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Regularity results for weak solutions of elliptic PDEs below the natural exponent

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Abstract We prove regularity estimates for the weak solutions to the Dirichlet problem for a divergence form elliptic operator. We give L^p estimates for the second derivative for $p < 2$. Our work generalizes results due to Miranda (Ann Mat Pura Appl 63(4):353–386, 1963).

Keywords Elliptic PDEs · Regularity · Weak solutions

Mathematics Subject Classification 35B45 · 35J15 · 42B37 · 46E35

1 Introduction

In this paper, we consider the regularity of solutions to the divergence form elliptic equation

$$\begin{cases} Lu = -\operatorname{div} A\nabla u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set whose boundary $\partial\Omega$ is C^2, and $A = A(x) = (a_{ij}(x))$ is an $n \times n$ matrix of real-valued, measurable functions that satisfies the ellipticity condition

$$\lambda|\xi|^2 \leq \langle A\xi, \xi\rangle \leq \Lambda|\xi|^2, \qquad 0 < \lambda < \Lambda, \quad \xi \in \mathbb{R}^n. \tag{1.2}$$

We derive L^p estimates, $p < 2$, for solutions of this equation when A has discontinuous coefficients and $f \in L^p(\Omega)$.

This and related problems have a long history. If A is continuous and $\partial\Omega$ is $C^{2,\alpha}$, then these results are classical: see Gilbarg and Trudinger [19]. Miranda [28] showed that if $n \geq 3$, $\partial\Omega$ is C^3, and $A \in W^{1,n}(\Omega)$, then any weak solution of $Lu = f$, $f \in L^q(\Omega)$, $q \geq 2$, is a strong solution and $\|D^2u\|_{L^2(\Omega)} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)})$. This result is false when $n = 2$: for a counter-example, see Example 1.5 below.

A similar problem for non-divergence form elliptic operators was considered by Chiarenza and Franciosi [5]. They proved that if $n \geq 3$, Ω is bounded and $\partial\Omega$ is C^2, then the non-divergence form equation $\operatorname{tr}(AD^2u) = f$, with $f \in L^2(\Omega)$ and A in a certain vanishing Morrey class (a generalization of VMO), has a unique solution u satisfying $\|u\|_{W^{2,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. This was generalized by Chiarenza et al. [6], who showed that if $f \in L^p$, $1 < p < \infty$, then the same equation has a unique solution satisfying $\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$. These results in turn were further generalized by Vitanza [31–33].

Divergence form equations of the form $\operatorname{div} A\nabla u = \operatorname{div} F$ were considered by Di Fazio [15] on bounded domains with $\partial\Omega \in C^{1,1}$ and Iwaniec and Sbordone [24] on $\mathbb{R}^n$; they showed that if $A \in VMO$, then there exists a unique weak solution that satisfies $\|\nabla u\|_{L^p(\Omega)} \leq C\|F\|_{L^p(\Omega)}$, $1 < p < \infty$. The results for bounded domains were improved by Auscher and Qafsaoui [4], who showed that it suffices to assume $\partial\Omega$ is C^1 and that A does not need to be real symmetric. For a generalization to nonlinear equations, see [18]. In [27], Meyers considered the more general equation $\operatorname{div} A\nabla u = \operatorname{div} F + f$ on a bounded domain with a smooth boundary. He showed that if A satisfies (1.2), then there exists $p_0 < 2$ such that for all $p_0 < p < p_0'$, there exists a weak solution that satisfies $\|\nabla u\|_{L^p(\Omega)} \leq C(\|F\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})$ (see Theorem 3.3 below).

Our main result is a generalization of the result of Miranda to $p < 2$ and $n \geq 2$.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set such that $\partial\Omega$ is C^1. Let A be an $n \times n$ real-valued matrix that satisfies (1.2). If $A \in W^{1,n}(\Omega)$, then there exists $p_0 \in (1, 2)$ so that for all $p \in (p_0, 2)$ and $f \in L^p(\Omega)$ there exists a unique solution u of (1.1) that satisfies a local regularity estimate: given any open set $\Omega' \Subset \Omega$,*

$$\|D^2u\|_{L^p(\Omega')} \leq D^{-1}C\|f\|_{L^p(\Omega)}, \tag{1.3}$$

where C is independent of both u and f and $D = d(\Omega', \partial\Omega)$. If we further assume that $\partial\Omega$ is C^2, then

$$\|D^2u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{1.4}$$

where C is independent of both u and f.

Remark 1.2 To compare Theorem 1.1 to the work of Di Fazio et al. described above, note that if $A \in W^{1,n}$ then $A \in VMO$: see, for instance, [9].

Remark 1.3 Our techniques actually allow us to assume that A is a complex matrix that satisfies

$$|\langle A\xi, \eta\rangle| \leq \Lambda|\xi||\eta|, \qquad \lambda|\xi|^2 \leq \operatorname{Re}\langle A\xi, \xi\rangle, \quad \xi, \eta \in \mathbb{C}^n.$$

Details are left to the interested reader.

The lower bound p_0 in Theorem 1.1 is intrinsic to our method of proof. It is an open question whether our results can be extended to the full range $1 < p < 2$. The stronger assumptions on the boundary to get global regularity in Theorem 1.1 are not unexpected: There exist examples that show that for $n \geq 2$ and $p > 1$, there exists a bounded C^1 domain Ω and $f \in C^\infty(\overline{\Omega})$ such that the solution u to $\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, is not in $W^{2,1}(\Omega)$ (see [14,25]).

When $n \geq 3$, an examination of the constants shows that we can take $p = 2$ in our proof. This lets us give a new proof of the result of Miranda referred to above, as well as a local regularity result.

Corollary 1.4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set such that $\partial\Omega$ is C^1. Let A be an $n \times n$ real-valued matrix that satisfies (1.2). If $A \in W^{1,n}(\Omega)$, then for all $f \in L^2(\Omega)$, there exists a unique solution u of (1.1) that satisfies*

$$\|D^2 u\|_{L^2(\Omega')} \leq D^{-1} C \|f\|_{L^2(\Omega)},$$

where C is independent of both u and f, $\Omega' \Subset \Omega$ and $D = d(\Omega', \partial\Omega)$. If we further assume that $\partial\Omega$ is C^2, then

$$\|D^2 u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

We now consider the case $p = n = 2$. In this case, Corollary 1.4 is false, as the next example shows.

