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Helix surfaces in the special linear group

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Abstract We characterize helix surfaces (constant angle surfaces) in the special linear group $SL(2, \mathbb{R})$. In particular, we give an explicit local description of these surfaces by means of a suitable curve and a 1-parameter family of isometries of $SL(2, \mathbb{R})$.

Keywords Special linear group · Helix surfaces · Constant angle surfaces · Homogeneous spaces

Mathematics Subject Classification 53B25

1 Introduction

In recent years much work has been done to understand the geometry of surfaces whose unit normal vector field forms a constant angle with a fixed field of directions of the ambient space. These surfaces are called *helix surfaces* or *constant angle surfaces* and they have been studied in most of the 3-dimensional geometries. In [2] Cermelli and Di Scala analyzed the case of constant angle surfaces in $\mathbb{R}^3$ obtaining a remarkable relation with a Hamilton–Jacobi equation and showing their application to equilibrium configurations of liquid crystals. Later, Dillen–Fastenakels–Van der Veken–Vrancken [4], and Dillen–Munteanu [3], classified the surfaces making a constant angle with the $\mathbb{R}$-direction in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, respectively. Moreover, helix submanifolds have been studied in higher dimensional euclidean spaces and product spaces in [5,6,10].

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The spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ can be seen as two particular cases of Bianchi–Cartan–Vranceanu spaces (BCV-spaces) which include all 3-dimensional homogeneous metrics whose group of isometries has dimension 4 or 6, except for those of constant negative sectional curvature. A crucial feature of BCV-spaces is that they admit a Riemannian submersion onto a surface of constant Gaussian curvature, called the Hopf fibration, that, in the cases of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, it is the natural projection onto the first factor. Consequently, one can consider the angle ϑ that the unit normal vector field of a surface in a BCV-space forms with the Hopf vector field, which is, by definition, the vector field tangent to the fibers of the Hopf fibration. This angle ϑ has a crucial role in the study of surfaces in BCV-spaces as shown by Daniel, in [1], where he proved that the equations of Gauss and Codazzi are given in terms of the function $\nu = \cos\vartheta$ and that this angle is one of the fundamental invariants for a surface in BCV-spaces. Consequently, in [7], the authors considered the surfaces in a BCV-space for which the angle ϑ is constant, giving a complete local classification in the case that the BCV-space is the Heisenberg space $\mathbb{H}_3$.

Later, López–Munteanu, in [8], defined and classified two types of constant angle surfaces in the homogeneous 3-manifold Sol_3, whose isometry group has dimension 3. Also, Montaldo–Onnis, in [9], characterized helix surfaces in the 1-parameter family of Berger spheres $\mathbb{S}^3_\epsilon$, with $\epsilon > 0$, proving that, locally, a helix surface is determined by a suitable 1-parameter family of isometries of the Berger sphere and by a geodesic of a 2-torus in the 3-dimensional sphere.

This paper is a continuation of our work [9] and it is devoted to the study and characterization of helix surfaces in the homogeneous 3-manifold given by the special linear group $\mathrm{SL}(2,\mathbb{R})$ endowed with a suitable 1-parameter family g_τ of metrics that we shall describe in sect. 2. Our study of helix surfaces in $(\mathrm{SL}(2,\mathbb{R}), g_\tau)$ will depend on a constant $B := (\tau^2+1)\cos^2\vartheta - 1$, where ϑ is the constant angle between the normal to the surface and the Hopf vector field of $\mathrm{SL}(2,\mathbb{R})$. A similar constant appeared also in the study of helix surfaces in the Berger sphere (see [9]) but in that case the constant was always positive. Thus we shall divide our study according to the three possibilities: $B > 0$, $B = 0$ and $B < 0$.

2 Preliminaries

Let $\mathbb{R}^4_2$ denote the 4-dimensional pseudo-Euclidean space endowed with the semi-definite inner product of signature (2,2) given by

$$\langle v, w\rangle = v_1 w_1 + v_2 w_2 - v_3 w_3 - v_4 w_4, \quad v, w \in \mathbb{R}^4.$$

We identify the special linear group with

$$\mathrm{SL}(2,\mathbb{R}) = \{(z,w) \in \mathbb{C}^2 \colon |z|^2 - |w|^2 = 1\} = \{v \in \mathbb{R}^4_2 \colon \langle v, v\rangle = 1\} \subset \mathbb{R}^4_2$$

and we shall use the Lorentz model of the hyperbolic plane with constant Gauss curvature -4, that is

$$\mathbb{H}^2(-4) = \{(x,y,z) \in \mathbb{R}^3_1 \colon x^2 + y^2 - z^2 = -1/4\},$$

where $\mathbb{R}^3_1$ is the Minkowski 3-space. Then the Hopf map $\psi : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{H}^2(-4)$ given by

$$\psi(z,w) = \frac{1}{2}\,(2z\bar{w}, |z|^2 + |w|^2)$$

is a submersion, with circular fibers, and if we put

$$X_1(z,w) = (iz, iw), \quad X_2(z,w) = (i\bar{w}, i\bar{z}), \quad X_3(z,w) = (\bar{w}, \bar{z}),$$

we have that X_1 is a vertical vector field while $X_2,\ X_3$ are horizontal. The vector field X_1 is called the *Hopf vector field*.

We shall endow $\mathrm{SL}(2,\mathbb{R})$ with the 1-parameter family of metrics $g_\tau,\ \tau>0$, given by

$$g_\tau(X_i,X_j)=\delta_{ij},\quad g_\tau(X_1,X_1)=\tau^2,\quad g_\tau(X_1,X_j)=0,\quad i,j\in\{2,3\},$$

which renders the Hopf map $\psi:(\mathrm{SL}(2,\mathbb{R}),g_\tau)\to\mathbb{H}^2(-4)$ a Riemannian submersion.

For those familiar with the notations of Daniel [1], we point out that $(\mathrm{SL}(2,\mathbb{R}),g_\tau)$ corresponds to a model for a homogeneous space $E(k,\tau)$ with curvature of the basis $k=-4$ and bundle curvature $\tau>0$.

With respect to the inner product in $\mathbb{R}^4_2$ the metric g_τ is given by

$$g_\tau(X,Y)=-\langle X,Y\rangle+(1+\tau^2)\langle X,X_1\rangle\langle Y,X_1\rangle. \tag{1}$$

From now on, we denote $(\mathrm{SL}(2,\mathbb{R}),g_\tau)$ with $\mathrm{SL}(2,\mathbb{R})_\tau$. Obviously

$$E_1=-\tau^{-1}X_1,\quad E_2=X_2,\quad E_3=X_3, \tag{2}$$

is an orthonormal basis on $\mathrm{SL}(2,\mathbb{R})_\tau$ and the Levi-Civita connection ∇^τ of $\mathrm{SL}(2,\mathbb{R})_\tau$ is given by (see, for example, [11]):

$$\begin{aligned}
&\nabla^\tau_{E_1}E_1=0,\quad \nabla^\tau_{E_2}E_2=0,\quad \nabla^\tau_{E_3}E_3=0,\\
&\nabla^\tau_{E_1}E_2=-\tau^{-1}(2+\tau^2)E_3,\quad \nabla^\tau_{E_1}E_3=\tau^{-1}(2+\tau^2)E_2,\\
&\nabla^\tau_{E_2}E_1=-\tau E_3,\quad \nabla^\tau_{E_3}E_1=\tau E_2,\quad \nabla^\tau_{E_3}E_2=-\tau E_1=-\nabla^\tau_{E_2}E_3.
\end{aligned} \tag{3}$$

Finally, we recall that the isometry group of $\mathrm{SL}(2,\mathbb{R})_\tau$ is the 4-dimensional indefinite unitary group $\mathrm{U}_1(2)$ that can be identified with:

$$\mathrm{U}_1(2)=\{A\in\mathrm{O}_2(4)\colon AJ_1=\pm J_1A\},$$

where J_1 is the complex structure of $\mathbb{R}^4$ defined by

$$J_1=\begin{pmatrix}J&0\\0&J\end{pmatrix},\quad J=\begin{pmatrix}0&-1\\1&0\end{pmatrix},$$

while

$$\mathrm{O}_2(4)=\{A\in\mathrm{GL}(4,\mathbb{R})\colon A^t=\epsilon A^{-1}\epsilon\},\qquad \epsilon=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

is the indefinite orthogonal group.

We observe that $\mathrm{O}_2(4)$ is the group of 4×4 real matrices preserving the semi-definite inner product of $\mathbb{R}^4_2$.

