Annali di Matematica (2018) 197:1855–1873
https://doi.org/10.1007/s10231-018-0753-9

The Laplacian coflow on almost-abelian Lie groups

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Received: 23 November 2017 / Accepted: 21 April 2018 / Published online: 30 April 2018

Abstract We find explicit solutions of the Laplacian coflow of G_2-structures on seven-dimensional almost-abelian Lie groups. Moreover, we construct new examples of solitons for the Laplacian coflow which are not eigenforms of the Laplacian and we exhibit a solution, which is not a soliton, having a bounded interval of existence.

Keywords G_2-structures · Laplacian coflow · Solitons · Lie groups

Mathematics Subject Classification 53C15 · 22E25

1 Introduction

A G_2-structure on a seven-dimensional manifold M is given by a 3-form φ on M with pointwise stabilizer isomorphic to the exceptional group $G_2 \subset SO(7)$. The 3-form φ induces a Riemannian metric g_φ, an orientation and so a Hodge star operator $\star_\varphi$ on M. It is well-known [7] that φ is parallel with respect to the Levi-Civita connection of g_φ if and only if φ is closed and coclosed and that when this happens the holonomy of g_φ is contained in G_2.

The different classes of G_2-structures can be described in terms of the exterior derivatives $d\varphi$ and $d\star_\varphi \varphi$ [4,7]. If $d\varphi= 0$, then the G_2-structure is called closed (or calibrated in the sense of Harvey and Lawson [13]) and if φ is coclosed, then the G_2-structure is called coclosed (or cocalibrated [13]).

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Flows of G_2-structures were first considered by Bryant in [4]. In particular, he considered the Laplacian flow of closed G_2-structures. Recently, Lotay and Wei investigated the properties of the Laplacian flow in the series of papers [19–21]. The Laplacian coflow has been originally proposed by Karigiannis, McKay and Tsui in [15], and, for an initial coclosed G_2-form φ_0 with $\star_{\varphi_0}\varphi_0 = \phi_0$, it is given by

$$\frac{\partial}{\partial t}\phi(t) = -\Delta_t\phi(t), \quad d\phi(t) = 0, \quad \phi(0) = \phi_0, \tag{1}$$

where $\phi(t)$ is the Hodge dual 4-form of a G_2-structure $\varphi(t)$ with respect to the Riemannian metric $g_{\varphi(t)}$. This flow preserves the condition of the G_2-structure being coclosed, and it was studied in [15] for warped products of an interval, or a circle, with a compact 6-manifold N which is taken to be either a nearly Kähler manifold or a Calabi–Yau manifold. No general result is known about the short-time existence of the coflow (1). In [2], the Laplacian coflow on the seven-dimensional Heisenberg group has been studied, showing that the solution is always ancient, that is it is defined in some interval $(-\infty, T)$, with $0 < T < +\infty$. Other examples of flows of G_2-structures are the modified Laplacian coflow [11,12] and Weiss and Witt's heat flow [24]. The first one is a flow of coclosed G_2-structures obtained by adding a fixing term to the Laplacian coflow in order to ensure weak parabolicity in the exact directions. The second one is the gradient flow associated with the functional which measures the full torsion tensor of a G_2-structure; generally it does not preserve any special class of G_2-structures, but it can be modified to fix the underlying metric (see [3]).

As for the Ricci flow (and other geometric flows), for the Laplacian coflow it is interesting to consider *self-similar solutions* which are evolving by diffeomorphisms and scalings. If x_t is a 1-parameter family of diffeomorphisms generated by a vector field X on M with $x_0 = \mathrm{Id}_M$ and c_t is a positive real function on M with $c_0 = 1$, then a coclosed G_2-structure $\phi(t) = c_t(x_t)^*\phi_0$ is a solution of the coflow (1) if and only if ϕ_0 satisfies

$$-\Delta_0\phi_0 = L_X\phi_0 + c_0'\phi_0 = d(X\neg\phi_0) + c_0'\phi_0,$$

where by L_X and $X\neg$ we denote, respectively, the Lie derivative and the contraction with the vector field X. A coclosed G_2-structure satisfying the previous equation is called *soliton*. As in the case of the Ricci flow, the soliton is said to be expanding, steady, or shrinking if c_0' is positive, zero, or negative, respectively. By Proposition 4.3 in [15], if M is compact, then there are no expanding or steady soliton solutions of (1), other than the trivial case of a torsion-free G_2-structure in the steady case. Examples of solitons for the Laplacian flow have been constructed in [5,8,16–18,22].

In this paper, we study the coflow (1) on almost-abelian Lie groups, i.e., on solvable Lie groups with a codimension-one abelian normal subgroup. Coclosed and closed left-invariant G_2-structures on almost-abelian Lie groups have been studied by Freibert in [9,10]. General obstructions to the existence of a coclosed G_2-structure on a Lie algebra of dimension seven with non-trivial center are given in [1].

By [16], the Laplacian coflow on homogeneous spaces can be completely described as a flow of Lie brackets on the ordinary euclidean space, the so-called *bracket flow*. In particular, Lauret showed in [16] that any left-invariant closed Laplacian flow solution $\varphi(t)$ on an almost-abelian Lie group is immortal, i.e., defined in the interval $[0, +\infty)$. Moreover, he proved that the scalar curvature of $g_{\varphi(t)}$ is strictly increasing and converges to zero as t goes to $+\infty$.

In Sect. 3, we find an explicit description of the left-invariant solutions to the Laplacian coflow on almost-abelian Lie groups under suitable assumptions on the initial data, showing that the solution is ancient.

In Sect. 4 we show sufficient conditions for a left-invariant coclosed G_2-structure on an almost-abelian Lie group to be a soliton for the Laplacian coflow. In particular, we construct new examples of solitons which are not eigenforms of the Laplacian.

2 Preliminaries

A k-form on an n-dimensional real vector space is stable if it lies in an open orbit of the linear group $GL(n, \mathbb{R})$. In this section, we review the theory of stable forms in dimensions six and seven. We refer to [6,14], and the references therein, for more details. Throughout the sections, we denote by ϑ and by $*$ the actions of the endomorphism group and the general linear group, respectively.

2.1 Linear G_2-structures

A 3-form φ on a seven-dimensional real vector space V is *stable* if the $\Lambda^7(V^*)$-valued bilinear form b_φ, defined by

$$b_\varphi(x, y) = \frac{1}{6}(x \neg \varphi) \wedge (y \neg \varphi) \wedge \varphi, \quad x, y \in V,$$

is nondegenerate. In this case, φ defines an orientation vol_φ by $\sqrt[9]{\det b_\varphi}$ and a bilinear form g_φ by $b_\varphi = g_\varphi vol_\varphi$. A stable 3-form φ is said to be *positive*, and we will write $\varphi \in \Lambda^3_+ V^*$, if, in addition, g_φ is positive definite.

It is a well-known fact that the action of $\mathrm{GL}(V)$ on $\Lambda^3_+ V^*$ is transitive and the stabiliser of every $\varphi \in \Lambda^3_+ V^*$ is a subgroup of $\mathrm{SO}(g_\varphi)$ isomorphic to G_2. Therefore, if we assume that $||\varphi||_{g_\varphi} = 7$ we get a one-to-one correspondence between normalized positive 3-forms on V and presentations of G_2 inside $\mathrm{GL}(V)$.

We denote by $\star_\varphi$ the Hodge operator induced by φ, and we will always write ϕ to indicate the Hodge dual form $\star_\varphi \varphi$ of φ. Precisely, ϕ belongs to the $\mathrm{GL}(V)$-orbit, denoted by $\Lambda^4_+ V^*$, of *positive* 4-forms. It is another basic fact that the stabilisers of φ and ϕ under $\mathrm{GL}^+(V)$ are equal, and therefore, the choice of ϕ and of an orientation vol is sufficient to define φ.

We will refer to a presentation of G_2 inside $\mathrm{GL}(V)$ as a *linear G_2-structure* on V, and we will call φ and ϕ the *fundamental* forms associated with the linear G_2-structure.

