

Constructions of H_r -hypersurfaces, barriers and Alexandrov theorem in $\mathbb{H}^n \times \mathbb{R}$

Maria Fernanda Elbert · Ricardo Sa Earp

Received: 1 April 2014 / Accepted: 26 August 2014 / Published online: 13 September 2014
© Fondazione Annali di Matematica Pura ed Applicata and Springer-Verlag Berlin Heidelberg 2014

Abstract In this paper, we are concerned with hypersurfaces in $\mathbb{H} \times \mathbb{R}$ with constant r -mean curvature, to be called H_r -hypersurfaces. We construct examples of complete H_r -hypersurfaces, which are invariant by parabolic screw motion or by rotation. We prove that there is a unique rotational strictly convex entire H_r -graph for each value $0 < H_r \leq \frac{n-r}{n}$. Also, for each value $H_r > \frac{n-r}{n}$, there is a unique embedded compact strictly convex rotational H_r -hypersurface. By using them as barriers, we obtain some interesting geometric results, including height estimates and an Alexandrov-type Theorem. Namely, we prove that an embedded compact H_r -hypersurface in $\mathbb{H}^n \times \mathbb{R}$ is rotational ($H_r > 0$).

Keywords r -Mean curvature · Alexandrov Theorem · H_r -Hypersurfaces · Barriers · Entire vertical graphs · Complete horizontal graphs

Mathematics Subject Classification 53C42 · 53A10 · 53C21

1 Introduction

The r -mean curvature, H_r , of an n -hypersurface is defined as the normalized r -symmetric function of the principal curvatures (see Sect. 2 for precise definitions). In this paper, we are concerned with hypersurfaces with constant r -mean curvature, to be called H_r -hypersurfaces. Although we sometimes work in a more general setting, we are particularly interested in the case the ambient is the product space $\mathbb{H}^n \times \mathbb{R}$, where \mathbb{H}^n denotes the hyperbolic space. Throughout the paper, we establish some key equations for the theory. For instance, in Sect. 8,

The authors are partially supported by CNPq of Brazil.

M. F. Elbert
Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil

R. Sa Earp (✉)
Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro, Brazil
e-mail: rsaearp@gmail.com

we deduce a suitable divergence formula for the r -mean curvature of a vertical graph over M in $M \times \mathbb{R}$.

We start, in Sects. 3, 4 and 5, by exploiting the geometry of the hypersurfaces with constant H_2 in $\mathbb{H}^n \times \mathbb{R}$. In Sect. 6, 7 and 8, we deal with the general case $r > 2$. When $n = 2$, we recall that H_2 is the extrinsic curvature K_{ext} of the surface.

We classify the complete H_r -hypersurfaces that are invariant by parabolic translation with $H_r = 0$ (see Theorem 6.2), and we classify some of the complete ones, which are invariant by rotation, for $H_r > 0$ (see Theorem 6.6). Surprisingly, they show a strong analogy with the study of the mean curvature case in $\mathbb{H}^n \times \mathbb{R}$. For the mean curvature, the behavior of the H -hypersurfaces, $H > 0$, depends on the value of H , and we distinguish the cases of H greater or less than the critical value $\frac{n-1}{n}$ (see [5, 10]). For the r -mean curvature, $H_r > 0$, the same happens for the value $\frac{n-r}{n}$. We should also notice that similarly to what happens for the mean curvature, there is a unique rotational strictly convex entire vertical graph for each constant value $0 < H_r \leq \frac{n-r}{n}$. Also, for each constant value $H_r > \frac{n-r}{n}$, there is a unique embedded compact strictly convex rotational H_r -hypersurface. Both are unique up to vertical or horizontal translations (see Theorem 6.6). The dependence on r and n is a distinguishing point from the theory of H_r -hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ ($H_r > 0$) from that of the euclidean or hyperbolic spaces, where the critical points are 0 and 1, respectively ([22]).

By using some of the constructed examples as barriers, we were able to obtain some interesting results (see Sect. 7). For instance, we prove that there is no compact without boundary immersion in $\mathbb{H}^n \times \mathbb{R}$ with prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$, $n > r$. We also obtain *a priori* height estimates for compact immersions of $\mathbb{H}^n \times \mathbb{R}$ with boundary in a slice and with prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$, $n > r$. An interesting question is if we could obtain a height estimate for the case $H_r > \frac{n-r}{n}$ and we ask: Would the maximum height of a compact graph with boundary in a slice with $H_r = \text{constant} > \frac{n-r}{n}$ be given by half of the total height of the compact rotational corresponding example? In fact, J. A. Aledo, J. M. Espinar and J. A. Gálvez (see [2]) proved that is true for H -graphs in $\mathbb{H}^2 \times \mathbb{R}$.

In Espinar et al. [12], address the case of surfaces with positive extrinsic curvature in some 3-dimensional product spaces. In particular, they show that a complete immersion in $\mathbb{H}^2 \times \mathbb{R}$ with $H_2 = \text{constant} > 0$ (i.e., $r = n = 2$) is a rotational sphere (see [12, Theorem (7.3)]). For $n > 2$, our Example (4.4) shows that there exist entire graphs with $H_2 > 0$ in $\mathbb{H}^n \times \mathbb{R}$. It is natural to ask what kind of complete H_r -immersions, $r = n$, $H_r > 0$, we can find. As one can see below, we have partial answer to this question.

In Theorem 7.6, we prove an Alexandrov-type Theorem (see [1] for the classical result) for compact embedded H_r -hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, i.e., we characterize the embedded compact H_r -hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, $H_r > 0$. Precisely, if $H_r > 0$, we prove that a compact H_r -hypersurface embedded in $\mathbb{H}^n \times \mathbb{R}$ is rotational (which is classified). Here, we notice that we only have compact rotational H_r -hypersurfaces for $H_r > \frac{n-r}{n}$ (see Theorem 6.6). In space forms, an Alexandrov-type result for the r -mean curvatures were obtained by Korevaar in [15] and by Montiel and Ros in [18].

On the other hand, for $H_r > \frac{n-r}{n}$, there exist no entire rotational H_r -graph (see Theorem 6.6). It is interesting to investigate the complete H_r -hypersurface for this case. We ask, for instance: Is there a non-compact complete embedded H_r -hypersurface in $\mathbb{H}^n \times \mathbb{R}$, $H_r > \frac{n-r}{n}$, with only one end? If $n=2$, in [12, Theorem (7.2)], the authors proved that if $K_{\text{ext}} > 0$ (or $H > 1/2$), there is no properly embedded K_{ext} -surface (or H -surface) in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.

We have then seen that for $H_r > \frac{n-r}{n}$, the only compact embedded immersion is a rotational n -sphere and that there exist no entire rotational H_r -graph. We ask if for the particular case $r = n$ we have the same behavior of the case $r = n = 2$, that is :

Question Is a complete immersion in $\mathbb{H}^n \times \mathbb{R}$, with $H_r = \text{constant} > 0$ and $r = n$ a rotational n -sphere?

