Annali di Matematica (2016) 195:1737–1748
DOI 10.1007/s10231-015-0545-4

Existence of solution for a general class of elliptic equations with exponential growth

Anderson L. A. de Araujo[1] · **Marcelo Montenegro**[2]

Received: 7 October 2014 / Accepted: 26 November 2015 / Published online: 12 December 2015

Abstract We use Galerkin approximations to show the existence of solution for a class of elliptic equations on bounded domains in $\mathbb{R}^2$ with subcritical or critical exponential nonlinearities. We are able to solve the problem under more general assumptions usually assumed in the variational the approach, but not in our paper.

Keywords Dirichlet problem · Galerkin approximation · Trudinger–Moser inequality · Exponential growth · Conformal geometry

Mathematics Subject Classification 35B38 · 35J92 · 35B33 · 35J62

1 Introduction

We prove the existence of solution of the problem

$$\begin{cases} -\Delta v = \lambda v^q + f(v) & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

This author was partially supported by FAPESP, Brazil, Grant 2013/22328-8. This author was partially supported by CNPq, Brazil.

✉ Anderson L. A. de Araujo
anderson.araujo@ufv.br

Marcelo Montenegro
msm@ime.unicamp.br

1 UFV, Viçosa, Brazil

2 Universidade Estadual de Campinas, Campinas, Brazil

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $\lambda > 0$ is a parameter, $0 < q < 1$, $f : [0, \infty) \to \mathbb{R}$ is a continuous function, and

$$0 \leq f(s)s \leq C|s|^p \exp(\alpha\, s^2) \tag{2}$$

where $2 < p < +\infty$ and $\alpha > 0$.

We state our main result.

Theorem 1.1 *Suppose that $f : [0, \infty) \to \mathbb{R}$ is a continuous function satisfying (2). Then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$, the problem (1) has a positive weak solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$.*

Elliptic problems of the type

$$\begin{cases} -\Delta v = g(x, v) \text{ in } \Omega \\ v = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{3}$$

in $\Omega \subset \mathbb{R}^2$ where $g(x, v)$ is continuous and behaves like $\exp(\alpha|v|^2)$ as $|v| \to +\infty$ have been studied by many authors, see [6,10–12,16,19]. One of the main ingredients is the Trudinger–Moser inequality introduced in [18,21], namely. Given $u \in H_0^1(\Omega)$, then

$$e^{\sigma|u|^2} \in L^1(\Omega) \text{ for every } \sigma > 0, \tag{4}$$

and there exists a positive constant L such that

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega e^{\sigma|u|^2} \mathrm{d}x \leq L \text{ for every } \sigma \leq 4\pi. \tag{5}$$

We say that g has subcritical growth at $+\infty$ if for every $\sigma > 0$

$$\lim_{s\to+\infty} \frac{|g(x, s)|}{e^{\sigma s^2}} = 0$$

and g has critical growth at $+\infty$ if there exists $\sigma_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{|g(x, s)|}{e^{\sigma s^2}} = 0\, \forall\, \sigma > \sigma_0 \quad \text{and} \quad \lim_{s\to+\infty} \frac{|g(x, s)|}{e^{\sigma s^2}} = +\infty\, \forall\, \sigma < \sigma_0.$$

The only assumptions we assume are that $0 < q < 1$, f is continuous and satisfies the growth assumption (2), and thus the nonlinearity $g(s) = \lambda s^q + f(s)$ of problem (1) can have subcritical or critical behavior at $+\infty$.

Most papers treat problem (3) by means of variational methods, and then usually it is assumed that g has subcritical or critical growth and sometimes $g(s) \geq c|s|^p$, where $c > 0$ is a constant, see [12]. Another common assumption on g is the so-called Ambrosetti–Rabinowitz condition

$$\exists R > 0 \text{ and } \theta > 2 \text{ such that } 0 < \theta G(x, s) \leq s g(x, s)\ \forall |s| \geq R \text{ and } x \in \Omega,$$

where $G(s) = \int_0^s g(t)\mathrm{d}t$, see [10–12].

Even when the Ambrosetti–Rabinowitz can be dropped, some conditions have to be assumed to give compactness of the Palais-Samle sequences or Cerami sequences, see for instance [16] where they assume

$$g : \overline{\Omega} \times \mathbb{R} \text{ is continuous and } g(x, 0) = 0;$$
$$\exists t_0 > 0 \text{ and } M > 0 \text{ such that } 0 < G(x, s) \leq M g(x, s)\ \forall |s| \geq t_0 \text{ and } x \in \Omega;$$
$$0 < 2G(x, s) \leq s g(x, s)\ \forall |s| \geq 0 \text{ and } x \in \Omega.$$

A problem in $\Omega = \mathbb{R}^2$ without Ambrosetti–Rabinowitz condition and exponential growth on g different from (2) has been addressed in [15].

