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The one-sided bounded slope condition in evolution problems

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Abstract

We establish a local Lipschitz regularity result of solutions to the Cauchy–Dirichlet problem associated with evolutionary partial differential equations

$$\begin{cases} \partial_t u - \operatorname{div} Df(\nabla u) = 0, & \text{in } \Omega_T, \\ u = u_0, & \text{on } \partial_{\mathcal{P}} \Omega_T. \end{cases}$$

We do not impose any growth assumptions from above on the function $f : \mathbb{R}^n \to \mathbb{R}$ and only require it to be convex and coercive. The domain Ω is required to be bounded and convex, and the time-independent boundary datum u_0 is supposed to be convex and Lipschitz continuous on $\overline{\Omega}$. It can be seen as an evolutionary analogue to the one-sided bounded slope condition. Additionally, assuming Ω to be uniformly convex, we establish global continuity on $\overline{\Omega_T}$ of the solution.

Keywords Parabolic equations · Continuity of solutions · One-sided bounded slope condition

Mathematics Subject Classification 35A15 · 35B65 · 35K55 · 49J40

1 Introduction

In this article, we are concerned with regularity properties of solutions to a class of Cauchy–Dirichlet problems of evolutionary partial differential equations of the form

$$\begin{cases} \partial_t u - \operatorname{div} Df(\nabla u) = 0, & \text{in } \Omega_T, \\ u = u_0, & \text{on } \partial_{\mathcal{P}} \Omega_T, \end{cases} \tag{1.1}$$

in a spacetime cylinder $\Omega_T := \Omega \times (0, T)$ with a bounded and convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and $T > 0$. As usual the parabolic boundary is defined by $\partial_{\mathcal{P}}\Omega_T := \overline{\Omega} \times \{0\} \cup$

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$\partial\Omega \times (0, T)$. We do not require our variational integrand f to satisfy any growth conditions from above but merely assume it to be convex and—to guarantee existence of solutions—also coercive (see 2.1). This approach follows a famous result in Hilbert–Haar theory for stationary problems, which—starting with a paper by Haar [14] and followed by papers from Hartman and Nirenberg [16], Stampacchia [23], Miranda [22], and Hartman and Stampacchia [17]—can now even be found in textbooks (see [13, Chapter 1] as one possible reference). Under the bounded slope condition (see Definition 5.2), it ensures the existence of Lipschitz continuous minimizers. For time-dependent problems, a related result for linear growth functionals can be found in [15]. Moreover, in [4] the existence of Lipschitz continuous solutions to parabolic Cauchy–Dirichlet problems is proven under the assumption that the time-independent initial and lateral boundary datum satisfies the bounded slope condition. More precisely, Lipschitz continuity with respect to the parabolic metric is obtained.

In recent years, there were different attempts to weaken the quite restrictive bounded slope condition. In [11], Clarke introduced the *one-sided bounded slope condition* and proved local Lipschitz continuity of minimizers to elliptic variational functionals. Later, a different generalization—named by Mariconda and Treu in [20] as *Cellina bounded slope condition*—was introduced by Cellina in [10] as a result of establishing a new kind of comparison principle working without the strict convexity assumption on the variational integrand in the elliptic case. Mariconda and Treu then combined the two approaches in [21] to receive a one-sided version of Cellina's condition. Further results on the one-sided and full bounded slope condition can be found in [7–9,19] and the references therein.

Our paper has its origin in the effort to weaken the bounded slope condition in the evolutionary setting. As in [11], we impose the one-sided bounded slope condition in order to compensate the lack of growth. As strict convexity of the variational integrand is not necessary to get unique solutions in the parabolic setting, we stay with Clarke's approach in this article and transfer the main ideas of his proof into the context of parabolic problems. To adapt the lower bounded slope condition to the new setting, we require existence of affine functions lying below the boundary datum u_0 at each point $x_0 \in \overline{\Omega}$ in contrast to just requiring it on $\partial\Omega$ as in [11, Section 1] in the elliptic case, so that we can cope with both the initial and lateral boundary parts at once. This condition is encoded into our structural assumptions on u_0 (see 2.3)—following an alternative characterization of Clarke's lower bounded slope condition in [11, Proposition 1.1]—and will be explicitly stated in Lemma 4.1.

Starting from the notion of variational solutions (see Definition 2.1), we prove local Lipschitz continuity with respect to space and time. The advantage of working with variational solutions rather than weak solutions is that they allow employing methods from the calculus of variations. If the integrand is sufficiently regular such that the Euler–Lagrange equation makes sense, it can be shown that variational solutions are weak solutions and vice versa, cf. [3, Sect. 1.3]. In the proof of the local Lipschitz continuity, we use a dilation method. Quite remarkably, this method allows to prove local Lipschitz continuity with respect to space and time. The convexity of the initial datum is needed to ensure the existence of an affine function lying below the solution in any parabolic boundary point (see Lemma 4.2). On the other hand, in [4] spatial Lipschitz continuity is proven under the two-sided bounded slope condition by a translation with respect to the spatial variable only. To prove Lipschitz continuity in time by a translation in time under the two-sided bounded slope condition would require the initial datum to be an affine function (u_0 as well as $-u_0$ have to be convex). In this respect, one of the advantages of the one-sided bounded slope condition in the parabolic setting is that for convex initial data solutions are Lipschitz continuous also with respect to the time variable.

Moreover, we are also interested in global regularity of variational solutions. Global Lipschitz continuity, however, cannot be expected. There are counterexamples already in the stationary setting [11, Sect. 1]. With this respect, it is more natural to investigate global continuity of solutions as in the stationary setting in [7, Theorem 5] and [11, Theorem 2.2]. In fact, as a second main result we establish that variational solutions are continuous on $\overline{\Omega_T}$, provided that the domain Ω is uniformly convex. In a certain sense, our results allow to weaken the assumptions in the main existence result in [4], which imposes the full bounded slope condition on the Cauchy–Dirichlet boundary datum and already starts in a suitably regular Lipschitz class of functions. As mentioned before, the consequence is that solutions are not anymore Lipschitz continuous up to the boundary, but merely locally Lipschitz and continuous up to the boundary in the uniformly convex case.

2 Notation and main results

Throughout this paper, $\Omega_T := \Omega \times (0, T)$ denotes a spacetime cylinder over a bounded and convex domain $\Omega \subset \mathbb{R}^n$ and $T > 0$. For short, we often write $v(t) := v(\cdot, t)$ for a function in $L^1(\Omega_T) \equiv L^1(0, T; L^1(\Omega))$. We consider the parabolic Cauchy–Dirichlet problem (1.1) with an integrand $f : \mathbb{R}^n \to \mathbb{R}$ that is only supposed to be convex and coercive. More precisely, we assume that

$$\begin{cases} f \text{ is convex,} \\ f(\xi) \geq \nu|\xi|^p + \mu, \quad \text{for any } \xi \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

with some $\nu > 0$, $p > 1$ and $\mu \in \mathbb{R}$. Since we do not impose a growth assumption from above on f, the notion of weak solution might in general not be well defined. Therefore, we introduce the notion of variational solution which goes back to a paper by Lichnewsky and Temam [18].