2 Preliminaries

Let M^n be an oriented Riemannian n -manifold, \bar{M}^{n+1} be an oriented Riemannian manifold and $X : M^n \rightarrow \bar{M}^{n+1}$ be an isometric immersion. For each $p \in M$, let $A : T_p M \rightarrow T_p M$ be the linear operator associated with the second fundamental form of X and k_1, \dots, k_n be its eigenvalues corresponding to the eigenvectors e_1, \dots, e_n . The r -mean curvature of X is then defined by

$$H_r(p) = \frac{1}{\binom{n}{r}} \sum_{i_1 < \dots < i_r} k_{i_1} \dots k_{i_r} = \frac{1}{\binom{n}{r}} S_r(p),$$

where S_r is the r -symmetric function of k_1, \dots, k_n . With this notation, H_1 is the mean curvature, H , of the immersion and H_n is the Gauss–Kronecker curvature. The Newton tensors associated with X are inductively defined by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - A \circ P_{r-1}, \quad r > 1 \end{aligned}$$

and it will be useful to recall that

$$(r + 1)S_{r+1} = \text{trace}(P_r A), \tag{1}$$

$$P_{r-1}(e_i) = \frac{\partial S_r}{\partial k_i}(e_i) \tag{2}$$

and that

$$\text{trace}(P_r) = (n - r)S_r. \tag{3}$$

For details and other properties, we suggest the paper from Barbosa and Colares [3].

If $H_{r+1} = 0$, we say that the immersion is r -minimal. In this context, the classical minimal immersions would be the 0-minimal ones.

We say that an immersion X is strictly convex (convex) at $p \in M$ if $k_i(p) > 0$ (respectively, $k_i(p) \geq 0$) for all $i = 1, \dots, n$, with respect to the normal orientation at p . In the literature, a strictly convex point is usually called an elliptic point.

Although we sometimes work in a more general setting, we are particularly interested in the case $\bar{M} = \mathbb{H}^n \times \mathbb{R}$, where \mathbb{H}^n denotes the hyperbolic space. We start with the case $r = 2$ and we recall that when $n = 2$, $S_2 = H_2$ is the extrinsic curvature K_{ext} of the surface.

3 2-Mean curvature for vertical graphs over M

Let $M = M_g$ denote a Riemannian n -manifold with metric g and consider on $\bar{M} = M \times \mathbb{R}$ the product metric $\langle \cdot, \cdot \rangle = g + dt^2$, where t is a global coordinate for \mathbb{R} .

If u is a real function defined over $\Omega \subset M$, the set $G = \{(p, u(p)) \in M \times \mathbb{R} \mid p \in \Omega\}$ is called the vertical graph of u , or more simply, the graph of u . We denote by $X : \Omega \subset M \rightarrow M \times \mathbb{R}$ the natural embedding of G in $M \times \mathbb{R}$. If u is C^2 and if we choose the orientation given by the upward unit normal, namely $N = \left(-\frac{\nabla_g u}{W}, \frac{1}{W}\right)$, it is proved in [8, Formula (3)] that

$$A(v) = \nabla^g_v \left(\frac{\nabla_g u}{W} \right), \text{ for all } v \in TM, \tag{4}$$

where ∇^g and ∇_g are, respectively, the connection and the gradient of M and $W = \sqrt{1 + |\nabla_g u|_g^2}$.

Let $\text{Ric}_g(v_1, v_2) = \text{trace}(z \rightarrow R_g(v_1, z)v_2)$ denote the Ricci tensor of M . Then, the following proposition holds.

Proposition 3.1 *The 2-mean curvature H_2 of the graph G is given by*

$$2S_2 = n(n - 1)H_2 = \text{div}_g \left(P_1 \frac{\nabla_g u}{W} \right) + \text{Ric}_g \left(\frac{\nabla_g u}{W}, \frac{\nabla_g u}{W} \right), \tag{5}$$

where div_g means the divergence in M .

Proof By (1) and (4) we have

$$2S_2 = \text{trace} \left(z \rightarrow P_1 \nabla^g_z \left(\frac{\nabla_g u}{W} \right) \right).$$

Claim:

$$\text{trace} (z \rightarrow P_1 \nabla^g_z v) = \text{trace} (z \rightarrow \nabla^g_z (P_1 v)) + \text{Ric}_g \left(v, \frac{\nabla_g u}{W} \right). \tag{6}$$

The proposition will be proved by taking $v = \frac{\nabla_g u}{W}$ in the latter. In order to prove the claim, we follow the proof of [8, Lemma 3.2] without assuming $\text{Ric}_g = 0$. We sketch it here for completeness.

By the definition of the curvature R_g and by (4.) we have

$$\nabla^g_v (A(z)) - \nabla^g_z (A(v)) = R(z, v) \left(\frac{\nabla_g u}{W} \right) + A([v, z]). \tag{7}$$

Let $p \in M$ and let $\{v_i\}_i$ be an orthonormal basis in a neighborhood of p in M , which is geodesic at p , that is, such that $\nabla^g_{v_j} v_i(p) = 0$. Let $v = \sum_i a_i v_i$.

Since $\{v_i\}_i$ is geodesic at p we have

$$\text{trace} (z \rightarrow P_1 \nabla^g_z (v)) (p) = \text{trace} \left(z \rightarrow \sum_i z(a_i) P_1 (v_i) \right) (p).$$

On the other hand,

$$\begin{aligned} \text{trace} (z \rightarrow \nabla^g_z (P_1(v))) (p) &= \text{trace} \left(z \rightarrow \sum_i z(a_i) P_1 (v_i) \right) (p) \\ &\quad + \sum_i a_i \text{trace} (z \rightarrow \nabla^g_z (P_1(v_i))) (p). \end{aligned}$$

Putting things together we have

$$\text{trace}(z \rightarrow P_1 \nabla^g_z (v))(p) = \text{trace} (z \rightarrow \nabla^g_z (P_1(v))) (p) - \sum_i a_i \text{trace} (z \rightarrow \nabla^g_z (P_1(v_i))) (p).$$

The proof will be completed if we prove that

$$\text{trace} \left(z \rightarrow \nabla^g_z (P_1(v_i)) \right) (p) = -\text{Ric}_g \left(v_i, \frac{\nabla^g u}{W} \right).$$

The latter holds as can be seen below, where in the third equality we use (7).

$$\begin{aligned} \text{trace} \left(z \rightarrow \nabla^g_z (P_1(v_i)) \right) (p) &= \text{trace} \left(z \rightarrow \nabla^g_z (S_1(v_i) - A(v_i)) \right) (p) \\ &= \text{trace} \left(z \rightarrow z(S_1)v_i \right) (p) - \text{trace} \left(z \rightarrow \nabla^g_z (A(v_i)) \right) (p) \\ &= v_i(S_1)(p) - \text{trace} \left(z \rightarrow \nabla^g_{v_i} (A(z)) \right) \\ &\quad + \text{trace} \left(z \rightarrow R_g(z, v_i) \left(\frac{\nabla^g u}{W} \right) \right) (p) \\ &= v_i(S_1)(p) - \text{trace} \left(z \rightarrow \nabla^g_{v_i} (A(z)) \right) \\ &\quad - \text{trace} \left(z \rightarrow R_g(v_i, z) \left(\frac{\nabla^g u}{W} \right) \right) (p) \\ &= -\text{Ric}_g \left(v_i, \frac{\nabla^g u}{W} \right). \end{aligned}$$

□

We remark that Eq. (5), which gives the 2-mean curvature of the graph, is elliptic iff P_1 is positive definite. If $S_2 > 0$, a standard argument shows that it is elliptic (see, for instance, [9, proof of Lemma 3.10]).