We are able to solve (1) under weaker assumptions by using the Galerkin method. For that matter, we approximate f by Lipschitz functions in Sect. 2. We solve the approximating problems (11) in Sect. 3. Section 4 is devoted to prove Theorem 1.1, and in doing so, we show that the solutions v_n of problem (11) are bounded away from zero and converge to a positive solution of (1).

Problem (1) is also studied in $\mathbb{R}^2$, see for instance [1,2,4,8,22]. Problems with nonlinearities with exponential growth are also important in conformal geometry [9,17].

2 Approximating functions

To prove Theorem 1.1, we approximate f by Lipschitz functions $f_k : \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(s) = \begin{cases} -k\left[G\left(-k-\frac{1}{k}\right)-G(-k)\right], & \text{if } s \le -k \\ -k\left[G\left(s-\frac{1}{k}\right)-G(s)\right], & \text{if } -k \le s \le -\frac{1}{k} \\ k^2 s\left[G\left(-\frac{2}{k}\right)-G\left(-\frac{1}{k}\right)\right], & \text{if } -\frac{1}{k} \le s \le 0 \\ k^2 s\left[G\left(\frac{2}{k}\right)-G\left(\frac{1}{k}\right)\right], & \text{if } 0 \le s \le \frac{1}{k}. \\ k\left[G\left(s+\frac{1}{k}\right)-G(s)\right], & \text{if } \frac{1}{k} \le s \le k \\ k\left[G\left(k+\frac{1}{k}\right)-G(k)\right], & \text{if } s \ge k. \end{cases} \tag{6}$$

where $G(s) = \int_0^s f(\xi)\mathrm{d}\xi$.

The following approximation result was proved in [20].

Lemma 2.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $sf(s) \ge 0$ for every $s \in \mathbb{R}$. Then there exists a sequence $f_k : \mathbb{R} \to \mathbb{R}$ of continuous functions satisfying*

(i) $sf_k(s) \ge 0$ for every $s \in \mathbb{R}$;
(ii) $\forall k \in \mathbb{N}\ \exists c_k > 0$ such that $|f_k(\xi) - f_k(\eta)| \le c_k|\xi - \eta|$ for every $\xi, \eta \in \mathbb{R}$;
(iii) f_k converges uniformly to f in bounded subsets of $\mathbb{R}$.

The sequence f_k of the previous lemma has some additional properties.

Lemma 2.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (2) for every $s \in \mathbb{R}$. Then the sequence f_k of Lemma 2.1 satisfies*

(i) $\forall k \in \mathbb{N}$, $0 \le sf_k(s) \le C_1|s|^p \exp(4\alpha s^2)$ for every $|s| \ge \frac{1}{k}$;
(ii) $\forall k \in \mathbb{N}$, $0 \le sf_k(s) \le C_2|s|^2 \exp(4\alpha s^2)$ for every $|s| \le \frac{1}{k}$,

where C_1 and C_2 are positive constants independent of k.

Proof of Lemma 2.2 Everywhere in this proof the constant C is the one of (2).

First step. Suppose that $-k \le s \le -\frac{1}{k}$.

By the mean value theorem, there exists $\eta \in \left(s-\frac{1}{k}, s\right)$ such that

$$f_k(s) = -k\left[G\left(s-\frac{1}{k}\right)-G(s)\right] = -kG'(\eta)\left(s-\frac{1}{k}-s\right) = f(\eta)$$

and

$$sf_k(s) = sf(\eta).$$

Since $s-\frac{1}{k}<\eta<s<0$ and $f(\eta)<0$, we have $sf(\eta)\leq \eta f(\eta)$. Therefore,

$$\begin{aligned} sf_k(s)\leq \eta f(\eta) &\leq C|\eta|^p\exp(\alpha\,|\eta|^2)\\ &\leq C|s-\frac{1}{k}|^p\exp\left(\alpha\,|s-\frac{1}{k}|^2\right)\\ &\leq C\left(|s|+\frac{1}{k}\right)^p\exp\left(\alpha\,\left(|s|+\frac{1}{k}\right)^2\right)\\ &\leq C(2|s|)^p\exp(\alpha\,(2|s|)^2)\\ &= C2^p|s|^p\exp(4\alpha\,|s|^2). \end{aligned}$$

Second step. Assume $\frac{1}{k}\leq s\leq k$.

