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A characterization of compact locally conformally hyperkähler manifolds

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Abstract

We give an equivalent definition of compact locally conformally hyperkähler manifolds in terms of the existence of a non-degenerate complex two-form with natural properties. This is a conformal analogue of Beauville's theorem stating that a compact Kähler manifold admitting a holomorphic symplectic form is hyperkähler

Keywords Locally conformally Kähler · Locally conformally hyperkähler · Weyl connection · Holonomy · Weitzenböck formula

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1 Introduction

A complex manifold (M, J) is called *locally conformally Kähler* (LCK for short) if it admits a Hermitian metric g such that the two-form $\omega(X, Y) := g(JX, Y)$ satisfies the integrability condition

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$$d\omega = \theta \wedge \omega$$

with respect to a *closed* one-form θ, called the *Lee form*.

It is then immediate that locally, the metric g is conformal to some local Kähler metrics $g'_U := e^{-f_U} g\big|_U$, where $\theta\big|_U = df_U$ on an open set U.

An equivalent definition requires that the universal cover $(\tilde{M}, J)$ of (M, J) admits a Kähler metric with respect to which the deck group acts by holomorphic homotheties; see [4]. This Kähler metric on $\tilde{M}$, which is globally conformal with the pull-back of the LCK metric g, is in fact obtained by gluing the pulled-back local Kähler metrics g'_U. Note that if θ is not exact, then the universal cover of an LCK manifold admits strict homotheties and thus is never compact.

There are many examples of LCK manifolds: diagonal and non-diagonal Hopf manifolds, Kodaira surfaces, Kato surfaces, some Oeljeklaus–Toma manifolds, etc. All complex submanifolds of LCK manifolds are LCK. See, for example, [4,12] and the bibliography therein.

The LCK condition is conformally invariant: if g is LCK with Lee form θ, then $e^f g$ is LCK with Lee form $\theta + df$. One may then speak about an *LCK structure* on (M, J) given by the couple $([g], [\theta])$, where [ ] denotes conformal class, respectively, de Rham cohomology class.

The complex structure J is parallel with respect to the Weyl connection D associated with θ and $[g]$, acting by $Dg = \theta \otimes g$. This implies that D is, in fact, obtained by gluing the Levi-Civita connections of the local Kähler metrics g'_U, and therefore, the Levi-Civita connection of the Kähler metric on $\tilde{M}$ is the pull-back of the Weyl connection on M.

The relation between the Weyl connection and the Levi-Civita connection is given by (see, for example, [4]):

$$D = \nabla^g - \frac{1}{2}(\theta \otimes \mathrm{id} + \mathrm{id} \otimes \theta - g \otimes \theta^\sharp). \tag{1.1}$$

On a compact LCK manifold, if the local Kähler metrics are Einstein, a well-known result by Gauduchon, [5], says that they are in fact Ricci flat. In this case, the LCK metric g is called *Einstein–Weyl* (see [11,13]) and has the property that in the conformal class of g there exists a metric with parallel Lee form, unique up to homotheties, known in the literature as Vaisman metric (see [5]).

A particular example of Einstein–Weyl metrics is given by the *locally conformally hyperkähler* (LCHK) ones. In this case, $\dim_{\mathbb{R}} M = 2n$ is a multiple of 4, and M admits a hyperhermitian structure (I, J, K, g) such that all three Hermitian couples (g, I), (g, J) and (g, K) are LCK with respect to the same Lee form θ. Again, this is a conformally invariant notion. The Kähler metric of the universal cover has then holonomy included in $\mathrm{Sp}(\frac{n}{2})$, thus being Calabi–Yau. See, for example, [3,10,13,14]. The quaternionic Hopf manifold is an important example. A complete list of compact, homogeneous LCHK manifolds is given in [10].