From now on, in this section, we consider the particular case where $M^n \subset \mathbb{R}^n$ and we denote by (x_1, x_2, \dots, x_n) the (euclidean) coordinates of M . On M , we consider the metric given by

$$g = \frac{1}{F^2} \langle \cdot, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle = (dx_1^2 + \dots + dx_n^2)$ is the euclidean metric. For such a particular M , Proposition 3.1 reads as follows.

Proposition 3.2 *The 2-mean curvature of the graph of u when M is as above is given by*

$$2S_2 = n(n-1)H_2 = F^2 \text{div} \left(P_1 \frac{\nabla u}{W} \right) + \frac{(2-n)F \langle P_1 \nabla u, \nabla F \rangle}{W} + \text{Ric}_g \left(\frac{\nabla_g u}{W}, \frac{\nabla_g u}{W} \right). \tag{8}$$

Here, div , ∇ , and $\|\cdot\|$ denote quantities in the euclidean metric and W can be written as $W = \sqrt{1 + F^2 \|\nabla u\|^2}$.

Sketch of the Proof Let $\{e_i\}_i$ be an orthonormal local field of M in the metric $\langle \cdot, \cdot \rangle$. Then

$$\text{div}_g \left(P_1 \frac{\nabla_g u}{W} \right) = \sum_i \left\langle e_i, \nabla^g_{e_i} \left(P_1 \left(\frac{\nabla_g u}{W} \right) \right) \right\rangle = \sum_i \left\langle e_i, \nabla^g_{e_i} \left(P_1 \left(\frac{F^2 \nabla u}{W} \right) \right) \right\rangle.$$

Now we use the relation between the connections and gradients of the two conformal metrics g and $\langle \cdot, \cdot \rangle$, Proposition 3.1 and some computation to obtain the result. □

Remark 3.3 An expression relating the Ricci Tensor of two conformal metrics can be obtained in [22, page 183]. By using it for our case, we have that

$$\text{Ric}_g(v_i, v_j) = (n-2) \frac{F_{ij}}{F} + \left(\frac{\Delta F}{F} - (n-1) \frac{|\nabla F|^2}{F^2} \right) \delta_{ij},$$

where Δ denotes the laplacian in the euclidean metric and F_{ij} denotes the second derivative of F . Doing this, one can express formula (8) in terms of the euclidean metric only.

Now we look forward to expressing the 2-mean curvature in the coordinates (x_1, x_2, \dots, x_n) . For this, let $X(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n, u(x_1, x_2, \dots, x_n))$ be the natural parametrization of G and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of vector fields of M in the euclidean metric. We write $X_i = dX(e_i) = (e_i, u_i)$. Then, a computation gives the following proposition.

Proposition 3.4 *Let $[I] = [\overline{I_{ij}}]$ and $[II] = [II_{ij}]$ denote the matrices of the first and the second fundamental forms of the (isometric) embedding X in the basis $\{e_i\}_i$. Let $[I]^{-1} = [I^{ij}]$ be the inverse matrix of $[I]$. Then we have*

$$I_{ij} = \frac{\delta_{ij}}{F^2} + u_i u_j$$

$$II_{ij} = -\langle \bar{\nabla}_{X_i} N, X_j \rangle = \frac{u_{ij}}{W} + \frac{1}{FW} \left[u_i F_j + u_j F_i - \left(\sum_m u_m F_m \right) \delta_{ij} \right],$$

and

$$I^{ij} = F^2 \delta_{ij} - F^4 \frac{u_i u_j}{W^2}$$

where we write u_i and u_{ij} for the corresponding first and second derivative of the function u and $\bar{\nabla}$ for the connection of $M \times \mathbb{R}$.

Remark 3.5 We can use the last proposition in order to re-obtain [10, formula(1.3)].

In the basis $\{e_i\}_i$, we can express the matrix of A as

$$[A] = [I^{ij}] \times [II_{ij}] \tag{9}$$

and in order to obtain an explicit expression for the 2-mean curvature of G in the coordinates (x_1, x_2, \dots, x_n) , one can use (1). The computation, in the general case, is a tough job but can be done by adapting the proof given by M. L. Leite in [17, Proposition (2.2)] and the result is the following.

Proposition 3.6 *The 2-mean curvature H_2 of the graph G of u considering in $M \subset \mathbb{R}^n$ the metric $g = \frac{1}{F^2} \langle \cdot, \cdot \rangle$ and the coordinates (x_1, x_2, \dots, x_n) is given by*

$$S_2 \frac{W^4}{F^4} = n(n-1)H_2 \frac{W^4}{F^4} = \sum_{i < j} \left(W^2 - F^2(u_i^2 + u_j^2) \right) \begin{vmatrix} V_{\{ii\}} & V_{\{ij\}} \\ V_{\{ji\}} & V_{\{jj\}} \end{vmatrix} - 2F^2 \sum_{i < k} u_i u_k \sum_{j \neq i,k} \begin{vmatrix} V_{\{ik\}} & V_{\{ij\}} \\ V_{\{jk\}} & V_{\{jj\}} \end{vmatrix}, \tag{10}$$

where $V_{\{ij\}} = u_{ij} + \frac{1}{F} [u_i F_j + u_j F_i - (\sum_m u_m F_m) \delta_{ij}] = II_{ij} \cdot W$ and the indices vary in $\{1, \dots, n\}$.

In this paper, we are in fact interested in the case where $M = \mathbb{H}^n$ is the hyperbolic space. Choosing F conveniently, we will be able to deal with different models for \mathbb{H}^n . For future use, we point out that by using that the Ricci curvature of \mathbb{H}^n is $-(n-1)$, we can rewrite the expression (8) for this case as

$$2S_2 = n(n-1)H_2 = F^2 \operatorname{div} \left(P_1 \frac{\nabla u}{W} \right) + \frac{(2-n)F \langle P_1 \nabla u, \nabla F \rangle}{W} - (n-1)F^2 \left\| \frac{\nabla u}{W} \right\|^2. \tag{11}$$

Of course, one can use the formula of Remark (3.3) to obtain (11).

