Annali di Matematica (2018) 197:1533–1545
https://doi.org/10.1007/s10231-018-0736-x

Continuity theorems for a class of convolution operators and applications to superoscillations

T. Aoki[1] · F. Colombo[2] · I. Sabadini[2] · D. C. Struppa[3]

Received: 7 December 2017 / Accepted: 2 March 2018 / Published online: 12 March 2018

Abstract We prove a new theorem on the continuity of convolution operators with variable coefficients, and we use it to deduce that the limit of a superoscillating sequence maintains the superoscillatory behaviour for all values of time, when evolved according to Schrödinger equations with time-dependent potentials.

Keywords Superoscillating functions · Convolution operators with variable coefficients · Holomorphic functions with growth conditions · Time-dependent potential

Mathematics Subject Classification 32A15 · 32A10 · 47B38

1 Introduction

In [1,7,8], Aharonov and his coauthors introduced a new concept in quantum mechanics, namely the notion of weak measurement of a quantum observable (see also [19,20,22,23]). This notion led to the discovery of an interesting family of functions, which had been observed as well in optical phenomena [12–17], and [24]. These functions, known as the Aharonov–Berry superoscillations, are band-limited functions with the apparently paradoxical property that they can oscillate faster than their fastest Fourier component.

An important question that was posed originally by both Aharonov and Berry is whether such superoscillatory behaviour can persist when we evolve a superoscillatory function according to some differential equations. In a series of papers [2–5,9,18] and the monograph [6], the authors have developed a powerful method to study the evolution of superoscillations propagated by the Schrödinger equation. This method employs essentially two steps: First

✉ I. Sabadini
irene.sabadini@polimi.it

1 Department of Mathematics, Kindai University, Higashi-Osaka 577-8502, Japan

2 Dipartimento di Matematica, Politecnico di Milano, Via E. Bonardi, 9, 20133 Milano, Italy

3 Schmid College of Science and Technology, Chapman University, Orange, CA 92866, USA

one uses Fourier analysis, or the relevant Green function, to solve the Cauchy problem associated with the Schrödinger equation, and then, one translates the problem in the complex plane (essentially by complexifying both the functions and the operators acting on them) and demonstrates the permanence of the superoscillatory behaviour as a consequence of a continuity theorem for suitable convolution operators. The key ingredient for the continuity theorem is the understanding of the growth of the associated symbol (usually an entire function).

In this paper, we consider a very large, and so far not yet considered, class of potentials for the Schrödinger equation, and we prove how a new theorem on the growth of analytic functions (which we prove in Sect. 2) can be used to demonstrate superoscillatory longevity in time for those equations.

The novelty of the present paper consists in the fact that we will now study convolution operators with non-constant coefficient of the form:

$$\mathcal{U}\left(t, \frac{\partial}{\partial x}\right) := \sum_{m=0}^{\infty} b_m(t, x) \frac{\partial^m}{\partial x^m}.$$

This situation will require a new continuity theorem, which we will prove in Sect. 2. This result will be applied in Sect. 3 to the study of the evolution of superoscillations by Schrödinger equations in which variable coefficients potential appear.

The first author would like to express his sincere thanks to Professor Susumu Yamasaki for valuable comments on the topology of the spaces of entire functions.

2 Theorem on continuity of a class of convolution operators

Let f be a non-constant entire function of a complex variable z. We define

$$M_f(r) = \max_{|z|=r} |f(z)|, \quad \text{for} \quad r \geq 0.$$

The non-negative real number ρ defined by

$$\rho = \limsup_{r\to\infty} \frac{\ln \ln M_f(r)}{\ln r}$$

is called the order of f. If ρ is finite, then f is said to be of finite order, and if $\rho = \infty$, the function f is said to be of infinite order.

