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The clamped plate in Gauss space

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Abstract In this paper, we study the analogue in Gauss space of Lord Rayleigh's conjecture for the clamped plate. We show that the first eigenvalue of the bi-Hermite operator in a bounded domain is bounded below by a constant C_V times the corresponding eigenvalue of a half-space with the same Gaussian measure V. Similar results are established on unbounded domains. We use rearrangement methods similar to Talenti's for the Euclidean clamped plate. We obtain our constant C_V following the Euclidean approach of Ashbaugh and Benguria, and we find a numerical bound $C_V \geq 0.91$ by solving an associated minimization problem in terms of parabolic cylinder functions.

Keywords Symmetrization · Comparison results · Clamped plate · Gauss space · Parabolic cylinder functions

Mathematics Subject Classification Primary 35P15; Secondary 35J40 · 73K10

1 Introduction

In 1877, Lord Rayleigh conjectured that among all clamped plates of a given area, the lowest frequency of vibration is minimized by a disk [28]. It took over one hundred years for this conjecture to be proved in dimensions $n = 2$ and 3. Partial results were first obtained by Szegő [31] and Talenti [32]. The proof for dimension $n = 2$ was attained by Nadirashvili [25,26] based on Talenti's rearrangement work, and Ashbaugh and Benguria [4] later extended the

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result to dimension $n = 3$. The problem remains open for dimensions $n \geq 4$, with a partial result by Ashbaugh and Laugesen [5]. For a survey of Euclidean plate problems, see [6].

For a clamped plate whose shape is a bounded region Ω, the frequencies and modes of vibration are solutions to the bi-Laplace eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

In this paper, we consider the Gauss space analogue of the clamped plate. In place of the bi-Laplace operator Δ^2, we consider the bi-Hermite operator $\mathscr{L}^2$. If g is the standard normal Gaussian over $\mathbb{R}^n$, we may write the Hermite operator as $\mathscr{L} u = \frac{1}{g}\nabla \cdot (g\nabla u)$. In addition, Gaussian measure $\mathrm{d}\gamma = g\,\mathrm{d}x$ replaces Lebesgue measure. See Sect. 2 for precise definitions. The eigenvalue problem we consider can then be written as

$$\begin{cases} \mathscr{L}^2 u = \Lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the domain Ω is possibly unbounded. We prove a lower bound on the first eigenvalue $\Lambda_1(\Omega)$ in terms of the first eigenvalue of a half-space of the same measure, namely:

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a C^∞ domain with Gaussian measure $V = \gamma(\Omega)$ satisfying $0 < V < 1$, and let $\Omega^\#$ be a half-space of the same Gaussian measure.*

(i) *If Ω is bounded, then there exists a constant $0 < C_V \leq 1$ depending only on the Gaussian measure V of Ω such that*

$$\Lambda_1(\Omega) \geq C_V \Lambda_1(\Omega^\#). \tag{2}$$

(ii) *If Ω is unbounded, then in terms of constants $C_{V'}$ from part (i),*

$$\Lambda_1(\Omega) \geq \left(\liminf_{V' \to V} C_{V'} \right) \Lambda_1(\Omega^\#),$$

where the lim inf *is taken over $V' < V$.*

Our proof of Theorem 1 follows the approach of Talenti [32] and Ashbaugh and Benguria [4]. Our value for C_V is found using symmetrization methods, calculations involving parabolic cylinder functions, and computations of eigenvalues of half-spaces. We plot our constant C_V numerically in Fig. 1 below. Observe that $C_V \geq 0.91$ for all Gaussian volumes $V \in (0, 1)$, and $C_V \geq 0.98$ for $V \geq 0.4$. We conjecture that inequality (2) holds with $C_V = 1$ for all Gaussian volumes.

Although balls are the usual extremal domains for Euclidean isoperimetric inequalities, the half-space often (though not always) plays this role in Gauss space. Indeed, the perimeter-minimizing domains in Gauss space are half-spaces, as was proved by Borell, Erhard, and Ledoux [13,18,23]. More recently, Bette, Chiacchio, and Ferone [10] proved the Gauss space analogue of the Faber–Krahn inequality, showing that minimizing domains for the first Dirichlet Hermite eigenvalue are likewise half-spaces.

Other closely related results have recently made spectral theory in Gauss space a lively area of research. Bounds on the first nonzero Neumann Hermitian eigenvalue $\mu_1(\Omega)$ include a universal lower bound $\mu_1 \geq 1$, proved by Brandolini, Chiacchio, Henrot, and Trombetti [14], and a sharp lower bound when the corresponding eigenfunction is odd with respect to

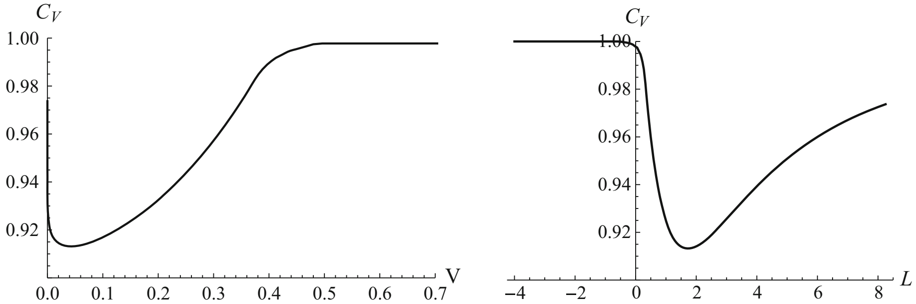

Fig. 1 *Left*, a graph of our C_V plotted against volume. *Right*, a graph of our C_V plotted against L, where $H_L = \{x \in \mathbb{R}^n \mid x_1 > L\}$ is the half-space with volume V. Note the axes cross at the point (0, 0.9) rather than the origin

a fixed axis, proved by Brandolini, Chiacchio, and Trombetti [15]. As a partial Gauss space version of the Kornhauser–Stäkgold inequality, Chiacchio and di Blasio [16] proved the ball maximizes $\mu_1(\Omega)$ for regions symmetric about the origin. Benguria and Linde [8] discussed the relationship between the Dirichlet Shrödinger operator with certain potentials and the Payne–Pólya–Weinberger inequality in both Gauss and inverse-Gauss space.

We finally remark that the rearrangement techniques used in this paper are similar to those employed by Betta, Brock, Mercaldo, and Posteraro in [11] and di Blasio, Feo, and Posteraro in [17], who proved comparison results involving the Hermite operator.

Our paper is organized as follows: Sects. 2, 3, and 4 lay out the foundation of the bi-Hermite Dirichlet problem, introducing notation, proving properties of the spectrum and eigenfunctions, and discussing more general plate problems in Gauss space. Section 5 contains our symmetrization argument, which allows us to bound $\Lambda_1(\Omega)$ below by the related $J_{a,b}$ minimization problem (notation to be defined). Sections 6, 7, and 8 are focused on explicit solutions of the $J_{a,b}$ minimization problem and half-space computations in terms of parabolic cylinder functions. Finally, we prove Theorem 1 in Sect. 9 and conclude with a discussion of our numerical work to obtain C_V.

2 Preliminaries and notation

In this section, we collect notation and definitions used throughout the paper.
First note that we only consider domains $\Omega \subseteq \mathbb{R}^n$ that are C^∞ and have measure $0 < \gamma(\Omega) < 1$.

2.1 The Hermite operator and Gaussian Sobolev spaces

We begin by noting that the Hermite operator is defined as

$$\mathscr{L} u = \frac{1}{g}\nabla \cdot (g\nabla u) \qquad \text{where} \qquad g = c_n\, \mathrm{e}^{-|x|^2/2},$$

with the normalizing constant c_n chosen so that $\int_{\mathbb{R}^n} g\, \mathrm{d}x = 1$.

Direct computation allows us to write the Hermite and bi-Hermite operators in terms of the more usual gradients and Laplacians, along with $x = (x_1, x_2, \ldots, x_n)$, as follows:

$$\mathscr{L} u = \Delta u - x \cdot \nabla u,$$
$$\mathscr{L}^2 = \Delta^2 u - 2x \cdot \nabla(\Delta u) - 2\Delta u + x^\top (D^2 u)x + x \cdot \nabla u.$$

The Gauss measure γ with

$$\mathrm{d}\gamma = g\,\mathrm{d}x$$

is the absolutely continuous measure on $\mathbb{R}^n$ with density g given above.

Given a domain $\Omega \subseteq \mathbb{R}^n$, we define the $L^p(\Omega, \gamma)$ norm in the expected manner:

$$\|u\|_{L^p(\Omega,\gamma)} = \left(\int_\Omega |u|^p\,\mathrm{d}\gamma\right)^{1/p}.$$

The space $L^p(\Omega, \gamma)$ consists of all measurable functions u with finite norm. Sobolev spaces are defined analogously to their Euclidean counterparts. For example, $H^1(\Omega, \gamma)$ denotes the collection of functions in $L^2(\Omega, \gamma)$ with weak first-order partials that also belong to $L^2(\Omega, \gamma)$. The corresponding norm on such functions is

$$\|u\|_{H^1(\Omega,\gamma)} = \left(\int_\Omega |\nabla u|^2 + |u|^2\,\mathrm{d}\gamma\right)^{1/2}.$$

Write $C_c^\infty(\Omega)$ for the collection of smooth functions compactly supported in Ω. Then $H_0^1(\Omega, \gamma)$ denotes the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega, \gamma)$ with respect to the above norm. Similarly, the Sobolev space $H^2(\Omega, \gamma)$ is defined as the collection of functions in $L^2(\Omega, \gamma)$ with weak partials up to second order, all of which belong to $L^2(\Omega, \gamma)$. The norm for such functions is

$$\|u\|_{H^2(\Omega,\gamma)} = \left(\int_\Omega |D^2u|^2 + |\nabla u|^2 + |u|^2\,\mathrm{d}\gamma\right)^{1/2}.$$

The space $H_0^2(\Omega, \gamma)$ is then the closure of $C_c^\infty(\Omega)$ in $H^2(\Omega, \gamma)$ with respect to the above norm.

2.2 Symmetrization

Given a real number L, write H_L for the half-space

$$H_L = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > L\}.$$

Let $\Phi : \mathbb{R} \to (0, 1)$ denote the function where $\Phi(L)$ is the Gaussian measure of H_L. Then

$$\Phi(L) = \gamma(H_L) = \frac{c_n}{c_{n-1}} \int_L^\infty \mathrm{e}^{-s^2/2}\,\mathrm{d}s.$$

Note that the fraction c_n/c_{n-1} is independent of dimension n.