On V there exists always a g_φ-orthonormal and positive oriented co-frame $\left(e^1, \ldots, e^7\right)$, called an *adapted* frame, such that

$$\begin{aligned}
\varphi &= e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \\
\phi &= e^{1234} + e^{3456} + e^{1256} - e^{2467} + e^{1367} + e^{1457} + e^{2357}.
\end{aligned}$$

2.2 Linear SU(3)-structures

Let U be a real vector space of dimension six. A 2-form ω on U is *stable* if it is nondegenerate, i.e., if $\omega^3 \neq 0$.

Given a 3-form ψ on U, the equivariant identification of $\Lambda^5 U^*$ with $U \otimes \Lambda^6 U^*$ allows us to define the operator

$$K_\psi : U \to U \otimes \Lambda^6 U^*, \quad x \mapsto (x \neg \psi) \wedge \psi.$$

We can consider the trace of its second iterate

$$\lambda(\psi) = \frac{1}{6}\mathrm{tr}(K_\psi^2) \in (\Lambda^6 U^*) \otimes (\Lambda^6 U^*),$$

where

$$K_\psi^2 : U \to U \otimes (\Lambda^6 U^*) \otimes (\Lambda^6 U^*).$$

Then ψ is *stable* if and only if $\lambda(\psi) \neq 0$. If $\lambda(\psi) < 0$, the 3-form ψ is called *negative*. In this case, we will write $\psi \in \Lambda^3_- U$. Here, the basic fact is that the action of $\mathrm{GL}^+(U)$ is transitive on $\Lambda^3_- U$ with stabiliser of ψ isomorphic to $\mathrm{SL}(3, \mathbb{C})$, where the associated complex structure J_ψ and complex volume form Ψ on U are given, respectively, by

$$J_\psi = \frac{1}{\sqrt{-\lambda}} K_\psi, \quad \Psi = -J_\psi^* \psi + i\psi.$$

It is important to note that the 3-form $J_\psi^* \psi$ is still negative and that it defines the same complex structure of ψ.

If ψ is a negative 3-form and ω a stable 2-form, then ω is of type $(1, 1)$ with respect to J_ψ, meaning that $J_\psi^* \omega = \omega$, if and only if $\psi \wedge \omega = 0$. In this case, we can define a symmetric bilinear form h on U by

$$h(x, y) = \omega(x, J_\psi y), \quad x, y \in U.$$

When h is positive definite, the couple (ω, ψ) is said to be a *positive couple* and it defines a *linear* SU(3)*-structure*, meaning that its stabiliser in $\mathrm{GL}(U)$ is isomorphic to SU(3). In this case, h is hermitian with respect to J_ψ and $\Psi = -J_\psi^* \psi + i\psi$ is a complex volume form. A positive couple is said to be *normalized* if

$$2\,\omega^3 = 3\,\psi \wedge J_\psi^* \psi.$$

If a positive couple (ω, ψ) is normalized, then there exists an h-orthonormal and positive oriented co-frame of U, called an *adapted* frame, $\left(f^1, J^* f^1, f^2, J^* f^2, f^3, J^* f^3\right)$ such that

$$\omega = f^1 \wedge J^* f^1 + f^2 \wedge J^* f^2 + f^3 \wedge J^* f^3,$$
$$\psi = -f^2 \wedge f^4 \wedge f^6 + f^1 \wedge J^* f^3 \wedge J^* f^6 + J^* f^1 \wedge f^4 \wedge J^* f^5 + J^* f^2 \wedge J^* f^3 \wedge f^5.$$

Therefore, if we denote by $*_h$ the Hodge operator on U associated with h, it follows that

$$*_h \omega = \frac{1}{2}\omega^2, \quad *_h \psi = J_\psi^* \psi.$$

2.3 From G_2 to SU(3)

Given a linear G_2-structure φ on V, with fundamental forms φ and ϕ, the six-dimensional sphere

$$S^6 = \{x \in V \mid g_\varphi(x, x) = 1\} \subset V$$

is G_2-homogeneous and, for any nonzero vector $v \in S^6$, there is an induced linear SU(3)-structure on the g_φ-orthogonal complement $U = (\mathrm{span} < v >)^\perp$. This structure is constructed as follows. Let

$$\omega = v \neg \varphi, \quad \psi = -v \neg \phi.$$

Then, (ω, ψ) is a positive couple on U defining the linear SU(3)-structure. It is then clear that the restriction of an adapted co-frame of (V, φ), with $v = e_7$, to U gives an adapted frame of (U, ω, ψ) and it follows that

$$\varphi = \omega \wedge e^7 - J_\psi^* \psi, \quad \phi = \frac{1}{2}\omega^2 + \psi \wedge e^7.$$

3 Explicit solutions to the Laplacian coflow on almost-abelian Lie groups

We recall that a Lie group G is said to be *almost-abelian* if its Lie algebra $\mathfrak{g}$ has a codimension-one abelian ideal $\mathfrak{h}$. Such a Lie algebra will be called almost-abelian, and it can be written as a semidirect product $\mathfrak{g} = \mathbb{R}x \ltimes_A \mathfrak{h}$. We point out that an almost-abelian Lie algebra is nilpotent if and only if the operator $ad_x|_{\mathfrak{h}}$ is nilpotent.

Freibert showed in [9] that if $\mathfrak{g}$ is a 7-dimensional almost-abelian Lie algebra, then, the following are equivalent:

1. $\mathfrak{g}$ admits a coclosed G_2-structure φ.
2. For any $x \in \mathfrak{g} \setminus \mathfrak{h}$, $ad(x)|_{\mathfrak{h}} \in \mathfrak{gl}(\mathfrak{h})$ belongs to $\mathfrak{sp}(\mathfrak{h}, \omega)$, where ω is a nondegenerate 2-form ω on $\mathfrak{h}$.
3. For any $x \in \mathfrak{g} \setminus \mathfrak{h}$, the complex Jordan normal form of $ad(x)|_{\mathfrak{h}}$ has the property that for all $m \in \mathbb{N}$ and all $\lambda \neq 0$ the number of Jordan blocks of size m with λ on the diagonal is the same as the number of Jordan blocks of size m with $-\lambda$ on the diagonal and the number of Jordan blocks of size $2m - 1$ with 0 on the diagonal is even.

In this section, we obtain an explicit description of the solutions to the Laplacian coflow on almost-abelian Lie groups under suitable assumptions on the initial data.

Let G be a seven-dimensional, simply connected, almost-abelian Lie group equipped with an invariant coclosed G_2-structure φ_0 with 4-form ϕ_0 and let $\mathfrak{h}$ be a codimension-one abelian ideal of the Lie algebra $\mathfrak{g}$ of G. By Proposition 4.5 in [23], if we choose a vector e_7 in the orthogonal complement of $\mathfrak{h}$ with respect to g_{φ_0} such that $g_{\varphi_0}(e_7, e_7) = 1$, the forms

$$\omega_0 = e_7 \lrcorner \varphi_0, \quad \psi_0 = -e_7 \lrcorner \phi_0, \tag{2}$$

define an SU(3)-structure (ω_0, ψ_0) on $\mathfrak{h}$. Let $\eta = e_7 \lrcorner g_{\varphi_0}$. Then, we can identify $\mathfrak{g}^*$ with $\mathfrak{h}^* \oplus \mathbb{R}\eta$ and we have $d\eta = 0$, since η vanishes on the commutator $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$. Moreover

$$d\alpha = \eta \wedge \vartheta(A)\alpha,$$

for every $\alpha \in \Lambda^p \mathfrak{h}^*$, where $A = ad_{e_7}|_{\mathfrak{h}}$. In particular, if ϕ is any 4-form on $\mathfrak{g}$, we can consider the decomposition

$$\phi = \phi^{(4)} + \phi^{(3)} \wedge \eta, \quad \phi^{(i)} \in \Lambda^i \mathfrak{h}^*, \; i = 3, 4. \tag{3}$$

So $(\phi_0)^{(4)} = 1/2\omega_0^2$ and $(\phi_0)^{(3)} = \psi_0$. Finally let us observe that $d\phi_0 = 0$ if and only if $\vartheta(A)(\omega_0^2) = 2\vartheta(A)(\omega_0) \wedge \omega_0 = 0$, which means that $A \in \mathfrak{sp}(\mathfrak{h}, \omega_0)$, since ω_0 is a nondegenerate 2-form.