Suppose now we are given a 1-parameter family $A(v),\ v\in(a,b)\subset\mathbb{R}$, consisting of 4×4 indefinite orthogonal matrices commuting (anticommuting, respectively) with J_1. In order to describe explicitly the family $A(v)$, we shall use two product structures of $\mathbb{R}^4$, namely

$$J_2=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\qquad J_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}.$$

Since $A(v)$ is an indefinite orthogonal matrix, the first row must be a unit vector $\mathbf{r}_1(v)$ of $\mathbb{R}^4_2$ for all $v\in(a,b)$. Thus, without loss of generality, we can take

$$\begin{aligned}\mathbf{r}_1(v)=(&\cosh\xi_1(v)\cos\xi_2(v),-\cosh\xi_1(v)\sin\xi_2(v),\sinh\xi_1(v)\cos\xi_3(v),\\&-\sinh\xi_1(v)\sin\xi_3(v)),\end{aligned}$$

for some real functions ξ_1, ξ_2 and ξ_3 defined in (a, b). Since $A(v)$ commutes (anticommutes, respectively) with J_1 the second row of $A(v)$ must be $\mathbf{r}_2(v) = \pm J_1\mathbf{r}_1(v)$. Now, the four vectors $\{\mathbf{r}_1, J_1\mathbf{r}_1, J_2\mathbf{r}_1, J_3\mathbf{r}_1\}$ form a pseudo-orthonormal basis of $\mathbb{R}^4_2$, thus the third row $\mathbf{r}_3(v)$ of $A(v)$ must be a linear combination of them. Since $\mathbf{r}_3(v)$ is unit and it is orthogonal to both $\mathbf{r}_1(v)$ and $J_1\mathbf{r}_1(v)$, there exists a function $\xi(v)$ such that

$$\mathbf{r}_3(v) = \cos\xi(v) J_2\mathbf{r}_1(v) + \sin\xi(v) J_3\mathbf{r}_1(v).$$

Finally the fourth row of $A(v)$ is $\mathbf{r}_4(v) = \pm J_1\mathbf{r}_3(v) = \mp\cos\xi(v) J_3\mathbf{r}_1(v) \pm \sin\xi(v) J_2\mathbf{r}_1(v)$. This means that any 1-parameter family $A(v)$ of 4×4 indefinite orthogonal matrices commuting (anticommuting, respectively) with J_1 can be described by four functions ξ_1, ξ_2, ξ_3 and ξ as

$$A(\xi, \xi_1, \xi_2, \xi_3)(v) = \begin{pmatrix} \mathbf{r}_1(v) \\ \pm J_1\mathbf{r}_1(v) \\ \cos\xi(v) J_2\mathbf{r}_1(v) + \sin\xi(v) J_3\mathbf{r}_1(v) \\ \mp\cos\xi(v) J_3\mathbf{r}_1(v) \pm \sin\xi(v) J_2\mathbf{r}_1(v) \end{pmatrix}. \tag{4}$$

3 Constant angle surfaces

We start this section giving the definition of constant angle surface in $\mathrm{SL}(2, \mathbb{R})_\tau$.

Definition 3.1 We say that a surface in the special linear group $\mathrm{SL}(2, \mathbb{R})_\tau$ is a *helix surface* or a *constant angle surface* if the angle $\vartheta \in [0, \pi)$ between the unit normal vector field and the unit Killing vector field E_1 (tangent to the fibers of the Hopf fibration) is constant at every point of the surface.

Let M^2 be an oriented helix surface in $\mathrm{SL}(2, \mathbb{R})_\tau$ and let N be a unit normal vector field. Then, by definition,

$$|g_\tau(E_1, N)| = \cos\vartheta,$$

for fixed $\vartheta \in [0, \pi/2]$. Note that $\vartheta \neq 0$. In fact, if it were zero then the vector fields E_2 and E_3 would be tangent to the surface M^2, which is absurd since the horizontal distribution of the Hopf map is not integrable. If $\vartheta = \pi/2$, we have that E_1 is always tangent to M and, therefore, M is a Hopf cylinder. Therefore, from now on we assume that the constant angle $\vartheta \neq \pi/2, 0$.

The Gauss and Weingarten formulas are

$$\begin{aligned} \nabla^\tau_X Y &= \nabla_X Y + \alpha(X, Y), \\ \nabla^\tau_X N &= -A(X), \end{aligned} \tag{5}$$

where with A we have indicated the shape operator of M in $\mathrm{SL}(2, \mathbb{R})_\tau$, with ∇ the induced Levi-Civita connection on M and by α the second fundamental form of M in $\mathrm{SL}(2, \mathbb{R})_\tau$. Projecting E_1 onto the tangent plane to M we have

$$E_1 = T + \cos\vartheta\, N,$$

where T is the tangent part which satisfies $g_\tau(T, T) = \sin^2\vartheta$.

For all $X \in TM$, we have that

$$\begin{aligned} \nabla^\tau_X E_1 &= \nabla^\tau_X T - \cos\vartheta\, A(X) \\ &= \nabla_X T + g_\tau(A(X), T)\, N - \cos\vartheta\, A(X). \end{aligned} \tag{6}$$

On the other hand, if $X = \sum X_i E_i$,

$$\begin{aligned} \nabla^\tau_X E_1 &= \tau\,(X_3 E_2 - X_2 E_3) = \tau\, X \wedge E_1 \\ &= \tau\, g_\tau(JX, T)\, N - \tau\, \cos\vartheta\, JX, \end{aligned} \tag{7}$$

where $JX = N \wedge X$ denotes the rotation of angle $\pi/2$ on TM. Identifying the tangent and normal component of (6) and (7) respectively, we obtain

$$\nabla_X T = \cos\vartheta\,(A(X) - \tau\, JX) \tag{8}$$

and

$$g_\tau(A(X) - \tau\, JX, T) = 0. \tag{9}$$

Lemma 3.2 *Let M^2 be an oriented helix surface in $\mathrm{SL}(2,\mathbb{R})_\tau$ with constant angle ϑ. Then, we have the followings properties.*

(i) *With respect to the basis $\{T, JT\}$, the matrix associates to the shape operator A takes the form*

$$A = \begin{pmatrix} 0 & -\tau \\ -\tau & \lambda \end{pmatrix},$$

for some function λ on M.

(ii) *The Levi-Civita connection ∇ of M is given by*

$$\begin{aligned} \nabla_T T &= -2\tau\cos\vartheta\; JT, \qquad & \nabla_{JT} T &= \lambda\cos\vartheta\; JT, \\ \nabla_T JT &= 2\tau\cos\vartheta\; T, \qquad & \nabla_{JT} JT &= -\lambda\cos\vartheta\; T. \end{aligned}$$

(iii) *The Gauss curvature of M is constant and satisfies*

$$K = -4(1+\tau^2)\,\cos^2\vartheta.$$

(iv) *The function λ satisfies the equation*

$$T\lambda + \lambda^2\,\cos\vartheta + 4B\,\cos\vartheta = 0, \tag{10}$$

where $B := (\tau^2+1)\cos^2\vartheta - 1$.

Proof Point (i) follows directly from (9). From (8) and using

$$g_\tau(T,T) = g_\tau(JT,JT) = \sin^2\vartheta, \quad g_\tau(T,JT) = 0,$$

we obtain (ii). From the Gauss equation in $\mathrm{SL}(2,\mathbb{R})_\tau$ (we refer to the equation in Corollary 3.2 of [1] with $\nu = \cos\theta$ and $k = -4$), and (i), we have that the Gauss curvature of M is given by

$$\begin{aligned} K &= \det A + \tau^2 - 4(1+\tau^2)\,\cos^2\vartheta \\ &= -4(1+\tau^2)\,\cos^2\vartheta. \end{aligned}$$

Finally, (10) follows from the Codazzi equation (see [1]):

$$\nabla_X A(Y) - \nabla_Y A(X) - A[X,Y] = -4(1+\tau^2)\,\cos\vartheta\,(g_\tau(Y,T)X - g_\tau(X,T)Y),$$

putting $X = T,\ Y = JT$ and using (ii). In fact, it is easy to check that

$$-4(1+\tau^2)\,\cos\vartheta\,(g_\tau(JT,T)T - g_\tau(T,T)JT)) = 4(1+\tau^2)\cos\vartheta\,\sin^2\vartheta\; JT$$

and

$$\begin{aligned}&\nabla_T A(JT) - \nabla_{JT} A(T) - A[T, JT]\\&= \nabla_T(-\tau\, T + \lambda\, JT) - \nabla_{JT}(-\tau\, JT) - A(2\tau \cos\vartheta\, T - \lambda \cos\vartheta\, JT)\\&= (4\tau^2 \cos\vartheta + T(\lambda) + \lambda^2 \cos\vartheta)\, JT.\end{aligned}$$

□

As $g_\tau(E_1, N) = \cos\vartheta$, there exists a smooth function φ on M such that

$$N = \cos\vartheta\, E_1 + \sin\vartheta \cos\varphi\, E_2 + \sin\vartheta \sin\varphi\, E_3.$$

