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A $\mathbb{Q}$-factorial complete toric variety is a quotient of a poly weighted space

Michele Rossi[1] · Lea Terracini[1]

Abstract We prove that every $\mathbb{Q}$-factorial complete toric variety is a finite abelian quotient of a poly weighted space (PWS), as defined in our previous work (Rossi and Terracini in Linear Algebra Appl 495:256–288, 2016. doi:10.1016/j.laa.2016.01.039). This generalizes the Batyrev–Cox and Conrads description of a $\mathbb{Q}$-factorial complete toric variety of Picard number 1, as a finite quotient of a weighted projective space (WPS) (Duke Math J 75:293–338, 1994, Lemma 2.11) and (Manuscr Math 107:215–227, 2002, Prop. 4.7), to every possible Picard number, by replacing the covering WPS with a PWS. By Buczyńska's results (2008), we get a universal picture of coverings in codimension 1 for every $\mathbb{Q}$-factorial complete toric variety, as topological counterpart of the $\mathbb{Z}$-linear universal property of the double Gale dual of a fan matrix. As a consequence, we describe the bases of the subgroup of Cartier divisors inside the free group of Weil divisors and the bases of the Picard subgroup inside the class group, respectively, generalizing to every $\mathbb{Q}$-factorial complete toric variety the description given in Rossi and Terracini (2016, Thm. 2.9) for a PWS.

Keywords $\mathbb{Q}$-factorial complete toric varieties · Connectedness in codimension 1 · Gale duality · Weighted projective spaces · Hermite normal form · Smith normal form

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Contents

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Introduction

The present paper is the second part of a long-standing tripartite study aimed at realizing, for $\mathbb{Q}$-factorial projective toric varieties, a classification inspired by what V. Batyrev did in 1991 for smooth complete toric varieties [1]. The first part of this study is given by [10], in which we studied Gale duality from the $\mathbb{Z}$-linear point of view and defined poly weighted spaces (PWS, for short: see the following Definition 1.5). The reader will often be referred to this work for notation, preliminaries and results.

In this paper, as a first application of $\mathbb{Z}$-linear Gale duality, we exhibit

(i) every $\mathbb{Q}$-factorial complete toric variety as a finite abelian quotient of a PWS (Theorem 2.2),

(ii) a universal covering in codimension 1 theorem for every $\mathbb{Q}$-factorial complete toric variety (Corollary 2.5),

(iii) explicit bases of the subgroup of Cartier divisors inside the free group of Weil divisors and of the Picard subgroup inside the class group, for every $\mathbb{Q}$-factorial complete toric variety (Theorem 3.2).

The first result (i) generalizes, to every $\mathbb{Q}$-factorial complete toric variety, results [2, Lemma 2.11] and [5, Prop. 4.7] by V. Batyrev–D. Cox and H. Conrads, respectively, in terms of PWS. Roughly speaking, by adopting the usual terminology for toric varieties of Picard number 1, called *fake WPS* as suitable finite quotients of a weighted projective space (WPS, for short), such a generalization can be stated by saying that *every $\mathbb{Q}$-factorial complete toric variety is a fake PWS*. Section 2 is entirely devoted to explaining this fact. The main result is given by Theorem 2.2: The group action realizing the quotient is determined by the torsion subgroup $\mathrm{Tors}(\mathrm{Cl}(X))$ of the classes group $\mathrm{Cl}(X)$ of the given $\mathbb{Q}$-factorial complete toric variety X, and it is represented by a suitable *torsion matrix* Γ. In other words, the codimension 1 structure of the $\mathbb{Q}$-factorial complete toric variety X can be completely assigned by a *weight matrix* Q, in the sense of Definition 1.4, describing the covering PWS as a Cox's quotient, and by the torsion matrix Γ.

The second result (ii) is obtained by the former (i), by making significant use of W. Buczynska's results presented in [3] and briefly summarized in 1.2, since a PWS can be characterized as a *1-connected in codimension 1* $\mathbb{Q}$-factorial complete toric variety. By $\mathbb{Z}$-linear Gale duality and the universal property (2), we get a complete description of any covering in codimension 1 of a given $\mathbb{Q}$-factorial complete toric variety: this is Corollary 2.5.

The so-given description, of a $\mathbb{Q}$-factorial complete toric variety as a fake PWS, allows us to extend the description of the bases of the subgroup of Cartier divisors inside the free group of Weil divisors and of the Picard subgroup inside the class group, given in [10, Thm. 2.9] for a PWS, to every $\mathbb{Q}$-factorial complete toric variety. This is the third result (iii): Sect. 3 is entirely devoted to this purpose. The main result in this context is given by Theorem 3.2, whose proof is essentially the same as that of [10, Thm. 2.9], although the torsion in $\mathrm{Cl}(X)$ makes the situation significantly more intricate.

For the most part, results stated in Theorem 3.2 are based on the algorithms giving Hermite and Smith normal forms of a matrix (HNF and SNF, respectively) and associated switching matrices, which are well-known algorithms (see, e.g., [4, Algorithms 2.4.4 and 2.4.14])

implemented in many computer algebra procedures. Theorem 3.2 gives quite effective and constructive methods for producing large amounts of interesting information characterizing a given $\mathbb{Q}$-factorial complete toric variety X (see Remark 3.3 for further details).

Finally, Sect. 4 is dedicated to reconstructing a fan matrix V, in the sense of Definition 1.2, of a $\mathbb{Q}$-factorial complete toric variety X, assigned by means of a weight matrix Q and a torsion matrix Γ. Here, the main result is given by Theorem 4.2, in which the fan matrix is obtained by Gale duality and by reconstructing the matrix β characterizing the universal property (2) and linking a fan matrix of X with a fan matrix of the covering PWS.

The last Sect. 5 is devoted to giving several examples and applications of all the techniques described in the previous sections: Examples are presented following the lines stated in Remark 3.3. Here it is rather important for the reader to be equipped with some computer algebra package, which has the ability to produce Hermite and Smith normal forms of matrices and their switching matrices. For example, using Maple, similar procedures are given by `HermiteForm` and `SmithForm` with their output options.

1 Preliminaries and notation

The present paper gives a first application of the $\mathbb{Z}$-linear Gale Duality developed in the previous paper [10], to which the reader is referred for notation and all the necessary preliminary results. In particular concerning toric varieties, cones and fans, the reader is referred to [10, § 1.1], for linear algebraic preliminaries about normal forms of matrices (Hermite and Smith normal forms—HNF and SNF for short) to [10, § 1.2]. $\mathbb{Z}$-linear Gale Duality and what is concerning fan matrices (F-matrices) and weight matrices (W-matrices) are developed in [10, § 3].

Every time the needed nomenclature will be recalled directly either by giving the necessary definition or by reporting the precise reference. Here is a list of main notation and relative references:

1.1 List of notation

Let $X(\Sigma)$ be a n-dimensional toric variety and $T \cong (\mathbb{C}^*)^n$ the acting torus, then

$M, N, M_{\mathbb{R}}, N_{\mathbb{R}}$	denote the *group of characters* of T, its dual group and their tensor products with $\mathbb{R}$, respectively;
$\Sigma \subseteq \mathfrak{P}(N_{\mathbb{R}})$	is the fan defining X; $\mathfrak{P}(N_{\mathbb{R}})$ denotes the power set of $N_{\mathbb{R}}$
$\Sigma(i)$	is the *i-skeleton* of Σ, that is, the collection of all the i-dimensional cones in Σ;
$\langle \mathbf{v}_1, \ldots, \mathbf{v}_s \rangle \subseteq N_{\mathbb{R}}$	denotes the cone generated by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in N_{\mathbb{R}}$; if $s = 1$, then this cone is also called the *ray* generated by $\mathbf{v}_1$;
$\mathcal{L}(\mathbf{v}_1, \ldots, \mathbf{v}_s) \subseteq N$	denotes the sublattice spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_s \in N$;
$\mathcal{W}_T(X), \ \mathcal{C}_T(X), \ \mathcal{P}_T(X)$	denote torus invariant Weil, Cartier, principal divisors of X, resp.

Let $A \in \mathbf{M}(d, m; \mathbb{Z})$ be a $d \times m$ integer matrix, then

$\mathcal{L}_r(A) \subseteq \mathbb{Z}^m$	denotes the sublattice spanned by the rows of A;
$\mathcal{L}_c(A) \subseteq \mathbb{Z}^d$	denotes the sublattice spanned by the columns of A;
$A_I, \ A^I$	$\forall I \subseteq \{1, \ldots, m\}$ the former is the submatrix of A given by the columns indexed by I and the latter is the submatrix of A whose columns are indexed by the complementary subset $\{1, \ldots, m\} \backslash I$;

$_sA$, ^sA	$\forall\, 1 \le s \le d$ the former is the submatrix of A given by the lower s rows and the latter is the submatrix of A given by the upper s rows of A;
HNF(A), SNF(A)	denote the Hermite and the Smith normal forms of A, respectively.
REF	Row Echelon Form of a matrix;
positive	a matrix (vector) whose entries are nonnegative.

Given a F-matrix $V = (\mathbf{v}_1, \ldots, \mathbf{v}_{n+r}) \in \mathbf{M}(n, n+r; \mathbb{Z})$ (see Definition 1.2 below), then

$\langle V \rangle = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{n+r} \rangle \subseteq N_{\mathbb{R}}$	denotes the cone generated by the columns of V;
$\mathcal{SF}(V) = \mathcal{SF}(\mathbf{v}_1, \ldots, \mathbf{v}_{n+r})$	is the set of all rational simplicial fans Σ such that $\Sigma(1) = \{\langle \mathbf{v}_1 \rangle, \ldots, \langle \mathbf{v}_{n+r} \rangle\} \subseteq N_{\mathbb{R}}$ and $|\Sigma| = \langle V \rangle$ [10, Def. 1.3].
$\mathcal{G}(V)$	Q is a *Gale dual* matrix of V [10, § 3.1];

Let us start by recalling four fundamental definitions:

Definition 1.1 A n-dimensional $\mathbb{Q}$-factorial complete toric variety $X = X(\Sigma)$ of rank r is the toric variety defined by a n-dimensional *simplicial* and *complete* fan Σ such that $|\Sigma(1)| = n + r$ [10, § 1.1.2]. In particular, the rank r coincides with the Picard number, i.e., $r = \mathrm{rk}(\mathrm{Pic}(X))$.