One easily verifies that on compact LCHK manifolds, the two-form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is non-degenerate of type (2,0) with respect to I, and produces a volume form by $\Omega^{\frac{n}{2}} \wedge \overline{\Omega}^{\frac{n}{2}} = c \cdot d\mathrm{vol}_g$, for a positive constant c. Moreover, Ω satisfies the equation $\overline{\partial}\Omega = \theta^{0,1} \wedge \Omega$, since it satisfies the stronger one $d\Omega = \theta \wedge \Omega$. The aim of this note is to prove that these conditions are also sufficient to define an LCHK structure:

Theorem A *Let (M, J, g) be a compact locally conformally Kähler manifold of real dimension $2n$ and let θ be the Lee form of g. Then g is locally conformally hyperkähler if and only if there exists a non-degenerate $(2, 0)$-form Ω such that*

$$\overline{\partial}\Omega = \theta^{0,1} \wedge \Omega \quad \textit{and} \quad \Omega^{\frac{n}{2}} \wedge \overline{\Omega}^{\frac{n}{2}} = c \cdot d\mathrm{vol}_g,$$

where $c \in \mathbb{R}^{>0}$, *and* $d\mathrm{vol}_g$ *is the volume form of* g.

Remark 1.1 Note that the existence of a non-degenerate $(2, 0)$-form implies that the real dimension is a multiple of 4.

Remark 1.2 A $(2, 0)$-form Ω satisfying $\overline{\partial}\Omega = \theta^{0,1} \wedge \Omega$ can be regarded as a holomorphic two-form valued in the line bundle $(L = M \times \mathbb{C}, \overline{\partial}_L = \overline{\partial} - \theta^{0,1}\wedge)$.

Theorem A can be viewed as a conformal version of the celebrated Beauville's theorem stating that a compact Kähler manifold, admitting a holomorphic symplectic form is hyperkähler, [1]. Our proof follows the line of Beauville's, which is to use the holonomy principle. The strategy is to move to the universal cover $\tilde{M}$, which has a Kähler metric $\tilde{g}$ and also a non-degenerate holomorphic two-form $\tilde{\Omega}$, as we shall see in the sequel. If we were able to prove that $\tilde{\Omega}$ is $\tilde{g}$-parallel, from the holonomy principle, we would get precisely that g is LCHK. Since there is no analogue of Yau's theorem for LCK metrics, which is in fact the main difficulty in adapting Beauville's result to the LCK setting, the rather strong condition $\Omega^{\frac{n}{2}} \wedge \overline{\Omega}^{\frac{n}{2}} = c \cdot d\mathrm{vol}_g$ is meant to prove that on $\tilde{M}$, $\tilde{g}$ is actually Ricci flat, replacing thus a Yau-like result. The advantage is that with a Ricci-flat metric, the Weitzenböck formula applied on $\tilde{M}$ to $\tilde{\Omega}$ simplifies and amounts to $(\nabla^{\tilde{g}})^{*_{\tilde{g}}}\nabla^{\tilde{g}}\tilde{\Omega} = 0$. As $\tilde{M}$ is not compact, one has to make a long detour to interpret this relation on the compact manifold M and ultimately conclude that $\tilde{\Omega}$ is in fact $\tilde{g}$-parallel, and thus, g is LCHK.

Remark 1.3 Note that the parallelism of $\tilde{\Omega}$ further gives on M the relation $d\Omega = \theta \wedge \Omega$. A complex manifold admitting a non-degenerate $(2, 0)$-form ω such that a closed one-form $\theta \in \Lambda^1(M, \mathbb{C})$ exists and $d\omega = \theta \wedge \omega$ is called *complex locally conformally symplectic* (CLCS). The Lee form θ can be real or complex. CLCS manifolds first appeared in [8, Section 5], motivated by the examples of even-dimensional leaves of the natural generalized foliation of a complex Jacobi manifold. Similarly, real LCS structures also appear as leaves of real Jacobi manifolds.

Remark 1.4 A generalization of Beauville's theorem, but in a different sense, namely when the manifold is compact Kähler, but admits a twisted holomorphic form, is presented in [6].

2 Proof of Theorem A

The following lemmas will be used in the proof.

