|n|^{2p+1}\left|\hat{k}(n,t)\right| \leq \delta\hat{\psi}(0)$, then there exists a $\gamma > 0$ that depends on $\delta > 0$, and a constant $C > 0$ such that the solution to (2) satisfies on $[0,\tau]$:

$$\left|\hat{k}(n,t)\right| \leq \frac{C\delta\hat{\psi}(0)e^{-\gamma|n|t}}{n^{2p+1}}.$$

Proof Notice that the set where the estimate is valid is obviously nonempty (it at least contains $t=0$) and closed, and it is also open by the previous lemma. The result follows.

□

Now we show that the average curvature $\hat{k}(0,t)$ remains controlled.

Lemma 8 *Given an initial condition ψ, there is a $\delta > 0$, which may depend on $\|\psi\|_\infty$, such that if the solution to (2) with the initial data ψ satisfies on the time interval $[0,\tau]$ that $(1-4\delta)\hat{\psi}(0) \leq \hat{k}(0,t) \leq (1+4\delta)\hat{\psi}(0)$, $n^{2p}\left|\hat{k}(n,t)\right| \leq \delta\hat{k}(0,t)$, and $\left|\hat{k}(\pm 1,t)\right| < \delta\hat{k}(0,t)$, then we actually have $(1-3\delta)\hat{\psi}(0) \leq \hat{k}(0,t) \leq (1+3\delta)\hat{\psi}(0)$ on the same time interval.*

Proof With the same proof of Lemma 3.5 of [5], it is easy to see that

$$\sum q_1^{j_1} q_2^{j_2} \cdots q_l^{j_l} e^{-\mu|q_1|} e^{-\mu|q_2|} \cdots e^{-\mu|q_l|} e^{-\mu|n-q_l-\cdots q_1|} \leq C_{\mu,l,n} e^{-\frac{\mu}{4^{j_1+j_2+\cdots+j_l}}|n|}.$$

Using the above equation, and because

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{k}(0,t) = -\sum_{q\neq 0} q^{2p}\hat{k}(q,t)\hat{k}^{*M}(-q,t) + \sum_{J}\sum_{Q\in\mathcal{A}_0} a_J i^{J\cdot R} Q^R \hat{k}(Q,t),$$

by Lemma 7, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{k}(0,t) = O\left(\delta^2\hat{\psi}(0)e^{-\beta\gamma t}\right),$$

and integrating (here taking into account that $\hat{k}(0,0) = \hat{\psi}(0)$), this equality for $\delta > 0$ small enough implies the result.

□

From the previous lemmas, we can conclude the following.

Proposition 9 (Trapping Lemma) *Given an initial condition* ψ, *there is a* $\delta > 0$, *which may depend on* $\|\psi\|_\infty$, *such that if* ψ *satisfies the following inequality*

$$\|\psi\|_{2p+1} \leq \delta\hat{\psi}(0),$$

then the solution to (2) *with the initial data* ψ *satisfies the following. There is a* $\gamma > 0$ *and a constant* $C > 0$ *such that*

$$\left|\hat{k}(n,t)\right| \leq \frac{C}{n^{2p+1}}e^{-\gamma t},$$

and $(1-3\delta)\hat{\psi}(0) \leq \hat{k}(0,t) \leq (1+3\delta)\hat{\psi}(0)$, *as long as the solution to* (2) *exists.*

As a consequence of the previous proposition, the curvature function k remains uniformly bounded as long as the solution to (2) exists, so the flow exists for all time. Also, it follows that k is converging towards a constant smoothly and exponentially. Indeed, by the fundamental theorem of calculus

$$\hat{k}(0,s) = \hat{k}(0,t) + \int_t^s \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{k}(0,\tau)\,\mathrm{d}\tau,$$

and hence, by Lemma 8 we can take the limit as $s \to \infty$. Denote

$$\lim_{s\to\infty}\hat{k}(0,s) = \hat{k}(0,\infty),$$

and then

$$\hat{k}(0,\infty) = \hat{k}(0,t) + \int_t^\infty \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{k}(0,\tau)\,\mathrm{d}\tau.$$

Therefore,

$$\left|\hat{k}(0,\infty) - \hat{k}(0,t)\right| \leq \int_t^\infty \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{k}(0,\tau)\right|\,\mathrm{d}\tau \leq Ce^{-\beta\gamma t},$$

which shows our claim.