Example 1.5 Let $B = B_{1/2}(0) \subset \mathbb{R}^2$ be the open ball of radius $1/2$ centered at the origin. Then there exists a matrix $A \in W^{1,2}(B)$ satisfying (1.2) and a solution to

$$-\operatorname{div}(A\nabla u) = 0$$

such that $u \in W^{2,p}(B)$ for all $p < 2$, but $u \notin W^{2,2}(B)$.

We can adapt the proofs of Theorem 1.1 to the case $p = n = 2$ if we assume that ∇A satisfies stronger integrability conditions. We state these in the scale of Orlicz spaces—for a precise definition, see Sect. 2 below. For brevity, we only state the global regularity result.

Theorem 1.6 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set such that $\partial\Omega$ is C^2. Let A be a 2×2 real-valued matrix that satisfies (1.2). Suppose further that for some $\delta > 0$,*

$$\|\nabla A\|_{L^2(\log L)^{1+\delta}(\Omega)} < \infty. \tag{1.5}$$

If $f \in L^2(\Omega)$ then there exists a unique solution u of (1.1) that satisfies

$$\|D^2 u\|_{L^2(\Omega)} \leq C \|\nabla A\|_{L^2(\log L)^{1+\delta}(\Omega)} \|f\|_{L^2(\Omega)}.$$

Our second result gives information in the end point case when $\delta = 0$. In this case, we need to impose an additional regularity condition. Recall (cf. [35]) that if $\Omega \subset \mathbb{R}^2$, a function u is contained in the Morrey space $L^{2,\lambda}(\Omega)$ if

$$\|u\|_{L^{2,\lambda}(\Omega)} = \sup_Q \left(|Q|^{-\frac{\lambda}{2}} \int_{Q\cap\Omega} u^2 \, dx \right)^{1/2} < \infty.$$

Theorem 1.7 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set such that $\partial\Omega$ is C^2. Let A be a 2×2 real-valued matrix that satisfies (1.2) and*

$$\|A\|_{L^2(\log L)(\Omega)} < \infty.$$

Suppose further that for some $1 < r < 2$, $\nabla A \in L^{2,\frac{4}{r}-2}(\Omega)$. *If* $f \in L^2(\Omega)$ *then there exists a unique solution u of* (1.1) *that satisfies*

$$\|D^2u\|_{L^2(\Omega)} \le C(r,\Omega)\|\nabla A\|_{L^{2,\frac{4}{r}-2}(\Omega)}\|f\|_{L^2(\Omega)}.$$

Unfortunately, both of these results are weaker than they appear. In two dimensions, (1.5) implies that ∇A is continuous: see Cianchi [7,8]. Similarly, if we assume that $\nabla A \in L^{2,\frac{4}{r}-2}(\Omega)$, then we also have that A is Hölder continuous: see [19, p. 298]. Thus, both of these results follow from classical Schauder estimates [19]. Nevertheless, since our proofs are different from the classical ones they are of some interest.

It remains open whether anything can be said when $p = n = 2$ and $A \in W^{1,2}(\Omega)$ or even when $\|\nabla A\|_{L^2(\log L)(\Omega)} < \infty$. We conjecture that in this endpoint case, $D^2u \in L^{2)}(\Omega)$, where $L^{2)}$ denotes the grand Lebesgue space with norm

$$\|f\|_{L^{2)}(\Omega)} = \sup_{0<\epsilon<1}\left(\epsilon \fint_\Omega |f(x)|^{2-\epsilon}\,dx\right)^{\frac{1}{2-\epsilon}}.$$

These spaces were introduced in [22] and have proved useful in the study of endpoint estimates in PDEs [20,21]. As evidence for this conjecture, we note that the solution u given in Example 1.5 is in $L^{2)}(B)$. A stronger conjecture, also satisfied by our example, is that D^2u lies in the Orlicz space $L^2(\log L)^{-1}(\Omega)$ (This space is a proper subset of $L^{2)}$: see [20]). In both cases, our proof techniques are not sharp enough to produce these estimates and a different approach will be required.

The remainder of this paper is organized as follows. In Sect. 2, we state some preliminary definitions and weighted Fefferman–Phong type inequalities that are central to our proofs. These results depend on recent work on two-weight norm inequalities for the Riesz potential [13]. In Sect. 3, we prove Theorem 1.1. Our proof uses ideas from [5]. In Sect. 4, we consider the special case when $n = 2$: We construct Example 1.5 and sketch the proofs of Theorems 1.6 and 1.7. Throughout our notation will be standard or defined as needed. Given a vector matrix function, if way say that it belongs to a scalar function space (e.g., $A \in W^{1,n}(\Omega)$), we mean that each component function is an element of the function space; to compute the norm we first take the ℓ^2 norm of the components. Constants C, $C(n)$, etc., may change in value at each appearance.

2 Preliminary results

In this section, we give conditions on a weight w for the two-weight Sobolev inequality

$$\|fw\|_{L^p(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)}$$

to hold. Such inequalities are sometimes referred to as Fefferman–Phong inequalities: see [17]. Given the classical pointwise inequality

$$|f(x)| \le C(n)I_1(|\nabla f|)(x), \qquad f \in C_0^\infty,$$

it suffices to prove two-weight estimates for the Riesz potential of order one:

$$I_1f(x) = \Delta^{-\frac{1}{2}}f(x) = c\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-1}}\,dy.$$

In Theorems 1.1 and 1.6, we will apply a sharp sufficient condition for the Riesz potential to be bounded that was proved by Pérez [29]; we will use the version from [13, Theorem 3.6] as this gives precise values for the constants. To state this result, we need to make some definitions; for additional information on Orlicz spaces and two-weight inequalities, see [12,13]. A convex, strictly increasing function $\Phi : [0,\infty] \to [0,\infty]$ is said to be a Young function if $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. Given a Young function, there exists another Young function, $\bar{\Phi}$, called the associate function, such that $\Phi^{-1}(t)\bar{\Phi}^{-1}(t) \simeq t$. For our purposes, there are two particularly important examples of Young functions that we will use. First, if $\Phi(t) = t^r$, $r > 1$, then $\bar{\Phi}(t) = t^{r'}$. If $\Phi(t) = t^r \log(e+t)^a$, then $\bar{\Phi}(t) \simeq t^{r'} \log(e+t)^{-\frac{a}{r-1}}$.

