

# Instability of equatorially trapped waves in stratified water

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**Abstract** In this paper, we consider the geometrical optics—or WKB—method for geophysical equatorial barotropic incompressible flows. The stability analysis of such flows is reduced to the stability analysis of an ordinary differential equation system along their trajectories. The analysis of this system in the case of equatorially trapped waves propagating eastward in stratified water shows that those waves for which the steepness parameter of the wave profile is higher than a specific value are unstable.

**Keywords** Short-wavelength method · Localized instability analysis · Equatorially trapped waves · Stratification · Explicit solutions

**Mathematics Subject Classification** 35Q86 · 76E20 · 34E20 · 76B70

## 1 Introduction

In the theoretical analysis of geophysical water waves in the Equator region, it is customary to use the  $\beta$ -plane approximation to the full water-wave problem (see the discussions in [8] and [14]). In this way, one obtains a variety of equatorially trapped waves, waves whose amplitude decays rapidly away from the Equator. In the shallow-water regime of a one-layer reduced-gravity model, one obtains the eastward propagating Kelvin waves (see [8] and [9]). An exact solution to the full water-wave problem in the  $\beta$ -plane approximation was constructed by Constantin [5]. This Gerstner<sup>1</sup>-type solution describes in the Lagrangian framework equatorially trapped waves propagating eastward in a stratified inviscid fluid. In Sect. 2, we present these explicit equatorially trapped waves in stratified water. We also show that these waves are barotropic, that is, their density is a function only of the pressure.

In Sect. 3, we make a detailed analysis of the short-wavelength instability method for general geophysical equatorial barotropic incompressible flow. Subsequently, in Sect. 4 we

<sup>1</sup> For modern detailed descriptions of Gerstner solution see [2, 16].

apply this method to the specific case of the equatorially trapped waves in stratified water presented in Sect. 2. The short-wavelength instability method was developed independently by Bayly [1], Friedlander and Vishik [11] and Lifschitz and Hameiri [24] (see also the surveys by Friedlander and Yudovich [13]; Friedlander and Lipton-Lifschitz [10]), in order to extend the stability analysis to more general non-steady three-dimensional fluids. This interesting approach is based on the geometrical optics method: the small-wavelength perturbations are considered in the WKB form, their evolution in time being governed up to the remainder terms by the eikonal equation for the wave phase and the transport equation for the wave amplitude of the velocity. Along the rays, that are, the basic flow trajectories, the system becomes an ordinary differential equation system. The remainder terms can be shown to be incapable of cancelling the growth of the leading-order terms. For a general geophysical equatorial barotropic incompressible flow, we give some estimates of the remainder terms by following the procedure presented by Ionescu-Kruse [21] in the non-geophysical case. Thus, we conclude that, in the geophysical case too, the stability analysis is reduced to the stability analysis of the ordinary differential equation system along the trajectories of the basic flow; if for some trajectories the corresponding transport equation has at least one solution growing in time without bound, then such flow is unstable.

This local stability analysis is convenient when the basic flow is described in the Lagrangian framework: it was made by Leblanc [23] for Gerstner’s waves, by Constantin and Germain [6] for the equatorially trapped waves [5] in the constant density case, by Ionescu-Kruse [20,21] for the edge waves in the constant density case and in the stratified water, respectively, by Genoud and Henry [15] for geophysical equatorial waves with an underlying current [18], by Henry and Hsu [19] for internal equatorial waves (with and without currents). For the equatorially trapped waves in stratified water, we find that, at the leading order, the wave phase and the wave amplitude of the velocity satisfy the same system of equations as in the constant density case (Constantin and Germain [6]), but the component of the pressure has a different expression. Such waves are unstable to short-wavelength perturbations if the vorticity in the meridional direction is smaller than  $-\frac{1}{4} \frac{(kc+4\Omega)^2}{kc(kc+2\Omega)}$  or if the steepness parameter of the wave profile is higher than  $\frac{kc+4\Omega}{3kc+4\Omega}$ . The exponential growth rate of instabilities is  $\frac{1}{2} \sqrt{\frac{(3kc+4\Omega)^2 e^{2\chi} - (kc+4\Omega)^2}{1-e^{2\chi}}}$ .

## 2 Equatorially trapped waves in stratified water

We take the Earth to be a perfect sphere of radius  $R = 6371$  km and with a constant rotational speed  $\Omega = 73 \cdot 10^{-6}$  rad s<sup>-1</sup> round the polar axis towards east. We consider a rotating framework with the origin located at a point on Earth’s surface and with the  $x$ -axis horizontally due east, the  $y$ -axis horizontally due north and the  $z$ -axis upward. The variable  $x$  will correspond to longitude, the variable  $y$  to latitude and the variable  $z$  to the local vertical. In the  $\beta$ -plane approximation, the governing equations for equatorially trapped waves are the Euler equations [14]

$$\begin{aligned}
 U_t + UU_x + VU_y + WU_z + 2\Omega W - \beta yV &= -\frac{1}{\rho} P_x \\
 V_t + UV_x + VV_y + WV_z + \beta yU &= -\frac{1}{\rho} P_y \\
 W_t + UW_x + VW_y + WW_z - 2\Omega U &= -\frac{1}{\rho} P_z - g,
 \end{aligned}
 \tag{1}$$

the condition of incompressibility

$$U_x + V_y + W_z = 0, \tag{2}$$

and the equation of mass conservation

$$\rho_t + U\rho_x + V\rho_y + W\rho_z = 0. \tag{3}$$

$\mathbf{U}(t, \mathbf{x}) = (U(t, x, y, z), V(t, x, y, z), W(t, x, y, z))$  is the velocity field,  $P(t, x, y, z)$  the pressure,  $\beta = \frac{2\Omega}{R} = 2.28 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$ ,  $g = 9.8 \text{ ms}^{-2}$  is the constant gravitational acceleration at the Earth’s surface and  $\rho(t, \mathbf{x})$  the water density. Subscripts denote partial derivatives.

The kinematic and the dynamic boundary conditions on the water’s free surface  $z = \eta(t, x, y)$  are given by (see the discussion in [4])

$$W = \eta_t + U\eta_x + V\eta_y \quad \text{on } z = \eta(t, x, y), \tag{4}$$

$$P = P_{\text{atm}} = \text{const} \quad \text{on } z = \eta(t, x, y). \tag{5}$$

We also require the velocity field to decay rapidly with depth.