By the mean value theorem, there exists $\eta\in\left(s,s+\frac{1}{k}\right)$ such that

$$f_k(s)=k\left[G\left(s+\frac{1}{k}\right)-G(s)\right]=kG'(\eta)\left(s+\frac{1}{k}-s\right)=f(\eta)$$

and

$$sf_k(s)=sf(\eta).$$

Since $0<s<\eta<s+\frac{1}{k}$ and $f(\eta)>0$, we have $sf(\eta)\leq\eta f(\eta)$. Therefore,

$$\begin{aligned} sf_k(s)\leq \eta f(\eta) &\leq C|\eta|^p\exp(\alpha\,|\eta|^2)\\ &\leq C|s+\frac{1}{k}|^p\exp\left(\alpha\,|s+\frac{1}{k}|^2\right)\\ &\leq C(2|s|)^p\exp(\alpha\,(2|s|)^2)\\ &= C2^p|s|^p\exp(4\alpha\,|s|^2). \end{aligned}$$

Third step. Suppose that $|s|\geq k$, then

$$f_k(s)=\begin{cases} -k\left[G\left(-k-\frac{1}{k}\right)-G(-k)\right], & \text{if } s\leq -k\\ k[G\left(k+\frac{1}{k}\right)-G(k)], & \text{if } s\geq k. \end{cases} \tag{7}$$

If $s\leq -k$, by the mean value theorem, there exists $\eta\in\left(-k-\frac{1}{k},-k\right)$ such that

$$f_k(s)=k\left[G\left(-k-\frac{1}{k}\right)-G(-k)\right]=-kG'(\eta)\left(-k-\frac{1}{k}-(-k)\right)=f(\eta)$$

and

$$sf_k(s)=sf(\eta).$$

Since $-k-\frac{1}{k}<\eta<-k<0$ and $k<|\eta|<k+\frac{1}{k}$, we conclude that

$$\begin{aligned} sf_k(s)=\frac{s}{\eta}\eta f(\eta) &\leq \frac{|s|}{|\eta|}C|\eta|^p\exp(\alpha\,|\eta|^2)=C|s||\eta|^{p-1}\exp(\alpha\,|\eta|^2)\\ &\leq C|s|\left(k+\frac{1}{k}\right)^{p-1}\exp\left(\alpha\,\left(k+\frac{1}{k}\right)^2\right)\\ &\leq C|s|\left(|s|+\frac{1}{k}\right)^{p-1}\exp\left(\alpha\,\left(|s|+\frac{1}{k}\right)^2\right) \end{aligned}$$

$$\leq C|s|(2|s|)^{p-1}\exp(\alpha\,(2|s|)^2)$$
$$\leq C2^{p-1}|s|^p\exp(4\alpha\,|s|^2). \tag{8}$$

If $s \geq k$, by the mean value theorem, there exists $\eta \in \left(k, k+\frac{1}{k}\right)$ such that

$$f_k(s) = k\left[G\left(k+\frac{1}{k}\right) - G(k)\right] = kG'(\eta)\left(k+\frac{1}{k}-k\right) = f(\eta).$$

By computations similar to conclude (8) one has

$$sf_k(s) = sf(\eta) = \frac{s}{\eta}\eta f(\eta) \leq \frac{|s|}{|\eta|}C|\eta|^p\exp(\alpha\,|\eta|^2) \leq C2^{p-1}|s|^p\exp(4\alpha\,|s|^2).$$

Fourth step. Assume $-\frac{1}{k} \leq s \leq \frac{1}{k}$, then

$$f_k(s) = \begin{cases} k^2 s\left[G\left(-\frac{2}{k}\right) - G\left(-\frac{1}{k}\right)\right], & \text{if } -\frac{1}{k} \leq s \leq 0 \\ k^2 s\left[G\left(\frac{2}{k}\right) - G\left(\frac{1}{k}\right)\right], & \text{if } \ 0 \leq s \geq \frac{1}{k}. \end{cases} \tag{9}$$

If $-\frac{1}{k} \leq s \leq 0$, by the mean value theorem, there exists $\eta \in (-\frac{2}{k}, -\frac{1}{k})$ such that

$$f_k(s) = k^2 s\left[G\left(-\frac{2}{k}\right) - G\left(-\frac{1}{k}\right)\right] = k^2 sG'(\eta)\left(-\frac{2}{k} - \left(-\frac{1}{k}\right)\right) = -ksf(\eta).$$

Therefore,

$$\begin{aligned} sf_k(s) &= -ks^2 f(\eta) = -k\frac{s^2}{\eta}\eta f(\eta) \leq k\frac{s^2}{|\eta|}\eta f(\eta) \\ &\leq Ck|s|^2|\eta|^{p-1}\exp(\alpha\,|\eta|^2) \leq Ck|s|^2(\frac{2}{k})^{p-1}\exp(\alpha\,|\eta|^2) \\ &\leq C2^{p-1}|s|^2\exp\left(\alpha\,\left(\frac{2}{k}\right)^2\right) \\ &\leq C2^{p-1}|s|^2\exp(4\alpha) \leq C2^{p-1}\exp(4\alpha)|s|^2\exp(4\alpha\,|s|^2). \end{aligned} \tag{10}$$

If $0 \leq s \leq \frac{1}{k}$, by the mean value theorem, there exists $\eta \in (\frac{1}{k}, \frac{2}{k})$ such that

$$f_k(s) = k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})] = k^2 sG'(\eta)(\frac{2}{k} - \frac{1}{k}) = ksf(\eta).$$