Lemma 2.1 *Let* (M, J, g) *be an LCK manifold endowed with a non-degenerate* $(2, 0)$*-form* Ω *such that* $\overline{\partial}\Omega = \theta^{0,1} \wedge \Omega$ *and* $\Omega^{\frac{n}{2}} \wedge \overline{\Omega}^{\frac{n}{2}} = c \cdot d\mathrm{vol}_g$, *where* θ *is the Lee form of* g, $c \in \mathbb{R}_+$ *and* $d\mathrm{vol}_g$ *is the volume form of* g. *Then* g *is Einstein–Weyl.*

Proof Let $\tilde{M}$ be the universal cover of M, endowed with the complex structure $\tilde{J} = \pi^*J$, where $\pi : \tilde{M} \to M$. Let $\pi^*\theta = df$ and let $\tilde{g}$ be the Kähler metric given by $e^{-f}\pi^*g$. Denote by $\tilde{\Omega} := e^{-f}\pi^*\Omega$. This is a holomorphic two-form, as a consequence of $\overline{\partial}\Omega = \theta^{0,1} \wedge \Omega$. Moreover, $\tilde{\Omega}^{\frac{n}{2}} \wedge \overline{\tilde{\Omega}}^{\frac{n}{2}} = c \cdot d\mathrm{vol}_{\tilde{g}}$, since $d\mathrm{vol}_{\tilde{g}} = e^{-nf}d\mathrm{vol}_g$.

Let $\tilde{K}$ be the canonical bundle of $\tilde{M}$. The metric $\tilde{g}$ induces a natural Hermitian product $\tilde{h}$ on $\tilde{K}$ which verifies the relation $\alpha \wedge *\bar{\beta} = \tilde{h}(\alpha, \beta)\, d\mathrm{vol}_{\tilde{g}}$. Note that because n is even, $*\bar{\beta} = \bar{\beta}$

(see [9, Exercise 18.2.1]); thus, $\tilde{h}(\tilde{\Omega}^{\frac{n}{2}}, \tilde{\Omega}^{\frac{n}{2}}) = c$. The curvature form of the Chern connection associated with $\tilde{h}$ is on the one hand $\mathrm{i}\,\partial\overline{\partial}\log\det(\tilde{g}_{ij})$, where $\det\tilde{g}_{ij} = \det\tilde{g}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial\overline{z_j}})$, and on the other hand, it is $-\mathrm{i}\,\partial\overline{\partial}\log\tilde{h}(\tilde{\Omega}^{\frac{n}{2}}, \tilde{\Omega}^{\frac{n}{2}}) = 0$. Since $\mathrm{i}\,\partial\overline{\partial}\log\det(\tilde{g}_{ij})$ is the local expression of the Ricci form $\rho(X, Y) = \mathrm{Ric}_{\tilde{g}}(\tilde{J}X, Y)$, we conclude that $\tilde{g}$ is Ricci flat, and hence, g is Einstein–Weyl. □

Remark 2.2 In particular, if M is compact, there exists in the conformal class of g a Vaisman metric, unique up to homotheties, as shown in [5, Théorème 3 (ii)].

Remark 2.3 Note that if instead of g we consider the metric $g_1 = e^f g$ with its corresponding Lee form $\theta_1 = \theta + df$, then taking $\Omega_1 = e^f\Omega$, we still obtain a non-degenerate form of type $(2, 0)$ satisfying $\overline{\partial}\Omega_1 = \theta_1^{0,1}\wedge\Omega_1$ and $\Omega_1^{\frac{n}{2}}\wedge\overline{\Omega_1}^{\frac{n}{2}} = c\cdot d\mathrm{vol}_{g_1}$; therefore, the statement of Lemma 2.1 is conformally invariant.

Lemma 2.4 *Let h be the Hermitian structure induced by g on $\Lambda^{2,0}_{\mathbb{C}}M$ (that is, $h(\omega, \eta) = g(\omega, \overline{\eta})$). The Weyl connection D on M satisfies $Dh = -2\theta\otimes h$.*

Proof This is because the Hermitian structure $\tilde{h}$ induced by $\tilde{g}$ on $\Lambda^{2,0}_{\mathbb{C}}\tilde{M}$ is given by $e^{2f}\pi^*h$. Then $\pi^*D = \nabla^{\tilde{g}}$ (see [4]) implies that $\pi^*D(e^{2f}\pi^*h) = 0$, which yields $(\pi^*D)(\pi^*h) = -2\pi^*\theta\otimes\pi^*h$ and our relation follows. □

The next computation may be interesting for its own sake, not only for the present proof:

Lemma 2.5 *The adjoint operator with respect to g of the Weyl connection D in (1.1), seen as a differential operator acting on $\Lambda^{2,0}_{\mathbb{C}}M$, is given by:*

$$D^*(\eta\otimes\sigma) = (\delta^g\eta)\sigma - D_{\eta^\sharp}\sigma + 2\eta(\theta^\sharp)\sigma,$$

where $\eta\otimes\sigma\in\Gamma(\Lambda^1_{\mathbb{C}}\otimes\Lambda^{2,0}_{\mathbb{C}})$.