Therefore

$$T = E_1 - \cos\vartheta\, N = \sin\vartheta\, [\sin\vartheta\, E_1 - \cos\vartheta \cos\varphi\, E_2 - \cos\vartheta \sin\varphi\, E_3] \tag{11}$$

and

$$JT = \sin\vartheta\, (\sin\varphi\, E_2 - \cos\varphi\, E_3).$$

Also

$$\begin{aligned}A(T) &= -\nabla^\tau_T N = (T\varphi - \tau^{-1}(2+\tau^2)\, \sin^2\vartheta + \tau \cos^2\vartheta)\, JT,\\A(JT) &= -\nabla^\tau_{JT} N = (JT\varphi)\, JT - \tau\, T.\end{aligned} \tag{12}$$

Comparing (12) with (i) of Lemma 3.2, it results that

$$\begin{cases} JT\varphi = \lambda,\\ T\varphi = -2\tau^{-1}\, B.\end{cases} \tag{13}$$

We observe that, as

$$[T, JT] = \cos\vartheta\, (2\tau\, T - \lambda\, JT),$$

the compatibility condition of system (13):

$$(\nabla_T JT - \nabla_{JT} T)\varphi = [T, JT]\varphi = T(JT\varphi) - JT(T\varphi)$$

is equivalent to (10).

We now choose local coordinates (u, v) on M such that

$$\partial_u = T. \tag{14}$$

Also, as ∂_v is tangent to M, it can be written in the form

$$\partial_v = a\, T + b\, JT, \tag{15}$$

for certain functions $a = a(u, v)$ and $b = b(u, v)$. As

$$0 = [\partial_u, \partial_v] = (a_u + 2\tau b \cos\vartheta)\, T + (b_u - b\lambda \cos\vartheta)\, JT,$$

then

$$\begin{cases} a_u = -2\tau b \cos\vartheta,\\ b_u = b\lambda \cos\vartheta.\end{cases} \tag{16}$$

Moreover, the Eq. (10) of Lemma 3.2 can be written as

$$\lambda_u + \cos\vartheta\, \lambda^2 + 4B\, \cos\vartheta = 0. \tag{17}$$

Depending on the value of B, by integration of (17), we have the following three possibilities.

(i) If $B = 0$

$$\lambda(u, v) = \frac{1}{u\,\cos\vartheta + \eta(v)},$$

for some smooth function η depending on v. Thus the solution of system (16) is given by

$$\begin{cases} a(u, v) = -\tau\, u\,\cos\vartheta\,(u\,\cos\vartheta + 2\,\eta(v)), \\ b(u, v) = u\,\cos\vartheta + \eta(v). \end{cases}$$

(ii) If $B > 0$

$$\lambda(u, v) = 2\sqrt{B}\tan(\eta(v) - 2\cos\vartheta\sqrt{B}\,u),$$

for some smooth function η depending on v and system (16) has the solution

$$\begin{cases} a(u, v) = \frac{\tau}{\sqrt{B}}\sin(\eta(v) - 2\cos\vartheta\sqrt{B}\,u), \\ b(u, v) = \cos(\eta(v) - 2\cos\vartheta\sqrt{B}\,u). \end{cases}$$

(iii) If $B < 0$

$$\lambda(u, v) = 2\sqrt{-B}\tanh(\eta(v) + 2\cos\vartheta\sqrt{-B}\,u),$$

for some smooth function η depending on v. Solving the system (16), we have

$$\begin{cases} a(u, v) = -\frac{\tau}{\sqrt{-B}}\sinh(\eta(v) + 2\cos\vartheta\sqrt{-B}\,u), \\ b(u, v) = \cosh(\eta(v) + 2\cos\vartheta\sqrt{-B}\,u). \end{cases}$$

Moreover, in the case (i) the system (13) becomes

$$\begin{cases} \varphi_u = 0, \\ \varphi_v = 1, \end{cases} \tag{18}$$

and so $\varphi(u, v) = v + c,\ c \in \mathbb{R}$. In the cases (ii) and (iii), the system (13) becomes

$$\begin{cases} \varphi_u = -2\tau^{-1}\,B, \\ \varphi_v = 0, \end{cases}$$

of which the general solution is given by

$$\varphi(u, v) = -2\tau^{-1}B\,u + c, \tag{19}$$

where c is a real constant.

With respect to the local coordinates (u, v) chosen above, we have the following characterization of the position vector of a helix surface.

Proposition 3.3 *Let M^2 be a helix surface in $\mathrm{SL}(2, \mathbb{R})_\tau \subset \mathbb{R}^4_2$ with constant angle ϑ. Then, with respect to the local coordinates (u, v) on M defined in (14), the position vector F of M^2 in $\mathbb{R}^4_2$ satisfies the following equation:*

(a) *if $B = 0$,*

$$\frac{\partial^2 F}{\partial u^2} = 0, \tag{20}$$

(b) *if $B \neq 0$,*

$$\frac{\partial^4 F}{\partial u^4} + (\tilde{b}^2 - 2\tilde{a})\,\frac{\partial^2 F}{\partial u^2} + \tilde{a}^2\,F = 0, \tag{21}$$

where

$$\tilde{a} = -\tau^{-2}\sin^2\vartheta\,B, \qquad \tilde{b} = -2\tau^{-1}\,B. \tag{22}$$

Proof Let M^2 be a helix surface and let F be the position vector of M^2 in $\mathbb{R}^4_2$. Then, with respect to the local coordinates (u, v) on M defined in (14), we can write $F(u, v) = (F_1(u, v), \ldots, F_4(u, v))$. By definition, taking into account (11), we have that

$$\begin{aligned} F_u &= (\partial_u F_1, \partial_u F_2, \partial_u F_3, \partial_u F_4) = T \\ &= \sin\vartheta\,[\sin\vartheta\, E_{1|F(u,v)} - \cos\vartheta\cos\varphi\, E_{2|F(u,v)} - \cos\vartheta\sin\varphi\, E_{3|F(u,v)}]. \end{aligned}$$

Using the expression of E_1, E_2 and E_3 with respect to the coordinates vector fields of $\mathbb{R}^4_2$, we obtain

$$\begin{cases} \partial_u F_1 = \sin\vartheta\,(\tau^{-1}\sin\vartheta\, F_2 - \cos\vartheta\cos\varphi\, F_4 - \cos\vartheta\sin\varphi\, F_3), \\ \partial_u F_2 = -\sin\vartheta\,(\tau^{-1}\sin\vartheta\, F_1 + \cos\vartheta\cos\varphi\, F_3 - \cos\vartheta\sin\varphi\, F_4), \\ \partial_u F_3 = \sin\vartheta\,(\tau^{-1}\sin\vartheta\, F_4 - \cos\vartheta\cos\varphi\, F_2 - \cos\vartheta\sin\varphi\, F_1), \\ \partial_u F_4 = -\sin\vartheta\,(\tau^{-1}\sin\vartheta\, F_3 + \cos\vartheta\cos\varphi\, F_1 - \cos\vartheta\sin\varphi\, F_2). \end{cases} \tag{23}$$

Therefore, if $B = 0$, taking the derivative of (23) with respect to u and using (18), we obtain that $F_{uu} = 0$.

If $B \neq 0$, taking the derivative of (23) with respect to u and using (19), we find two constants $\tilde{a}$ and $\tilde{b}$ such that

$$\begin{cases} (F_1)_{uu} = \tilde{a}\, F_1 + \tilde{b}\,(F_2)_u, \\ (F_2)_{uu} = \tilde{a}\, F_2 - \tilde{b}\,(F_1)_u, \\ (F_3)_{uu} = \tilde{a}\, F_3 + \tilde{b}\,(F_4)_u, \\ (F_4)_{uu} = \tilde{a}\, F_4 - \tilde{b}\,(F_3)_u, \end{cases} \tag{24}$$

where

$$\tilde{a} = \frac{\tau^{-1}\sin^2\vartheta}{2}\varphi_u = -\tau^{-2}\sin^2\vartheta\, B, \qquad \tilde{b} = \varphi_u.$$

Finally, taking twice the derivative of (24) with respect to u and using (23) and (24) in the derivative we obtain the desired Eq. (21). □