Lemma 3.1 *Let U be a real vector space of dimension 6 endowed with a linear SU(3)-structure (ω, ψ) and $A \in \mathfrak{sp}(\omega)$ be normal with respect to the inner product h defined by (ω, ψ). Denote by J the complex structure induced by ψ and by S and L the symmetric and skew-symmetric part of A, respectively. Then, there exist $\theta \in [0, 2\pi]$ and a basis*

$(e_1, e_2, e_3, Je_1, Je_2, Je_3)$ of U such that

$$\begin{aligned} \omega &= e^1 \wedge J^*e^1 + e^2 \wedge J^*e^2 + e^3 \wedge J^*e^3, \\ \Psi &= (e^1 + iJ^*e^1) \wedge (e^2 + iJ^*e^2) \wedge (e^3 + iJ^*e^3) \end{aligned} \tag{4}$$

and

$$S(e_i) = s_i(\cos(\theta)e_i + \sin(\theta)Je_i), \quad S(Je_i) = -s_i(-\sin(\theta)e_i + \cos(\theta)Je_i), \quad i = 1, 2, 3, \tag{5}$$

where the real numbers $\{\pm s_i, i = 1, 2, 3\}$ are the eigenvalues of S (counted with their multiplicities), and $JV_{s_i} = V_{-s_i}$, where V_{s_i} denotes the eigenspace of S associated with the eigenvalue s_i. Moreover,

1. *if $s_j = 0$, then $Le_j = l_j Je_j$ and $LJe_j = -l_j e_j$, for $l_j \in \mathbb{R}$;*
2. *if $s_j \neq 0$ with multiplicity m_j, then $L|_{V_{s_j} \oplus V_{-s_j}}$ is given by the block matrix*

$$L = \begin{bmatrix} L' & 0 \\ 0 & L' \end{bmatrix}, \quad L' \in \mathfrak{so}(m_j),$$

with respect to the basis $(e_{i_1}, \ldots, e_{i_{m_j}}, Je_{i_1}, \ldots, Je_{i_{m_j}})$ of $V_{s_j} \oplus V_{-s_j}$,

Proof Clearly S and L belong to $\mathfrak{sp}(\omega)$ since A does. Therefore we have

$$h(x, SJy) = h(Sx, Jy) = -\omega(Sx, y) = \omega(x, Sy) = -h(x, JSy), \quad x, y \in V.$$

Thus $SJ = -JS$ and, similarly, $LJ = JL$.

The spectrum of S must be real and centrally symmetric, since S is symmetric and anti-commutes with J. Let $\{\pm s_i, i = 1, 2, 3\}$ be the spectrum of S. Denote by V_{s_i} the eigenspace of S associated with the eigenvalue s_i, and by $m(s_i)$ its multiplicity. It is then clear that, since $[S, L] = 0$ and $SJ = -JS$, L preserves each eigenspace V_{s_i} and $JV_{s_i} = V_{-s_i}$.

Now we show that on each J-invariant subspace $W_{s_i} = V_{s_i} + V_{-s_i}$, both S and L are given as in the statement with respect to some orthonormal basis. First, let us consider the case when $s_i = 0$ is an eigenvalue of S. Clearly, its multiplicity $m_0 = m(0)$ is even and the restriction $L|_{V_0}$ of L to the eigenspace V_0 belongs to $\mathfrak{sp}(m_0, \mathbb{R}) \cap \mathfrak{so}(m_0) = \mathfrak{u}(m_0/2)$. Therefore, we can diagonalize L over $\mathbb{C}$ as a complex matrix finding the desired expression; indeed, its eigenvalues are all imaginary numbers.

Now let $s_i \neq 0$ and $m(s_i) = m_i$. Then, W_{s_i} has real dimension $2m_i$ and there exists an orthonormal basis $\left(e_{r_1}, \ldots, e_{r_{m_i}}\right)$ of V_{s_i} such that, $L|_{W_i}$ has the following expression with respect to the orthonormal basis $\left(e_{r_1}, \ldots, e_{r_{m_i}}, Je_{r_1}, \ldots, Je_{r_{m_i}}\right)$

$$L = \begin{bmatrix} L_1 & L_3 \\ -L_3^{\dagger} & L_2 \end{bmatrix}, \; L_1, L_2 \in \mathfrak{so}(m_i), \;\; L_3 \in \mathfrak{gl}(m_i, \mathbb{R}),$$

where by $\dagger$ we denote the transpose. So, by $LJ = JL$ and $LS = SL$ we get $L_3 = 0$ and $L_1 = L_2$.

Putting all together the basis of W_i we get an orthonormal basis $(e_1, e_2, e_3, Je_1, Je_2, Je_3)$ of U but, generally, the basis is not an adapted frame with respect to the linear $\mathrm{SU}(3)$-structure (ω, ψ). Indeed $\Psi_0 = (e^1 + iJ^*e^1) \wedge (e^2 + iJ^*e^2) \wedge (e^3 + iJ^*e^3)$ does not necessarily coincide with Ψ. However, there exists a complex number z of modulus 1 such that $z^{-1}\Psi_0 = \Psi$. If we take a cubic root w of z and we consider the linear map Q defined by $Q = \mathrm{Re}(w)\mathrm{id} + \mathrm{Im}(w)J$

we get that $Q^*\Psi_0 = \Psi$. The transformation Q commutes with J and preserves each vector subspace W_{s_i}. Moreover,

$$\begin{aligned}
Q^*S = QSQ^{-1} &= (\mathrm{Re}(w)\mathrm{id} + \mathrm{Im}(w)J)S(\mathrm{Re}(w)\mathrm{id} - \mathrm{Im}(w)J)\\
&= S(\mathrm{Re}(w)\mathrm{id} - \mathrm{Im}(w)J)(\mathrm{Re}(w)\mathrm{id} - \mathrm{Im}(w)J)\\
&= S\left\{\left(\mathrm{Re}(w)^2 - \mathrm{Im}(w)^2\right)\mathrm{id} - 2\left(\mathrm{Re}(w)\mathrm{Im}(w)\right)J\right\}\\
&= \cos(\theta)S + \sin(\theta)JS,\\
Q^*L = QLQ^{-1} &= (\mathrm{Re}(w)\mathrm{id} + \mathrm{Im}(w)J)L(\mathrm{Re}(w)\mathrm{id} - \mathrm{Im}(w)J)\\
&= L(\mathrm{Re}(w)\mathrm{id} + \mathrm{Im}(w)J)(\mathrm{Re}(w)\mathrm{id} - \mathrm{Im}(w)J)\\
&= L\left(\mathrm{Re}(w)^2 + \mathrm{Im}(w)^2\right)\mathrm{id}\\
&= L.
\end{aligned}$$

Therefore, the new basis $(Qe_1, Qe_2, Qe_3, JQe_1, JQe_2, JQe_3)$ satisfies all the requested properties. □

Lemma 3.2 *Let* $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ *be an almost-abelian Lie algebra endowed with a coclosed* G_2*-structure* φ_0*. Let* (ω_0, ψ_0) *be the induced* SU(3)*-structure on* $\mathfrak{h}$ *defined by* (2)*, with* $\eta(e_7) \neq 0$*,* $\eta|_{\mathfrak{h}} = 0$ *and* $\|\eta\|_{g_{\varphi_0}} = 1$*. The solution* ϕ_t *of the Laplacian coflow on* $\mathfrak{g}$

$$\begin{cases} \dot{\phi}_t = -\Delta_t\phi_t, \\ d\phi_t = 0, \\ \phi_0 = \star_0\varphi_0, \end{cases} \tag{6}$$

is given by

$$\phi_t = \frac{1}{2}\omega_0^2 + p_t \wedge \eta,$$

where p_t *is a time-dependent negative* 3*-form on* $\mathfrak{h}$ *solving*

$$\begin{cases} \dot{p}_t = -\varepsilon(p_t)^2\,\vartheta(A)\vartheta(B_t)p_t, \\ p_0 = \psi_0, \end{cases} \tag{7}$$

where $\varepsilon(p_t)$ *is a function such that* $(\omega_0, \varepsilon(p_t)p_t)$ *defines an* SU(3)*-structure on* $\mathfrak{h}$ *and* B_t *is the adjoint of* $A = ad_{e_7}|_{\mathfrak{h}}$ *with respect to the scalar product* h_t *induced by the* SU(3)*-structure* $(\omega_0, \varepsilon(p_t)p_t)$*.*