In the case f is of finite order ρ, we define the non-negative real number

$$\sigma = \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho},$$

which is called the type of f. If $\sigma \in (0, \infty)$, we call f of normal type, while we say that f is of minimal type if $\sigma = 0$ and of maximal type if $\sigma = \infty$. The constant functions are said to be of minimal type and order zero. In the sequel, we make use of the notions given in the next definitions. These notions are classical, see e.g. [11], and go back to Hörmander see [21]:

Definition 2.1 Let p be a positive number. We define the class A_p to be the set of entire functions such that there exists $C > 0$ and $B > 0$ for which

$$|f(z)| \leq C \exp(B|z|^p), \qquad \forall z \in \mathbb{C}.$$

The class $A_{p,0}$ consists of those entire functions such that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(z)| \leq C_\varepsilon \exp(\varepsilon |z|^p), \quad \forall z \in \mathbb{C}.$$

To define a topology in these spaces, we follow [11, Section 2.1]: For $p > 0$, $c > 0$ and for any entire function f, we set

$$\|f\|_c := \sup_{z \in \mathbb{C}} \{|f(z)| \exp(-c|z|^p)\}.$$

Let A_p^c denote the linear space of entire functions satisfying $\|f\|_c < \infty$. Then, $\|\cdot\|$ defines a norm in A_p^c which makes this space a Banach space. The natural inclusion mapping $A_p^c \hookrightarrow A_p^{c'}$ is a compact operator for any $0 < c < c'$.
For any sequence $\{c_n\}_{n \geq 1}$ of positive numbers, strictly increasing to infinity, we can introduce an LF topology on A_p given by the inductive limit

$$A_p := \varinjlim A_p^{c_n}.$$

Since this topology is stronger than the topology of the pointwise convergence, it is independent of the choice of the sequence $\{c_n\}_{n \geq 1}$. In this inductive limit topology, a sequence $\{f_k\}$ in A_p converges to f in A_p if and only if there exists n such that $f_j \in A_p^{c_n}$ for all j, $f \in A_p^{c_n}$ and $\|f_j - f\|_{c_n} \to 0$ for $j \to \infty$. The topology in $A_{p,0}$ is given by the projective limit

$$A_{p,0} := \varprojlim A_p^{c_n}.$$

It can be proved, see [11, Section 6.1], that A_p is a DFS space and $A_{p,0}$ is an FS space, respectively.
To prove our main results, we need an important lemma that characterizes the coefficients of entire functions with growth conditions.

Lemma 2.2 *The function*

$$f(z) = \sum_{j=0}^{\infty} f_j z^j$$

belongs to A_p if and only if there exists $C_f > 0$ and $b > 0$ such that

$$|f_j| \leq C_f \frac{b^j}{\Gamma\left(\frac{j}{p} + 1\right)}.$$

Proof First suppose that $f(z) \in A_p$ and let us prove that the estimate on the coefficients f_j follows by the Cauchy formula. In fact, we have

$$f^{(j)}(z) = \frac{j!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{j+1}} \, \mathrm{d}w,$$

where the path of integration γ is the circle $|w - z| = s|z|$, where s is a positive real number and $z \neq 0$. Then, we have

$$\begin{aligned} |f^{(j)}(z)| &\leq \frac{j!}{(s|z|)^j} \max_{|w-z|=s|z|} |f(w)| \\ &\leq \frac{C_f \, j!}{(s|z|)^j} \exp(B(1+s)^p |z|^p) \end{aligned}$$

for all $s > 0$, where we have used the fact that $f \in A_p$ and $|w| \leq (1+s)|z|$. The well-known estimate

$$(a+b)^p \leq 2^p(a^p + b^p), \quad a > 0, \quad b > 0, \quad p > 0$$

gives

$$(1+s)^p \leq 2^p(s^p + 1)$$

for all $s > 0$. Hence we have

$$|f^{(j)}(z)| \leq C_f \frac{j!}{(s|z|)^j} \exp(B \cdot 2^p s^p |z|^p) \exp(B \cdot 2^p |z|^p)$$

for all $z \in \mathbb{C}$ and $s > 0$. We now take the minimum of the right-hand side of the above estimate with respect to s, i.e. the minimum of the function

$$g(s) := \frac{1}{(s|z|)^j} \exp(B \cdot 2^p s^p |z|^p)$$

in $(0, \infty)$. The minimum is at the point

$$s_{\min} = \left(\frac{j}{2^p Bp}\right)^{1/p} \frac{1}{|z|}$$

so that we obtain

$$|f^{(j)}(z)| \leq C_f j! \left(\frac{2^p Bp}{j}\right)^{j/p} e^{j/p} \exp(A 2^p |z|^p).$$