Definition 2.1 Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a variational integrand satisfying (2.1) and consider a Cauchy–Dirichlet datum

$$u_0 \in L^p\big(0, T; W^{1,p}(\Omega)\big) \quad \text{with } \partial_t u_0 \in L^2(\Omega_T) \text{ and } u_0(0) \in L^2(\Omega).$$

We identify

$$u \in C^0\big([0, T]; L^2(\Omega)\big) \cap u_0 + L^p\big(0, T; W_0^{1,p}(\Omega)\big)$$

as a *variational solution* associated with the Cauchy–Dirichlet problem (1.1) if and only if the variational inequality

$$\begin{aligned} \int_{\Omega_T} f(\nabla u)\, \mathrm{d}x\mathrm{d}t \leq \int_{\Omega_T} &\big[\partial_t v(v-u) + f(\nabla v)\big]\, \mathrm{d}x\mathrm{d}t \\ &+ \tfrac{1}{2}\|v(0) - u_0\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)} \end{aligned} \tag{2.2}$$

holds true, for any comparison function $v \in u_0 + L^p(0, T; W_0^{1,p}(\Omega))$ with $v(0) \in L^2(\Omega)$ and $\partial_t v \in L^2(\Omega_T)$. □

If the integrand f satisfies additional assumptions like p-growth from above, then variational solutions are weak solutions in the usual sense, cf. [3]. Moreover, the variational inequality (2.2) ensures that the initial datum $u(0) = u_0(0)$ is assumed. In this paper, we are mainly interested in time-independent initial and lateral boundary data $u_0 : \overline{\Omega} \to \mathbb{R}$. We assume that

$$\begin{cases} u_0 \text{ is convex,} \\ u_0 \text{ is Lipschitz continuous with constant } L > 0. \end{cases} \tag{2.3}$$

From [3, Theorem 1.2], we have the following existence result for variational solutions.

Theorem 2.2 *Suppose that $f:\mathbb{R}^n\to\mathbb{R}$ is a variational integrand satisfying* (2.1) *and that the Cauchy–Dirichlet datum u_0 fulfills the second condition in* (2.3). *Then, there exists a unique variational solution u in the sense of Definition* 2.1. *Moreover, u satisfies*

$$\partial_t u\in L^2(\Omega_T)\qquad \text{and}\qquad u\in C^{0,\frac12}\big([0,T];L^2(\Omega)\big).$$

Note that the second condition of (2.3) together with the convexity of f ensures that $\int_\Omega f(\nabla u_o)\,\mathrm{d}x<\infty$ and $u_o\in L^2(\Omega)$, which are the hypotheses in [3, Theorem 1.2]. Contrary to the elliptic setting, in the parabolic setting strict convexity is not a necessary assumption for uniqueness of solutions. In fact, as stated above, convexity is enough, cf. [4] and Lemma 3.4. Our first main result ensures that variational solutions are locally Lipschitz continuous with respect to space and time.

Theorem 2.3 *Suppose that $f:\mathbb{R}^n\to\mathbb{R}$ is a variational integrand satisfying* (2.1), *that the Cauchy–Dirichlet datum u_0 fulfills* (2.3) *and that u is a variational solution in the sense of Definition* 2.1. *Then, u is locally Lipschitz continuous with respect to space and time. More precisely, for any subcylinder $C\Subset\Omega_T$ there holds*

$$|u(x,s)-u(y,t)|\le\frac{2M}{\delta}\big(|x-y|+|s-t|\big)\quad \textit{for a.e. } (x,s),(y,t)\in C,$$

where $\delta:=\operatorname{dist}(C,\partial\Omega_T)$ denotes the Euclidean distance of C to the boundary of Ω_T and

$$M:=2\|u_0\|_{L^\infty(\overline{\Omega})}+L\operatorname{diam}(\Omega).$$

The next natural question concerns global regularity of variational solutions. Global Lipschitz continuity cannot be expected due to the counterexample in [11, page 5] and [7]. Nevertheless, under the stronger uniform convexity condition on the domain Ω we are able to prove continuity of variational solutions up to the lateral boundary. We first give the definition of uniform convexity from [22, Definizione 6.1].

Definition 2.4 A bounded, open subset $\Omega\subset\mathbb{R}^n$ is called *uniformly convex* if for every boundary point $x_0\in\partial\Omega$ there exists a hyperplane H_{x_0} passing through that point satisfying

$$\operatorname{dist}(y,H_{x_0})\ge\varepsilon|y-x_0|^2,\qquad \text{for any } y\in\partial\Omega. \tag{2.4}$$

Note that this assumption is stronger than strict convexity. As shown in [22, Proposizione 6.2], this condition ensures that any C^2-function satisfies the bounded slope condition on $\partial\Omega$. Our second main result can then be stated as follows:

Theorem 2.5 *Suppose that Ω is uniformly convex, that the variational integrand $f:\mathbb{R}^n\to\mathbb{R}$ satisfies* (2.1), *that the Cauchy–Dirichlet datum u_0 fulfills* (2.3) *and that u is a variational solution in the sense of Definition* 2.1. *Then, u is continuous on $\overline{\Omega_T}$.*

3 Setup and preliminaries

Throughout the paper, we will use some important results for variational solutions that will be stated here for the convenience of the reader. We start with localization principles for variational solutions with respect to space and time. The proofs are straight forward and can for instance be deduced as in [4, Lemma 3.2 and Remark 4.2].

Lemma 3.1 (Spatial localization principle) *Let u be a variational solution on Ω_T according to Definition 2.1 with $\partial_t u \in L^2(\Omega_T)$ and let $\Omega' \subset \Omega$ be a subdomain. Then, u is also a variational solution on the subcylinder $\Omega'_T := \Omega' \times (0, T)$.*

Lemma 3.2 (Temporal localization principle) *Let u be a variational solution on Ω_T according to Definition 2.1 with $\partial_t u \in L^2(\Omega_T)$ and let $0 \le t_1 < t_2 \le T$. Then, u is also a variational solution on the subcylinder $\Omega \times (t_1, t_2)$.*

Next, we observe that the variational inequality (2.2) can be reformulated by the use of the time derivative of u.