Definition 1.2 ([10], Def. 3.10) An *F-matrix* is a $n \times (n + r)$ matrix V with integer entries, satisfying the conditions:

(a) $\mathrm{rk}(V) = n$;
(b) V is *F-complete*, i.e., $\langle V \rangle = N_{\mathbb{R}} \cong \mathbb{R}^n$ [10, Def. 3.4];
(c) all the columns of V are nonzero;
(d) if $\mathbf{v}$ is a column of V, then V does not contain another column of the form $\lambda\mathbf{v}$ where $\lambda > 0$ is real number.

A *CF-matrix* is a F-matrix satisfying the further requirement

(e) the sublattice $\mathcal{L}_c(V) \subseteq \mathbb{Z}^n$ is cotorsion-free, that is, $\mathcal{L}_c(V) = \mathbb{Z}^n$ or, equivalently, $\mathcal{L}_r(V) \subseteq \mathbb{Z}^{n+r}$ is cotorsion-free.

A F-matrix V is called *reduced* if every column of V is composed by coprime entries [10, Def. 3.13].

Example 1.3 The most significant example of F-matrix is given by a matrix V whose columns are integral vectors generating the rays of the 1-skeleton $\Sigma(1)$ of a rational fan Σ. In the following V will be called a *fan matrix* of Σ; when every column of V is composed by coprime entries, it will be called a *reduced fan matrix*. For a detailed definition, see [10, Def. 1.3].

Definition 1.4 ([10], Def. 3.9) A *W-matrix* is an $r \times (n + r)$ matrix Q with integer entries, satisfying the following conditions:

(a) $\mathrm{rk}(Q) = r$;
(b) $\mathcal{L}_r(Q)$ does not have cotorsion in $\mathbb{Z}^{n+r}$;
(c) Q is *W-positive*, that is, $\mathcal{L}_r(Q)$ admits a basis consisting of positive vectors [10, Def. 3.4].
(d) Every column of Q is nonzero.
(e) $\mathcal{L}_r(Q)$ does not contain vectors of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$.
(f) $\mathcal{L}_r(Q)$ does not contain vectors of the form $(0, a, 0, \ldots, 0, b, 0, \ldots, 0)$, with $ab < 0$.

A W-matrix is called *reduced* if $V = \mathcal{G}(Q)$ is a reduced F-matrix [10, Def. 3.14, Thm. 3.15]

Definition 1.5 ([10], Def. 2.7) A *poly weighted space* (PWS) is a n-dimensional $\mathbb{Q}$-factorial complete toric variety $X(\Sigma)$ of rank r, whose reduced fan matrix V is a CF-matrix, i.e., if

- $V = (\mathbf{v}_1, \ldots, \mathbf{v}_{n+r})$ is a $n \times (n + r)$ CF-matrix,
- $\Sigma \in \mathcal{SF}(V)$.

1.2 The fundamental group in codimension 1

This subsection is devoted to recall notation and results introduced in [3, § 3] to which the interested reader is referred for any further detail.

Definition 1.6 The *fundamental group in codimension 1* of an irreducible, complex, algebraic variety X is the inverse limit

$$\pi_1^1(X) := \varprojlim_{U \subseteq X} \pi_1(U)$$

where U is an open, non-empty, algebraic subset of X such that $\operatorname{codim}_X(X \setminus U) \geq 2$. X is called *1-connected in codimension 1* if $\pi_1^1(X)$ is trivial.

Theorem 1.7 ([3], Thm. 3.4) *Assume X be a smooth variety and $V \subseteq X$ be a closed subset with $\operatorname{codim}_X V \geq 2$. Then, $\pi_1(X) = \pi_1(X \setminus V)$.*

Corollary 1.8 ([3], Cor.3.9 and Cor.3.10) *If X is a normal variety, then*

$$\pi_1^1(X) = \pi_1(X \setminus \operatorname{Sing}(X)).$$

In particular, if X is smooth, then $\pi_1^1(X) = \pi_1(X)$.

In fact, by Theorem 1.7, the inverse limit in the Definition 1.6 has a realization on the smooth subset $X \setminus \operatorname{Sing}(X)$.

Definition 1.9 A finite surjective morphism $\varphi : Y \to X$ is called a *covering in codimension 1* (or simply a *1-covering*) if it is unramified in codimension 1, that is, there exists a subvariety $V \subseteq X$ such that $\operatorname{codim}_X V \geq 2$ and $\varphi|_{Y_V}$ is a topological covering, where $Y_V := \varphi^{-1}(X \setminus V)$. Moreover, a *universal covering in codimension 1* is a 1-covering $\varphi : Y \to X$ such that for any 1-covering $\phi : X' \to X$ of X there exists a 1-covering $f : Y \to X'$ such that $\varphi = \phi \circ f$.

Proposition 1.10 ([3], Rem. 3.14) *A 1-covering $\varphi : Y \to X$ is universal if and only if Y is 1-connected in codimension 1, i.e., $\pi_1^1(Y) = 0$.*

Theorem 1.11 (π_1^1 for toric varieties [3], Thm. 4.8) *Let $X = X(\Sigma)$ with Σ be a fan in $N_{\mathbb{R}}$. For any ray $\rho \in \Sigma(1)$ let $\mathbf{v}_\rho$ be the generator of the monoid $\rho \cap N$ and consider the $\mathbb{Z}$-module*

$$N_{\Sigma(1)} := \mathcal{L}\left(\mathbf{v}_\rho \mid \rho \in \Sigma(1)\right) \tag{1}$$

as a subgroup of the lattice N. Then, $\pi_1^1(X) \cong N/N_{\Sigma(1)}$.

This result is an application of Van Kampen Theorem to the open covering $\{X_\rho\}_{\rho \in \Sigma(1)}$ of X, where $X_\rho = X(\rho) \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$ is the toric variety associated with the fan given by the single ray $\rho \in \Sigma$.

2 Universal 1-coverings

Jointly with [10, Thm. 2.4, Prop. 2.6], the previous Theorem 1.11 gives the following equivalence between a 1-connected $\mathbb{Q}$-factorial complete variety and a PWS:

Theorem 2.1 *Let $X = X(\Sigma)$ be a $\mathbb{Q}$-factorial complete n-dimensional toric variety of rank r. Then, the following are equivalent:*

1. *X is 1-connected in codimension 1,*
2. $\pi_1^1(X) \cong N/N_{\Sigma(1)} \cong \mathrm{Tors}(\mathrm{Cl}(X)) = 0$,
3. *the reduced fan matrix V of X has coprime $n \times n$ minors,*
4. *the* HNF *of the transposed matrix V^T is given by* $\begin{pmatrix} I_n \\ \mathbf{0}_{r,n} \end{pmatrix}$,
5. *X is a PWS.*

The previous results allow us to sketch a nice geometric picture associated with any $\mathbb{Q}$-factorial, complete toric variety: the following Theorem 2.2 is a generalization on the rank r of a well-known result holding for $r = \mathrm{rk}(\mathrm{Pic}(X)) = 1$ [2, Lemma 2.11], [5, Prop. 4.7]. In fact a 1-connected in codimension 1, $\mathbb{Q}$-factorial, complete n-dimensional toric variety of rank 1 is necessarily a *weighted projective space* (WPS) whose weights are given by the $1 \times (n+1)$ weight matrix Q.

Theorem 2.2 *A $\mathbb{Q}$-factorial, complete toric variety X admits a canonical universal 1-covering, Y which is a PWS and such that the 1-covering morphism $\varphi : Y \to X$ is equivariant with respect to the torus actions. In particular every $\mathbb{Q}$-factorial, complete toric variety X can be canonically described as a finite geometric quotient $X \cong Y/\pi_1^1(X)$ of a PWS Y by the torus-equivariant action of $\pi_1^1(X)$.*

Proof Given a complete fan $\Sigma \subseteq \mathfrak{P}(N_{\mathbb{R}})$, let $X(\Sigma)$ be the associated toric variety. Let $\widehat{N}$ be the sublattice $N_{\Sigma(1)} \subseteq N$ defined in (1); it has finite index since Σ is complete. Let $\widehat{\Sigma}$ be the fan defined by $\Sigma \subseteq \mathfrak{P}(\widehat{N}_{\mathbb{R}})$. Consider the toric variety $Y(\widehat{\Sigma})$. The inclusion $i : \widehat{N} \hookrightarrow N$ induces a surjection $i_* : Y \to X$ which turns out to be the canonical projection on the quotient of Y by the action of $N/\widehat{N}$. Furthermore, part 2 of Theorem 2.1 shows that $\mathrm{Cl}(Y)$ is torsion-free, so that the results of Section 1.2 imply that the canonical morphism i_* is the universal 1-covering. Finally, if X is simplicial, so is Y, and then Y is a PWS by Theorem 2.1. This completes the proof.   □

Remark 2.3 Let us notice that the given proof of Theorem 2.2 shows that a $\mathbb{Q}$-factorial complete toric variety X uniquely determines its universal 1-covering Y which is the PWS defined by the following data:

– a weight matrix $Q = \mathcal{G}(V)$, where V is a reduced fan matrix of X: in fact a fan matrix of Y is given by $\mathcal{G}(Q) = \widehat{V}$,
– the choice of a fan $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$ uniquely determined by the given choice of the fan $\Sigma \in \mathcal{SF}(V)$ defining X.

Remark 2.4 Given a fan matrix V, consider $\widehat{V} = \mathcal{G}(\mathcal{G}(V))$, which is a CF-matrix by [10, Prop. 3.11]. Let us recall that:

$$\exists \beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z}) \; : \; V = \beta \cdot \widehat{V} \tag{2}$$

[10, Prop. 3.1(3)]. In a sense, the universality property of the 1-covering $Y(\widehat{\Sigma})$ is summarized by the linear algebraic property (2). In fact, the proof of Theorem 2.2 given above shows that any further 1-covering of the $\mathbb{Q}$-factorial complete toric variety $X(\Sigma)$ admitting V as a fan matrix is given by the choice of an integer matrix $\gamma \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ *dividing* the matrix β defined in (2), i.e., such that $\beta \cdot \gamma^{-1} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$. Namely:

– under our hypothesis, the fan $\Sigma \in \mathcal{SF}(V)$ is determined as the fan of all the faces of every cone in $\Sigma(n)$,
– $\Sigma(n)$ is assigned by the fan matrix V and the following subset of the power set $\mathfrak{P}(\{1, \ldots, n+r\})$: $\mathcal{I}_\Sigma := \{I \subseteq \{1, \ldots, n+r\} \,|\, \langle V_I \rangle \in \Sigma(n)\}$,

– consider the set of *divisors of* β:

$$\mathfrak{D}(\beta) := \{\eta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z}) \mid \beta \cdot \eta^{-1} \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})\},$$

then, for every $\eta \in \mathfrak{D}(\beta)$, $\mathcal{I}_\Sigma$ determines a fan $\Sigma_\eta \in \mathcal{SF}(V_\eta)$, with $V_\eta := \eta \cdot \widehat{V}$, giving a ℚ-factorial complete toric variety $X_\eta(\Sigma_\eta)$.