For a domain $\Omega \subseteq \mathbb{R}^n$, we shall write $\Omega^\#$ for the half-space

$$\Omega^\# = H_L,$$

where L is chosen so that $\gamma(H_L) = \gamma(\Omega)$. If Per denotes the Gaussian perimeter of a set, then for sufficiently regular sets $\Omega \subseteq \mathbb{R}^n$, we have

$$\mathrm{Per}(\Omega) = \int_{\partial\Omega} g\,\mathrm{d}\mathcal{H}^{n-1}.$$

The isoperimetric inequality in Gauss space [13,18,23] then states

$$\mathrm{Per}(\Omega) \geq \mathrm{Per}(\Omega^\#).$$

Given $u \in L^1(\Omega, \gamma)$, we write u^* for the decreasing rearrangement of u, defined on $[0, \gamma(\Omega)]$ by

$$u^*(t) = \begin{cases} \operatorname*{ess\,sup}_{\Omega} u & \text{if } t = 0, \\ \inf\{s \mid \gamma(\{x \in \Omega \mid s < u(x)\}) \le t\} & \text{if } 0 < t < \gamma(\Omega), \\ \operatorname*{ess\,inf}_{\Omega} u & \text{if } t = \gamma(\Omega). \end{cases}$$

We define the Gauss symmetrization of u as the function $u^\# \in L^1(\Omega^\#, \gamma)$ given by

$$u^\#(x) = u^*(\Phi(x_1)),$$

where x_1 is the first coordinate of x. Note that $u^\#$ is increasing in x_1 and that the upper level sets $\{u^\# > t\}$ are half-spaces. Further information on rearrangements can be found in [21,22,24].

3 Existence of the spectrum and regularity of solutions

The goal of this section is to establish properties of the eigenvalues and eigenfunctions for the problem

$$\begin{cases} \mathscr{L}^2 u = \Lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where we recall $\Omega \subseteq \mathbb{R}^n$ is a C^∞ domain with $0 < \gamma(\Omega) < 1$, and $\mathscr{L}^2$ is the bi-Hermite operator. In particular, we show that the above PDE problem admits a sequence of eigenvalues $\Lambda_1 \le \Lambda_2 \le \cdots$ whose corresponding eigenfunctions are smooth up to the boundary $\partial\Omega$ (Proposition 5). These results essentially follow from the coercivity and continuity of the bilinear form

$$a(u, v) = \int_\Omega (\mathscr{L} u)(\mathscr{L} v)\, \mathrm{d}\gamma, \qquad u, v \in H_0^2(\Omega, \gamma). \tag{4}$$

Our choice of form parallels the usual form for the Euclidean clamped plate problem, but this choice is a simplification of a more general characterization of plates and cannot be used for other boundary conditions. We discuss this issue and more general Gauss space plate problems in Sect. 4.

It is not immediately clear that the above form is well defined, i.e., that the integral converges for all u and v. It turns out that convergence of $a(u, v)$ follows from, and is closely linked with, the logarithmic Sobolev inequality of Gross [20]. Related results and generalizations may be found in the work of Adams [2] and Feissner [19]; we need only the following, narrower result.

Proposition 2 *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and say $u \in H_0^1(\Omega, \gamma)$. Then*

$$\int_\Omega |u|^2 \log |u|\, d\gamma \le \int_\Omega |\nabla u|^2\, d\gamma + \|u\|^2_{L^2(\Omega,\gamma)} \log \|u\|_{L^2(\Omega,\gamma)}.$$

Proof Let $u \in H_0^1(\Omega, \gamma)$ and choose a sequence of test functions $\phi_k \in C_c^\infty(\Omega)$ where $\phi_k \to u$ in the H^1 norm. Then according to the classical logarithmic Sobolev inequality due to Gross [20],

$$\int_\Omega |\phi_k|^2 \log |\phi_k|\, \mathrm{d}\gamma \le \int_\Omega |\nabla \phi_k|^2\, \mathrm{d}\gamma + \|\phi_k\|^2_{L^2(\Omega,\gamma)} \log \|\phi_k\|_{L^2(\Omega,\gamma)}.$$

By passing to a subsequence, we may assume the ϕ_k converge pointwise γ−a.e. to u. Since the function $x^2 \log x$ is bounded below on $[0, \infty)$, we may apply Fatou's lemma to deduce

$$\begin{aligned}\int_\Omega |u|^2 \log |u| \, \mathrm{d}\gamma &= \int_\Omega \liminf_{k\to\infty} |\phi_k|^2 \log |\phi_k| \, \mathrm{d}\gamma \\ &\leq \liminf_{k\to\infty} \int_\Omega |\phi_k|^2 \log |\phi_k| \, \mathrm{d}\gamma \\ &\leq \liminf_{k\to\infty} \left(\int_\Omega |\nabla \phi_k|^2 \, \mathrm{d}\gamma + \|\phi_k\|^2_{L^2(\Omega,\gamma)} \log \|\phi_k\|_{L^2(\Omega,\gamma)} \right) \\ &= \int_\Omega |\nabla u|^2 \, \mathrm{d}\gamma + \|u\|^2_{L^2(\Omega,\gamma)} \log \|u\|_{L^2(\Omega,\gamma)}.\end{aligned}$$

□

The following result is an immediate consequence of the logarithmic Sobolev inequality.

Proposition 3 *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and say $u \in H^1_0(\Omega, \gamma)$. Then $x_i u \in L^2(\Omega, \gamma)$ for $1 \leq i \leq n$.*

Proof Define sets E and F by

$$E = \left\{ x \in \Omega \;\middle|\; \log |u(x)| \geq x_i^2/8 \right\},$$
$$F = \left\{ x \in \Omega \;\middle|\; \log |u(x)| < x_i^2/8 \right\}.$$

Then

$$\frac{1}{8} \int_\Omega x_i^2 u^2 \, \mathrm{d}\gamma \leq \int_E u^2 \log |u| \, \mathrm{d}\gamma + \frac{1}{8} \int_F x_i^2 \mathrm{e}^{x_i^2/4} \, \mathrm{d}\gamma.$$

In the inequality above, the first integral converges by Proposition 2. The second integral is bounded by the convergent integral $\int_{\mathbb{R}^n} |x|^2 \mathrm{e}^{-|x|^2/4} \, \mathrm{d}x$. □

We next deduce the following corollary, which together with Cauchy–Schwarz gives convergence of our form $a(\cdot, \cdot)$ on $H^2_0(\Omega, \gamma)$.

Corollary 1 *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and say $u \in H^2_0(\Omega, \gamma)$. Then $\mathscr{L} u \in L^2(\Omega, \gamma)$.*

Proof Observe that $\mathscr{L} u = \Delta u - x \cdot \nabla u$. Now, $\Delta u \in L^2(\Omega, \gamma)$ by assumption. By Proposition 3, $x_i u_{x_i} \in L^2(\Omega, \gamma)$ for each $1 \leq i \leq n$, since $u_{x_i} \in H^1_0(\Omega, \gamma)$. We deduce $x \cdot \nabla u \in L^2(\Omega, \gamma)$, which completes our proof. □

We next collect several related results used in the remainder of the paper.

Proposition 4 *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and suppose $u, u_k \in H^1_0(\Omega, \gamma)$ with $u_k \to u$ in the H^1 norm. Then for each $1 \leq i \leq n$, there exists a subsequence of $x_i u_k$ that converges to $x_i u$ in $L^2(\Omega, \gamma)$.*

Proof First, pass to a subsequence with $u_k \to u$ pointwise γ−a.e. Fix an index i and define the sets

$$E_k = \left\{ x \in \Omega \;\middle|\; \log |u(x) - u_k(x)| \geq x_i^2/8 \right\},$$
$$F_k = \left\{ x \in \Omega \;\middle|\; \log |u(x) - u_k(x)| < x_i^2/8 \right\}.$$

Then as before, we have

$$\frac{1}{8}\int_{\Omega} x_i^2(u-u_k)^2\,\mathrm{d}\gamma \le \int_{E_k}(u-u_k)^2\log|u-u_k|\mathrm{d}\gamma + \frac{1}{8}\int_{F_k} x_i^2|u-u_k|\mathrm{e}^{x_i^2/8}\,\mathrm{d}\gamma.$$

Now observe that

$$\int_{E_k}(u-u_k)^2\log|u-u_k|\,\mathrm{d}\gamma = \int_{\Omega}(u-u_k)^2\log|u-u_k|\,\mathrm{d}\gamma - \int_{F_k}(u-u_k)^2\log|u-u_k|\,\mathrm{d}\gamma.$$

In the limit as $k\to\infty$, the integral over Ω vanishes by Proposition 2. By Egoroff's Theorem,

$$\limsup_{k\to\infty}\left(-\int_{F_k}(u-u_k)^2\log|u-u_k|\,\mathrm{d}\gamma\right) < \epsilon$$

for every $\epsilon > 0$. By Cauchy–Schwarz,

$$\int_{F_k} x_i^2|u-u_k|\mathrm{e}^{x_i^2/8}\,\mathrm{d}\gamma \le \left(\int_{\Omega} x_i^4\mathrm{e}^{x_i^2/4}\,\mathrm{d}\gamma\right)^{1/2}\left(\int_{\Omega}|u-u_k|^2\,\mathrm{d}\gamma\right)^{1/2},$$

which vanishes as $k\to\infty$ by assumption. We deduce $x_i u_k \to x_i u$ in $L^2(\Omega,\gamma)$. □

Corollary 2 *Let $\Omega\subseteq\mathbb{R}^n$ be a domain and suppose $u, u_k \in H_0^2(\Omega,\gamma)$ with $u_k\to u$ in the H^2 norm. Then there is a subsequence of $\mathscr{L}u_k$ that converges to $\mathscr{L}u$ in $L^2(\Omega,\gamma)$.*

Proof Note $\Delta u_k \to \Delta u$ in $L^2(\Omega,\gamma)$ by assumption. Since the partials of u_k belong to $H_0^1(\Omega,\gamma)$, Proposition 4 allows us to pass to a subsequence for which $x\cdot\nabla u_k \to x\cdot\nabla u$ in $L^2(\Omega,\gamma)$. □

We may finally address the spectrum of the bi-Hermite eigenvalue problem.

Proposition 5 *Let $\Omega\subseteq\mathbb{R}^n$ be a C^∞ domain with $\gamma(\Omega)<1$. The spectrum of the operator associated with the form*

$$a(u,v) = \int_{\Omega}(\mathscr{L}u)(\mathscr{L}v)\,d\gamma, \qquad u,v\in H_0^2(\Omega,\gamma)$$

consists entirely of isolated eigenvalues of finite multiplicity satisfying

$$\Lambda_1\le\Lambda_2\le\cdots\le\Lambda_n\le\cdots\to\infty \qquad \text{as } n\to\infty,$$

and whose associated weak eigenfunctions form a complete orthonormal basis of $L^2(\Omega,\gamma)$. Moreover, the eigenfunctions belong to $C^\infty(\overline{\Omega})$ and are real analytic in Ω.