Remark 3.3 Let u be a variational solution according to Definition 2.1 with $\partial_t u \in L^2(\Omega_T)$. Adding and subtracting $\int_{\Omega_T} \partial_t u(v-u)\mathrm{d}x\mathrm{d}t$ on the right-hand side of the variational inequality (2.2) and using integration by parts with respect to time, we find that

$$\int_{\Omega_T} f(\nabla u)\,\mathrm{d}x\mathrm{d}t \le \int_{\Omega_T} \big[\partial_t u(v-u) + f(\nabla v)\big]\,\mathrm{d}x\mathrm{d}t \tag{3.1}$$

for any $v \in u_0 + L^p(0, T; W_0^{1,p}(\Omega))$ with $v(0) \in L^2(\Omega)$ and $\partial_t v \in L^2(\Omega_T)$. By an approximation argument, we infer that the preceding inequality is valid for any $v \in L^2(\Omega_T) \cap u_0 + L^p(0, T; W_0^{1,p}(\Omega))$.

Another very important result, which will be used frequently throughout the paper, is the following version of the comparison principle for variational solutions. Its proof can be deduced as in [4, Lemma 4.3].

Lemma 3.4 *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a variational integrand satisfying (2.1) and that*

$$u_i \in L^p\big(0, T; W^{1,p}(\Omega)\big) \quad \text{with } u_i(0) \in L^2(\Omega) \text{ and } \partial_t u_i \in L^2(\Omega_T)$$

for $i \in \{1, 2\}$ are two variational solutions on Ω_T according to Definition 2.1 with $u_1 \le u_2$ on $\partial_{\mathcal{P}}\Omega_T$ in the sense of traces. Then, we have $u_1 \le u_2$ a.e. on Ω_T.

4 Local Lipschitz continuity

In this section, we prove our first main result, the local Lipschitz continuity of variational solutions stated in Theorem 2.3. We start with a simple observation for convex functions. For the sake of completeness, we include the proof.

Lemma 4.1 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex and let $\varphi\colon \overline{\Omega} \to \mathbb{R}$ be a convex and Lipschitz continuous function with Lipschitz-constant $L > 0$. Then, for every point $x_0 \in \overline{\Omega}$ there exists $p \in \mathbb{R}^n$ with $|p| \le L$ such that*

$$\varphi(x) \ge \varphi(x_0) + \langle p, x - x_0\rangle$$

for all $x \in \overline{\Omega}$.

Proof We first consider the case $x_0 \in \Omega$. From the convexity of the function φ, we deduce that the subdifferential $\partial\varphi(x_0)$ of φ in x_0 is nonempty. For any subgradient $p \in \partial\varphi(x_0) \subset \mathbb{R}^n$ of φ at x_0, we have

$$\varphi(x) \ge \varphi(x_0) + \langle p, x - x_0\rangle, \quad \text{for any } x \in \overline{\Omega}.$$

Due to the Lipschitz continuity of φ with constant $L > 0$, we know that

$$\langle p, x - x_0 \rangle \leq \varphi(x) - \varphi(x_0) \leq L|x - x_0|, \quad \text{for any } x \in \overline{\Omega},$$

which implies $|p| \leq L$. This proves the result for $x_0 \in \Omega$.

If $x_0 \in \partial\Omega$, we consider a sequence $(x_i)_{i\in\mathbb{N}} \in \Omega$ with $x_i \to x_0$ as $i \to \infty$. We have already shown that for any $i \in \mathbb{N}$ there exists $p_i \in \mathbb{R}^n$ with $|p_i| \leq L$ such that $\varphi(x) \geq \varphi(x_i) + \langle p_i, x - x_i \rangle$ for any $x \in \overline{\Omega}$. Then, there exists a (not re-labeled) subsequence and $p \in \mathbb{R}^n$ such that $p_i \to p$. Since $|p_i| \leq L$ for any $i \in \mathbb{N}$ we also have $|p| \leq L$ and due to the continuity of φ we conclude that $\varphi(x) \geq \varphi(x_0) + \langle p, x - x_0 \rangle$. This finishes the proof of the lemma. □

Combining this result with the comparison principle in Lemma 3.4, we obtain the following lemma.

Lemma 4.2 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a variational integrand satisfying (2.1), that the Cauchy–Dirichlet datum u_0 fulfills (2.3) and that u is a variational solution in the sense of Definition 2.1. Then, for any $(x_0, t_0) \in \partial_{\mathcal{P}}\Omega_T$ there exists $p \in \mathbb{R}^n$ with $|p| \leq L$ such that $u \geq \ell_{x_0}$ a.e. in Ω_T, where ℓ_{x_0} denotes the affine function defined by $\ell_{x_0}(x) := u_0(x_0) + \langle p, x - x_0 \rangle$ for $x \in \overline{\Omega}$.*

Proof We first note that $x_0 \in \overline{\Omega}$, since $(x_0, t_0) \in \partial_{\mathcal{P}}\Omega_T$. From Lemma 4.1, we infer that there exists $p \in \mathbb{R}^n$ with $|p| \leq L$ such that $u_0(x) \geq \ell_{x_0}(x)$ for every $x \in \overline{\Omega}$, where the affine function ℓ_{x_0} is defined in the statement of the lemma. In particular, this shows that $u \geq \ell_{x_0}$ on $\partial_{\mathcal{P}}\Omega_T$. Note that the constant in time extension of ℓ_{x_0} to Ω_T is a variational solution in the sense of Definition 2.1 with initial and lateral boundary values ℓ_{x_0} and that $\partial_t u \in L^2(\Omega_T)$ due to Theorem 2.2. Therefore, the application of the comparison principle in Lemma 3.4 yields that $u \geq \ell_{x_0}$ a.e. in Ω_T. □

We are now in the position to prove Theorem 2.3. The idea to use a dilation of the domain in order to prove local Lipschitz continuity of minimizers to elliptic variational integrals is due to Clarke [11]. Here, we introduce a parabolic version of this technique. Quite surprisingly, we obtain a stronger result than local Lipschitz continuity with respect to the parabolic metric. We prove that variational solutions are locally Lipschitz continuous in space and time.

Proof of Theorem 2.3 We start by introducing some notation used throughout the proof. Let $\lambda \in (0, 1)$. For a fixed point $\xi \in \overline{\Omega}$, we define the space-dilated domain

$$\Omega^{\xi;\lambda} := \lambda\Omega + (1 - \lambda)\xi$$

and space-dilated points

$$x^{\xi;\lambda} := \lambda x + (1 - \lambda)\xi, \quad \text{for } x \in \overline{\Omega}. \tag{4.1}$$

Similarly, for a fixed time $\tau \in [0, T]$ we let

$$(0, T)^{\tau;\lambda} := \big((1 - \lambda^2)\tau, \lambda^2 T + (1 - \lambda^2)\tau\big).$$

and

$$t^{\tau;\lambda} := \lambda^2 t + (1 - \lambda^2)\tau, \quad \text{for } t \in [0, T]. \tag{4.2}$$

Furthermore, the dilated spacetime cylinder is given by

$$\Omega_T^{\xi,\tau;\lambda} := \Omega^{\xi;\lambda} \times (0, T)^{\tau;\lambda},$$

and the dilated variational solution $u^{\xi,\tau;\lambda}\colon \Omega_T^{\xi,\tau;\lambda} \to \mathbb{R}$ is defined via

$$u^{\xi,\tau;\lambda}(x,t) := \lambda u\left(\xi + \tfrac{x-\xi}{\lambda},\ \tau + \tfrac{t-\tau}{\lambda^2}\right). \tag{4.3}$$

Note that $u^{\xi,\tau;\lambda}(x^{\xi;\lambda}, t^{\tau;\lambda}) = \lambda u(x,t)$ for any $x \in \overline{\Omega}$ and $t \in [0,T]$ by definition. In the following, we proceed in four steps.

