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Infinitely many sign-changing solutions for the nonlinear Schrödinger–Poisson system

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Abstract In this paper, we consider the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

We investigate the existence of multiple bound state solutions, in particular sign-changing solutions. By using the method of invariant sets of descending flow, we prove that this system has infinitely many sign-changing solutions. In particular, the nonlinear term includes the power-type nonlinearity $f(u) = |u|^{p-2}u$ for the well-studied case $p \in (4, 6)$, and the less studied case $p \in (3, 4)$, and for the latter case, few existence results are available in the literature.

Mathematics Subject Classification 35J20 · 35J60

1 Introduction and main results

In this paper, we are concerned with the existence of bound state solutions, in particular sign-changing solutions, to the following nonlinear Schrödinger–Poisson system

Dedicated to Jiaquan Liu on the occasion of his 70th birthday.

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$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.1}$$

In the last two decades, system (1.1) has been studied extensively due to its strong physical background. From a physical point of view, it describes systems of identical charged particles interacting each other in the case that magnetic effects could be ignored and its solution is a standing wave for such a system. The nonlinear term f models the interaction between the particles [28]. The first equation of (1.1) is coupled with a Poisson equation, which means that the potential is determined by the charge of the wave function. The term ϕu is nonlocal and concerns the interaction with the electric field. For more detailed physical aspects of systems like (1.1) and for further mathematical and physical interpretation, we refer to [3,12,13] and the references therein.

In recent years, there has been increasing attention to systems like (1.1) on the existence of positive solutions, ground states, radial and non-radial solutions and semiclassical states. Ruiz [26] considered the following problem

$$\begin{cases} -\Delta u + u + \lambda \phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.2}$$

and gave existence and nonexistence results, depending on the parameters $p \in (2, 6)$ and $\lambda > 0$. In particular, if $\lambda \geq \frac{1}{4}$, the author showed that $p = 3$ is a critical value for the existence of positive solutions. By using the concentration compactness principle, Azzollini and Pomponio [5] proved the existence of a ground state solution of (1.1) when $f(u) = |u|^{p-2}u$ and $p \in (3, 6)$. But no symmetry information concerning, this ground state solution was given. In [27], Ruiz studied the profile of the radial ground state solutions to (1.2) as $\lambda \to 0$ for $p \in \left(\frac{18}{7}, 3\right)$. Using variational method together with a perturbation argument, Ambrosetti [2] investigated the multiplicity of solutions and semiclassical states to systems like (1.1). Here, we would also like to mention the papers [4,14,15,17,21,29] for related topics.

Another topic which has increasingly received interest in recent years is the existence of sign-changing solutions of systems like (1.1). Recall that a solution (u, ϕ) to (1.1) is called a sign-changing solution if u changes its sign. Using a Nehari-type manifold and gluing solution pieces together, Kim and Seok [20] proved the existence of radial sign-changing solutions with prescribed numbers of nodal domains for (1.1) in the case where $V(x) = 1$, $f(u) = |u|^{p-2}u$, and $p \in (4, 6)$. Ianni [16] obtained a similar result to [20] for $p \in [4, 6)$, via a heat flow approach together with a limit procedure. Recently, with a Lyapunov–Schmidt reduction argument, Ianni and Vaira [18] constructed non-radial multi-peak solutions with arbitrary large numbers of positive peaks and arbitrary large numbers of negative peaks to the Schrödinger–Poisson system

$$\begin{cases} -\varepsilon^2 \Delta u + u + \phi u = f(u) & \text{in } \mathbb{R}^N, \\ -\Delta \phi = a_N u^2 & \text{in } \mathbb{R}^N \end{cases} \tag{1.3}$$

for $\epsilon > 0$ small, where $3 \leq N \leq 6$ and a_N is a positive constant. All the sign-changing solutions obtained in [16,18,20] have certain types of symmetries; they are either $O(N)$-invariant or G-invariant for some finite subgroup G of $O(N)$, and thus the system is required to have a certain group invariance. Based on variational method and Brouwer degree theory, Wang and Zhou [30] obtained a least energy sign-changing solution to (1.1) without any symmetry by seeking minimizer of the energy functional on the sign-changing Nehari manifold when $f(u) = |u|^{p-2}u$ and $p \in (4, 6)$. More recently, in the case where the system is considered on bounded domains $\Omega \subset \mathbb{R}^3$, Alves and Souto [1] obtained a similar result to [30] for a more general nonlinear term f.

To the best of our knowledge, there is no result in the literature on the existence of multiple sign-changing solutions as bound states to problem (1.1) without any symmetry and thus to prove the existence of infinitely many sign-changing solutions to problem (1.1) without any symmetry is the first purpose of the present paper. Since the approaches in [1,16,20,30], when applied to the monomial nonlinearity $f(u) = |u|^{p-2}u$, are only valid for $p \geq 4$, we want to provide an argument which covers the case $p \in (3, 4)$ and this is the second purpose of the present paper. Moreover, our method does not depend on existence of the Nehari manifold.

In what follows, we assume $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ satisfies the following condition.

(V_0) V is coercive, i.e., $\lim_{|x|\to\infty} V(x) = \infty$.

Moreover, we assume f satisfies the following hypotheses.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} \frac{f(s)}{s} = 0$.
(f_2) $\limsup_{|s|\to+\infty} \frac{|f(s)|}{|s|^{p-1}} < \infty$ for some $p \in (3, 6)$.
(f_3) There exists $\mu > 3$ such that $tf(t) \geq \mu F(t) > 0$ for all $t \neq 0$, where $F(t) = \int_0^t f(s)ds$.

As a consequence of (f_2) and (f_3), one has $3 < \mu \leq p < 6$. Our first result reads as

Theorem 1.1 *If* (V_0) *and* (f_1)–(f_3) *hold and* $\mu > 4$, *then problem* (1.1) *has one sign-changing solution. If moreover* f *is odd, then problem* (1.1) *has infinitely many sign-changing solutions.*

Remark 1.1 Assumption (V_0) is used only in deriving compactness (the (PS) condition) of the energy functional associated with (1.1). If $\mathbb{R}^3$ in problem (1.1) is replaced with a smooth bounded domain $\Omega \subset \mathbb{R}^3$, Theorem 1.1 without (V_0) and any symmetry assumption on Ω still holds.

(f_3) is the so-called Ambrosetti–Rabinowitz condition ((AR) for short). Since the nonlocal term $\int_{\mathbb{R}^3} \phi_u u^2$ in the expression of I (see Sect. 2) is homogeneous of degree 4, if μ from (f_3) satisfies $\mu > 4$ then (AR) guarantees boundedness of (PS)-sequences as well as existence of a mountain pass geometry in the sense that $I(tu) \to -\infty$ as $t \to \infty$ for each $u \neq 0$. If $\mu < 4$, (PS)-sequences may not be bounded and one has $I(tu) \to \infty$ as $t \to \infty$ for each $u \neq 0$. To overcome these difficulties, in the case $\mu < 4$, we impose on V an additional condition

(V_1) V is differentiable, $\nabla V(x) \cdot x \in L^r(\mathbb{R}^3)$ for some $r \in [\frac{3}{2}, \infty]$ and

$$2V(x) + \nabla V(x) \cdot x \geq 0 \quad \text{for a.e.} \quad x \in \mathbb{R}^3.$$

This assumption was introduced in [31,32] in order to prove compactness with the monotonicity trick of Jeanjean [19]. That $\nabla V(x) \cdot x \in L^r(\mathbb{R}^3)$ for some $r \in [\frac{3}{2}, \infty]$ plays a role only in deriving the Pohozǎev identity for solutions of (4.1) in Sect. 4, and it can clearly be weakened since solutions of (4.1) decay at infinity. Nevertheless, we do not want to go further in that direction. We state our second result as follows.

Theorem 1.2 *If* (V_0)–(V_1) *and* (f_1)–(f_3) *hold, then problem* (1.1) *has one sign-changing solution. If in addition* f *is odd, then problem* (1.1) *has infinitely many sign-changing solutions.*

Remark 1.2 The class of nonlinearities f satisfying the assumptions of Theorem 1.2 includes the monomial nonlinearity $f(u) = |u|^{p-2}u$ with $p \in (3, 4)$. Even in this special case, Theorem 1.2 seems to be the first attempt in finding sign-changing solutions to (1.1).