Proof In order to find an explicit expression of D^*, we use the same method as in the proof of [9, Lemma 20.1]. Since there is no risk of confusion, in the following computation we shall denote equally by h the Hermitian structure on $\Lambda^1_{\mathbb{C}}\otimes\Lambda^{2,0}_{\mathbb{C}}M$. Note that this h is the product of the Hermitian structure induced by g on $\Lambda^1_{\mathbb{C}}M$ and h defined in Lemma 2.4.

Let s be a $(2, 0)$-form on M and consider a ∇^g-parallel frame $\{e_i\}$ at a point $x\in M$. We perform the following computations at x. Define the one-form $\alpha(X) := h(\eta(X)\sigma, s)$. Then, using Lemma 2.4 and the fact that θ is real, we derive:

$$\begin{aligned}
-\delta^g\alpha &= \sum_{i=1}^{2n} e_i(\alpha(e_i)) = \sum_{i=1}^{2n} e_i(h(\eta(e_i)\sigma, s))\\
&= \sum_{i=1}^{2n}(D_{e_i}h)(\eta(e_i)\sigma, s) + h(D_{e_i}(\eta(e_i)\sigma), s) + h(\eta(e_i)\sigma, D_{e_i}s)\\
&= \sum_{i=1}^{2n}(-2\theta(e_i)h(\eta(e_i)\sigma, s) + h(e_i(\eta(e_i))\sigma, s) + h(\eta(e_i)D_{e_i}\sigma, s)) + h(\eta\otimes\sigma, Ds)\\
&= h(D_{\eta^\sharp}\sigma - (\delta^g\eta)\sigma, s) + h(\eta\otimes\sigma, Ds - 2\theta\otimes s).
\end{aligned}$$

After integration on M, this implies:

$$\int_M h(\eta\otimes\sigma, Ds - 2\theta\otimes s)d\mathrm{vol}_g = \int_M h(-D_{\eta^\sharp}\sigma + (\delta^g\eta)\sigma, s)d\mathrm{vol}_g,$$

and hence, the adjoint of $D - 2\theta\otimes$ acts as follows:

$$(D - 2\theta\otimes)^*(\eta \otimes \sigma) = (\delta^g \eta)\sigma - D_{\eta^\sharp}\sigma.$$

Then $D^* = (D - 2\theta\otimes)^* + 2(\theta\otimes)^*$, which gives:

$$D^*(\eta \otimes \sigma) = (\delta^g \eta)\sigma - D_{\eta^\sharp}\sigma + 2\eta(\theta^\sharp)\sigma. \tag{2.1}$$

□

We proceed with the proof of Theorem A.

In terms of holonomy, the metric g is LCHK if and only if the holonomy of $\tilde{g}$ on $\tilde{M}$ is contained in $\mathrm{Sp}(\frac{n}{2})$, which is equivalent to $\tilde{g}$ being hyperkähler. According to the holonomy principle (see, for example, [1, Page 758]), this is equivalent to the existence of a complex structure $\tilde{J}$ on $\tilde{M}$, with respect to which $\tilde{g}$ is Kähler and a holomorphic, non-degenerate two-form $\tilde{\Omega}$, parallel with respect to the Levi-Civita connection $\nabla^{\tilde{g}}$ of $\tilde{g}$. These turn out to be $\tilde{J}$ and $\tilde{\Omega}$ from the proof of Lemma 2.1. As we already saw that $\tilde{\Omega}$ is holomorphic, the non-trivial part is to prove:

Lemma 2.6 *$\tilde{\Omega}$ is $\tilde{g}$-parallel.*

Proof Since $\nabla^{\tilde{g}} = \pi^* D$, the equality $\nabla^{\tilde{g}}\tilde{\Omega} = 0$ is equivalent to

$$D\Omega = \theta \otimes \Omega. \tag{2.2}$$

This is a conformally invariant relation on M: for any smooth f on M, we have

$$D(e^f \Omega) = (\theta + df) \otimes e^f \Omega.$$

Thus, by the compactness of M, Lemma 2.1, Remarks 2.2 and 2.3, we may suppose without loss of generality that g is a Vaisman metric. In this case, the Lee form is harmonic, has constant norm, and moreover, we can choose the Vaisman metric with the Lee form of norm 1. We shall use these facts in the following computations.