Proof The standard approach to such a problem is to first prove our form is coercive and continuous. That is, we need to show there exist constants $C_1, C_2, K>0$ such that:

$$C_1\|u\|^2_{H^2(\Omega,\gamma)} \le a(u,u) + K\|u\|^2_{L^2(\Omega,\gamma)} \le C_2\|u\|^2_{H^2(\Omega,\gamma)}.$$

Let $\phi \in C_c^\infty(\Omega)$, and let $\widetilde{\Omega}$ denote a bounded C^∞ domain which is precompact in Ω and contains the support of ϕ. A straightforward calculation with integration by parts reveals

$$\begin{aligned} a(\phi, \phi) &= \sum_{i,j=1}^{n} \int_{\widetilde{\Omega}} (g\phi_{x_i})_{x_i} (g\phi_{x_j})_{x_j} \frac{1}{g} \, \mathrm{d}x \\ &= \int_{\widetilde{\Omega}} |\nabla \phi|^2 \, \mathrm{d}\gamma + \sum_{i,j=1}^{n} \int_{\widetilde{\Omega}} \phi_{x_i x_j}^2 \, \mathrm{d}\gamma \\ &= \int_{\Omega} |\nabla \phi|^2 \, \mathrm{d}\gamma + \sum_{i,j=1}^{n} \int_{\Omega} \phi_{x_i x_j}^2 \, \mathrm{d}\gamma. \end{aligned}$$

Now with $u \in H_0^2(\Omega, \gamma)$, choose a sequence $\phi_k \in C_c^\infty(\Omega)$ with $\phi_k \to u$ in the H^2 norm and which, by Corollary 2, has the additional property $\mathscr{L}\phi_k \to \mathscr{L}u$ in $L^2(\Omega, \gamma)$. As $k \to \infty$ we have

$$a(\phi_k, \phi_k) \to a(u, u).$$

By the definition of H^2 convergence, we then have

$$\int_\Omega |\nabla \phi_k|^2 \, \mathrm{d}\gamma + \sum_{i,j=1}^{n} \int_\Omega \phi_{k_{x_i x_j}}^2 \, \mathrm{d}\gamma \to \int_\Omega |\nabla u|^2 \, \mathrm{d}\gamma + \sum_{i,j=1}^{n} \int_\Omega u_{x_i x_j}^2 \, \mathrm{d}\gamma.$$

We deduce

$$a(u, u) + \|u\|_{L^2(\Omega,\gamma)}^2 = \|u\|_{H^2(\Omega,\gamma)}^2$$

whenever $u \in H_0^2(\Omega)$. The coercivity and continuity of the form is now established.

Next, the space $H_0^2(\Omega, \gamma)$ is compactly embedded in $L^2(\Omega, \gamma)$. This follows since $H_0^1(\Omega, \gamma)$ is compactly embedded in $L^2(\Omega, \gamma)$ (see [16, Proposition 2.2]), and $u \in H_0^1(\Omega, \gamma)$ whenever $u \in H_0^2(\Omega, \gamma)$ with $\|u\|_{H^1} \le \|u\|_{H^2}$. By coercivity of our form and the compact embedding of our space, we may apply [29, Corollary 7.8, p. 88] and conclude the form a has a set of weak eigenfunctions which form an orthonormal basis of $L^2(\Omega, \gamma)$. Furthermore, the corresponding eigenvalues are of finite multiplicity and satisfy the desired inequalities.

Smoothness of the eigenfunctions follows from standard regularity results [27, p. 668] and the Trace Theorem [33, Prop 4.3, p. 286 and Prop 4.5, p. 287]. Real analyticity of the eigenfunctions follows from the Analyticity Theorem [9, p. 136]. It follows that our weak eigenfunctions are classical (strong) solutions to problem (3). □

We close this section with a result essential to our proof of Theorem 1.

Proposition 6 *Suppose* $\Omega, \Omega_k \subset \mathbb{R}^n$ *are* C^∞ *domains with* $\gamma(\Omega) < 1$ *satisfying*

$$\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \cdots \subseteq \Omega \quad \text{and} \quad \bigcup_{k=1}^{\infty} \Omega_k = \Omega.$$

Let $\Lambda_1(\Omega)$, $\Lambda_1(\Omega_k)$ *denote the lowest eigenvalues for problem* (3). *Then*

$$\lim_{k\to\infty} \Lambda_1(\Omega_k) = \Lambda_1(\Omega).$$

Proof We adapt the techniques of [7]. By Rayleigh–Ritz,

$$\Lambda_1(\Omega_k) = \inf_{u \in H_0^2(\Omega_k, \gamma)} \frac{\int_{\Omega_k} (\mathscr{L} u)^2 \, d\gamma}{\int_{\Omega_k} u^2 \, d\gamma},$$

and similarly for Ω. Thus,

$$\Lambda_1(\Omega) \le \Lambda_1(\Omega_{k+1}) \le \Lambda_1(\Omega_k) \le \cdots,$$

and so the limit

$$\widetilde{\Lambda} := \lim_{k \to \infty} \Lambda_1(\Omega_k)$$

exists with $\Lambda_1(\Omega) \le \widetilde{\Lambda}$. We claim that the reverse inequality is also true. Let $\epsilon > 0$ and use Corollary 2 to choose $\phi \in C_c^\infty(\Omega)$ with $\|\phi\|_{L^2(\Omega,\gamma)} = 1$ such that

$$\left| \int_\Omega (\mathscr{L} \phi)^2 \, d\gamma - \Lambda_1(\Omega) \right| < \epsilon.$$

Choose N so that $\operatorname{supp} \phi \subset \Omega_k$ for $k \ge N$. Then for $k \ge N$,

$$\Lambda_1(\Omega_k) \le \int_\Omega (\mathscr{L} \phi)^2 \, d\gamma \le \Lambda_1(\Omega) + \epsilon.$$

Letting $k \to \infty$, and then sending $\epsilon \to 0$, we conclude $\widetilde{\Lambda} \le \Lambda_1(\Omega)$. Our proof is complete. □

4 Gaussian plate problems

The Rayleigh quotient for the Euclidean clamped plate is most frequently presented with numerator $\int_\Omega (\Delta u)^2 \, dx$ and corresponding form $A(u, v) = \int_\Omega (\Delta u)(\Delta v) \, dx$. However, this is a simplification made possible due to the boundary conditions and does not generalize to other boundary conditions. In fact, this form is not coercive when considered over the domain $H^2(\Omega)$ (for the free plate).

The most general form for a plate in Euclidean space with uniform tension/rigidity controlled by a parameter τ is

$$A(u, v) = \int_\Omega \left(\sum_{i,j=1}^n u_{x_i x_j} v_{x_i x_j} + \tau (\nabla u \cdot \nabla v) \right) dx.$$

Taken over $H_0^2(\Omega)$, this form corresponds to the clamped plate; extending our domain to $H^2(\Omega)$ gives the free (unconstrained) plate. Taking $\tau = 0$ in the clamped case recovers the familiar Dirichlet bi-Laplace eigenvalue problem.

Therefore, for the general plate problem in Gauss space, one should fix a constant B and begin with the bilinear form

$$\tilde{a}(u, v) = \int_\Omega \left(\sum_{i,j=1}^n u_{x_i x_j} v_{x_i x_j} + B (\nabla u \cdot \nabla v) \right) d\gamma.$$

For the clamped plate, we should take our form domain to be $H_0^2(\Omega, \gamma)$; if we were to consider the free plate, our domain would be $H^2(\Omega, \gamma)$.

As long as the constant $B > 0$, coercivity and continuity of this form over both $H_0^2(\Omega, \gamma)$ (clamped plate) and $H^2(\Omega, \gamma)$ (free plate) is immediate, and so we have the expected discrete spectrum and orthonormal eigenbasis. The Euler–Lagrange equation for this form is easily shown to be

$$\mathscr{L}^2 u + (1 - B)\mathscr{L} u = \Lambda u,$$

although of course the boundary conditions depend on our chosen form domain.

When we consider the form domain $H_0^2(\Omega, \gamma)$, the boundary conditions $u = \partial u/\partial \nu = 0$ and a straightforward integration-by-parts argument allow us to rewrite the form $\tilde{a}(\cdot, \cdot)$ as

$$\int_\Omega \Big((\mathscr{L} u)(\mathscr{L} v) + (B - 1)(\nabla u \cdot \nabla v) \Big)\, \mathrm{d}\gamma.$$

Note that taking $B = 1$ recovers our chosen form (4) for the bi-Hermite problem $\mathscr{L}^2 u = \Lambda u$. Thus, we are justified in using our simplified form for the clamped plate problem.

5 Symmetrization

In this section, we assume $\Omega \subset \mathbb{R}^n$ is a C^∞ domain with measure $0 < \gamma(\Omega) < 1$. We take $\Lambda_1 = \Lambda_1(\Omega)$ to be the first eigenvalue and u an associated principle eigenfunction for the bi-Hermite problem

$$\begin{cases} \mathscr{L}^2 u = \Lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We additionally assume that u has the additional property:

$$\mathscr{L} u \text{ belongs to } L^\infty(\Omega).$$

From the weak characterization of the above problem, we have

$$\Lambda_1 = \frac{\int_\Omega (\mathscr{L} u)^2\, \mathrm{d}\gamma}{\int_\Omega u^2\, \mathrm{d}\gamma}.$$

We remind the reader that H_L denotes the half-space $\{x \in \mathbb{R}^n \mid x_1 > L\}$.

Talenti's approach to the clamped plate problem in Euclidean space begins with a symmetrization argument [32]. In what follows, we adapt Talenti's approach to the Gaussian setting and establish comparison results for the positive and negative parts of a principle eigenfunction u using symmetrization:

Theorem 7 *Let Ω and u be as above. Then there exist $a, b \in \mathbb{R}$ such that $\gamma(H_a) + \gamma(H_b) = \gamma(\Omega)$ and functions $v \in H_0^1(H_a, \gamma) \cap H^2(H_a, \gamma)$ and $w \in H_0^1(H_b, \gamma) \cap H^2(H_b, \gamma)$ such that*

$$u_+^\#(x) \le v(x) \quad on\ H_a \quad and \quad u_-^\#(x) \le w(x) \quad on\ H_b,$$

where $u_+^\#$ and $u_-^\#$ denote the Gaussian symmetrizations of the positive and negative parts of u, respectively. Moreover,

$$\frac{\int_\Omega (\mathscr{L} u)^2\, d\gamma}{\int_\Omega u^2\, d\gamma} \ge \frac{\int_{H_a} (\mathscr{L} v)^2\, d\gamma + \int_{H_b} (\mathscr{L} w)^2\, d\gamma}{\int_{H_a} v^2\, d\gamma + \int_{H_b} w^2\, d\gamma}.$$

The functions v and w will be explicitly constructed in the proof that follows.

In what follows, we use comparison results to formulate a related minimization problem, which we shall solve in later sections using special functions.

Before proceeding, the reader might find it useful to review the symmetrization notation and definitions introduced in Sect. 2.

Proof The following argument is rather long, so for the sake of digestibility, we break it down into several steps. □

Step 1: The identity $\int_\Omega \mathscr{L}\, u \,\mathrm{d}\gamma = 0$

Using Corollary 2, choose a sequence of test functions $\phi_k \in C_c^\infty(\Omega)$ where $\phi_k \to u$ in the H^2 norm and $\mathscr{L}\,\phi_k \to \mathscr{L}\,u$ in $L^2(\Omega, \gamma)$. Fix k and choose a bounded C^∞ domain $\widetilde{\Omega}$ that contains the support of ϕ_k and is precompact in Ω. By the Divergence Theorem,

$$\int_\Omega \mathscr{L}\,\phi_k \,\mathrm{d}\gamma = \int_{\widetilde{\Omega}} \mathscr{L}\,\phi_k \,\mathrm{d}\gamma = \int_{\widetilde{\Omega}} \nabla \cdot (g \nabla \phi_k) \,\mathrm{d}x = \int_{\partial\widetilde{\Omega}} g \nabla \phi_k \cdot \nu \,\mathrm{d}\,\mathcal{H}^{n-1} = 0.$$

$$\text{We conclude } \int_\Omega \mathscr{L}\,u \,\mathrm{d}\gamma = \lim_{k\to\infty} \int_\Omega \mathscr{L}\,\phi_k \,\mathrm{d}\gamma = 0.$$

Step 2: An inequality involving perimeter

We introduce the following notation for level sets of u:

$$\begin{aligned} \{u > t\} &= \{x \in \Omega \mid u(x) > t\}, \\ \{u = t\} &= \{x \in \Omega \mid u(x) = t\}, \\ \{t < u \le s\} &= \{x \in \Omega \mid t < u(x) \le s\}. \end{aligned}$$

Let α denote the distribution function of u, so that $\alpha(t) = \gamma\,(\{u > t\})$. Note that α is a.e. differentiable and that the equality of sets

$$\partial\{u > t\} = \{u = t\}$$

holds for almost every $t > 0$ by Sard's Theorem [30, p.342].