In particular, $Y(\widehat{\Sigma}) = X_{I_n}(\Sigma_{I_n})$ and $X(\Sigma) = X_\beta(\Sigma_\beta)$.

Then, we get the following

Corollary 2.5 (Universal 1-covering Theorem) *In the notation introduced in Remark 2.4, the following facts hold:*

1. $Y = X_{I_n}$ *is the universal 1-covering PWS of X given by Theorem 2.2, meaning that X is a finite abelian quotient of Y by the action of $\pi_1^1(X)$,*
2. $X_\eta \to X$ *is a 1-covering and there is a natural factorization of 1-coverings*
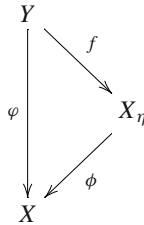
as in Definition 1.9; in particular, X is a finite abelian quotient of X_η by the action of $\pi_1^1(X)/\pi_1^1(X_\eta)$.

Consequently, writing $\eta \sim \eta'$ if they are related by means of left multiplication of an element in $\mathrm{GL}_n(\mathbb{Z})$, *the quotient set $\mathfrak{D}(\beta)/\sim$ parameterizes all the topologically distinct 1-coverings of X.*

3 Weil versus Cartier

The aim of the present section is that of extending [10, Thm. 2.9] to every ℚ-factorial complete toric variety. By Theorem 2.2, this means extending this result, currently holding for PWS, to their quotients.

Let us start by the following preliminary

Proposition 3.1 *Let $X(\Sigma)$ be a Q-factorial complete toric variety and $Y(\widehat{\Sigma})$ be its universal 1-covering. Let $\{D_\rho\}_{\rho \in \Sigma(1)}$ and $\{\widehat{D}_\rho\}_{\rho \in \widehat{\Sigma}(1)}$ be the standard bases of $\mathcal{W}_T(X)$ and $\mathcal{W}_T(Y)$, respectively, given by the torus orbit closures of the rays. Then*

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \in \mathcal{C}_T(X) \iff \widehat{D} = \sum_{\rho \in \widehat{\Sigma}(1)} a_\rho \widehat{D}_\rho \in \mathcal{C}_T(Y).$$

Therefore, $\mathcal{C}_T(X) \cong \mathcal{C}_T(Y)$ is a maximal rank subgroup of $\mathbb{Z}^{|\Sigma(1)|} = \mathbb{Z}^{|\widehat{\Sigma}(1)|}$ and $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$ is a maximal rank subgroup of the free part $\mathrm{Cl}(Y)$ of $\mathrm{Cl}(X)$.

Proof The result follows by writing down the condition of being locally principal for a Weil divisor and observing that

$$\mathcal{I}_\Sigma = \{I \subseteq \{1, \ldots, n+r\} : \left\langle V^I \right\rangle \in \Sigma(n)\}$$

$$= \{I \subseteq \{1, \ldots, n+r\} : \left\langle \widehat{V}^I \right\rangle \in \widehat{\Sigma}(n)\} = \mathcal{I}_{\widehat{\Sigma}} \qquad (3)$$

by the construction of $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$, given the choice of $\Sigma \in \mathcal{SF}(V)$. Then, the assertion follows by the same argument proving part (2) of [10, Thm 2.9]. □

We are then in a position of stating and proving the main result of this section

Theorem 3.2 *Let $X = X(\Sigma)$ be a n-dimensional $\mathbb{Q}$-factorial complete toric variety of rank r and $Y = Y(\widehat{\Sigma})$ be its universal 1-covering. Let V be a reduced fan matrix of X, $Q = \mathcal{G}(V)$ a weight matrix of X and $\widehat{V} = \mathcal{G}(Q)$ be a CF-matrix giving a fan matrix of Y.*

1. *Consider the matrix*

$$U_Q = (u_{ij}) \in \mathrm{GL}_{n+r}(\mathbb{Z}) \ : \ U_Q \cdot Q^T = \mathrm{HNF}\left(Q^T\right)$$

   then the rows of $^r U_Q$ (recall notation in 1.1) describe the following set of generators of the free part $F (\cong \mathrm{Cl}(Y))$ of $\mathrm{Cl}(X)$

$$\forall\, 1 \leq i \leq r \quad L_i := \sum_{j=1}^{n+r} u_{ij} D_j \in \mathcal{W}_T(X) \quad and \quad F = \bigoplus_{i=1}^{r} \mathbb{Z}[d_X(L_i)] \qquad (4)$$

   where $d_X : \mathcal{W}_T(X) \to \mathrm{Cl}(X)$ is the morphism giving to a Weil divisor its linear equivalence class.
2. *Define $\mathcal{I}_\Sigma$ as in (3). Through the identification $F \cong \mathbb{Z}^r$, just fixed in (4), one gets*

$$\mathrm{Pic}(X) = \bigcap_{I \in \mathcal{I}_\Sigma} \mathcal{L}_c(Q_I).$$

   Therefore, a basis of $\mathrm{Pic}(X) \subseteq F \cong \mathbb{Z}^r$ can be computed by applying the procedure described in [10, § 1.2.3].
3. *Let $\mathbf{b}_1, \ldots, \mathbf{b}_r$ be a basis of $\mathrm{Pic}(X)$ in $\mathbb{Z}^r \cong F$, and let B be the $r \times r$ matrix having $\mathbf{b}_1, \ldots, \mathbf{b}_r$ on the rows. Then a basis of $\mathcal{C}_T(X) \subseteq \mathbb{Z}^{n+r} \cong \mathcal{W}_T(X)$ is given by the rows of the matrix*

$$C_X = \begin{pmatrix} B & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \beta \end{pmatrix} \cdot U_Q = \begin{pmatrix} B \cdot {}^r U_Q \\ V \end{pmatrix},$$

   where $\beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ is such that $\beta \cdot \widehat{V} = V$ (recall (2)).
4. *There exists a choice of the fan matrices V and $\widehat{V} = \mathcal{G}(\mathcal{G}(V))$ and a diagonal matrix $\Delta = \mathrm{diag}(c_1, \ldots, c_n) \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ such that*

   (a) $1 = c_1 \mid \ldots \mid c_n$,
   (b) $V = \Delta \cdot \widehat{V}$,
   (c) $\mathrm{Tors}(\mathrm{Cl}(X)) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/c_i\mathbb{Z} = \bigoplus_{k=1}^{s} \mathbb{Z}/\tau_k\mathbb{Z}$,
       according with the decomposition of $\mathrm{Cl}(X)$ given by the fundamental theorem of finitely generated abelian groups,

$$\mathrm{Cl}(X) = F \oplus \mathrm{Tors}(\mathrm{Cl}(X)) \cong \mathbb{Z}^r \oplus \bigoplus_{k=1}^{s} \mathbb{Z}/\tau_k\mathbb{Z} \qquad (5)$$

   that is, $s < n$, $\tau_k = c_{n-s+k} > 1$, $c_1 = \cdots = c_{n-s} = 1$.

5. *Given the choice of $\widehat{V} = (\widehat{v}_{ij})$ and V as in the previous part (4), then the rows of the submatrix $_s\widehat{V}$ describe the following set of generators of $\mathrm{Tors}(\mathrm{Cl}(X))$*

$$\forall\, 1 \leq k \leq s \quad T_k := \sum_{j=1}^{n+r} \widehat{v}_{n-s+k,j} D_j \in \mathcal{W}_T(X) \quad and$$

$$\mathrm{Tors}(\mathrm{Cl}(X)) = \mathcal{L}(d_X(T_1), \ldots, d_X(T_s)) \tag{6}$$

6. *Given the choice of $\widehat{V}$ and V as in the previous parts (4) and (5), consider*

$$W \in \mathrm{GL}_{n+r}(\mathbb{Z}) \;:\; W \cdot ({}^{n-s}V)^T = \mathrm{HNF}\left(({}^{n-s}V)^T\right)$$

$$G := {}_s\widehat{V} \cdot ({}_{r+s}W)^T$$

$$U_G \in \mathrm{GL}_{r+s}(\mathbb{Z}) \;:\; U_G \cdot G^T = \mathrm{HNF}(G^T).$$

Then, a "torsion matrix" representing the "torsion part" of the morphism d_X, that is, $\tau_X : \mathcal{W}_T(X) \to \mathrm{Tors}(\mathrm{Cl}(X))$, is given by

$$\Gamma = {}^s U_G \cdot {}_{r+s}W \quad \mathrm{mod}\ \tau \tag{7}$$

where this notation means that the (k, j)-entry of Γ is given by the class in $\mathbb{Z}/\tau_k\mathbb{Z}$ represented by the corresponding (k, j)-entry of ${}^s U_G \cdot {}_{r+s}W$, for every $1 \leq k \leq s$, $1 \leq j \leq n+r$.

7. *Setting $\delta_\Sigma := \mathrm{lcm}\,(\det(Q_I) : I \in \mathcal{I}_\Sigma)$ then*

$$\delta_\Sigma \mathcal{W}_T(X) \subseteq \mathcal{C}_T(X) \text{ and } \delta_\Sigma \mathcal{W}_T(Y) \subseteq \mathcal{C}_T(Y)$$

and there are the following divisibility relations

$$\delta_\Sigma \mid [F : \mathrm{Pic}(X)] = [\mathrm{Cl}(Y) : \mathrm{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)] \mid [\mathcal{W}_T(X) : \mathcal{C}_T(X)].$$

Proof Once for all, let us recall the following commutative diagram of short exact sequences

where, fixing all the needed basis, V is a reduced fan matrix of $X(\Sigma)$.