Constantin [5] found an exact Gerstner-like solution of the nonlinear system (1)–(5) which describes equatorially trapped waves propagating eastward in a stratified water. Adequate for the nonlinear exact solution is the Lagrangian framework. In this framework, the variables  $(x, y, z)$  that denote the position of the fluid particles at time,  $t$ , are given as functions of  $t$  and some labels  $(q, s, r)$  which mark individual fluid particles by the following expressions [5]

$$\begin{cases} x = q - \frac{1}{k}e^{k[r-f(s)]} \sin[k(q - ct)] \\ y = s \\ z = r + \frac{1}{k}e^{k[r-f(s)]} \cos[k(q - ct)]. \end{cases} \tag{6}$$

Here,  $k$  is the wave number,

$$c = \frac{\sqrt{\Omega^2 + kg} - \Omega}{k} \tag{7}$$

is the wave speed, and

$$f(s) = \frac{c\beta}{2g}s^2 \tag{8}$$

is a function describing the latitudinal variation of the particle oscillation. The labels  $q$  and  $s \in \mathbf{R}$ , while  $r \in (-\infty, r_0]$ , for some fixed  $r_0 < 0$ . In order for the transformation (6) to be well defined and to ensure that the flow has the appropriate decay properties, in the vertical direction as well as in the latitudinal direction, one requires that

$$r - \frac{c\beta}{2g}s^2 \leq r_0, \quad \text{for some fixed } r_0 < 0. \tag{9}$$

With the notations

$$\chi := k \left[ r - \frac{c\beta}{2g}s^2 \right], \quad \theta := k(q - ct), \tag{10}$$

the Jacobian matrix of the transformation (6) has the form

$$F := \begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 - e^\chi \cos \theta & 0 & -e^\chi \sin \theta \\ s \frac{c\beta}{g} e^\chi \sin \theta & 1 & -s \frac{c\beta}{g} e^\chi \cos \theta \\ -e^\chi \sin \theta & 0 & 1 + e^\chi \cos \theta \end{pmatrix}, \tag{11}$$

the determinant of the Jacobian matrix is

$$J = \det F = 1 - e^{2\chi}, \tag{12}$$

and its inverse has the expression

$$G := \begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial r}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1+e^\chi \cos \theta}{1-e^{2\chi}} & 0 & \frac{e^\chi \sin \theta}{1-e^{2\chi}} \\ -s \frac{c\beta}{g} \frac{e^\chi \sin \theta}{1-e^{2\chi}} & 1 & s \frac{c\beta}{g} \frac{e^\chi (\cos \theta - e^\chi)}{1-e^{2\chi}} \\ \frac{e^\chi \sin \theta}{1-e^{2\chi}} & 0 & \frac{1-e^\chi \cos \theta}{1-e^{2\chi}} \end{pmatrix}. \tag{13}$$

The determinant of the Jacobian matrix is time independent; thus, the flow is volume preserving and (2) is satisfied. For a stratified water with the density defined by

$$\rho = \mathfrak{F} \left( \frac{e^{2k \left[ r - \frac{c\beta}{2g} s^2 \right]}}{2k} - r \right), \tag{14}$$

where  $\mathfrak{F} : (0, \infty) \rightarrow (0, \infty)$  is a continuously differentiable non-decreasing function (in order to have a gravitationally stable water stratification), and for a suitably defined value of the pressure function, that is,

$$P = P_0 + g\mathcal{F} \left( \frac{e^{2k \left[ r - \frac{c\beta}{2g} s^2 \right]}}{2k} - r \right) - g\mathcal{F} \left( \frac{e^{2r_0}}{2k} - r_0 \right), \tag{15}$$

where  $\mathcal{F}' = \mathfrak{F}$ ,  $\mathcal{F}(0) = 0$ , the Euler Eqs. (1) are satisfied too [5].

In Lagrangian variables, the kinematic boundary condition (4) holds if at each fixed  $s$ , the free surface at latitude  $y = s$  is given by setting  $r = r(s)$  in (6), where  $r(s) < r_0$  is the unique solution of the equation

$$\frac{e^{2k \left[ r - \frac{c\beta}{2g} s^2 \right]}}{2k} - r - \frac{e^{2kr_0}}{2k} + r_0 = 0, \tag{16}$$

the label  $q$  being the free parameter of the curve that represents the wave profile at this latitude. At the equator, where  $s = 0$ , the free surface is the curve determined by setting  $r = r_0$  in (6). Then, from the form (15) of the pressure, the dynamic condition (5) is also satisfied.

The significant properties of the solution (6) are presented in [5]. At a fixed latitude, all particles beneath the surface wave move on circles. These particle paths are quite different from those beneath an irrotational periodic travelling wave (see [3,7,17]). By direct calculation, the vorticity vector  $\boldsymbol{\gamma} := \text{curl } \mathbf{U}$  has the following expression

$$\boldsymbol{\gamma} := (\gamma_1, \gamma_2, \gamma_3) = \left( -s \frac{kc^2\beta}{g} \frac{e^\chi \sin \theta}{1-e^{2\chi}}, -\frac{2kce^{2\chi}}{1-e^{2\chi}}, s \frac{kc^2\beta}{g} \frac{e^\chi (\cos \theta - e^\chi)}{1-e^{2\chi}} \right). \tag{17}$$

Thus, the flow corresponding to the solution (6) is rotational. The second component of the vorticity is identical to the vorticity of a Gerstner wave. Away from the equator, for  $s \neq 0$ , the first and the third component of the vorticity are both non-zero.

Let us now prove that *the equatorially trapped waves (6) in stratified water are barotropic*, that is, the density is a function only of the pressure,

$$\rho = f(P), \quad f'(P) \neq 0. \tag{18}$$

Differentiating the third equation of (1) with respect to  $y$  and the second equation of (1) with respect to  $z$ , then subtracting the results, we get

$$(\gamma_1)_t + [(\mathbf{U} \cdot \nabla)\boldsymbol{\gamma}]_1 - [(\boldsymbol{\gamma} \cdot \nabla)\mathbf{U}]_1 - 2\Omega U_y - \beta y U_z = \frac{1}{\rho^2}(\nabla\rho \times \nabla P)_1. \tag{19}$$

Differentiating the first equation of (1) with respect to  $z$  and the third equation of (1) with respect to  $x$ , then subtracting the results, we have

$$(\gamma_2)_t + [(\mathbf{U} \cdot \nabla)\boldsymbol{\gamma}]_2 - [(\boldsymbol{\gamma} \cdot \nabla)\mathbf{U}]_2 - 2\Omega V_y - \beta y V_z = \frac{1}{\rho^2}(\nabla\rho \times \nabla P)_2, \tag{20}$$

In the same manner, from the second and the first equation of (1), we get

$$(\gamma_3)_t + [(\mathbf{U} \cdot \nabla)\boldsymbol{\gamma}]_3 - [(\boldsymbol{\gamma} \cdot \nabla)\mathbf{U}]_3 - 2\Omega W_y - \beta y W_z + \beta V = \frac{1}{\rho^2}(\nabla\rho \times \nabla P)_3. \tag{21}$$

In the Eqs. (19)–(21),  $[(\mathbf{U} \cdot \nabla)\boldsymbol{\gamma}]_i$ ,  $[(\boldsymbol{\gamma} \cdot \nabla)\mathbf{U}]_i$  and  $(\nabla\rho \times \nabla P)_i$ , for  $i = \overline{1, 3}$ , are the  $i$ -components of the corresponding vectors.