In order to improve our estimates, we need to consider the integral form of the equation

$$\partial_t\hat{k}(n,t) = \left[-n^{2p}\hat{k}^M(0,t) + \sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{k}^{j_l}(0,t)\right]\hat{k}(n,t) - \sum_{q\in\mathcal{Z}-\{n\}} q^{2p}\hat{k}(q,t)\hat{k}^{*M}(n-q,t) + \sum_J\sum_{A_n} a_J i^{J\cdot R}Q^R\hat{k}(Q,t).$$

For that, we can use the integrating factor method to obtain

$$
\begin{aligned}
&\hat{k}(n,t)\\
&=\left[\int_\tau^t\left[\sum_J\sum_{A_n} a_J i^{J\cdot R} Q^R \hat{k}(Q,\sigma)-\sum_{q\in\mathcal{Z}-\{n\}} q^{2p}\hat{k}(q,\sigma)\hat{k}^{*M}(n-q,\sigma)\right]e^{-\int_\tau^\sigma \Xi(s)\mathrm{d}s}\mathrm{d}\sigma\right.\\
&\qquad\left.+\hat{k}(n,\tau)\right]e^{\int_\tau^t \Xi(s)\,\mathrm{d}s},
\end{aligned}
$$

where $\Xi(s)$ is given by

$$
-n^{2p}\hat{k}^M(0,s)+\sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\hat{k}^{j_l}(0,s).
$$

Using the bounds proved above yields

$$
\begin{aligned}
&\left|\hat{k}(n,t)\right|\\
&\le\left[\int_\tau^t\left|\sum_J\sum_{A_n} a_J i^{J\cdot R} Q^R \hat{k}(Q,\sigma)-\sum_{q\in\mathcal{Z}-\{n\}} q^{2p}\hat{k}(q,\sigma)\hat{k}^{*M}(n-q,\sigma)\right|\left|e^{-\int_\tau^\sigma \Xi(s)\mathrm{d}s}\right|\mathrm{d}\sigma\right.\\
&\quad\left.+\left|\hat{k}(n,\tau)\right|\right]\left|e^{\int_\tau^t \Xi(s)\mathrm{d}s}\right|\\
&\le Ke^{-n^{2p}\int_\tau^t \hat{k}^M(0,\tau)\,\mathrm{d}\tau+\sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\int_\tau^t \hat{k}^{j_l}(0,\tau)\,\mathrm{d}\tau}.
\end{aligned}
$$

The bound K on the outermost integral can be shown to exist due to the decay estimates (which are exponential) that we have shown for the Fourier coefficients $\hat{k}(n,t)$.

Hence, by our previous considerations, given $\epsilon>0$, there is τ such that if $t>\tau$ then

$$
(1-\epsilon)\hat{k}(0,\infty)\le\hat{k}(0,t)\le(1+\epsilon)\hat{k}(0,\infty).
$$

Thence, we can bound

$$
\left|\hat{k}(n,t)\right|\le Ke^{P_{n,\epsilon}[t-\tau]},
$$

where $P_{n,\epsilon}$ is given by

$$
\begin{aligned}
P_{n,\epsilon}&=-n^{2p}[1-\epsilon]^M\hat{k}^M(0,\infty)\\
&\quad+\sum_{l=1}^{p-1}\sum_{j_l}(-1)^l n^{2l} a_{j_l}\left(1+\operatorname{sgn}\left(\left((-1)^l a_{j_l}\right)\epsilon\right)^{j_l}\hat{k}^{j_l}(0,\infty).
\end{aligned}
$$

This finishes the proof of Theorem 1, as for $\epsilon>0$ small enough, with $n=2$ we have assumed that $P_{2,\epsilon}\ge P_{n,\epsilon}$.