So if we set

$$b := (2^p Bpe)^{1/p},$$

we obtain

$$|f^{(j)}(z)| \leq C_f j! \frac{b^j}{j^{j/p}} \exp(B \cdot 2^p |z|^p)$$

for all $z \in \mathbb{C}$. Since $f_j = \frac{f^{(j)}(0)}{j!}$, we have, by the maximum modules principle applied in a disc centred at the origin and with radius $\epsilon > 0$ sufficiently small,

$$\begin{aligned}
|f_j| &\leq C_f \frac{b^j}{j^{j/p}} \exp(B \cdot 2^p \epsilon^p) \\
&\leq 2C_f \frac{b^j}{j^{j/p}} \\
&= C'_f \frac{b^j}{(j!)^{1/p}} \\
&\leq C'_f \frac{b^j}{\Gamma(\frac{j}{p} + 1)}.
\end{aligned}$$

The other direction follows form the properties of the Mittag–Leffler function. In fact,

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

is an entire function of order $1/\alpha$ (and of type 1) for $\alpha > 0$ and $Re(\beta) > 0$, see [10]. So, in our case, f is entire of order p. $\square$

In order to prove our main results, we need some more notations and definitions:

Definition 2.3 Let $p > 0$. The set $\mathcal{D}_{p,0}$ consists of operators of the form

$$P(z, \partial_z) = \sum_{n=0}^{\infty} a_n(z)\partial_z^n$$

satisfying:

(i) $a_n(z)$ $(n = 0, 1, 2, \dots)$ are entire functions.
(ii) There exists a constant $B > 0$ such that for every $\varepsilon > 0$ one can take a constant $C_\varepsilon > 0$ for which

$$|a_n(z)| \leq C_\varepsilon \frac{\varepsilon^n}{(n!)^{1/q}} \exp(B|z|^p),$$

holds, where

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and $1/q = 0$ when $p = 1$.

Theorem 2.4 *Let $P(z, \partial_z) \in \mathcal{D}_{p,0}$ and let $f \in A_p$. Then, $P(z, \partial_z)f \in A_p$ and $P(z, \partial_z)$ is continuous on A_p, that is $P(z, \partial_z)f \to 0$ as $f \to 0$.*

Proof We apply the operator $P(z, \partial_z)$ to the function $f \in A_p$ and we get

$$\begin{aligned} P(z, \partial_z)f(z) &= \sum_{n=0}^{\infty} a_n(z)\partial_z^n \sum_{j=0}^{\infty} f_j z^j \\ &= \sum_{n=0}^{\infty} a_n(z) \sum_{j=n}^{\infty} f_j \frac{j!}{(j-n)!} z^{j-n} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n(z) f_{n+k} \frac{(k+n)!}{k!} z^k. \end{aligned}$$

Now we observe that

$$|P(z, \partial_z)f(z)| \leq \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |a_n(z)|\, |f_{n+k}| \frac{(k+n)!}{k!} |z|^k.$$

Using the hypothesis on P and f, which translate into conditions on the a_n and on f_j, we get

$$|P(z, \partial_z)f(z)| \leq C_f C_\varepsilon \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\varepsilon^n}{(n!)^{1/q}} \exp(B|z|^p)\ \frac{b^{n+k}}{\Gamma\left(\frac{n+k}{p}+1\right)} \frac{(k+n)!}{k!} |z|^k.$$

Since

$$(n!)^{1/q} \geq \Gamma\left(\frac{n}{q}+1\right) \quad \text{and} \quad (k+n)! \leq 2^{k+n} k! n!$$

we get

$$
\begin{aligned}
|P(z,\partial_z)f(z)| &\le C_f C_\varepsilon \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\varepsilon^n}{\Gamma\left(\frac{n}{q}+1\right)} \frac{b^{n+k}}{\Gamma\left(\frac{n+k}{p}+1\right)} \frac{2^{k+n}k!n!}{k!}|z|^k \exp(B|z|^p) \\
&\le C_f C_\varepsilon \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\varepsilon^n}{\Gamma\left(\frac{n}{q}+1\right)} \frac{b^{n+k}}{\Gamma\left(\frac{n+k}{p}+1\right)} 2^{k+n}n!|z|^k \exp(B|z|^p) \\
&\le C_f C_\varepsilon \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (2b)^k(2\varepsilon b)^n \frac{1}{\Gamma\left(\frac{n}{q}+1\right)} \frac{n!}{\Gamma\left(\frac{n+k}{p}+1\right)}|z|^k \exp(B|z|^p).
\end{aligned}
\tag{1}
$$