Proof By Cauchy theorem the system of ODEs (6) admits a unique solution. Let ϕ_t be the solution of (6) and ε_t be the norm $||\eta||_t$ with respect to the scalar product g_t induced by ϕ_t. Then, we can write

$$\phi_t = \frac{1}{2}\omega_t^2 + \psi_t \wedge \frac{1}{\varepsilon_t}\eta,$$

where the couple (ω_t, ψ_t) defines an SU(3)-structure on $\mathfrak{h} = \mathrm{Ker}(\eta)$. To see this observe that if we define x_t by $g_t(x_t, y) = \eta(y)$, for any $y \in \mathfrak{g}$, then $\mathfrak{h} = \mathrm{Ker}(\eta) = \{y \in \mathfrak{g} \mid g_t(x_t, y) = 0\}$. Therefore, for every t, the 4-form ϕ_t defines an SU(3)-structure (ω_t, ψ_t) on $\mathfrak{h}$.

With respect to the decomposition (3), we can write ϕ_t as $\phi_t = \phi_t^{(4)} + \phi_t^{(3)} \wedge \eta$ with

$$\phi_t^{(4)} = \frac{1}{2}\omega_t^2, \quad \phi_t^{(3)} = \frac{1}{\varepsilon_t}\psi_t.$$

Since the cohomology class of ϕ_t is fixed by the flow, i.e., $\phi_t = \phi_0 + \mathrm{d}\alpha_t$ it turns out that

$$\dot{\phi}_t = \dot{\phi}_t^{(4)} + \dot{\phi}_t^{(3)} \wedge \eta = \mathrm{d}\dot{\alpha}_t \in d\Lambda^3\mathfrak{g}^* \subseteq \Lambda^3\mathfrak{h}^* \wedge \mathbb{R}\eta.$$

Therefore, $\dot{\phi}_t^{(4)} = 0$, i.e., $\omega_t \equiv \omega_0$, and consequently,

$$\phi_t = \frac{1}{2}\omega_0^2 + \psi_t \wedge \frac{1}{\varepsilon_t}\eta.$$

Now define $\eta_t = \frac{1}{\varepsilon_t}\eta$ and denote by $\star_{g_t}$ and $\star_{h_t}$ the star Hodge operators on $\mathfrak{g}$ and $\mathfrak{h}$ with respect to g_t and h_t, respectively. Note that

$$\star_{g_t}\phi_t = \omega_0 \wedge \eta_t - \star_{h_t}\psi_t,$$

since

$$\star_{g_t}\beta = \star_{h_t}\beta \wedge \eta_t, \quad \star_{g_t}(\beta \wedge \eta_t) = (-1)^k \star_{h_t} \beta,$$

for every k-form β on $\mathfrak{h}$.

Then,

$$\begin{aligned}
\Delta_t\phi_t &= d\star_{g_t} d\star_{g_t} \phi_t = d\star_t d\,(\omega \wedge \eta_t - *_t\psi_t) \\
&= -d\star_t (\eta \wedge \vartheta(A) *_t \psi_t) = -d\star_t (\varepsilon_t\eta_t \wedge \vartheta(A) *_t \psi_t) \\
&= -\varepsilon_t d *_t (\vartheta(A) *_t \psi_t) \\
&= \varepsilon_t \,(\vartheta(A) *_t \vartheta(A) *_t \psi_t) \wedge \eta \\
&= \varepsilon_t^2 \,(\vartheta(A) *_t \vartheta(A) *_t \psi_t) \wedge \eta_t.
\end{aligned}$$

On the other hand, we have

$$\dot{\phi}_t = \dot{\psi}_t \wedge \frac{1}{\varepsilon_t}\eta - \psi_t \wedge \frac{\dot{\varepsilon}_t}{\varepsilon_t^2}\eta = \dot{\psi}_t \wedge \eta_t - \frac{\dot{\varepsilon}_t}{\varepsilon_t}\psi_t \wedge \eta_t.$$

Imposing $\dot{\phi}_t = -\Delta_t\phi_t$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t - \frac{d}{dt}(\varepsilon_t)\varepsilon_t^{-1}\psi_t = -\varepsilon_t^2 \,(\vartheta(A) *_t \vartheta(A) *_t \psi_t)\,. \tag{8}$$

Consider the 3-form $p_t = \varepsilon_t^{-1}\psi_t$. It is clear that p_t is a negative 3-form, compatible with ω_0 and defining the same complex structure J_t induced by ψ_t. Moreover, it satisfies the condition

$$-6\, p_t \wedge J_t^* p_t = 4\,\varepsilon_t^{-2}{\omega_0}^3.$$

Then, by (8) we obtain

$$\varepsilon_t\dot{p}_t + \dot{\varepsilon}_t p_t - \dot{\varepsilon}_t p_t = -\varepsilon_t^3\,(\vartheta(A) *_t \vartheta(A) *_t p_t)\,,$$

and thus, the following equation in terms of the 3-form p_t

$$\dot{p}_t = -\varepsilon(p_t)^2\,(\vartheta(A) *_t \vartheta(A) *_t p_t)\,, \quad p_0 = \psi_0, \tag{9}$$

where the function $\varepsilon(p_t) = \varepsilon_t = \|\eta\|_t$ is defined in terms of the 3-form p_t by

$$6\, p_t \wedge J_t^* p_t = 4\,\varepsilon(p_t)^{-2}{\omega_0}^3.$$

It is easy to see that $*_t\vartheta(A)*_t$ is the h_t-adjoint operator of $\vartheta(A)$ on $\Lambda^3\mathfrak{h}^*$. Indeed, if $\alpha, \beta \in \Lambda^3\mathfrak{h}^*$, then

$$\langle((*_t\vartheta(A)*_t)\alpha, \beta\rangle_t\,\omega_0^3/6 = -\beta \wedge \vartheta(A)(*_t\alpha) = \vartheta(A)(\beta) \wedge *_t(\alpha) = \langle\alpha, \vartheta(A)(\beta)\rangle_t\,\omega_0^3/6,$$

where in the second equality we have used that A is traceless and consequently that $\vartheta(A)$ acts trivially on 6-forms.