By similar computations to conclude (10) one obtains

$$sf_k(s) = ks^2 f(\eta) = k\frac{s^2}{|\eta|}\eta f(\eta) \leq C2^{p-1}\exp(4\alpha)|s|^2\exp(4\alpha\,|s|^2).$$

The proof of the lemma follows by taking $C_1 = C2^p$ ad $C_2 = C2^{p-1}$, where C is given in (2). □

3 Approximate equation

To prove Theorem 1.1, we first show the existence of a solution of the following auxiliary problem

$$\begin{cases} -\Delta v = \lambda v^q + f_n(v) + \frac{1}{n} & \text{in } \ \Omega \\ v > 0 & \text{in } \ \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where f_n are given by Lemma 2.1 and Lemma 2.2.

We will use the Galerkin method together with the following fixed point theorem, see [20] and [14, Theorem5.2.5]. A similar approach was already used in [3].

Lemma 3.1 *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function such that $\langle F(\xi), \xi\rangle \geq 0$ for every $\xi \in \mathbb{R}^d$ with $|\xi| = r$ for some $r > 0$. Then, there exists z_0 in the closed ball $\overline{B}_r(0)$ such that $F(z_0) = 0$.*

The main result in this section is the following.

Lemma 3.2 *There exist $\lambda^* > 0$ and $n^* \in \mathbb{N}$ such that (11) has a weak nonnegative and nontrivial solution for every $\lambda \in (0, \lambda^*)$ and $n \geq n^*$.*

Proof of Lemma 3.2 Let $\mathcal{B} = \{w_1, w_2, \dots, w_m, \dots\}$ be an orthonormal basis of $H_0^1(\Omega)$ and define

$$W_m = [w_1, w_2, \dots, w_m],$$

to be the space generated by $\{w_1, w_2, \dots, w_m\}$. Define the function $F : \mathbb{R}^m \to \mathbb{R}^m$ such that $F(\xi) = (F_1(\xi), F_2(\xi), \dots, F_m(\xi))$, where $\xi = (\xi_1, \xi_2, ..., \xi_m) \in \mathbb{R}^m$,

$$F_j(\xi) = \int_\Omega \nabla v \nabla w_j - \lambda \int_\Omega (v_+)^q w_j - \int_\Omega f_n(v_+) w_j - \frac{1}{n}\int_\Omega w_j, \quad j = 1, 2, \dots, m$$

and $v = \sum_{i=1}^m \xi_i w_i$ belongs to W_m. Therefore,

$$\langle F(\xi), \xi\rangle = \int_\Omega |\nabla v|^2 - \lambda \int_\Omega (v_+)^{q+1} - \int_\Omega f_n(v_+) v_+ - \frac{1}{n}\int_\Omega v, \tag{12}$$

where $v_+ = \max\{v, 0\}$ and $v_- = v_+ - v$.

Given $v \in W_m$, we define

$$\Omega_n^+ = \left\{x \in \Omega : |v(x)| \geq \frac{1}{n}\right\}$$

and

$$\Omega_n^- = \left\{x \in \Omega : |v(x)| < \frac{1}{n}\right\}.$$

Thus, we rewrite (12) as

$$\langle F(\xi), \xi\rangle = \langle F(\xi), \xi\rangle_P + \langle F(\xi), \xi\rangle_N ,$$

where

$$\langle F(\xi), \xi\rangle_P = \int_{\Omega_n^+} |\nabla v|^2 - \lambda \int_{\Omega_n^+} (v_+)^{q+1} - \int_{\Omega_n^+} f_n(v_+) v_+ - \frac{1}{n}\int_{\Omega_n^+} v$$

and

$$\langle F(\xi), \xi\rangle_N = \int_{\Omega_n^-} |\nabla v|^2 - \lambda \int_{\Omega_n^-} (v_+)^{q+1} - \int_{\Omega_n^-} f_n(v_+) v_+ - \frac{1}{n}\int_{\Omega_n^-} v.$$

Step 1. Since $0 < q < 1$, then

$$\int_{\Omega_n^+} (v_+)^{q+1} \leq \int_\Omega |v|^{q+1} = \|v\|^{q+1}_{L^{q+1}(\Omega)} \leq C_1 \|v\|^{q+1}_{H_0^1(\Omega)}. \tag{13}$$

By virtue of Lemma 2.2 (i), we get

$$
\begin{aligned}
\int_{\Omega_n^+} f_n(v_+)v_+ &\le C_1 \int_{\Omega_n^+} |v_+|^p \exp(4\alpha|v_+|^2)\mathrm{d}x \\
&\le C_1 \left(\int_{\Omega} |v_+|^{p+1}\right)^{\frac{p}{p+1}} \left(\int_{\Omega} \exp(4\alpha(p+1)|v_+|^2)\mathrm{d}x\right)^{\frac{1}{p+1}} \\
&\le C_1 \|v\|^p_{L^{p+1}(\Omega)} \left(\int_{\Omega} \exp(4\alpha(p+1)|v|^2)\mathrm{d}x\right)^{\frac{1}{p+1}}. \qquad (14)
\end{aligned}
$$