Remark 3.4 As $\langle F, F\rangle = 1$, using (21), (23) and (24), we find that the position vector $F(u, v)$ and its derivatives must satisfy the relations:

$$\begin{array}{lll} \langle F, F\rangle = 1, & \langle F_u, F_u\rangle = \tilde{a}, & \langle F, F_u\rangle = 0, \\ \langle F_u, F_{uu}\rangle = 0, & \langle F_{uu}, F_{uu}\rangle = D, & \langle F, F_{uu}\rangle = -\tilde{a}, \\ \langle F_u, F_{uuu}\rangle = -D, & \langle F_{uu}, F_{uuu}\rangle = 0, & \langle F, F_{uuu}\rangle = 0, \\ \langle F_{uuu}, F_{uuu}\rangle = E, & & \end{array} \tag{25}$$

where

$$D = \tilde{a}\,\tilde{b}^2 - 3\tilde{a}^2, \qquad E = (\tilde{b}^2 - 2\tilde{a})\, D - \tilde{a}^3.$$

In addition, as

$$J_1 F(u, v) = X_{1|F(u,v)} = -\tau\, E_{1|F(u,v)} = -\tau\,(F_u + \cos\vartheta\, N),$$

using (21)–(25), we obtain the following identities

$$\begin{aligned} &\langle J_1 F, F_u\rangle = -\tau^{-1}\sin^2\vartheta, \\ &\langle J_1 F, F_{uu}\rangle = 0, \\ &\langle F_u, J_1 F_{uu}\rangle = \tilde{a}\,(\tilde{b} - \tau^{-1}\sin^2\vartheta) := I, \\ &\langle J_1 F_u, F_{uuu}\rangle = 0, \\ &\langle J_1 F_u, F_{uu}\rangle + \langle J_1 F, F_{uuu}\rangle = 0, \\ &\langle J_1 F_{uu}, F_{uuu}\rangle + \langle J_1 F_u, F_{uuuu}\rangle = 0. \end{aligned} \tag{26}$$

Using Remark 3.4 we can prove the following proposition that gives the conditions under which an immersion defines a helix surface.

Proposition 3.5 *Let* $F : \Omega \to \mathrm{SL}(2,\mathbb{R})_\tau \subset \mathbb{R}^4_2$ *be an immersion from an open set* $\Omega \subset \mathbb{R}^2$*, with local coordinates* (u, v)*, such that the projection of* $E_1 = -\tau^{-1} J_1 F$ *to the tangent space of* $F(\Omega) \subset \mathrm{SL}(2,\mathbb{R})_\tau$ *is* F_u*. Then* $F(\Omega) \subset \mathrm{SL}(2,\mathbb{R})_\tau$ *defines a helix surface of constant angle* ϑ *if and only if*

$$g_\tau(F_u, F_u) = g_\tau(E_1, F_u) = \sin^2\vartheta, \tag{27}$$

and

$$g_\tau(F_v, E_1) - g_\tau(F_u, F_v) = 0. \tag{28}$$

Proof Suppose that F is a helix surface of constant angle ϑ. Then

$$\begin{aligned} g_\tau(F_u, F_u) &= -\langle F_u, F_u\rangle + (1+\tau^2)\langle F_u, J_1 F\rangle^2 \\ &= \tau^{-2}\sin^2\vartheta\, B + (1+\tau^2)(\tau^{-2}\sin^4\vartheta) \\ &= \sin^2\vartheta. \end{aligned}$$

Similarly

$$\begin{aligned} g_\tau(E_1, F_u) &= \tau^{-1}\langle J_1 F, F_u\rangle - \tau^{-1}(1+\tau^2)\langle J_1 F, F_u\rangle\langle J_1 F, J_1 F\rangle \\ &= \tau^{-1}\langle J_1 F, F_u\rangle[1 - (1+\tau^2)] = \sin^2\vartheta. \end{aligned}$$

Finally, using (15), we have

$$\begin{aligned} g_\tau(F_v, E_1) - g_\tau(F_u, F_v) &= -\frac{a}{\tau} g_\tau(F_u, J_1 F) - \frac{b}{\tau} g_\tau(J_1 F_u, J_1 F) \\ &\quad - a g_\tau(F_u, F_u) - b g_\tau(J_1 F_u, F_u) \\ &= a\sin^2\vartheta - 0 - a\sin^2\vartheta - 0 = 0. \end{aligned}$$

For the converse, put

$$T_2 = F_v - \frac{g_\tau(F_v, F_u) F_u}{g_\tau(F_u, F_u)}.$$

Then, if we denote by N the unit normal vector field to the surface $F(\Omega)$, $\{F_u, T_2, N\}$ is an orthogonal bases of the tangent space of $\mathrm{SL}(2,\mathbb{R})_\tau$ along the surface $F(\Omega)$. Now, using (28), we get $g_\tau(E_1, T_2) = 0$, thus $E_1 = a\, F_u + c\, N$. Moreover, using (27) and that $g_\tau(E_1, F_u) = a\, g_\tau(F_u, F_u)$, we conclude that $a = 1$. Finally,

$$1 = g_\tau(E_1, E_1) = g_\tau(F_u + c\, N, F_u + c\, N) = \sin^2\vartheta + c^2,$$

which implies that $c^2 = \cos^2\vartheta$. Thus the angle between E_1 and N is

$$g_\tau(E_1, N) = g_\tau(F_u + \cos\vartheta\, N, N) = \cos\vartheta.$$

□

4 The case $B = 0$

Theorem 4.1 *Let* M^2 *be a helix surface in the* $\mathrm{SL}(2,\mathbb{R})_\tau \subset \mathbb{R}^4_2$ *with constant angle* ϑ *such that* $B = 0$*. Then* $\cos\vartheta = \frac{1}{\sqrt{1+\tau^2}}$ *and, locally, the position vector of* M^2 *in* $\mathbb{R}^4_2$*, with respect* *to the local coordinates* (u, v) *on* M *defined in* (14)*, is given by*

$$F(u, v) = A(v)\Big(1, -\frac{\tau\, u}{1+\tau^2}, \frac{\tau\, u}{1+\tau^2}, 0\Big), \tag{29}$$

where $A(v) = A(\xi, \xi_1, \xi_2, \xi_3)(v)$ *is a 1-parameter family of* 4×4 *indefinite orthogonal matrices commuting with* J_1*, as described in* (4)*, with*

$$\begin{aligned}&[\xi'(v) + \xi_2'(v) + \xi_3'(v)]\, \sin(\xi_2(v) - \xi_3(v))\, \sinh(2\xi_1(v))\\ &-2(\xi'(v) - \xi_3'(v))\, \sinh^2 \xi_1(v) + 2\,[\xi_1'(v)\, \cos(\xi_2(v) - \xi_3(v)) + \xi_2'(v)\, \cosh^2 \xi_1(v)] = 0.\end{aligned} \tag{30}$$

Conversely, a parametrization

$$F(u, v) = A(v)\Big(1, -\frac{\tau\, u}{1+\tau^2}, \frac{\tau\, u}{1+\tau^2}, 0\Big),$$

with $A(v)$ *as above, defines a helix surface in the special linear group with constant angle* $\vartheta = \arccos \frac{1}{\sqrt{1+\tau^2}}$.

Proof Since $B = 0$ we have immediately that $\cos^2\vartheta = 1/(1+\tau^2)$. Integrating (20), we obtain that

$$F(u, v) = h^1(v) + u\, h^2(v), \tag{31}$$

where $h^i(v),\ i = 1, 2$, are vector fields in $\mathbb{R}^4_2$, depending only on v.

Evaluating in $(0, v)$ the identities:

$$\begin{aligned}&\langle F, F\rangle = 1, \quad \langle F_u, F_u\rangle = 0,\\ &\langle F, F_u\rangle = 0, \quad \langle J_1 F, F_u\rangle = -\tau^{-1}\, \sin^2\vartheta = -\frac{\tau}{1+\tau^2},\end{aligned}$$

it results that

$$\begin{aligned}&\langle h^1(v), h^1(v)\rangle = 1, \quad \langle h^1(v), h^2(v)\rangle = 0,\\ &\langle h^2(v), h^2(v)\rangle = 0, \quad \langle J_1 h^1(v), h^2(v)\rangle = -\frac{\tau}{1+\tau^2}.\end{aligned} \tag{32}$$

Moreover, using (23) in $(0, v)$, we have that

$$h^2(v) = -\frac{\tau}{1+\tau^2}\,(J_1 h^1(v) - h^3(v)),$$

where $h^3(v)$ is a vector field of $\mathbb{R}^4_2$ satisfying

$$\langle h^3(v), h^3(v)\rangle = -1, \quad \langle h^1(v), h^3(v)\rangle = 0, \quad \langle J_1 h^1(v), h^3(v)\rangle = 0. \tag{33}$$

Consequently, if we fix the orthonormal basis $\{\hat{E}_i\}_{i=1}^4$ of $\mathbb{R}^4_2$ given by

$$\hat{E}_1 = (1, 0, 0, 0), \quad \hat{E}_2 = (0, 1, 0, 0), \quad \hat{E}_3 = (0, 0, 1, 0), \quad \hat{E}_4 = (0, 0, 0, 1),$$

there must exists a 1-parameter family of matrices $A(v) \in O_2(4)$, with $J_1\, A(v) = A(v)\, J_1$, such that

$$h^1(v) = A(v)\hat{E}_1, \quad J_1 h^1(v) = A(v)\hat{E}_2, \quad h^3(v) = A(v)\hat{E}_3, \quad J_1 h^3(v) = A(v)\hat{E}_4.$$

Then (31) becomes

$$F(u, v) = h^1(v) - \frac{\tau\, u}{1+\tau^2}\,(J_1 h^1(v) - h^3(v)) = A(v)\Big(1, -\frac{\tau\, u}{1+\tau^2}, \frac{\tau\, u}{1+\tau^2}, 0\Big).$$

Finally, the 1-parameter family $A(v)$ depends, according to (4), on four functions $\xi_1(v)$, $\xi_2(v)$, $\xi_3(v)$ and $\xi(v)$ and, in this case, condition (28) reduces to $\langle F_u, F_v\rangle = 0$ which is equivalent to (30).