(1) Recall that Q is a representative matrix of the "free part" of the morphism $d_X : \mathcal{W}_T(X) \to \mathrm{Cl}(X)$, that is, the morphism $f_X : \mathcal{W}_T(X) \to F \cong \mathrm{Cl}(Y)$. Then, the proof goes on as proving part (1) of [10, Thm. 2.9]: namely

$$U_Q \cdot Q^T = \mathrm{HNF}\left(Q^T\right) = \begin{pmatrix} \mathbf{I}_r \\ \mathbf{0}_{n,r} \end{pmatrix} \implies Q \cdot {}^r U_Q = \mathbf{I}_r.$$

(2) Recalling Proposition 3.1, the proof proceeds as for part (2) in [10, Thm. 2.9].

(3) As in the proof of part (3) in [10, Thm. 2.9], observe that

$$Q \cdot C_X^T = Q \cdot \left(({}^r U_Q)^T \cdot B^T \mid V^T \right) = \left(B^T \mid \mathbf{0}_{r,n} \right),$$

so that $\mathcal{L}_r(C_X) \subseteq \mathcal{C}_T(X)$, where $\mathcal{L}_r(C_X)$ is the subgroup spanned by the rows of C_X; moreover

$$[\mathcal{W}_T(X) : \mathcal{L}_r(C_X)] = \det(B) \det(\beta) = [F : \mathrm{Pic}(X)] \cdot [\mathrm{Cl}(X) : F]$$
$$= [\mathrm{Cl}(X) : \mathrm{Pic}(X)] = [\mathcal{W}_T(X) : \mathcal{C}_T(X)];$$

therefore $\mathcal{L}_r(C_X) = \mathcal{C}_T(X)$.

(4) By (2), there exists a matrix $\beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ is such that $\beta \cdot \widehat{V} = V$. Define $\Delta := \mathrm{SNF}(\beta)$. It is then a well-known fact (see, e.g., [4, Algorithm 2.4.14]) the existence of matrices $\mu, \nu \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$\Delta = \mu \cdot \beta \cdot \nu \implies \Delta \cdot (\nu^{-1} \cdot \widehat{V}) = \mu \cdot V. \tag{9}$$

Notice that $\nu^{-1} \cdot \widehat{V} \sim \widehat{V}$ are equivalent CF-matrices and $\mu \cdot V \sim V$ are equivalent F-matrices giving a choice of the fan matrices of Y and X, respectively, satisfying conditions (a) and (b): notice that $c_1 = \gcd(c_1, \ldots, c_n) = 1$ since V is a reduced F-matrix. Then, (c) follows by recalling that

$$\mathrm{Tors}(\mathrm{Cl}(X)) \cong \mathbb{Z}^n / \mathcal{L}_r(T_n),$$

where T_n is the upper $n \times n$ submatrix of $\mathrm{HNF}(V^T)$ (see relations (10) in [10, Thm. 2.4]).

(5) If $\widehat{V}$ and V are such that condition (b) in (4) holds, then the kth row of ${}_s\widehat{V}$ is actually the $(n - s + k)$th row of V divided by the gcd of its entries, i.e., by $c_{n-s+k} = \tau_k > 1$. Since the rows of V span $\ker(d_X)$, the kth row of ${}_s\widehat{V}$ gives then rise to the torsion Weil divisor $T_k = \sum_{j=1}^{n+r} \widehat{v}_{n-s+k,j} D_j$. Recalling (4)(c) the classes $d_X(T_1), \ldots, d_X(T_s)$ suffice to generate $\mathrm{Tors}(\mathrm{Cl}(X))$.

(6) A representative matrix of the torsion morphism $\tau_X : \mathcal{W}_T(X) \to \mathrm{Cl}(X)$ in (8) is any matrix satisfying the following properties:

(i) $\Gamma = (\gamma_{kj})$ with $\gamma_{kj} \in \mathbb{Z}/\tau_k \mathbb{Z}$,

(ii) $\Gamma \cdot V^T = \mathbf{0}_{s,n} \mod \boldsymbol{\tau}$, where V is a fan matrix satisfying condition (4)(b): this is due to the fact that the rows of V span $\ker(d_X)$,

(iii) $\Gamma \cdot ({}_s\widehat{V})^T = \mathbf{I}_s \mod \boldsymbol{\tau}$, since the rows of ${}_s\widehat{V}$ give the generators of $\mathrm{Tors}(\mathrm{Cl}(X))$, as in (6).

Therefore, it suffices to show that the matrix ${}^s U_G \cdot {}_{r+s} W$ in (7) satisfies the previous conditions (ii) and (iii) without any reduction mod $\boldsymbol{\tau}$, that is,

$${}^s U_G \cdot {}_{r+s} W \cdot ({}^{n-s} V)^T = \mathbf{0}_{s,n-s}, \quad {}^s U_G \cdot {}_{r+s} W \cdot ({}_s\widehat{V})^T = \mathbf{I}_s.$$

The first equation follows by the definition of W, in fact

$$W \cdot ({}^{n-s} V)^T = \mathrm{HNF}\left(({}^{n-s} V)^T \right)$$
$$= \mathrm{HNF}\left(({}^{n-s} \widehat{V})^T \right) = \begin{pmatrix} \mathbf{I}_{n-s} \\ \mathbf{0}_{r+s,n-s} \end{pmatrix} \implies {}_{r+s} W \cdot ({}^{n-s} V)^T = \mathbf{0}_{r+s,n-s}$$

The second equation follows by the definition of U_G, in fact

$$U_G \cdot {}_{r+s} W \cdot ({}_s\widehat{V})^T = U_G \cdot G^T = \mathrm{HNF}(G^T) = \begin{pmatrix} \mathbf{I}_s \\ \mathbf{0}_{r,s} \end{pmatrix}.$$

(7) Part (4) of [10, Thm. 2.9] gives that $\delta_\Sigma \mid [\mathrm{Cl}(Y) : \mathrm{Pic}(Y)]$. On the other Proposition 3.1 gives that $[F : \mathrm{Pic}(X)] = [\mathrm{Cl}(Y) : \mathrm{Pic}(Y)] = [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)]$. Finally, recalling the central vertical exact sequence in (8), one clearly gets

$$[\mathcal{W}_T(X) : \mathcal{C}_T(X)] = [\mathcal{W}_T(X) : \alpha(\mathcal{W}_T(Y))] \cdot [\alpha(\mathcal{W}_T(Y)) : \mathcal{C}_T(X)]$$
$$= [\mathcal{W}_T(X) : \alpha(\mathcal{W}_T(Y))] \cdot [\mathcal{W}_T(Y) : \mathcal{C}_T(Y)]$$

where the last equality is still guaranteed by Proposition 3.1. □

Remark 3.3 This is the generalization of [10, Rem. 2.10]. The most part of results stated in Theorem 3.2 are based on the algorithms giving the HNF and the SNF of a matrix, and associated switching matrices, which are well-known algorithms (see, e.g., [4, Algorithms 2.4.4 and 2.4.14]) implemented in many computer algebra procedures. Then, Theorem 3.2 gives quite effective and constructive methods to produce a lot of interesting information characterizing a given ℚ-factorial complete toric variety X. In particular

(i) given a fan matrix V of X, a switching matrix $U_V \in \mathrm{GL}_{n+r}(\mathbb{Z})$, such that $\mathrm{HNF}(V^T) = U_V \cdot V^T$, encodes a weight matrix $Q = \mathcal{G}(V)$ of X in the last r rows [10, Prop. 4.3];

(ii) given a weight matrix Q of X, a switching matrix $U_Q \in \mathrm{GL}_{n+r}(\mathbb{Z})$, such that $\mathrm{HNF}(Q^T) = U_Q \cdot Q^T$, encodes both a basis $\{d_X(L_i)\}_{i=1}^r$ of the free part $F \cong \mathbb{Z}^r$ of the divisor class group $\mathrm{Cl}(X)$, given by the upper r rows by (4), and a fan matrix $\widehat{V}$ of the universal 1-covering Y of X, given by the lower n rows [10, Prop. 4.3];

(iii) by (2) there exists $\beta \in \mathrm{GL}_n(\mathbb{Z})$ such that $\beta \cdot \widehat{V} = V$; an effective procedure producing β is the following:

– set

$$H = (h_{i,j}) := \mathrm{HNF}(V), \ \widehat{H} = (\hat{h}_{i,j}) := \mathrm{HNF}(\widehat{V})$$
$$U \in \mathrm{GL}(n, \mathbb{Z}) : U \cdot V = H$$
$$\widehat{U} \in \mathrm{GL}(n, \mathbb{Z}) : \widehat{U} \cdot \widehat{V} = \widehat{H}$$
$$\beta_H = (b_{i,j}) := U \cdot \beta \cdot \widehat{U}^{-1},$$

– then $V = \beta \cdot \widehat{V} \ \Rightarrow \ H = \beta_H \cdot \widehat{H}$: since, up to a simultaneous permutations of columns of V and $\widehat{V}$, one can assume that both the submatrices $H_{\{1,\ldots,n\}}$ and $\widehat{H}_{\{1,\ldots,n\}}$ are upper triangular, then also β_H turns out to be upper triangular,

– one can then get β_H from H and $\widehat{H}$, by the following recursive relations

$$\forall\, 1 \le i \le n \quad b_{i,i} = \frac{h_{i,i}}{\hat{h}_{i,i}}$$
$$\forall\, i+1 \le j \le n \quad b_{i,j} = \frac{1}{\hat{h}_{i,i}} \left(h_{i,j} - \sum_{k=i}^{j-1} b_{i,k} \hat{h}_{k,j} \right) \tag{10}$$

(iv) let $\Delta := \mathrm{SNF}(\beta)$ and $\mu, \nu \in \mathrm{GL}_n(\mathbb{Z})$ be matrices realizing relations (9); setting $V' := \nu^{-1} \cdot \widehat{V}$, construct the submatrix $_sV'$ as in part (5) of Theorem 3.2; then, the rows of $_sV'$ describe a set of generators $\{d_X(T_k)\}_{k=1}^s$ of $\mathrm{Tors}(\mathrm{Cl}(X))$, as in (6);

(v) get matrices W, G and U_G as in part (6) of Theorem 3.2 and construct the torsion matrix Γ as in (7), giving a representative matrix of the torsion part τ_X of the morphism d_X;

(vi) apply the procedure [10, § 1.2.3], also based on the HNF algorithm, to get a $r \times r$ matrix B whose rows give a basis of $\mathrm{Pic}(X) \subseteq F = \bigoplus_{i=1}^r \mathbb{Z}[d_X(L_i)]$;

(vii) construct the matrix C_X as in part (3) of Theorem 3.2 whose rows give a basis of $\mathcal{C}_T(X) \subseteq \mathcal{W}_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z}[D_j]$.