3 Applications

3.1 Polyharmonic flows

If s is the arclength parameter and k the curvature, the polyharmonic flow is given by

$$\frac{\partial}{\partial t}\gamma = (-1)^p \left[\frac{\partial^{2p} k}{\partial s^{2p}}\right] \nu.$$

The curvature then evolves by the equation

$$\frac{\partial k}{\partial t} = (-1)^p \left[\frac{\partial^{2p+2} k}{\partial s^{2p+2}} + k^2 \frac{\partial^{2p} k}{\partial s^{2p}}\right] - k\left(\frac{\partial k}{\partial \theta}\right)^2,$$

where

$$\frac{\partial}{\partial s} = \frac{\partial \theta}{\partial s}\frac{\partial}{\partial \theta} = k\frac{\partial}{\partial \theta}.$$

It is easy to show that the evolution equation satisfied by the curvature in the case of the polyharmonic flow satisfies an equation of the form of (2) plus the structure conditions (I) and (II). In fact, if all the operators $\frac{\partial}{\partial \theta}$ jump over all the k's we obtain a term

$$(-1)^p k^{2p+2}\left(\frac{\partial^{2p+2} k}{\partial \theta^{2p+2}} + \frac{\partial^{2p} k}{\partial \theta^{2p}}\right),$$

whereas, if any of the operators $\frac{\partial}{\partial \theta}$ act on any of the k's, we obtain a term where there is at least a product of two derivatives of k (of perhaps different orders). So in general, the curvature function in the polyharmonic flow satisfies an equation

$$\frac{\partial k}{\partial t} = (-1)^p k^{2p+2}\left(\frac{\partial^{2p+2} k}{\partial \theta^{2p+2}} + \frac{\partial^{2p} k}{\partial \theta^{2p}}\right) + H\left(k, \ldots, \frac{\partial^{2p+1} k}{\partial \theta^{2p+1}}\right), \tag{8}$$

where any term $z_0^{\alpha_0} \ldots z_{2p+1}^{\alpha_{2p+1}}$ of the polynomial $H\left(z_0, \ldots, z_{2p+1}\right)$ satisfies $\alpha_1 + \alpha_2 + \cdots + \alpha_{2p+1} \geq 2$. In this case, we have then that

$$G = (-1)^p z_0^{2p+2}\left(z_{2p+2} + z_{2p}\right) + H,$$

and G satisfies the structure conditions (I) and (II). For instance, for $p = 1$ the flow takes the form

$$\begin{aligned}
\frac{\partial k}{\partial t} &= (-1)\left[\frac{\partial^4 k}{\partial s^4} + k^2\frac{\partial^2 k}{\partial s^2}\right] - k\left(\frac{\partial k}{\partial \theta}\right)^2 \\
&= -k^4\frac{\partial^4 k}{\partial \theta^4} - k^4\frac{\partial^2 k}{\partial \theta^2} - k\left(\frac{\partial k}{\partial \theta}\right)^4 - 11k^2\frac{\partial k}{\partial \theta}\frac{\partial^2 k}{\partial \theta^2} \\
&\quad - 4k^3\left(\frac{\partial^2 k}{\partial \theta^2}\right)^2 - 7k^3\frac{\partial k}{\partial \theta}\frac{\partial^3 k}{\partial \theta^3} - k^3\left(\frac{\partial k}{\partial \theta}\right)^2 - k\left(\frac{\partial k}{\partial \theta}\right)^2.
\end{aligned}$$

It is then clear that G satisfies the conditions required in the introduction. The polynomial that determines the stability is then

$$\hat{\psi}(0)^{2p+2}\left(-x^{2p+2} + x^{2p}\right),$$

which is clearly negative for $x \geq 2$.

Applying Theorem 1, if we start close to the unit circle, in the sense of the seminorms $\|\cdot\|_{2p+5}$, the solution to the polyharmonic flow exists for all time and converges smoothly and exponentially fast to a circle (and as shown in [12] to a unit circle if the area enclosed by the initial curve is π).