4 Examples of complete H_2 -graphs in $\mathbb{H}^n \times \mathbb{R}$

In this section, we consider the half-space model for \mathbb{H}^n , that is, we consider

$$\mathbb{H}^n = \{x = (x_1, \dots, x_{n-1}, x_n = y) \in \mathbb{R}^n \mid y > 0\}$$

endowed with the metric

$$\frac{dx_1^2 + \dots + dx_n^2}{F^2} = \frac{dx_1^2 + \dots + dx_n^2}{y^2}.$$

Searching for examples, here we consider for each $(l_1, \dots, l_{n-1}) \in \mathbb{R}^{n-1}$, the graph G given by

$$t = u(x_1, \dots, x_{n-1}, y) = \lambda(y) + l_1x_1 + \dots + l_{n-1}x_{n-1} \tag{12}$$

and we have the following.

Proposition 4.1 *The 2-mean curvature H_2 of the graph G of $t = \lambda(y) + l_1x_1 + \dots + l_{n-1}x_{n-1}$ is given by*

$$\frac{2S_2}{y^2} = \frac{n(n-1)H_2}{y^2} = \left(-\frac{y(l^2 + (n-1)\dot{\lambda}^2)}{W^2} \right)' + \frac{(n-1)(n-3)\dot{\lambda}^2 - l^2}{W^2}, \tag{13}$$

where $l^2 = l_1^2 + \dots + l_{n-1}^2$ and $W^2 = 1 + y^2l^2 + y^2\dot{\lambda}^2$.

Sketch of the Proof By using Propositions 3.4 and Eq. (9), we can obtain the matrix $[A]$ and, a fortiori, $[P_1]$. Then, we obtain the result by computing formula (11). \square

Remark 4.2 By using Proposition (3.6) for graphs given by (12), we obtain the equation

$$2S_2W^4 = (n-1)(n-4)\dot{\lambda}^2y^2 + (n-1)(n-2)\dot{\lambda}^4y^4 + (n-1)(n-2)\dot{\lambda}^2y^4l^2 + 2\dot{\lambda}\ddot{\lambda}y^3[(2-n)l^2y^2 - (n-1)] - 2l^2y^2,$$

that turns out to be equivalent to (13). The latter can also be obtained by using (1).

We want to solve Eq. (13) for some particular data finding then some interesting graphs with constant 2-mean curvature. For simplifying purposes, we introduce in (13) the variable $z = l^2 + (n-1)\dot{\lambda}^2$ obtaining

$$\frac{2S_2}{y^2} = -\left(\frac{z}{W^2}y\right)' + (n-3)\frac{z}{W^2} + (2-n)\frac{l^2}{W^2},$$

or equivalently,

$$\frac{2S_2}{y^{n-1}} = -\left(\frac{z}{W^2}y^{4-n}\right)' + (2-n)\frac{l^2y^{3-n}}{W^2}. \tag{14}$$

We recall that a parabolic translation can be identified with a horizontal euclidean translation in this model for \mathbb{H}^n . Then, when $l = 0$, the graph G given by (12) is invariant by parabolic translation. When $l \neq 0$, the parabolic translation is composed with a vertical translation and we say that G is invariant by parabolic screw motion.

We notice that in $\mathbb{H}^n \times \mathbb{R}$, we can also define the notion of a horizontal graph, $y = g(x_1, \dots, x_{n-1}, t)$, of a real and positive function g .

We first deal with the case $l = 0$ and $S_2 = 0$, and we obtain the following classification result.

Theorem 4.3 (1-minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic translations) *Despite the slice, 1-minimal vertical graphs invariant by parabolic translations are, up to vertical translations or reflections, of the following type:*

- (a) If $n = 2$, $\lambda(y) = c \ln(y)$, for all $y > 0$ and for $c > 0$.
- (b) If $n > 2$, $\lambda(y) = \frac{2}{n-2} \arcsin(\frac{y^{\frac{n-2}{2}}}{c})$, for a positive constant c and $y \in (0, c^{\frac{2}{n-2}})$.

The function $\lambda(y)$ in (a) gives rise to an entire vertical graph. The function $\lambda(y)$ in (b) generate a family of non-entire horizontal 1-minimal complete graphs in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic translations. The asymptotic boundary of each graph of this family is formed by two parallel $(n - 1)$ -planes.

Proof Equation (14) for $l = 0$ and $S_2 = 0$ yields

$$\frac{z}{W^2}y^{4-n} = d, \quad \text{for a constant } d \text{ that must be nonnegative.}$$

Replacing the values for z and W and integrating we obtain, up to reflection,

$$\lambda(y) = \int_0^y \frac{\sqrt{d} \xi^{\frac{n-4}{2}}}{\sqrt{(n-1) - d\xi^{n-2}}} d\xi.$$

If $n = 2$, we obtain (a) with $c = \sqrt{\frac{d}{1-d}}$ and if $n > 2$, we set $c = \sqrt{\frac{n-1}{d}}$ and we obtain (b).

The function $\lambda(y)$ given by b) is increasing in the interval $(0, c^{\frac{2}{n-2}})$ and is vertical at $y = c^{\frac{2}{n-2}}$. Since the induced metric in the yt -plane is euclidean, a simple computation shows that the euclidean curvature is finite and strictly positive at this point. If we set $t_0 = \lambda\left(c^{\frac{2}{n-2}}\right)$, we can then glue together the graph $t = \lambda(y)$ with its reflection given by

$$2t_0 - \lambda(y)$$

in order to obtain a horizontal 1-minimal complete graph invariant by parabolic screw motion defined over $\{(x_1, \dots, x_{n-1}, t) \in \mathbb{H}^{n-1} \times \mathbb{R} \mid 0 \leq t \leq 2t_0\}$. The asymptotic boundary of this example is formed by two parallel hyperplanes. □

Remark 4.4 Recalling that when $n = 2$, S_2 is the extrinsic curvature of the surface, we see that each solution of Theorem (4.3) a) gives rise to an entire graph with null extrinsic curvature. By considering the result given in [10, Proposition (4.1)], one can see that this solution has constant mean curvature $H = \frac{1}{2} \frac{c}{\sqrt{1+c^2}}$. Then, the solution $\lambda(y) = c \ln(y)$, $c \in \mathbb{R}$, is an entire graph with null extrinsic curvature and constant mean curvature $|H| < \frac{1}{2}$. We wonder if this is the only example of an entire graph with null extrinsic curvature and constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$.

Question Is there any entire graph with null extrinsic curvature and constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$ other than $\lambda(y) = c \ln(y)$, $c \in \mathbb{R}$?