The idea of the proofs of Theorems 1.1 and 1.2 is to use suitable minimax arguments in the presence of invariant sets of a descending flow for the variational formulation. In particular, we make use of an abstract critical point theory developed by Liu et al. [23]. The method of invariant sets of descending flow plays an important role in the study of sign-changing solutions of elliptic problems; we refer to [6–11,24,25] and the references therein. However, with the presence of the coupling term ϕu, the techniques of constructing invariant sets of descending flow in [6–11,24,25] cannot be directly applied to system (1.1), which makes the problem more complicated. The reason is that ϕu is a non-local term and the decomposition

$$\int_{\mathbb{R}^3} \phi_u |u|^2 = \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^2 + \int_{\mathbb{R}^3} \phi_{u^-} |u^-|^2$$

does not hold in general for $u \in H^1(\mathbb{R}^3)$. To overcome this difficulty, we adopt an idea from [23] to construct an auxiliary operator A (see Sect. 2), which is the starting point in constructing a pseudo-gradient vector field guaranteeing existence of the desired invariant sets of the flow. Since $f \in C(\mathbb{R}, \mathbb{R})$ and A is merely continuous, A itself cannot be used to define the flow. Instead, A is used in a similar way to [8] to construct a locally Lipschitz continuous operator B inheriting the main properties of A, and we use B to define the flow. Finally, by minimax arguments in the presence of invariant sets, we obtain the existence of sign-changing solutions to (1.1), proving Theorem 1.1. For the proof of Theorem 1.2, the above framework is not directly applicable due to changes of geometric nature of the variational formulation. We use a perturbation approach by adding a term growing faster than monomial of degree 4 with a small coefficient $\lambda > 0$. For the perturbed problems, we apply the program above to establish the existence of multiple sign-changing solutions, and a convergence argument allows us to pass limit to the original system.

The paper is organized as follows. Section 2 contains the variational framework of our problem and some preliminary properties of ϕ_u. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we use a perturbation approach to prove Theorem 1.2.

2 Preliminaries and functional setting

In this paper, we make use of the following notations.

- $\|u\|_p := \left(\int_{\mathbb{R}^3} |u|^p\right)^{1/p}$ for $p \in [2, \infty)$ and $u \in L^p(\mathbb{R}^3)$;
- $\|u\| := \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{1/2}$ for $u \in H^1(\mathbb{R}^3)$;
- C, C_j denote (possibly different) positive constants.

For any given $u \in H^1(\mathbb{R}^3)$, the Lax–Milgram theorem implies that there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u = u^2$. It is well known that

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} \mathrm{d}y.$$

We now summarize some properties of ϕ_u, which will be used later. See, for instance, [26] for a proof.

Lemma 2.1 (1) $\phi_u(x) \geq 0,\ x \in \mathbb{R}^3$;

(2) *there exists* $C > 0$ *independent of* u *such that*

$$\int_{\mathbb{R}^3} \phi_u u^2 \leq C\|u\|^4;$$

(3) *if* u *is a radial function, then so is* ϕ_u*;*

(4) *if* $u_n \to u$ *strongly in* $L^{\frac{12}{5}}(\mathbb{R}^3)$*, then* $\phi_{u_n} \to \phi_u$ *strongly in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

Define the Sobolev space

$$E = \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 < \infty\right\}$$

with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right)\right)^{\frac{1}{2}}.$$

This is a Hilbert space, and its inner product is denoted by $(\cdot, \cdot)_E$.

Remark 2.1 By (V_0), the embedding $E \hookrightarrow L^q(\mathbb{R}^3)$ $(2 \leq q < 6)$ is compact. This fact implies the (PS) condition; see, e.g., [10]. As in [9], (V_0) can be replaced with the weaker condition:

$(V_0)'$ There exists $r > 0$ such that for any $b > 0$,

$$\lim_{|y|\to\infty} m\left(\{x \in \mathbb{R}^3 : V(x) \leq b\} \cap B_r(y)\right) = 0,$$

where $B_r(y) = \{x \in \mathbb{R}^3 : |x - y| < r\}$ and m is the Lebesgue measure in $\mathbb{R}^3$.

Let us define

$$D(f, g) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{f(x)g(y)}{4\pi|x - y|}\mathrm{d}x\mathrm{d}y.$$

In particular, for $u \in H^1(\mathbb{R}^3)$, $D(u^2, u^2) = \int_{\mathbb{R}^3} \phi_u u^2$. Moreover, we have the following properties. For a proof, we refer to [22, p. 250] and [27].

Lemma 2.2 (1) $D(f, g)^2 \leq D(f, f)D(g, g)$ *for any* $f, g \in L^{\frac{6}{5}}(\mathbb{R}^3)$*;*
(2) $D(uv, uv)^2 \leq D(u^2, u^2)D(v^2, v^2)$ *for any* $u, v \in L^{\frac{12}{5}}(\mathbb{R}^3)$.

Substituting $\phi = \phi_u$ into system (1.1), we can rewrite system (1.1) as the single equation

$$-\Delta u + V(x)u + \phi_u u = f(u), \quad u \in E. \tag{2.1}$$

We define the energy functional I on E by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} F(u).$$

It is standard to show that $I \in C^1(E, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv + \phi_u uv - f(u)v), \quad u, v \in E.$$

It is easy to verify that $(u, \phi_u) \in E \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in E$ is a critical point of I.

3 Proof of Theorem 1.1

In this section, we prove the existence of sign-changing solutions to system (1.1) in the case $\mu > 4$, working with (2.1).

3.1 Properties of operator A

We introduce an auxiliary operator A, which will be used to construct the descending flow for the functional I. Precisely, the operator A is defined as follows: for any $u \in E$, $v = A(u) \in E$ is the unique solution to the equation

$$-\Delta v + V(x)v + \phi_u v = f(u), \quad v \in E. \tag{3.1}$$

Clearly, the three statements are equivalent: u is a solution of (2.1), u is a critical point of I, and u is a fixed point of A.

Lemma 3.1 *The operator A is well defined and is continuous and compact.*

Proof Let $u \in E$ and define

$$J_0(v) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla v|^2 + (V(x) + \phi_u)v^2\right) - \int_{\mathbb{R}^3} f(u)v, \quad v \in E.$$

Then $J_0 \in C^1(E, \mathbb{R})$. By (f_1)–(f_2) and Remark 2.1, J_0 is coercive, bounded below, weakly lower semicontinuous, and strictly convex. Thus, J_0 admits a unique minimizer $v = A(u) \in E$, which is the unique solution to (3.1). Moreover, A maps bounded sets into bounded sets.

In the following, we prove that A is continuous. Let $\{u_n\} \subset E$ with $u_n \to u \in E$ strongly in E. Let $v = A(u)$ and $v_n = A(u_n)$. We need to prove $\|v_n - v\|_E \to 0$. We have

$$\begin{aligned}\|v - v_n\|_E^2 &= \int_{\mathbb{R}^3} (\phi_{u_n} v_n - \phi_u v)(v - v_n) + \int_{\mathbb{R}^3} (f(u) - f(u_n))(v - v_n)\\ &= I_1 + I_2.\end{aligned}$$

By Lemmas 2.1 and 2.2,

$$\begin{aligned} I_1 &\le \int_{\mathbb{R}^3} (\phi_{u_n} v - \phi_u v)(v - v_n)\\ &= D(u_n^2 - u^2, v(v - v_n))\\ &\le D\left(u_n^2 - u^2, u_n^2 - u^2\right)^{\frac{1}{2}} D\left(v(v - v_n), v(v - v_n)\right)^{\frac{1}{2}}\\ &\le D\left((u_n - u)^2, (u_n - u)^2\right)^{\frac{1}{4}} D\left((u_n + u)^2, (u_n + u)^2\right)^{\frac{1}{4}}\\ &\quad \times D\left(v^2, v^2\right)^{\frac{1}{4}} D\left((v - v_n)^2, (v - v_n)^2\right)^{\frac{1}{4}}\\ &\le C_1\|u_n - u\|\|u_n + u\|\|v\|\|v - v_n\|\\ &\le C_1\|u_n - u\|_E\|v - v_n\|_E.\end{aligned}$$

Now, we estimate the second term I_2. Let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi(t) \in [0, 1]$ for $t \in \mathbb{R}$, $\phi(t) = 1$ for $|t| \le 1$ and $\phi(t) = 0$ for $|t| \ge 2$. Setting

$$g_1(t) = \phi(t)f(t), \quad g_2(t) = f(t) - g_1(t).$$

By (f_1)–(f_2), there exists $C_2 > 0$ such that $|g_1(s)| \leq C_2|s|$ and $|g_2(s)| \leq C_2|s|^5$ for $s \in \mathbb{R}$. Then,

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^3} (g_1(u) - g_1(u_n))(v - v_n) + \int_{\mathbb{R}^3} (g_2(u) - g_2(u_n))(v - v_n) \\ &\leq \left(\int_{\mathbb{R}^3} |g_1(u_n) - g_1(u)|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |v - v_n|^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{\mathbb{R}^3} |g_2(u_n) - g_2(u)|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |v - v_n|^6 \right)^{\frac{1}{6}} \\ &\leq C_3 \|v - v_n\|_E \left[\left(\int_{\mathbb{R}^3} |g_1(u_n) - g_1(u)|^2 \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^3} |g_2(u_n) - g_2(u)|^{\frac{6}{5}} \right)^{\frac{5}{6}} \right]. \end{aligned}$$

Thus,

$$\begin{aligned} \|v - v_n\|_E \leq C_4 \Bigg[\|u - u_n\|_E &+ \left(\int_{\mathbb{R}^3} |g_1(u_n) - g_1(u)|^2 \right)^{\frac{1}{2}} \\ &+ \left(\int_{\mathbb{R}^3} |g_2(u_n) - g_2(u)|^{\frac{6}{5}} \right)^{\frac{5}{6}} \Bigg]. \end{aligned}$$

Therefore, by the dominated convergence theorem, $\|v - v_n\|_E \to 0$ as $n \to \infty$.