Step 1: Bound for the difference $u^\lambda - u$ on the parabolic boundary $\partial_{\mathcal{P}}\Omega_T^\lambda$. Let $\lambda \in (0,1)$, $\xi \in \overline{\Omega}$, and $\tau \in [0,T]$. We claim that

$$u^{\xi,\tau;\lambda} - u \le (1-\lambda)M_0 \quad \text{on } \partial_{\mathcal{P}}\Omega_T^{\xi,\tau;\lambda} \tag{4.4}$$

in the sense of traces, where

$$M_0 := \|u_0\|_{L^\infty(\overline{\Omega})} + L\,\mathrm{diam}(\Omega). \tag{4.5}$$

We consider $(\bar{x},\bar{t}) \in \partial_{\mathcal{P}}\Omega_T^{\xi,\tau;\lambda}$. Then, there exists a point $(x_0,t_0) \in \partial_{\mathcal{P}}\Omega_T$ such that $(x_0^{\xi;\lambda}, t_0^{\tau;\lambda}) = (\bar{x},\bar{t})$. For convenience in notation, we abbreviate $x_0^\lambda := x_0^{\xi;\lambda}$, $t_0^\lambda := t_0^{\tau;\lambda}$, $\Omega_T^\lambda := \Omega_T^{\xi,\tau;\lambda}$, and $u^\lambda := u^{\xi,\tau;\lambda}$. From Lemma 4.2, we infer that there exists $p \in \mathbb{R}^n$ with $|p| \le L$ such that $u \ge \ell_{x_0}$ a.e. in Ω_T, where ℓ_{x_0} denotes the affine function $\ell_{x_0}(x) := u_0(x_0) + \langle p, x - x_0\rangle$. Since $\partial_{\mathcal{P}}\Omega_T^\lambda \subset \Omega_T$, this implies that $u \ge \ell_{x_0}$ on $\partial_{\mathcal{P}}\Omega_T^\lambda$ in the sense of traces. Recalling that $(\bar{x},\bar{t}) = (x_0^\lambda, t_0^\lambda) \in \partial_{\mathcal{P}}\Omega_T^\lambda$, this allows to estimate

$$\begin{aligned}
u^\lambda(\bar{x},\bar{t}) - u(\bar{x},\bar{t}) &= \lambda u(x_0,t_0) - u(x_0^\lambda, t_0^\lambda)\\
&= \lambda u_0(x_0) - u(x_0^\lambda, t_0^\lambda)\\
&\le \lambda u_0(x_0) - \ell_{x_0}(x_0^\lambda)\\
&= \lambda u_0(x_0) - u_0(x_0) - \langle p, x_0^\lambda - x_0\rangle\\
&= (\lambda - 1)\big[u_0(x_0) - \langle p, x_0 - \xi\rangle\big]\\
&\le (1-\lambda)\Big[\|u_0\|_{L^\infty(\overline{\Omega})} + L\,\mathrm{diam}(\Omega)\Big].
\end{aligned}$$

This completes the proof of the claimed inequality (4.4). In the next step, we extend inequality (4.4) to the whole dilated spacetime cylinder Ω_T^λ.

Step 2: Bound for the difference $u^\lambda - u$ on the spacetime cylinder Ω_T^λ. Let $\lambda \in (0,1)$, $\xi \in \overline{\Omega}$, and $\tau \in [0,T]$. We claim that

$$u^{\xi,\tau;\lambda} - u \le (1-\lambda)M_0 \quad \text{a.e. on } \Omega_T^{\xi,\tau;\lambda}, \tag{4.6}$$

where M_0 is defined in (4.5).

As before, we use the shorthand notations $u^\lambda := u^{\xi,\tau;\lambda}$, and $\Omega_T^\lambda := \Omega_T^{\xi,\tau;\lambda}$ and recall that $\partial_t u \in L^2(\Omega_T)$ by Theorem 2.2. Due to the spatial and temporal localization principles for variational solutions from Lemmas 3.1 and 3.2, we know that $(u + M_0(1-\lambda))|_{\Omega_T^\lambda}$ is a variational solution on $\Omega_T^\lambda \subset \Omega_T$. We claim that also u^λ is a variational solution on Ω_T^λ. To prove this, we consider a comparison function $v \in u_0 + L^p\big((1-\lambda^2)\tau, \lambda^2 T + (1-\lambda^2)\tau; W_0^{1,p}(\Omega^\lambda)\big)$ with $v((1-\lambda^2)\tau) \in L^2(\Omega^\lambda)$ and $\partial_t v \in L^2(\Omega_T^\lambda)$ and let

$$\widetilde{v}(y,s) := \tfrac{1}{\lambda}\, v\big(\xi + \lambda(y-\xi), \tau + \lambda^2(s-\tau)\big), \quad \text{for } (y,s) \in \Omega_T.$$

Keeping in mind that $0^\lambda = (1-\lambda^2)\tau$ and $T^\lambda = \lambda^2 T + (1-\lambda^2)\tau$ by (4.2), the computation

$$\begin{aligned}
&\iint_{\Omega_T^\lambda} f\big(\nabla_x u^\lambda(x,t)\big)\mathrm{d}x\mathrm{d}t \\
&= \lambda^{n+2}\iint_{\Omega_T} f\big(\nabla_y u(y,s)\big)\mathrm{d}y\mathrm{d}s \\
&\le \lambda^{n+2}\iint_{\Omega_T}\Big[\partial_s\widetilde{v}(y,s)(\widetilde{v}-u)(y,s) + f\big(\nabla_y\widetilde{v}(y,s)\big)\Big]\mathrm{d}y\mathrm{d}s \\
&\quad + \lambda^{n+2}\Big[\tfrac12\|\widetilde{v}(0)-u_0\|^2_{L^2(\Omega)} - \tfrac12\|(\widetilde{v}-u)(T)\|^2_{L^2(\Omega)}\Big] \\
&= \iint_{\Omega_T^\lambda}\Big[\partial_t v(x,t)(v-u^\lambda)(x,t) + f\big(\nabla_x v(x,t)\big)\Big]\mathrm{d}x\mathrm{d}t \\
&\quad + \tfrac12\|(v-u^\lambda)(0^\lambda)\|^2_{L^2(\Omega^\lambda)} - \tfrac12\|(v-u^\lambda)(T^\lambda)\|^2_{L^2(\Omega^\lambda)}
\end{aligned}$$

shows the claim that u^λ is a variational solution on Ω_T^λ. Thus, so far we have shown that $u + M_0(1-\lambda)$ and u^λ are variational solutions on Ω_T^λ with weak time derivatives in $L^2(\Omega_T^\lambda)$. Moreover, from (4.4) we know that

$$u^\lambda \le u + M_0(1-\lambda) \quad \text{on } \partial_{\mathcal{P}}\Omega_T^\lambda$$

in the sense of traces. At this point, the comparison principle from Lemma 3.4 yields the claimed inequality (4.6).