From (6), the velocity field  $\mathbf{U}$  has the components

$$U = ce^\chi \cos \theta, \quad V = 0, \quad W = ce^\chi \sin \theta. \tag{22}$$

Therefore, with (13) in view,

$$\begin{aligned} U_x &= -kc \frac{e^\chi \sin \theta}{1 - e^{2\chi}}, \quad U_y = s \frac{kc^2\beta}{g} \frac{e^\chi (e^\chi - \cos \theta)}{1 - e^{2\chi}}, \quad U_z = kc \frac{e^\chi (\cos \theta - e^\chi)}{1 - e^{2\chi}} \\ V_x &= 0, \quad V_y = 0, \quad V_z = 0 \\ W_x &= kc \frac{e^\chi (\cos \theta + e^\chi)}{1 - e^{2\chi}}, \quad W_y = -s \frac{kc^2\beta}{g} \frac{e^\chi \sin \theta}{1 - e^{2\chi}}, \quad W_z = kc \frac{e^\chi \sin \theta}{1 - e^{2\chi}}. \end{aligned} \tag{23}$$

From (17) and (13), we also get

$$\begin{aligned} (\gamma_1)_t &= -c(\gamma_1)_x, \quad (\gamma_2)_t = -c(\gamma_2)_x, \quad (\gamma_3)_t = -c(\gamma_3)_x \\ (\gamma_1)_x &= s \frac{k^2c^2\beta}{g} \frac{e^\chi}{(1 - e^{2\chi})^2} \left[ -\cos \theta (1 + e^\chi \cos \theta) - e^\chi \sin^2 \theta \frac{1 + e^{2\chi}}{1 - e^{2\chi}} \right] \\ (\gamma_1)_z &= -s \frac{k^2c^2\beta}{g} \frac{e^\chi \sin \theta}{(1 - e^{2\chi})^2} \left[ e^\chi \cos \theta + (1 - e^\chi \cos \theta) \frac{1 + e^{2\chi}}{1 - e^{2\chi}} \right] \\ (\gamma_2)_x &= -4k^2c \frac{e^{3\chi} \sin \theta}{(1 - e^{2\chi})^3} \\ (\gamma_2)_z &= -4k^2c \frac{e^{2\chi}}{(1 - e^{2\chi})^3} (1 - e^\chi \cos \theta) \\ (\gamma_3)_x &= s \frac{k^2c^2\beta}{g} \frac{e^\chi \sin \theta}{(1 - e^{2\chi})^2} \left[ -(1 + e^\chi \cos \theta) + e^\chi \cos \theta \frac{1 + e^{2\chi}}{1 - e^{2\chi}} - \frac{2e^{2\chi}}{1 - e^{2\chi}} \right] \\ (\gamma_3)_z &= s \frac{k^2c^2\beta}{g} \frac{e^\chi}{(1 - e^{2\chi})^2} \left[ -e^\chi \sin^2 \theta + \cos \theta (1 - e^\chi \cos \theta) \frac{1 + e^{2\chi}}{1 - e^{2\chi}} \right. \\ &\quad \left. - (1 - e^\chi \cos \theta) \frac{2e^\chi}{1 - e^{2\chi}} \right]. \end{aligned} \tag{24}$$

Substituting (22)–(24) into the left-hand side of the Eqs. (19)–(21), and taking into account the dispersion relation

$$c^2k + 2\Omega c - g = 0, \tag{25}$$

we get that all the components of the vector  $(\nabla\rho \times \nabla P)$  are equal to zero, that is,

$$\nabla\rho \times \nabla P \equiv 0. \tag{26}$$

Therefore, the equatorially trapped waves (6) in stratified water have to be barotropic with a density  $\rho$  of the form (18). We observe that the structure of the particle trajectories (6) is indeed such that the lines of constant pressure are identical with the lines of constant density. At the equator, where  $s = 0$ , the pressure is constant on the curve obtained by setting  $r = r_0$  in (6). At planes parallel to the plane of the equator,  $s = \text{const}$ , the pressure is constant on the curve obtained by setting  $r = r(s)$  in (6), where  $r(s)$  is the solution of the equation (16).

### 3 Short-wavelength instability method for barotropic geophysical flows

We suppose now that a geophysical flow  $(\mathbf{U}(t, \mathbf{x}), P(t, \mathbf{x}))$  which satisfies the system (1)–(3), with a density  $\rho$  of the form (18) is disturbed by a small perturbation

$$(\mathbf{u}(t, \mathbf{x}), p(t, \mathbf{x})). \tag{27}$$

By the incompressibility condition (2) and by the relation (18), the equation (3) becomes

$$P_t + \mathbf{U} \cdot \nabla P = 0. \tag{28}$$

We substitute  $\mathbf{U} + \mathbf{u}$  and  $P + p$  into the equations (1), (2) and (28), we develop

$$\frac{1}{f(P + p)} = \frac{1}{f(P)} - \frac{f'(P)}{f^2(P)} p + \mathcal{O}(p^2),$$

and by neglecting the terms that are quadratic in  $\mathbf{u}$ , quadratic in  $p$  and the terms of the form  $\mathbf{u} \cdot \nabla p$ , we get the linearized equations governing the dynamics of the small perturbation

$$\mathbf{u}_t + (\mathbf{U} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta,\Omega}\mathbf{u} = -\nabla \left( \frac{p}{f(P)} \right) \tag{29}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{30}$$

$$p_t + \mathbf{u} \cdot \nabla P + \mathbf{U} \cdot \nabla p = 0, \tag{31}$$

with  $\mathcal{L}_{\beta,\Omega}$  given by

$$\mathcal{L}_{\beta,\Omega} := \begin{pmatrix} 0 & -\beta y & 2\Omega \\ \beta y & 0 & 0 \\ -2\Omega & 0 & 0 \end{pmatrix}. \tag{32}$$