Finally, we show that A is compact. Let $\{u_n\} \subset E$ be a bounded sequence. Then $\{v_n\} \subset E$ is a bounded sequence, where, as above, $v_n = A(u_n)$. Passing to a subsequence, by Remark 2.1, we may assume that $u_n \to u$ and $v_n \to v$ weakly in E and strongly in $L^q(\mathbb{R}^3)$ as $n \to \infty$ for $q \in [2, 6)$. Consider the identity

$$\int_{\mathbb{R}^3} \left(\nabla v_n \cdot \nabla \xi + V v_n \xi + \phi_{u_n} v_n \xi \right) = \int_{\mathbb{R}^3} f(u_n)\xi, \quad \xi \in E. \tag{3.2}$$

Since $u_n \to u$ strongly in $L^{\frac{12}{5}}(\mathbb{R}^3)$, it follows from Lemma 2.1 (4) and the Sobolev imbedding theorem that $\phi_{u_n} \to \phi_u$ strongly in $L^6(\mathbb{R}^3)$. Since, in addition, $v_n \to v$ strongly in $L^{\frac{12}{5}}(\mathbb{R}^3)$, using the Hölder inequality, we have

$$\left| \int_{\mathbb{R}^3} (\phi_{u_n} v_n - \phi_u v)\xi \right| \leq \|\phi_{u_n}\|_6 \|v_n - v\|_{\frac{12}{5}} \|\xi\|_{\frac{12}{5}} + \|\phi_{u_n} - \phi_u\|_6 \|v\|_{\frac{12}{5}} \|\xi\|_{\frac{12}{5}} \to 0$$

for any $\xi \in E$. Taking limit as $n \to \infty$ in (3.2) yields

$$\int_{\mathbb{R}^3} (\nabla v \cdot \nabla \xi + V v \xi + \phi_u v \xi) = \int_{\mathbb{R}^3} f(u)\xi, \quad \xi \in E.$$

This means $v = A(u)$ and thus

$$\|v - v_n\|_E^2 = \int_{\mathbb{R}^3} \left(\phi_u v(v_n - v) - \phi_{u_n} v_n(v_n - v) \right) + \int_{\mathbb{R}^3} (f(u_n) - f(u))(v_n - v).$$

Hence, in the same way as above, $\|v - v_n\|_E \to 0$, i.e., $A(u_n) \to A(u)$ in E as $n \to \infty$. □

Remark 3.1 Obviously, if f is odd then A is odd.

Lemma 3.2 (1) $\langle I'(u), u - A(u) \rangle \geq \|u - A(u)\|_E^2$ *for all* $u \in E$;
(2) $\|I'(u)\| \leq \|u - A(u)\|_E (1 + C\|u\|_E^2)$ *for some* $C > 0$ *and all* $u \in E$.

Proof Since $A(u)$ is the solution of Eq. (3.1), we see that

$$\langle I'(u), u - A(u)\rangle = \|u - A(u)\|_E^2 + \int_{\mathbb{R}^3} \phi_u (u - A(u))^2, \tag{3.3}$$

which implies $\langle I'(u), u - A(u)\rangle \geq \|u - A(u)\|_E^2$ for all $u \in E$. For any $\varphi \in E$, we have

$$\begin{aligned}\langle I'(u), \varphi\rangle &= (u - A(u), \varphi)_E + \int_{\mathbb{R}^3} \phi_u (u - A(u))\varphi \\ &= (u - A(u), \varphi)_E + D(u^2, (u - A(u))\varphi).\end{aligned}$$

By Lemmas 2.1 and 2.2,

$$\left|D(u^2, (u - A(u))\varphi)\right| \leq C\|u\|_E^2 \|u - A(u)\|_E \|\varphi\|_E.$$

Thus, $\|I'(u)\| \leq \|u - A(u)\|_E (1 + C\|u\|_E^2)$ for all $u \in E$. □

Lemma 3.3 *For $a < b$ and $\alpha > 0$, there exists $\beta > 0$ such that $\|u - A(u)\|_E \geq \beta$ if $u \in E$, $I(u) \in [a, b]$ and $\|I'(u)\| \geq \alpha$.*

Proof For $u \in E$, by (f_3), we have

$$\begin{aligned}& I(u) - \frac{1}{\mu}(u, u - A(u))_E \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \phi_u u^2 \\ &\quad + \frac{1}{\mu}\int_{\mathbb{R}^3} \phi_u u(u - A(u)) + \int_{\mathbb{R}^3} \left(\frac{1}{\mu} f(u)u - F(u)\right) \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \phi_u u^2 + \frac{1}{\mu}\int_{\mathbb{R}^3} \phi_u u(u - A(u)).\end{aligned}$$

Then,

$$\|u\|_E^2 + \int_{\mathbb{R}^3} \phi_u u^2 \leq C_1 \left(|I(u)| + \|u\|_E \|u - A(u)\|_E + \left|\int_{\mathbb{R}^3} \phi_u u(u - A(u))\right|\right). \tag{3.4}$$

By Hölder's inequality and Lemmas 2.1 and 2.2,

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \phi_u u(u - A(u))\right| &\leq \left(\int_{\mathbb{R}^3} \phi_u (u - A(u))^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \phi_u u^2\right)^{\frac{1}{2}} \\ &\leq C_2 \|u\|_E \|u - A(u)\|_E \left(\int_{\mathbb{R}^3} \phi_u u^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, it follows from (3.4) that

$$\|u\|_E^2 \leq C_3 \left(|I(u)| + \|u\|_E \|u - A(u)\|_E + \|u\|_E^2 \|u - A(u)\|_E^2\right). \tag{3.5}$$

If there exists $\{u_n\} \subset E$ with $I(u_n) \in [a, b]$ and $\|I'(u_n)\| \geq \alpha$ such that $\|u_n - A(u_n)\|_E \to 0$ as $n \to \infty$, then it follows from (3.5) that $\{\|u_n\|_E\}$ is bounded, and by Lemma 3.2, we see that $\|I'(u_n)\| \to 0$ as $n \to \infty$, which is a contradiction. Thus, the proof is completed. □

3.2 Invariant subsets of descending flow

To obtain sign-changing solutions, we make use of the positive and negative cones as in many references such as [7,8,11,23]. Precisely, define

$$P^+ := \{u \in E : u \geq 0\} \text{ and } P^- := \{u \in E : u \leq 0\}.$$

Set for $\varepsilon > 0$,

$$P_\varepsilon^+ := \{u \in E : \operatorname{dist}(u, P^+) < \varepsilon\} \text{ and } P_\varepsilon^- := \{u \in E : \operatorname{dist}(u, P^-) < \varepsilon\},$$

where $\operatorname{dist}(u, P^\pm) = \inf_{v \in P^\pm} \|u - v\|_E$. Obviously, $P_\varepsilon^- = -P_\varepsilon^+$. Let $W = P_\varepsilon^+ \cup P_\varepsilon^-$. Then, W is an open and symmetric subset of E and $E \backslash W$ contains only sign-changing functions. On the other hand, the next lemma shows that for ε small, all sign-changing solutions to (2.1) are contained in $E \backslash W$.