Given $1 < p < \infty$ and a Young function Φ, define

$$\alpha_{p,\Phi} = \left(\int_1^\infty \frac{\Phi(t)}{t^p}\frac{\mathrm{d}t}{t}\right)^{1/p}. \tag{2.1}$$

Our conditions on weights are defined using a normalized Orlicz norm: given Young function Φ and a cube Q, let

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \fint_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\}.$$

Given a pair of weights (u,v) (i.e., non-negative, locally integrable functions) and Young functions Φ and Ψ, let

$$[u,v]_{A^1_{p,\Psi,\Phi}} = \sup_Q |Q|^{\frac{1}{n}} \|u^{1/p}\|_{\Psi,Q}\|v^{-1/p}\|_{\Phi,Q}.$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

Theorem 2.1 [13, Theorem 3.6] *Given $1 < p < \infty$, a pair of weights (u,v), and Young functions Φ and Ψ, we have that*

$$\|I_1\|_{L^p(v)\to L^p(u)} \le C(n,p)[u,v]_{A^1_{p,\Psi,\Phi}}\,\alpha_{p,\bar{\Phi}}\,\alpha_{p',\bar{\Psi}}.$$

Remark 2.2 In Theorem 2.1, we need to apply the integral condition in (2.1) to the associate functions $\bar{\Phi}$, $\bar{\Psi}$. If Φ and Ψ are doubling (i.e., $\Phi(2t) \le C\Phi(t)$, $t > 0$, and similarly for Ψ), then by a change of variables, this condition can be restated in terms of Φ and Ψ. See [12, Prop. 5.10] for further information.

We can now give the Sobolev inequalities needed for our results.

Lemma 2.3 *Fix $n \ge 2$ and $1 < p < n$. Let $\Omega \subset \mathbb{R}^n$. Then, for any $f \in W_0^{1,p}(\Omega)$ and $w \in L^n(\Omega)$,*

$$\|fw\|_{L^p(\Omega)} \le C(n)(p'-n')^{-1/p'}\|w\|_{L^n(\Omega)}\|\nabla f\|_{L^p(\Omega)}. \tag{2.2}$$

Proof Extend w to a function on all of $\mathbb{R}^n$ by setting it equal to 0 outside of Ω. Let $\Psi(t) = t^n$ and $\Phi(t) = t^r$, $1 < r < p$; the exact value of r is not significant. Then

$$\alpha_{p',\bar{\Psi}} = (p'-n')^{-1/p'}, \qquad \alpha_{p,\bar{\Phi}} = (p-r)^{-1/p},$$

and so we have that

$$\begin{aligned}[][w^p,1]_{A^1_{p,\Psi,\Phi}}\,\alpha_{p,\bar{\Phi}}\,\alpha_{p',\bar{\Psi}} \\ &= (p'-n')^{-1}(p-r)^{-1}\sup_Q |Q|^{1/n}\left(\fint_Q w^n\,\mathrm{d}x\right)^{1/n} \le (p'-n')^{-1}(p-r)^{-1}\|w\|_{L^n(\Omega)}.\end{aligned}$$

Therefore, by Lemma 2.1 we have that for all $f \in C_0^\infty(\Omega)$,

$$\|fw\|_{L^p(\mathbb{R}^n)} \le \|I_1(|\nabla f|)w\|_{L^p(\mathbb{R}^n)} \le C(n,p,r)(p'-n')^{-1/p'}\|w\|_{L^n(\Omega)}\|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

The desired inequality follows for all f by a standard approximation argument. □

When $n \ge 3$, we see that $w \in L^n(\Omega)$ implies the Sobolev inequality for $p = 2$. When $n = 2$, we only get the Sobolev inequality for $1 < p < 2$, and the constant blows up as p tends to 2 (and also as it tends to 1). In general, $w \in L^2(\Omega)$ will not be a sufficient condition for the Sobolev inequality when $p = n = 2$.

To prove Theorem 1.6, we can use the full power of Theorem 2.1 to prove a substitute for Lemma 2.3. To state it, we define the non-normalized Orlicz norm: given an open set Ω and an Orlicz function Ψ,

$$\|f\|_{L^\Psi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Psi\left(\frac{|f(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\}.$$

When $\Psi(t) = t^2\log(e+t)^{1+\delta}$, then we write $L^\Phi(\Omega) = L^2(\log L)^{1+\delta}(\Omega)$.

Lemma 2.4 *Given a bounded open set $\Omega \subset \mathbb{R}^2$ and $w \in L^2(\log L)^{1+\delta}(\Omega)$, if $f \in W_0^{1,2}(\Omega)$, then*

$$\|fw\|_{L^2(\Omega)} \le C\delta^{-1/2}[1+\mathrm{diam}(\Omega)]\|w\|_{L^2(\log L)^{1+\delta}(\Omega)}\|\nabla f\|_{L^2(\Omega)}. \tag{2.3}$$

Proof We begin as in the proof of Lemma 2.3, but we now take $\Psi(t) = t^2\log(e+t)^{1+\delta}$. Then

$$\alpha_{2,\bar{\Psi}} = \left(\int_1^\infty \frac{\mathrm{d}t}{t\log(e+t)^{1+\delta}}\right)^{1/2} = C\delta^{-1/2} < \infty,$$

and

$$[w^2,1]_{A^1_{2,\Phi,\Psi}} = \sup_Q |Q|^{1/2}\|w\|_{\Psi,Q}.$$

Since we may assume $\mathrm{supp}(w) \subset \Omega$, we may restrict the supremum to cubes Q such that $|Q| \le \mathrm{diam}(\Omega)^2$. Then by the definition of the norm, we have that

$$\begin{aligned}
|Q|^{1/2}\|w\|_{\Psi,Q} &= \inf\left\{\lambda > 0 : \fint_Q \frac{|Q|w(x)^2}{\lambda^2}\log\left(e + \frac{|Q|^{1/2}w(x)}{\lambda}\right)^{1+\delta}\mathrm{d}x \le 1\right\}\\
&\le \inf\left\{\lambda > 0 : \int_\Omega \frac{w(x)^2}{\lambda^2}\log\left(e + \frac{\mathrm{diam}(\Omega)w(x)}{\lambda}\right)^{1+\delta}\mathrm{d}x \le 1\right\}\\
&\le [1+\mathrm{diam}(\Omega)]\|w\|_{L^\Psi(\Omega)}.
\end{aligned}$$

The desired inequality now follows as before. □

To prove Theorem 1.7, we need an "off-diagonal" inequality for the Riesz potential. There is a version of Theorem 2.1 in this case, but we will use a stronger result due to D. R. Adams [1] (see also [34, Theorem 4.7.2]).