For $t, h > 0$ fixed, Cauchy–Schwarz implies

$$\left(\frac{1}{h}\int_{\{t<u\le t+h\}} |\nabla u| \,\mathrm{d}\gamma\right)^2 \le \frac{\alpha(t) - \alpha(t+h)}{h} \frac{1}{h} \int_{\{t<u\le t+h\}} |\nabla u|^2 \,\mathrm{d}\gamma.$$

The coarea formula gives us

$$\begin{aligned} \int_{\{t<u\le t+h\}} |\nabla u| \,\mathrm{d}\gamma &= \int_t^{t+h} \left(\int_{\{u=s\}} g \,\mathrm{d}\mathcal{H}^{n-1}\right) \mathrm{d}s, \\ \int_{\{t<u\le t+h\}} |\nabla u|^2 \,\mathrm{d}\gamma &= \int_t^{t+h} \left(\int_{\{u=s\}} |\nabla u| g \,\mathrm{d}\mathcal{H}^{n-1}\right) \mathrm{d}s. \end{aligned}$$

Writing Per for the Gaussian perimeter and letting $h \to 0$ in the first inequality of this step, we have

$$(\operatorname{Per}\{u > t\})^2 \le -\alpha'(t) \int_{\{u=t\}} |\nabla u| g \,\mathrm{d}\,\mathcal{H}^{n-1}\,. \tag{6}$$

Since $u \in H_0^2(\Omega, \gamma)$, we apply Corollary 2 to conclude

$$\int_\Omega (-\mathscr{L} u)\phi \,\mathrm{d}\gamma = \int_\Omega \nabla u \cdot \nabla \phi \,\mathrm{d}\gamma \qquad \text{for all } \phi \in H_0^1(\Omega, \gamma). \tag{7}$$

Following the proof of Theorem 3.1 in [11], we fix $t, h > 0$ and define functions ϕ_h on Ω as

$$\phi_h(x) = \begin{cases} 1 & \text{on } \{u > t + h\}, \\ \frac{u(x)-t}{h} & \text{on } \{t < u \le t + h\}, \\ 0 & \text{otherwise.} \end{cases}$$

Using these ϕ_h in (7) and letting $h \to 0$, we deduce that for almost every $t > 0$

$$\int_{\{u>t\}} (-\mathscr{L} u) \,\mathrm{d}\gamma = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\{u>t\}} |\nabla u|^2 \,\mathrm{d}\gamma = \int_{\{u=t\}} |\nabla u| \,\mathrm{d}\gamma.$$

Let $u^\# : \Omega^\# \to \mathbb{R}$ be the Gauss symmetrization of u. Then by the isoperimetric inequality for Gauss space, the inequality (6) involving perimeter can be written as

$$\left(\mathrm{Per}\{u^\# > t\}\right)^2 \le (\mathrm{Per}\{u > t\})^2 \le -\alpha'(t) \int_{\{u>t\}} (-\mathscr{L} u) \,\mathrm{d}\gamma.$$

Step 3: Expressing Per$\{u^\# > t\}$ in terms of $\alpha(t)$

Let $r = r(t)$ be a function such that

$$\{u^\# = t\} = \{x \in \mathbb{R}^n : x_1 = r(t)\}.$$

Recalling the notation

$$H_L = \{x \in \mathbb{R}^n \mid x_1 > L\},$$

the above definition of r is equivalent to saying

$$\{u^\# > t\} = H_{r(t)}.$$

That is,

$$\alpha(t) = \Phi(r(t)),$$

or equivalently

$$\Phi^{-1}(\alpha(t)) = r(t).$$

By definition, we have $\mathrm{Per}\{u^\# > t\} = \mathrm{Per}\, H_r = \frac{c_n}{c_{n-1}} \mathrm{e}^{-r^2/2} = \frac{c_n}{c_{n-1}} \mathrm{e}^{-[\Phi^{-1}(\alpha(t))]^2/2}$. Squaring gives

$$\left(\mathrm{Per}\{u^\# > t\}\right)^2 = \frac{c_n^2}{c_{n-1}^2} \mathrm{e}^{-[\Phi^{-1}(\alpha(t))]^2}.$$

Hence, the final inequality from Step 2 becomes

$$1 \le \frac{c_{n-1}^2}{c_n^2} \mathrm{e}^{[\Phi^{-1}(\alpha(t))]^2} \left(-\alpha'(t)\right) \int_{\{u>t\}} (-\mathscr{L} u) \,\mathrm{d}\gamma.$$

Step 4: The positive and negative parts of $\mathscr{L} u$ and their rearrangements

Write $(\mathscr{L} u)_-$ and $(\mathscr{L} u)_+$ for the positive and negative parts of $\mathscr{L} u$, and $(\mathscr{L} u)^*_-$ and $(\mathscr{L} u)^*_+$ for the decreasing rearrangements of $(\mathscr{L} u)_-$ and $(\mathscr{L} u)_+$, respectively. Then

$$\begin{aligned}\int_{\{u>t\}} (-\mathscr{L} u)\,\mathrm{d}\gamma &= \int_{\{u>t\}} ((\mathscr{L} u)_- - (\mathscr{L} u)_+)\ \mathrm{d}\gamma \\ &\le \int_0^{\alpha(t)} \big((\mathscr{L} u)^*_-(s) - (\mathscr{L} u)^*_+(\gamma(\Omega)-s)\big)\ \mathrm{d}s.\end{aligned}$$

If we define

$$f(s) = (\mathscr{L} u)^*_-(s) - (\mathscr{L} u)^*_+(\gamma(\Omega)-s), \quad 0 \le s \le \gamma(\Omega),$$

then we have

$$1 \le \frac{c_{n-1}^2}{c_n^2} \mathrm{e}^{[\Phi^{-1}(\alpha(t))]^2}(-\alpha'(t)) \int_0^{\alpha(t)} f(s)\,\mathrm{d}s.$$

Observe that f is the sum of two decreasing functions, and so is itself decreasing. Integrating the above inequality from 0 to t gives

$$t \le -\frac{c_{n-1}^2}{c_n^2} \int_0^t \alpha'(s) \mathrm{e}^{[\Phi^{-1}(\alpha(s))]^2} \left(\int_0^{\alpha(s)} f(r)\,\mathrm{d}r \right) \mathrm{d}s.$$

Next make the change of variable $z = \alpha(s)$. Then we have

$$t \le \frac{c_{n-1}^2}{c_n^2} \int_{\alpha(t)}^{\alpha(0)} \mathrm{e}^{\Phi^{-1}(z)^2} \left(\int_0^z f(r)\,\mathrm{d}r \right) \mathrm{d}z.$$

The above inequality holds for every $t \ge 0$. For $u^\#_+(x) > 0$, take $t = u^\#_+(x) - \epsilon$, where $u^\#_+$ denotes the Gauss symmetrization of the positive part of u. We then obtain

$$u^\#_+(x) - \epsilon \le \frac{c_{n-1}^2}{c_n^2} \int_{\alpha(u^\#_+(x)-\epsilon)}^{\alpha(0)} \mathrm{e}^{\Phi^{-1}(z)^2} \left(\int_0^z f(r)\,\mathrm{d}r \right) \mathrm{d}z.$$

Fix a real number a so that

$$\gamma(\{u>0\}) = \gamma(H_a).$$

If $x_1 > a$, then $u^\#$ is strictly positive, so that

$$\begin{aligned}\alpha(u^\#_+(x) - \epsilon) &= \alpha(u^\#(x) - \epsilon) \\ &= \alpha\left(u^*\left(\Phi(x_1)\right) - \epsilon\right) \\ &\ge \Phi(x_1).\end{aligned}$$

The last inequality follows from the definition of decreasing rearrangement. Now, the function $F(z) = \int_0^z f(r)\,\mathrm{d}r$ is nonnegative on $[0, \gamma(\Omega)]$, a consequence of F being concave with $F(0) = F(\gamma(\Omega)) = 0$. Thus, letting $\epsilon \to 0$,

$$u^\#_+(x) \le \frac{c_{n-1}^2}{c_n^2} \int_{\Phi(x_1)}^{\alpha(0)} \mathrm{e}^{\Phi^{-1}(z)^2} \left(\int_0^z f(r)\,\mathrm{d}r \right) \mathrm{d}z. \tag{8}$$

Define a function $v : H_a \to \mathbb{R}$ depending only on x_1, the first component of x, by

$$v(x) = \frac{c_{n-1}^2}{c_n^2} \int_{\Phi(x_1)}^{\alpha(0)} e^{\Phi^{-1}(z)^2} \left(\int_0^z f(r)\, dr \right) dz.$$

Since $u^\#$ and v are functions that only depend on x_1, in what follows we often write $u^\#(x_1)$ for $u^\#(x)$ and similarly for v. We also write $g(x_1)$ for $g(x_1, 0, \ldots, 0)$.

Step 5: A comparison result for $u_+^\#$

By direct computation of the derivatives, we obtain

$$-\mathscr{L} v = -\frac{1}{g(x_1)} \frac{\partial}{\partial x_1} \left(g(x_1) \frac{\partial v}{\partial x_1} \right) = f(\Phi(x_1)) \text{ for } x_1 > a.$$

Note that since $\Phi(a) = \alpha(0)$, we have $v(a) = 0$. Then the work from this step and Eq. (8) from Step 4 combine to give the comparison result

$$u_+^\#(x) = u_+^\#(x_1) \leq v(x) = v(x_1),$$

where v depends only on x_1 and is the solution to

$$\begin{cases} -\mathscr{L} v = f(\Phi(x_1)) & \text{in } H_a, \\ v = 0 & \text{on } x_1 = a. \end{cases}$$

Note that by construction $\mathscr{L} v \in L^2(H_a, \gamma)$.