It follows from (13) and (14) that

$$
\begin{aligned}
\langle F(\xi),\xi\rangle_P \ge& \int_{\Omega_n^+} |\nabla v|^2 - \lambda C_0 \|v\|^{q+1}_{H_0^1(\Omega)} \\
&- C_1\|v\|^p_{H_0^1(\Omega)} \left(\int_{\Omega} \exp(4\alpha(p+1)|v|^2)\mathrm{d}x\right)^{\frac{1}{p+1}} - \frac{C_3}{n}\|v\|_{H_0^1(\Omega)}, \qquad (15)
\end{aligned}
$$

where C_0, C_1, and C_3 are constants depending only on C, p, and $|\Omega|$.

Step 2. Since $0 < q < 1$, then

$$
\int_{\Omega_n^-} (v_+)^{q+1} \le \int_{\Omega_n^-} |v|^{q+1} \le |\Omega|\frac{1}{n^{q+1}}. \qquad (16)
$$

By virtue of Lemma 2.2 (ii), we get

$$
\int_{\Omega_n^-} f_n(v_+)v_+ \le C_2 \int_{\Omega_n^-} |v_+|^2 \exp(4\alpha|v_+|^2)\mathrm{d}x \le C_2 \exp(4\alpha)|\Omega|\frac{1}{n^2}. \qquad (17)
$$

It follows from (16) and (17) that

$$
\langle F(\xi),\xi\rangle_N \ge \int_{\Omega_n^-} |\nabla v|^2 - \lambda|\Omega|\frac{1}{n^{q+1}} - C_2\exp(4\alpha)|\Omega|\frac{1}{n^2} - |\Omega|\frac{1}{n^2}. \qquad (18)
$$

Thus, (15) and (18) imply

$$
\begin{aligned}
\langle F(\xi),\xi\rangle \ge& \|v\|^2_{H_0^1(\Omega)} - \lambda C_0\|v\|^{q+1}_{H_0^1(\Omega)} - C_1\|v\|^p_{H_0^1(\Omega)} \left(\int_{\Omega} \exp(4\alpha(p+1)|v|^2)\mathrm{d}x\right)^{\frac{1}{p+1}} \\
&- \frac{C_3}{n}\|v\|_{H_0^1(\Omega)} - \lambda|\Omega|\frac{1}{n^{q+1}} - C_2\exp(4\alpha)|\Omega|\frac{1}{n^2} - |\Omega|\frac{1}{n^2}. \qquad (19)
\end{aligned}
$$

Assume now that $\|v\|_{H_0^1(\Omega)} = r$ for some $r > 0$ to be chosen later. We have

$$
\int_{\Omega} \exp(4\alpha(p+1)|v|^2)\mathrm{d}x = \int_{\Omega} \exp\left(4\alpha(p+1)r^2\left(\frac{v}{\|v\|_{H_0^1(\Omega)}}\right)^2\right)\mathrm{d}x \qquad (20)
$$

and in order to apply the Trudinger–Moser inequality (5), we must have $4\alpha(p+1)r^2 \le 4\pi$. Consequently,

$$
r \le \left(\frac{\pi}{\alpha(p+1)}\right)^{\frac{1}{2}}.
$$

Then

$$\sup_{\|v\|_{H_0^1(\Omega)}\le 1}\int_\Omega \exp\left(4\alpha(p+1)r^2\left(\frac{v}{\|v\|_{H_0^1(\Omega)}}\right)^2\right)\mathrm{d}x \le L.$$

Hence,

$$\langle F(\xi),\xi\rangle \ge r^2-\lambda C_0 r^{q+1}-C_1 L^{\frac{1}{p+1}} r^p-\frac{C_3}{n}r-\lambda|\Omega|\frac{1}{n^{q+1}}-C_2\exp(4\alpha)|\Omega|\frac{1}{n^2}-|\Omega|\frac{1}{n^2}.$$

We need to choose r such that

$$r^2-C_1 L^{\frac{1}{p+1}} r^p \ge \frac{r^2}{2};$$

in other words,

$$r \le \frac{1}{(2C_1 L^{\frac{1}{p+1}})^{\frac{1}{p-2}}};$$

thus, let $r=\min\left\{\frac{1}{2(2C_1 L^{\frac{1}{p+1}})^{\frac{1}{p-2}}},\left(\frac{\pi}{\alpha(p+1)}\right)^{\frac{1}{2}}\right\}$, and hence

$$\langle F(\xi),\xi\rangle \ge \frac{r^2}{2}-\lambda C_0 r^{q+1}-\frac{C_3}{n}r-\lambda|\Omega|\frac{1}{n^{q+1}}-C_2\exp(4\alpha)|\Omega|\frac{1}{n^2}-|\Omega|\frac{1}{n^2}.$$