Theorem 2.5 *Given $1 < p < n$, $p < q < \infty$ and a weight u, if*

$$u(Q) \le K|Q|^{\frac{a}{n}},$$

where $a = \frac{q(n-p)}{p}$, then

$$\|I_1 f\|_{L^q(u)} \le C(p,q,n)K^{1/q}\|f\|_{L^p}.$$

Lemma 2.6 *Given an open set* $\Omega \subset \mathbb{R}^2$*, suppose that for* $1 < r < 2$*,* $w \in L^{2,\frac{4}{r}-2}(\Omega)$ *If* $f \in W_0^{1,2}(\Omega)$*, then*

$$\|fw\|_{L^2(\Omega)} \le C(r)\|w\|_{L^{2,\frac{4}{r}-2}(\Omega)}\|\nabla f\|_{L^r(\Omega)}. \tag{2.4}$$

Proof Extend w to a function on all of $\mathbb{R}^2$ by setting it equal to 0 outside of Ω. Define a as in Theorem 2.5 with $p = r$, $q = n = 2$. Then for all cubes Q that intersect Ω,

$$|Q|^{-\frac{a}{2}} \int_Q w(x)^2\,\mathrm{d}x = |Q|^{1-\frac{2}{r}} \int_{Q\cap\Omega} w(x)^2\,\mathrm{d}x \le \|w\|^2_{L^{2,\frac{4}{r}-2}(\Omega)}.$$

Inequality (2.4) now follows from Theorem 2.5 and an approximation argument. □

3 Proof of Theorem 1.1

We begin with two results due to Meyers [27]. The first is a coercivity condition.

Theorem 3.1 *Given a bounded open set* $\Omega \subset \mathbb{R}^n$ *with* C^1 *boundary, let* A *be an* $n \times n$ *real-valued matrix that satisfies* (1.2)*. Define the sesquilinear form*

$$\mathfrak{a}(u, v) = \int_\Omega A\nabla u \cdot \nabla v\, dx.$$

Then there exists $p_0 = p_0(n, \lambda, \Lambda, \Omega)$*,* $1 < p_0 < 2$*, such that for all* p*,* $p_0 < p \le 2$*, and all* $u \in W_0^{1,p}(\Omega)$*,*

$$\|u\|_{W_0^{1,p}(\Omega)} \approx \sup_{\|v\|_{W_0^{1,p'}(\Omega)}=1} |\mathfrak{a}(u, v)|. \tag{3.1}$$

Moreover, the constants in this equivalence depend on λ*,* Λ*,* p*,* n*, and* Ω*. They are independent of the specific matrix* A*.*

Proof The upper estimate for $\mathfrak{a}(u, v)$ is just Hölder's inequality; it is the lower estimate that is non-trivial. From the proof of Theorem 1 in [27] we have the existence of $p_0 < 2$ and $\kappa = \kappa(\lambda, \Lambda, p, \Omega) > 0$ such that

$$\inf_{\|u\|_{W_0^{1,p}}=1} \sup_{\|v\|_{W_0^{1,p'}}=1} |\mathfrak{a}(u, v)| \ge \kappa.$$

A key hypothesis in the proof is the existence of $q > p_0'$ such that for every $F \in L^q(\Omega)$, there exists a unique weak solution Φ to the equation $\Delta\Phi = \operatorname{div} F$ on Ω and the estimate

$$\|\nabla\Phi\|_q \le C\|F\|_q, \tag{3.2}$$

holds. Since $\partial\Omega$ is C^1, by Auscher and Qafsaoui [4], we have that such a solution exists and (3.2) holds for all q, $1 < q < \infty$. □

Remark 3.2 The value of p_0 is difficult to estimate from Meyer's proof. There is an elegant proof of this result in [30] that uses the Hodge decomposition. In [23] a careful estimate is given for the resulting constants; though again the exact value is not easy to determine. In passing we note that in [30], Theorem 3.1 is proved for "regular" domains, which are defined abstractly in [23]. However, regular domains include Lipschitz domains: see [21].

For our existence results, we also need the following result of Meyers [27, Theorem 1].

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set such that $\partial\Omega$ is C^1, and let A be an $n \times n$ real-valued matrix that satisfies (1.2). Then the equation*

$$Lu = \operatorname{div}(A\nabla u) = f$$

has a unique solution in $W_0^{1,p}(\Omega)$ for every $f \in L^p(\Omega)$, provided $p_0 < p < p_0'$, where p_0 is the constant from Theorem 3.1. The solution satisfies the estimate

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)},$$

where $C = C(n, \lambda, \Lambda, p, \Omega)$.

Proof of Theorem 1.1 Fix a matrix A satisfying (1.2), and fix p, $p_0 < p < 2$, where p_0 is as in Theorem 3.1. By Theorem 3.3, for any $f \in L^p(\Omega)$ the equation $Lu = f$ has a unique solution $u \in W_0^{1,p}(\Omega)$ such that

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{3.3}$$

with C independent of f.

We first prove the desired estimate on $D^2 u$ in the special case when $f \in C^\infty(\Omega)$ and $A \in C^\infty(\Omega)$; afterward, we will prove the general case by a double approximation argument. Let u be the solution of (1.1). Then $u \in C^\infty(\Omega)$: see Evans [16, Th. 3,Sec. 6.3.1]. (Note that in this result, there is an implicit assumption on the regularity of the boundary because of an appeal to a Poincaré-Sobolev type inequality for functions without compact support in Ω; C^1 is more than sufficient for this purpose.) We now have the pointwise identity

$$f = -\operatorname{div} A\nabla u = -\sum_{i,j} \left(a_{ij} u_{x_j}\right)_{x_i}.$$

Fix s with $1 \leq s \leq n$ and $\eta \in C_0^\infty(\Omega)$ with $0 \leq \eta \leq 1$. Then $\eta u_{x_s} \in W_0^{1,p}(\Omega)$, so by Theorem 3.1 there exists $v \in C_0^2(\Omega)$, $\|v\|_{W_0^{1,p'}} = 1$, and $\kappa = \kappa(n, \lambda, \Lambda, \Omega) > 0$ such that

$$\begin{aligned} |\mathfrak{a}(\eta u_{x_s}, v)| &\geq \kappa\|\eta u_{x_s}\|_{W_0^{1,p}} \geq \kappa\|\nabla\left(\eta u_{x_s}\right)\|_{L^p(\Omega)} \\ &\geq \kappa\|\eta\nabla u_{x_s}\|_{L^p(\Omega)} - \kappa\|u_{x_s}\nabla\eta\|_{L^p(\Omega)}. \end{aligned} \tag{3.4}$$