Lemma 3.4 *There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,*

(1) $A(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ *and every nontrivial solution $u \in P_\varepsilon^-$ is negative,*
(2) $A(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ *and every nontrivial solution $u \in P_\varepsilon^+$ is positive.*

Proof Since the two conclusions are similar, we only prove the first one. By (f_1)–(f_2), for any fixed $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f(t)| \leq \delta|t| + C_\delta|t|^p, \quad t \in \mathbb{R}.$$

Let $u \in E$ and $v = A(u)$. By Remark 2.1, for any $q \in [2, 6]$, there exists $m_q > 0$ such that

$$\|u^\pm\|_q = \inf_{w \in P^\mp} \|u - w\|_q \leq m_q \inf_{w \in P^\mp} \|u - w\|_E = m_q \operatorname{dist}(u, P^\mp). \tag{3.6}$$

Obviously, $\operatorname{dist}(v, P^-) \leq \|v^+\|_E$. Then, by (f_3), we estimate

$$\begin{aligned}
\operatorname{dist}(v, P^-)\|v^+\|_E &\leq \|v^+\|_E^2 = (v, v^+)_E \\
&= \int_{\mathbb{R}^3} \left(f(u)v^+ - \phi_u v v^+\right) \\
&\leq \int_{\mathbb{R}^3} f(u)v^+ \leq \int_{\mathbb{R}^3} f(u^+)v^+ \\
&\leq \int_{\mathbb{R}^3} \left(\delta|u^+| + C_\delta|u^+|^{p-1}\right)|v^+| \\
&\leq \delta\|u^+\|_2\|v^+\|_2 + C_\delta\|u^+\|_p^{p-1}\|v^+\|_p \\
&\leq C\left(\delta \operatorname{dist}(u, P^-) + C_\delta \operatorname{dist}(u, P^-)^{p-1}\right)\|v^+\|_E.
\end{aligned}$$

It follows that

$$\operatorname{dist}(A(u), P^-) \leq C\left(\delta \operatorname{dist}(u, P^-) + C_\delta \operatorname{dist}(u, P^-)^{p-1}\right).$$

Thus, choosing δ small enough, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\operatorname{dist}(A(u), P^-) \leq \frac{1}{2}\operatorname{dist}(u, P^-) \quad \text{for any } u \in P_\varepsilon^-.$$

This implies that $A(\partial P_\varepsilon^-) \subset P_\varepsilon^-$. If there exists $u \in P_\varepsilon^-$ such that $A(u) = u$, then $u \in P^-$. If $u \not\equiv 0$, by the maximum principle, $u < 0$ in $\mathbb{R}^3$. □

Denote the set of fixed points of A by K, which is exactly the set of critical points of I. Since A is merely continuous, A itself is not applicable to construct a descending flow for I, and we have to construct a locally Lipschitz continuous operator B on $E_0 := E\backslash K$ which inherits the main properties of A.

Lemma 3.5 *There exists a locally Lipschitz continuous operator $B : E_0 \to E$ such that*

(1) $B(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ *and* $B(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ *for* $\varepsilon \in (0, \varepsilon_0)$*;*
(2) $\frac{1}{2}\|u - B(u)\|_E \le \|u - A(u)\|_E \le 2\|u - B(u)\|_E$ *for all* $u \in E_0$*;*
(3) $\langle I'(u), u - B(u)\rangle \ge \frac{1}{2}\|u - A(u)\|_E^2$ *for all* $u \in E_0$*;*
(4) *if* f *is odd then* B *is odd.*

Proof The proof is similar to the proofs of [6, Lemma 4.1] and [8, Lemma 2.1]. We omit the details. □

3.3 Existence of one sign-changing solution

In this subsection, we will find one sign-changing solution of (2.1) via minimax method incorporated with invariant sets of descending flow. First of all, we introduce the critical point theorem [23, Theorem 2.4]. For more details, we refer to [23].

Let X be a Banach space, $J \in C^1(X, \mathbb{R})$, $P, Q \subset X$ be open sets, $M = P \cap Q$, $\Sigma = \partial P \cap \partial Q$ and $W = P \cup Q$. For $c \in \mathbb{R}$, $K_c = \{x \in X : J(x) = c, J'(x) = 0\}$ and $J^c = \{x \in X : J(x) \le c\}$. In [23], a critical point theory on metric spaces was given, but here we only need a Banach space version of the theory.

Definition 3.1 ([23]) $\{P, Q\}$ is called an admissible family of invariant sets with respect to J at level c provided that the following deformation property holds: if $K_c\backslash W = \emptyset$, then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, there exists $\eta \in C(X, X)$ satisfying

(1) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$;
(2) $\eta \mid_{J^{c-2\varepsilon}} = id$;
(3) $\eta(J^{c+\varepsilon}\backslash W) \subset J^{c-\varepsilon}$.

Theorem A ([23]) *Assume that $\{P, Q\}$ is an admissible family of invariant sets with respect to J at any level $c \ge c_* := \inf_{u\in\Sigma} J(u)$ and there exists a map $\varphi_0 : \Delta \to X$ satisfying*

(1) $\varphi_0(\partial_1\Delta) \subset P$ *and* $\varphi_0(\partial_2\Delta) \subset Q$*,*
(2) $\varphi_0(\partial_0\Delta) \cap M = \emptyset$*,*
(3) $\sup_{u\in\varphi_0(\partial_0\Delta)} J(u) < c_*$*,*

where $\Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \ge 0,\ t_1 + t_2 \le 1\}$*,* $\partial_1\Delta = \{0\} \times [0, 1]$*,* $\partial_2\Delta = [0, 1] \times \{0\}$ *and* $\partial_0\Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \ge 0,\ t_1 + t_2 = 1\}$*. Define*

$$c = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\Delta)\backslash W} J(u),$$

where $\Gamma := \left\{\varphi \in C(\Delta, X) : \varphi(\partial_1\Delta) \subset P,\ \varphi(\partial_2\Delta) \subset Q,\ \varphi|_{\partial_0\Delta} = \varphi_0|_{\partial_0\Delta}\right\}$*. Then* $c \ge c_*$ *and* $K_c\backslash W \ne \emptyset$*.*

Now, we use Theorem A to prove the existence of a sign-changing solution to problem (2.1), and for this, we take $X = E$, $P = P_\varepsilon^+$, $Q = P_\varepsilon^-$, and $J = I$. We will show that $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional I at any level $c \in \mathbb{R}$. Indeed, if $K_c\backslash W = \emptyset$, then $K_c \subset W$. Since $\mu > 4$, by Remark 2.1, it is easy to see that I satisfies the (PS)-condition and therefore K_c is compact. Thus, $2\delta := \text{dist}(K_c, \partial W) > 0$.

Lemma 3.6 *If $K_c \backslash W = \emptyset$, then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon' < \varepsilon_0$, there exists a continuous map $\sigma : [0, 1] \times E \to E$ satisfying*

(1) $\sigma(0, u) = u$ *for* $u \in E$;
(2) $\sigma(t, u) = u$ *for* $t \in [0, 1]$, $u \notin I^{-1}[c - \varepsilon', c + \varepsilon']$;
(3) $\sigma(1, I^{c+\varepsilon} \backslash W) \subset I^{c-\varepsilon}$;
(4) $\sigma(t, \overline{P_\varepsilon^+}) \subset \overline{P_\varepsilon^+}$ *and* $\sigma(t, \overline{P_\varepsilon^-}) \subset \overline{P_\varepsilon^-}$ *for* $t \in [0, 1]$.

Proof The proof is similar to the proof of [23, Lemma 3.5]. For the sake of completeness, we give the details here. For $G \subset E$ and $a > 0$, let $N_a(G) := \{u \in E : \text{dist}(u, G) < a\}$. Then $N_\delta(K_c) \subset W$. Since I satisfies the (PS)-condition, there exist $\varepsilon_0, \alpha > 0$ such that

$$\|I'(u)\| \geq \alpha \quad \text{for} \;\; u \in I^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \backslash N_{\frac{\delta}{2}}(K_c).$$

By Lemmas 3.3 and 3.5, there exists $\beta > 0$ such that

$$\|u - B(u)\|_E \geq \beta \quad \text{for} \;\; u \in I^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \backslash N_{\frac{\delta}{2}}(K_c).$$

Without loss of generality, assume that $\varepsilon_0 \leq \frac{\beta\delta}{32}$. Let

$$V(u) = \frac{u - B(u)}{\|u - B(u)\|_E} \quad \text{for} \;\; u \in E_0 = E \backslash K,$$

and take a cut-off function $g : E \to [0, 1]$, which is locally Lipschitz continuous, such that

$$g(u) = \begin{cases} 0, & \text{if} \;\; u \notin I^{-1}[c - \varepsilon', c + \varepsilon'] \;\; \text{or} \;\; u \in N_{\frac{\delta}{4}}(K_c), \\ 1, & \text{if} \;\; u \in I^{-1}[c - \varepsilon, c + \varepsilon] \;\; \text{and} \;\; u \notin N_{\frac{\delta}{2}}(K_c). \end{cases}$$

Decreasing ε_0 if necessary, one may find a $\nu > 0$ such that $I^{-1}[c - \varepsilon_0, c + \varepsilon_0] \cap N_\nu(K) \subset N_{\delta/4}(K_c)$, and this can be seen as a consequence of the (PS) condition. Thus, $g(u) = 0$ for any $u \in N_\nu(K)$. By Lemma 3.5, $g(\cdot)V(\cdot)$ is locally Lipschitz continuous on E.