Now let B_t be the h_t-adjoint of A on $\mathfrak{h}$. We claim that $(*_t\vartheta(A)*_t)\alpha = \vartheta(B_t)\alpha$ for any 3-form α on $\mathfrak{h}$. To see this let $(e_1 \ldots, e_6)$ be an h_t-orthonormal basis of $\mathfrak{h}$, so[1]

$$(B_t)^i{}_j = \sum_{a,b}(A)^a{}_b\,(h_t)_{aj}(h_t)^{bi} = (A)^j{}_i, \quad i, j = 1, \ldots, 6.$$

On the other hand, for any choice of ordered triples (i, j, k) and (a, b, c), we get

$$\begin{aligned} h_t\left(\vartheta(A)e^{ijk}, e^{abc}\right) &= -h_t \sum_{l,m,n}\left(A^i{}_l e^{ljk} + A^j{}_m e^{imk} + A^k{}_n e^{ijn}, e^{abc}\right)\\ &= -\sum_{l,m,n} h_t\left(A^i{}_{i'}e^{i'j'k'} + A^j{}_{j'}e^{i'j'k'} + A^k{}_{k'}e^{i'j'k'}, e^{abc}\right)\\ &= -\left(A^i{}_a + A^j{}_b + A^k{}_c\right) \end{aligned}$$

and

$$\begin{aligned} h_t\left(e^{ijk}, \vartheta(B_t)e^{abc}\right) &= -\sum_{l,m,n} h_t\left(e^{ijk}, B^a_l e^{lbc} + B^b_m e^{amc} + B^c_n e^{abn}\right)\\ &= -\sum_{l,m,n} h_t\left(e^{ijk}, B^a_{a'}e^{a'b'c'} + B^b_{b'}e^{a'b'c'} + B^c_{c'}e^{a'b'c'}\right)\\ &= -\left(B^a_i + B^b_j + B^c_k\right)\\ &= -\left(A^i{}_a + A^j{}_b + A^k{}_c\right), \end{aligned}$$

since $h_t\left(e^{ijk}, e^{abc}\right) = \delta^{ia}\delta^{bj}\delta^{kc}$. Therefore,

$$h_t\left(\vartheta(A)\alpha, \beta\right) = h_t\left(\alpha, \vartheta(B_t)\beta\right), \quad \alpha, \beta \in \Lambda^3\mathfrak{h}^*,$$

as we claimed and (7) holds. □

Theorem 3.3 *Let* $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ *be an almost-abelian Lie algebra endowed with a coclosed* G_2*-structure* φ_0*. Let* (ω_0, ψ_0) *be the induced* SU(3)*-structure on* $\mathfrak{h}$ *defined by* (2)*, with* $\eta(e_7) \neq 0$*,* $\eta|_{\mathfrak{h}} = 0$ *and* $\|\eta\|_{g_{\varphi_0}} = 1$*, and let* $J_0 = J_{\psi_0}$*. Suppose that* $A = ad_{e_7}|_{\mathfrak{h}}$ *is symmetric with respect to the inner product* $h_0 = g_0|_{\mathfrak{h}}$ *and fix an adapted frame* $(e_1, J_0e_1, e_2, J_0e_2, e_3, J_0e_3)$ *of* $(\mathfrak{h}, \omega_0, \psi_0)$ *such that* ω_0 *and* ψ_0 *are given by* (4) *and* A *has the normal form* (5)*. Furthermore, assume that* A *satisfies* $\theta = 0$*. Then, the solution* p_t *of* (7) *is ancient and it is given by*

$$p_t = -b_1(t)e^{246} + b_2(t)e^{136} + b_3(t)e^{145} + b_4(t)e^{235}, \quad t \in \left(-\infty, \frac{1}{8\left(s_1^2 + s_2^2 + s_3^2\right)}\right),$$

where $b_i(t) = e^{-\sigma_i\epsilon(t)}$ *for suitable constants* σ_i *and*

$$\epsilon(t) = \int_0^t \frac{1}{1 - 8\left(s_1^2 + s_2^2 + s_3^2\right)u}\mathrm{d}u.$$

[1] Note that we are not using the Einstein notation.

Proof Consider the following system

$$\begin{cases} \dot{\chi}_t = -f(t)^2 \vartheta(A)\vartheta(A)\chi_t, \\ \chi_0 = \psi_0, \end{cases} \tag{10}$$

where $f(t)$ is a positive function which will be defined later. Moreover, let

$$(f_1, f_2, f_3, f_4, f_5, f_6) = (e_1, J_0e_1, e_2, J_0e_2, e_3, J_0e_3)$$

be an adapted frame of $\mathfrak{h}$ such that ω_0 and ψ_0 are given by (4) and A has the normal form (5). It is clear that

$$\begin{aligned} \vartheta(A)\vartheta(A)\psi_0 = &-(s_1+s_2+s_3)^2 f^{246} \\ &+ (s_1+s_2-s_3)^2 f^{136} \\ &+ (s_1-s_2+s_3)^2 f^{145} \\ &+ (-s_1+s_2+s_3)^2 f^{235}. \end{aligned}$$

So

$$\vartheta(A)\vartheta(A)\psi_0 = -\sigma_1 f^{246} + \sigma_2 f^{136} + \sigma_3 f^{145} + \sigma_4 f^{235},$$

for suitable constants $\sigma_1, \sigma_2, \sigma_3$ and σ_4. The solution of (10) is then given by

$$\chi_t = -b_1(t) f^{246} + b_2(t) f^{136} + b_3(t) f^{145} + b_4(t) f^{235}, \tag{11}$$

where $b_i(t) = e^{-\sigma_i \epsilon(t)}$ for a function $\epsilon(t)$ satisfying $\dot{\epsilon}(t) = f(t)^2$. In order to determine the function $f(t)$, note that, for every t where it is defined, the 3-form χ_t is negative, compatible with ω_0 and it defines a complex structure J_t, given by

$$J_t = \frac{2}{\sqrt{-\nu_t}} \begin{bmatrix} 0 & -b_4b_1 & 0 & 0 & 0 & 0 \\ b_2b_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_3b_1 & 0 & 0 \\ 0 & 0 & b_2b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_2b_1 \\ 0 & 0 & 0 & 0 & b_3b_4 & 0 \end{bmatrix}, \quad \nu_t = -4b_1^2b_2^2b_3^2b_4^2, \tag{12}$$

with respect to the adapted frame $(f_1, f_2, f_3, f_4, f_5, f_6)$. Moreover,

$$6\,\chi_t \wedge J_t^* \chi_t = 4b_1^2b_2^2b_3^2b_4^2\,\omega_0^3.$$

The previous condition is satisfied if we choose $f(t)$ such that

$$f(t)^{-2} = b_1^2b_2^2b_3^2b_4^2 = e^{-2(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\int_0^t f(u)^2 \mathrm{d}u} = e^{-8(s_1^2+s_2^2+s_3^2)\int_0^t f(u)^2 \mathrm{d}u}.$$

The above identity is satisfied if and only if the function $F_t = \int_0^t f(u)^2 \mathrm{d}u$ solves the following Cauchy problem

$$\begin{cases} \dot{F}_t = e^{8\delta F_t}, \\ F_0 = 0, \end{cases}$$

where $\delta = s_1^2 + s_2^2 + s_3^2$. Integrating we get

$$t = \frac{1 - e^{-8\delta F_t}}{8\delta}.$$

Therefore,

$$F_t = \frac{\ln(1-8\delta t)}{-8\delta},$$

and consequently, $f(t) = \frac{1}{\sqrt{1-8\delta t}}$. Finally, we observe that the metric h_t defined by (ω_0, J_t) is positive definite. Moreover, the endomorphism H_t, defined by $g_0(x, H_t y) = h_t(x, y)$ for any $x, y \in \mathfrak{h}$, has the following matrix representation

$$H_t = \frac{2}{\sqrt{-\nu_t}} \begin{bmatrix} b_2b_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1b_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_3b_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1b_2 \end{bmatrix}, \tag{13}$$

with respect to the adapted frame $(f_1, f_2, f_3, f_4, f_5, f_6)$. Now we claim that χ_t, given by (11), with

$$\epsilon(t) = \int_0^t f(u)^2 \, \mathrm{d}u = \int_0^t \frac{1}{1 - 8\left(s_1^2 + s_2^2 + s_3^2\right) u} \mathrm{d}u,$$

is the solution of (7). To see this, first observe that the choice of $f(t)$ ensures that $\varepsilon(\chi_t)^2 = f(t)^2$. The only thing we have to prove is that the adjoint C_t of A with respect to h_t is constant. It is clear that $C_t = H_t^{-1} B_0 H_t$ and then the claim is equivalent to show that $[H_t, B_0] = 0$. With respect to the adapted frame $(f_1, f_2, f_3, f_4, f_5, f_6)$, the endomorphism H_t is diagonal as well as $B_0 = A$, and then the claim follows. Thus, the solution p_t of (7) is given by χ_t, and in particular $B_t \equiv B_0$. ◻

Remark 3.4 The previous proof can be adapted to the case $\theta = \pi$. Indeed, if $\theta = \pi$ then $\vartheta(A)\vartheta(A)\psi_0$ is again a linear combination of elements of the form $e^a \wedge e^b \wedge J_t^* e^c$ with coefficients given by a suitable choice of $\pm(s_a + s_b - s_c)$. On the other hand, when θ is different from 0 and π, it turns out that $\vartheta(A)\vartheta(A)\psi_0$ is a linear combination of elements $e^a \wedge e^b \wedge J_t^* e^c$ and $e^a \wedge J_t^* e^b \wedge J_t^* e^c$. Therefore, the derivative of J_t^* at $t = 0$ is much more complicated than in the other cases (see Remark 3.6).