If we multiply f by ηv_{x_s}, integrate over Ω and then integrate by parts twice we get

$$\begin{aligned} \int_\Omega f\eta v_{x_s}\,\mathrm{d}x &= -\int_\Omega \sum_{i,j} \left(a_{ij}u_{x_j}\right)_{x_i} \eta v_{x_s}\,\mathrm{d}x \\ &= \int_\Omega \sum_{i,j} a_{ij}u_{x_j}\left(\eta v_{x_s}\right)_{x_i}\,\mathrm{d}x \\ &= -\int_\Omega \sum_{i,j} \left(\eta a_{ij}u_{x_j}\right)_{x_s} v_{x_i}\,\mathrm{d}x + \int_\Omega v_{x_s} A\nabla u \cdot \nabla\eta\,\mathrm{d}x \\ &= -\int_\Omega \sum_{i,j} \eta\left(a_{ij}\right)_{x_s} u_{x_j} v_{x_i}\,\mathrm{d}x - \int_\Omega \eta_{x_s} A\nabla u \cdot \nabla v\,\mathrm{d}x \\ &\quad - \int_\Omega \eta A\nabla\left(u_{x_s}\right) \cdot \nabla v\,\mathrm{d}x + \int_\Omega v_{x_s} A\nabla u \cdot \nabla\eta\,\mathrm{d}x. \end{aligned}$$

Therefore, if we take absolute values, rearrange terms, and combine this with inequality (3.4), we get

$$\begin{aligned}
\kappa\|\eta\nabla(u_{x_s})\|_{L^p(\Omega)} &\le |\mathfrak{a}(\eta u_{x_s}, v)| + \kappa\|u_{x_s}\nabla\eta\|_{L^p(\Omega)} \\
&\le \left|\int_\Omega \eta A\nabla(u_{x_s})\cdot\nabla v\,\mathrm{d}x\right| + \left|\int_\Omega u_{x_s}A\nabla\eta\cdot\nabla v\,\mathrm{d}x\right| + \kappa\|u_{x_s}\nabla\eta\|_{L^p(\Omega)} \\
&\le \int_\Omega \left|\sum_{i,j}\eta\big(a_{ij}\big)_{x_s}u_{x_j}v_{x_i}\right|\mathrm{d}x + \int_\Omega |\eta_{x_s}||A\nabla u\cdot\nabla v|\,\mathrm{d}x \\
&\quad + \int_\Omega |v_{x_s}||A\nabla u\cdot\nabla\eta|\,\mathrm{d}x + \int_\Omega |u_{x_s}||A\nabla\eta\cdot\nabla v|\,\mathrm{d}x \\
&\quad + \int_\Omega |f\eta v_{x_s}|\,\mathrm{d}x + \kappa\|u_{x_s}\nabla\eta\|_{L^p(\Omega)} \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

We estimate each separately. The bound for I_5 is straightforward: by Hölder's inequality,

$$I_5 \le \|f\|_{L^p(\Omega)}\|v_{x_s}\|_{L^{p'}(\Omega)} \le \|f\|_{L^p(\Omega)}\|v\|_{W_0^{1,p'}(\Omega)} = \|f\|_{L^p(\Omega)}.$$

Similarly, using Hölder's inequality together with (1.2) we see that

$$I_2 + I_3 + I_4 + I_6 \le C(\kappa,\Lambda)(\sup_\Omega|\nabla\eta|)\|\nabla u\|_{L^p(\Omega)}.$$

The key estimate is for I_1. Define

$$A_s = \big(\big(a_{ij}\big)_{x_s}\big), \qquad U = |\nabla A| = \left(\sum_{i,j,s}\big(a_{ij}\big)_{x_s}^2\right)^{1/2},$$

and fix $\epsilon > 0$; the exact value of ϵ will be given below. Since $U \in L^n(\Omega)$, there exists $K = K(\epsilon, U)$ such that

$$\left(\int_{\{x:U(x)>K\}} U(x)^n\,\mathrm{d}x\right)^{1/n} < \epsilon. \tag{3.5}$$

Let $U_1 = U\chi_{\{x:U(x)>K\}}$ and $U_2 = U - U_1$. Then, by Hölder's inequality and Lemma 2.3, here using that $\eta\nabla u \in W_0^{1,p}(\Omega)$, we can estimate as follows:

$$\begin{aligned}
I_1 &= \int_\Omega |\eta A_s\nabla u\cdot\nabla v|\,\mathrm{d}x \\
&\le \int_\Omega U|\eta\nabla u||\nabla v|\,\mathrm{d}x \\
&\le \left(\int_\Omega |\eta\nabla u\,U|^p\,\mathrm{d}x\right)^{1/p}\left(\int_\Omega |\nabla v|^{p'}\,\mathrm{d}x\right)^{1/p'} \\
&\le \left(\int_\Omega |\eta\nabla u\,U_1|^p\,\mathrm{d}x\right)^{1/p} + \left(\int_\Omega |\eta\nabla u\,U_2|^p\,\mathrm{d}x\right)^{1/p} \\
&\le C(n)(p'-n')^{-1/p'}\epsilon\left(\int_\Omega |\nabla(\eta\nabla u)|^p\,\mathrm{d}x\right)^{1/p} + K(\epsilon,U)\left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)^{1/p} \\
&\le C(n,p)\epsilon\left(\int_\Omega |\eta D^2u|^p\,\mathrm{d}x\right)^{1/p} + C(n,p)\epsilon\left(\int_\Omega |\nabla\eta\cdot\nabla u|^p\,\mathrm{d}x\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&+K\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{1/p}\\
&\le C(n,p)\epsilon\left(\int_\Omega |\eta D^2 u|^p \,\mathrm{d}x\right)^{1/p} + \tilde{K}(1+\|\nabla\eta\|_\infty)\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)^{1/p},
\end{aligned}
$$

where $\tilde{K} = \tilde{K}(n, p, \varepsilon, K)$.