Consider the following initial value problem

$$\begin{cases} \frac{\mathrm{d}\tau}{\mathrm{d}t} = -g(\tau)V(\tau), \\ \tau(0, u) = u. \end{cases} \tag{3.7}$$

For any $u \in E$, one sees that problem (3.7) admits a unique solution $\tau(\cdot, u) \in C(\mathbb{R}^+, E)$. Define $\sigma(t, u) = \tau(\frac{16\varepsilon}{\beta} t, u)$. It suffices to check (3) and (4) since (1) and (2) are obvious.

To verify (3), we let $u \in I^{c+\varepsilon} \backslash W$. By Lemma 3.5, $I(\tau(t, u))$ is decreasing for $t \geq 0$. If there exists $t_0 \in [0, \frac{16\varepsilon}{\beta}]$ such that $I(\tau(t_0, u)) < c - \varepsilon$ then $I(\sigma(1, u)) = I\left(\tau\left(\frac{16\varepsilon}{\beta}, u\right)\right) < c - \varepsilon$. Otherwise, for any $t \in [0, \frac{16\varepsilon}{\beta}]$, $I(\tau(t, u)) \geq c - \varepsilon$. Then, $\tau(t, u) \in I^{-1}[c - \varepsilon, c + \varepsilon]$ for $t \in [0, \frac{16\varepsilon}{\beta}]$. We claim that for any $t \in \left[0, \frac{16\varepsilon}{\beta}\right]$, $\tau(t, u) \notin N_{\frac{\delta}{2}}(K_c)$. If, for some $t_1 \in \left[0, \frac{16\varepsilon}{\beta}\right]$, $\tau(t_1, u) \in N_{\frac{\delta}{2}}(K_c)$, then, since $u \notin N_\delta(K_c)$,

$$\frac{\delta}{2} \leq \|\tau(t_1, u) - u\|_E \leq \int_0^{t_1} \|\tau'(s, u)\|_E ds \leq t_1 \leq \frac{16\varepsilon}{\beta},$$

which contradicts the fact that $\varepsilon < \varepsilon_0 \le \frac{\beta\delta}{32}$. So $g(\tau(t,u)) \equiv 1$ for $t \in \left[0, \frac{16\varepsilon}{\beta}\right]$. Then by (2) and (3) of Lemma 3.5,

$$\begin{aligned} I\left(\tau\left(\frac{16\varepsilon}{\beta}, u\right)\right) &= I(u) - \int_0^{\frac{16\varepsilon}{\beta}} \left\langle I'(\tau(s,u)), V(\tau(s,u))\right\rangle \\ &\le I(u) - \int_0^{\frac{16\varepsilon}{\beta}} \frac{1}{8}\|\tau(s,u) - B\tau(s,u)\|_E \\ &\le c + \varepsilon - \frac{16\varepsilon}{\beta}\frac{\beta}{8} = c - \varepsilon. \end{aligned}$$

Finally, (4) is a consequence of (1) of Lemma 3.5 (see [24] for a detailed proof). □

Corollary 3.1 *$\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional I at any level $c \in \mathbb{R}$.*

Proof The conclusion follows from Lemma 3.6. □

In the following, we will construct φ_0 satisfying the hypotheses in Theorem A. Choose $v_1, v_2 \in C_0^\infty(\mathbb{R}^3)\backslash\{0\}$ satisfying $\mathrm{supp}(v_1)\cap\mathrm{supp}(v_2) = \emptyset$ and $v_1 \le 0$, $v_2 \ge 0$. Let $\varphi_0(t,s) := R(tv_1 + sv_2)$ for $(t,s) \in \Delta$, where R is a positive constant to be determined later. Obviously, for $t, s \in [0,1]$, $\varphi_0(0,s) = Rsv_2 \in P_\varepsilon^+$ and $\varphi_0(t,0) = Rtv_1 \in P_\varepsilon^-$.

Lemma 3.7 *For $q \in [2,6]$, there exists $m_q > 0$ independent of ε such that $\|u\|_q \le m_q\varepsilon$ for $u \in M = P_\varepsilon^+ \cap P_\varepsilon^-$.*

Proof This follows from (3.6). □

Lemma 3.8 *If $\varepsilon > 0$ is small enough, then $I(u) \ge \frac{\varepsilon^2}{2}$ for $u \in \Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, that is, $c_* \ge \frac{\varepsilon^2}{2}$.*

Proof For $u \in \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, we have $\|u^\pm\|_E \ge \mathrm{dist}(u, P^\mp) = \varepsilon$. By (f_1)–(f_2), we have $F(t) \le \frac{1}{3m_2^2}|t|^2 + C_1|t|^p$ for all $t \in \mathbb{R}$. Then, using Lemma 3.7, we see that

$$I(u) \ge \varepsilon^2 - \frac{1}{3}\varepsilon^2 - C_2\varepsilon^p \ge \frac{\varepsilon^2}{2},$$

for ε small enough. □

Proof of Theorem 1.1 (Existence part) It suffices to verify assumptions (2)–(3) in applying Theorem A. Observe that $\rho = \min\{\|tv_1 + (1-t)v_2\|_2 : 0 \le t \le 1\} > 0$. Then, $\|u\|_2 \ge \rho R$ for $u \in \varphi_0(\partial_0\Delta)$ and it follows from Lemma 3.7 that $\varphi_0(\partial_0\Delta) \cap M = \emptyset$ for R large enough. By (f_3), we have $F(t) \ge C_1|t|^\mu - C_2$ for any $t \in \mathbb{R}$. For any $u \in \varphi_0(\partial_0\Delta)$, by Lemma 2.1,

$$\begin{aligned} I(u) &\le \frac{1}{2}\|u\|_E^2 + C_3\|u\|_E^4 - \int_{\mathrm{supp}(v_1)\cup\mathrm{supp}(v_2)} F(u) \\ &\le \frac{1}{2}\|u\|_E^2 + C_3\|u\|_E^4 - C_1\|u\|_\mu^\mu + C_4, \end{aligned}$$

which together with Lemma 3.8 implies that, for R large enough and ε small enough,

$$\sup_{u\in\varphi_0(\partial_0\Delta)} I(u) < 0 < c_*.$$

According to Theorem A, I has at least one critical point u in $E\backslash(P_\varepsilon^+ \cup P_\varepsilon^-)$, which is a sign-changing solution of Eq. (2.1). Then (u, ϕ_u) is a sign-changing solution of system (1.1). □

3.4 Existence of infinitely many sign-changing solutions

In this section, we prove the existence of infinitely many sign-changing solutions to system (1.1). For this, we will make use of [23, Theorem 2.5], which we recall below.

We will use the notations from Sect. 3.3. Assume $G : X \to X$ to be an isometric involution, that is, $G^2 = id$ and $d(Gx, Gy) = d(x, y)$ for $x, y \in X$. We assume J is G-invariant on X in the sense that $J(Gx) = J(x)$ for any $x \in X$. We also assume $Q = GP$. A subset $F \subset X$ is said to be symmetric if $Gx \in F$ for any $x \in F$. The genus of a closed symmetric subset F of $X\backslash\{0\}$ is denoted by $\gamma(F)$.

Definition 3.2 ([23]) P is called a G-admissible invariant set with respect to J at level c, if the following deformation property holds, there exist $\varepsilon_0 > 0$ and a symmetric open neighborhood N of $K_c\backslash W$ with $\gamma(\overline{N}) < \infty$, such that for $\varepsilon \in (0, \varepsilon_0)$ there exists $\eta \in C(X, X)$ satisfying

(1) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$;
(2) $\eta \circ G = G \circ \eta$;
(3) $\eta\,|_{J^{c-2\varepsilon}} = id$;
(4) $\eta(J^{c+\varepsilon}\backslash(N \cup W)) \subset J^{c-\varepsilon}$.

Theorem B ([23]) *Assume that P is a G-admissible invariant set with respect to J at any level $c \geq c^* := \inf_{u\in\Sigma} J(u)$ and for any $n \in \mathbb{N}$, there exists a continuous map $\varphi_n : B_n := \{x \in \mathbb{R}^n : |x| \leq 1\} \to X$ satisfying*

(1) $\varphi_n(0) \in M := P \cap Q$, $\varphi_n(-t) = G\varphi_n(t)$ *for* $t \in B_n$,
(2) $\varphi_n(\partial B_n) \cap M = \emptyset$,
(3) $\sup_{u\in \mathrm{Fix}_G \cup \varphi_n(\partial B_n)} J(u) < c^*$, *where* $\mathrm{Fix}_G := \{u \in X : Gu = u\}$.