We can also give rates of convergence. For instance, in the case of $p = 1$, for any $\epsilon > 0$ and t large enough (assuming that the initial curve encloses an area of π),

$$\left|\hat{k}(n,t)\right| \leq K e^{[-n^4[1-\epsilon]^4 + n^2[1+\epsilon]^4][t-\tau]}.$$

Also, as it is known that if convergent, when the initial curve encloses a region of area π, the polyharmonic flow satisfies $k \to 1$, then we have that for any $\epsilon > 0$ and t large enough the following estimate holds

$$\|k(\theta,t) - 1\|_{C^l(\mathbb{S}^1)} \leq C_{l,\epsilon} e^{[-16[1-\epsilon]^4 + 4(1+\epsilon)^4]t},$$

which improves on known results (see [12]). The general case of this estimate, as described in Corollary 1, follows from the structure of the evolution equation of the curvature given by (8).

3.2 Other flows: a very specific example

As another example, let us consider a simple plane curve γ evolving by the following flow

$$\frac{\partial \gamma}{\partial t} = -k \left[\frac{\partial^2 k}{\partial s^2}\right] N, \tag{9}$$

where s is the arclength parameter, k the curvature and N the exterior normal. A calculation shows that the curvature evolves by the following equation

$$\begin{aligned}
\frac{\partial k}{\partial t} &= -k\left(\frac{\partial^4 k}{\partial s^4}\right) - k^3\left(\frac{\partial^2 k}{\partial s^2}\right) - \left(\frac{\partial^2 k}{\partial s^2}\right)^2 - 2\left(\frac{\partial k}{\partial s}\right)\left(\frac{\partial^3 k}{\partial s^3}\right) - k\left(\frac{\partial k}{\partial \theta}\right)^2 \\
&= -k^5\left(\frac{\partial^4 k}{\partial \theta^4}\right) - k^5\left(\frac{\partial^2 k}{\partial \theta^2}\right) - 5k^4\left(\frac{\partial^2 k}{\partial \theta^2}\right)^2 - k^4\left(\frac{\partial k}{\partial \theta}\right)^2 \\
&\quad - 2k^2\left(\frac{\partial k}{\partial \theta}\right)^4 - 21k^3\left(\frac{\partial k}{\partial \theta}\right)^2\left(\frac{\partial^2 k}{\partial \theta^2}\right) - 9k^4\left(\frac{\partial k}{\partial \theta}\right)\left(\frac{\partial^3 k}{\partial \theta^3}\right) - k\left(\frac{\partial k}{\partial \theta}\right)^2.
\end{aligned}$$

First of all, notice that for a given $\delta > 0$ we have that

$$\begin{aligned}
P_{x,4\delta} &= -x^4(1-4\delta)^5\left(\hat{\psi}(0)\right)^5 + x^2(1+4\delta)^5\left(\hat{\psi}(0)\right)^5 \\
&= \left(-x^4(1-4\delta)^5 + x^2(1+4\delta)^5\right)\left(\hat{\psi}(0)\right)^5 \leq -0.5\left(\hat{\psi}(0)\right)^5,
\end{aligned}$$

whenever $0 < \delta < \frac{1}{32}$.

As a consequence, our computations give us for this case that for the initial conditions near the circle of radius 1, with $\hat{\psi}(0) = 1$, the solution to (9) converges towards a curve whose curvature converges exponentially towards a constant, say k_∞, and for any $\epsilon > 0$, there is a time $T_\epsilon > 0$ and a constant K_ϵ such that for $t > T_\epsilon$ the following estimate holds

$$\left|\hat{k}(n,t)\right| \leq K_\epsilon e^{[-n^4[1-\epsilon]^5 + n^2[1+\epsilon]^5]k_\infty^5[t-\tau]},$$

and a convergence rate of the curvature towards k_∞ is given by (that is, for any $\epsilon > 0$ there is a constant $C_{l,\epsilon}$ such that)

$$\|k(\theta,t) - k_\infty\|_{C^l(\mathbb{S}^1)} \leq C_{l,\epsilon} e^{k_\infty^5\left[-16[1-\epsilon]^5 + 4[1+\epsilon]^5\right]t},$$

which means that the rate of convergence is as close to $\exp\left(-12k_\infty^5\right)$ as wished.