Theorem 3.5 *Let $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ be an almost-abelian Lie algebra endowed with a coclosed G_2-structure φ_0. Let (ω_0, ψ) be the induced* SU(3)*-structure on $\mathfrak{h}$ defined by* (2)*, with $\eta(e_7) \neq 0$, $\eta|_{\mathfrak{h}} = 0$ and $\|\eta\|_{g_{\varphi_0}} = 1$. Suppose that $A = ad_{e_7}|_{\mathfrak{h}}$ is skew-symmetric with respect to the inner product $h_0 = g_0|_{\mathfrak{h}}$ and define $l = l_1 + l_2 + l_3$, where l_1, l_2 and l_3 are as in Lemma* 3.1*. Then, the solution p_t of* (7) *is given by*

$$p_t = b(t)\psi_0,$$

where $b(t) = e^{-l^2 \int_0^t \varepsilon_u^2 \mathrm{d}u}$ and ε_t is a positive function given by

$$\varepsilon_t = \frac{1}{\sqrt{1 - 2l^2 t}}.$$

In particular, p_t is an ancient solution, defined for every t in $\left(-\infty, \frac{1}{2l^2}\right)$.

Proof Let f_t be a positive function which will be fixed later and let us consider the following system

$$\begin{cases} \dot{\chi}_t = -f_t^2 \vartheta(A)\vartheta(-A)\chi_t, \\ \chi_0 = \psi_0. \end{cases} \tag{14}$$

Moreover, let $(f_1, \dots, f_6) = (e_1, J_0e_1, e_2, J_0e_2, e_3, J_0e_3)$ be an adapted frame such that ω_0 and ψ_0 are given by (4) and A has the normal form (5). It is clear that

$$\vartheta(A)\vartheta(A)\psi_0 = -l^2\psi_0.$$

Therefore, the solution of (14) is given by

$$\chi_t = b(t)\psi_0,$$

where $b(t) = e^{-l^2 \int_0^t f_u^2 \mathrm{d}u}$.

The 3-form χ_t is negative, compatible with ω_0 and it defines a constant complex structure $J_t \equiv J_0$. Moreover, it satisfies

$$6\, \chi_t \wedge I_t \chi_t = 4b^2(t)\, \omega^3.$$

Now we choose f_t so that

$$f_t^{-2} = b(t)^2 = e^{-2l^2 \int_0^t f_u^2 \mathrm{d}u}.$$

To do this, we solve the system

$$\begin{cases} \dot{F}_t = e^{2l^2 F_t \mathrm{d}u}, \\ F_0 = 0, \end{cases}$$

and then we put $f_t = \sqrt{\dot{F}_t}$. Integrating by t we get

$$t = \frac{1 - e^{2l^2 F_t}}{2l^2}.$$

Thus,

$$F_t = \frac{\ln(1 - 2l^2 t)}{-2l^2},$$

and consequently, $\varepsilon_t = \frac{1}{\sqrt{1-2l^2 t}}$.

Now it is easy to show that χ_t is a solution of (7). Indeed, the choice of f_t ensures that $\varepsilon(\chi_t)^2 = f_t^2$, and moreover, that the metric h_t induced by ω_0 and χ_t is constant. Therefore, the adjoint of A is constantly equal to $-A$ and, as a consequence, the solution p_t of (7) is given by χ_t. □

Remark 3.6 It is not hard to prove that if A is normal with respect to h_0, then the solution p_t of (7) is given by

$$\begin{aligned} p_t = &-b_1(t) f^{246} + b_2(t) f^{136} + b_3(t) f^{145} + b_4(t) f^{235} + c_1(t) f^{135} \\ &- c_2(t) f^{245} - c_3(t) f^{236} - c_4(t) f^{145}, \end{aligned}$$

where $(f_1, \dots, f_6) = (e_1, J_0e_1, e_2, J_0e_2, , e_3, J_0e_3)$ is an adapted frame of $\mathfrak{h}$ and A is given by (5). Unfortunately in this case, we cannot find an explicit solution of (7).

However, note that if we write $p_t = (x_t)^*\psi_0$, for $[x_t] \in \mathrm{GL}(\mathfrak{h})/\mathrm{SL}(3, \mathbb{C})$, then x_t belongs to $\mathrm{GL}^+(2, \mathbb{R})^3$ acting on $< e_1, J_0e_1 > \oplus < e_2, J_0e_2 > \oplus < e_3, J_0e_3 >$. Therefore, $[x_t] = ([x_t^{(1)}], [x_t^{(2)}], [x_t^{(3)}]) \in \left((\mathrm{GL}^+(2, \mathbb{R})/\mathrm{SO}(2)\right)^3$.

4 Solitons for the Laplacian coflow on almost-abelian Lie groups

In this section, we find sufficient conditions for a left-invariant coclosed G_2-structure on an almost-abelian Lie group G to be a soliton for the Laplacian coflow.

Let $\mathfrak{g}$ be a Lie algebra. We recall the following

Definition 4.1 Let $\mathfrak{g}$ be a seven-dimensional Lie algebra endowed with a coclosed G_2-structure φ_0. A solution ϕ_t to the Laplacian coflow (6) on $\mathfrak{g}$ is *self-similar* if

$$\phi_t = c_t(x_t)^*\phi_0,$$

for a real-valued function c_t and a $\mathrm{GL}(\mathfrak{g})$-valued function x_t.

It is well-known that a solution ϕ_t of (6) is self-similar if and only if the Cauchy datum ϕ_0 at $t = 0$ is a *soliton*, namely if it satisfies

$$-\Delta_0\phi_0 = -4c\phi_0 + \vartheta(D)\phi_0,$$

for some real number c and some derivation D of the Lie algebra $\mathfrak{g}$ (see [16]). A soliton is said to be *expanding* if $c < 0$, *shrinking* if $c > 0$ and *steady* if $c = 0$.

Let K_t be the stabiliser of ϕ_t and fix a K_t-invariant decomposition of $\mathrm{End}(\mathfrak{g})$. Since ϕ_t is stable at any time t, there exists a time-dependent endomorphism X_t of $\mathfrak{g}$, transversal to the Lie algebra of K_t (in the sense that, for every t, X_t takes values in an ad-invariant complement of the Lie algebra of K_t), such that

$$-\Delta_t\phi_t = \vartheta(X_t)\phi_t.$$

Therefore, ϕ_0 is a soliton on $\mathfrak{g}$ if and only if

$$X_0 = c\,\mathrm{Id} + D.$$

Suppose now that $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ is an almost-abelian Lie algebra endowed with a coclosed G_2-structure φ_0. In Lemma 3.2, we have seen that, with no further assumptions on $A = ad_{e_7}|_{\mathfrak{h}}$, the Laplacian coflow reads as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\phi_t = -\varepsilon_t\vartheta(A)\vartheta(B_t)\psi_t \wedge \eta, \\ \phi_0 = \star_0\varphi_0. \end{cases}$$

We can show that the term $-\varepsilon_t\vartheta(A)\vartheta(B_t)\psi_t \wedge \eta$ can be rewritten as

$$-\varepsilon_t\,(\vartheta(A)\vartheta(B_t)\psi_t) \wedge \eta = \vartheta(X_t)\phi_t, \tag{15}$$

for a time-dependent endomorphism X_t of $\mathfrak{g}$ in the following way.