We could handle to find solutions for Eq. (14) for some values of n, l and S_2 . We do not aim to exhaust the cases here, but we choose some particular data in order to present some interesting examples.

Example 4.1 (Graphs with null extrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$ invariant by parabolic screw motion) Equation (14) for $n = 2, l \neq 0$ and $S_2 = 0$ yields

$$\frac{z}{W^2}y^2 = d$$

which implies that $d > 0$ and gives

$$\lambda = \pm \int_*^y \frac{\sqrt{c - \xi^2 l^2}}{\xi} d\xi, \text{ where } c = \frac{d}{1 - d} > 0.$$

The integration gives, up to vertical translations or reflection, the explicit solution

$$\lambda = -\sqrt{c} \ln(\sqrt{c} + \sqrt{c - l^2 y^2}) + \sqrt{c} \ln(l y) + \sqrt{c - l^2 y^2}, \text{ for } y \in (0, \sqrt{\frac{c}{l}}).$$

□

Example 4.2 (Entire H_2 -graphs in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic translations, $0 < H_2 < \frac{n-2}{n}$, $n > 2$)

Equation (14) for $l = 0$ and $H_2 = k \neq 0$ yields

$$\frac{n(n-1)k}{y^{n-1}} = -\left(\frac{z}{W^2} y^{4-n}\right)',$$

which, for $n > 2$, gives

$$\frac{z}{W^2} = \left(\frac{n(n-1)k}{n-2}\right) \frac{1}{y^2}.$$

The latter implies that $k > 0$ and can be written as

$$((n-2) - nk) \dot{\lambda}^2 = (nk) \frac{1}{y^2}.$$

Then, we must have $0 < k < \frac{(n-2)}{n}$. Integration gives, up to vertical translations, the solution

$$\lambda = c \ln(y), c \in \mathbb{R}, y > 0.$$

□

In [12, Theorem (7.3)], the authors proved that, when $n = 2$, a complete immersion with $H_2 = \text{constant} > 0$ is a rotational sphere. In contrast, when $n > 2$, the last example shows that there exist entire graphs with $H_2 = \text{constant} > 0$.

Remark 4.5 One can easily see that when $n = 2$ or $l = 0$, we can obtain explicit solutions of (14) by integration. For each case, a careful analysis of the behavior should be taken.

5 Examples of rotational H_2 -hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$

Now, we search for rotational hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ with $H_2 = \text{constant}$, and we use the ball model of the hyperbolic space \mathbb{H}^n ($n \geq 2$), i.e., we consider

$$\mathbb{H}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$$

endowed with the metric

$$g_{\mathbb{H}} := \frac{dx_1^2 + \dots + dx_n^2}{F^2} = \frac{1}{F^2} \langle \cdot, \cdot \rangle,$$

where $F = \left(\frac{1 - |x|^2}{2}\right)$.

We notice that in [12, Section 5], the authors classify the complete rotational surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant positive extrinsic curvature. With our method, we could re-obtain their examples (see Example (4.3) below). In [7, 16, 21] and references therein, one can find examples of rotational hypersurfaces with constant H_r in space forms, as well as classification results.

In the vertical plane $V := \{(x_1, \dots, x_n, t) \in \mathbb{H}^n \times \mathbb{R} \mid x_1 = \dots = x_{n-1} = 0\}$, we consider a generating curve $(\tanh(\rho/2), \lambda(\rho))$, for positive values of ρ , the hyperbolic distance to the axis \mathbb{R} .

For our purpose, we define a *rotational hypersurface* in $\mathbb{H}^n \times \mathbb{R}$ by the parametrization

$$X : \begin{cases} \mathbb{R}_+ \times S^{n-1} \rightarrow \mathbb{H}^n \times \mathbb{R} \\ (\rho, \xi) \rightarrow (\tanh(\rho/2), \lambda(\rho)). \end{cases}$$

The normal field to the immersion and the associated principal curvatures will be given by

$$N = (1 + \dot{\lambda}^2)^{-1/2} \left(\frac{-\dot{\lambda}}{2 \cosh^2(\rho/2)} \xi, 1 \right),$$

$$k_1 = k_2 = \dots = k_{n-1} = \operatorname{cotgh}(\rho) \dot{\lambda} (1 + \dot{\lambda}^2)^{-1/2} \text{ and } k_n = \ddot{\lambda} (1 + \dot{\lambda}^2)^{-3/2}.$$

(See [4, Section 3.1] with a slight modification writing functions in terms of ρ instead of t). Then, the 2-mean curvature

$$\binom{n}{2} H_2 = S_2 = \sum_{i_1 < i_2} k_{i_1} k_{i_2}$$

is given by

$$nH_2 \frac{(\sinh^{n-1}(\rho))}{\cosh(\rho)} = \frac{\partial}{\partial \rho} \left[\sinh^{n-2}(\rho) \left(\frac{\dot{\lambda}^2}{1 + \dot{\lambda}^2} \right) \right]. \tag{15}$$

By setting $I(\xi) = \int_0^\xi \frac{(\sinh^{(n-1)}(s))}{\cosh(s)} ds$ and integrating twice we obtain, up to vertical translation or reflection,

$$\lambda(\rho) = \int_*^\rho \sqrt{\frac{nH_2 I + d}{\sinh^{(n-2)}(\xi) - (nH_2 I + d)}} d\xi, \tag{16}$$

where the constant d comes from the first integration.

Now we set

$$p(\xi) = nH_2 I + d \quad \text{and} \quad q(\xi) = \sinh^{(n-2)}(\xi) - (nH_2 I + d) = \sinh^{(n-2)}(\xi) - p(\xi)$$

and we write

$$\lambda(\rho) = \int_*^\rho \sqrt{\frac{p(\xi)}{q(\xi)}} d\xi.$$

We notice that we must have $p(\xi) \geq 0$ and $q(\xi) > 0$. We also notice that when $d \neq 0$, λ could not be defined for $\rho = 0$, since, in this case, $\frac{p(0)}{q(0)} = -1$.

Differentiation gives

$$\begin{aligned} \dot{\lambda} &\geq 0, \\ \dot{p} &= nH_2 \frac{\sinh^{(n-1)}(\xi)}{\cosh(\xi)}, \end{aligned}$$

$$\dot{q} = \frac{\sinh^{n-3}(\xi)}{\cosh(\xi)} [(n-2) \cosh^2(\xi) - nH_2 \sinh^2(\xi)] = \frac{\sinh^{n-3}(\xi)}{\cosh^3(\xi)} \left[1 - \frac{nH_2}{(n-2)} \operatorname{tgh}^2(\xi) \right] \tag{17}$$

and

$$\ddot{\lambda} = \frac{1}{2} \left(\frac{q}{p} \right)^{1/2} \left[\frac{\dot{p}q - p\dot{q}}{q^2} \right] = \frac{1}{2q^2} \left(\frac{q}{p} \right)^{1/2} \frac{\sinh^{n-3}(\xi)}{\cosh(\xi)} [nH_2 \sinh^n(\xi) - (n-2) \cosh^2(\xi)p], \text{ for } \xi > 0.$$

Now, exploring (16) and analyzing the behavior of the functions p and q , we highlight some interesting examples of rotational H_2 -hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and their geometric properties.