Each of the above estimates hold for all values of s. Therefore, by Minkowski's inequality, if we sum over all s and combine these estimates, we get that

$$
\begin{aligned}
\kappa\|\eta D^2 u\|_{L^p(\Omega)} &\le \sum_s \kappa\|\eta\nabla(u_{x_s})\|_{L^p(\Omega)}\\
&\le C(n,p)\epsilon\|\eta D^2 u\|_{L^p(\Omega)} + \overline{K}(1+\|\nabla\eta\|_\infty)\|\nabla u\|_{L^p(\Omega)} + n\|f\|_{L^p(\Omega)},
\end{aligned}
$$

where $\overline{K} = \overline{K}((n, p, \Lambda, \epsilon, K)$. Since $\epsilon > 0$ is arbitrary, we can fix $\epsilon = \kappa/2C(n, p)$ and then rearrange terms to get

$$
\begin{aligned}
&\|\eta D^2 u\|_{L^p(\Omega)}\\
&\quad\le 2\overline{K}\kappa^{-1}(1+\|\nabla\eta\|_\infty)\|\nabla u\|_{L^p(\Omega)} + 2n\kappa^{-1}\|f\|_{L^p(\Omega)} \le C_0(1+\|\nabla\eta\|_\infty)\|f\|_{L^p(\Omega)},
\end{aligned}
\tag{3.6}
$$

where the last inequality follows from (3.3), and $C_0 = C_0(p, n, \lambda, \Lambda, \Omega, K)$.

To complete the proof, fix $\Omega' \Subset \Omega$ and choose $\eta \in C_0^\infty(\Omega)$ such that $\eta(x) = 1$ in Ω' and so that $\|\nabla\eta\| \approx D^{-1}$, where $D = \mathrm{dist}(\Omega', \partial\Omega)$. Without loss of generality, we may assume $D > 1$. With this choice of η, inequality (3.6) yields the local $W^{2,p}(\Omega)$ estimate

$$
\|D^2 u\|_{L^p(\Omega')} \le D^{-1}C\|f\|_{L^p(\Omega)}, \tag{3.7}
$$

where $C = C(n, p, \lambda, \Lambda, K)$. Finally, if we assume that $\partial\Omega$ is C^2, we can apply the argument given in [19, p. 187] to obtain a constant $C > 0$ depending on K, p, n, λ, and Λ so that

$$
\|D^2 u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{3.8}
$$

This completes the proof of inequality (1.4) when f and A are sufficiently smooth.

We will now prove that inequalities (3.7) and (3.8) hold for general f and A satisfying the hypotheses. We will only consider the latter equation as the proof of the former is essentially the same.

We will first show that we can take an arbitrary f. Fix $f \in L^p(\Omega)$, and fix a sequence of functions $\{f_j\}$ in $C^\infty(\Omega)$ that converge to f in $L^p(\Omega)$. Fix $A \in C^\infty(\Omega)$ and let $u_j \in W_0^{1,p}(\Omega)$ be the solution to $Lu_j = f_j$, and let $u \in W_0^{1,p}$ be the solution to $Lu = f$. By inequality (3.3) and the Sobolev inequality, we have that

$$
\|u - u_j\|_{L^p(\Omega)} \le C\|\nabla(u - u_j)\|_{L^p(\Omega)} \le C\|f - f_j\|_{L^p(\Omega)}.
$$

Therefore, $u_j \to u$ in $W_0^{1,p}(\Omega)$.

Since f_j and A have the requisite smoothness, we can apply (3.8) to $u_i - u_j$ to get

$$
\|D^2(u_i - u_j)\|_{L^p(\Omega)} \le C\|f_i - f_j\|_{L^p(\Omega)}.
$$

Thus, the sequence $\{u_j\}$ is Cauchy in $W^{2,p}(\Omega)$. For $1 \le r,\ s \le n$, let $v_{r,s}$ denote the limit of $\{(u_j)_{x_r,x_s}\}$. Then for any $\phi \in C_0^\infty(\Omega)$,

$$
\int_\Omega u_{x_s}\phi_{x_r}\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega (u_j)_{x_s}\phi_{x_r}\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega (u_j)_{x_r,x_s}\phi\,\mathrm{d}x = \int_\Omega v_{r,s}\phi\,\mathrm{d}x. \tag{3.9}
$$

Therefore, $u \in W^{2,p}(\Omega)$ and $u_j \to u$ in $W^{2,p}(\Omega)$. Inequality (1.4) for u now follows immediately.

Finally, we prove that we can take arbitrary $A \in W^{1,n}(\Omega)$. Fix such an A, and let $\{A_j\}$ be a sequence of matrices in $C^\infty(\Omega)$ that converges to A in $W^{1,n}(\Omega)$. It follows at once from the standard construction of the A_j (cf. Adams and Fournier [2]) that we may assume that the A_j are elliptic with the same ellipticity constants as A. Finally, let $U_j = |\nabla A_j|$; then $U_j \to U = |\nabla A|$ in $L^n(\Omega)$. By the converse to the dominated convergence theorem (see Lieb and Loss [26, Th. 2.7]), if we pass to a subsequence, then we may assume that $U_j \to U$ pointwise a.e., and there exists $g \in L^n(\Omega)$ such that $U_j(x) \le g(x)$ a.e. Therefore, by the dominated convergence theorem (again passing to a subsequence), we may assume that (3.5) holds (with fixed ϵ) for each U_j with a constant K independent of j.

Fix $f \in L^p(\Omega)$ and let $u_j \in W_0^{1,p}(\Omega)$ be the solution of $-\operatorname{div} A_j\nabla u_j = f$ and let $u \in W_0^{1,p}(\Omega)$ be the solution of $Lu = -\operatorname{div} A\nabla u = f$. Then for any $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega A_j\nabla u_j \cdot \nabla\phi \,\mathrm{d}x = -\int_\Omega f\phi \,\mathrm{d}x = \int_\Omega A\nabla u \cdot \nabla\phi \,\mathrm{d}x.$$

Therefore,

$$\int_\Omega \big(A\nabla u - A\nabla u_j + A\nabla u_j - A_j\nabla u_j\big)\nabla\phi \,\mathrm{d}x = 0,$$

and so by rearranging terms we have that

$$|\mathfrak{a}(u - u_j, \phi)| = \left|\int_\Omega A(\nabla u - \nabla u_j)\cdot\nabla\phi \,\mathrm{d}x\right| \le \int_\Omega |(A - A_j)\nabla u_j \cdot \nabla\phi| \,\mathrm{d}x.$$

By Theorem 3.1 there exists ϕ such that $\|\phi\|_{W_0^{1,p'}(\Omega)} = 1$ and $\kappa > 0$ such that

$$\begin{aligned}\kappa\|u - u_j\|_{W_0^{1,p}} &\le \int_\Omega |(A - A_j)\nabla u_j \cdot \nabla\phi| \,\mathrm{d}x \\ &\le \|A - A_j\|_{L^n(\Omega)}\|\nabla u_j\|_{L^{\frac{np}{n-p}}(\Omega)}\|\nabla\phi\|_{L^{p'}(\Omega)}.\end{aligned} \tag{3.10}$$

The last estimate follows by Hölder's inequality, since

$$\frac{1}{n} + \frac{n-p}{np} + \frac{1}{p'} = 1.$$

The last term on the right-hand side of (3.10) is at most 1. By our choice of the A_j, the first term tends to 0 as $j \to \infty$. And by the Sobolev inequality,

$$\|\nabla u_j\|_{L^{\frac{np}{n-p}}(\Omega)} \le C\|D^2u_j\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)};$$

the final inequality holds since by our choice of the A_j, inequality (1.4) holds for each u_j with a constant independent of j. Therefore, the middle term on the righthand side of (3.10) is uniformly bounded. Hence, $u_j \to u$ in $W_0^{1,p}(\Omega)$.