For $j \in \mathbb{N}$, define

$$c_j = \inf_{B\in\Gamma_j} \sup_{u\in B\backslash W} J(u),$$

where

$$\Gamma_j := \left\{ B \middle| \begin{array}{l} B = \varphi(B_n\backslash Y) \text{ for some } \varphi \in G_n,\ n \geq j, \text{ and open } Y \subset B_n \\ \text{such that } -Y = Y \text{ and } \gamma(\bar{Y}) \leq n - j \end{array} \right\}$$

and

$$G_n := \left\{ \varphi \middle| \begin{array}{l} \varphi \in C(B_n, X),\ \varphi(-t) = G\varphi(t) \text{ for } t \in B_n, \\ \varphi(0) \in M \text{ and } \varphi|_{\partial B_n} = \varphi_n|_{\partial B_n} \end{array} \right\}.$$

Then for $j \geq 2$, $c_j \geq c_*$, $K_{c_j}\backslash W \neq \emptyset$ and $c_j \to \infty$ as $j \to \infty$.

To apply Theorem B, we take $X = E$, $G = -id$, $J = I$ and $P = P_\varepsilon^+$. Then $M = P_\varepsilon^+ \cap P_\varepsilon^-$, $\Sigma = \partial P_\varepsilon^+ \cap \partial P_\varepsilon^-$, and $W = P_\varepsilon^+ \cup P_\varepsilon^-$. In this subsection, f is assumed to be odd, and, as a consequence, I is even. Now, we show that P_ε^+ is a G-admissible invariant set for the functional I at any level c. Since K_c is compact, there exists a symmetric open neighborhood N of $K_c\backslash W$ such that $\gamma(\overline{N}) < \infty$.

Lemma 3.9 *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon' < \varepsilon_0$, there exists a continuous map $\sigma : [0, 1] \times E \to E$ satisfying*

(1) $\sigma(0, u) = u$ *for* $u \in E$.
(2) $\sigma(t, u) = u$ *for* $t \in [0, 1]$, $u \notin I^{-1}[c - \varepsilon', c + \varepsilon']$.

(3) $\sigma(t,-u)=-\sigma(t,u)$ *for* $(t,u)\in[0,1]\times E$.
(4) $\sigma(1,I^{c+\varepsilon}\backslash(N\cup W))\subset I^{c-\varepsilon}$.
(5) $\sigma(t,\overline{P_\varepsilon^+})\subset\overline{P_\varepsilon^+}$, $\sigma(t,\overline{P_\varepsilon^-})\subset\overline{P_\varepsilon^-}$ *for* $t\in[0,1]$.

Proof The proof is similar to the proof of Lemma 3.6. Since I is even, B is odd and thus σ is odd in u.

Corollary 3.2 *P_ε^+ is a G-admissible invariant set for the functional I at any level c.*

Proof of Theorem 1.1 (Multiplicity part) According to Theorem B, if φ_n exists and satisfies the assumptions in Theorem B then I has infinitely many critical points in $E\backslash(P_\varepsilon^+\cup P_\varepsilon^-)$, which are sign-changing solutions to (2.1) and thus yield sign-changing solution to (1.1). It suffices to construct φ_n. For any $n\in\mathbb{N}$, choose $\{v_i\}_1^n\subset C_0^\infty(\mathbb{R}^3)\backslash\{0\}$ such that $\text{supp}(v_i)\cap\text{supp}(v_j)=\emptyset$ for $i\neq j$. We define $\varphi_n\in C(B_n,E)$ as

$$\varphi_n(t)=R_n\sum_{i=1}^n t_iv_i,\quad t=(t_1,t_2,\cdots,t_n)\in B_n,$$

where $R_n>0$. For R_n large enough, it is easy to check that all the assumptions of Theorem B are satisfied. □

4 Proof of Theorem 1.2

In this section, we do not assume $\mu>4$ and thus the argument of Sect. 3 which essentially depends on the assumption $\mu>4$ is not valid in the present case. This obstacle will be overcome via a perturbation approach which is originally due to [23]. The method from Sect. 3 can be used for the perturbed problem. By passing to the limit, we then obtain sign-changing solutions of the original problem (1.1).

Fix a number $r\in(\max\{4,p\},6)$. For any fixed $\lambda\in(0,1]$, we consider the modified problem

$$-\Delta u+V(x)u+\phi_uu=f(u)+\lambda|u|^{r-2}u,\quad u\in E \tag{4.1}$$

and its associated functional

$$I_\lambda(u)=I(u)-\frac{\lambda}{r}\int_{\mathbb{R}^N}|u|^r.$$

It is standard to show that $I_\lambda\in C^1(E,\mathbb{R})$ and

$$\langle I_\lambda'(u),v\rangle=\langle I'(u),v\rangle-\lambda\int_{\mathbb{R}^N}|u|^{r-2}uv,\quad u,\ v\in E.$$

For any $u\in E$, we denote by $v=A_\lambda(u)\in E$ the unique solution to the problem

$$-\Delta v+V(x)v+\phi_uv=f(u)+\lambda|u|^{r-2}u,\quad v\in E.$$

As in Sect. 3, one verifies that the operator $A_\lambda:E\to E$ is well defined and is continuous and compact. In the following, if the proof of a result is similar to its counterpart in Sect. 3, it will not be written out.

Lemma 4.1 (1) *$\langle I_\lambda'(u),u-A_\lambda(u)\rangle\geq\|u-A_\lambda(u)\|_E^2$ for all $u\in E$;*
(2) *there exists $C>0$ independent of λ such that $\|I_\lambda'(u)\|\leq\|u-A_\lambda(u)\|_E(1+C\|u\|_E^2)$ for all $u\in E$.*

Lemma 4.2 *For any $\lambda \in (0,1)$, $a < b$ and $\alpha > 0$, there exists $\beta(\lambda) > 0$ such that $\|u - A_\lambda(u)\|_E \geq \beta(\lambda)$ for any $u \in E$ with $I_\lambda(u) \in [a,b]$ and $\|I'_\lambda(u)\| \geq \alpha$.*

Proof Fix a number $\gamma \in (4, r)$. For $u \in E$,

$$\begin{aligned} &I_\lambda(u) - \frac{1}{\gamma}(u, u - A_\lambda(u))_E \\ &= \left(\frac{1}{2} - \frac{1}{\gamma}\right)\|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^3}\phi_u u^2 \\ &\quad + \frac{1}{\gamma}\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u)) + \int_{\mathbb{R}^3}\left(\frac{1}{\gamma}f(u)u - F(u)\right) + \lambda\left(\frac{1}{\gamma} - \frac{1}{r}\right)\|u\|_r^r. \end{aligned}$$

Then, by (f_1)–(f_2),

$$\begin{aligned} &\|u\|_E^2 + \int_{\mathbb{R}^3}\phi_u u^2 + \lambda\|u\|_r^r \\ &\leq C_1\left(|I_\lambda(u)| + \|u\|_E\|u - A_\lambda(u)\|_E + \|u\|_p^p + \left|\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u))\right|\right). \end{aligned}$$

Since

$$\left|\int_{\mathbb{R}^3}\phi_u u(u - A_\lambda(u))\right| \leq C_2\|u\|_E\|u - A_\lambda(u)\|_E\left(\int_{\mathbb{R}^3}\phi_u u^2\right)^{\frac{1}{2}},$$

one sees that

$$\begin{aligned} &\|u\|_E^2 + \int_{\mathbb{R}^3}\phi_u u^2 + \lambda\|u\|_r^r \\ &\leq C_3\left(|I_\lambda(u)| + \|u\|_p^p + \|u\|_E\|u - A_\lambda(u)\|_E + \|u\|_E^2\|u - A_\lambda(u)\|_E^2\right). \end{aligned} \tag{4.2}$$

If there exists $\{u_n\} \subset E$ with $I_\lambda(u_n) \in [a,b]$ and $\|I'_\lambda(u_n)\| \geq \alpha$ such that $\|u_n - A_\lambda(u_n)\|_E \to 0$ as $n \to \infty$, then it follows from (4.2) that, for large n,

$$\|u_n\|_E^2 + \int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \lambda\|u_n\|_r^r \leq C_4(1 + \|u_n\|_p^p).$$

Claim: $\{u_n\}$ **is bounded in** E. Otherwise, assume that $\|u_n\|_E \to \infty$ as $n \to \infty$. Then

$$\|u_n\|_E^2 + \int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \lambda\|u_n\|_r^r \leq C_5\|u_n\|_p^p. \tag{4.3}$$