For the converse, let

$$F(u, v) = A(v)\Big(1, -\frac{\tau\, u}{1+\tau^2}, \frac{\tau\, u}{1+\tau^2}, 0\Big),$$

be a parametrization where $A(v) = A(\xi(v), \xi_1(v), \xi_2(v), \xi_3(v))$ is a 1-parameter family of indefinite orthogonal matrices with functions $\xi(v), \xi_1(v), \xi_2(v), \xi_3(v)$ satisfying (30). Since $A(v)$ satisfies (30), then F satisfies (28), thus, in virtue of Proposition 3.5, we only have to show that (27) is satisfied for some constant angle ϑ. For this we put

$$\gamma(u) = \Big(1, -\frac{\tau\, u}{1+\tau^2}, \frac{\tau\, u}{1+\tau^2}, 0\Big).$$

Now, using (1) and taking into account that $A(v)$ commutes with J_1, we get

$$\begin{aligned} g_\tau(F_u, F_u) &= -\langle A(v)\gamma'(u), A(v)\gamma'(u)\rangle + (1+\tau^2)\langle A(v)\gamma'(u), J_1 A(v)\gamma(u)\rangle^2 \\ &= (1+\tau^2)\langle \gamma'(u), J_1\gamma(u)\rangle^2 = \frac{\tau^2}{1+\tau^2}, \end{aligned}$$

and we can choose ϑ such that $\tau^2/(1+\tau^2) = \sin^2\vartheta$. Similarly,

$$\begin{aligned} g_\tau(E_1, F_u) &= -\langle E_1, F_u\rangle + (1+\tau^2)\langle E_1, J_1 F\rangle\langle F_u, J_1 F\rangle \\ &= \frac{\langle F_u, J_1 F\rangle}{\tau}\big[1-(1+\tau^2)\big] \\ &= (-\tau)\langle\gamma'(u), J_1\gamma(u)\rangle = \frac{\tau^2}{1+\tau^2}. \end{aligned}$$

□

Example 4.2 If we take $\xi_2 = \xi_3 =$ constant, (30) becomes $\xi'(1-\sinh^2\xi_1) = 0$. Thus, if also $\xi =$ constant, we find, from (4), a 1-parameter family $A(v) = A(\xi_1(v))$ of indefinite orthogonal matrices such that (29) defines a helix surface for any function ξ_1.

5 The case $B > 0$

Supposing $B > 0$, integrating (21) we have the following

Proposition 5.1 *Let M^2 be a helix surface in $\mathrm{SL}(2,\mathbb{R})_\tau$ with constant angle ϑ so that $B > 0$. Then, with respect to the local coordinates (u, v) on M defined in (14), the position vector F of M^2 in $\mathbb{R}^4_2$ is given by*

$$F(u, v) = \cos(\alpha_1\, u)\, g^1(v) + \sin(\alpha_1\, u)\, g^2(v) + \cos(\alpha_2\, u)\, g^3(v) + \sin(\alpha_2\, u)\, g^4(v),$$

where

$$\alpha_{1,2} = \frac{1}{\tau}(\tau\sqrt{B}\cos\vartheta \pm B)$$

are positive real constants, while the $g^i(v),\ i \in \{1, \dots, 4\}$, *are mutually orthogonal vector fields in* $\mathbb{R}^4_2$, *depending only on* v, *such that*

$$\begin{aligned} g_{11} &= \langle g^1(v), g^1(v)\rangle = g_{22} = \langle g^2(v), g^2(v)\rangle = -\frac{\tau}{2B}\,\alpha_2, \\ g_{33} &= \langle g^3(v), g^3(v)\rangle = g_{44} = \langle g^4(v), g^4(v)\rangle = \frac{\tau}{2B}\,\alpha_1. \end{aligned} \tag{34}$$

Proof First, a direct integration of (21), gives the solution

$$F(u, v) = \cos(\alpha_1 u)\, g^1(v) + \sin(\alpha_1 u)\, g^2(v) + \cos(\alpha_2 u)\, g^3(v) + \sin(\alpha_2 u)\, g^4(v),$$

where

$$\alpha_{1,2} = \sqrt{\frac{\tilde{b}^2 - 2\tilde{a} \pm \sqrt{\tilde{b}^4 - 4\tilde{a}\tilde{b}^2}}{2}}$$

are two constants, while the $g^i(v),\ i \in \{1, \dots, 4\}$, are vector fields in $\mathbb{R}^4_2$ which depend only on v. Now, taking into account the values of $\tilde{a}$ and $\tilde{b}$ given in (22), we get

$$\alpha_{1,2} = \frac{1}{\tau}(\tau\sqrt{B}\cos\vartheta \pm B). \tag{35}$$

Putting $g_{ij}(v) = \langle g^i(v), g^j(v)\rangle$, and evaluating the relations (25) in $(0, v)$, we obtain:

$$g_{11} + g_{33} + 2g_{13} = 1, \tag{36}$$
$$\alpha_1^2\, g_{22} + \alpha_2^2\, g_{44} + 2\alpha_1\alpha_2\, g_{24} = \tilde{a}, \tag{37}$$
$$\alpha_1\, g_{12} + \alpha_2\, g_{14} + \alpha_1\, g_{23} + \alpha_2\, g_{34} = 0, \tag{38}$$
$$\alpha_1^3\, g_{12} + \alpha_1\alpha_2^2\, g_{23} + \alpha_1^2\alpha_2\, g_{14} + \alpha_2^3 g_{34} = 0, \tag{39}$$
$$\alpha_1^4\, g_{11} + \alpha_2^4\, g_{33} + 2\alpha_1^2\alpha_2^2\, g_{13} = D, \tag{40}$$
$$\alpha_1^2\, g_{11} + \alpha_2^2\, g_{33} + (\alpha_1^2 + \alpha_2^2)\, g_{13} = \tilde{a}, \tag{41}$$
$$\alpha_1^4\, g_{22} + \alpha_1^3\alpha_2\, g_{24} + \alpha_1\alpha_2^3\, g_{24} + \alpha_2^4\, g_{44} = D, \tag{42}$$
$$\alpha_1^5\, g_{12} + \alpha_1^3\alpha_2^2\, g_{23} + \alpha_1^2\alpha_2^3\, g_{14} + \alpha_2^5\, g_{34} = 0, \tag{43}$$
$$\alpha_1^3\, g_{12} + \alpha_1^3\, g_{23} + \alpha_2^3\, g_{14} + \alpha_2^3\, g_{34} = 0, \tag{44}$$
$$\alpha_1^6\, g_{22} + \alpha_2^6\, g_{44} + 2\alpha_1^3\alpha_2^3\, g_{24} = E. \tag{45}$$

From (38), (39), (43), (44), it follows that

$$g_{12} = g_{14} = g_{23} = g_{34} = 0.$$

Also, from (36), (40) and (41), we obtain

$$g_{11} = \frac{\tau^2\,(D + \alpha_2^4) + 2B\sin^2\vartheta\,\alpha_2^2}{\tau^2(\alpha_1^2 - \alpha_2^2)^2}, \qquad g_{13} = 0, \qquad g_{33} = \frac{\tau^2\,(D + \alpha_1^4) + 2B\sin^2\vartheta\,\alpha_1^2}{\tau^2(\alpha_1^2 - \alpha_2^2)^2}.$$

Finally, using (37), (42) and (45), we obtain

$$g_{22} = \frac{\tau^2\,(E - 2D\,\alpha_2^2) - B\sin^2\vartheta\,\alpha_2^4}{\tau^2\alpha_1^2\,(\alpha_1^2 - \alpha_2^2)^2}, \quad g_{24} = 0, \qquad g_{44} = \frac{\tau^2\,(E - 2D\,\alpha_1^2) - B\sin^2\vartheta\,\alpha_1^4}{\tau^2\alpha_2^2\,(\alpha_1^2 - \alpha_2^2)^2}.$$

We observe that

$$g_{11} = g_{22} = \frac{\sqrt{B} - \tau \cos\vartheta}{2\sqrt{B}} < 0, \qquad g_{33} = g_{44} = \frac{\sqrt{B} + \tau \cos\vartheta}{2\sqrt{B}} > 0.$$

Therefore, taking into account (35), we obtain the expressions (34). □

We are now in the right position to state the main result of this section.