3.3 A more general family of examples I

In this section, we will consider flows of form (1) with

$$F\left(k, k_s, \ldots, k_s^{(2p)}\right) = \sum_{j=1}^{2p} a_j \frac{\partial^{2j} k}{\partial s^{2j}} + H, \tag{10}$$

where H is again a polynomial on the derivatives of k of order less than $2p$ and where we only allow terms which contain products of two or more derivatives of k of order bigger or equal than 1. To give an example, H may contain terms of the form $k_s k_{ss}$ (as long as $2p \geq 4$) or $k(k_s)^2$.

For a constant W such that given ψ there exists a $\delta > 0$ for which whenever

$$W = \hat{\psi}(0) \geq \delta \|\psi\|_{2p+5}$$

then the conclusion of Theorem 1 holds, then we say that *flow* (1) *is stable around* W, *which in turn corresponds to the solution of the original flow being stable around a circle of radius* W^{-1} *in the sense discussed above*.

Again using the fact that the curvature k satisfies an equation

$$\frac{\partial k}{\partial t} = \frac{\partial^2 F}{\partial s^2} + k^2 F - k\left(\frac{\partial k}{\partial \theta}\right)^2, \quad \text{where} \quad \frac{\partial}{\partial s} = k\frac{\partial}{\partial \theta},$$

it can be shown that the polynomial that determines the stability of the flow is

$$\begin{gathered} (-1)^{p+1} a_p \hat{\psi}(0)^{2p+2} n^{2p+2} \\ + \\ \sum_{j=2}^{p} (-1)^j \left(a_j \hat{\psi}(0)^2 + a_{j-1}\right) \hat{\psi}(0)^{2j} n^{2j} \\ + \\ (-1) a_1 \hat{\psi}(0)^4 n^2, \end{gathered}$$

since the negativity of this polynomial for $n \geq 2$ implies the condition on $P_{n,4\delta}$, for $\delta > 0$ small enough, needed in the hypothesis of Theorem 1. Notice then that if $W = \hat{\psi}(0)$ is large enough, the previous polynomial is strictly negative for $n \geq 2$. Hence, we have from Theorem 1 the following.

Proposition 10 *Assume that* $sgn(a_p) = (-1)^p$. *There exists an* M *such that if* $W \geq M$ *then flow* (1) *with* F *of form* (10) *is stable around* W.

This proposition basically tells us that in the case of a flow with F of form (10), its stability depends on the sign of the derivative of the largest order, and this holds around circles that are small enough.

3.4 A more general family of examples II

Now we consider

$$F\left(k, k_s, \ldots, k_s^{(2p)}\right) = \sum_{j=0}^{p-1} a_{p-j} k^{2j} \frac{\partial^{2p-2j} k}{\partial s^{2p-2j}} + H, \tag{11}$$

with H as in the previous section. This family of possible F's includes as an example the case of

$$V = k_{ssss} + k^2 k_{ss} - \frac{1}{2} k (k_s)^2,$$

which is associated with the (normal) evolution of a curve, which is related to the steepest gradient flow of the functional (see [2])

$$E[\gamma] = \int_\gamma (k_s)^2 \, \mathrm{d}s.$$

In the case that F has form (11), the polynomial that determines stability is given by

$$(-1)^{p+1} a_p \hat{\psi}(0)^{2p+2} n^{2p+2}$$
$$+$$
$$\sum_{j=0}^{p-2} (-1)^{p-j} \left(a_{p-j} + a_{p-j-1}\right) \hat{\psi}(0)^{2p+2} n^{2p-2j}$$
$$+$$
$$(-1) a_1 \hat{\psi}(0)^{2p+2} n^2,$$

and hence, a conclusion we can draw without much difficulty is the following.