From the inequality

$$
\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \le 1,
$$

we deduce

$$
\Gamma\left(\frac{k+n}{p}+1\right) \ge \Gamma\left(\frac{k}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)
$$

and so we can write (1) as

$$
\begin{aligned}
|P(z,\partial_z)f(z)| \le C_f C_\varepsilon \sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p}+\frac{1}{2}\right)}|z|^k \exp(B|z|^p) \\
\times \sum_{n=0}^{\infty}(2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)}.
\end{aligned}
$$

Finally, consider the series

$$
\sum_{n=0}^{\infty}(2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)}
$$

and observe that

$$
(2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)} \sim \frac{n^n(2\varepsilon b)^n}{\left(\frac{n}{p}\right)^{n/p}\left(\frac{n}{q}\right)^{n/q}},
$$

and

$$
\frac{n^n(2\varepsilon b)^n}{\left(\frac{n}{p}\right)^{n/p}\left(\frac{n}{q}\right)^{n/q}} = \frac{n^n(2\varepsilon b)^n[p^{1/p}q^{1/q}]^n}{n^n} = (2\varepsilon b)^n[p^{1/p}q^{1/q}]^n.
$$

Since ε is arbitrary small, the series converges, i.e.

$$
\sum_{n=0}^{\infty}(2\varepsilon b)^n \frac{n!}{\Gamma\left(\frac{n}{p}+\frac{1}{2}\right)\Gamma\left(\frac{n}{q}+1\right)} = C'.
$$

This finally gives

$$|P(z,\partial_z)f(z)| \le C'C_f C_\varepsilon \sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p}+\frac{1}{2}\right)} |z|^k \exp(B|z|^p)$$

and, by the properties of the Mittag–Leffler function, we have

$$\sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\left(\frac{k}{p}+\frac{1}{2}\right)} |z|^k \le C' \exp(B'|z|^p).$$

We conclude that there exists $B'' > 0$ such that

$$|P(z,\partial_z)f(z)| \le C'C_f C_\varepsilon \exp(B''|z|^p)$$

which means that $P(z,\partial_z)f(z) \in A_p$ and for $C_f \to 0$ the same estimate proves the continuity, i.e. $|P(z,\partial_z)f(z)| \to 0$ when $f \to 0$. □

We conclude this section with a result which is the integral counterpart of the previous theorem and is of independent interest.

We define the action of the operator denoted by ∂_z^{-n} $(n = 1, 2, 3, \dots)$ on the space of entire functions by the Riemann–Liouville integral

$$\partial_z^{-n} f(z) = \frac{1}{(n-1)!} \int_0^z (z-t)^{n-1} f(t)\mathrm{d}t.$$

Definition 2.5 Let $\mathcal{E}_p$ denote the set of all formal power series

$$P(z,\partial_z^{-1}) = \sum_{n=0}^{\infty} a_n(z)\partial_z^{-n}$$

of ∂_z^{-1} satisfying

(i) $a_n(z)$ $(n = 0, 1, 2, \dots)$ are entire functions.
(ii) There exist constants $B > 0$ and $C > 0$ for which

$$|a_n(z)| \le C^{n+1} n!^{\frac{1}{q}} \exp(B|z|^p)$$

holds for $n = 0, 1, 2, \dots$ and where

$$\frac{1}{p} + \frac{1}{q} = 1,$$

with $1/q = 0$ when $p = 1$.