We apply the Weitzenböck formula to the holomorphic form $\tilde{\Omega}$. According to [9] (see Theorem 20.2 and the beginning of the proof of Theorem 20.5), as $\tilde{g}$ is Ricci flat (by Lemma 2.1) the curvature term vanishes identically and the Weitzenböck formula reduces to:

$$(\nabla^{\tilde{g}})^{\tilde{*}}\nabla^{\tilde{g}}\,\tilde{\Omega} = 0 \tag{2.3}$$

where we denote by $\tilde{*}$ the adjoint with respect to $\tilde{g}$. However, $\tilde{M}$ is not compact and we cannot deduce by integration that $\nabla^{\tilde{g}}\tilde{\Omega} = 0$. Instead, we show in the sequel that (2.3) can be read on the compact manifold M and by using integration on M we shall deduce the $\tilde{g}$-parallelism of $\tilde{\Omega}$.

For simplicity, from now on we write ∇ for $\nabla^{\tilde{g}}$. By [9, Lemma 20.1],

$$\nabla^{\tilde{*}}\nabla\tilde{\Omega} = \sum_{i=1}^{2n} \nabla_{\nabla_{f_i} f_i}\tilde{\Omega} - \nabla_{f_i}\nabla_{f_i}\tilde{\Omega}, \tag{2.4}$$

where $\{f_i\}$ is a local $\tilde{g}$—orthonormal frame. We may choose $f_i = e^{\frac{f}{2}}\pi^* e_i$, where $\{e_i\}$ is a local g—orthonormal frame on M. Then (2.4) implies

$$\nabla^{\tilde{*}}\nabla\tilde{\Omega} = \sum_{i=1}^{2n} e^f \left(\nabla_{\nabla_{\pi^* e_i}\pi^* e_i}\tilde{\Omega} - \nabla_{\pi^* e_i}\nabla_{\pi^* e_i}\tilde{\Omega}\right),$$

and hence,

$$\sum_{i=1}^{2n} \nabla_{\nabla_{\pi^* e_i}\pi^* e_i}\tilde{\Omega} - \nabla_{\pi^* e_i}\nabla_{\pi^* e_i}\tilde{\Omega} = 0.$$

Writing now $\tilde{\Omega} = e^{-f}\pi^*\Omega$, the above relation gives:

$$\begin{aligned} 0 = \sum_{i=1}^{2n} e^{-f}(&\nabla_{\nabla_{\pi^* e_i}\pi^* e_i}\pi^*\Omega - \nabla_{\pi^* e_i}\nabla_{\pi^* e_i}\pi^*\Omega - \pi^*\theta(\nabla_{\pi^* e_i}\pi^* e_i)\pi^*\Omega \\ &+ 2\pi^*\theta(\pi^* e_i)\nabla_{\pi^* e_i}\pi^*\Omega - (\pi^*\theta(\pi^* e_i))^2\pi^*\Omega + \pi^* e_i(\pi^*\theta(\pi^* e_i))\pi^*\Omega). \end{aligned} \tag{2.5}$$

As $\nabla = \pi^* D$, (2.5) descends on M to the following equality:

$$\sum_{i=1}^{2n} D_{D_{e_i} e_i}\Omega - D_{e_i} D_{e_i}\Omega - \theta(D_{e_i} e_i)\Omega + 2\theta(e_i) D_{e_i}\Omega - (\theta(e_i))^2\Omega + e_i(\theta(e_i))\Omega = 0. \tag{2.6}$$

We notice that $\sum_{i=1}^{2n}(\theta(e_i))^2\Omega = \|\theta\|_g^2\Omega = \Omega$. Using that θ is harmonic, $\|\theta\|_g = 1$ and (1.1), we get:

$$0 = -\delta^g\theta = \sum_{i=1}^{2n} e_i(\theta(e_i)) - \theta(\nabla^g_{e_i} e_i) = \sum_{i=1}^{2n}(e_i(\theta(e_i)) - \theta(D_{e_i} e_i)) + n - 1.$$