Example 5.1 (The slice) We easily see that the slice, $\lambda = \text{constant}$, is a solution of (15) for $H_2 = 0$ and any dimension. □

Example 5.2 (1-minimal cones in $\mathbb{H}^2 \times \mathbb{R}$) Setting $n = 2$ and $H_2 = 0$ in (15) we obtain the solution

$$\lambda(\rho) = \pm \sqrt{\frac{d}{1-d}} \rho, \quad 0 < d < 1.$$

Thus, we see that the cone $\lambda(\rho) = c\rho, c \in \mathbb{R}, \rho > 0$, is a 1-minimal surface in $\mathbb{H}^2 \times \mathbb{R}$, i.e., a surface with null extrinsic curvature. In Euclidean coordinates, we can write them as

$$t = 2c \operatorname{arctgh}(\sqrt{x^2 + y^2}), \quad c \in \mathbb{R}, \quad x^2 + y^2 < 1.$$

□

Example 5.3 (Compact rotational surface with positive extrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$) Setting $n = 2, d = 0$ and $H_2 > 0$, the expression in (16) reads as follows

$$\lambda(\rho) = \int_0^\rho \sqrt{\frac{2H_2 \ln(\cosh(\xi))}{1 - (2H_2 \ln(\cosh(\xi)))}} d\xi$$

and we must have $\rho < \rho_0 := \operatorname{arcosh}(e^{\frac{1}{2H_2}})$. We want to analyze the convergence of the integral at ρ_0 . By using that $\ln(\cosh(\xi))$ and $\operatorname{tgh}(\xi)$ are increasing functions we have, for all $a \in (0, \rho_0)$,

$$\begin{aligned} \int_a^{\rho_0} \sqrt{\frac{2H_2 \ln(\cosh(\xi))}{1 - (2H_2 \ln(\cosh(\xi)))}} d\xi &\leq \int_a^{\rho_0} \sqrt{\frac{1}{1 - (2H_2 \ln(\cosh(\xi)))}} d\xi \\ &\leq \frac{1}{2H_2 \operatorname{tgh}(a)} \int_a^{\rho_0} \frac{2H_2 \operatorname{tgh}(\xi)}{\sqrt{1 - (2H_2 \ln(\cosh(\xi)))}} d\xi. \end{aligned}$$

By setting $v(\xi) = 2H_2 \ln(\cosh(\xi))$ and using the inequalities above, we can see that

$$\int_a^{\rho_0} \sqrt{\frac{2H_2 \ln(\cosh(\xi))}{1 - (2H_2 \ln(\cosh(\xi)))}} d\xi \leq \frac{1}{2H_2 \operatorname{tgh}(a)} \int_{v(a)}^1 \sqrt{\frac{1}{1-v}} dv$$

and then the integral is convergent at ρ_0 and $\lambda(\rho_0)$ is well defined.

Now, we use (17) to see that, for $\xi > 0$,

$$\dot{p} > 0, \quad \dot{q} < 0 \quad \text{and} \quad \ddot{\lambda} = \frac{1}{2} \left(\frac{q}{p} \right)^{1/2} \left[\frac{\dot{p}q - p\dot{q}}{q^2} \right] > 0.$$

Bringing all together, we see that λ is increasing and strictly convex in the interval $(0, \rho_0)$ and is vertical and assume a finite value at $\rho = \rho_0$.

Since the induced metric in the ρt -plane is Euclidean, a simple computation shows that the Euclidean curvature is finite at this point and strictly positive at this point. If we set $t_0 = \lambda(\rho_0)$, we can then glue together the graph $t = \lambda(y)$ with its reflection given by

$$2t_0 - \lambda(y)$$

in order to obtain a compact rotational surface. □

We now choose $d = 0$ and $n > 2$ in the expression (16). We notice that, for this case, $\lim_{\xi \rightarrow 0} \frac{p(\xi)}{q(\xi)} = 0$ then we can set $\frac{p(0)}{q(0)} = 0$ and we have

$$\lambda(\rho) = \int_0^\rho \sqrt{\frac{nH_2 I}{\sinh^{(n-2)}(\xi) - nH_2 I}} \, d\xi. \tag{18}$$

We also notice that, with these conditions, there is no solution for $H_2 < 0$, since we would have $\frac{p(\xi)}{q(\xi)} < 0$. Then, it remains to treat the case $H_2 > 0$.

Lemma 5.1 *The function λ given by (18), for $H_2 > 0$ and $n > 2$:*

- (i) *is increasing.*
- (ii) *satisfies $\dot{\lambda} > 0$, for all $\xi > 0$.*
- (iii) *satisfies $\lambda = \frac{\sqrt{H_2}}{2} \rho^2 + o(\rho^3)$, near 0.*

In particular, the corresponding hypersurface is strictly convex.

Proof (i) is clear. We proceed with the proof of (ii).

For all $\xi > 0$, we write the last equation in (17) as

$$\ddot{\lambda} = \frac{1}{2q^2} \left(\frac{q}{p}\right)^{1/2} \frac{\sinh^{n-3}(\xi)}{\cosh(\xi)} S(\xi),$$

where $S(\xi) = nH_2 \sinh^n(\xi) - (n - 2) \cosh^2(\xi)p$. A computation gives

$$\dot{S}(\xi) = 2 \sinh(\xi) \cosh(\xi)(nH_2 \sinh^{n-2}(\xi) - (n - 2)p).$$

Now, we set $R(\xi) = nH_2 \sinh^{n-2}(\xi) - (n - 2)p$ and we have

$$\dot{S}(\xi) = 2 \sinh(\xi) \cosh(\xi)R(\xi).$$

We easily see that

$$\dot{R}(\xi) = (n - 2)nH_2 \frac{\sinh^{n-3}(\xi)}{\cosh(\xi)} > 0 \text{ for all } \xi > 0.$$

Then, we can see that R , and *a fortiori* S , vanish at $\xi = 0$ and is positive for $\xi > 0$. This finishes the proof of (ii).