We make now the Wentzel–Kramers–Brillouin (WKB) ansatz for the perturbation, that is, a rapidly oscillating initial disturbance  $\mathbf{u}_0$  of the form

$$\mathbf{u}_0 := \mathbf{u}(0, \mathbf{x}) = \mathbf{A}(0, \mathbf{x}) e^{i\tilde{z}\phi(0,\mathbf{x})} =: \mathbf{A}_0(\mathbf{x}) e^{i\tilde{z}\phi_0(\mathbf{x})}, \tag{33}$$

is supposed to have in time the following evolution

$$\mathbf{u}(t, \mathbf{x}) = [\mathbf{A}(t, \mathbf{x}) + \epsilon \mathcal{A}(t, \mathbf{x})] e^{i\tilde{z}\phi(t,\mathbf{x})} + \epsilon \mathbf{u}_{\text{rem}}(t, \mathbf{x}, \epsilon) \tag{34}$$

$$p(t, \mathbf{x}) = [B(t, \mathbf{x}) + \epsilon \mathcal{B}(t, \mathbf{x})] e^{i\tilde{z}\phi(t,\mathbf{x})} + \epsilon p_{\text{rem}}(t, \mathbf{x}, \epsilon), \tag{35}$$

where  $\epsilon$  is a small parameter,  $\mathbf{A}, \mathcal{A}$  are vector functions, and  $\phi, B, \mathcal{B}$  are scalar functions. Substituting (34) into (30) yields

$$\mathbf{A} \cdot \nabla \phi = 0 \tag{36}$$

$$\mathcal{A} \cdot \nabla\phi = i\nabla \cdot \mathbf{A} \tag{37}$$

$$\nabla \cdot \mathbf{u}_{\text{rem}} = -e^{\frac{i}{\varepsilon}\phi}(\nabla \cdot \mathcal{A}). \tag{38}$$

By (37),  $\mathcal{A}$  can be expressed in terms of  $\mathbf{A}$  and  $\nabla\phi$ ,

$$\mathcal{A} = i \frac{\nabla \cdot \mathbf{A}}{|\nabla\phi|^2} \nabla\phi. \tag{39}$$

Substituting (34) and (35) into (29) we get

$$(\phi_t + \mathbf{U} \cdot \nabla\phi) \mathbf{A} = -\frac{B}{f(P)} \nabla\phi \tag{40}$$

$$\begin{aligned} &\mathbf{A}_t + (\mathbf{U} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta,\Omega} \mathbf{A} + i\mathcal{A} (\phi_t + \mathbf{U} \cdot \nabla\phi) \\ &= -\left[ \nabla \left( \frac{B}{f(P)} \right) + \frac{i\mathcal{B}}{f(P)} \nabla\phi \right] \end{aligned} \tag{41}$$

$$\begin{aligned} &(\mathbf{u}_{\text{rem}})_t + (\mathbf{U} \cdot \nabla)\mathbf{u}_{\text{rem}} + (\mathbf{u}_{\text{rem}} \cdot \nabla)\mathbf{U} + \nabla \left( \frac{P_{\text{rem}}}{f(P)} \right) \\ &= -e^{\frac{i}{\varepsilon}\phi(t,\mathbf{x})} \left[ \nabla \left( \frac{\mathcal{B}}{f(P)} \right) + \mathcal{L}_{\beta,\Omega} \mathcal{A} \right. \\ &\quad \left. + \mathcal{A}_t + (\mathbf{U} \cdot \nabla)\mathcal{A} + (\mathcal{A} \cdot \nabla)\mathbf{U} \right] \end{aligned} \tag{42}$$

By virtue of (36), for  $\mathbf{A} \neq 0$  and  $\nabla\phi \neq 0$ , (40) gives the eikonal equation

$$\phi_t + \mathbf{U} \cdot \nabla\phi = 0 \tag{43}$$

and

$$B = 0. \tag{44}$$

Therefore, (41) becomes

$$\mathbf{A}_t + (\mathbf{U} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta,\Omega} \mathbf{A} = -\frac{i\mathcal{B}}{f(P)} \nabla\phi \tag{45}$$

Taking the scalar product of (45) with the vector  $\nabla\phi$ , we get

$$\mathcal{B} = if(P) \frac{\nabla\phi \cdot [\mathbf{A}_t + (\mathbf{U} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta,\Omega} \mathbf{A}]}{\|\nabla\phi\|^2}. \tag{46}$$

The total time derivative of the orthogonality condition (36) gives

$$\nabla\phi \cdot [\mathbf{A}_t + (\mathbf{U} \cdot \nabla)\mathbf{A}] = -\mathbf{A} \cdot [(\nabla\phi)_t + (\mathbf{U} \cdot \nabla)(\nabla\phi)], \tag{47}$$

and the gradient of the Eq. (43) yields the equation

$$(\nabla\phi)_t + (\mathbf{U} \cdot \nabla)(\nabla\phi) + (\nabla\mathbf{U})^T(\nabla\phi) = 0, \tag{48}$$

where

$$\nabla\mathbf{U} := \begin{pmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{pmatrix}. \tag{49}$$

Hence, the expression (46) of  $\mathcal{B}$  can be simplified to

$$\mathcal{B} = if(P) \frac{\nabla\phi \cdot [2(\mathbf{A} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta,\Omega} \mathbf{A}]}{\|\nabla\phi\|^2}. \tag{50}$$

Finally, substituting (34) and (35) into (31) and taking into account (43) and (44), we also have

$$\mathbf{A} \cdot \nabla P = 0 \tag{51}$$

$$(p_{\text{rem}})_t + \mathbf{u}_{\text{rem}} \cdot \nabla P = -e^{\frac{i}{\epsilon}\phi} (\mathcal{B}_t + \mathcal{A} \cdot \nabla P + \mathbf{U} \cdot \nabla \mathcal{B}). \tag{52}$$