Consequently, (2.6) is rewritten as:

$$\sum_{i=1}^{2n} D_{D_{e_i} e_i}\Omega - D_{e_i} D_{e_i}\Omega + 2\theta(e_i) D_{e_i}\Omega = n\Omega. \tag{2.7}$$

The goal is to prove that $\nabla\tilde{\Omega} = 0$, that is $D\Omega = \theta \otimes \Omega$. Hence, if we define $\mathcal{D} := D - \theta\,\otimes$, we need to show that $\mathcal{D}\Omega = 0$. If $\mathcal{D}^*$ is the adjoint of $\mathcal{D}$, as M is compact, $\mathcal{D}\Omega = 0$ will follow from:

$$\int_M h(\mathcal{D}^*\mathcal{D}\,\Omega, \Omega) d\mathrm{vol}_g = 0.$$

We compute now $\mathcal{D}^*\mathcal{D}\,\Omega$. By Lemma 2.5,

$$\begin{aligned} \mathcal{D}^*\Omega &= D^* - (\theta\otimes)^* \\ \mathcal{D}^*\mathcal{D}\,\Omega &= \mathcal{D}^*(D\Omega - \theta\otimes\Omega) = \mathcal{D}^* D\Omega - \mathcal{D}^*(\theta\otimes\Omega). \end{aligned} \tag{2.8}$$

Relation (2.8) and the fact that θ is harmonic of g-norm 1 yield:

$$\mathcal{D}^*(\theta\otimes\Omega) = (\delta^g\theta)\Omega - D_{\theta^\sharp}\Omega + \Omega = \Omega - D_{\theta^\sharp}\Omega. \tag{2.9}$$

The first term is equal to:

$$\mathcal{D}^* D\,\Omega = \sum_{i=1}^{2n} \mathcal{D}^*(e^i \otimes D_{e_i}\Omega) = \sum_{i=1}^{2n}(\delta^g e^i) D_{e_i}\Omega - D_{e_i} D_{e_i}\Omega + e^i(\theta^\sharp) D_{e_i}\Omega,$$

where $\{e^i\}$ is the dual frame to $\{e_i\}$. But (1.1) implies:

$$\delta^g e^i = \sum_{k=1}^{2n} -e_k(e^i(e_k)) + e^i(\nabla^g_{e_k} e_k) = \sum_{k=1}^{2n} g(D_{e_k} e_k, e_i) + (1-n)\theta(e_i),$$

and thus:

$$\mathcal{D}^* D\Omega = \sum_{i=1}^{2n}\left(\sum_{k=1}^{2n} g(D_{e_k}e_k, e_i)D_{e_i}\Omega + (1-n)\theta(e_i)D_{e_i}\Omega - D_{e_i}D_{e_i}\Omega\right) + D_{\theta^\sharp}\Omega$$
$$= \sum_{i=1}^{2n}(D_{D_{e_i}e_i}\Omega - D_{e_i}D_{e_i}\Omega) + (2-n)D_{\theta^\sharp}\Omega. \tag{2.10}$$

Combining (2.10) and (2.9), we arrive at:

$$\mathcal{D}^*\mathcal{D}\Omega = \sum_{i=1}^{2n}(D_{D_{e_i}e_i}\Omega - D_{e_i}D_{e_i}\Omega) + (3-n)D_{\theta^\sharp}\Omega - \Omega,$$

which, together with (2.7), leads to the final result:

$$\mathcal{D}^*\mathcal{D}\Omega = (n-1)(\Omega - D_{\theta^\sharp}\Omega). \tag{2.11}$$

Integrating (2.11) on M, we find:

$$\int_M h(\mathcal{D}\Omega, \mathcal{D}\Omega)d\mathrm{vol}_g = \int_M h(\mathcal{D}^*\mathcal{D}\Omega, \Omega)d\mathrm{vol}_g$$
$$= (n-1)\left(\int_M h(\Omega, \Omega)d\mathrm{vol}_g - \int_M h(D_{\theta^\sharp}\Omega, \Omega)d\mathrm{vol}_g\right). \tag{2.12}$$