Theorem 2.6 *Let* $P(z,\partial_z^{-1}) = \sum_{n=0}^{\infty} a_n(z)\partial_z^{-n} \in \mathcal{E}_p$, *and define the action of* $P(z,\partial_z^{-1})$ *on* A_p *by*

$$P(z,\partial_z^{-1})f(z) = a_0(z)f(z) + \sum_{n=1}^{\infty} a_n(z)\partial_z^{-n} f(z).$$

This action is well defined, that is, if $P(z,\partial_z^{-1}) \in E_p$ *and* $f \in A_p$, *then* $P(z,\partial_z^{-1})f \in A_p$ *and continuous.*

Proof Suppose that

$$|f(z)| \leq C_1 \exp(B_1|z|^p)$$

holds for $C_1 > 0,\ B_1 > 0$. We rewrite the Riemann–Liouville integral in the form

$$\partial_z^{-n} f(z) = \frac{z^n}{(n-1)!} \int_0^1 (1-s)^{n-1} f(zs)\mathrm{d}s.$$

Using this expression, we have

$$\begin{aligned} |\partial_z^{-n} f(z)| &\leq \frac{|z|^n}{(n-1)!} \int_0^1 (1-s)^{n-1} C_1 \exp(B_1|zs|^p)\mathrm{d}s \\ &\leq C_1 \frac{|z|^n}{n!} \exp(B_1|z|^p). \end{aligned}$$

By the condition (b), we have

$$\begin{aligned} |a_n(z)\partial_z^{-n} f(z)| &\leq C_1 C^{n+1} |z|^n \frac{n!^{\frac{1}{q}}}{n!} \exp((B+B_1)|z|^p) \\ &\leq C_1 C^{n+1} \frac{|z|^n}{n!^{\frac{1}{p}}} \exp(B_2|z|^p) \end{aligned}$$

for $n \in \mathbb{N}$. Here we set $B_2 = B + B_1$. Thus, there exist $C_2 > 0$ and $B_3 > 0$ for which

$$\sum_{n=1}^{\infty} |a_n(z)\partial_z^{-n} f(z)| \leq C_2 \exp(B_3|z|^p)$$

hold for all $z \in \mathbb{C}$. This implies $Pf \in A_p$. The continuity of P can be proved as in the proof of the previous theorem. □

3 Evolution of superoscillations for a class of potentials

The prototypical superoscillating sequence is

$$F_n(x,a) := \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n, \tag{2}$$

where x is a real variable and $a > 1$ is a parameter. By using Euler's identity for the exponential, and the Newton binomial formula, it is immediate to show that

$$F_n(x,a) = \sum_{j=0}^{n} C_j(n,a) e^{ix(1-\frac{2j}{n})}, \tag{3}$$

where

$$C_j(n,a) = \binom{n}{j} \left(\frac{1+a}{2}\right)^{n-j} \left(\frac{1-a}{2}\right)^j. \tag{4}$$

The reason for the term *superoscillations* is easily understood if one considers that all the frequencies that appear in (3) are in modulus less than one, but that the sequence $F_n(x,a)$ itself converges uniformly (on compact subsets of $\mathbb{R}$) to e^{iax}. In our works, see e.g. [3,4,6], we considered some Cauchy problems in which the datum when the time t equals 0 is $F_n(x,a)$, and we asked whether the solution $\psi_n(x,t)$ to the problem maintained superoscillatory characteristics. In order to show that superoscillations perpetually persist in time, i.e. when the time tends to infinity, we need to explicitly compute the limit

$$\lim_{n\to\infty} \psi_n(x,t).$$

We now consider a potential $V(t,x)$ and the Cauchy problem for the Schrödinger equation

$$i\frac{\partial\psi(t,x)}{\partial t} = \Big(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(t,x)\Big)\psi(t,x), \quad \psi(0,x) = F_n(x,a). \tag{5}$$

According to the type of potential, in some cases it is possible to determine explicitly the Green function $G_V(t,x,y)$, but in most of the cases this is not possible. The solution of Cauchy problem (5) is given by

$$\psi(t,x) = \int_{\mathbb{R}} G_V(t,x,y)\psi(0,y)\mathrm{d}y. \tag{6}$$

Since the superoscillatory functions such as $F_n(x,a)$ are linear combinations of exponential functions, we determine the solution of the Cauchy problem

$$i\frac{\partial\varphi(t,x)}{\partial t} = \Big(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(t,x)\Big)\varphi(t,x), \quad \varphi(0,x) = e^{iax}, \tag{7}$$

that is

$$\varphi_a(t,x) = \int_{\mathbb{R}} G_V(t,x,y)e^{iay}\mathrm{d}y. \tag{8}$$

The solution of the Cauchy problem is obtained by linearity

$$\psi_n(t,x) = \sum_{k=0}^{n} C_k(n,a)\varphi_{(1-2k/n)}(t,x). \tag{9}$$

We will consider the following classes of potentials such that a very general structure of the Green function is of the form

$$G_V(t,x,y) = e^{ig_1(t)x^2+2ig_2(t)xy+ig_3(t)y^2} f(x,y,t) \tag{10}$$

where $g_j, g : \mathbb{R} \to \mathbb{R}$ are given functions of the time and f is an analytic function; all those functions depend on the potential V.