It remains to show D^2u exists and estimate its norm. By inequality (1.4), the sequence $\{D^2u_j\}$ is uniformly bounded in $L^p(\Omega)$, and so has a weakly convergent subsequence. Passing to this subsequence, we can repeat the argument at (3.9) to conclude that $u \in W^{2,p}(\Omega)$ and D^2u_j converges weakly to D^2u. But then we have that

$$\|D^2u\|_{L^p(\Omega)} \le \liminf_{j\to\infty}\|D^2u_j\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)},$$

and this completes the proof. □

4 The case $n = 2$

In this section, we consider the two-dimensional case. We first construct Example 1.5 and then prove Theorems 1.6 and 1.7.

Construction of Example 1.5 Our example is adapted from one given by Clop et al. [10, p. 205] and is based on the theory of quasiregular mappings. Let $B = B_{1/2}(0)$ and let $z = x + iy$. Define $f(z) = z(1 - 2\log|z|)$. Then

$$\partial f(z) = -2\log|z| \quad \text{and} \quad \bar{\partial} f(z) = \frac{z}{\bar{z}},$$

and so f satisfies the Beltrami equation $\bar{\partial} f = \mu\, \partial f$ with Beltrami coefficient

$$\mu(z) = \frac{z}{\bar{z}\log(|z|^{-2})} = \frac{z^2}{|z|^2\log(|z|^{-2})}.$$

If we let let $u = \operatorname{Re} f$, that is,

$$u(x, y) = x(1 - \log(x^2 + y^2)),$$

then u satisfies the equation

$$-\operatorname{div}(A\nabla u) = 0$$

where A is the symmetric, real-valued matrix

$$A = \begin{bmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} \\ \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{bmatrix} = \frac{1+\sigma^2}{1-\sigma^2}\mathbf{Id} - \frac{2}{1-\sigma^2}\begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix},$$

and

$$\sigma = |\mu| = \frac{-1}{\log(x^2+y^2)}, \quad \alpha = \operatorname{Re}\mu = \frac{x^2-y^2}{x^2+y^2}\sigma, \quad \beta = \operatorname{Im}\mu = \frac{2xy}{x^2+y^2}\sigma.$$

This follows from a straightforward calculation: for the details, see [3, p. 412].

We claim that A is elliptic and in $W^{1,2}(B)$, and that $u \in W^{2,p}(B)$ for $p < 2$ but not when $p = 2$. By our choice of domain, $0 \le \sigma \le k = (\log 4)^{-1}$. Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$; then

$$\langle A\xi, \xi\rangle = \frac{1+\sigma^2}{1-\sigma^2}|\xi|^2 - \frac{2\alpha(\xi_1^2 - \xi_2^2) + 4\beta\xi_1\xi_2}{1-\sigma^2}. \tag{4.1}$$

Since

$$\alpha(\xi_1^2 - \xi_2^2) + 4\beta\xi_1\xi_2 = 2(\alpha, \beta)\cdot(\xi_1^2 - \xi_2^2, 2\xi_1\xi_2),$$

by the Cauchy-Schwarz inequality we have that

$$|2\alpha(\xi_1^2 - \xi_2^2) + 4\beta\xi_1\xi_2| \le 2\sqrt{\alpha^2+\beta^2}\sqrt{(\xi_1^2-\xi_2^2)^2 + 4\xi_1^2\xi_2^2} = 2\sigma|\xi|^2.$$

Hence,

$$-2\sigma|\xi|^2 \le 2\alpha(\xi_1^2 - \xi_2^2) + 4\beta\xi_1\xi_2 \le 2\sigma|\xi|^2,$$

and if we combine this with inequality (4.1), we get

$$\frac{1-k}{1+k}|\xi|^2 \le \frac{1-\sigma}{1+\sigma}|\xi|^2 \le \langle A\xi, \xi\rangle \le \frac{1+\sigma}{1-\sigma}|\xi|^2 \le \frac{1+k}{1-k}|\xi|^2.$$

Thus, A is elliptic with $\lambda = \frac{1-k}{1+k}$ and $\Lambda = \frac{1+k}{1-k}$.

To see that $A = (a_{ij}) \in W^{1,2}(B)$, a lengthy (and *Mathematica* assisted) calculation shows that

$$\frac{\partial a_{11}}{\partial x} = -\frac{4x\left[x^2-y^2-2y^2\log^3(x^2+y^2)+(x^2-y^2)\log^2(x^2+y^2)+2(x^2+2y^2)\log(x^2+y^2)\right]}{(x^2+y^2)^2(\log^2(x^2+y^2)-1)^2}$$

and the derivatives $\frac{\partial}{\partial x}a_{ij}$ and $\frac{\partial}{\partial y}a_{ij}$ are similar. It follows that

$$\left|\frac{\partial}{\partial x}a_{ij}\right|, \left|\frac{\partial}{\partial y}a_{ij}\right| \le C\frac{|\log^3(x^2+y^2)|}{(x^2+y^2)^{\frac{1}{2}}(\log^2(x^2+y^2)-1)^2} \in L^2(B).$$

Finally to see that $u \in W^{2,p}(B)$ for $p < 2$ but not in $W^{2,2}(B)$, another calculation shows that

$$u_{xx}(x,y) = \frac{-2x(x^2+3y^2)}{(x^2+y^2)^2}, \quad u_{xy}(x,y) = \frac{-2y(y^2-x^2)}{(x^2+y^2)^2}, \quad u_{yy}(x,y) = \frac{-2x(x^2-y^2)}{(x^2+y^2)^2}.$$