Now, defining $\rho=\frac{r^2}{2}-\lambda C_0 r^{q+1}$, we choose $\lambda^*>0$ such that $\rho>0$ for $\lambda<\lambda^*$. Therefore, we choose

$$\lambda^*=\frac{r^{1-q}}{4C_0}.$$

Now we choose $n^*\in\mathbb{N}$ such that

$$\frac{C_3}{n}r+\lambda|\Omega|\frac{1}{n^{q+1}}+C_2\exp(4\alpha)|\Omega|\frac{1}{n^2}+|\Omega|\frac{1}{n^2}<\frac{\rho}{2},$$

for every $n\ge n^*$. Let $\xi\in\mathbb{R}^m$, such that $|\xi|=r$, then for $\lambda<\lambda^*$ and $n\ge n^*$ we obtain

$$\langle F(\xi),\xi\rangle \ge \frac{\rho}{2}>0. \tag{21}$$

For every $n\in\mathbb{N}$, f_n is a Lipschitz function, and then by Lemma 3.1 for every $m\in\mathbb{N}$ there exists $y\in\mathbb{R}^m$ with $|y|\le r$ such that $F(y)=0$, that is, there exists $v_m\in W_m$ verifying

$$\|v_m\|_{H_0^1(\Omega)}\le r \text{ for every } m\in\mathbb{N}$$

and such that

$$\int_\Omega \nabla v_m\nabla w=\lambda\int_\Omega (v_{m+})^q w+\int_\Omega f_n(v_{m+})w+\frac{1}{n}\int_\Omega w,\quad \forall\, w\in W_m. \tag{22}$$

Since $W_m\subset H_0^1(\Omega)\ \forall\, m\in\mathbb{N}$ and r does not depend on m, then (v_m) is a bounded sequence in $H_0^1(\Omega)$. Then, for some subsequence, there exists $v\in H_0^1(\Omega)$ such that

$$v_m\rightharpoonup v \text{ weakly in } H_0^1(\Omega) \tag{23}$$

and

$$v_m\to v \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{24}$$

Let $k \in \mathbb{N}$, then for every $m \geq k$ we obtain

$$\int_\Omega \nabla v_m \nabla w_k = \lambda \int_\Omega (v_{m+})^q w_k + \int_\Omega f_n(v_{m+}) w_k + \frac{1}{n} \int_\Omega w_k, \quad \forall\, w_k \in W_k. \tag{25}$$

It follows from (23) that

$$\int_\Omega \nabla v_m \nabla w_k \to \int_\Omega \nabla v \nabla w_k \;\text{ as }\; m \to \infty \tag{26}$$

and by (24) one obtains

$$\int_\Omega f_n(v_{m+}) w_k \to \int_\Omega f_n(v_+) w_k \;\text{ as }\; m \to \infty. \tag{27}$$

Indeed, by Lemma 2.1 (ii) it follows that $|f_n(v_{m+}) - f_n(v_+)| \leq c_n |v_{m+} - v_+|$; hence,

$$\left| \int_\Omega f_n(v_{m+}) w_k - \int_\Omega f_n(v_+) w_k \right| \leq c_n \|w_k\|_{L^2(\Omega)} \|v_m - v\|_{L^2(\Omega)} \;\text{ as }\; m \to \infty$$

and then (24) implies (27). By (23), (27), and Sobolev compact imbedding, letting $m \to \infty$ one has

$$\lambda \int_\Omega (v_{m+})^q w_k + \int_\Omega f_n(v_{m+}) w_k + \frac{1}{n} \int_\Omega w_k \to \lambda \int_\Omega (v_+)^q w_k + \int_\Omega f_n(v_+) w_k + \frac{1}{n} \int_\Omega w_k. \tag{28}$$

By (25), (26), and (28)

$$\int_\Omega \nabla v \nabla w_k = \lambda \int_\Omega (v_+)^q w_k + \int_\Omega f_n(v_+) w_k + \frac{1}{n} \int_\Omega w_k, \quad \forall\, w_k \in W_k. \tag{29}$$

Since $[W_k]_{k \in \mathbb{N}}$ is dense in $H_0^1(\Omega)$, we conclude that

$$\int_\Omega \nabla v \nabla w = \lambda \int_\Omega (v_+)^q w + \int_\Omega f_n(v_+) w + \frac{1}{n} \int_\Omega w, \quad \forall\, w \in H_0^1(\Omega). \tag{30}$$

Furthermore, $v \geq 0$ in Ω. In fact, since $v_- \in H_0^1(\Omega)$, then from (30) we obtain

$$\int_\Omega \nabla v \nabla v_- = \lambda \int_\Omega (v_+)^q v_- + \int_\Omega f_n(v_+) v_- + \frac{1}{n} \int_\Omega v_-.$$