Step 6: A comparison result for $u_-^\#$

Write $u_-^\#$ for the Gauss symmetrization of the negative part of u. Let $y = -u$ and note y is also a principle eigenfunction of the bi-Hermite problem (5). Hence, we may apply the above symmetrization process to the eigenfunction y. We then define the number b so that $\Phi(b) = \gamma(\{y \geq 0\}) = \gamma(\{u \leq 0\})$ and functions $h : [0, \gamma(\Omega)] \to \mathbb{R}$ and $w : H_b \to \mathbb{R}$ as follows:

$$h(s) = (\mathscr{L} y)_-^*(s) - (\mathscr{L} y)_+^*(\gamma(\Omega) - s), \quad 0 \leq s \leq \gamma(\Omega),$$
$$w(x) = \frac{c_{n-1}^2}{c_n^2} \int_{\Phi(x_1)}^{\Phi(b)} e^{\Phi^{-1}(z)^2} \left(\int_0^z h(r)\, dr \right) dz.$$

By construction,

$$\Phi(a) + \Phi(b) = \gamma(\Omega).$$

Then by our prior work in Steps 4 and 5, we see w solves the boundary value problem

$$\begin{cases} -\mathscr{L} w = h(\Phi(x_1)) & \text{in } H_b, \\ w = 0 & \text{on } x_1 = b, \end{cases}$$

and also

$$y_+^\#(x) \leq w(x).$$

By construction, $\mathscr{L} w \in L^2(H_b, \gamma)$. Now consider the fact that $y = -u$. Then by properties of positive and negative parts, $y_+ = u_-$, and so the Gaussian symmetrizations satisfy $y_+^\#(x) = u_-^\#(x)$. For $0 \leq s \leq \gamma(\Omega)$, we also have that

$$\begin{aligned} h(s) &= (\mathscr{L}\, y)^*_-(s) - (\mathscr{L}\, y)^*_+(\gamma(\Omega) - s) \\ &= (-\mathscr{L}\, u)^*_-(s) - (-\mathscr{L}\, u)^*_+(\gamma(\Omega) - s) \\ &= (\mathscr{L}\, u)^*_+(s) - (\mathscr{L}\, u)^*_-(\gamma(\Omega) - s), \end{aligned}$$

and so $h(\gamma(\Omega) - s) = -f(s)$.

Step 7: The identity $\int_{H_a} -\mathscr{L}\, v \,\mathrm{d}\gamma = \int_{H_b} -\mathscr{L}\, w \,\mathrm{d}\gamma$

We compute directly

$$\int_{H_a} -\mathscr{L}\, v \,\mathrm{d}\gamma = \int_{H_a} f\,(\Phi(x_1))\,\mathrm{d}\gamma = \int_0^{\Phi(a)} f(s)\,\mathrm{d}s.$$

On the other hand,

$$\int_{H_b} -\mathscr{L}\, w \,\mathrm{d}\gamma = \int_{H_b} h\,(\Phi(x_1))\,\mathrm{d}\gamma = \int_0^{\Phi(b)} h(s)\,\mathrm{d}s.$$

Making a change of variable and using the identity $\Phi(a) + \Phi(b) = \gamma(\Omega)$, the previous line becomes

$$\int_0^{\Phi(b)} h(s)\,\mathrm{d}s = -\int_{\gamma(\Omega)}^{\Phi(a)} h\,(\gamma(\Omega) - s)\,\mathrm{d}s = -\int_{\Phi(a)}^{\gamma(\Omega)} f(s)\,\mathrm{d}s = \int_0^{\Phi(a)} f(s)\,\mathrm{d}s,$$

where the last equality follows since $\int_0^{\gamma(\Omega)} f(s)\,\mathrm{d}s = 0$, a consequence of $\int_\Omega \mathscr{L}\, u\,\mathrm{d}\gamma = 0$.

We now make the following two computations by integrating first in the x_1-variable:

$$\begin{aligned} \int_{H_a} -\mathscr{L}\, v\,\mathrm{d}\gamma &= -\frac{c_n}{c_{n-1}} \mathrm{e}^{-a^2/2} \frac{\partial v}{\partial x_1}(a), \\ \int_{H_b} -\mathscr{L}\, w\,\mathrm{d}\gamma &= -\frac{c_n}{c_{n-1}} \mathrm{e}^{-b^2/2} \frac{\partial w}{\partial x_1}(b). \end{aligned}$$

By the work in this step, the two expressions above are equal, so

$$\mathrm{e}^{-a^2/2} \frac{\partial v}{\partial x_1}(a) = \mathrm{e}^{-b^2/2} \frac{\partial w}{\partial x_1}(b).$$

Step 8: The identity $\int_{H_a} (\mathscr{L}\, v)^2 \,\mathrm{d}\gamma + \int_{H_b} (\mathscr{L}\, w)^2 \,\mathrm{d}\gamma = \int_\Omega (\mathscr{L}\, u)^2 \,\mathrm{d}\gamma$

We compute directly

$$\begin{aligned} \int_{H_a} (\mathscr{L}\, v)^2\,\mathrm{d}\gamma + \int_{H_b} (\mathscr{L}\, w)^2\,\mathrm{d}\gamma &= \int_{H_a} \left[(\mathscr{L}\, u)^*_-\,(\Phi(x_1)) - (\mathscr{L}\, u)^*_+\,(\gamma(\Omega) - \Phi(x_1))\right]^2\,\mathrm{d}\gamma \\ &\quad + \int_{H_b} \left[(\mathscr{L}\, u)^*_-\,(\gamma(\Omega) - \Phi(x_1)) - (\mathscr{L}\, u)^*_+\,(\Phi(x_1))\right]^2\,\mathrm{d}\gamma \\ &= \int_0^{\Phi(a)} \left[(\mathscr{L}\, u)^*_-\,(s) - (\mathscr{L}\, u)^*_+\,(\gamma(\Omega) - s)\right]^2\,\mathrm{d}s \\ &\quad + \int_0^{\Phi(b)} \left[(\mathscr{L}\, u)^*_-\,(\gamma(\Omega) - s) - (\mathscr{L}\, u)^*_+\,(s)\right]^2\,\mathrm{d}s \\ &= \int_0^{\gamma(\Omega)} \left[(\mathscr{L}\, u)^*_-\,(s) - (\mathscr{L}\, u)^*_+\,(\gamma(\Omega) - s)\right]^2\,\mathrm{d}s. \end{aligned}$$

Since the functions $(\mathscr{L} u)^*_-(s)$ and $(\mathscr{L} u)^*_+(\gamma(\Omega) - s)$ are never simultaneously nonzero, we conclude the above equals

$$\int_0^{\gamma(\Omega)} \left[(\mathscr{L} u)^*_-(s)\right]^2 + \left[(\mathscr{L} u)^*_+(s)\right]^2 \, \mathrm{d}s = \int_\Omega (\mathscr{L} u)^2 \, \mathrm{d}\gamma.$$

Step 9: The identity $\mathscr{L} v(a) + \mathscr{L} w(b) = 0$

We see

$$\begin{aligned} \mathscr{L} v(a) + \mathscr{L} w(b) &= f(\Phi(a)) + h(\Phi(b)) \\ &= f(\Phi(a)) - f(\gamma(\Omega) - \Phi(b)) \\ &= 0. \end{aligned}$$

Step 10: A bound for Λ_1 and the minimization problem

Since $\mathscr{L} u \in L^\infty(\Omega)$, easy estimates show that v and w belong to $H^2(H_a, \gamma)$ and $H^2(H_b, \gamma)$, respectively. The comparison results from Steps 5 and 6 above give

$$\int_\Omega u^2 \, \mathrm{d}\gamma \leq \int_{H_a} v^2 \, \mathrm{d}\gamma + \int_{H_b} w^2 \, \mathrm{d}\gamma.$$

Now from the Rayleigh quotient for Λ_1, Step 8, and the inequality immediately above,

$$\Lambda_1 = \frac{\int_\Omega (\mathscr{L} u)^2 \, \mathrm{d}\gamma}{\int_\Omega u^2 \, \mathrm{d}\gamma} \geq \frac{\int_{H_a} (\mathscr{L} v)^2 \, \mathrm{d}\gamma + \int_{H_b} (\mathscr{L} w)^2 \, \mathrm{d}\gamma}{\int_{H_a} v^2 \, \mathrm{d}\gamma + \int_{H_b} w^2 \, \mathrm{d}\gamma}.$$

We are thus led to define

$$J_{a,b} = \inf_{(v,w)} \frac{\int_{H_a} (\mathscr{L} v)^2 \, \mathrm{d}\gamma + \int_{H_b} (\mathscr{L} w)^2 \, \mathrm{d}\gamma}{\int_{H_a} v^2 \, \mathrm{d}\gamma + \int_{H_b} w^2 \, \mathrm{d}\gamma},$$

where the infimum is taken over all pairs of functions (v, w) that only depend on x_1 such that

$$v \in H^2(H_a, \gamma) \cap H_0^1(H_a, \gamma) \text{ and } w \in H^2(H_b, \gamma) \cap H_0^1(H_b, \gamma),$$

and where

$$\begin{aligned} \mathrm{e}^{-a^2/2} \frac{\partial v}{\partial x_1}(a) &= \mathrm{e}^{-b^2/2} \frac{\partial w}{\partial x_1}(b), \\ \mathscr{L} u &\in L^2(H_a, \gamma), \\ \mathscr{L} w &\in L^2(H_b, \gamma). \end{aligned}$$

The solutions to the $J_{a,b}$ minimization problem can be expressed in terms of parabolic cylinder functions.

6 Parabolic cylinder functions

In this section, we gather a collection of useful properties of parabolic cylinder functions, which are used in the solution of the $J_{a,b}$ minimization problem arising from our symmetrization argument. All properties of these functions in this section stated without proof may be found in [1,12].

6.1 Definitions and notation

Given a parameter a, the Weber functions $U(a, z)$ and $V(a, z)$ are a particular pair of linearly independent solutions to the ODE

$$\frac{d^2w}{dz^2} - \left(\frac{1}{4}z^2 + a\right) w = 0.$$

The Weber functions can be written in terms of power series or confluent hypergeometric functions, although neither representation is needed in our application. In special cases of the parameter a, the Weber functions can be expressed in terms of the physicists' Hermite polynomials as follows:

$$U(-n - 1/2, z) = 2^{-n/2}e^{-z^2/4}H_n(z/\sqrt{2}) \qquad \text{for } n \in \mathbb{N} \cup \{0\}.$$

In particular, we have $U(-1/2, z) = e^{-z^2/4}$. We will use this special case in Sect. 8 to find the eigenfunctions of $\mathscr{L}^2$ on half-spaces.

For derivatives with respect to the variable z, we will use Newton's notation:

$$U'(a, z) := \frac{\partial}{\partial z}U(a, z).$$

Derivatives with respect to the parameter a will not appear in this paper.

6.2 Recurrence relations, derivative formulae, and special cases

We have the following recurrence relations:

$$zU(a, z) - U(a - 1, z) + (a + 1/2)U(a + 1, z) = 0,$$
$$U'(a, z) + \frac{1}{2}zU(a, z) + (a + 1/2)U(a + 1, z) = 0.$$

For $m \in \mathbb{N}$, we have

$$\frac{d^m}{dz^m}\left(e^{z^2/4}U(a, z)\right) = (-1)^m \left(\frac{1}{2} + a\right)_m e^{z^2/4}U(a + m, z).$$

In terms of our one-dimensional Hermite operator $\mathscr{L} = \frac{d^2}{dz^2} - z\frac{d}{dz}$, we can show directly that for any $\lambda \in \mathbb{R}$,

$$\mathscr{L}\left(e^{z^2/4}U(\lambda - 1/2, z)\right) = \lambda e^{z^2/4}U(\lambda - 1/2, z).$$

6.3 Asymptotics

For all $a \in \mathbb{R}$, we have that as $z \to \infty$,

$$U(a, z) = e^{-z^2/4}z^{-a-1/2}\left(1 + O(z^{-2})\right),$$
$$V(a, z) = e^{z^2/4}z^{a-1/2}\left(\sqrt{\frac{2}{\pi}} + O(z^{-2})\right),$$
$$U'(a, z) = e^{-z^2/4}z^{-a+1/2}\left(-\frac{1}{2} + O(z^{-2})\right).$$

6.4 Zeros

The function $U(a, z)$ has no real zeros when $a \geq -1/2$. The number of real zeros increases as a decreases. When $a < -1/2$, the number of positive real zeros is n, where $-2n - 3/2 < a < -2n + 1/2$. When $a = -n - 1/2$, $U(a, z)$ is a Hermite polynomial with exactly n real zeros, all in the interval $[-2\sqrt{|a|}, 2\sqrt{|a|}]$.