Definition 3.1 Let $V = V(t,x)$ be the potential and let G_V be the Green function of the Schrödinger equation associated with V and let $a \in \mathbb{R}$.

(T1) We say that G_V is of type (I) if

$$\int_{\mathbb{R}} G_V(t,x,y)e^{iay}\mathrm{d}y = h_1(t,x)\sum_{\ell=0}^{\infty} b_\ell(t,x)a^\ell e^{iat} \tag{11}$$

where h_1 is a given function and the coefficients $b_\ell(t,x)$ extend analytically to an entire function $b_\ell(z,x)$ that satisfy the growth condition: There exists $A > 0$, and for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$, for all $x \in \mathbb{R}$, such that

$$|b_\ell(z,x)| \le C_\varepsilon \frac{A^\ell}{(\ell!)^{1/q}} \exp(\varepsilon|z|^p). \tag{12}$$

(T2) We say that G_V is of type (II) if

$$\int_{\mathbb{R}} G_V(t,x,y)e^{iay}\mathrm{d}y = h_2(t,x)\sum_{\ell=0}^{\infty} c_\ell(t,x)a^\ell e^{iax} \tag{13}$$

where h_2 is a given function and the coefficients $c_\ell(t,x)$ extend analytically to an entire function that satisfy the growth condition: There exists $A>0$, and for every $\varepsilon>0$, there exists $C_\varepsilon>0$, for all $t\in\mathbb{R}$, such that

$$|c_\ell(t,z)| \le C_\varepsilon \frac{A^\ell}{(\ell!)^{1/q}} \exp(\varepsilon|z|^p). \tag{14}$$

The following result is now immediate to prove.

Lemma 3.2 *Let $a\in\mathbb{R}$.*

(I) Let G_V be a Green function of type (I). Then, the solution of Cauchy problem (7) can be represented by

$$\varphi_a(t,x) = h_1(t,x)\sum_{\ell=0}^{\infty} b_\ell(t,x)i^{-\ell}\partial_t^\ell e^{iat}. \tag{15}$$

(II) Let G_V be a Green function of type (II). Then, the solution of Cauchy problem (7) can be represented by

$$\varphi_a(t,x) = h_2(t,x)\sum_{\ell=0}^{\infty} c_\ell(t,x)i^{-\ell}\partial_x^\ell e^{iax}. \tag{16}$$

Inspired by the above lemma, we define the operators

$$\mathcal{U}_1(t,x,\partial_t) = h_1(t,x)\sum_{\ell=0}^{\infty} b_\ell(t,x)i^{-\ell}\partial_t^\ell, \tag{17}$$

and

$$\mathcal{U}_2(t,x,\partial_x) = h_2(t,x)\sum_{\ell=0}^{\infty} c_\ell(t,x)i^{-\ell}\partial_x^\ell. \tag{18}$$

So we can prove the main result.

Theorem 3.3 *Under the hypothesis of the previous lemma, the solution can be written as*

$$\psi_n(t,x) = \mathcal{U}_1(t,x,\partial_x)F_n(x,a), \tag{19}$$

or

$$\phi_n(t,x) = \mathcal{U}_2(t,x,\partial_t)F_n(t,a). \tag{20}$$

Moreover, we have

$$\lim_{n\to\infty}\psi_n(t,x) = \varphi_a(t,x), \tag{21}$$

or

$$\lim_{n\to\infty}\phi_n(t,x) = \varphi_a(t,x). \tag{22}$$

Proof We consider just one case, since the second is analogous

$$\begin{aligned}\lim_{n\to\infty}\psi_n(t,x) &= \lim_{n\to\infty}\mathcal{U}_1(t,x,\partial_t)F_n(x,a)\\ &= \mathcal{U}_1(t,x,\partial_t)\lim_{n\to\infty}F_n(x,a) = \varphi_a(t,x).\end{aligned}$$