Summing up, for barotropic incompressible geophysical flows, the WKB evolution of short-wavelength perturbations are governed, at leading order in powers of  $\epsilon$ , by the system

$$\begin{aligned} \phi_t + \mathbf{U} \cdot \nabla \phi &= 0 \\ \mathbf{A}_t + (\mathbf{U} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta, \Omega} \mathbf{A} &= \frac{\nabla \phi \cdot [2(\mathbf{A} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta, \Omega} \mathbf{A}]}{\|\nabla \phi\|^2} \nabla \phi. \end{aligned} \tag{53}$$

with the initial conditions

$$\phi(0, \mathbf{x}) = \phi_0(\mathbf{x}), \quad \mathbf{A}(0, \mathbf{x}) = \mathbf{A}_0(\mathbf{x}) \tag{54}$$

satisfying

$$\mathbf{A}_0 \cdot \nabla \phi_0 = 0. \tag{55}$$

In (34)–(35), the component  $\mathcal{B}$  is zero, the components  $\mathcal{A}$  and  $\mathcal{B}$  can be expressed in terms of  $\mathbf{A}$  and  $\nabla \phi$ , see (39) and (50), respectively. The remainder terms  $\mathbf{u}_{\text{rem}}$ ,  $p_{\text{rem}}$  satisfy the following system

$$(\mathbf{u}_{\text{rem}})_t + (\mathbf{U} \cdot \nabla)\mathbf{u}_{\text{rem}} + (\mathbf{u}_{\text{rem}} \cdot \nabla)\mathbf{U} + \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) = \mathbf{T} \tag{56}$$

$$\nabla \cdot \mathbf{u}_{\text{rem}} = g, \tag{57}$$

where

$$\begin{aligned} \mathbf{T}(t, \mathbf{x}) &:= -e^{\frac{i}{\epsilon}\phi(t, \mathbf{x})} \mathbb{T}(t, \mathbf{x}) \\ \mathbb{T}(t, \mathbf{x}) &:= \nabla \left( \frac{\mathcal{B}}{f(P)} \right) + \mathcal{L}_{\beta, \Omega} \mathcal{A} + \mathcal{A}_t + (\mathbf{U} \cdot \nabla)\mathcal{A} + (\mathcal{A} \cdot \nabla)\mathbf{U} \end{aligned} \tag{58}$$

and

$$g(t, \mathbf{x}) := -e^{\frac{i}{\epsilon}\phi(t, \mathbf{x})} \mathcal{G}(t, \mathbf{x}), \quad \mathcal{G}(t, \mathbf{x}) := \nabla \cdot \mathcal{A}, \tag{59}$$

are functions which depend only on  $\mathbf{A}$  and  $\nabla \phi$ .

In the constant density case, Constantin and Germain [6] obtained for the amplitude  $\mathbf{A}$  and the wave phase  $\phi$  the same system (53)–(55). In the barotropic case considered here, the component  $\mathcal{B}$  which appears in the pressure has a different expression, namely, it is multiplied by  $f(P)$ , see (50).

The system of partial differential equations (53) can be written as a system of ordinary differential equations along the trajectories of the basic flow  $\mathbf{U}(t, \mathbf{x})$ .

Consider the trajectory passing through a point  $\mathbf{x}_0$ ,

$$\frac{d\mathbf{x}}{dt} = \mathbf{U}(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{60}$$

The eikonal equation for the wave vector  $\boldsymbol{\xi} := \nabla \phi$  can be written as

$$\frac{d\boldsymbol{\xi}}{dt} = -(\nabla \mathbf{U})^T \boldsymbol{\xi}, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0. \tag{61}$$



This equation will give the wave vector  $\xi(t; \mathbf{x}_0, \xi_0)$  having the initial orientation  $\xi_0$ . The velocity amplitude  $\mathbf{A}(t; \mathbf{x}_0, \xi_0, \mathbf{A}_0)$ , which at  $t = 0$  satisfies the condition

$$\mathbf{A}(0) = \mathbf{A}_0, \quad \mathbf{A}_0 \cdot \xi_0 = 0, \tag{62}$$

is a solution of the transport equation (53)<sub>2</sub>

$$\frac{d\mathbf{A}}{dt} = -(\mathbf{A} \cdot \nabla)\mathbf{U} - \mathcal{L}_{\beta, \Omega} \mathbf{A} + \frac{\xi \cdot [2(\mathbf{A} \cdot \nabla)\mathbf{U} + \mathcal{L}_{\beta, \Omega} \mathbf{A}]}{\|\xi\|^2} \xi. \tag{63}$$

By (39) and (50), the expressions of  $\mathcal{A}$  and  $\mathcal{B}$  depend only on  $\xi$  and  $\mathbf{A}$ ; they do not depend on  $\epsilon$ , and hence, in (34)–(35), these terms can be made as small as we want by multiplication with  $\epsilon$ .

$\mathbf{u}_{\text{rem}}(t, \mathbf{x}, \epsilon)$  and  $p_{\text{rem}}(t, \mathbf{x}, \epsilon)$  can not be found explicitly from the system (56)–(57), but following the approach presented by Ionescu-Kruse [21] for barotropic incompressible fluids in the non-geophysical case, their  $L^2$ -norms are bounded at any time  $t$  by functions that can depend on  $t$  but are independent on  $\epsilon$ .

In order to find an estimate for the squared norm of  $\mathbf{u}_{\text{rem}}$ , that is,

$$\|\mathbf{u}_{\text{rem}}\|^2 = \langle \mathbf{u}_{\text{rem}}, \mathbf{u}_{\text{rem}} \rangle := \left( \int_{\mathcal{D}} \mathbf{u}_{\text{rem}} \cdot \overline{\mathbf{u}_{\text{rem}}} d\mathbf{x} \right), \tag{64}$$

where  $\langle \cdot, \cdot \rangle$  is the  $L^2$ -scalar product, the bar meaning complex conjugation and  $\mathcal{D}$  a time-dependent domain, we first give an estimate for the time derivative of the squared norm of  $\mathbf{u}_{\text{rem}}$ . The domain  $\mathcal{D}$  being time dependent and the fluid incompressible, from (64) we have

$$\frac{d}{dt} \|\mathbf{u}_{\text{rem}}\|^2 = \int_{\mathcal{D}} [\partial_t + \mathbf{U} \cdot \nabla] (\mathbf{u}_{\text{rem}} \cdot \overline{\mathbf{u}_{\text{rem}}}) d\mathbf{x}. \tag{65}$$

Now  $\mathbf{u}_{\text{rem}}$  and  $\overline{\mathbf{u}_{\text{rem}}}$  satisfy the system (56)–(57) and its complex conjugate, respectively. Therefore, (65) becomes

$$\begin{aligned} \frac{d}{dt} \|\mathbf{u}_{\text{rem}}\|^2 = & - \left\langle [(\nabla\mathbf{U}) + (\nabla\mathbf{U})^T] \mathbf{u}_{\text{rem}}, \mathbf{u}_{\text{rem}} \right\rangle + \left\langle \frac{p_{\text{rem}}}{f(P)}, g \right\rangle \\ & + \left\langle g, \frac{p_{\text{rem}}}{f(P)} \right\rangle + \langle \mathbf{T}, \mathbf{u}_{\text{rem}} \rangle + \langle \mathbf{u}_{\text{rem}}, \mathbf{T} \rangle. \end{aligned} \tag{66}$$

The pressure perturbation was considered zero on the boundary  $\partial\mathcal{D}$ .