Moreover:

- recall the commutative diagram of short exact sequences (8),
- recall that, for the universal 1-covering Y of X, once fixed the basis $\{\widehat{D}_j\}_{j=1}^{n+r}$ of $\mathcal{W}_T(Y) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_Y(\widehat{L}_i)\}_{i=1}^r$ of $\mathrm{Cl}(Y) \cong \mathbb{Z}^r$, constructed as in (4) by replacing D_j with $\widehat{D}_j$ (see (11) in [10, Thm. 2.9]), one gets the following commutative diagram
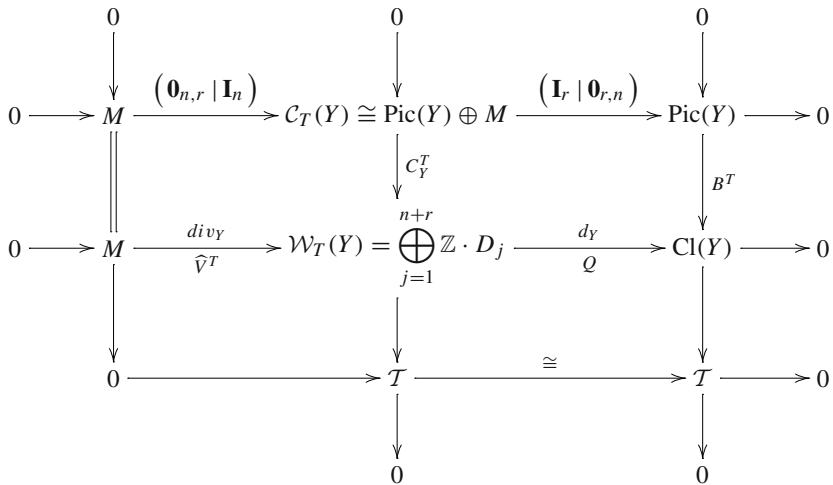
where B is the $r \times r$ matrix constructed in [10, Thm. 2.9(3)] and which is actually the same of the matrix B constructed in part (3) of Theorem 3.2, due to Proposition 3.1, and

$$C_Y = \begin{pmatrix} B & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} \cdot U_Q = \begin{pmatrix} B \cdot {}^r U_Q \\ \widehat{V} \end{pmatrix},$$

- once fixed the basis $\{D_j\}_{j=1}^{n+r}$ for $\mathcal{W}_T(X) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_X(L_i)\}_{i=1}^r$ of the free part $F \cong \mathbb{Z}^r$ of $\mathrm{Cl}(X)$, constructed in (4), one gets the following commutative diagram
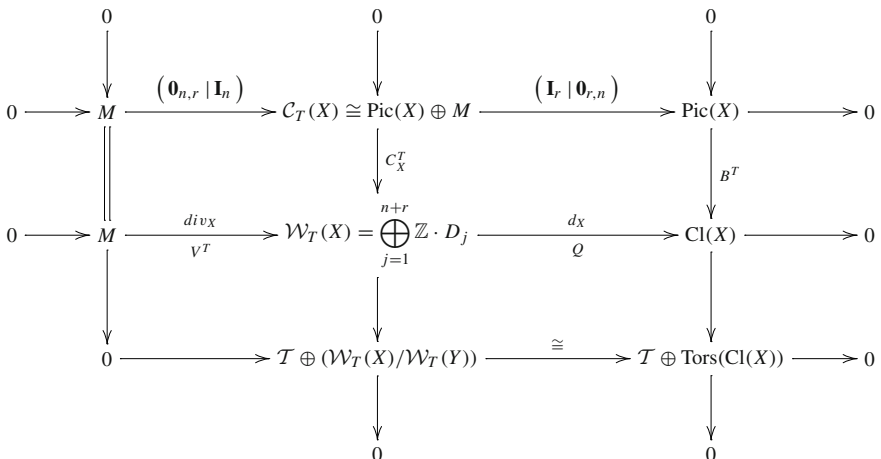
then, putting all together, one gets the following 3-dimensional commutative diagram

$$(11)$$
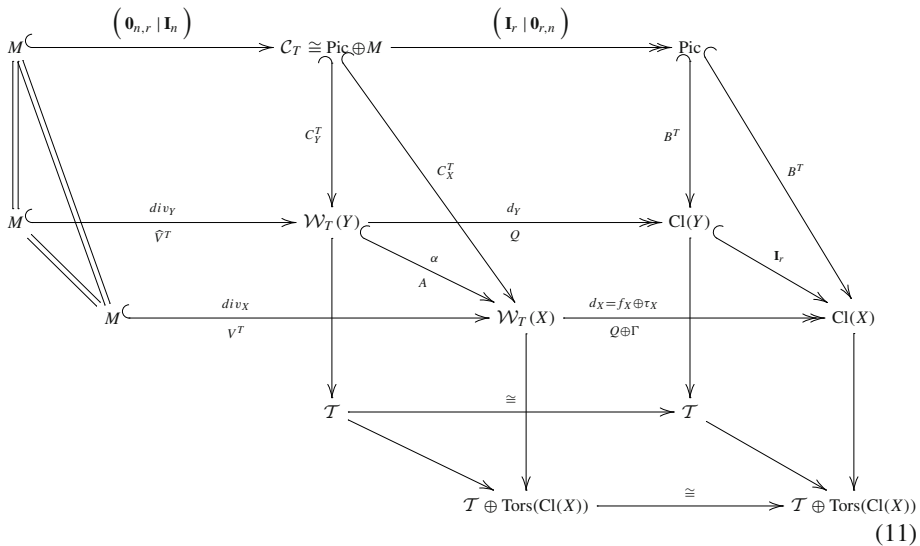
where $\mathcal{C}_T(X) = \mathcal{C}_T(Y) =: \mathcal{C}_T$ and $\mathrm{Pic}(X) = \mathrm{Pic}(Y) =: \mathrm{Pic}$ have been identified, by Proposition 3.1. In fact, diagram (11) follows by checking that

$$A \cdot C_Y^T = U_Q^T \cdot \begin{pmatrix} \mathbf{I}_r & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \beta^T \end{pmatrix} \cdot (U_Q^T)^{-1} \cdot U_Q^T \cdot \begin{pmatrix} B^T & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \mathbf{I}_n \end{pmatrix} = U_Q^T \cdot \begin{pmatrix} B^T & \mathbf{0}_{r,n} \\ \mathbf{0}_{n,r} & \beta^T \end{pmatrix} = C_X^T.$$

4 Going back: from the geometric quotient to the fan

In the present section, we want to reverse our point of view. Namely assume that a $\mathbb{Q}$-factorial complete n-dimensional toric variety X of rank r is presented as a geometric quotient as follows:

– let Z be an algebraic subset of $\mathbb{C}^{n+r}$ defined by a suitable monomial ideal $B \subseteq \mathbb{C}[x_1, \ldots, x_{n+r}]$,
– consider the reductive subgroup $G = \mathrm{Hom}(A, \mathbb{C}^*) \subseteq (\mathbb{C}^*)^{n+r}$ where A is a finitely generated abelian group of rank r; this means that

$$G \cong (\mathbb{C}^*)^r \oplus \bigoplus_{k=1}^{s} \mu_{\tau_k}$$

where μ_{τ_k} is the cyclic group of τ_kth roots of unity with $1 < \tau_1 \mid \cdots \mid \tau_s$;
– assume that the action of G over $\mathbb{C}^{n+r}$ is equivariant with respect to the usual multiplication of $(\mathbb{C}^*)^{n+r}$, meaning that for every $g = (\mathbf{t}, \boldsymbol{\varepsilon}) \in G$, with $\mathbf{t} = (t_1, \ldots, t_r) \in (\mathbb{C}^*)^r$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_s) \in \bigoplus_{k=1}^{s} \mu_{\tau_k}$, there exist matrices

$$Q = (q_{ij}) \in \mathbf{M}(r, n+r; \mathbb{Z}), \quad C = (c_{kj}) \in \mathbf{M}(s, n+r; \mathbb{Z}) \qquad (12)$$

such that the action is given by the usual multiplication as follows

$$\forall \mathbf{x} \in \mathbb{C}^{n+r} \quad (\mathbf{t}, \boldsymbol{\varepsilon}) \cdot \mathbf{x} = \left(\prod_{i=1}^{r} t_i^{q_{ij}} \cdot \prod_{k=1}^{s} \varepsilon_k^{a_k c_{kj}} \cdot x_j \right)_{1 \le j \le n+r} \qquad (13)$$

– assume that the given action $G \times \mathbb{C}^{n+r} \to \mathbb{C}^{n+r}$ defines a geometric quotient $(\mathbb{C}^{n+r} \setminus Z)/G$ giving precisely the toric variety X; this is possible for every $\mathbb{Q}$-factorial complete toric variety by the well-known Cox' result [6].

Remark 4.1 By (12) and (13), X is completely assigned by the following three data:

(i) a reduced W-matrix $Q = (q_{ij})$ defining the action of the free part $(\mathbb{C}^*)^r$ of G: the fact that Q can be assumed to be a W-matrix follows by the completeness of X [10, Thm. 3.8]; moreover, a W-matrix and its reduction define isomorphic actions of the free part of G;

(ii) the *torsion* matrix $\Gamma = (\gamma_{kj})$, with $\gamma_{kj} = [c_{k,j}]_{\tau_k} \in \mathbb{Z}/\tau_k\mathbb{Z}$, defining the action of the torsion part of G: in fact such an action is invariant with respect to the choice of different representatives of the class $\gamma_{k,j}$;

(iii) the algebraic subset $Z \subseteq \mathbb{C}^{n+r}$.

Wanted *a fan matrix V of X and a fan $\Sigma \in \mathcal{SF}(V)$ defining X.*

Let us first of all notice that Theorem 2.2 gives immediately the PWS Y which is the universal 1-covering of X: this is obtained by Gale duality since $\widehat{V} = \mathcal{G}(Q)$ is a CF-matrix (by [10, Prop. 3.12(1)]), hence giving a fan matrix of Y. Moreover, the *irrelevant ideal B* defining Z reconstructs the n-skeleton of a fan $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$ obtained as the fan of all the faces of every cone in $\widehat{\Sigma}(n)$.

Therefore, to get V and a fan Σ of X it suffices to recover a matrix

$$\beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z}) \; : \; \beta \cdot \widehat{V} = V$$

as in (2). Then, the fan $\Sigma \in \mathcal{SF}(V)$ is obtained from $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$, by reversing the process described in the proof of Theorem 2.2, that is, by means of the subset $I_{\widehat{\Sigma}} \subseteq \mathfrak{P}(\{1, \ldots, n+r\})$, as defined in Remark 2.4.