Theorem 5.2 *Let M^2 be a helix surface in the $\mathrm{SL}(2,\mathbb{R})_\tau \subset \mathbb{R}^4_2$ with constant angle $\vartheta \neq \pi/2$ so that $B > 0$. Then, locally, the position vector of M^2 in $\mathbb{R}^4_2$, with respect to the local coordinates (u, v) on M defined in (14), is*

$$F(u, v) = A(v)\,\gamma(u), \tag{46}$$

where

$$\gamma(u) = (\sqrt{g_{33}}\,\cos(\alpha_2\,u), -\sqrt{g_{33}}\,\sin(\alpha_2\,u), \sqrt{-g_{11}}\,\cos(\alpha_1\,u), \sqrt{-g_{11}}\,\sin(\alpha_1\,u)) \tag{47}$$

is a curve in $\mathrm{SL}(2,\mathbb{R})_\tau$, g_{11}, g_{33}, α_1, α_2 are the four constants given in Proposition 5.1, and $A(v) = A(\xi, \xi_1, \xi_2, \xi_3)(v)$ is a 1-parameter family of 4×4 indefinite orthogonal matrices commuting with J_1, as described in (4), with $\xi = \text{constant}$ and

$$\cosh^2(\xi_1(v))\,\xi_2'(v) + \sinh^2(\xi_1(v))\,\xi_3'(v) = 0. \tag{48}$$

Conversely, a parametrization $F(u, v) = A(v)\,\gamma(u)$, with $\gamma(u)$ and $A(v)$ as above, defines a constant angle surface in $\mathrm{SL}(2,\mathbb{R})_\tau$ with constant angle $\vartheta \neq \pi/2$.

Proof With respect to the local coordinates (u, v) on M defined in (14), Proposition 5.1 implies that the position vector of the helix surface in $\mathbb{R}^4_2$ is given by

$$F(u, v) = \cos(\alpha_1 u)\,g^1(v) + \sin(\alpha_1 u)\,g^2(v) + \cos(\alpha_2 u)\,g^3(v) + \sin(\alpha_2 u)\,g^4(v),$$

where the vector fields $\{g^i(v)\}_{i=1}^4$ are mutually orthogonal and

$$||g^1(v)|| = ||g^2(v)|| = \sqrt{-g_{11}} = \text{constant},$$
$$||g^3(v)|| = ||g^4(v)|| = \sqrt{g_{33}} = \text{constant}.$$

Thus, if we put $e_i(v) = g^i(v)/||g^i(v)||$, $i \in \{1, \ldots, 4\}$, we can write:

$$\begin{aligned} F(u, v) = &\sqrt{-g_{11}}\,(\cos(\alpha_1\,u)\,e_1(v) + \sin(\alpha_1\,u)\,e_2(v)) \\ &+\sqrt{g_{33}}\,(\cos(\alpha_2\,u)\,e_3(v) + \sin(\alpha_2\,u)\,e_4(v)). \end{aligned} \tag{49}$$

Now, the identities (26), evaluated in $(0, v)$, become respectively:

$$\begin{aligned} &\alpha_2\,g_{33}\langle J_1e_3, e_4\rangle - \alpha_1 g_{11}\langle J_1e_1, e_2\rangle \\ &\quad+\sqrt{-g_{11}g_{33}}\,(\alpha_1\langle J_1e_3, e_2\rangle + \alpha_2\langle J_1e_1, e_4\rangle) = -\tau^{-1}\sin^2\vartheta, \end{aligned} \tag{50}$$

$$\langle J_1e_1, e_3\rangle = 0, \tag{51}$$

$$\begin{aligned} &\alpha_2^3\,g_{33}\langle J_1e_3, e_4\rangle - \alpha_1^3\,g_{11}\langle J_1e_1, e_2\rangle \\ &\quad+\sqrt{-g_{11}g_{33}}\,(\alpha_1\alpha_2^2\langle J_1e_3, e_2\rangle + \alpha_1^2\alpha_2\langle J_1e_1, e_4\rangle) = -I, \end{aligned} \tag{52}$$

$$\langle J_1e_2, e_4\rangle = 0, \tag{53}$$

$$\alpha_1\langle J_1e_2, e_3\rangle + \alpha_2\langle J_1e_1, e_4\rangle = 0, \tag{54}$$

$$\alpha_2\langle J_1e_2, e_3\rangle + \alpha_1\langle J_1e_1, e_4\rangle = 0. \tag{55}$$

We point out that to obtain the previous identities we have divided by $\alpha_1^2 - \alpha_2^2 = 4\tau^{-1}\sqrt{B^3}\cos\vartheta$ which is, by the assumption on ϑ, always different from zero. From (54) and (55), taking into account the $\alpha_1^2 - \alpha_2^2 \neq 0$, it results that

$$\langle J_1 e_3, e_2\rangle = 0, \qquad \langle J_1 e_1, e_4\rangle = 0. \tag{56}$$

Therefore

$$|\langle J_1 e_1, e_2\rangle| = 1 = |\langle J_1 e_3, e_4\rangle|.$$

Substituting (56) in (50) and (52), we obtain the system

$$\begin{cases} \alpha_1\, g_{11}\langle J_1 e_1, e_2\rangle - \alpha_2\, g_{33}\langle J_1 e_3, e_4\rangle = \tau^{-1}\sin^2\vartheta \\ \alpha_1^3\, g_{11}\langle J_1 e_1, e_2\rangle - \alpha_2^3\, g_{33}\langle J_1 e_3, e_4\rangle = I, \end{cases}$$

a solution of which is

$$\langle J_1 e_1, e_2\rangle = \frac{\tau I - \alpha_2^2 \sin^2\vartheta}{\tau g_{11}\,\alpha_1(\alpha_1^2 - \alpha_2^2)}, \qquad \langle J_1 e_3, e_4\rangle = \frac{\tau I - \alpha_1^2 \sin^2\vartheta}{\tau g_{33}\,\alpha_2(\alpha_1^2 - \alpha_2^2)}.$$

Now, as

$$g_{11}\, g_{33} = -\frac{\sin^2\vartheta}{4B}, \qquad \alpha_1\,\alpha_2 = \frac{B}{\tau^2}\sin^2\vartheta, \qquad \alpha_1^2 - \alpha_2^2 = \frac{4\sqrt{B^3}}{\tau}\cos\vartheta,$$

it results that

$$\langle J_1 e_1, e_2\rangle\langle J_1 e_3, e_4\rangle = 1.$$

Moreover, as

$$\tau I - \alpha_2^2\sin^2\vartheta = 2\tau^{-1}\sqrt{B^3}\cos\vartheta\,\sin^2\vartheta,$$

it results that $\langle J_1 e_1, e_2\rangle < 0$. Consequently, $\langle J_1 e_1, e_2\rangle = \langle J_1 e_3, e_4\rangle = -1$ and $J_1 e_1 = e_2,\ J_1 e_3 = -e_4$.

Then, if we fix the orthonormal basis of $\mathbb{R}^4_2$ given by

$$\tilde{E}_1 = (0,0,1,0), \quad \tilde{E}_2 = (0,0,0,1), \quad \tilde{E}_3 = (1,0,0,0), \quad \tilde{E}_4 = (0,-1,0,0),$$

there must exists a 1-parameter family of 4×4 indefinite orthogonal matrices $A(v) \in \mathrm{O}_2(4)$, with $J_1 A(v) = A(v) J_1$, such that $e_i(v) = A(v)\tilde{E}_i$. Replacing $e_i(v) = A(v)\tilde{E}_i$ in (49) we obtain

$$F(u,v) = A(v)\gamma(u),$$

where

$$\gamma(u) = (\sqrt{g_{33}}\,\cos(\alpha_2\, u), -\sqrt{g_{33}}\,\sin(\alpha_2\, u), \sqrt{-g_{11}}\,\cos(\alpha_1\, u), \sqrt{-g_{11}}\,\sin(\alpha_1\, u))$$

is a curve in $\mathrm{SL}(2,\mathbb{R})_\tau$.