Let (ω_t, ψ_t) be the SU(3)-structure on $\mathfrak{h}$ induced by ϕ_t. By Lemma 3.1 there exist $\theta(t) \in [0, 2\pi]$ and an adapted frame of $\mathfrak{h}$ such that $\eta = \varepsilon_t e^7$ and the symmetric part $S(t)$ of A has the normal form (5). More precisely, let

$$a(t) = \cos(\theta(t)), \quad b(t) = \sin(\theta(t)).$$

With respect to the adapted frame at time t, $S(t)$ has the form (5), so it is given by

$$S(t) = \begin{bmatrix} S_1(t) & 0 & 0 & 0 \\ 0 & S_2(t) & 0 & 0 \\ 0 & 0 & S_3(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$S_i(t) = \begin{bmatrix} a(t)s_i(t) & b(t)s_i(t) \\ b(t)s_i(t) & -a(t)s_i(t) \end{bmatrix}.$$

Define $l(t)$ to be the imaginary part of the complex trace of the skew-symmetric part $L(t)$ of A at time t, and let

$$\Sigma(t) = \begin{bmatrix} \Sigma_1(t) & 0 & 0 & 0 \\ 0 & \Sigma_2(t) & 0 & 0 \\ 0 & 0 & \Sigma_3(t) & 0 \\ 0 & 0 & 0 & -s^2(t) \end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l(t)^2 \end{bmatrix} \tag{16}$$

where

$$\Sigma_1(t) = \begin{bmatrix} -2s_2(t)s_3(t) + 4a(t)^2 s_2(t)s_3(t) & -4a(t)b(t)s_2(t)s_3(t) \\ -4a(t)b(t)s_2(t)s_3(t) & 2s_2(t)s_3(t) - 4a(t)^2 s_2(t)s_3(t) \end{bmatrix},$$
$$\Sigma_2(t) = \begin{bmatrix} -2s_1(t)s_3(t) + 4a(t)^2 s_1(t)s_3(t) & -4a(t)b(t)s_1(t)s_3(t) \\ -4a(t)b(t)s_1(t)s_3(t) & 2s_1(t)s_3(t) - 4a(t)^2 s_1(t)s_3(t) \end{bmatrix},$$
$$\Sigma_3(t) = \begin{bmatrix} -2s_1(t)s_2(t) + 4a^2(t)s_1(t)s_2(t) & -4a(t)b(t)s_1(t)s_2(t) \\ -4a(t)b(t)s_1(t)s_2(t) & 2s_1(t)s_2(t) - 4a(t)^2 s_1(t)s_2(t) \end{bmatrix},$$

and $s(t)^2 = s_1(t)^2 + s_2(t)^2 + s_3(t)^2$. We claim that

$$X_t = -\varepsilon_t\left(\Sigma(t) + \Lambda(t) - [S(t), L(t)]\right). \tag{17}$$

To prove this first observe that

$$\begin{aligned} \vartheta(A)\vartheta(B_t)\psi_t &= \vartheta(S(t) + L(t))\vartheta(S(t) - L(t))\psi_t \\ &= \left(\vartheta(S(t))\vartheta(S(t)) - \vartheta(L(t))\vartheta(L(t)) - \vartheta([S(t), L(t)])\right)\psi_t. \end{aligned}$$

Then, a direct computation shows that

$$\vartheta(\Sigma(t))\phi_t = \vartheta(\Sigma(t))(\psi_t \wedge \eta) = (\vartheta(S(t))\vartheta(S(t))\psi_t) \wedge \eta.$$

On the other hand, if we change the adapted frame so that L_t has the form (5) we see that, as already seen in the proof of Theorem 3.5, $l(t) = l_1(t) + l_2(t) + l_3(t)$ and

$$-\vartheta(L(t))\vartheta(L(t))\psi_t = l^2(t)\psi_t,$$

which is an expression independent on the choice of the adapted frame. Thus,

$$\vartheta(\Lambda(t))\phi_t = \vartheta(\Lambda(t))(\psi_t \wedge \eta) = l(t)^2\psi_t \wedge \eta = -(\vartheta(L(t))\vartheta(L(t))\psi_t) \wedge \eta,$$

proving the claim.

Theorem 4.2 *Let $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ be an almost-abelian Lie algebra endowed with a coclosed G_2-structure φ_0. The 4-form $\phi_0 = \star_0\varphi_0$ is a soliton for the Laplacian coflow if and only if A satisfies*

$$[-\Sigma(0) + 1/2[A, A^\dagger], A] = \delta A, \tag{18}$$

where $A^\dagger$ denotes the transpose of A with respect to the underlying metric on $\mathfrak{h}$, $\delta = l_0^2 + s_0^2 - c$ for a constant $c \in \mathbb{R}$ and $\Sigma(0)$ the endomorphism (16). If ϕ_0 is a soliton, then the solution ϕ_t to the Laplacian coflow is given by

$$\phi_t = c(t)e^{f(t)D}\phi_0, \quad c(t) = (1 - 2ct)^2, \quad f(t) = -\frac{1}{2c}\ln(1 - 2ct), \quad t < \frac{1}{2c}, \tag{19}$$

where the derivation D of $\mathfrak{g}$ is given by $X_0 - c\,\mathrm{Id}$, with X_0 as in (17)*.*

Proof In the light of the previous results, we can write down the soliton equation for the Laplacian coflow as follows. Suppose that ϕ_0 is a soliton, that is, $X_0 = c\mathrm{Id} + D$ for some $c \in \mathbb{R}$ and a derivation D of $\mathfrak{h}$. Then, by [16, Theorem 4.10], (19) holds. Therefore, A corresponds to a soliton if and only if there exists $c \in \mathbb{R}$ such that $D = X_0 - c\,\mathrm{Id}$ is a derivation of $\mathfrak{g}$.

This condition can be read as a system of algebraic equations for c and the elements of the matrix associated with A. Note that $De_7 = \delta e_7$, with $\delta = l_0^2 + s_0^2 - c$. Hence, denoting by μ_A the Lie bracket structure defined by A,

$$[D, A]v = DAv - ADv = D\mu_A(e_7, v) - \mu_A(e_7, Dv) = \mu_A(De_7, v) = \delta Av, \quad v \in \mathfrak{h}.$$

This reads as

$$[D, A] = \delta A. \tag{20}$$

Finally, writing $D = X_0 - c\,\mathrm{Id}$ for X_0 as in (17) and recalling that $[A, A^\dagger] = -2[S(0), L(0)]$, we derive (18) from (20). □

We call (18) the *soliton equation* of the almost-abelian Laplacian coflow.

Notice that we can split the soliton equation into two coupled equations, for the symmetric and skew-symmetric parts of A, in the following way. Since the commutator of two symmetric matrices is skew-symmetric and the commutator of a symmetric matrix and a skew-symmetric one is symmetric, we find

$$[-\Sigma(0) + [S(0), L(0)], L(0)] = \delta S(0), \quad [-\Sigma(0) + [S(0), L(0)], S(0)] = \delta L(0). \tag{21}$$

We have just proved the following result.