Hence,

$$-\|v_-\|^2_{H_0^1(\Omega)} = \int_\Omega \nabla v \nabla v_- = \int_\Omega f_n(v_+) v_- + \frac{1}{n} \int_\Omega v_- \geq 0,$$

then $v_- \equiv 0$ a.e. in Ω. □

4 Proof of the main result

In this section, we prove Theorem 1.1. We will use the unique solution $\widetilde{w}$ of the problem

$$\begin{cases} -\Delta \widetilde{w} = \widetilde{w}^q & \text{in } \;\Omega \\ \widetilde{w} > 0 & \text{in } \;\Omega \\ \widetilde{w} = 0 & \text{on } \partial\Omega, \end{cases} \tag{31}$$

for $0 < q < 1$, see for instance [7]. The solution $\widetilde{w}$ allows us to bound from below the solutions v_n of (11).

The following lemma of [20, Theorem1.1] is used to show that v_n converges to a solution v of (1).

Lemma 4.1 *Let Ω be a bounded open set in $\mathbb{R}^N$, $u_k : \Omega \to \mathbb{R}$ be a sequence function, and $g_k : \mathbb{R} \to \mathbb{R}$ be a sequence of functions such that $g_k(u_k)$ are measurable in Ω for every $k \in \mathbb{N}$. Assume that $g_k(u_k) \to v$ a.e. in Ω and $\int_\Omega |g_k(u_k)u_k|dx < C$ for a constant C independent of k. And suppose that for every $B \subset \mathbb{R}$, B bounded, there is a constant C_B depending only on B such that $|g_k(x)| \leq C_B$, for all $x \in B$ and $k \in \mathbb{N}$. Then $v \in L^1(\Omega)$ and $g_k(u_k) \to v$ in $L^1(\Omega)$.*

Proof of Theorem 1.1 By Lemma 3.2, equation (11) has a weak solution $v_n \in H_0^1(\Omega)$ for each $n \in \mathbb{N}$. Since $0 < q < 1$ and f_n is Lipschitz, then $\lambda v_n^q + f_n(v_n) + \frac{1}{n} \in L^p(\Omega)$ with $p > 2$. Hence, $v_n \in C^{1,\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$, see [13]. Therefore, $v_n \in H_0^1(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

We have by (23) that

$$v_m \rightharpoonup v_n \ \text{ weakly in } \ H_0^1(\Omega) \text{ as } m \to \infty. \tag{32}$$

Therefore,

$$\|v_n\|_{H_0^1(\Omega)} \leq \liminf_{m\to\infty} \|v_m\|_{H_0^1(\Omega)} \leq r, \ \ \forall n \in \mathbb{N},$$

and r does not depend on n. Thus, there exists $v \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v \ \text{ weakly in } \ H_0^1(\Omega) \text{ as } n \to \infty. \tag{33}$$

By Sobolev compact imbedding for $1 \leq s < +\infty$,

$$v_n \to v \text{ in } L^s(\Omega) \text{ and a.e. in } \Omega.$$

Note that

$$\begin{cases} -\Delta v_n \geq \lambda v_n^q, & \text{in } \ \Omega \\ v_n > 0 & \text{in } \ \Omega \\ v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{34}$$

By rescaling, thus $w_n = \lambda^{\frac{1}{q-1}} v_n$ and we obtain

$$-\Delta\left(\frac{w_n}{\lambda^{\frac{1}{q-1}}}\right) \geq \lambda\left(\frac{w_n}{\lambda^{\frac{1}{q-1}}}\right)^q$$

implying

$$-\Delta\, w_n \geq w_n^q. \tag{35}$$

By Lemma 3.3 of [5], it follows that $w_n \geq \widetilde{w}\ \forall n \in \mathbb{N}$, that is,

$$v_n \geq \lambda^{\frac{1}{1-q}} \widetilde{w} \text{ a.e. in } \Omega, \ \ \forall n \in \mathbb{N}. \tag{36}$$

Letting $n \to +\infty$ in (36), we obtain

$$v \geq \lambda^{\frac{1}{1-q}} \widetilde{w} \text{ a.e. in } \Omega$$

showing that $v > 0$ in Ω.

We prove now that v is a solution of (1). Since

$$v_n \to v \text{ a.e. in } \Omega,$$

we have

$$f_n(v_n(x)) \to f(v(x)) \text{ a.e. in } \Omega, \tag{37}$$

by the uniform convergence of Lemma 2.1 (iii).