Step 3: Local spatial Lipschitz continuity. Here, we prove that

$$|u(x,t)-u(y,t)| \le \frac{M}{\min\{\mathrm{d}_{\partial\Omega}(x|y),\mathrm{d}_{\partial\Omega}(y|x)\}}|x-y|, \tag{4.7}$$

for a.e. $x, y \in \Omega$ and a.e. $t \in [0,T)$, where for $x \ne y$ $\mathrm{d}_{\partial\Omega}(x|y)$ denotes the distance in space of x to the lateral boundary $\partial\Omega$ in the direction of y and M is defined as

$$M := 2\|u_0\|_{L^\infty(\overline{\Omega})} + L\,\mathrm{diam}(\Omega). \tag{4.8}$$

For $x \ne y$, we denote by $\pi_{\partial\Omega}(x|y)$ the spatial projection of x onto $\partial\Omega$ in the direction of y, which is the unique point of the form $\pi_{\partial\Omega}(x|y) = x + \alpha(y-x) \in \partial\Omega$ for some $\alpha > 1$. Then $d_{\partial\Omega}(x|y) = |x - \pi_{\partial\Omega}(x|y)|$. We now choose $\xi = \pi_{\partial\Omega}(x|y)$ and $\tau = t$ in the dilations defined in (4.1) – (4.3). Then, y can be written in the form

$$y = x^{\xi;\lambda} \in \Omega^{\xi;\lambda} \qquad \text{with } \lambda = 1 - \frac{|x-y|}{\mathrm{d}_{\partial\Omega}(x|y)}$$

and $t = t^{\tau;\lambda}$. From (4.6), we infer that

$$\lambda u(x,t) - u(y,t) = u^{\xi,\tau;\lambda}\big(x^{\xi;\lambda}, t^{\tau;\lambda}\big) - u\big(x^{\xi;\lambda}, t^{\tau;\lambda}\big) \le (1-\lambda)M_0$$

holds true for a.e. $x, y \in \Omega$ and a.e. $t \in (0,T)$. The comparison principle in Lemma 3.4 ensures that $|u| \le \|u_0\|_{L^\infty(\overline{\Omega})}$ a.e. in Ω_T. Hence, adding $(1-\lambda)u(x,t)$ on both sides of the preceding inequality, we obtain

$$u(x,t) - u(y,t) \le (1-\lambda)u(x,t) + (1-\lambda)M_0 \le (1-\lambda)M = \frac{M}{\mathrm{d}_{\partial\Omega}(x|y)}|x-y|$$

for a.e. $x, y \in \Omega$ and a.e. $t \in [0,T)$. Since the argument can be repeated with x and y reversed and choosing $\xi = \pi_{\partial\Omega}(y|x)$, the claimed local Lipschitz estimate (4.7) follows.

Step 4: Lipschitz continuity with respect to time. In the final step, we prove that

$$|u(x,t)-u(x,s)| \le \frac{M}{\min\{t, T-s\}}(t-s), \tag{4.9}$$

for a.e. $x \in \Omega$ and a.e. $s, t \in (0, T)$ with $s < t$, where M is defined as in (4.8).

For the dilations defined in (4.1)–(4.3), we choose $\xi = x$ and $\tau = 0$, so that $x = x^{\xi;\lambda}$ and

$$s = t^{\tau;\lambda} \in (0,T)^{\tau;\lambda} \qquad \text{with } \lambda = \sqrt{\frac{s}{t}} \in (0,1).$$

From (4.6), we infer that

$$\lambda u(x,t) - u(x,s) = u^{\xi,\tau;\lambda}\big(x^{\xi;\lambda}, t^{\tau;\lambda}\big) - u\big(x^{\xi;\lambda}, t^{\tau;\lambda}\big) \le (1-\lambda)M_0$$

holds true for a.e. $x \in \Omega$ and a.e. $0 < s < t < T$. As before, we add $(1-\lambda)u(x,t)$ on both sides of the preceding inequality and recall that $|u| \le \|u_0\|_{L^\infty(\overline{\Omega})}$ in Ω_T. In this way, we obtain

$$\begin{aligned} u(x,t) - u(x,s) &\le (1-\lambda)u(x,t) + (1-\lambda)M_0 \le (1-\lambda)M \\ &= \frac{M}{\sqrt{t}}\left(\sqrt{t}-\sqrt{s}\right) = \frac{M}{\sqrt{t}\big(\sqrt{t}+\sqrt{s}\big)}(t-s) \le \frac{M}{t}(t-s) \end{aligned}$$

for a.e. $x \in \Omega$ and a.e. $0 < s < t < T$.

Next, we keep $\xi = x$ and choose $\tau = T$ in the dilations (4.1)–(4.3). Then, we have $x = x^{\xi;\lambda}$ and

$$t = s^{\tau;\lambda} \in (0,T)^{\tau;\lambda} \qquad \text{with } \lambda = \sqrt{\frac{T-t}{T-s}} \in (0,1).$$

In an analogous way as done above, we infer that

$$u(x,s) - u(x,t) \le (1-\lambda)M = \frac{M}{\sqrt{T-s}\left(\sqrt{T-s}+\sqrt{T-t}\right)}(t-s) \le \frac{M}{T-s}(t-s).$$

Combining these two estimates, we receive the claimed inequality (4.9). Note that (4.9) is equivalent with

$$|u(x,t)-u(x,s)| \le \frac{M}{\min\{\max\{t,s\}, \max\{T-t, T-s\}\}}|t-s|,$$

for a.e. $x \in \Omega$ and a.e. $s, t \in (0, T)$. This finishes the proof of Theorem 2.3. □

5 Continuity up to the boundary

In this section, we analyze global continuity of variational solutions. We start with a result ensuring lower Lipschitz semi-continuity up to the parabolic boundary.