Let now examine the 1-parameter family $A(v)$ that, according to (4), depends on four functions $\xi_1(v), \xi_2(v), \xi_3(v)$ and $\xi(v)$. From (15), it results that $\langle F_v, F_v\rangle = -\sin^2\vartheta =$ constant. The latter implies that

$$\frac{\partial}{\partial u}\langle F_v, F_v\rangle_{|u=0} = 0. \tag{57}$$

Now, if we denote by $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ the four colons of $A(v)$, (57) implies that

$$\begin{cases} \langle \mathbf{c}_2{}', \mathbf{c}_3{}' \rangle = 0 \\ \langle \mathbf{c}_2{}', \mathbf{c}_4{}' \rangle = 0, \end{cases} \tag{58}$$

where with $'$ we means the derivative with respect to v. Replacing in (58) the expressions of the $\mathbf{c}_i$'s as functions of $\xi_1(v), \xi_2(v), \xi_3(v)$ and $\xi(v)$, we obtain

$$\begin{cases} \xi' \, h(v) = 0 \\ \xi' \, k(v) = 0, \end{cases} \tag{59}$$

where $h(v)$ and $k(v)$ are two functions such that

$$h^2 + k^2 = 4(\xi_1')^2 + \sinh^2(2\xi_1)\,(-\xi' + \xi_2' + \xi_3')^2.$$

From (59) we have two possibilities:

(i) $\xi = \text{constant}$;
 or
(ii) $4(\xi_1')^2 + \sinh^2(2\xi_1)\,(-\xi' + \xi_2' + \xi_3')^2 = 0$.

We will show that case (ii) cannot occur, more precisely we will show that if (ii) happens then the parametrization $F(u, v) = A(v)\gamma(u)$ defines a Hopf tube, that is the Hopf vector field E_1 is tangent to the surface. To this end, we write the unit normal vector field N as

$$N = \frac{N_1 E_1 + N_2 E_2 + N_3 E_3}{\sqrt{N_1^2 + N_2^2 + N_3^2}}.$$

A long but straightforward computation (that can be also made using a software of symbolic computations) gives

$$\begin{aligned} N_1 = {} & 1/2(\alpha_1 + \alpha_2)\sqrt{-g_{11}}\sqrt{g_{33}}\,[2\xi_1' \cos(\alpha_1 u + \alpha_2 u - \xi_2 + \xi_3) \\ & + \sinh(2\xi_1)\sin(\alpha_1 u + \alpha_2 u - \xi_2 + \xi_3)(-\xi' + \xi_2' + \xi_3')]. \end{aligned}$$

Now case (ii) occurs if and only if $\xi_1 = \text{constant} = 0$, or if $\xi_1 = \text{constant} \neq 0$ and $-\xi' + \xi_2' + \xi_3' = 0$. In both cases $N_1 = 0$ and this implies that $g_\tau(N, J_1 F) = -\tau g_\tau(N, E_1) = 0$, i.e. the Hopf vector field is tangent to the surface. Thus we have proved that $\xi = \text{constant}$.

Finally, in this case, (28) is equivalent to

$$\tau \cos\vartheta \sqrt{B}[\cosh^2(\xi_1(v))\,\xi_2'(v) + \sinh^2(\xi_1(v))\,\xi_3'(v)] = 0.$$

Since $\vartheta \neq \pi/2$ we conclude that condition (48) is satisfied.

The converse follows immediately from Proposition 3.5 since a direct calculation shows that $g_\tau(F_u, F_u) = g_\tau(E_1, F_u) = \sin^2\vartheta$ which is (27), while (48) is equivalent to (28). □

6 The case $B < 0$

In this section we study the case $B < 0$. Integrating (21) we have the following result:

Proposition 6.1 *Let M^2 be a helix surface in $\mathrm{SL}(2, \mathbb{R})_\tau$ with constant angle ϑ and $B < 0$. Then, with respect to the local coordinates (u, v) on M defined in (14), the position vector F of M^2 in $\mathbb{R}^4_2$ is given by*

$$F(u,v) = \cos\left(\frac{\tilde b}{2}\,u\right)[\cosh(\beta\,u)\,w^1(v) + \sinh(\beta\,u)\,w^3(v)]$$
$$+\sin\left(\frac{\tilde b}{2}\,u\right)[\cosh(\beta\,u)\,w^2(v) + \sinh(\beta\,u)\,w^4(v)], \tag{60}$$

where

$$\beta = \sqrt{-B}\,\cos\vartheta$$

is a real constant, $\tilde b = -2\tau^{-1}B$*, while the* $w^i(v),\ i \in \{1,\dots,4\}$*, are vector fields in* $\mathbb{R}^4_2$*, depending only on* v*, such that*

$$\begin{aligned}
&\langle w^1(v), w^1(v)\rangle = \langle w^2(v), w^2(v)\rangle = -\langle w^3(v), w^3(v)\rangle = -\langle w^4(v), w^4(v)\rangle = 1,\\
&\langle w^1(v), w^2(v)\rangle = \langle w^1(v), w^3(v)\rangle = \langle w^2(v), w^4(v)\rangle = \langle w^3(v), w^4(v)\rangle = 0,\\
&\langle w^1(v), w^4(v)\rangle = -\langle w^2(v), w^3(v)\rangle = -\frac{2\beta}{\tilde b}.
\end{aligned} \tag{61}$$

Proof A direct integration of (21), gives the solution

$$F(u,v) = \cos\left(\frac{\tilde b}{2}\,u\right)[\cosh(\beta\,u)\,w^1(v) + \sinh(\beta\,u)\,w^3(v)]$$
$$+\sin\left(\frac{\tilde b}{2}\,u\right)[\cosh(\beta\,u)\,w^2(v) + \sinh(\beta\,u)\,w^4(v)],$$

where

$$\beta = \frac{\sqrt{4\tilde a - \tilde b^2}}{2} = \sqrt{-B}\,\cos\vartheta$$

is a constant, while the $w^i(v),\ i \in \{1,\dots,4\}$, are vector fields in $\mathbb{R}^4$ which depend only on v. If $w_{ij}(v) := \langle w^i(v), w^j(v)\rangle$, evaluating the relations (25) in $(0,v)$, we obtain

$$w_{11} = 1, \tag{62}$$

$$\frac{\tilde b^2}{4}\,w_{22} + \beta^2\,w_{33} + \beta\,\tilde b\,w_{23} = \tilde a, \tag{63}$$

$$\frac{\tilde b}{2}\,w_{12} + \beta\,w_{13} = 0, \tag{64}$$

$$\frac{\tilde b}{2}\left(\beta^2 - \frac{\tilde b^2}{4}\right)w_{12} + \beta^2\,\tilde b\,w_{34} + \beta\,\frac{\tilde b^2}{2}\,w_{24} + \beta\left(\beta^2 - \frac{\tilde b^2}{4}\right)w_{13} = 0, \tag{65}$$

$$\left(\beta^2 - \frac{\tilde b^2}{4}\right)^2 w_{11} + \beta^2\,\tilde b^2\,w_{44} + 2\beta\,\tilde b\left(\beta^2 - \frac{\tilde b^2}{4}\right)w_{14} = D, \tag{66}$$

$$\left(\beta^2 - \frac{\tilde b^2}{4}\right)w_{11} + \beta\,\tilde b\,w_{14} = -\tilde a, \tag{67}$$

$$\frac{\tilde b^2}{4}\left(3\beta^2 - \frac{\tilde b^2}{4}\right)w_{22} + \beta^2\left(\beta^2 - 3\frac{\tilde b^2}{4}\right)w_{33} + \beta\,\frac{\tilde b}{2}\,(4\beta^2 - \tilde b^2)\,w_{23} = -D, \tag{68}$$

$$\begin{aligned}
&\frac{\tilde b}{2}\left(3\beta^2 - \frac{\tilde b^2}{4}\right)\left(\beta^2 - \frac{\tilde b^2}{4}\right)w_{12} + \tilde b\,\beta^2\left(\beta^2 - 3\frac{\tilde b^2}{4}\right)w_{34}\\
&+\beta\left(\beta^2 - 3\frac{\tilde b^2}{4}\right)\left(\beta^2 - \frac{\tilde b^2}{4}\right)w_{13} + \beta\,\frac{\tilde b^2}{2}\left(3\beta^2 - \frac{\tilde b^2}{4}\right)w_{24} = 0,
\end{aligned} \tag{69}$$

$$\frac{\tilde b}{2}\left(3\beta^2 - \frac{\tilde b^2}{4}\right)w_{12} + \beta\left(\beta^2 - 3\frac{\tilde b^2}{4}\right)w_{13} = 0, \tag{70}$$

$$\frac{\tilde{b}^2}{4}\left(3\beta^2-\frac{\tilde{b}^2}{4}\right)^2 w_{22}+\beta^2\left(\beta^2-3\frac{\tilde{b}^2}{4}\right)^2 w_{33}$$
$$+\beta\,\tilde{b}\left(3\beta^2-\frac{\tilde{b}^2}{4}\right)\left(\beta^2-3\frac{\tilde{b}^2}{4}\right) w_{23}=E. \tag{71}$$