For (iii), we consider the integrand $\sqrt{\frac{p(\xi)}{q(\xi)}}$ of λ in (18) and we recall that $\sqrt{\frac{p(0)}{q(0)}} = 0$. Now, we compute its derivative at $\xi = 0$ obtaining $\sqrt{H_2}$. Its Taylor approximation near zero is then given by

$$\sqrt{\frac{p(\xi)}{q(\xi)}} = \sqrt{H_2} \xi + o(\xi^2)$$

which gives the result by integration. □

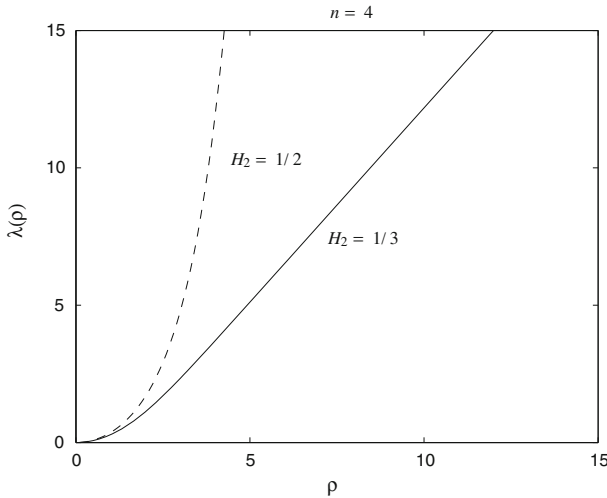


Fig. 1 Entire rotational strictly convex H_2 -graph with $0 < H_2 \leq \frac{n-2}{n}, n = 4$

Example 5.4 (Entire rotational strictly convex H_2 -graph in $\mathbb{H}^n \times \mathbb{R}$ with $0 < H_2 \leq \frac{n-2}{n}, n > 2$)

We have $q(0) = 0$ and $\lambda = \frac{\sqrt{H_2}}{2} \rho^2 + o(\rho^3)$ near 0. Since $\text{tgh}(\xi) < 1$, the expression for \dot{q} in (17) gives that $\dot{q} > 0$ for $\xi > 0$. Then, the generating curve is given by a function $\lambda(\rho)$ defined for $\rho > 0$. By applying Lemma 5.1 it follows that the generating curve is strictly convex (Fig. 1). □

Example 5.5 (Compact embedded strictly convex rotational H_2 -hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with $H_2 > \frac{n-2}{n}, n > 2$) (Fig. 2)

We have $q(0) = 0$ and as before, near 0, λ has the following behavior

$$\lambda = \frac{\sqrt{H_2}}{2} \rho^2 + o(\rho^3).$$

Claim: The asymptotic behavior of $q(\xi), n > 2$ when ξ tends to infinity is as follows

$$q(\xi) = \frac{e^{(n-2)\xi}}{2^{n-2}} \left[(1 - e^{-2\xi})^{(n-2)} - \left(\frac{nH_2}{n-2} \right) + O(e^{-2\xi}) \right]. \tag{19}$$

Proof of the Claim: We start by analyzing the function $\frac{\sinh^{(n-1)}(s)}{\cosh(s)}$, the integrand of I . We have

$$\frac{\sinh^{(n-1)}(s)}{\cosh(s)} = \frac{e^{(n-2)s}}{2^{n-2}} \frac{(1 - z^2)^{n-1}}{(1 + z^2)}, \text{ where } z = e^{-s}.$$

The behavior of $\frac{(1-z^2)^{n-1}}{(1+z^2)}$ near $z = 0$ given by its Taylor approximation helps us to see the behavior of $\frac{\sinh^{(n-1)}(s)}{\cosh(s)}$ at infinity, namely,

$$\frac{\sinh^{(n-1)}(s)}{\cosh(s)} = \frac{e^{(n-2)\xi}}{2^{n-2}} + O(e^{(n-4)\xi}).$$

Then, a computation leads us to the asymptotic behavior of q when ξ tends to infinity. This concludes the proof of the claim.

By using the expression in (19), we can see that when $H_2 > \frac{n-2}{n} > 0$, $q(\xi)$ becomes negative when ξ tends to infinity. Also, the behavior of the function $\text{tgh}(\xi)$ gives, by using the expression for \dot{q} in (17), that \dot{q} is positive near zero, vanishes at $\xi = \text{arctgh}(\sqrt{\frac{n-2}{nH_2}})$ and becomes negative later. This shows that q is positive near 0, attains a maximum at $\text{arctgh}(\sqrt{\frac{n-2}{nH_2}})$, has a positive root $\rho_0 > \text{arctgh}(\sqrt{\frac{n-2}{nH_2}})$ and is negative for $\xi > \rho_0$.

Then $\lambda(\rho)$ is defined for $\rho \in [0, \rho_0]$ and at ρ_0 we have $nH_2I(\rho_0) = \sinh^{(n-2)}(\rho_0)$. Also, since $nH_2I(\xi)$ is increasing we obtain

$$\int_0^{\rho_0} \sqrt{\frac{nH_2I}{\sinh^{(n-2)}(\xi) - nH_2I}} d\xi \leq \int_0^{\rho_0} \sqrt{\frac{\sinh^{(n-2)}(\rho_0)}{\sinh^{(n-2)}(\xi) - \sinh^{(n-2)}(\rho_0)}} d\xi$$

By setting $v = \frac{\sinh(\xi)}{\sinh(\rho_0)}$ we see that

$$\begin{aligned} \int_0^{\rho_0} \sqrt{\frac{nH_2I}{\sinh^{(n-2)}(\xi) - nH_2I}} d\xi &\leq \int_0^1 \frac{\sinh(\rho_0)}{\sqrt{1 + v^2 \sinh^2(\rho_0)}} \cdot (v^{n-2} - 1)^{-1/2} dv \\ &\leq \int_0^1 v^{-1} (v^{n-2} - 1)^{-1/2} dv. \end{aligned}$$

Since $\int v^{-1} (v^{n-2} - 1)^{-1/2} = \frac{2}{n-2} \arctan(v^{n-2} - 1)^{1/2} + \text{constant}$, we conclude that the integral converges to a finite value $t_0 = \lambda(\rho_0)$ at ρ_0 .

Then, we see that for each value of H_2 greater than $\frac{n-2}{n}$, we obtain that λ is increasing and strictly convex in the interval $(0, \rho_0)$ and is vertical and assume a finite value at $\rho = \rho_0$.

Since the induced metric in the ρt -plane is Euclidean, a simple computation shows that the Euclidean curvature is finite and strictly positive at this point. If we set $t_0 = \lambda(\rho_0)$, we can then glue together the graph $t = \lambda(y)$ with its reflection given by

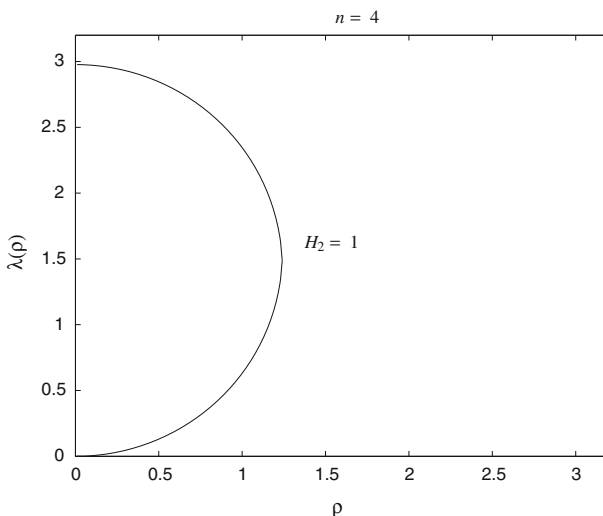


Fig. 2 Embedded compact strictly convex rotational H_2 -hypersurface with $H_2 > \frac{n-2}{n}$, $n = 4$

$$2t_0 - \lambda(y)$$

in order to obtain an embedded compact rotational surface. □

In order to highlight the geometric properties of the examples with $d = 0$, we notice that careful analysis of (18) gives the following proposition that we will need later.