The estimate for the first term in the right-hand side of (66) is (see Joseph [22] sect. 4)

$$- \left\langle [(\nabla\mathbf{U}) + (\nabla\mathbf{U})^T] \mathbf{u}_{\text{rem}}, \mathbf{u}_{\text{rem}} \right\rangle \leq \mathcal{C}_0 \|\mathbf{u}_{\text{rem}}\|^2, \tag{67}$$

where  $\mathcal{C}_0 = -\text{vol}(\mathcal{D}) \min_{\mathbf{x} \in \mathcal{D}} (\lambda_1, \lambda_2, \lambda_3) > 0$ ,  $\lambda_1, \lambda_2, \lambda_3$  being the eigenvalues of the symmetric matrix  $(\nabla\mathbf{U}) + (\nabla\mathbf{U})^T$ .

By the complex form of the Cauchy–Schwarz inequality

$$\left\langle \frac{p_{\text{rem}}}{f(P)}, g \right\rangle + \left\langle g, \frac{p_{\text{rem}}}{f(P)} \right\rangle \leq \left\| \frac{p_{\text{rem}}}{f(P)} \right\|^2 + \|g\|^2, \tag{68}$$

and

$$\langle \mathbf{T}, \mathbf{u}_{\text{rem}} \rangle + \langle \mathbf{u}_{\text{rem}}, \mathbf{T} \rangle \leq \|\mathbf{T}\|^2 + \|\mathbf{u}_{\text{rem}}\|^2. \tag{69}$$

From (59), (58), (39) and (50),

$$\|g\|^2 = \|\nabla \cdot \mathcal{A}\|^2 =: \mathcal{K}_1(t, \mathbf{A}, \xi) \tag{70}$$

and

$$\|\mathbf{T}\|^2 = \|\mathbb{T}\|^2 =: \mathcal{K}_2(t, \mathcal{L}_{\beta, \Omega}, \mathbf{A}, \boldsymbol{\xi}). \tag{71}$$

By Poincaré’s inequality we find an estimate for the first term in the right-hand side of (68):

$$\left\| \frac{p_{\text{rem}}}{f(P)} \right\|^2 \leq \mathcal{C}_1 \left\| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\|^2, \tag{72}$$

$\mathcal{C}_1$  being a constant which can depend on the time-dependent domain  $\mathcal{D}$ . Since  $p_{\text{rem}}$  is vanishing on  $\partial\mathcal{D}$ ,

$$\left\| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\|^2 = - \left\langle \Delta \left( \frac{p_{\text{rem}}}{f(P)} \right), \frac{p_{\text{rem}}}{f(P)} \right\rangle. \tag{73}$$

The divergence of the Eq. (56) implies the following Poisson equation

$$\begin{aligned} \Delta \left( \frac{p_{\text{rem}}}{f(P)} \right) &= \nabla \cdot \mathbf{T} - \nabla \cdot [(\mathbf{u}_{\text{rem}})_t + (\mathbf{U} \cdot \nabla)\mathbf{u}_{\text{rem}}] - \nabla \cdot [(\mathbf{u}_{\text{rem}} \cdot \nabla)\mathbf{U}] \\ &= \nabla \cdot \{\mathbf{T} - 2[(\mathbf{u}_{\text{rem}} \cdot \nabla)\mathbf{U}]\} + e^{\frac{i}{\varepsilon}\phi(t, \mathbf{x})} [\mathcal{G}_t + \mathbf{U} \cdot \nabla\mathcal{G}], \end{aligned} \tag{74}$$

the last equality being a consequence of the incompressibility condition, the Eq. (57) and the eikonal Eq. (43). Replacing (74) into (73) and integrating by parts yields

$$\begin{aligned} \left\| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\|^2 &= \left\langle \mathbf{T} - 2[(\mathbf{u}_{\text{rem}} \cdot \nabla)\mathbf{U}], \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\rangle \\ &\quad - \left\langle e^{\frac{i}{\varepsilon}\phi(t, \mathbf{x})} [\mathcal{G}_t + \mathbf{U} \cdot \nabla\mathcal{G}], \frac{p_{\text{rem}}}{f(P)} \right\rangle. \end{aligned} \tag{75}$$

By the triangle inequality, we are thus led to

$$\begin{aligned} \left\| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\|^2 &\leq \left\langle |\mathbb{T}|, \left| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right| \right\rangle + \mathcal{C}_3 \left\langle |\mathbf{u}_{\text{rem}}|, \left| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right| \right\rangle \\ &\quad + \left\langle |\mathcal{G}_t + \mathbf{U} \cdot \nabla\mathcal{G}|, \left| \frac{p_{\text{rem}}}{f(P)} \right| \right\rangle, \end{aligned} \tag{76}$$

$\mathcal{C}_3$  being a time-dependent constant. Finally, we apply the inequality

$$2ab \leq \frac{1}{\mu} a^2 + \mu b^2, \tag{77}$$

which is valid for any positive real numbers  $a, b, \mu$ . Thus, (76) becomes

$$\begin{aligned} \left\| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\|^2 &\leq \frac{1}{2\mu} \{ \|\mathbb{T}\|^2 + \mathcal{C}_3 \|\mathbf{u}_{\text{rem}}\|^2 + \|\mathcal{G}_t + \mathbf{U} \cdot \nabla\mathcal{G}\|^2 \} \\ &\quad + \frac{1}{2}\mu \left\{ (1 + \mathcal{C}_3) \left\| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\|^2 + \left\| \frac{p_{\text{rem}}}{f(P)} \right\|^2 \right\}, \end{aligned} \tag{78}$$

which for a  $\mu$  such that  $\mu(1 + \mathcal{C}_3) < 2$ , yields the inequality

$$\left\| \nabla \left( \frac{p_{\text{rem}}}{f(P)} \right) \right\|^2 \leq \mathcal{C}_4 \mathcal{K}_3(t, \mathcal{L}_{\beta, \Omega}, \mathbf{A}, \boldsymbol{\xi}) + \mathcal{C}_5 \|\mathbf{u}_{\text{rem}}\|^2 + \mathcal{C}_6 \mu \left\| \frac{p_{\text{rem}}}{f(P)} \right\|^2, \tag{79}$$

$$\mathcal{K}_3(t, \mathcal{L}_{\beta, \Omega}, \mathbf{A}, \boldsymbol{\xi}) := \|\mathbb{T}\|^2 + \|\mathcal{G}_t + \mathbf{U} \cdot \nabla\mathcal{G}\|^2. \tag{80}$$