Theorem 4.2 *Given a reduced W-matrix Q and a torsion matrix Γ, as in parts* i *and* ii *of Remark 4.1, describing a Q-factorial complete toric variety X as a geometric quotient, and setting $\widehat{V} = \mathcal{G}(Q)$, then a matrix $\beta \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathbf{M}_n(\mathbb{Z})$ such that $V = \beta \cdot \widehat{V}$ is a fan matrix of X can be reconstructed as follows:*

1. *define a matrix $C \in \mathbf{M}(s, n+r; \mathbb{Z})$ by choosing a representative c_{kj} for any entry γ_{kj} of Γ,*

2. *consider the matrix* $K := \begin{pmatrix} \widehat{V} \cdot C^T \\ \tau_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_s \end{pmatrix} \in \mathbf{M}(n+s, s; \mathbb{Z})$

3. *let $U \in \mathrm{GL}_{n+s}(\mathbb{Z}) \; : \; \mathrm{HNF}(K) = U \cdot K$*

then, recalling notation in 1.1,

$$\beta = ({}_nU)_{\{\mathbf{1}, \ldots, n\}}$$

Proof A fan matrix $V = (v_{ij})$ of X has to satisfy the relations

$$V \cdot Q^T = 0, \quad V \cdot \Gamma^T = 0 \mod \tau \tag{14}$$

where the latter means that $\sum_{j=1}^{n+r} v_{ij}\gamma_{kj} = [0]_{\tau_k}$, for every $1 \leq i \leq n$, $1 \leq k \leq s$. Since by (2) there exists a matrix β such that $V = \beta \cdot \widehat{V}$, the former relation is a consequence of

the fact that $\widehat{V} = \mathcal{G}(Q)$. Setting $\beta = (x_{l,i})$ and $\widehat{V} = (\widehat{v}_{ij})$, the second relation in (14) can be rewritten as follows

$$\forall\, 1 \leq l \leq n,\ 1 \leq k \leq s \qquad \sum_{i=1}^{n}\left(\sum_{j=1}^{n+r} v_{ij}\gamma_{kj}\right) x_{l,i} = [0]_{\tau_k}$$

$$\Longleftrightarrow \sum_{i=1}^{n}\left(\sum_{j=1}^{n+r} v_{ij}c_{kj}\right) x_{l,i} + \tau_k(-x_{l+k}) = 0$$

where c_{kj} is a representative of γ_{kj}. Then, we are looking for n independent integer common solutions of a system of s linear homogeneous equations in $n + s$ variables whose matrix is given by K^T, where K is defined as in part (2) of the statement. These solutions are given by the lower n rows of a matrix

$$U \in \mathrm{GL}_{n+s}(\mathbb{Z}) : \quad U \cdot K = \mathrm{HNF}(K) = \begin{pmatrix} T_s \\ 0_{n,s} \end{pmatrix}$$

where T_s is an upper triangular matrix. In particular, the first n columns of $_nU$ give a $n \times n$ matrix β such that

$$\beta \cdot \widehat{V} \cdot \Gamma^T = 0 \mod \tau$$

ending up the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.3 The previous Theorem 4.2 gives also a method to check whether two geometric quotients giving ℚ-factorial complete toric varieties are actually isomorphic. In fact by recovering the fan matrices, this problem is turned into the problem of checking whether the fan matrices are equivalent, up to a permutation on columns, in the sense of [10, § 3], and if their respective fans are each other related by such a fan matrices equivalence. If only the latter fact is not true, then the two given toric geometric quotients are still birationally isomorphic (and isomorphic in codimension 1 [7, § 15.3]).

5 Examples

In this section, we are going to present concrete applications of Theorems 2.2, 3.2 and 4.2.

Example 5.1 The following example is the same given in [8, Ex. 1.3], which is here re-discussed to introduce the reader to the use of all the above illustrated techniques in the easier case of an already known example with $r = 1$.

Let us consider the 3-dimensional toric variety $X = X(\Sigma)$, $\Sigma = \mathrm{fan}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ and $\mathbf{v}_i$ is the ith column of the 3×4 fan matrix

$$V = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}. \qquad\qquad (15)$$

Let us follow the list from (i) to (vii) in Remark 3.3.
(i) A matrix $U_V \in \mathrm{GL}_4(\mathbb{Z})$ such that $\mathrm{HNF}(V^T) = U_V \cdot V^T$ is given by

$$U_V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

whose last row gives the weight matrix $Q = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ of X: then the universal 1-covering Y of X is the projective space $Y = \mathbb{P}^3$.

(ii) A switching matrix $U_Q \in \mathrm{GL}_4(\mathbb{Z})$ such that $\mathrm{HNF}(Q^T) = U_Q \cdot Q^T$ is given by

$$U_Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

encoding a basis $\{d_X(L)\}$ of the free part $F \cong \mathbb{Z}$ of $\mathrm{Cl}(X)$, given by the first row and meaning that $L = D_1$, and a fan matrix $\widehat{V}$ of the universal 1-covering $Y = \mathbb{P}^3$, given by the further rows, that is,

$$\widehat{V} = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

(iii) One can then immediately get a matrix β such that $\beta \cdot \widehat{V} = V$, that is, $\beta = \mathrm{diag}(1, 1, 5)$ and already in SNF.

(iv) Therefore, $s = 1$ and the last row of $\widehat{V}$ gives a generator of

$$\mathrm{Tors}(\mathrm{Cl}(X)) = \mathbb{Z}/5\mathbb{Z},$$

namely $d_X(T)$ with $T = D_3 - D_4$. Hence

$$\mathrm{Cl}(X) = \mathcal{L}\left(f_X(D_1), \tau_X(D_3 - D_4)\right).$$

(v) A matrix $W \in \mathrm{GL}_4(\mathbb{Z})$ such that $\mathrm{HNF}\left(({}^2V)^T\right) = W \cdot ({}^2V)^T$ is given by

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix}$$

giving

$$G := {}_1\widehat{V} \cdot ({}_2W)^T = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$U_G := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \ : \ U_G \cdot G^T = \mathrm{HNF}(G^T)$$

Therefore,

$$\Gamma = {}^1U_G \cdot {}_2W \mod 5 = \begin{pmatrix} [4]_5 & [3]_5 & [1]_5 & [0]_5 \end{pmatrix}.$$

(vi) Applying the procedure [10, § 1.2.3], one gets $B = \begin{pmatrix} 1 \end{pmatrix}$. This fact can also deduced by observing that $L = D_1$ is Cartier, meaning that $\mathrm{Cl}(Y) = \mathrm{Pic}(Y) = \mathrm{Pic}(X) = F$.

(vii) A basis of $\mathcal{C}_T(X) \subseteq \mathcal{W}_T(X)$ is then given by the rows of the matrix

$$C_X = \mathrm{diag}(1, 1, 1, 5) \cdot U_Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}$$

meaning that

$$\mathcal{C}_T(X) = \mathcal{L}\left(D_1, D_1 + D_3 - 2D_4, D_2 - 3D_3 + 2D_4, 5D_3 - 5D_4\right).$$

On the other hand, a basis of $\mathcal{C}_T(Y) \subseteq \mathcal{W}_T(Y)$ is given by the rows of

$$C_Y = \mathbf{I}_4 \cdot U_Q = U_Q \in \mathrm{GL}_n(\mathbb{Z})$$

giving $\mathcal{C}_T(Y) = \mathcal{W}_T(Y)$, as expected for $Y = \mathbb{P}^3$.

Since $|\mathcal{SF}(V)| = 1$, matrices Q and Γ suffice to give the geometric description of X as Cox geometric quotient. Namely, saying Z_Σ the exceptional subset of $\mathbb{C}^{n+r}$ determined by Σ, i.e., $Z_\Sigma = \{\mathbf{0}\} \subseteq \mathbb{C}^4$, the columns of the weight matrix Q describe the exponents of the action of $\mathrm{Hom}(F, \mathbb{C}^*) \cong \mathbb{C}^*$ on $\mathbb{C}^4 \backslash \{\mathbf{0}\}$ clearly giving $\mathbb{P}^3$, while the columns of the torsion matrix Γ describe the exponents of the action of $\mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*) \cong \mu_5$ on $Y = \mathbb{P}^3$ whose quotient gives X:

$$
\begin{array}{ccc}
\mu_5 \times \mathbb{P}^3 & \longrightarrow & \mathbb{P}^3 \\
(\varepsilon, [x_1 : \ldots : x_4]) & \mapsto & \left[\varepsilon^4 x_1 : \varepsilon^3 x_2 : \varepsilon x_3 : x_4 \right].
\end{array}
\tag{16}
$$

As M. Kasprzyk observes, this example has been firstly presented by M. Reid as the quotient of $\mathbb{P}^3$ by the action

$$
\begin{array}{ccc}
\mu_5 \times \mathbb{P}^3 & \longrightarrow & \mathbb{P}^3 \\
(\varepsilon, [x_1 : \ldots : x_4]) & \mapsto & \left[\varepsilon x_1 : \varepsilon^2 x_2 : \varepsilon^3 x_3 : \varepsilon^4 x_4 \right]
\end{array}
\tag{17}
$$

(see [9] Ex. (4.15)). Recalling Remark 4.3, to show that actions (16) and (17) give isomorphic quotients, one can go back obtaining the fan matrix of the geometric quotient (17) and check that, up to a permutation on columns, it is equivalent to the fan matrix V. Then, let us follow steps (1), (2) and (3) of Theorem 4.2. The data of the Reid's quotient (17) are the weight matrix $Q = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ and the torsion matrix $\Gamma_R := \begin{pmatrix} [1]_5 & [2]_5 & [3]_5 & [4]_5 \end{pmatrix}$. Let $C = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ be a matrix of representative integers for Γ_R, as in step (1) of Theorem 4.2. The the matrix K defined in step (2) of the same Theorem is given by

$$
K = \begin{pmatrix} \widehat{V} \cdot C^T \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -1 \\ 5 \end{pmatrix}
$$

Step (3) then gives

$$
\mathrm{HNF}(K) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = U \cdot K \quad \text{with} \quad U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}
$$

Therefore,

$$
\beta_R = (_3U)_{\{1,2,3\}} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 0 \end{pmatrix}
$$

giving the fan matrix

$$
V_R := \beta_R \cdot \widehat{V} = \begin{pmatrix} 1 & 4 & -11 & 6 \\ 0 & 1 & -2 & 1 \\ 2 & 3 & -7 & 2 \end{pmatrix}
$$

The latter F-matrix turns out to be equivalent to the fan matrix V in (15), up to a permutation on columns, since $V_R = R \cdot V \cdot S$ with $R \in \mathrm{GL}_3(\mathbb{Z})$ and $S \in \mathrm{GL}_4(\mathbb{Z})$ a permutation matrix, given by

$$R = \begin{pmatrix} 1 & -11 & -6 \\ 0 & -2 & -1 \\ 2 & -7 & -4 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Example 5.2 This is a less trivial example for which the procedure of Remark 3.3 turns out to be quite useful to get all the necessary information.