Proposition 11 *Assume that* $sgn(a_p) = (-1)^p$, *and*

$$\frac{3}{4} \left|a_p\right| \geq \sum_{j=1}^{p-1} \frac{5}{4^{j+1}} \left|a_{p-j}\right|,$$

then flow (1) *with* F *of form* (11) *is stable around any* $\hat{\psi}(0) = W > 0$.

Finally, notice that V above satisfies the hypothesis of the proposition.

3.5 Regularity

Here we discuss the sharpness of requirement (5) in the statement of Theorem 1. We start with the case of polyharmonic flows, and we shall centre our discussion in the case $p = 1$.

We will show that condition (5) which amounts to

$$\hat{\psi}(0) \geq \delta \|\psi\|_5$$

can be replaced by a milder

$$\hat{\psi}(0) \geq \delta \|\psi\|_{\frac{5}{2}+\eta}$$

where $\eta > 0$ is small, and hence (in view of the theorem in appendix), we only need to assume in Theorem 1 that$\psi \in H^{\frac{5}{2}+2}\left(\mathbb{S}^1\right)$.

Let us justify, without entering into the technicalities, the reasons behind our claim. As the reader has seen so far, in the evolution of a Fourier wave number there are good and bad terms. By a good term, we mean a term whose existence, if it were alone, affects the evolution of a Fourier wave number $\hat{k}(n,t)$ by making its norm decrease. The best of these terms is given by

$$-\hat{k}(0,t)^4\hat{k}(n,t), \tag{12}$$

which comes from taking the Fourier transform of the term

$$-k^4\frac{\partial^4 k}{\partial\theta^4}.$$

Regarding bad terms, those that could offset the evolution of a Fourier wave number towards values of smaller norm, there is what we might call a very bad term (it has the largest possible order), which in this case is given by a sum

$$\sum_{q_1,q_2,q_3,q_4}\hat{k}\left(q_1,t\right)\hat{k}\left(q_2,t\right)\hat{k}\left(q_3,t\right)\hat{k}\left(q_4,t\right) \\ \times\left(n-q_1-q_2-q_3-q_4\right)^4\hat{k}\left(n-q_1-q_2-q_3-q_4,t\right),$$

but in this sum, at least there are two q's which are nonzero. If we assume that

$$\hat{k}(q,t)\sim\frac{\delta}{|q|^{\frac{5}{2}+\eta}},$$

then one can show that the sum above is of order $\delta^2 n^{\frac{3}{2}-\eta}$. But then the term (12) is of order $\delta n^{\frac{3}{2}-\eta}$, so it will dominate the behaviour of the evolution of $\hat{k}(n,t)$, and then, there will be stability for the flow. In the general case, in order to obtain stability for the polyharmonic flow, we seem to require

$$\hat{\psi}(0)\geq\delta\|\psi\|_{\frac{2p+3}{2}+\eta},$$

with $\eta>0$ small.

As yet another example, working with this heuristics and which can be made rigourous without much difficulty, it can be shown that the flow studied in the recent paper [2], for which

$$F=k_{ssss}+k^2k_{ss}-\frac{1}{2}k(k_s)^2,$$

it is enough to ask for a condition, for a well chosen $\delta>0$,

$$\hat{\psi}(0)\geq\delta\|\psi\|_{\frac{7}{2}+\eta},$$

instead of the stronger (required by the hypotheses of Theorem 1)

$$\hat{\psi}(0)\geq\delta\|\psi\|_7,$$

to obtain stability.

Appendix A: Convergence of the finite-dimensional approximations

Here we discuss how the finite-dimensional approximations $k_{\mathcal{Z}}$ converge towards the full solution k of (2). We will prove the following.