Lemma 5.1 *Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a variational integrand satisfying (2.1), that the Cauchy–Dirichlet datum u_0 fulfills (2.3) and that u is a variational solution in the sense of Definition 2.1. Then, for any $(x_0, t_0) \in \partial_{\mathcal{P}}\Omega_T$ we have the inequality*

$$u_0(x_0) \le u(x,t) + L|x - x_0|, \qquad \text{for a.e. } (x,t) \in \Omega_T.$$

In particular, u is lower Lipschitz semi-continuous when approaching the parabolic boundary of Ω_T.

Proof Let $(x_0, t_0) \in \partial_{\mathcal{P}}\Omega_T$. From Lemma 4.2, we know that there exists $p \in \mathbb{R}^n$ with $|p| \le L$ such that $u \ge \ell_{x_0}$ a.e. in Ω_T, where ℓ_{x_0} denotes the affine function defined by $\ell_{x_0}(x) := u_0(x_0) + \langle p, x - x_0\rangle$ for $x \in \overline{\Omega}$. This implies the claim, since

$$u(x,t) \ge u_0(x_0) + \langle p, x - x_0\rangle \ge u_0(x_0) - L|x - x_0|$$

for a.e. $(x, t) \in \Omega_T$. □

Concerning global Lipschitz continuity, Lemma 5.1 is the best result we can expect. Already in the elliptic setting the counterexample in [11, Sect. 1] shows that global Lipschitz continuity is in general not true, i.e., variational solutions might fail to be upper Lipschitz semi-continuous when approaching the parabolic boundary. Therefore, we have to restrict ourselves to the proof of global continuity as stated in Theorem 2.5. This will be achieved in the remaining part of this section.

As mentioned before, due to a result by Miranda [22] we know that the uniform convexity of Ω ensures that every C^2-function satisfies the bounded slope condition on $\partial\Omega$. For the sake of completeness, we shortly recall the classical bounded slope condition.

Definition 5.2 We say that a function $\varphi\colon \partial\Omega \to \mathbb{R}$ satisfies the *bounded slope condition (B.S.C.)* with constant $Q > 0$ if for any $x_0 \in \partial\Omega$ there exist two affine functions $\ell^-_{x_0}$ and $\ell^+_{x_0}$ with $[\ell^-_{x_0}]_{0,1} \le Q$ and $[\ell^+_{x_0}]_{0,1} \le Q$ such that

$$\ell^-_{x_0}(x) \le \varphi(x) \le \ell^+_{x_0}(x), \qquad \text{for any } x \in \partial\Omega$$

and $\ell^-_{x_0}(x_0) = \varphi(x_0) = \ell^+_{x_0}(x_0)$ hold true. □

From [22, Proposizione 6.2], we have the following result.

Proposition 5.3 *Let Ω be a uniformly convex domain in $\mathbb{R}^n$. Then, every function $\varphi \in C^2(\mathbb{R}^n)$ satisfies the B.S.C. on $\partial\Omega$.*

We now come to the proof of our second main result, stated in Theorem 2.5. We first prove the result under the stronger condition that u_0 satisfies the full bounded slope condition.

Proposition 5.4 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a variational integrand satisfying (2.1) and let $u_0 \in W^{1,\infty}(\Omega)$ be a Cauchy–Dirichlet datum such that $u_0|_{\partial\Omega}$ fulfills the B.S.C. with constant $Q > 0$. Then, there exists a variational solution $u\colon \Omega_T \to \mathbb{R}$ in the sense of Definition 2.1 which in addition is Lipschitz continuous with respect to the parabolic metric on $\overline{\Omega_T}$, i.e., $u \in C^{0;1,1/2}(\overline{\Omega_T})$.*

Proof Following the notation in [4, Chapter 1.1], we define

$$K(\Omega_T) := \left\{v \in L^\infty(\Omega_T) \cap C^0([0,T]; L^2(\Omega))\colon \nabla v \in L^\infty(\Omega_T; \mathbb{R}^n)\right\}$$

and denote by $K_{u_0}(\Omega_T)$ the subclass of those $v \in K(\Omega_T)$ coinciding with u_0 on the lateral boundary $\partial\Omega \times (0, T)$. From here on, we will continue in two steps, first proving existence of a variational solution under C^1-regularity of f and afterward dealing with general convex integrands via an approximation procedure.

Step 1: Integrands of class $\mathbf{C^1}$. We assume that f satisfies (2.1) and is continuously differentiable on $\mathbb{R}^n$. An application of [4, Theorem 1.2] yields the existence of a unique variational solution $u \in K_{u_0}(\Omega_T)$ in the sense of [4, Definition 1.1] satisfying

$$\|\nabla u\|_{L^\infty(\Omega_T;\mathbb{R}^n)} \le M := \max\left\{Q, \|\nabla u_0\|_{L^\infty(\Omega,\mathbb{R}^n)}\right\} \tag{5.1}$$

and the variational inequality

$$\iint_{\Omega_T} f(\nabla u)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} \big[\partial_t v(v-u) + f(\nabla v)\big]\mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)} \tag{5.2}$$

for all $v \in K_{u_0}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$. Our next aim is to show that u is a variational solution in the sense of Definition 2.1 with initial and lateral boundary values u_0. Since

$$K_{u_0}(\Omega_T) \subset C^0\big([0,T]; L^2(\Omega)\big) \cap u_0 + L^p\big(0,T; W^{1,p}_0(\Omega)\big),$$

we are left with showing that u fulfills the variational inequality (5.2) for all comparison functions $v \in u_0 + L^p(0,T; W^{1,p}_0(\Omega))$ with $v(0) \in L^2(\Omega)$ and $\partial_t v \in L^2(\Omega_T)$.

To this aim, we derive the weak Euler–Lagrange equation associated with (5.2) by choosing the comparison function $v = u + s\varphi$ with $s \in (0,1)$ and $\varphi \in C^\infty(\Omega_T)$ such that $\varphi = 0$ on $\partial\Omega \times (0,T)$; we note that $\partial_t u \in L^2(\Omega_T)$ due to [4, Theorem 1.3]. In this way, we obtain

$$\iint_{\Omega_T} f(\nabla u)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} \big[s\partial_t(u+s\varphi)\varphi + f(\nabla u + s\nabla\varphi)\big]\mathrm{d}x\mathrm{d}t + \tfrac{s^2}{2}\|\varphi(0)\|^2_{L^2(\Omega)} - \tfrac{s^2}{2}\|\varphi(T)\|^2_{L^2(\Omega)}.$$

Due to convexity of f and the fact that f is of class C^1, we have

$$f(\nabla u) \ge f(\nabla u + s\nabla\varphi) - s\big\langle Df(\nabla u + s\nabla\varphi), \nabla\varphi\big\rangle.$$