From (62), (66) and (67), it follows that

$$w_{11}=-w_{44}=1, \qquad w_{14}=-\frac{2\,\beta}{\tilde{b}}.$$

Also, from (64) and (70), we obtain

$$w_{12}=w_{13}=0$$

and, therefore, from (65) and (69),

$$w_{24}=w_{34}=0.$$

Moreover, using (63), (68) and (71), we get

$$w_{22}=-w_{33}=1, \qquad w_{23}=\frac{2\,\beta}{\tilde{b}}.$$

□

Theorem 6.2 *Let M^2 be a helix surface in $\mathrm{SL}(2,\mathbb{R})_\tau$ with constant angle $\vartheta\neq\pi/2$ so that $B<0$. Then, locally, the position vector of M^2 in $\mathbb{R}^4_2$, with respect to the local coordinates (u,v) on M defined in (14), is given by*

$$F(u,v)=A(v)\,\gamma(u),$$

where the curve $\gamma(u)=(\gamma_1(u),\gamma_2(u),\gamma_3(u),\gamma_4(u))$ is given by

$$\begin{cases}\gamma_1(u)=\cos\left(\frac{\tilde{b}}{2}u\right)\cosh(\beta\,u)-\frac{2\beta}{\tilde{b}}\sin\left(\frac{\tilde{b}}{2}u\right)\sinh(\beta\,u),\\ \gamma_2(u)=\sin\left(\frac{\tilde{b}}{2}u\right)\cosh(\beta\,u)+\frac{2\beta}{\tilde{b}}\cos\left(\frac{\tilde{b}}{2}u\right)\sinh(\beta\,u),\\ \gamma_3(u)=\frac{\sin\vartheta}{\sqrt{-B}}\cos\left(\frac{\tilde{b}}{2}u\right)\sinh(\beta\,u),\\ \gamma_4(u)=\frac{\sin\vartheta}{\sqrt{-B}}\sin\left(\frac{\tilde{b}}{2}u\right)\sinh(\beta\,u),\end{cases} \tag{72}$$

$\beta=\sqrt{-B}\cos\vartheta$, $\tilde{b}=-2\tau^{-1}B$ and $A(v)=A(\xi,\xi_1,\xi_2,\xi_3)(v)$ is a 1- parameter family of 4×4 indefinite orthogonal matrices anticommuting with J_1, as described in (4), with $\xi=$ constant and

$$\sin\vartheta\,[2\cos(\xi_2(v)-\xi_3(v))\,\xi_1'(v)+(\xi_2'(v)+\xi_3'(v))\sin(\xi_2(v)-\xi_3(v))\,\sinh(2\xi_1(v))]$$
$$-2\tau\cos\vartheta\,[\cosh^2(\xi_1(v))\,\xi_2'(v)+\sinh^2(\xi_1(v))\,\xi_3'(v)]=0. \tag{73}$$

Conversely, a parametrization $F(u,v)=A(v)\,\gamma(u)$, with $\gamma(u)$ and $A(v)$ as above, defines a helix surface in $\mathrm{SL}(2,\mathbb{R})_\tau$ with constant angle $\vartheta\neq\pi/2$.

Proof From (61), we can define the following orthonormal basis in $\mathbb{R}^4_2$:

$$\begin{cases}e_1(v)=w^1(v),\\ e_2(v)=w^2(v),\\ e_3(v)=\frac{1}{\sin\vartheta}[\sqrt{-B}\,w^3(v)-\tau\cos\vartheta\;w^2(v)],\\ e_4(v)=\frac{1}{\sin\vartheta}[\sqrt{-B}\,w^4(v)+\tau\cos\vartheta\;w^1(v)],\end{cases} \tag{74}$$

with $\langle e_1,e_1\rangle=1=\langle e_2,e_2\rangle$ and $\langle e_3,e_3\rangle=-1=\langle e_4,e_4\rangle$.

Evaluating the identities (26) in $(0, v)$, and taking into account that:

$$
\begin{aligned}
&F(0, v) = w^1(v),\\
&F_u(0, v) = \frac{\tilde{b}}{2}\, w^2(v) + \beta\, w^3(v),\\
&F_{uu}(0, v) = \Big(\beta^2 - \frac{\tilde{b}^2}{4}\Big)\, w^1(v) + \beta\, \tilde{b}\, w^4(v),\\
&F_{uuu}(0, v) = \frac{\tilde{b}}{2}\, \Big(3\beta^2 - \frac{\tilde{b}^2}{4})\, w^2(v) + \beta\, \Big(\beta^2 - \frac{3}{4}\tilde{b}^2\Big)\, w^3(v),\\
&F_{uuuu}(0, v) = \Big(\beta^4 - \frac{3}{2}\beta^2\, \tilde{b}^2 + \frac{\tilde{b}^4}{16}\Big)\, w^1(v) + 2\beta\, \tilde{b}\, \Big(\beta^2 - \frac{\tilde{b}^2}{4}\Big)\, w^4(v),
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
\langle J_1 w^3, w^4\rangle &= -\langle J_1 w^1, w^2\rangle = 1,\\
\langle J_1 w^3, w^2\rangle &= \langle J_1 w^1, w^4\rangle = 0,\\
\langle J_1 w^2, w^4\rangle &= \langle J_1 w^1, w^3\rangle = -\frac{\tau\, \cos\vartheta}{\sqrt{-B}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
-\langle J_1 e_1, e_2\rangle &= \langle J_1 e_3, e_4\rangle = 1,\\
\langle J_1 e_1, e_4\rangle = \langle J_1 e_1, e_3\rangle &= \langle J_1 e_2, e_3\rangle = \langle J_1 e_2, e_4\rangle = 0.
\end{aligned}
$$

Therefore, we obtain that

$$
J_1 e_1 = -e_2, \qquad J_1 e_3 = -e_4.
$$

Consequently, if we consider the orthonormal basis $\{\hat{E}_i\}_{i=1}^4$ of $\mathbb{R}^4_2$ given by

$$
\hat{E}_1 = (1, 0, 0, 0), \quad \hat{E}_2 = (0, 1, 0, 0), \quad \hat{E}_3 = (0, 0, 1, 0), \quad \hat{E}_4 = (0, 0, 0, 1),
$$

there must exists a 1-parameter family of matrices $A(v) \in O_2(4)$, with $J_1 A(v) = -A(v) J_1$, such that $e_i(v) = A(v)\hat{E}_i,\ i \in \{1, \ldots, 4\}$. As

$$
F = \langle F, e_1\rangle\, e_1 + \langle F, e_2\rangle\, e_2 - \langle F, e_3\rangle\, e_3 - \langle F, e_4\rangle\, e_4,
$$

computing $\langle F, e_i\rangle$ and substituting $e_i(v) = A(v)\hat{E}_i$, we obtain that $F(u, v) = A(v)\,\gamma(u)$, where the curve $\gamma(u)$ of $SL(2, \mathbb{R})_\tau$ is given in (72).

Let now examine the 1-parameter family $A(v)$ that, according to (4), depends on four functions $\xi_1(v), \xi_2(v), \xi_3(v)$ and $\xi(v)$. Similarly to what we have done in the proof of Theorem 5.2 we have that the condition

$$
\frac{\partial}{\partial u}\langle F_v, F_v\rangle_{|u=0} = 0
$$

implies that the functions $\xi_1(v), \xi_2(v), \xi_3(v)$ and $\xi(v)$ satisfy the equation

$$
\xi'\, [2\sin(\xi_2 - \xi_3)\, \xi_1' - (\xi_2' + \xi_3' - \xi')\cos(\xi_2 - \xi_3)\ \sinh(2\,\xi_1)] = 0.
$$

Then we have two possibilities:

(i) $\xi = \text{constant}$;
or
(ii) $2\sin(\xi_2 - \xi_3)\, \xi_1' - (\xi_2' + \xi_3' - \xi')\cos(\xi_2 - \xi_3)\ \sinh(2\,\xi_1) = 0.$

Also in this case, using the same argument as in Theorem 5.2, condition (ii) would implies that the surface is a Hopf tube, thus we can assume that $\xi =$ constant.

Finally, a long but straightforward computation shows that, in the case $\xi =$ constant, (28) is equivalent to (73).

The converse of the theorem follows immediately from Proposition 3.5 since a direct calculation shows that $g_\tau(F_u, F_u) = g_\tau(E_1, F_u) = \sin^2\vartheta$ which is (27) while (73) is equivalent to (28). □

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