In particular, the above equality proves that $\int_M h(D_{\theta^\sharp}\Omega, \Omega)d\mathrm{vol}_g$ is a real number. Moreover,

$$\int_M (D_{\theta^\sharp}h)(\Omega, \Omega)d\mathrm{vol}_g = \int_M \theta^\sharp(h(\Omega, \Omega))d\mathrm{vol}_g$$
$$- \int_M h(D_{\theta^\sharp}\Omega, \Omega)d\mathrm{vol}_g - \int_M h(\Omega, D_{\theta^\sharp}\Omega)d\mathrm{vol}_g. \tag{2.13}$$

The first integral in the right-hand side vanishes, since by Stokes' formula and the fact that $\theta^\sharp$ is Killing (see [4]) we have:

$$\int_M \theta^\sharp(h(\Omega, \Omega))d\mathrm{vol}_g = \int_M \mathcal{L}_{\theta^\sharp}(h(\Omega, \Omega)d\mathrm{vol}_g) - \int_M h(\Omega, \Omega)\mathcal{L}_{\theta^\sharp}d\mathrm{vol}_g = 0. \tag{2.14}$$

Using once more Lemma 2.4, $\|\theta\|_g = 1$, (2.13) and (2.14), we derive:

$$-2\int_M h(\Omega, \Omega)d\mathrm{vol}_g = \int_M (D_{\theta^\sharp}h)(\Omega, \Omega)d\mathrm{vol}_g$$
$$= -2\mathrm{Re}\int_M h(D_{\theta^\sharp}\Omega, \Omega)d\mathrm{vol}_g = -2\int_M h(D_{\theta^\sharp}\Omega, \Omega)d\mathrm{vol}_g,$$

which, together with (2.12), implies that

$$\int_M h(\mathcal{D}\Omega, \mathcal{D}\Omega)d\mathrm{vol}_g = 0,$$

and thus $\mathcal{D}\Omega = 0$, completing the proof. □

3 Final remarks

From the above proof, we see that Theorem A can be reformulated in the following way, in terms of non-compact Kähler manifolds:

Theorem B *Let (M, J, g) be a non-compact Kähler–Ricci-flat manifold, endowed with a non-degenerate holomorphic two-form Ω and the action of a cocompact discrete group Γ such that for any $\gamma \in \Gamma$, $\gamma^* g = c_\gamma g$ and $\gamma^* \Omega = c_\gamma \Omega$, where c_γ are positive real numbers, not all of them equal to* 1. *Then g is hyperkähler.*

As an application, we obtain the following corollary concerning holomorphic contact structures, which we explain more in detail below (see also [7] for other constructions relating holomorphic contact structures on Kähler manifolds and holomorphic symplectic forms on cones over Sasakian manifolds). We recall that by a holomorphic contact structure on a complex manifold M, we understand a codimension 1 holomorphic sub-bundle of $T^{1,0}M$, which is maximally non-integrable.

Corollary 3.1 *Let (X, J, g) be a compact Kähler–Einstein manifold and S the S^1 principal bundle corresponding to $-\frac{c_1(X)}{I}$, where I is the largest positive integer such that $\frac{c_1(X)}{I}$ is an integral class. We consider the action of $\mathbb{Z}$ on $\mathbb{R}^{>0} \times S$ given by $n \cdot (t, s) = (2^n t, s)$. If on $\mathbb{R}^{>0} \times S$ there exists a non-degenerate holomorphic two-form Ω such that $n^*\Omega = 2^{2n}\Omega$, then X carries a holomorphic contact structure.*

Proof In [2], it is proved that assuming without loss of generality that g is normalized by $\mathrm{Ric}_g = 2(n+1)g$, then $\mathbb{R}^{>0} \times S$ carries a conical Kähler–Ricci-flat metric $\tilde{g} = dt^2 + t^2 g_S$ (where g_S is a Riemannian metric of S) which under the action of $\mathbb{Z}$ behaves as $n^*\tilde{g} = 2^{2n}\tilde{g}$. Then we are in the situation described in Theorem B and we obtain that $\tilde{g}$ is in fact hyperkähler. By [2, Section 2], we conclude that M carries a holomorphic contact structure. □

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