Using this estimate in the previous inequality and letting $s \downarrow 0$, which is allowed since $f(\nabla u + s\nabla\varphi)$ is uniformly bounded on Ω_T, we receive

$$\iint_{\Omega_T} \big[\langle Df(\nabla u), \nabla\varphi\rangle + \partial_t u\,\varphi\big]\mathrm{d}x\mathrm{d}t \ge 0 \tag{5.3}$$

for all $\varphi \in C^\infty(\overline{\Omega_T})$ with $\varphi = 0$ on $\partial\Omega \times (0,T)$. Finally, replacing φ by $-\varphi$, we also get the reverse inequality. Hence, (5.3) holds as an identity. By approximation, we observe that this identity holds for a larger class of test functions. More precisely, we have

$$\iint_{\Omega_T} \big[\langle Df(\nabla u), \nabla\varphi\rangle + \partial_t u\,\varphi\big]\mathrm{d}x\mathrm{d}t = 0$$

for all $\varphi \in L^p(0,T; W^{1,p}_0(\Omega))$ with $\varphi(0) \in L^2(\Omega)$ and $\partial_t\varphi \in L^2(\Omega_T)$. Now, we let $v \in u_0 + L^p(0,T; W^{1,p}_0(\Omega))$ satisfying $v(0) \in L^2(\Omega)$ and $\partial_t v \in L^2(\Omega_T)$ and choose $\varphi = v - u$. Using again the convexity of f in the form

$$\langle Df(\nabla u), \nabla\varphi\rangle \le f(\nabla u + \nabla\varphi) - f(\nabla u) = f(\nabla v) - f(\nabla u)$$

yields the inequality

$$\iint_{\Omega_T} f(\nabla u)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} \big[f(\nabla v) + \partial_t u(v-u)\big]\mathrm{d}x\mathrm{d}t.$$

In the second term on the right-hand side, we write $u = v-(v-u)$ and perform an integration by parts with respect to time. Simplifying the appearing expressions leaves us with

$$\iint_{\Omega_T} f(\nabla u)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} \big[f(\nabla v) + \partial_t v(v-u)\big]\mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)},$$

for all $v \in u_0 + L^p(0,T;W_0^{1,p}(\Omega))$ with $v(0) \in L^2(\Omega)$ and $\partial_t v \in L^2(\Omega_T)$. This shows that u is a variational solution in the sense of Definition 2.1. Finally, applying [4, Theorem 1.3] in the case of a C^1-integrand f, we have u being Lipschitz continuous with respect to the parabolic metric. More precisely, as in [4, Sect. 8] we have for any parabolic cylinder $Q_\varrho(z_0) := B_\varrho(x_0) \times (t_0-\varrho^2, t_0+\varrho^2)$ with $z_0 = (x_0,t_0)$ such that $B_\varrho(x_0) \subset \Omega$ and $t_0 \in [0,T)$ the Poincaré type inequality

$$\begin{aligned}
&⨏⨏_{Q_\varrho(z_0)\cap\Omega_T} \big|u-(u)_{Q_\varrho(z_0)\cap\Omega_T}\big|^2\,\mathrm{d}x\mathrm{d}t \\
&\le c(n)\,\varrho^2\bigg[⨏⨏_{Q_\varrho(z_0)\cap\Omega_T} |\nabla u|^2\,\mathrm{d}x\mathrm{d}t + \sup_{B_M(0)} |Df|^2\bigg] \\
&\le c(n)\,\varrho^2\Big[M^2 + \sup_{B_M(0)} |Df|^2\Big],
\end{aligned}$$

where in the last line we used (5.1). The proof of this inequality can be deduced as in [6, Lemma 3.1]. Observe that the argument continues to work for cylinders $Q_\varrho(z_0)$ intersecting the initial boundary $\Omega\times\{0\}$, cf. [5, Lemma 4.13]. We note that Ω is convex by assumption, and therefore, it is a Lipschitz domain. This allows to apply Poincaré's inequality for parabolic cylinders with center $z_0 = (x_0,t_0)$ on the lateral boundary, i.e., with $x_0 \in \partial\Omega$. In this way, we obtain

$$\begin{aligned}
&⨏⨏_{Q_\varrho(z_0)\cap\Omega_T} \big|u-(u)_{Q_\varrho(z_0)\cap\Omega_T}\big|^2\,\mathrm{d}x\mathrm{d}t \\
&\le 6⨏⨏_{Q_\varrho(z_0)\cap\Omega_T} |u-u_0|^2\,\mathrm{d}x\mathrm{d}t + 3⨏_{B_\varrho(x_0)\cap\Omega} \big|u_0-(u_0)_{B_\varrho(x_0)\cap\Omega}\big|^2\,\mathrm{d}x \\
&\le c(n,\Omega)\,\varrho^2\bigg[⨏⨏_{Q_\varrho(z_0)\cap\Omega_T} |\nabla u-\nabla u_0|^2\,\mathrm{d}x\mathrm{d}t + ⨏_{B_\varrho(x_0)\cap\Omega} |\nabla u_0|^2\,\mathrm{d}x\bigg] \\
&\le c(n,\Omega)\,\varrho^2\big[M^2 + \|\nabla u_0\|^2_{L^\infty(\Omega)}\big]
\end{aligned}$$

for any parabolic cylinder $Q_\varrho(z_0)$ with $x_0 \in \partial\Omega$ and $t_0 \in [0,T)$. Combining both cases, we find that

$$\begin{aligned}
&⨏⨏_{Q_\varrho(z_0)\cap\Omega_T} \big|u-(u)_{Q_\varrho(z_0)\cap\Omega_T}\big|^2\,\mathrm{d}x\mathrm{d}t \\
&\le c\,\varrho^2\Big[M^2 + \sup_{B_M(0)} |Df|^2 + \|\nabla u_0\|^2_{L^\infty(\Omega,\mathbb{R}^n)}\Big] \qquad (5.4)
\end{aligned}$$

holds true with a constant $c = c(n,\Omega)$ and for any parabolic cylinder $Q_\varrho(z_0)$ with $z_0 \in \overline{\Omega_T}$. Due to the parabolic version of Campanato's characterization of Hölder continuity by Da Prato [12, Teorema 3.1], this implies the Lipschitz continuity of u with respect to the parabolic metric, i.e., $u \in C^{0;1,1/2}(\overline{\Omega_T})$.