Let us consider the following matrix

$$V = \begin{pmatrix} 18 & -21 & -9 & 333 & -492 & 120 \\ -3 & 8 & 4 & -14 & 13 & -4 \\ -23 & 33 & 14 & -404 & 588 & -144 \\ -20 & 26 & 12 & -337 & 493 & -121 \end{pmatrix}. \tag{18}$$

(i) A matrix $U_V \in \mathrm{GL}_6(\mathbb{Z})$ such that $\mathrm{HNF}(V^T) = U_V \cdot V^T$ is given by

$$U_V = \begin{pmatrix} 20 & 18 & 11 & 24 & 16 & 0 \\ 5 & 4 & 4 & 6 & 4 & 0 \\ 20 & 19 & 9 & 24 & 16 & 0 \\ 12 & 10 & 9 & 15 & 10 & 0 \\ 2 & 4 & 1 & 5 & 4 & 3 \\ 1 & 1 & 3 & 2 & 3 & 7 \end{pmatrix}$$

whose last two rows give the W-matrix

$$Q = \begin{pmatrix} 2 & 4 & 1 & 5 & 4 & 3 \\ 1 & 1 & 3 & 2 & 3 & 7 \end{pmatrix}.$$

Therefore, V is a F-matrix [10, Prop. 3.12(2)]. Consider $X(\Sigma)$ with $\Sigma \in \mathcal{SF}(V)$. Then, X is a 4-dimensional $\mathbb{Q}$-factorial complete toric variety of rank 2. One can check that $|\mathcal{SF}(V)| = 3$, meaning that we have 3 choices for the fan Σ, i.e., for the toric variety $X(\Sigma)$.

(ii) A switching matrix $U_Q \in \mathrm{GL}_6(\mathbb{Z})$ such that $\mathrm{HNF}(Q^T) = U_Q \cdot Q^T$ is given by

$$U_Q = \begin{pmatrix} 5 & -2 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 11 & -5 & -2 & 0 & 0 & 0 \\ 4 & -3 & -1 & 1 & 0 & 0 \\ 7 & -4 & -2 & 0 & 1 & 0 \\ 15 & -7 & -5 & 0 & 0 & 1 \end{pmatrix}$$

encoding a basis $\{d_X(L_1), d_X(L_2)\}$ of the free part $F \cong \mathbb{Z}^2$ of $\mathrm{Cl}(X)$, given by the upper two rows and meaning that

$$L_1 = 5D_1 - 2D_2 - D_3, \quad L_2 = 2D_1 - D_2,$$

and a fan matrix $\widehat{V}$ of the universal 1-covering Y, given by the further rows, that is,

$$\widehat{V} = \begin{pmatrix} 11 & -5 & -2 & 0 & 0 & 0 \\ 4 & -3 & -1 & 1 & 0 & 0 \\ 7 & -4 & -2 & 0 & 1 & 0 \\ 15 & -7 & -5 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

(iii) To get a matrix β such that $\beta \cdot \widehat{V} = V$, we need the HNF of both $\widehat{V}$ and V:

$$\widehat{H} := \mathrm{HNF}\left(\widehat{V}\right) = \begin{pmatrix} 1 & 0 & 0 & 16 & -25 & 6 \\ 0 & 1 & 0 & 7 & -12 & 3 \\ 0 & 0 & 1 & 4 & -6 & 1 \\ 0 & 0 & 0 & 19 & -29 & 7 \end{pmatrix}$$

$$\widehat{U} = \begin{pmatrix} 2 & 16 & -25 & 6 \\ 1 & 7 & -12 & 3 \\ 1 & 4 & -6 & 1 \\ 2 & 19 & -29 & 7 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}) : \quad \widehat{U} \cdot \widehat{V} = \widehat{H}.$$

$$H := \mathrm{HNF}(V) = \begin{pmatrix} 1 & 0 & 2 & 100 & -153 & 36 \\ 0 & 1 & 2 & 53 & -82 & 19 \\ 0 & 0 & 3 & 69 & -105 & 24 \\ 0 & 0 & 0 & 285 & -435 & 105 \end{pmatrix}$$

$$U = \begin{pmatrix} -2 & 3 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 3 & -1 & -1 & 4 \\ -40 & 34 & -14 & -25 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}) : \quad U \cdot V = H.$$

The given recursive relations (10) define

$$\beta_H = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 15 \end{pmatrix} \Rightarrow \beta = U^{-1} \cdot \beta_H \cdot \widehat{U} = \begin{pmatrix} 30 & 333 & -492 & 120 \\ 2 & -14 & 13 & -4 \\ -33 & -404 & 588 & -144 \\ -28 & -337 & 493 & -121 \end{pmatrix}.$$

(iv) Therefore, $\Delta = \mathrm{SNF}(\beta)$ and $\mu, \nu \in \mathrm{GL}_4(\mathbb{Z}) : \Delta = \mu \cdot \beta \cdot \nu$ are given by

$$\Delta = \mathrm{diag}(1, 1, 3, 15)$$

$$\mu = \begin{pmatrix} 1410 & -1138 & 551 & 780 \\ 1140 & -916 & 420 & 661 \\ -1623 & 1304 & -598 & -941 \\ 8425 & -6769 & 3104 & 4885 \end{pmatrix}$$

$$\nu = \begin{pmatrix} 1 & 58 & 2224 & 2022 \\ 0 & 1 & 27 & 24 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & -78 & -71 \end{pmatrix}$$

Then, $s = 2$ and the last two rows of

$$\widehat{V}' = \nu^{-1}\widehat{V} = \begin{pmatrix} 521 & -251 & -168 & -2 & 14 & 28 \\ 388 & -222 & -112 & 7 & 45 & 3 \\ -184 & 105 & 53 & -2 & -23 & -1 \\ 191 & -109 & -55 & 2 & 24 & 1 \end{pmatrix}$$

give the generators of

$$\mathrm{Tors}(\mathrm{Cl}(X)) = \mathbb{Z}/90\mathbb{Z},$$

namely $d_X(T_1)$ and $d_X(T_2)$ with

$$T_1 = -184D_1 + 105D_2 + 53D_3 - 2D_4 - 23D_5 - D_6$$
$$T_2 = 191D_1 - 109D_2 - 55D_3 + 2D_4 + 24D_5 + D_6$$

Hence $\mathrm{Cl}(X) = \mathcal{L}\left(f_X(L_1), f_X(L_2), \tau_X(T_1), \tau_X(T_2)\right)$.

(v) Set

$$V' = \mu \cdot V = \begin{pmatrix} 521 & -251 & -168 & -2 & 14 & 28 \\ 388 & -222 & -112 & 7 & 45 & 3 \\ -552 & 315 & 159 & -6 & -69 & -3 \\ 2865 & -1635 & -825 & 30 & 360 & 15 \end{pmatrix}$$

A matrix $W \in \mathrm{GL}_6(\mathbb{Z})$ such that $\mathrm{HNF}((^2V')^T) = W \cdot ((^2V')^T)$ is given by

$$W = \begin{pmatrix} 5343 & 5282 & 8576 & 8576 & 0 & 0 \\ 2626 & 2596 & 4215 & 4215 & 0 & 0 \\ 11385 & 11255 & 18274 & 18274 & 0 & 0 \\ 3689 & 3647 & 5921 & 5922 & 0 & 0 \\ 573 & 567 & 919 & 919 & 1 & 0 \\ 1908 & 1886 & 3063 & 3063 & 0 & 1 \end{pmatrix}$$

then

$$G = {}_2\widehat{V}' \cdot ({}_4W)^T = \begin{pmatrix} 18909 & 6128 & 949 & 3170 \\ -20782 & -6735 & -1043 & -3484 \end{pmatrix}$$

A matrix $U_G \in \mathrm{GL}_4(\mathbb{Z})$ such that $\mathrm{HNF}(G^T) = U_G \cdot G^T$ is given by

$$U_G = \begin{pmatrix} 351 & -1084 & 6 & 0 \\ 315 & -974 & 13 & 0 \\ -11 & 31 & 19 & 0 \\ -8 & 22 & 14 & 1 \end{pmatrix}$$

hence giving

$$\begin{aligned}
\Gamma = {}^2U_G \cdot {}_4W \quad &\mathrm{mod}\ \tau \\
= \begin{pmatrix} 697 & 559 & 1324 & 240 & 6 & 0 \\ 638 & 518 & 1203 & 229 & 13 & 0 \end{pmatrix} &\ \mathrm{mod} \begin{pmatrix} 3 \\ 15 \end{pmatrix} \\
= \begin{pmatrix} [1]_3 & [1]_3 & [1]_3 & [0]_3 & [0]_3 & [0]_3 \\ [8]_{15} & [8]_{15} & [3]_{15} & [4]_{15} & [13]_{15} & [0]_{15} \end{pmatrix} &
\end{aligned}$$

vi. Applying the procedure [10, § 1.2.3], one gets the following three possibilities

$$B_1 = \begin{pmatrix} 5909475 & 0 \\ -238040 & 165 \end{pmatrix},\ B_2 = \begin{pmatrix} 5805800 & 0 \\ -4648596 & 1 \end{pmatrix},\ B_3 = \begin{pmatrix} 5805800 & 0 \\ -217580 & 55 \end{pmatrix}$$

depending on the choice of the fan $\Sigma \in \mathcal{SF}(V)$.