By (4.3), there exists $C(\lambda) > 0$ such that for large n,

$$\|u_n\|_2^2 + \|u_n\|_r^r \leq C(\lambda)\|u_n\|_p^p.$$

Let $t \in (0,1)$ be such that

$$\frac{1}{p} = \frac{t}{2} + \frac{1-t}{r}.$$

Then, by the interpolation inequality,

$$\|u_n\|_2^2 + \|u_n\|_r^r \leq C(\lambda)\|u_n\|_p^p \leq C(\lambda)\|u_n\|_2^{tp}\|u_n\|_r^{(1-t)p},$$

from which it follows that there exist $C_1(\lambda), C_2(\lambda) > 0$ such that, for large n,

$$C_1(\lambda)\|u_n\|_2^{\frac{2}{r}} \leq \|u_n\|_r \leq C_2(\lambda)\|u_n\|_2^{\frac{2}{r}}.$$

Thus $\|u_n\|_p^p \leq C_3(\lambda)\|u_n\|_2^2$ and, by (4.3) again,

$$\|u_n\|_E^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 + \lambda\|u_n\|_r^r \leq C_4(\lambda)\|u_n\|_2^2.$$

Let $w_n = \frac{u_n}{\|u_n\|_E}$. The last inequality implies that

$$\|w_n\|_2^2 \geq (C_4(\lambda))^{-1} \tag{4.4}$$

and

$$\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 \leq C_5(\lambda)\|u_n\|_E^{-2}. \tag{4.5}$$

From (4.5), we have $\int_{\mathbb{R}^3} \phi_{w_n} w_n^2 \to 0$ as $n \to \infty$. Since $\|w_n\|_E = 1$, we assume that $w_n \to w$ weakly in E and strongly both in $L^{\frac{12}{5}}(\mathbb{R}^3)$ and in $L^2(\mathbb{R}^3)$. Note that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} (\phi_{w_n} w_n^2 - \phi_w w^2)\right| &\leq \int_{\mathbb{R}^3} |\phi_{w_n} - \phi_w| w_n^2 + \int_{\mathbb{R}^3} \phi_w |w_n^2 - w^2| \\ &\leq \|\phi_{w_n} - \phi_w\|_6 \|w_n\|_{\frac{12}{5}}^2 + \|\phi_w\|_6 \|w_n - w\|_{\frac{12}{5}} \|w_n + w\|_{\frac{12}{5}}.\end{aligned}$$

Since $w_n \to w$ strongly in $L^{\frac{12}{5}}(\mathbb{R}^3)$ and, by Lemma 2.1, $\phi_{w_n} \to \phi_w$ strongly in $L^6(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \phi_w w^2 = \lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 = 0,$$

which implies $w = 0$. But (4.4) implies $\|w\|_2^2 \geq (C_4(\lambda))^{-1}$, and thus we have a contradiction and finish the proof of the claim. The claim combined with Lemma 4.1 implies $\|I'_\lambda(u_n)\| \to 0$ as $n \to \infty$, which is again a contradiction. □

Lemma 4.3 *There exists $\varepsilon_1 > 0$ independent of λ such that for $\varepsilon \in (0, \varepsilon_1)$,*

(1) *$A_\lambda(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ and every nontrivial solution $u \in P_\varepsilon^-$ is negative.*
(2) *$A_\lambda(\partial P_\varepsilon^+) \subset P_\varepsilon^+$ and every nontrivial solution $u \in P_\varepsilon^+$ is positive.*

Lemma 4.4 *There exists a locally Lipschitz continuous map $B_\lambda : E\backslash K_\lambda \to E$, where $K_\lambda := \mathrm{Fix}(A_\lambda)$, such that*

(1) *$B_\lambda(\partial P_\varepsilon^+) \subset P_\varepsilon^+$, $B_\lambda(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ for $\varepsilon \in (0, \varepsilon_1)$;*
(2) *$\frac{1}{2}\|u - B_\lambda(u)\|_E \leq \|u - A_\lambda(u)\|_E \leq 2\|u - B_\lambda(u)\|_E$ for all $u \in E\backslash K_\lambda$;*
(3) *$\langle I'_\lambda(u), u - B_\lambda(u)\rangle \geq \frac{1}{2}\|u - A_\lambda(u)\|_E^2$ for all $u \in E\backslash K_\lambda$;*
(4) *if f is odd then B_λ is odd.*

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2 (Existence part) **Step 1.** We use Theorem A for $J = I_\lambda$. We claim that $\{P_\varepsilon^+, P_\varepsilon^-\}$ is an admissible family of invariant sets for the functional I_λ at any level c. In view of the approach in Sect. 3 and the fact that we have already had Lemmas 4.1–4.4, we need only to prove that for any fixed $\lambda \in (0, 1)$, I_λ satisfies the (PS)-condition. Assume that there exist $\{u_n\} \subset E$ and $c \in \mathbb{R}$ such that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Similar to the proof of Lemma 4.2, we have, for $\gamma \in (4, r)$,

$$
\begin{aligned}
& I_\lambda(u_n) - \frac{1}{\gamma}\langle I'_\lambda(u_n), u_n\rangle \\
&= \left(\frac{1}{2}-\frac{1}{\gamma}\right)\|u_n\|_E^2 + \left(\frac{1}{4}-\frac{1}{\gamma}\right)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 \\
&\quad + \int_{\mathbb{R}^3}\left(\frac{1}{\gamma}f(u_n)u_n - F(u_n)\right) + \lambda\left(\frac{1}{\gamma}-\frac{1}{r}\right)\|u_n\|_r^r.
\end{aligned}
$$

By (f_1)–(f_2),

$$
\|u_n\|_E^2 + \int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \lambda\|u_n\|_r^r \le C_1\left(|I_\lambda(u_n)| + \|u_n\|_E\|I'_\lambda(u_n)\| + \|u_n\|_p^p\right).
$$

Hence, for large n,

$$
\|u_n\|_E^2 + \int_{\mathbb{R}^3}\phi_{u_n}u_n^2 + \lambda\|u_n\|_r^r \le C_2(1+\|u_n\|_p^p).
$$

As in the proof of Lemma 4.2, one sees that $\{u_n\}$ is bounded in E. Then, by Remark 2.1, one can show that $\{u_n\}$ has a convergent subsequence, verifying the (PS)-condition.

Step 2. Choose $v_1, v_2 \in C_0^\infty(B_1(0))\backslash\{0\}$ such that $\operatorname{supp}(v_1)\cap\operatorname{supp}(v_2)=\emptyset$ and $v_1 \le 0$, $v_2 \ge 0$, where $B_r(0) := \{x\in\mathbb{R}^3 :\ |x|<r\}$. For $(t,s)\in\Delta$, let

$$
\varphi_0(t,s)(\cdot) := R^2\left(tv_1(R\cdot) + sv_2(R\cdot)\right), \tag{4.6}
$$

where R is a positive constant to be determined later. Obviously, for $t, s\in[0,1]$, $\varphi_0(0,s)(\cdot) = R^2sv_2(R\cdot)\in P_\varepsilon^+$ and $\varphi_0(t,0)(\cdot) = R^2tv_1(R\cdot)\in P_\varepsilon^-$. Similar to Lemma 3.8, for small $\varepsilon>0$,

$$
I_\lambda(u) \ge I_1(u) \ge \frac{\varepsilon^2}{2} \text{ for } u\in\Sigma := \partial P_\varepsilon^+\cap\partial P_\varepsilon^-,\ \lambda\in(0,1),
$$

which implies that $c_\lambda^* := \inf_{u\in\Sigma} I_\lambda(u) \ge \frac{\varepsilon^2}{2}$ for $\lambda\in(0,1)$. Let $u_t = \varphi_0(t,1-t)$ for $t\in[0,1]$. Then a direct computation shows that

(i) $\int_{\mathbb{R}^3}|\nabla u_t|^2 = R^3\int_{\mathbb{R}^3}\left(t^2|\nabla v_1|^2 + (1-t)^2|\nabla v_2|^2\right)$,
(ii) $\int_{\mathbb{R}^3}|u_t|^2 = R\int_{\mathbb{R}^3}\left(t^2v_1^2 + (1-t)^2v_2^2\right)$,
(iii) $\int_{\mathbb{R}^3}|u_t|^\mu = R^{2\mu-3}\int_{\mathbb{R}^3}(t^\mu|v_1|^\mu + (1-t)^\mu|v_2|^\mu)$,
(iv) $\int_{\mathbb{R}^3}\phi_{u_t}|u_t|^2 = R^3\int_{\mathbb{R}^3}\phi_{\tilde u_t}|\tilde u_t|^2$, where $\tilde u_t = tv_1 + (1-t)v_2$.