Theorem A1 *Assume* (2) *has a unique smooth solution in* $[0, \tau')$. *Then for any compact subinterval* $[0, \tau]$, $0 < \tau < \tau'$, *given* $\epsilon > 0$ *there exists a finite symmetric set* $\mathcal{Z}_0$ *such that if* $\mathcal{Z} \supset \mathcal{Z}_0$, *then*

$$\sup_{t\in[0,\tau]} \sup_{q\in\mathcal{Z}} |q|^{2p+1} \left|\hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t)\right| < \epsilon.$$

For the purpose of the proof of this theorem, we have assumed that our equation has a solution on an interval $[0, \tau]$ which is smooth; however, we can require less regularity and prove similar statements. In fact, we can ask for k to belong to $L^\infty\left([0,\tau]; H^{2p+3}\left(\mathbb{S}^1\right)\right)$, which, taking into account the regularity theory of parabolic equations, would require the initial data ψ to belong to $H^{2p+3}\left(\mathbb{S}^1\right)$.

Now, we begin with the proof. The Fourier transform of k, $\hat{k}$, satisfies the (infinite-dimensional) system described in Section 2, whereas the finite approximation satisfies a truncated system as described in the same section. By definition, for $q \notin \mathcal{Z}$ we have $\hat{k}_{\mathcal{Z}}(q,t) = 0$. Notice that for $q \in \mathcal{Z}$ $\hat{\psi}_{\mathcal{Z}}(q) = \hat{\psi}(q)$ and $\hat{\psi}_{\mathcal{Z}}(q) = 0$ otherwise.

We can write a system of ODEs for the difference $\hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t)$ with $q \in \mathcal{Z}$.

$$\partial_t \left(\hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t)\right) = F_{\mathcal{Z}}\left(\hat{k}_{\mathcal{Z}}, q\right) - F_{\mathcal{Z}}\left(\hat{k}, q\right) + R_{\mathcal{Z}}\left(\hat{k}, q\right).$$

A solution to this system can be written as

$$\hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t) = \int_0^t F_{\mathcal{Z}}\left(\hat{k}_{\mathcal{Z}}, q\right) - F_{\mathcal{Z}}\left(\hat{k}, q\right) + R_{\mathcal{Z}}\left(\hat{k}, q\right) \,\mathrm{d}t,$$

which we shall rewrite in the following convenient way, using the fact that the $F_{\mathcal{Z}}$, with $\mathcal{Z}$ fixed, is multilinear,

$$\varphi := \hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t) = \int_0^t G_{\mathcal{Z}}\left(\varphi, \hat{k}, \hat{k} + \varphi, q\right) + R_{\mathcal{Z}}\left(\hat{k}, q\right) \,\mathrm{d}t.$$

To estimate φ, we set up the following iteration scheme

$$\varphi_{n+1} = \int_0^t G_{\mathcal{Z}}\left(\varphi_n, \hat{k}, \hat{k} + \varphi_n, q\right) + R_{\mathcal{Z}}\left(\hat{k}, q\right) \,\mathrm{d}t, \tag{13}$$

with $\varphi_0 = 0$.

Since k belongs uniformly to the Sobolev space $H^{2p+3}\left(\mathbb{S}^1\right)$, for any $p > 0$, there exists an A such that on $[0, \tau]$

$$\left|\hat{k}(q,t)\right| \leq \frac{A}{|q|^{2p+3}},$$

and hence, we can bound (independently of q) on $[0, \tau]$

$$R_{\mathcal{Z}}\left(\hat{k}, q\right) = O\left(\frac{1}{|B|^{2p+2}}\right),$$

where $B = \min_{q\notin\mathcal{Z}} |q|$. We shall write

$$D_n = \sup_{t\in[0,\delta]} \max_{q\in\mathcal{Z}} |q|^{2p+1} |\varphi_n(q,t)|$$

and $\delta > 0$ will be chosen below.