□

We conclude this article by showing how non-constant coefficients differential operator may naturally appear in the study of these problems. Consider the Cauchy problem

$$\begin{cases} i\dfrac{\partial\phi(t,x)}{\partial t} = -a(t)\dfrac{\partial^2\phi(t,x)}{\partial x^2} + b(t)x^2\phi(t,x) - i\Big(c(t)x\dfrac{\partial\phi(t,x)}{\partial x} + d(t)\phi(t,x)\Big), \\ \phi(0,x) = e^{iax}. \end{cases} \tag{23}$$

The solution of this problem is given by

$$\phi_a(t,x) = \frac{1}{\sqrt{2\pi i\mu(t)}}\int_{\mathbb{R}} e^{i(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2)}e^{iay}\mathrm{d}y$$

and also

$$\phi_a(t,x) = \frac{1}{\sqrt{2\pi i\mu(t)}}e^{i\alpha(t)x^2}\int_{\mathbb{R}} e^{i(\gamma(t)y^2+(a+\beta(t)x)y)}\mathrm{d}y.$$

Using the integral

$$\int_{\mathbb{R}} e^{i(\delta y^2+2\xi y)}\,dy = \left(\frac{i\pi}{\delta}\right)^{1/2} e^{-i\xi^2/\delta}, \quad Im(\delta)\geq 0,$$

we obtain

$$\begin{aligned} \phi_a(t,x) &= \frac{1}{\sqrt{2\pi i\mu(t)}}e^{i\alpha(t)x^2}\left(\frac{i\pi}{\gamma(t)}\right)^{1/2} e^{-i(a+\beta(t)x)^2/(4\gamma(t))} \\ \phi_a(t,x) &= \frac{1}{\sqrt{2\mu(t)\gamma(t)}}e^{i[\alpha(t)+\beta^2(t)/(4\gamma(t))]x^2}\,e^{-ia^2/(4\gamma(t))}\,e^{-iax\beta(t)/(4\gamma(t))}. \end{aligned}$$

Consider the term

$$\mathcal{G}(t,x) := e^{-ia^2/(4\gamma(t))}\,e^{-iax\beta(t)/(4\gamma(t))};$$

with a change of variables, we have

$$\beta(t)/(4\gamma(t)) := z.$$

If we suppose that this change of variable is invertible, we can write

$$t = K(z).$$

Replacing this expression in $\frac{1}{4\gamma(t)}$, we can make the substitution

$$\eta(z) = \frac{1}{4\gamma(K(z))},$$

obtaining

$$G(z,x) := e^{-ia^2\eta(z)}\,e^{-iaxz} = \sum_{m\geq 0}\frac{(-i\eta(z))^m}{m!}\,a^{2m}\,e^{-iaxz}.$$

It is clear that a non-constant coefficient operator appears, in fact

$$a^{2m}\,e^{-iaxz} = (-ix)^{-2m}\partial_z^{2m}e^{-iaxz}$$

and so we have

$$G(z,x) := e^{-ia^2\eta(z)}\,e^{-iaxz} = \sum_{m\geq 0}\frac{(-i\eta(z))^m}{m!}\,(-ix)^{-2m}\partial_z^{2m}\,e^{-iaxz}$$

and the operator

$$\mathcal{G}(z, x, \partial_z) := \sum_{m\geq 0} (-ix)^{-2m} \frac{(-i\eta(z))^m}{m!} \partial_z^{2m}.$$

To summarize, the solution of the Cauchy problem

$$i\frac{\partial \psi(t,x)}{\partial t} = -a(t)\frac{\partial^2 \psi(t,x)}{\partial x^2} + b(t)x^2\psi(t,x) \\ -i\Big(c(t)x\frac{\partial \psi(t,x)}{\partial x} + d(t)\psi(t,x)\Big), \qquad \psi(0,x) = F_n(x,a). \tag{24}$$

can be written as

$$\psi_n(t,x) = \mathcal{G}(z,x,\partial_z)F_n(z,a),$$

where $\mathcal{G}(z, x, \partial_z)$ is a non-constant coefficients differential operator.