vii. A basis of $\mathcal{C}_T(X) \subseteq \mathcal{W}_T(X)$ is then given, depending on the choice of the fan $\Sigma \in \mathcal{SF}(V)$, by the rows of the matrices

$$\forall\, 1 \le i \le 3 \quad C_{X,i} = \begin{pmatrix} B_i & \mathbf{0}_{2,4} \\ \mathbf{0}_{4,2} & \beta \end{pmatrix} \cdot U_Q$$

which are

$$
C_{X,1} = \begin{pmatrix}
29547375 & -11818950 & -5909475 & 0 & 0 & 0 \\
-1189870 & 475915 & 238040 & 0 & 0 & 0 \\
18 & -21 & -9 & 333 & -492 & 120 \\
-3 & 8 & 4 & -14 & 13 & -4 \\
-23 & 33 & 14 & -404 & 588 & -144 \\
-20 & 26 & 12 & -337 & 493 & -121
\end{pmatrix}
$$

$$
C_{X,2} = \begin{pmatrix}
29029000 & -11611600 & -5805800 & 0 & 0 & 0 \\
-23242978 & 9297191 & 4648596 & 0 & 0 & 0 \\
18 & -21 & -9 & 333 & -492 & 120 \\
-3 & 8 & 4 & -14 & 13 & -4 \\
-23 & 33 & 14 & -404 & 588 & -144 \\
-20 & 26 & 12 & -337 & 493 & -121
\end{pmatrix}
$$

$$
C_{X,3} = \begin{pmatrix}
29029000 & -11611600 & -5805800 & 0 & 0 & 0 \\
-1087790 & 435105 & 217580 & 0 & 0 & 0 \\
18 & -21 & -9 & 333 & -492 & 120 \\
-3 & 8 & 4 & -14 & 13 & -4 \\
-23 & 33 & 14 & -404 & 588 & -144 \\
-20 & 26 & 12 & -337 & 493 & -121
\end{pmatrix}
$$

On the other hand, a basis of $\mathcal{C}_T(Y) \subseteq \mathcal{W}_T(Y)$ is given, depending on the choice of the fan $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$, by the rows of the matrices

$$
\forall\, 1 \leq i \leq 3 \quad C_{Y,i} = \begin{pmatrix} B_i & \mathbf{0}_{2,4} \\ \mathbf{0}_{4,2} & \mathbf{I}_4 \end{pmatrix} \cdot U_Q
$$

which are

$$
C_{Y,1} = \begin{pmatrix}
29547375 & -11818950 & -5909475 & 0 & 0 & 0 \\
-1189870 & 475915 & 238040 & 0 & 0 & 0 \\
11 & -5 & -2 & 0 & 0 & 0 \\
4 & -3 & -1 & 1 & 0 & 0 \\
7 & -4 & -2 & 0 & 1 & 0 \\
15 & -7 & -5 & 0 & 0 & 1
\end{pmatrix}
$$

$$
C_{Y,2} = \begin{pmatrix}
29029000 & -11611600 & -5805800 & 0 & 0 & 0 \\
-23242978 & 9297191 & 4648596 & 0 & 0 & 0 \\
11 & -5 & -2 & 0 & 0 & 0 \\
4 & -3 & -1 & 1 & 0 & 0 \\
7 & -4 & -2 & 0 & 1 & 0 \\
15 & -7 & -5 & 0 & 0 & 1
\end{pmatrix}
$$

$$
C_{Y,3} = \begin{pmatrix}
29029000 & -11611600 & -5805800 & 0 & 0 & 0 \\
-1087790 & 435105 & 217580 & 0 & 0 & 0 \\
11 & -5 & -2 & 0 & 0 & 0 \\
4 & -3 & -1 & 1 & 0 & 0 \\
7 & -4 & -2 & 0 & 1 & 0 \\
15 & -7 & -5 & 0 & 0 & 1
\end{pmatrix}
$$

Finally, matrices Q and Γ give the geometric description of $X(\Sigma)$ as a Cox geometric quotient. Namely, saying $Z = Z_\Sigma = Z_{\widehat{\Sigma}}$ the exceptional subset of $\mathbb{C}^{n+r}$ determined by $\Sigma \in \mathcal{SF}(V)$ (or equivalently by $\widehat{\Sigma} \in \mathcal{SF}(\widehat{V})$), whose explicit determination is left to

the reader, the columns of the weight matrix Q describe the exponents of the action of $\mathrm{Hom}(F, \mathbb{C}^*) \cong (\mathbb{C}^*)^2$ on $\mathbb{C}^6 \backslash Z$, whose quotient gives Y; on the other hand the columns of the torsion matrix Γ describe the exponents of the action of $\mathrm{Hom}(\mathrm{Tors}(\mathrm{Cl}(X)), \mathbb{C}^*) \cong \mu_3 \oplus \mu_{15}$ on Y, whose quotient gives X. Putting all together one gets the following action of $G(X) = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^2 \oplus \mu_3 \oplus \mu_{15}$

$$g : \left((\mathbb{C}^*)^2 \oplus \mu_3 \oplus \mu_{15} \right) \times \left(\mathbb{C}^6 \backslash Z \right) \longrightarrow \mathbb{C}^6 \backslash Z$$

defined by setting

$$g\left(((t_1, t_2), \varepsilon, \eta), (x_1, \ldots : x_6) \right) :=$$
$$\left(t_1^2 t_2 \varepsilon \eta^8 \, x_1, t_1^4 t_2 \varepsilon \eta^8 \, x_2, t_1 t_2^3 \varepsilon \eta^3 \, x_3, t_1^5 t_2^2 \eta^4 \, x_4, t_1^4 t_2^3 \eta^{13} \, x_5, t_1^3 t_2^7 \, x_6 \right) \quad (20)$$

Example 5.3 This example is devoted to give an application of Theorem 4.2 by reversing the previous Example 5.2.

Let X be the $\mathbb{Q}$-factorial complete toric variety defined by the toric geometric quotient

$$X = \left(\mathbb{C}^6 \backslash Z \right) \Big/_g \left((\mathbb{C}^*)^2 \oplus \mu_3 \oplus \mu_{15} \right)$$

where g is the action defined in (20). Our aim is recovering a fan matrix of X.

From the exponents of the action g, we get the two matrices:

$$\text{the } W\text{-matrix } Q = \begin{pmatrix} 2 & 4 & 1 & 5 & 4 & 3 \\ 1 & 1 & 3 & 2 & 3 & 7 \end{pmatrix}$$

$$\text{the torsion matrix } \Gamma = \begin{pmatrix} [1]_3 & [1]_3 & [1]_3 & [0]_3 & [0]_3 & [0]_3 \\ [8]_{15} & [8]_{15} & [3]_{15} & [4]_{15} & [13]_{15} & [0]_{15} \end{pmatrix}$$

By Gale duality, one gets immediately the CF-matrix $\widehat{V} = \mathcal{G}(Q)$ given in (19), which is a fan matrix of the PWS Y giving the universal 1-covering of X. Hence V is obtained by recovering a matrix β such that $V = \beta \cdot \widehat{V}$, as in (2). At this purpose, let us apply Theorem 4.2. Let

$$C = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 8 & 8 & 3 & 4 & 13 & 0 \end{pmatrix}$$

be a matrix of representative integers of Γ and construct the matrix

$$K := \begin{pmatrix} \widehat{V} \cdot C^T \\ 3 & 0 \\ 0 & 15 \end{pmatrix} = \begin{pmatrix} 4 & 42 \\ 0 & 9 \\ 1 & 31 \\ 3 & 49 \\ 3 & 0 \\ 0 & 15 \end{pmatrix}$$

Then,

$$\mathrm{HNF}(K) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = U \cdot K \text{ with } U = \begin{pmatrix} -5 & -49 & 21 & 0 & 0 & 0 \\ -1 & -9 & 4 & 0 & 0 & 0 \\ -9 & -82 & 36 & 0 & 0 & 0 \\ -8 & -68 & 29 & 1 & 0 & 0 \\ -3 & -17 & 9 & 0 & 1 & 0 \\ -3 & -29 & 12 & 0 & 0 & 1 \end{pmatrix}$$

giving

$$\beta = (_4U)_{\{1,2,3,4\}} = \begin{pmatrix} -9 & -82 & 36 & 0 \\ -8 & -68 & 29 & 1 \\ -3 & -17 & 9 & 0 \\ -3 & -29 & 12 & 0 \end{pmatrix}$$

Therefore, a fan matrix of X is given by

$$V'' := \beta \cdot \widehat{V} = \begin{pmatrix} -175 & 147 & 28 & -82 & 36 & 0 \\ -142 & 121 & 21 & -68 & 29 & 1 \\ -38 & 30 & 5 & -17 & 9 & 0 \\ -65 & 54 & 11 & -29 & 12 & 0 \end{pmatrix}$$

As a final remark, let us notice that the so-obtained matrix V'' is actually equivalent to the matrix V in (18): in fact $R \cdot V = V''$ for

$$R = \begin{pmatrix} -646 & 512 & -203 & -416 \\ -533 & 422 & -166 & -345 \\ -143 & 113 & -45 & -92 \\ -237 & 188 & -75 & -152 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}).$$

References

1. Batyrev, V.: On the classification of smooth projective toric varieties. Tohoku Math. J. **43**, 569–585 (1991)
2. Batyrev, V., Cox, D.: On the Hodge structure of projective hypersurfaces in toric varieties. Duke Math. J. **75**, 293–338 (1994)
3. Buczyńska, W.: Fake weighted projective spaces (2008). arXiv:0805.1211
4. Cohen, H.: A course in computational algebraic number theory graduate text in mathematics, vol. 138. Springer, Berlin (1993)
5. Conrads, H.: Weighted projective spaces and reflexive simplices. Manuscr. Math. **107**, 215–227 (2002)
6. Cox, D.A.: The homogeneous coordinate ring of a toric variety. J. Algebr. Geom. **4**, 17–50 (1995). arXiv:alg-geom/9210008
7. Cox, D.A., Little, J.B., Schenck, H.K.: Toric varieties graduate studies in mathematics, vol. 124. American Mathematical Society, Providence (2011)
8. Kasprzyk, A.M.: Bounds on fake weighted projective space. Kodai Math. J. **32**, 197–208 (2009)
9. Reid, M.: Young person's guide to canonical singularities. In: Algebraic geometry, Bowdoin, 1985, Brunswick, Maine, 1985, Proceedings on Symposia in Pure Mathematics, vol. 46, pp. 345–414. American Mathematical Society, Providence (1987)
10. Rossi, M., Terracini, L.: ℤ-linear Gale duality and poly weighted spaces (PWS). Linear Algebra Appl. **495**, 256–288 (2016). doi:10.1016/j.laa.2016.01.039. arXiv:1501.05244