Corollary 4.3 *If a soliton ϕ_0 of the Laplacian coflow on the almost-abelian Lie algebra $\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}$ is an eigenform of the Laplacian then it is harmonic, namely $A \in \mathrm{su}(6)$.*

Proof Clearly, if ϕ_0 is an eigenform of the Laplacian, then $D = 0$, hence $X_0 = c\,\mathrm{Id}$. Taking the trace of $X_0|_{\mathfrak{h}} = c\mathrm{Id}|_{\mathfrak{h}}$ we find $c = 0$. □

Corollary 4.4 *Let $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ be an almost-abelian Lie algebra endowed with a coclosed G_2-structure φ_0. Assume that A is normal with respect to the underlying metric. Then, $\phi_0 = \star_0\varphi_0$ is soliton on $\mathfrak{g}$ if and only if $4b(1 - 4a^2)s_1 s_2 s_3 = 0$ and $[\Sigma(0), L(0)] = 0$. In such a case $\delta = 0$ and hence $c \geq 0$.*

Proof By hypotheses, Eq. (21) reduce to

$$-[\Sigma(0), S(0)] = \delta L(0), \quad -[\Sigma(0), L(0)] = \delta S(0).$$

Using the normal form (5), a direct computation shows that

$$[\Sigma(0), S(0)] = 4b(1 - 4a^2)s_1 s_2 s_3 J_0.$$

If δ was different from zero, then $S(0)$ would be invertible (each s_i should be nonzero), and therefore, $L(0)$ would be a nonzero multiple of J_0, contradicting Lemma 3.1. Thus, $\delta = 0$, that is $4b(1 - 4a^2)s_1 s_2 s_3 = 0$. Clearly, if $L(0) \neq 0$, equation $[\Sigma(0), L(0)] = 0$ is not generically satisfied. □

Corollary 4.5 *Let* $(\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}, \varphi_0)$ *be an almost-abelian Lie algebra endowed with a coclosed* G_2*-structure* φ_0 *and suppose that* $A = ad_{e_7}|_{\mathfrak{h}}$ *is skew-symmetric with respect to the underlying metric. Then, the solution to the Laplacian coflow obtained in Theorem 3.5 is a soliton.*

Remark 4.6 Differently from the Laplacian flow studied in [16] there exist almost-abelian Lie algebras $\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}$ with A symmetric and admitting coclosed G_2-structures that are no solitons for the Laplacian coflow. Indeed, in the light of Corollary 4.4 it is enough to choose a symmetric matrix A and a suitable G_2-structure for which the constant $4b(1-4a^2)s_1s_2s_3$ is nonzero. For instance, we can consider the G_2-structure

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

on the Lie algebra $\mathfrak{g} = \mathbb{R}e_7 \ltimes_A \mathfrak{h}$, where $\mathfrak{h} = \mathbb{R} < e_1, \dots, e_6 >$ and

$$A = ad_{e_7}|_{\mathfrak{h}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Moreover, we are able to prove that the interval of existence of the corresponding solution is bounded. To this aim, and in analogy with the proof of Theorem 3.3, observe that the solution to (10) has the following expression:

$$\chi_t = -b_1(t)e^{246} + b_2(t)e^{136} + b_3(t)e^{145} + b_4(t)e^{235}.$$

Indeed,

$$\begin{aligned} \vartheta(A)\vartheta(A)\chi_t = & -(3b_1(t) - 2b_2(t) - 2b_3(t) - 2b_4(t))e^{246} \\ & + (-2b_1(t) + 3b_2(t) + 2b_3(t) + 2b_4(t))e^{136} \\ & + (-2b_1(t) + 2b_2(t) + 3b_3(t) + 2b_4(t))e^{145} \\ & + (-2b_1(t) + 2b_2(t) + 2b_3(t) + 3b_4(t))e^{235}, \end{aligned}$$

and therefore, the vector-valued function $(b_1(t), b_2(t), b_3(t), b_4(t))$ satisfies a linear ODE whose matrix is

$$-f_t^2 \begin{pmatrix} 3 & -2 & -2 & -2 \\ -2 & 3 & 2 & 2 \\ -2 & 2 & 3 & 2 \\ -2 & 2 & 2 & 3 \end{pmatrix}.$$

Taking into account that this matrix is symmetric, with eigenvalues $-9f_t^2, -f_t^2, -f_t^2, -f_t^2$ and eigenvectors $(-1, 1, 1, 1)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$ and $(1, 0, 0, 1)$, it follows that

$$2b_1(t) = -e^{-9\int_0^t f_u^2 \mathrm{d}u} + 3e^{-\int_0^t f_u^2 \mathrm{d}u}, \quad 2b_2(t) = 2b_3(t) = 2b_4(t) = e^{-9\int_0^t f_u^2 \mathrm{d}u} + e^{-\int_0^t f_u^2 \mathrm{d}u}.$$

The function $F_t = \int_0^t f_u^2 \mathrm{d}u$ can be fixed, as we did in Theorem 3.3, by imposing

$$1 = \dot{F}_t b_1^2(t) b_2^2(t) b_3^2(t) b_4^2(t) = \dot{F}_t \left(-e^{-9F_t} + 3e^{-F_t}\right)^2 \left(e^{-9F_t} + e^{-F_t}\right)^6 \frac{1}{32}.$$

This guarantees that χ_t actually solves (7) (note also that A is symmetric for any time).

Notice that the previous equation, after integration, ensures that, since $F_t \geq 0$ if and only if $t \geq 0$, the solution extinguishes in finite time. With an analogous argument, we see that F_t cannot exist for any negative time. To be more precise, let I be the maximal interval of existence of F, then

$$32t = \int_0^{F_t} (-e^{-9x} + 3e^{-x})^2 (e^{-9x} + e^{-x})^6 \mathrm{d}x, \quad t \in I.$$

We immediately see that $\sup_I < +\infty$. On the other hand, if $\inf_I = -\infty$ then F_t should be unbounded near $-\infty$: indeed when $M < F_t < 0$ it turns out that

$$32t = \int_0^{F_t} (-e^{-9x} + 3e^{-x})^2 (e^{-9x} + e^{-x})^6 \mathrm{d}x > \int_0^{M} (-e^{-9x} + 3e^{-x})^2 (e^{-9x} + e^{-x})^6 \mathrm{d}x.$$

Therefore, it would exist a sufficiently large negative time t such that $0 = -e^{-9F_t} + 3e^{-F_t} = 2b_1(t)$. Clearly, this cannot happen because χ_t must be a stable and negative form. By these considerations, we also deduce that the only negative singular time τ for the monotone function F_t satisfies $b_1(\tau) = 0$, that is $F_\tau = -1/8\ln(3)$.

We will now construct an explicit example of soliton on a nilpotent almost-abelian Lie group.

Example 4.7 Let $\mathfrak{g}$ be the nilpotent almost-abelian Lie algebra with structure equations

$$\begin{aligned} de^1 &= e^{27}, \\ de^j &= 0, \quad j = 2, 4, 6, 7, \\ de^3 &= e^{47}, \\ de^5 &= e^{67}. \end{aligned}$$

Then in this case, we have $\mathfrak{h} = \mathbb{R} < e_1, \ldots, e_6 >$ and

$$A = ad_{e_7}|_{\mathfrak{h}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the G_2-structure $\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$. The 4-form

$$\phi_0 = \star_{\varphi_0} \varphi_0 = e^{1234} + e^{3456} + e^{1256} - e^{2467} + e^{1367} + e^{1457} - e^{2357}$$

is closed, and thus, φ defines a coclosed G_2-structure. The basis $(e_1, \ldots, e_7)$ is orthonormal with respect to g_{φ_0}, and one can check that A is not normal. We will apply Theorem 4.2 to show that ϕ_0 is a soliton for the Laplacian coflow. First observe that $S(0)$ and $L(0)$, on $\mathfrak{h}$, restrict to

$$\begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \end{pmatrix}.$$

So $S(0)$ is in normal form (5), and therefore, the matrix $\Sigma(0)$, restricted to $\mathfrak{h}$, turns out to be

$$\begin{pmatrix} -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

A direct computation then shows that $[S(0), L(0)] = \Sigma(0)$ on $\mathfrak{h}$, which leads to $[-\Sigma(0) + [S(0), L(0)], A] = 0$, so A solves the soliton equation for $\delta = 0$. In particular, we have $s_1(0) = s_2(0) = s_3(0) = 1/2$ and $l(0) = l_1(0) + l_2(0) + l_3(0) = 3/2$. Thus,

$$s^2(0) = 3/4, \quad l^2(0) = 9/4$$

and $c = 3$. Then, the associated derivation D is given by $D = X - 3\,\mathrm{Id}$, with

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

and the existence interval is $(-\infty, 1/6)$. Note that ϕ_0 is not an eigenform of the Laplacian since

$$\vartheta(X)\phi_0 = -3(-e^{2467} + e^{1367} + e^{1457} - e^{2357}).$$

Acknowledgements The authors are supported by the project PRIN " varietà reali e complesse: geometria, topologia e analisi armonica" and by GNSAGA. of I.N.d.A.M. The authors also thank the anonymous referees for all the useful improvements they suggested.

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