We can bound the right-hand side of (13) after multiplication by $|q|^{2p+1}$ by

$$\delta M D_n (D_n + A)^m + \delta\theta,$$

where θ is a bound on $|B|^{2p+1} \left| R_{\mathcal{Z}} \left(\hat{k}, q \right) \right|$, with B as above, M is a constant independent of $\mathcal{Z}$ and q, and m depends only on $\mathcal{F}$. Therefore,

$$D_{n+1} \leq \delta M D_n (D_n + A)^m + \delta\theta.$$

From this, we propose the recurrence

$$E_{n+1}^{(0)} = \delta M E_n^{(0)} \left(E_n^{(0)} + A \right)^m + \delta\theta, \quad E_0^{(0)} = 0.$$

Clearly $D_n \leq E_n^{(0)}$, and

$$\sup_{t \in [0,\delta]} \sup_{q \in \mathcal{Z}} |q|^{2p+1} \left| \hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t) \right| \leq \sup_n E_n^{(0)} =: E^{(0)}.$$

Now we specify our choices. Given $\epsilon > 0$, we choose $\mathcal{Z}$ such that $\theta = \epsilon / 3^{2M(2+A)^m \tau + 2}$ and we choose $\delta = \dfrac{1}{2M(2+A)^m}$. With this, it is easily shown that, given $\epsilon > 0$ small enough, $E_n^{(1)} \leq 2$ and then

$$E^{(0)} \leq 2\delta\theta = \frac{2\delta\epsilon}{3^{2M(2+A)^m \tau + 2}} \quad \text{on} \quad [0, \delta].$$

To proceed to $[\delta, 2\delta]$, we set for $t \in [\delta, 2\delta]$

$$\varphi := \hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t) = U_1(q) + \int_\delta^t G_{\mathcal{Z}} \left(\varphi, \hat{k}, \hat{k} + \varphi, q \right) + R_{\mathcal{Z}} \left(\hat{k}, q \right) \mathrm{d}t,$$

where

$$U_1(q) = \hat{k}_{\mathcal{Z}}(q,\delta) - \hat{k}(q,\delta),$$

and then, by following the same arguments as before (in particular setting up an iteration scheme as in (13)), given the solution to the recurrence

$$E_{n+1}^{(1)} = \delta M E_n^{(1)} \left(E_n^{(1)} + A \right)^m + \delta\theta + 2\delta\theta, \quad E_0^{(1)} = 0,$$

the number $\limsup_{n \to \infty} E_n^{(1)} = E^{(1)}$ gives a bound on

$$\sup_{t \in [\delta, 2\delta]} \sup_{q \in \mathcal{Z}} |q|^{2p+1} \left| \hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t) \right|,$$

and again, from our choices it can be shown that $E_n^{(1)} \leq 2$, so we have that

$$\limsup_{n \to \infty} E_n^{(1)} \leq 6\delta\theta = \frac{6\delta\epsilon}{3^{2M(2+A)^m \tau + 2}}.$$

In general, having established a bound

$$\sup_{t \in [k\delta, (k+1)\delta]} \sup_{q \in \mathcal{Z}} |q|^{2p+1} \left| \hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t) \right| \leq E^{(k)} \leq C_k \delta\theta, \quad \text{with} \quad C_k \leq 3^{k+1},$$

to obtain a bound on $[(k+1)\delta, (k+2)\delta]$, we get a recurrence

$$E_{n+1}^{(k+1)} = \delta M E_n^{(k+1)} \left(E_n^{(k+1)} + A\right)^m + \delta\theta + C_k \delta\theta, \quad E_0^{(k+1)} = 0.$$

Again, it can be shown that $E_n^{(k+1)} \leq 2$ (as a consequence of our choices), from which we get, writing $E^{(k+1)} = \limsup_{n\to\infty} E_n^{(k+1)}$,

$$\sup_{t\in[(k+1)\delta,(k+2)\delta]} \sup_{q\in\mathcal{Z}} |q|^{2p+1} \left|\hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t)\right| \leq E^{(k+1)} \leq C_{k+1}\delta\theta,$$

with $C_{k+1} = 2(C_k + 1)$, $C_0 = 2$, and we can show that $C_{k+1} \leq 3^{k+2}$. The whole interval will be covered in $k = \lfloor 2M(2+A)^m\tau \rfloor + 1$ steps, and hence, we will have that

$$\sup_{t\in[0,\tau]} \sup_{q\in\mathcal{Z}} |q|^{2p+1} \left|\hat{k}_{\mathcal{Z}}(q,t) - \hat{k}(q,t)\right| \leq \epsilon.$$

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