Since $F(t)\ge C_3|t|^\mu - C_4$ for any $t\in\mathbb{R}$, by (i)–(iv) we have, for $\lambda\in(0,1)$ and $t\in[0,1]$,

$$
\begin{aligned}
I_\lambda(u_t) &\le \frac{1}{2}\|u_t\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_t}|u_t|^2 - \int_{B_{R^{-1}}(0)}F(u_t) \\
&\le \frac{R^3}{2}\int_{\mathbb{R}^3}\left(t^2|\nabla v_1|^2 + (1-t)^2|\nabla v_2|^2\right) + \frac{R^3}{4}\int_{\mathbb{R}^3}\phi_{\tilde u_t}|\tilde u_t|^2 \\
&\quad + \frac{R}{2}\max_{|x|\le 1}V(x)\int_{\mathbb{R}^3}\left(t^2v_1^2 + (1-t)^2v_2^2\right) \\
&\quad - C_3R^{2\mu-3}\int_{\mathbb{R}^3}\left(t^\mu|v_1|^\mu + (1-t)^\mu|v_2|^\mu\right) + C_5R^{-3}.
\end{aligned}
$$

Since $\mu>3$, one sees that $I_\lambda(u_t)\to-\infty$ as $R\to\infty$ uniformly for $\lambda\in(0,1)$, $t\in[0,1]$. Hence, choosing R independent of λ and large enough, we have

$$
\sup_{u\in\varphi_0(\partial_0\Delta)} I_\lambda(u) < c_\lambda^* := \inf_{u\in\Sigma} I_\lambda(u), \quad \lambda\in(0,1).
$$

Since $\|u_t\|_2 \to \infty$ as $R \to \infty$ uniformly for $t \in [0, 1]$, it follows from Lemma 3.7 that $\varphi_0(\partial_0\Delta) \cap M = \emptyset$ for R large enough. Thus, φ_0 with a large R independent of λ satisfies the assumptions of Theorem A. Therefore, the number

$$c_\lambda = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\Delta)\setminus W} I_\lambda(u),$$

is a critical value of I_λ satisfying $c_\lambda \geq c_\lambda^*$, and there exists $u_\lambda \in E\setminus(P_\varepsilon^+ \cup P_\varepsilon^-)$ such that $I_\lambda(u_\lambda) = c_\lambda$ and $I'_\lambda(u_\lambda) = 0$.

Step 3. Passing to the limit as $\lambda \to 0$. By the definition of c_λ, we see that for $\lambda \in (0, 1)$,

$$c_\lambda \leq c(R) := \sup_{u\in\varphi_0(\Delta)} I(u) < \infty.$$

We claim that $\{u_\lambda\}_{\lambda\in(0,1)}$ is bounded in E. We first have

$$c_\lambda = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u_\lambda|^2 + V(x)u_\lambda^2\right) + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 - \int_{\mathbb{R}^3} \left(F(u_\lambda) + \frac{\lambda}{r}|u_\lambda|^r\right) \tag{4.7}$$

and

$$\int_{\mathbb{R}^3} \left(|\nabla u_\lambda|^2 + V(x)u_\lambda^2 + \phi_{u_\lambda} u_\lambda^2 - u_\lambda f(u_\lambda) - \lambda|u_\lambda|^r\right) = 0. \tag{4.8}$$

Moreover, we have the Pohožaev identity

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 + \frac{3}{2}\int_{\mathbb{R}^3} V(x)u_\lambda^2 + \frac{1}{2}\int_{\mathbb{R}^3} u_\lambda^2 \nabla V(x)\cdot x \\ + \frac{5}{4}\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 - \int_{\mathbb{R}^3} \left(3F(u_\lambda) + \frac{3\lambda}{r}|u_\lambda|^r\right) = 0.\end{aligned} \tag{4.9}$$

Multiplying (4.7) by $3 - \frac{\mu}{2}$, (4.8) by -1 and (4.9) by $\frac{\mu}{2} - 1$ and adding them up, we obtain

$$\begin{aligned}\left(3 - \frac{\mu}{2}\right) c_\lambda = &\left(\frac{\mu}{4} - \frac{1}{2}\right)\int_{\mathbb{R}^3} (2V(x) + \nabla V(x)\cdot x)u_\lambda^2 \\ &+ \left(\frac{\mu}{2} - \frac{3}{2}\right)\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 + \left(1 - \frac{\mu}{r}\right)\lambda\int_{\mathbb{R}^3} |u_\lambda|^r \\ &+ \int_{\mathbb{R}^3} (u_\lambda f(u_\lambda) - \mu F(u_\lambda)).\end{aligned} \tag{4.10}$$

Using (V_1), (f_3) and the fact that $3 < \mu \leq p < r$, one sees that $\{\int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2\}_{\lambda\in(0,1)}$ is bounded. From this fact, it can be deduced from (f_3), (4.7), and (4.8) that $\{u_\lambda\}_{\lambda\in(0,1)}$ is bounded in E.

Assume that up to a subsequence, $u_\lambda \to u$ weakly in E as $\lambda \to 0^+$. By Remark 2.1, $u_\lambda \to u$ strongly in $L^q(\mathbb{R}^3)$ for $q \in [2, 6)$. Then, by Lemma 2.1, $\phi_{u_\lambda} \to \phi_u$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. By a standard argument, we see that $I'(u) = 0$ and $u_\lambda \to u$ strongly in E as $\lambda \to 0^+$. Moreover, the fact that $u_\lambda \in E\setminus(P_\varepsilon^+ \cup P_\varepsilon^-)$ and $c_\lambda \geq \frac{\varepsilon^2}{2}$ for $\lambda \in (0, 1)$ implies $u \in E\setminus(P_\varepsilon^+ \cup P_\varepsilon^-)$ and $I(u) \geq \frac{\varepsilon^2}{2}$. Therefore, u is a sign-changing solution of (2.1). □

In the following, we prove the existence of infinitely many sign-changing solutions to (2.1). We assume that f is odd. Thanks to Lemmas 4.1–4.4, we have seen that P_ε^+ is a $G-$admissible invariant set for the functional I_λ $(0 < \lambda < 1)$ at any level c.

Proof Theorem 1.2 (Multiplicity part) **Step 1.** We construct φ_n satisfying the assumptions in Theorem B. For any $n \in \mathbb{N}$, we choose $\{v_i\}_1^n \subset C_0^\infty(\mathbb{R}^3)\setminus\{0\}$ such that

$\text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset$ for $i \neq j$. Define $\varphi_n \in C(B_n, E)$ as

$$\varphi_n(t)(\cdot) = R_n^2 \sum_{i=1}^{n} t_i v_i(R_n \cdot), \quad t = (t_1, t_2, \cdots, t_n) \in B_n, \tag{4.11}$$

where $R_n > 0$ is a large number independent of λ such that $\varphi_n(\partial B_n) \cap (P_\varepsilon^+ \cap P_\varepsilon^-) = \emptyset$ and

$$\sup_{u \in \varphi_n(\partial B_n)} I_\lambda(u) < 0 < \inf_{u \in \Sigma} I_\lambda(u).$$

Obviously, $\varphi_n(0) = 0 \in P_\varepsilon^+ \cap P_\varepsilon^-$ and $\varphi_n(-t) = -\varphi_n(t)$ for $t \in B_n$.

Step 2. For any $j \in \mathbb{N}$ and $\lambda \in (0, 1)$, we define

$$c_j(\lambda) = \inf_{B \in \Gamma_j} \sup_{u \in B \setminus W} I_\lambda(u),$$

where $W := P_\varepsilon^+ \cup P_\varepsilon^-$ and Γ_j is as in Theorem B. According to Theorem B, for any $0 < \lambda < 1$ and $j \geq 2$,

$$0 < \inf_{u \in \Sigma} I_\lambda(u) := c^*(\lambda) \leq c_j(\lambda) \to \infty \text{ as } j \to \infty$$

and there exists $\{u_{\lambda,j}\}_{j \geq 2} \subset E \setminus W$ such that $I_\lambda(u_{\lambda,j}) = c_j(\lambda)$ and $I_\lambda'(u_{\lambda,j}) = 0$.

Step 3. In a similar way to the above, for any fixed $j \geq 2$, $\{u_{\lambda,j}\}_{\lambda \in (0,1)}$ is bounded in E. Without loss of generality, we assume that $u_{\lambda,j} \to u_j$ weakly in E as $\lambda \to 0^+$. Observe that $c_j(\lambda)$ is decreasing in λ. Let $c_j = \lim_{\lambda \to 0^+} c_j(\lambda)$. Clearly $c_j(\lambda) \leq c_j < \infty$ for $\lambda \in (0, 1)$. Then, we may assume that $u_{\lambda,j} \to u_j$ strongly in E as $\lambda \to 0^+$ for some $u_j \in E \setminus W$ such that $I'(u_j) = 0$, $I(u_j) = c_j$. Since $c_j \geq c_j(\lambda)$ and $\lim_{j \to \infty} c_j(\lambda) = \infty$, $\lim_{j \to \infty} c_j = \infty$. Therefore, equation (2.1) and thus system (1.1) has infinitely many sign-changing solutions. The proof is completed. □

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