Thus, each second derivative is bounded by a constant multiple of $(x^2+y^2)^{-\frac{1}{2}} \in L^p(B)$, so $u \in W^{2,p}$. On the other hand,

$$\int_B |u_{xx}|^2 \,\mathrm{d}x\mathrm{d}y = \infty,$$

so $u \notin W^{2,2}(B)$. □

Proof of Theorem 1.6 Most of the proof is identical to the proof of Theorem 1.1, setting $n = p = 2$. However, in two places, we need to make specific changes to the proof. The proof for f and A smooth is the same up to inequality (3.5). We again split U, but now we fix ϵ (to be determined below) and find K such that

$$\|U\chi_{\{U>K\}}\|_{L^\Psi(\Omega)} < \epsilon, \tag{4.2}$$

where $\Psi(t) = t^2\log(e+t)^{1+\delta}$. (This is again possible by the dominated convergence theorem in the context of Orlicz spaces.) Let $U = U_1 + U_2 = U\chi_{\{U>K\}} + U\chi_{\{U\le K\}}$; then by Lemma 2.4,

$$\begin{aligned}\left(\int_\Omega (|\eta\nabla u|U)^2\,\mathrm{d}x\right)^{1/2} &\le \left(\int_\Omega (|\eta\nabla u|U_1)^2\,\mathrm{d}x\right)^{1/2} + K\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{1/2}\\ &\le \epsilon\, C(\delta,\Omega)\left(\int_\Omega |D^2u|^2\,\mathrm{d}x\right)^{1/2} + \overline{K}(1+\|\nabla\eta\|_\infty)\|f\|_{L^2(\Omega)}. \end{aligned}\tag{4.3}$$

The argument now proceeds as before, yielding

$$\|\eta D^2u\|_{L^2(\Omega)} \le C_0(1+\|\nabla\eta\|_\infty)\|f\|_{L^2(\Omega)},$$

where again the constant $C_0 = C_0(n, p, \lambda, \Lambda, \Omega, K)$.

The proof for arbitrary $f \in L^2(\Omega)$ goes through exactly as before. For the proof for arbitrary $\nabla A \in L^\Psi(\Omega)$, note first that by the Sobolev embedding theorem, we have $A \in L^\Psi(\Omega)$. We now fix smooth $A_j \to A$ in $W^{1,\Psi}(\Omega)$ (the Sobolev space defined with respect to the L^Ψ norm), and we may again assume that the A_j have the same ellipticity constants and that we may choose K such that (4.2) holds for all $U_j = |\nabla A_j|$ with a constant K

independent of j. This is possible since all the arguments for $W^{1,p}(\Omega)$ extend to $W^{1,\Psi}(\Omega)$ with almost no change. Smooth functions are dense (see [2]), and the proof of density again shows that ellipticity constants are preserved. The converse of dominated convergence also holds in this setting; the proof is implicit in the literature. For a proof in a different context that readily adapts to Orlicz spaces, see [11, Prop. 2.67].

The proof now continues as before until inequality (3.10). Here, we need to apply the generalized Hölder's inequality in the scale of Orlicz spaces (see [12, Lemma 5.2]). If we let $\Phi(t) = \exp(t^{\frac{2}{1+\delta}}) - 1$, then

$$\Psi^{-1}(t)\Phi^{-1}(t) \approx \frac{t^{1/2}}{\log(e+t)^{\frac{1+\delta}{2}}} \log(e+t)^{\frac{1+\delta}{2}} \lesssim t^{1/2}.$$

Therefore, we can estimate as follows:

$$\left|\int_\Omega A(\nabla u - \nabla u_j)\cdot\nabla\phi \,\mathrm{d}x\right| \le \int_\Omega |(A-A_j)\nabla u_j\cdot\nabla\phi|\,\mathrm{d}x$$
$$\le \|(A-A_j)\nabla u_j\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)} \le C\|A-A_j\|_{L^\Psi(\Omega)}\|\nabla u_j\|_{L^\Phi(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}.$$

As in the previous argument, we have chosen ϕ so that $\|\nabla\phi\|_{L^2(\Omega)} \le 1$. We also have that $\|A-A_j\|_{L^\Psi(\Omega)} \to 0$ as $j \to \infty$. Therefore, we could complete the proof as before if we can show that

$$\|\nabla u_j\|_{L^\Phi(\Omega)} \le C\|f\|_{L^2(\Omega)}$$

with a constant independent of j.

Let $\Phi_0(t) = \exp(t^2) - 1$. Then for $t \ge 1$, $\Phi(t) \le \Phi_0(t)$, and so by the properties of Orlicz norms (see [12, Sec.5.2]) there exists a constant depending on δ and Ω such that $\|\nabla u_j\|_{L^\Phi(\Omega)} \le C\|\nabla u_j\|_{L^{\Phi_0}(\Omega)}$. But by Trudinger's inequality [34, Thm. 2.9.1] we have the endpoint Sobolev inequality:

$$\|\nabla u_j\|_{L^{\Phi_0}(\Omega)} \le C\|D^2u_j\|_{L^2(\Omega)}.$$

By the first part of the proof, we have that $\|D^2u_j\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$ with a constant independent of j; combining these inequalities, we get the desired estimate and this completes the proof. □

Proof of Theorem 1.7 The proof is nearly identical to the proof of Theorem 1.6. Let $\Psi(t) = t^2\log(e+t)$. The first half of the proof for smooth f and A is the same until (4.3). Here, we use the off-diagonal estimate in Lemma 2.6 and Hölder's inequality to get

$$\left(\int_\Omega (|\eta\nabla u|U)^2\,\mathrm{d}x\right)^{1/2}$$
$$\le \left(\int_\Omega (|\eta\nabla u|U_1)^2\,\mathrm{d}x\right)^{1/2} + K\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{1/2}$$
$$\le \epsilon\, C(\delta,\Omega)\left(\int_\Omega |\eta D^2u|^r\,\mathrm{d}x\right)^{1/r} + \overline{K}(1+\|\nabla\eta\|_\infty)\|f\|_{L^2(\Omega)}$$
$$\le \epsilon\, C(\delta,\Omega)|\Omega|^{\frac{1}{(2/r)'}}\left(\int_\Omega |\eta D^2u|^2\,\mathrm{d}x\right)^{1/2} + \overline{K}(1+\|\nabla\eta\|_\infty)\|f\|_{L^2(\Omega)}.$$

We can now complete the proof of the smooth case as before.

The remainder of the proof goes through as before, only now we apply the generalized Hölder's inequality with $\Psi(t)$ and $\Phi(t) = \exp(t^2) - 1$ and then directly apply Trudinger's inequality. □

Remark 4.1 Note that in the proof of Theorem 1.7 we use the regularity assumption on ∇A in the proof of the smooth case, and use the higher integrability assumption on A in the density argument to prove the general case.

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