Hence, Poincaré’s inequality (72) yields (after applying one more time (77) in the same manner as above)

$$\left\| \frac{P_{\text{rem}}}{f(P)} \right\|^2 \leq \mathcal{C}_7 \mathcal{K}_3(t, \mathcal{L}_{\beta, \Omega}, \mathbf{A}, \boldsymbol{\xi}) + \mathcal{C}_8 \|\mathbf{u}_{\text{rem}}\|^2. \tag{81}$$

We therefore conclude with the following estimate for the time derivative of the squared norm of  $\mathbf{u}_{\text{rem}}$

$$\frac{d}{dt} \|\mathbf{u}_{\text{rem}}\|^2 \leq \mathcal{C} \|\mathbf{u}_{\text{rem}}\|^2 + \mathcal{K}(t, \mathcal{L}_{\beta, \Omega}, \mathbf{A}, \boldsymbol{\xi}), \tag{82}$$

where  $\mathcal{C}$  is a constant that can depend on time by means of the volume integral over the domain  $\mathcal{D}$ . For any fixed time interval  $0 \leq t \leq T$ , the volume integral can be bounded; thus, we can consider  $\mathcal{C}$  as constant on this interval. The leading-order amplitude  $\mathbf{A}(t, \mathbf{x}(t))$  and the wave vector  $\boldsymbol{\xi}(t, \mathbf{x}(t))$  are the solutions of the ODE’s system (63). The matrix  $\mathcal{L}_{\beta, \Omega}$  depends on  $\mathbf{x}(t)$  and the constants  $\Omega$  and  $\beta$ . Therefore, we think of the above inequality as

$$\frac{d}{dt} \|\mathbf{u}_{\text{rem}}\|^2 \leq \mathcal{C} \|\mathbf{u}_{\text{rem}}\|^2 + \mathcal{K}(t). \tag{83}$$

By Gronwall’s inequality, for any fixed time interval  $0 \leq t \leq T$ ,

$$\|\mathbf{u}_{\text{rem}}\|^2 \leq \int_0^t e^{\mathcal{C}(t-s)} \mathcal{K}(s) ds, \tag{84}$$

$\mathbf{u}_{\text{rem}}(0)$  being  $\mathbf{0}$ . This type of estimate was obtained in the non-geophysical case by Lifschitz and E. Hameiri [24] in the constant density case and by Ionescu-Kruse [21] in the barotropic case.

Thus, on any finite interval of time  $[0, T]$ , the term

$$\epsilon (\|\mathcal{A}\| + \|\mathbf{u}_{\text{rem}}\|) \tag{85}$$

can be made as small as we want by multiplication with  $\epsilon$ . We conclude with the following *sufficient instability criterion with respect to short-wavelength perturbations: a barotropic incompressible geophysical flow is unstable in the velocity  $L^2$ -norm near the trajectory passing through a point  $\mathbf{x}_0$ , if for some  $\boldsymbol{\xi}_0, \mathbf{A}_0$  satisfying  $\mathbf{A}_0 \cdot \boldsymbol{\xi}_0 = 0$ , the corresponding amplitude  $\mathbf{A}(t; \mathbf{x}_0, \boldsymbol{\xi}_0, \mathbf{A}_0)$  increases unboundedly in time.*

### 4 Short-wavelength instabilities of equatorially trapped waves in stratified water

In order to apply the sufficient instability criterion obtained above to the equatorially trapped waves (6) in stratified water, we analyse the ODE system (60)–(63) for these waves. The equation (60) has the solution (6). Differentiating with respect to  $t$  the Jacobian matrix (11) of the transformation (6), we get the following equation

$$\frac{dF}{dt} = F(\nabla \mathbf{U})^T \tag{86}$$

with  $\nabla \mathbf{U}$  the matrix defined in (49). Therefore, by differentiating with respect to  $t$  the relation

$$FG = \text{Id}, \tag{87}$$

$G$  being the inverse matrix of  $F$ , we get the equation

$$\frac{dG}{dt} = -(\nabla \mathbf{U})^T G. \tag{88}$$

In this way, the solution of the initial value equation (61) is

$$\xi(t) = G(t)F(0)\xi_0. \tag{89}$$

Taking into account the expressions (11) and (13) of the matrices  $F$  and  $G$ , respectively, (89) becomes

$$\xi(t) = \begin{pmatrix} \frac{1+e^\chi [\cos \theta - \cos(kq)] - e^{2\chi} \cos(kct)}{1-e^{2\chi}} & 0 & \frac{e^\chi [\sin \theta - \sin(kq)] - e^{2\chi} \sin(kct)}{1-e^{2\chi}} \\ -s \frac{c\beta}{g} \frac{e^\chi [\sin \theta - \sin(kq)] + e^{2\chi} \sin(kct)}{1-e^{2\chi}} & 1 & s \frac{c\beta}{g} \frac{e^\chi [\cos \theta - \cos(kq)] + e^{2\chi} \cos(kct) - e^{2\chi}}{1-e^{2\chi}} \\ \frac{e^\chi [\sin \theta - \sin(kq)] + e^{2\chi} \sin(kct)}{1-e^{2\chi}} & 0 & \frac{1-e^\chi [\cos \theta - \cos(kq)] - e^{2\chi} \cos(kct)}{1-e^{2\chi}} \end{pmatrix} \xi_0.$$

We thus get, for the initial condition

$$\xi_0 = (0 \ 1 \ 0)^T, \tag{90}$$

the solution

$$\xi(t) = (0 \ 1 \ 0)^T, \text{ for all } t \geq 0. \tag{91}$$

By replacing the solution (91) into the equation (63) and taking into account (23), (32), it follows that the evolution of the amplitude vector  $\mathbf{A}$  is governed by

$$\frac{d\mathbf{A}}{dt} = \left[ -\nabla\mathbf{U} - \begin{pmatrix} 0 & -\beta s & 2\Omega \\ 0 & 0 & 0 \\ -2\Omega & 0 & 0 \end{pmatrix} \right] \mathbf{A}(t), \tag{92}$$

where

$$\nabla\mathbf{U} = kc \frac{e^\chi}{1-e^{2\chi}} \begin{pmatrix} -\sin \theta & s \frac{c\beta}{g} (e^\chi - \cos \theta) & \cos \theta - e^\chi \\ 0 & 0 & 0 \\ \cos \theta + e^\chi & -s \frac{c\beta}{g} \sin \theta & \sin \theta \end{pmatrix}, \tag{93}$$

$\chi$  and  $\theta$  having the expressions (10).