We finally state the following proposition, which will prove useful in Sect. 7.

Proposition 8 *For all $z \in \mathbb{R}$ and parameters $a \in \mathbb{R}$, we have:*

(1) *The functions $U(a, z)$ and $U(a-1, z)$ cannot be simultaneously zero.*
(2) *The functions $U(a, z)$ and $U'(a, z)$ cannot be simultaneously zero.*

Part (1) of Proposition 8 follows from the recurrence relations and properties of Hermite polynomials. Part (2) follows from (1) and the recurrence relations.

7 The $J_{a,b}$ minimization problem

In this section, we return to the $J_{a,b}$ minimization problem introduced at the end of Sect. 5. Using the parabolic cylinder functions introduced in Sect. 6, we construct a function whose smallest positive root is precisely $\sqrt{J_{a,b}}$.

7.1 The Euler–Lagrange system

Recall that H_a denotes the half-space $H_a = \{x \in \mathbb{R}^n \mid x_1 > a\}$, and that

$$J_{a,b} = \inf_{(v,w)} \frac{\int_{H_a} (\mathscr{L} v)^2 \, \mathrm{d}\gamma + \int_{H_b} (\mathscr{L} w)^2 \, \mathrm{d}\gamma}{\int_{H_a} v^2 \, \mathrm{d}\gamma + \int_{H_b} w^2 \, \mathrm{d}\gamma}, \tag{9}$$

where the inf ranges over all pairs of functions (v, w) such that:

$$\begin{aligned} &v \in H^2(H_a, \gamma) \cap H_0^1(H_a, \gamma), \quad w \in H^2(H_b, \gamma) \cap H_0^1(H_b, \gamma), \\ &v(x) = v(x_1), \quad w(x) = w(x_1) \quad (v \text{ and } w \text{ depend only on } x_1), \\ &\mathrm{e}^{-a^2/2} \frac{\partial v}{\partial x_1}(a) = \mathrm{e}^{-b^2/2} \frac{\partial w}{\partial x_1}(b), \\ &\mathscr{L} v \in L^2(H_a, \gamma), \quad \mathscr{L} w \in L^2(H_b, \gamma). \end{aligned}$$

Observe that all of these conditions hold true for the v and w arising in our symmetrization argument.

A straightforward modification of the argument in Appendix 2 of [4] yields the existence of a minimizing pair (v, w) in (9); we collect details in the "Appendix" at the end of our paper.

By the usual calculus of variations argument, these minimizers satisfy the following Euler–Lagrange system:

$$\begin{cases} \mathscr{L}^2 v = \mu v \quad \text{on} \quad \{x_1 > a\}, \\ \mathscr{L}^2 w = \mu w \quad \text{on} \quad \{x_1 > b\}, \\ v(a) = w(b) = 0, \\ \mathrm{e}^{-a^2/2} v'(a) = \mathrm{e}^{-b^2/2} w'(b), & (10) \\ \mathscr{L} v(a) + \mathscr{L} w(b) = 0, & (11) \end{cases}$$

where the scalar μ is $J_{a,b}$. Although the above system has eigenvalues μ other than $J_{a,b}$, the smallest eigenvalue, by construction, is precisely $J_{a,b}$. The solutions to the system can be written in terms of parabolic cylinder functions; relevant properties of these functions were collected in Sect. 6.

7.2 The equation $\mathscr{L}^2 y = \mu y$ with solutions in $H^2(H_L, \gamma) \cap H_0^1(H_L, \gamma)$

Let us first examine the general ODE problem $\mathscr{L}^2 y = \mu y$, where $y = y(x)$ is a function of a single variable. This fourth-order ODE can be factored as

$$(\mathscr{L} - \sqrt{\mu})(\mathscr{L} + \sqrt{\mu})y = 0.$$

Solutions will then be linear combinations of solutions to the second-order problems $\mathscr{L} y = \pm\sqrt{\mu} y$. Direct computation shows that $\mathscr{L} y = y'' - xy'$; our factored equations can then be written in the form

$$y'' - xy' = \lambda y \qquad \text{where } \lambda = \pm\sqrt{\mu}.$$

By writing $y(x) = \mathrm{e}^{x^2/4}u(x)$, we can transform the second-order ODE above to the parabolic cylinder differential equation

$$u'' - (\lambda - 1/2 + x^2/4)u = 0.$$

We will take as our pair of linearly independent solutions the Weber parabolic cylinder functions $U(\lambda - 1/2, x)$ and $V(\lambda - 1/2, x)$. Thus, the solutions to our second-order ODE can be written in the form

$$y(x) = \mathrm{e}^{x^2/4}\Big(BU(\lambda - 1/2, x) + DV(\lambda - 1/2, x)\Big),$$

where B and D are real constants.

Fixing $\lambda = +\sqrt{\mu}$, we may write the general solution to our fourth-order problem as

$$y(x) = \mathrm{e}^{x^2/4}\Big(AU(-\lambda - 1/2, x) + BU(\lambda - 1/2, x) + CV(-\lambda - 1/2, x) + DV(\lambda - 1/2, x)\Big),$$

where A, B, C, and D are real constants. Note that y is continuous on its domain.

We are seeking solutions y in $H^2(H_L, \gamma) \cap H_0^1(H_L, \gamma)$, and so we must require $y \in L^2(H_L, \gamma)$ and $y(L) = 0$.

7.2.1 *Integrability of y*

We now impose the requirement $y \in L^2(H_L, \gamma)$. This will allow us to simplify our solutions significantly. Since our general solution $y(x)$ is continuous on H_L, in order that $\|y\|_{L^2(H_L,\gamma)}$ be finite, we must have

$$\lim_{x\to\infty} \frac{|y(x)|^2}{\mathrm{e}^{x^2/2}} \neq \infty.$$

Examining the asymptotics of the U and V functions, we see the terms involving $V(\pm\lambda - 1/2, x)$ go to infinity at different orders. Thus for the above to hold, the coefficients C and D must both vanish. We conclude

$$y(x) = \mathrm{e}^{x^2/4}\Big(AU(-\lambda - 1/2, x) + BU(\lambda - 1/2, x)\Big).$$

Moving forward, it will be convenient to develop some shorthand. For a fixed λ, we will write

$$U_{-}(z) := U(-\lambda - 1/2, z) \qquad \text{and} \qquad U_{+}(z) := U(+\lambda - 1/2, z).$$

Our solution y can now be written as

$$y(x) = \mathrm{e}^{x^2/4}\Big(AU_{-}(x) + BU_{+}(x)\Big).$$

7.2.2 *The boundary condition* $y(L) = 0$

Our requirement that $y \in H_0^1(H_L, \gamma)$ gives us the boundary condition $y(L) = 0$. We will use this to obtain information on the remaining constants A and B.

By our knowledge of zeros of the Weber functions U, the term $U_{+}(x)$ is never zero since $\lambda - 1/2 \geq -1/2$; thus, in order that $y \neq 0$, the coefficient A of the *other* term must be nonzero.

Solving $y(L) = 0$ for B/A yields

$$G_L := \frac{B}{A} = \frac{-U_{-}(L)}{U_{+}(L)}.$$

In the general case, our solutions are now of the form

$$y(x) = A\mathrm{e}^{x^2/4}\Big(U_{-}(x) + G_L U_{+}(x)\Big).$$

Note that it is possible that $G_L = 0$.

In the case that $\lambda = 0$, then $U_{-}(x) = U_{+}(x) = \mathrm{e}^{-x^2/4}$, and so $y(x)$ is constant. In order to satisfy the boundary condition $y(L) = 0$, we must have the trivial solution $y(x) = 0$ for all $x \geq L$. Thus, we may assume that $\lambda > 0$ in what follows.

7.2.3 *Computations with* $y(x)$

We collect here some computations to simplify our future work.

Note that by derivative properties of U and the differential equation used to find y, we have

$$\mathscr{L}\, y(L) = A\mathrm{e}^{L^2/4}\Big(- \lambda U_{-}(L) + \lambda G_L U_{+}(L)\Big),$$

which by our definition of G_L simplifies to

$$\mathscr{L}\, y(L) = -2\lambda A\mathrm{e}^{L^2/4} U_{-}(L).$$

By differentiating $y(x)$ directly, we obtain

$$y'(x) = \frac{x}{2} y(x) + A\mathrm{e}^{x^2/4}\left(U_{-}'(x) + G_L U_{+}'(x)\right),$$

and since $y(L) = 0$, we write:

$$g(L)y'(L) = c_n A\mathrm{e}^{-L^2/4}\left(U_{-}'(L) + G_L U_{+}'(L)\right).$$

7.3 The functions v and w

Applying our work in the general case, the solutions v and w of the Euler–Lagrange system take the form:

$$v(x) = A_v e^{x^2/4}\Big(U_-(x) + G_a U_+(x)\Big),$$
$$w(x) = A_w e^{x^2/4}\Big(U_-(x) + G_b U_+(x)\Big),$$

where A_v and A_w are constants. Note that multiplying both v and w by the same nonzero constant gives a new pair of solutions satisfying the Euler–Lagrange equations but produces the same value of the quotient

$$\frac{\int_{H_a} (\mathscr{L} v)^2 \, d\gamma + \int_{H_b} (\mathscr{L} w)^2 \, d\gamma}{\int_{H_a} v^2 \, d\gamma + \int_{H_b} w^2 \, d\gamma}.$$

Thus, the two constants A_v and A_w together give us only one degree of freedom. The two additional boundary conditions are therefore enough to determine the ratio $A_v : A_w$ and values of λ.

With u and v as above, nontrivially satisfying our Euler–Lagrange system, we claim that the constants A_v, A_w are both nonzero. Without loss of generality, assume $A_w = 0$ but $A_v \neq 0$. The other two boundary conditions may then be written as

$$0 = \mathscr{L} v(a) = -2A_v \lambda e^{a^2/4} U_-(a),$$
$$0 = v'(a) = A_v e^{a^2/4}(U_-{}'(a) + G_a U_+{}'(a)).$$

Since the quantities A_v, λ, and $e^{a^2/4}$ are nonzero, our first boundary condition implies $U_-(a) = 0$. But then the constant $G_a = 0$. The second boundary condition reduces to

$$0 = v'(a) = A_v e^{a^2/4} U_-{}'(a) \qquad \text{and so} \qquad U_-{}'(a) = 0.$$

However, from Proposition 8 we know $U'(a, z)$ and $U(a, z)$ cannot simultaneously vanish. This contradiction implies that A_v and A_w must both be nonzero.

7.3.1 Imposing the remaining boundary conditions

We shall impose the boundary condition $\mathscr{L} v(a) + \mathscr{L} w(b) = 0$ to determine the ratio A_v/A_w. Recall that for the solution $y(x)$ with $y(L) = 0$, we found

$$\mathscr{L} y(L) = -2\lambda A e^{L^2/4} U_-(L).$$

Thus in order to satisfy the boundary condition (11), we must have

$$A_v e^{a^2/4} U_-(a) = -A_w e^{b^2/4} U_-(b). \tag{12}$$

Since A_v, A_w, $e^{a^2/4}$, and $e^{b^2/4}$ are all nonzero, we have two possibilities:

(1) Both $U_-(a) = 0$ and $U_-(b) = 0$.
(2) Neither $U_-(a)$ nor $U_-(b)$ vanish, and the constants A_v and A_w satisfy Eq. (12) nontrivially.