Step 2: General convex integrands. Now we consider the case where f satisfies only assumption (2.1). From the convexity assumption, we infer that f is Lipschitz continuous on $B_{2M}(0)$, where M is defined in (5.1). We let $K := \sup_{B_{2M}(0)} |Df| < \infty$. For $\varepsilon \in (0, 1)$, we infer from [1, Corollary 1.3] the existence of a convex function $f_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}$ of class C^1 with $f - \varepsilon \leq f_\varepsilon \leq f$ and $\sup_{B_M(0)} |Df_\varepsilon| \leq K$. Note that the coercivity of f_ε follows immediately from the one of f and the fact that $f_\varepsilon \geq f - \varepsilon$. Applying our previously obtained result from Step 1 with integrand f_ε, we infer that there exist variational solutions

$$u_\varepsilon \in C^0\big([0, T]; L^2(\Omega)\big) \cap u_0 + L^p\big(0, T; W_0^{1,p}(\Omega)\big)$$

in the sense of Definition 2.1 with integrand f_ε instead of f satisfying (5.1) and (5.4), i.e.,

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega_T;\mathbb{R}^n)} \leq M \tag{5.5}$$

and

$$\iint_{Q_\varrho(z_0)\cap\Omega_T} \left|u_\varepsilon - (u_\varepsilon)_{Q_\varrho(z_0)\cap\Omega_T}\right|^2 \mathrm{d}x\mathrm{d}t \leq c\,\varrho^2\Big[M^2 + K^2 + \|\nabla u_0\|^2_{L^\infty(\Omega,\mathbb{R}^n)}\Big]$$

hold true for any parabolic cylinder $Q_\varrho(z_0)$ with $z_0 \in \overline{\Omega_T}$. Recall that the preceding inequality is a bound on the parabolic Campanato seminorm of u_ε, which ensures—using Da Prato's characterization [12, Teorema 3.1] again—that u_ε is Lipschitz continuous with respect to the parabolic metric. Moreover, we have the quantitative estimate

$$|u_\varepsilon(x, s) - u_\varepsilon(y, t)| \leq C\sqrt{|x - y|^2 + |s - t|}, \quad \text{for a.e. } x, y, \in \overline{\Omega}, s, t \in [0, T], \tag{5.6}$$

with a constant C independent of ε. Moreover, from the comparison principle in Lemma 3.4 we know that

$$\|u_\varepsilon\|_{L^\infty(\overline{\Omega}_T)} \leq \|u_0\|_{L^\infty(\overline{\Omega})}. \tag{5.7}$$

Our next aim is to pass to the limit $\varepsilon \downarrow 0$. Inequalities (5.6) and (5.7) ensure that the family of functions $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded and equicontinuous with respect to the Euclidean metric on $\mathbb{R}^{n+1}$. Therefore, Arzelà–Ascoli's theorem ensures the existence of a subsequence $(u_{\varepsilon_i})_{i\in\mathbb{N}}$ with $\varepsilon_i \downarrow 0$ as $i \to \infty$ and a continuous function $u \in C^0(\overline{\Omega_T})$ such that $u_{\varepsilon_i} \to u$ uniformly on $\overline{\Omega}_T$ as $i \to \infty$. Since we can now pass to the limit $\varepsilon_i \downarrow 0$ in inequality (5.6), we infer that $u \in C^{0;1,1/2}(\overline{\Omega_T})$. Particularly, we have that $u \in C^0([0, T]; L^2(\Omega)) \cap u_0 + L^p(0, T; W_0^{1,p}(\Omega))$. Our last step is to show that u fulfills the variational inequality (2.2) associated with the original variational integrand f. To this aim, we make use of the lower semi-continuity of $u \mapsto \iint_{\Omega_T} f(\nabla u)\,\mathrm{d}x\mathrm{d}t$ with respect to strong convergence in $L^1(\Omega_T)$ (cf. [2, Lemma 3.1]), the variational inequality for u_ε, $f - \varepsilon \leq f_\varepsilon \leq f$, as well as the uniform convergence of u_{ε_i} to u. For $v \in u_0 + L^p(0, T; W_0^{1,p}(\Omega))$ with $v(0) \in L^2(\Omega)$ and $\partial_t v \in L^2(\Omega_T)$, we receive

$$\begin{aligned}
\iint_{\Omega_T} f(\nabla u)\,\mathrm{d}x\mathrm{d}t &\leq \liminf_{i\to\infty} \iint_{\Omega_T} f(\nabla u_{\varepsilon_i})\,\mathrm{d}x\mathrm{d}t \\
&= \liminf_{i\to\infty} \iint_{\Omega_T} f_{\varepsilon_i}(\nabla u_{\varepsilon_i})\,\mathrm{d}x\mathrm{d}t \\
&\leq \liminf_{i\to\infty} \bigg[\iint_{\Omega_T} \big[f_{\varepsilon_i}(\nabla v) + \partial_t v(v - u_{\varepsilon_i})\big]\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$+\tfrac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega)}-\tfrac{1}{2}\|(v-u_{\varepsilon_i})(T)\|^2_{L^2(\Omega)}\Big]$$
$$=\iint_{\Omega_T}\big[f(\nabla v)+\partial_t v(v-u)\big]\,\mathrm{d}x\mathrm{d}t$$
$$+\tfrac{1}{2}\|v(0)-u_0\|^2_{L^2(\Omega)}-\tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)},$$

which finishes the proof of Proposition 5.4. □

Observe that under the full B.S.C. we have obtained global Lipschitz continuity with respect to the parabolic metric in Proposition 5.4. As explained before, this cannot be expected under the assumption of the one-sided bounded slope condition.

Proof of Theorem 2.5 Here, we assume that Ω is uniformly convex. Since u_0 is Lipschitz continuous on $\overline{\Omega}$, there exists a sequence of C^2-functions $(u_0^{(i)})_{i\in\mathbb{N}}$ such that $u_0^{(i)}\to u_0$ uniformly on Ω as $i\to\infty$. From Proposition 5.3, we know that our approximating functions $u_0^{(i)}$ satisfy the full bounded slope condition on $\partial\Omega$ with constants $Q_i>0$. Using Proposition 5.4 with $u_0^{(i)}$ as boundary data and f as variational integrand yields the existence of variational solutions $u^{(i)}$ in the sense of Definition 2.1 with $u^{(i)}\in C^{0,1;1/2}(\overline{\Omega_T})$. Considering now the quantity $|u-u^{(i)}|$ on $\partial_{\mathcal{P}}\Omega_T$, we get

$$|u(x_0,t_0)-u^{(i)}(x_0,t_0)|=|u_0(x_0)-u_0^{(i)}(x_0)|\le\|u_0-u_0^{(i)}\|_{L^\infty(\Omega)}=:\varepsilon_i.$$

Due to the uniform convergence $u_0^{(i)}\to u_0$, we have that $\varepsilon_i\to 0$ as $i\to\infty$. Using the comparison principle from Lemma 3.4, we find that $|u-u^{(i)}|\le\varepsilon_i$ a.e. on Ω_T. Therefore, we know u to be a uniform limit of a sequence of continuous functions on $\overline{\Omega_T}$; hence, it is continuous itself on $\overline{\Omega_T}$. This proves the claim of Theorem 2.5. □

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