References

1. Aharonov, Y., Albert, D., Vaidman, L.: How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100. Phys. Rev. Lett. **60**, 1351–1354 (1988)
2. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Some mathematical properties of superoscillations. J. Phys. A **44**, 365304 (2011). (16pp)
3. Aharonov, Y., Colombo, F., Nussinov, S., Sabadini, I., Struppa, D.C., Tollaksen, J.: Superoscillation phenomena in $SO(3)$. Proc. R. Soc. A. **468**, 3587–3600 (2012)
4. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: On the Cauchy problem for the Schrödinger equation with superoscillatory initial data. J. Math. Pures Appl. **99**, 165–173 (2013)
5. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Superoscillating sequences as solutions of generalized Schrödinger equations. J. Math. Pures Appl. **103**, 522–534 (2015)
6. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: The mathematics of superoscillations, Memoirs of the American Mathematical Society, vol. 247, no. 1174 (2017)
7. Aharonov, Y., Rohrlich, D.: Quantum Paradoxes: Quantum Theory for the Perplexed. Wiley, Weinheim (2005)
8. Aharonov, Y., Vaidman, L.: Properties of a quantum system during the time interval between two measurements. Phys. Rev. A **41**, 11–20 (1990)
9. Aoki, T., Colombo, F., Sabadini, I., Struppa, D.C.: Continuity of some operators arising in the theory of superoscillations. Quantum Stud.: Math. Found. (2018). https://doi.org/10.1007/s40509-018-0159-9
10. Bateman, H.: Bateman Manuscript Project, Higher Transcendental Functions, vol. 3. McGraw-Hill Book Company Inc, Pennsylvania (1955)
11. Berenstein, C.A., Gay, R.: Complex Analysis and Special Topics in Harmonic Analysis. Springer, New York (1995)
12. Berry, M.V.: Evanescent and real waves in quantum billiards and Gaussian beams. J. Phys. A. **27**, 391 (1994)
13. Berry, M.: Exact nonparaxial transmission of subwavelength detail using superoscillations. J. Phys. A. **46**, 205203 (2013)
14. Berry, M.V.: Faster than Fourier, 1994. In: Anandan, J.S., Safko, J.L. (eds.) Quantum Coherence and Reality; in Celebration of the 60th Birthday of Yakir Aharonov, pp. 55–65. World Scientific, Singapore (1994)
15. Berry, M., Dennis, M.R.: Natural superoscillations in monochromatic waves in D dimension. J. Phys. A **42**, 022003 (2009)
16. Berry, M.V., Popescu, S.: Evolution of quantum superoscillations, and optical superresolution without evanescent waves. J. Phys. A **39**, 6965–6977 (2006)
17. Berry, M.V., Shukla, P.: Pointer supershifts and superoscillations in weak measurements. J. Phys. A **45**, 015301 (2012)
18. Buniy, R., Colombo, F., Sabadini, I., Struppa, D.C.: Quantum Harmonic Oscillator with superoscillating initial datum. J. Math. Phys. **55**, 113511 (2014)

19. Ferreira, P.J.S.G., Kempf, A.: Unusual properties of superoscillating particles. J. Phys. A **37**, 12067–76 (2004)
20. Ferreira, P.J.S.G., Kempf, A.: Superoscillations: faster than the Nyquist rate. IEEE Trans. Signal. Process. **54**, 3732–40 (2006)
21. Hörmander, L.: Generators for some rings of analytic functions. Bull. Am. Math. Soc. **73**, 943–949 (1967)
22. Lee, D.G., Ferreira, P.J.S.G.: Superoscillations of prescribed amplitude and derivative. IEEE Trans. Signal Process. **62**(13), 3371–3378 (2014)
23. Lee, D.G., Ferreira, P.J.S.G.: Superoscillations with optimal numerical stability. IEEE Sign. Proc. Lett. **21**(12), 1443–1447 (2014)
24. Lindberg, J.: Mathematical concepts of optical superresolution. J. Opt. **14**, 083001 (2012)
25. Taylor, B.A.: Some locally convex spaces of entire functions, 1968 entire functions and related parts of analysis. In: Proceedings of Symposia in Pure Mathematics, La Jolla, California, 1966) pp. 431–467. Amererican Mathematical Society, Providence, RI (1966)