7.3.2 Case I: $U_-(a) = 0$ and $U_-(b) = 0$

In this case, we have both $G_a = 0$ and $G_b = 0$, so our solutions simplify to

$$v(x) = A_v e^{x^2/4} U_-(x) \quad \text{and} \quad w(x) = A_w e^{x^2/4} U_-(x).$$

The boundary condition $g(a)v'(a) = g(b)w'(b)$ then becomes

$$e^{-a^2/2}\left(A_v e^{a^2/4} U_-'(a)\right) = e^{-b^2/2}\left(A_w e^{b^2/4} U_-'(b)\right),$$

or, more simply,

$$A_v e^{-a^2/4} U_-'(a) = A_w e^{-b^2/4} U_-'(b).$$

Since $U_-(a) = U_-(b) = 0$, by Proposition 8, both $U_-'(a)$ and $U_-'(b)$ must be nonzero. None of the above terms can vanish, and we may therefore write the quotient

$$\frac{A_v}{A_w} = \frac{e^{-b^2/4} U_-'(b)}{e^{-a^2/4} U_-'(a)}.$$

The eigenvalue λ is then determined by the equations

$$U_-(a) = 0 \quad \text{and} \quad U_-(b) = 0.$$

These roots occur at isolated points in the (a, λ) plane, a fact that can easily be verified numerically for any particular L by plotting the level sets $U_-(a) = 0$ and $U_-(b) = 0$.

7.3.3 Case II: $U_-(a) \neq 0$ and $U_-(b) \neq 0$

The boundary condition (10) requires investigating the derivatives $v'(x)$ and $w'(x)$. Again, recall

$$g(L)y'(L) = c_n A e^{-L^2/4}\left(U_-'(L) + G_L U_+'(L)\right).$$

Thus, the boundary condition $g(a)v'(a) = g(b)w'(b)$ can be written

$$A_v e^{-a^2/4}\left(U_-'(a) + G_a U_+'(a)\right) = A_w e^{-b^2/4}\left(U_-'(b) + G_b U_+'(b)\right).$$

The boundary condition $v(a) = w(b) = 0$ was used to determine the constants G_a, G_b, so the final boundary condition $\mathscr{L}\, v(a) + \mathscr{L}\, w(b) = 0$ gives us

$$-2\lambda A_v e^{a^2/4} U_-(a) - 2\lambda A_w e^{b^2/4} U_-(b) = 0.$$

These two conditions are linear equations in A_v, A_w. Since A_v and A_w are nonzero, the determinant of the system of equations above must vanish. After some algebra, this condition can be expressed as

$$g(a)\left(\frac{U_-'(a)}{U_-(a)} - \frac{U_+'(a)}{U_+(a)}\right) + g(b)\left(\frac{U_-'(b)}{U_-(b)} - \frac{U_+'(b)}{U_+(b)}\right) = 0,$$

since we have assumed $U_-(a)$, $U_-(b)$ are nonzero. Finally, we define

$$h_\lambda(x) := U_-'(x)U_+(x) - U_+'(x)U_-(x)$$

and

$$f_\lambda(x) := \frac{g(x)h_\lambda(x)}{U_+(x)U_-(x)} = g(x)\left(\frac{U_-'(x)}{U_-(x)} - \frac{U_+'(x)}{U_+(x)}\right).$$

This allows us to write our final condition as $f_\lambda(a) + f_\lambda(b) = 0$, or equivalently,

$$g(a)h_\lambda(a)U_+(b)U_-(b) + g(b)h_\lambda(b)U_+(a)U_-(a) = 0. \tag{13}$$

To summarize, if $\mu = \lambda^2$ is an eigenvalue of our Euler–Lagrange system, then λ is a root in Eq. (13); the smallest positive root is precisely $\lambda = \sqrt{J_{a,b}}$. The equation above allows us to numerically determine λ in terms of a and b (or equivalently a and L when $\gamma(H_a) + \gamma(H_b) = \gamma(H_L)$) and hence compute $J_{a,b} = \mu = \lambda^2$. Finally note that if $U(\lambda - 1/2, a) = U(\lambda - 1/2, b) = 0$ (Case I), condition (13) is satisfied, so we have a method for finding the values of λ that covers both cases.

8 Eigenfunctions of the half-space

In this section, we shall find the eigenfunctions of $\mathscr{L}^2$ on half-spaces using a separation of variables approach. Our solutions will involve both parabolic cylinder functions and Hermite polynomials. For $m \in \mathbb{N} \cup \{0\}$, we let $H_m(x)$ denote the mth physicists' Hermite polynomial.

Proposition 9 *If $L \in \mathbb{R}$ and H_L is the half-space $H_L = \{x \in \mathbb{R}^n \mid x_1 > L\}$, then the solutions to the eigenvalue problem*

$$\begin{cases} \mathscr{L}^2 u = \Lambda u & \text{in } H_L, \\ u = \frac{\partial u}{\partial x_1} = 0 & \text{when } x_1 = L, \end{cases}$$

can be written in the form

$$u(x_1, x_2, \ldots, x_n) = AY(x_1)H_{k_2}(x_2/\sqrt{2})H_{k_3}(x_3/\sqrt{2}) \cdots H_{k_n}(x_n/\sqrt{2}),$$

with corresponding eigenvalues

$$\Lambda = \mu^2 + k_2^2 + \cdots + k_n^2.$$

Here the k_ℓ are nonnegative integers, and Y and μ are given by

$$Y(x) = e^{x^2/4}\left(U(-\mu - 1/2, x) + G_L U(\mu - 1/2, x)\right),$$

where G_L is chosen so that $Y(L) = 0$ and the constant μ satisfies $Y'(L) = 0$.

Proof We first show that eigenfunctions can be written as the product of a function $Y(x_1)$ in x_1 and Hermite polynomials in the other variables.

Note that the operator $\mathscr{L}$ can be written as the sum of Hermite operators in each variable:

$$\mathscr{L} u = \left(\frac{\partial^2 u}{\partial x_1^2} - x_1 \frac{\partial u}{\partial x_1}\right) + \cdots + \left(\frac{\partial^2 u}{\partial x_n^2} - x_n \frac{\partial u}{\partial x_n}\right) =: L_1 u + (L_2 + \cdots + L_n)u.$$

By Proposition 5, each eigenvalue Λ has finite multiplicity and so its corresponding eigenspace $X_\Lambda \subseteq H_0^2(H_L, \gamma)$ is finite-dimensional. The Hermite operators L_k, $k = 1, \ldots, n$ operate with respect to independent variables and so commute with each other, and hence with $\mathscr{L}^2$. The single-variable Hermite operators L_k are also symmetric on the space $L^2(\mathbb{R}, \gamma)$, with eigenfunctions $F_k(x_k)$. Thus, by the standard argument, the operators $\mathscr{L}^2$ and the L_k, $k = 2, \ldots, n$ are all simultaneously diagonalizable.

Thus, the eigenfunctions of $\mathscr{L}^2$ can be written in the form $u(x) = Y(x_1)F_2(x_2) \cdots F_n(x_n)$, where Y is an eigenfunction of L_1^2 on $H_0^2(H_L, \gamma)$ and F_k is an eigenfunction of the Hermite operator L_k on $H_0^2(\mathbb{R}, \gamma)$.

The Hermite differential equation is well studied; in our case, the eigenfunctions are the Hermite polynomials $H_m(x/\sqrt{2})$ with corresponding eigenvalues $m \in \mathbb{N} \cup \{0\}$. (This is consistent with our earlier work in Sect. 7, as the Weber function $U(-m-1/2, x)$ can be written in terms of H_m.) From here we have the desired form of the F_k's.

The function $Y(x_1)$ must satisfy the ODE problem

$$\begin{cases} L_1^2 Y = \mu^2 Y \quad \text{when } \ x_1 > L \\ Y(L) = Y'(L) = 0. \end{cases}$$

That Y has the desired form follows immediately from our work in Sect. 7. □

Note that $H_0(x/\sqrt{2})$ is the constant function 1, so the smallest eigenvalue Λ_1 is obtained by taking the Hermite orders $k_2 = \cdots = k_n = 0$ and finding the smallest μ satisfying $y'(L) = 0$. We use this observation to numerically compute Λ_1 for half-spaces.

9 Proof of the main theorem

In this section, we prove our main result (Theorem 1) and discuss the numerical support for our conjecture that inequality (2) holds with $C_V = 1$.

Proof of Theorem 1 We take Ω to be a C^∞ domain with Gaussian measure $\gamma(\Omega) = V$ such that $0 < V < 1$.

First, suppose Ω is bounded. By regularity of the eigenfunctions of problem (1) established in Propositon 5, the principal eigenfunction u is C^∞ on the closure $\overline{\Omega}$. Since Ω is bounded, we have that u and all of its derivatives are bounded on $\overline{\Omega}$, and so in particular $\mathscr{L} u \in L^\infty(\Omega)$. Then the symmetrization argument (Theorem 7) in Sect. 5 shows the existence of numbers a and b satisfying $\gamma(H_a) + \gamma(H_b) = \gamma(\Omega)$ and where $\Lambda_1(\Omega) \geq J_{a,b}$.

These precise values of a, b are impossible to find for most, if not all, choices of Ω, so to find a computable lower bound on $\Lambda_1(\Omega)$, we must take the infimum of the $J_{a,b}$ over a set that contains these particular a, b. We therefore have

$$\Lambda_1(\Omega) \geq \inf\left\{J_{a,b} \mid \gamma(H_a) + \gamma(H_b) = \gamma(\Omega)\right\} =: \widetilde{J}.$$

Note that $\widetilde{J}$ is independent of dimension; it depends only on the measure of Ω. Likewise, from Proposition 9, we know the value $\Lambda_1(\Omega^\#)$ depends only on $\gamma(\Omega^\#) = \gamma(\Omega)$. We then observe that

$$\Lambda_1(\Omega) \geq \frac{\widetilde{J}}{\Lambda_1(\Omega^\#)} \Lambda_1(\Omega^\#).$$

Writing $C_V = \widetilde{J}/\Lambda_1(\Omega^\#)$ completes the proof in this case. Note that by positivity of both $J_{a,b}$ and the principle half-space eigenvalue $\Lambda_1(\Omega^\#)$, we have $0 < C_V \leq 1$ by construction.