We denote by  $(A^1(t), A^2(t), A^3(t))^T$  the components of the vector  $\mathbf{A}(t)$  in the canonical basis. For the initial vector  $\xi_0$  given by (90) and the condition (62), we must have  $A^2(0) = 0$ . Thus, (92) yields

$$A^2(t) = 0, \text{ for all } t \geq 0. \tag{94}$$

This shows that the velocity perturbation lies in the plane perpendicular to the vector  $(0 \ 1 \ 0)^T$  for all time. According to (92), the amplitude vector  $\mathbf{A}(t) = (A^1(t), 0, A^3(t))^T$  satisfies

$$\frac{d\mathbf{A}}{dt} = \left\{ -kc \frac{e^\chi}{1-e^{2\chi}} \mathcal{M}(t) + \left[ -kc \frac{e^{2\chi}}{1-e^{2\chi}} + 2\Omega \right] \mathcal{N} \right\} \mathbf{A}(t), \tag{95}$$

where

$$\mathcal{M}(t) = \begin{pmatrix} -\sin[k(q-ct)] & 0 & \cos[k(q-ct)] \\ 0 & 0 & 0 \\ \cos[k(q-ct)] & 0 & \sin[k(q-ct)] \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{96}$$

The Eq. (95) written in the canonical basis is non-autonomous.

By rotating the canonical basis with the angle  $(-\frac{kct}{2})$  around the axis  $(0\ 1\ 0)^T$ , the rotation matrix being

$$\mathcal{R}(t) = \begin{pmatrix} \cos(\frac{kct}{2}) & 0 & \sin(\frac{kct}{2}) \\ 0 & 1 & 0 \\ -\sin(\frac{kct}{2}) & 0 & \cos(\frac{kct}{2}) \end{pmatrix}, \tag{97}$$

the Eq. (95) takes the form

$$\begin{aligned} \frac{d\mathbf{A}}{dt} &= \left\{ \frac{d}{dt} [\mathcal{R}^{-1}(t)] \mathcal{R}(t) - kc \frac{e^\chi}{1 - e^{2\chi}} \mathcal{R}^{-1}(t) \mathcal{M}(t) \mathcal{R}(t) \right. \\ &\quad \left. + \left[ -kc \frac{e^{2\chi}}{1 - e^{2\chi}} + 2\Omega \right] \mathcal{R}^{-1}(t) \mathcal{N} \mathcal{R}(t) \right\} \mathbf{A}(t) \\ &= \left\{ \frac{kc}{2} \mathcal{N} - kc \frac{e^\chi}{1 - e^{2\chi}} \begin{pmatrix} -\sin(kq) & 0 & \cos(kq) \\ 0 & 0 & 0 \\ \cos(kq) & 0 & \sin(kq) \end{pmatrix} \right. \\ &\quad \left. + \left[ -kc \frac{e^{2\chi}}{1 - e^{2\chi}} + 2\Omega \right] \mathcal{N} \right\} \mathbf{A}(t). \end{aligned} \tag{98}$$

We thus get in the rotating basis an autonomous system:

$$\frac{d\mathbf{A}}{dt} = \mathcal{Q} \mathbf{A}(t), \tag{99}$$

with  $\mathcal{Q}$  the time-independent matrix

$$\mathcal{Q} = \frac{kc}{1 - e^{2\chi}} \begin{pmatrix} e^\chi \sin(kq) & 0 & -e^\chi \cos(kq) + \frac{kc(3e^{2\chi}-1)-4\Omega(1-e^{2\chi})}{2kc} \\ 0 & 0 & 0 \\ -e^\chi \cos(kq) - \frac{kc(3e^{2\chi}-1)-4\Omega(1-e^{2\chi})}{2kc} & 0 & -e^\chi \sin(kq) \end{pmatrix}.$$

The solution to the non-autonomous equation (95) is obtained by multiplying the rotation matrix  $\mathcal{R}(t)$  with the solution to the autonomous Eq. (99). The matrix  $\mathcal{R}(t)$  being time periodic, the behaviour in time of the amplitude vector  $\mathbf{A}$  is determined by the eigenvalues of the matrix  $\mathcal{Q}$ . The eigenvalues of the matrix  $\mathcal{Q}$  satisfy the following equation

$$\begin{aligned} \lambda^2 &= \left( \frac{kc}{1 - e^{2\chi}} \right)^2 \left\{ e^{2\chi} - \left[ \frac{(3kc + 4\Omega)e^{2\chi} - (kc + 4\Omega)}{2kc} \right]^2 \right\} \\ &= \frac{1}{4} \frac{(3kc + 4\Omega)^2 e^{2\chi} - (kc + 4\Omega)^2}{1 - e^{2\chi}}. \end{aligned} \tag{100}$$

From (9)

$$0 \leq e^{2\chi} < 1, \tag{101}$$

thus, if

$$e^\chi > \frac{kc + 4\Omega}{3kc + 4\Omega} \stackrel{(7)}{=} \frac{\sqrt{\Omega^2 + kg} + 3\Omega}{3\sqrt{\Omega^2 + kg} + \Omega}, \tag{102}$$

then the amplitude  $\mathbf{A}$  increases unboundedly in time, the exponential growth rate being

$$\lambda = \frac{1}{2} \sqrt{\frac{(3kc + 4\Omega)^2 e^{2\chi} - (kc + 4\Omega)^2}{1 - e^{2\chi}}}. \tag{103}$$

Constantin and Germain [6] found the condition (102) in their instability analysis of equatorially trapped waves in the constant density case.

The condition (102) is satisfied for the equatorially trapped waves with the second component of the vorticity (17) smaller than

$$-\frac{1}{4} \frac{(kc + 4\Omega)^2}{kc(kc + 2\Omega)}.$$

The steepness parameter  $\tau$  of the wave profile is defined as the amplitude of the wave multiplied by the wave vector  $k$ , that is,

$$\tau(s) = e^\chi. \quad (104)$$

Thus, we conclude that also in stratified water the eastward propagating equatorially trapped waves (6), with wave number  $k > 0$ , are unstable to short-wavelength perturbations if the vorticity (17) in the  $y$ -direction is smaller than  $-\frac{1}{4} \frac{(kc+4\Omega)^2}{kc(kc+2\Omega)}$  or if the steepness parameter  $\tau$  is higher than  $\frac{kc+4\Omega}{3kc+4\Omega}$ . The exponential growth rate of instabilities is  $\frac{1}{2} \sqrt{\frac{(3kc+4\Omega)^2 e^{2\chi} - (kc+4\Omega)^2}{1-e^{2\chi}}}$ .

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