Recall from (30) that

$$\int_\Omega \nabla v_n \nabla w = \lambda \int_\Omega (v_n)^q w + \int_\Omega f_n(v_n) w + \frac{1}{n} \int_\Omega w, \quad \forall\, w \in H_0^1(\Omega). \tag{38}$$

Taking $w = v_n$ in (38) and since v_n is bounded in $H_0^1(\Omega)$, we obtain

$$\int_\Omega f_n(v_n) v_n \mathrm{d}x \leq C, \tag{39}$$

for every $n \in \mathbb{N}$, where $C > 0$ is a constant independent of n. By (37), (39), and by the expression of f_n defined in (6), the assumptions of Lemma 4.1 are satisfied, implying

$$f_n(v_n) \to f(v) \text{ in } L^1(\Omega). \tag{40}$$

It follows from (4) that $e^{v^2} \in L^1(\Omega)$, and in view of (2) and Hölder inequality, we conclude that $f(v) \in L^2(\Omega)$.

By (38), we have

$$\int_\Omega \nabla v \nabla w = \lambda \int_\Omega v^q w + \int_\Omega f(v) w, \quad \forall\, w \in H_0^1(\Omega). \tag{41}$$

Since $f(v) \in L^2(\Omega)$ and $\lambda\, v^q \in L^2(\Omega)$, we conclude from (41) that $v \in H^2(\Omega)$ and

$$-\Delta v = \lambda\, v^q + f(v).$$

The proof of the theorem is complete. □

References

1. Adimurthi, A.: Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian. Annali della Scuola Normale Superiore di Pisa. Classe di Scienze **17**, 393–413 (1990)
2. Adimurthi, A., Yadava, S.L.: Multiplicity results for semilinear elliptic equations in a bounded domain of R2 involving critical exponent. Annali della Scuola Normale Superiore di Pisa. Classe di Scienze **17**, 481–504 (1990)
3. Alves, C.O., de Figueiredo, D.G.: Nonvariational elliptic systems via Galerkin methods. In: Haroske, D., Runst, T., Schmeisser, H.J. (eds.) Function Spaces, Differential Operators and Nonlinear Analysis. The Hans Triebel Anniversary Volume, 2003
4. Alves, C.O., Figueiredo, G.M.: On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$. J. Differ. Equ. **246**, 1288–1311 (2009)
5. Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)
6. Atkinson, F.V., Peletier, L.A.: Elliptic equations with critical growth. Math. Inst. Univ. Leiden, Rep. 21 (1986)
7. Brezis, H., Oswald, L.: Remarks on sublinear elliptic equations. Nonlinear Anal. TMA **10**, 55–64 (1986)
8. Cao, D.: Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$. Commun. Partial Differ. Equ. **17**, 407–435 (1992)
9. Chang, A., Yang, P.: The inequality of Moser and Trudinger and applications to conformal geometry. Commun. Pure Appl. Math. **56**, 1135–1150 (2003)
10. de Figueiredo, D.G., Miyagaki, O.H., Ruf, B.: Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. Calc. Var. Partial Differ. Equ. **3**, 139–153 (1995)
11. de Figueiredo, D.G., do Ó, J.M., Ruf, B.: On an inequality by N. Trudinger and J. Moser and related elliptic equations. Commun. Pure Appl. Math. **55**, 135–152 (2002)

12. de Freitas, L.R.: Multiplicity of solutions for a class of quasilinear equations with exponential critical growth. Nonlinear Anal. TMA **95**, 607–624 (2014)
13. Gilbarg, D., Trundiger, N.S.: Elliptic Partial Differential Equations of Second Order, 3rd edn. Springer, New York (2001)
14. Kesavan, S.: Topics in Functional Analysis and Applications. Wiley, New Jersey (1989)
15. Lam, N., Lu, G.: Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in $\mathbb{R}^N$. J. Funct. Anal. **262**, 1132–1165 (2012)
16. Lam, N., Lu, G.: Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti–Rabinowitz condition. J. Geom. Anal. **24**, 118–143 (2014)
17. Li, Y.X., Liu, P.: A Moser–Trudinger inequality on the boundary of a compact Riemann surface. Math. Z. **250**, 363–386 (2005)
18. Moser, J.: A sharp form of an inequality by Trudinger. Indiana Univ. Math. J. **20**, 1077–1092 (1971)
19. Silva, E.A.B., Soares, S.H.M.: Liouville-Gelfand type problems for the N-Laplacian on bounded domains of $\mathbb{R}^N$. Annali della Scuola Normale Superiore di Pisa. Classe di Scienze **4**, 1–30 (1999)
20. Strauss, W.A.: On weak solutions of semilinear hyperbolic equations. An. Acad. Brasil. Ciênc. **42**, 645–651 (1970)
21. Trudinger, N.S.: On the imbeddings into Orlicz spaces and applications. J. Math. Mech. **17**, 473–484 (1967)
22. Wang, Y., Yang, J., Zhang, Y.: Quasilinear elliptic equations involving the N-Laplacian with critical exponential growth in $\mathbb{R}^N$. Nonlinear Anal. TMA **71**, 6157–6169 (2009)