Next, we consider the case when Ω is an unbounded C^∞ domain. Choose a nested sequence Ω_k of bounded C^∞ domains where

$$\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \cdots \subseteq \Omega \quad \text{and} \quad \bigcup_{k=1}^{\infty} \Omega_k = \Omega.$$

Writing $V_k = \gamma(\Omega_k)$, our work above shows

$$\Lambda_1(\Omega_k) \geq C_{V_k} \Lambda_1(\Omega_k^\#).$$

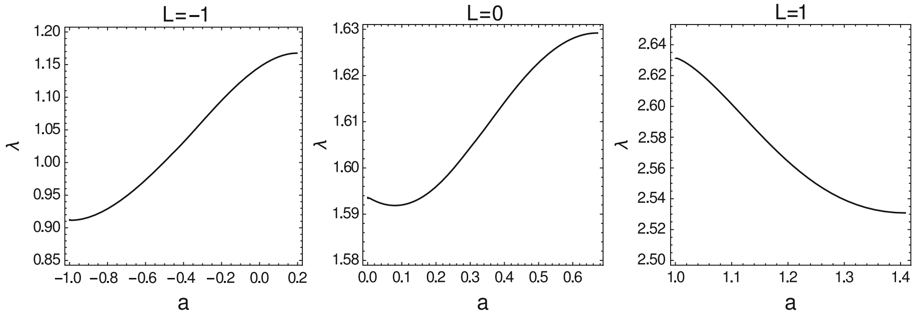

Fig. 2 Values of the minimal $\lambda = \sqrt{J_{a,b}}$ as a function of a for $L = -1$, $L = 0$, and $L = 1$

Invoking Proposition 6, we see

$$\Lambda_1(\Omega) \geq \left(\liminf_{k\to\infty} C_{V_k}\right) \Lambda_1(\Omega^{\#}).$$

Since our initial domain Ω_1 may be taken arbitrarily close to Ω in measure, the result follows. □

Our C_V values are plotted in Fig. 1. Note that C_V is at least .91 for all Gaussian volumes V. By the time the Gaussian volume reaches .5, the value C_V appears almost indistinguishable from 1.

We end with a discussion of the constant $\widetilde{J}$, i.e., the minimal $J_{a,b}$, defined in the proof of Theorem 1. For a given domain Ω with $\gamma(\Omega) = \gamma(H_L)$, assume a and b are numbers satisfying $\gamma(H_a) + \gamma(H_b) = \gamma(H_L)$. We define the function $F_\lambda(a)$ as

$$F_\lambda(a) = g(a)h_\lambda(a)U_-(b)U_+(b) + g(b)h_\lambda(b)U_-(a)U_+(a).$$

Our Eq. (13) can be written as $F_\lambda(a) = 0$, so for any choice of L and $a > L$ (and hence b), we may use this function to numerically compute the eigenvalues of the Euler–Lagrange system studied in Sect. 7. Because of the symmetry of the problem in a and b, we need only consider the values of a ranging from $a = L, b = \infty$ (where all of the volume is given to one half-space) to the symmetric case $a = b$ (where the half-spaces have equal measure). The contour plots of $F_\lambda(a) = 0$ give us the (square roots of) eigenvalues $J_{a,b} = \lambda^2$ as functions of a. Three are shown in Fig. 2. From these we see that the location of the minimum $J_{a,b}$ varies with L. We need only consider the values of a satisfying $L \leq a \leq a^*$, where a^* is the value at which $a = b$ and our two half-spaces have equal measure.

As $L \to -\infty$ (volume $V \to 1$), we see the a minimizing $J_{a,b}$ tends very quickly to L; that is, a single half-space is very nearly the minimal case. However, as $L \to +\infty$, we see $J_{a,b}$ is minimized very close to the symmetric case $a = a^*$. The transition from one regime to the other occurs very quickly.

In the corresponding $J_{a,b}$ minimization problem for the Euclidean clamped plate, which distributes volume across two balls, the a minimizing $J_{a,b}$ occurs in the case $a = L, b = 0$ (all of the volume is given to one ball) in dimensions $n = 2, 3$, but for $n \geq 4$ $J_{a,b}$ is minimized in the symmetric case $a = b$ (the two balls have equal volume) [5]. However, our $J_{a,b}$ minimization problem in Gauss space is independent of the original dimension, so it is not so strange to see both types of Euclidean behaviors reflected.

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Appendix: Existence of the minimizer for $J_{a,b}$

In this appendix, we establish the existence of a minimizer to the $J_{a,b}$ minimization problem of Sect. 7. Following the approach of [4], we consider the equivalent problem

$$J_{a,b} = \inf_{(v,w)} \int_{H_a} (\mathscr{L} v)^2 \,\mathrm{d}\gamma + \int_{H_b} (\mathscr{L} w)^2 \,\mathrm{d}\gamma,$$

where the inf is taken over all pairs (v, w) satisfying the conditions stated at the beginning of Sect. 7, together with the normalization requirement

$$\int_{H_a} v^2 \,\mathrm{d}\gamma + \int_{H_b} w^2 \,\mathrm{d}\gamma = 1.$$

In particular, the functions under consideration depend only on the first coordinate x_1, so we sometimes write $v(x_1)$ in place of $v(x)$ and the standard derivative v' in place of $\frac{\partial v}{\partial x_1}$.

Let (v_m, w_m) be a minimizing sequence for $J_{a,b}$, so

$$\int_{H_a} (\mathscr{L} v_m)^2 \,\mathrm{d}\gamma + \int_{H_b} (\mathscr{L} w_m)^2 \,\mathrm{d}\gamma \to J_{a,b}.$$

We first show that the sequences v_m and w_m are bounded in $H^2(H_a, \gamma)$ and $H^2(H_b, \gamma)$, respectively. Since we chose minimizing sequences, the real sequences $\int_{H_a} (\mathscr{L} v_m)^2 \,\mathrm{d}\gamma$ and $\int_{H_a} v_m^2 \,\mathrm{d}\gamma$ are bounded. Since $x_1 v_m' \in L^1(H_a, \gamma)$, it must be the case that $|x_1 v_m'(x_1) g(x_1)|$ does not diverge to infinity as $x_1 \to \infty$. Thus, there exists a sequence x_k^m in (a, ∞) with $x_k^m \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} v_m'(x_k^m) g(x_k^m) = 0. \tag{14}$$

Since $\| \mathscr{L} v_m \|_{L^1(H_a,\gamma)}$ is a bounded sequence, we also have

$$\int_{H_a} \mathscr{L} v_m \,\mathrm{d}\gamma = \lim_{b\to\infty} [g(b) v_m'(b) - g(a) v_m'(a)],$$

so we deduce by (14) that the sequence

$$v_m'(a) \text{ is bounded.} \tag{15}$$

We next set $\tilde{v}_m(x_1) := v_m(x_1) - v_m'(a)(x_1 - a)$ and note that $\tilde{v}_m(a) = \tilde{v}_m'(a) = 0$, so $\tilde{v}_m \in H_0^2(H_a, \gamma)$. Our proof of coercivity (Proposition 5) gives

$$\| \mathscr{L} \tilde{v}_m \|^2_{L^2(H_a,\gamma)} + \| \tilde{v}_m \|^2_{L^2(H_a,\gamma)} = \| \tilde{v}_m \|^2_{H^2(H_a)}. \tag{16}$$

Since $\mathscr{L} \tilde{v}_m = \mathscr{L} v_m + x_1 v_m'(a)$, the triangle inequality and Eqs. (15) and (16) may be used to show the sequence $\| v_m \|_{H^2(H_a,\gamma)}$ is bounded. An analogous argument yields boundedness of $\| w_m \|_{H^2(H_b,\gamma)}$.

By Banach–Alaoglu, we may pass to a subsequence and assume (v_m, w_m) converges to a pair (v, w) weakly in H^2. We claim that

$$\liminf_{m\to\infty} \int_{H_a} (\mathscr{L} v_m)^2 \, \mathrm{d}\gamma \geq \int_{H_a} (\mathscr{L} v)^2 \, \mathrm{d}\gamma.$$

To prove the claim, we first write $H_{a,K}$ for the truncated half-space (vertical strip) $\{x \in \mathbb{R}^n : a < x_1 < a + K\}$. Note that

$$\begin{aligned}\int_{H_a} (\mathscr{L} v_m)^2 \, \mathrm{d}\gamma &\geq \int_{H_{a,K}} (\mathscr{L} v_m)^2 \, \mathrm{d}\gamma \\ &\geq 2 \int_{H_{a,K}} (\mathscr{L} v)(\mathscr{L} v_m) \, \mathrm{d}\gamma - \int_{H_{a,K}} (\mathscr{L} v)^2 \, \mathrm{d}\gamma.\end{aligned}$$

Taking the liminf as $m \to \infty$ and noting $H_{a,K}$ is bounded in the x_1 direction, we deduce

$$\liminf_{m\to\infty} \int_{H_a} (\mathscr{L} v_m)^2 \, \mathrm{d}\gamma \geq \int_{H_{a,K}} (\mathscr{L} v)^2 \, \mathrm{d}\gamma.$$

Taking $K \to \infty$ gives the claim.

We now have

$$\liminf_{m\to\infty} \int_{H_a} (\mathscr{L} v_m)^2 \, \mathrm{d}\gamma \geq \int_{H_a} (\mathscr{L} v)^2 \, \mathrm{d}\gamma \quad \text{and} \quad \liminf_{m\to\infty} \int_{H_b} (\mathscr{L} w_m)^2 \, \mathrm{d}\gamma \geq \int_{H_b} (\mathscr{L} w)^2 \, \mathrm{d}\gamma.$$

We therefore see

$$J_{a,b} \geq \int_{H_a} (\mathscr{L} v)^2 \, \mathrm{d}\gamma + \int_{H_b} (\mathscr{L} w)^2 \, \mathrm{d}\gamma.$$

By [16, Proposition 2.2], $H_0^1(\Omega, \gamma)$ embeds compactly in $L^2(\Omega, \gamma)$, and hence, we may assume v_m and w_m converge to v and w in $L^2(H_a, \gamma)$ and $L^2(H_b, \gamma)$, respectively. The normalization condition

$$\int_{H_a} v^2 \, \mathrm{d}\gamma + \int_{H_b} w^2 \, \mathrm{d}\gamma = 1$$

is therefore preserved.

The space of functions in $H^2(H_a, \gamma)$ that depend on a single variable x_1 embeds compactly in $C^1[a, a + N]$ (see, e.g., [3, Theorem 6.3]), from which it follows that v and w satisfy in the classical sense the condition

$$\mathrm{e}^{-\frac{1}{2}a^2} \frac{\partial v}{\partial x_1}(a) = \mathrm{e}^{-\frac{1}{2}b^2} \frac{\partial w}{\partial x_1}(b).$$

Writing $\mu = J_{a,b}$, the minimizer (v, w) also satisfies

$$\begin{aligned}&\int_{H_a} (\mathscr{L}^2 v - \mu v)\phi \, \mathrm{d}\gamma + \int_{H_b} (\mathscr{L}^2 w - \mu w)\psi \, \mathrm{d}\gamma \\ &\quad + \mathscr{L} v(a) g(a) \frac{\partial \phi}{\partial x_1}(a) + \mathscr{L} w(b) g(b) \frac{\partial \psi}{\partial x_1}(b) = 0\end{aligned}$$

for all C^2 functions ϕ and ψ that depend on x_1, equal zero at $x_1 = a$ and $x_1 = b$, vanish for sufficiently large x_1, and satisfy the condition

$$\mathrm{e}^{-\frac{1}{2}a^2} \frac{\partial \phi}{\partial x_1}(a) = \mathrm{e}^{-\frac{1}{2}b^2} \frac{\partial \psi}{\partial x_1}(b).$$

Choosing $\psi = 0$ and $\phi = 0$ one at a time, we deduce $\mathscr{L}^2 v = \mu v$ and $\mathscr{L}^2 w = \mu w$, respectively, along with the final boundary condition

$$\mathscr{L} v(a) + \mathscr{L} w(b) = 0.$$

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