Proposition 5.2 *For fixed n , the profile curves of the rotational H_2 -hypersurfaces obtained by (18) satisfy:*

- (i) *For a fixed ρ , λ increases when H_2 increases.*
- (ii) *For $H_2 > \frac{n-2}{n}$, we have that ρ_0 tends to infinity when H_2 tends to $\frac{n-2}{n}$.*
- (iii) *$\lambda(\rho) \rightarrow 0$ uniformly in compact subsets of $[0, \infty]$ when $H_2 \rightarrow 0$. This means that when $H_2 \rightarrow 0$, the rotational H_2 -hypersurfaces converge to the slice, uniformly in compact subsets of $\mathbb{H}^n \times \{0\}$.*

6 Generalizations for H_r -hypersurfaces, $r > 2$

In this section, we give an insight into the case $r > 2$. Again, by adapting the proof of [17, Proposition (2.2)] given by M. L. Leite, we obtain

Proposition 6.1 *The r -mean curvature H_r of the graph G of u considering in $M \subset \mathbb{R}^n$ the metric $g = \frac{1}{F^2} \langle \cdot, \cdot \rangle$ and the coordinates (x_1, x_2, \dots, x_n) is given by*

$$S_r \frac{W^{r+2}}{F^{2r}} = \sum_{j_1 < \dots < j_r} \left(W^2 - F^2 \left(u_{j_1}^2 + \dots + u_{j_r}^2 \right) \right) \begin{vmatrix} V_{\{j_1 j_1\}} & V_{\{j_1 j_2\}} & \dots & V_{\{j_1 j_r\}} \\ V_{\{j_1 j_2\}} & V_{\{j_2 j_2\}} & \dots & V_{\{j_2 j_r\}} \\ \dots & \dots & \dots & \dots \\ V_{\{j_1 j_r\}} & V_{\{j_2 j_r\}} & \dots & V_{\{j_r j_r\}} \end{vmatrix} \tag{20}$$

$$- 2F^2 \sum_{i < k} u_i u_k \left(\sum_{j_2, \dots, j_r \neq i, k} \begin{vmatrix} V_{\{i k\}} & V_{\{i j_2\}} & \dots & V_{\{i j_r\}} \\ V_{\{j_2 k\}} & V_{\{j_2 j_2\}} & \dots & V_{\{j_2 j_r\}} \\ \dots & \dots & \dots & \dots \\ V_{\{j_r k\}} & V_{\{j_r j_2\}} & \dots & V_{\{j_r j_r\}} \end{vmatrix} \right),$$

where $V_{\{ij\}} = u_{ij} + \frac{1}{F} \left[u_i F_j + u_j F_i - \left(\sum_m u_m F_m \right) \delta_{ij} \right] = II_{ij} \cdot W$ and the indices vary in $\{1, \dots, n\}$, where we recall that $W = \sqrt{1 + |\nabla_g u|_g^2}$.

For $r = 1$, formula (20) is equivalent to [23, formula (3)] and for $r = 2$ it reduces to formula (10) of Proposition 3.6. We can use it to construct many examples of H_r -hypersurfaces. For instance, we can consider the half-space model for \mathbb{H}^n and the graph G of differentiable functions of the form $t = v(x_1)$ in order to obtain hypersurfaces with $H_r = 0$. Now, as in Sect. 4, we consider the graph G of functions of the form $t = u(x_1, \dots, x_{n-1}, y) = \lambda(y)$. We have the following.

Theorem 6.2 (Hypersurfaces with $H_r = 0$ in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic translations) *Despite the slice, vertical graphs G with $H_r = 0$ are, up to vertical translations or reflections, of the following types:*

- a) If $n = r$, $\lambda(y) = c \ln(y)$, for all $y > 0$ and for $c > 0$.
- b) If $n > r$, $\lambda(y) = \frac{r}{n-r} \arcsin\left(\frac{y^{\frac{n-r}{r}}}{c}\right)$, for a positive constant c and $y \in (0, c^{\frac{r}{n-r}})$.

The function $\lambda(y)$ in a) gives rise to an entire vertical graph. The function $\lambda(y)$ in b) generate a family of non-entire horizontal $(r - 1)$ -minimal complete graphs in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic translations. The asymptotic boundary of each graph of this family is formed by two parallel $(n - 1)$ -planes.

Proof For G , we have the following

$$V_{ij} = -\frac{\dot{\lambda}(y)}{y} \delta_{ij}, \text{ for } i, j \leq n, \quad V_{nn} = \ddot{\lambda}(y) + \frac{\dot{\lambda}(y)}{y}$$

and

$$W^2 = 1 + y^2 \dot{\lambda}^2(y).$$

The last proposition then yields

$$S_r \frac{W^{r+2}}{F^{2r}} = \binom{n-1}{r} W^2 (-1)^r \left(\frac{\dot{\lambda}}{y}\right)^r + \binom{n-1}{r-1} (-1)^{r-1} \left(\frac{\dot{\lambda}}{y}\right)^{r-1} \left(\ddot{\lambda}(y) + \frac{\dot{\lambda}(y)}{y}\right) \quad (21)$$

Rearranging the terms and imposing $S_r = 0$, we obtain the slice as a solution or

$$ry\ddot{\lambda} = (n - 2r)\dot{\lambda} + (n - r)\dot{\lambda}^3 y^2.$$

The latter can be rewritten as

$$\left(\dot{\lambda}^{-2} \cdot y^{\left(\frac{2(n-2r)}{r}\right)}\right)' = \frac{2(r-n)}{r} \cdot y^{\left(\frac{2n-3r}{r}\right)}.$$

For $r = n$, integrating twice, we obtain, up to vertical translations, the one parameter family of solutions

$$\lambda(y) = c \ln(y), \text{ for all } y > 0 \text{ and for } c \in \mathbb{R}.$$

Each solution gives then rise to a entire graph with $H_r = 0$.

For $n > r$, the first integration gives

$$\dot{\lambda}^{-2} y^{\left(\frac{2(n-2r)}{r}\right)} = c^2 - y^{\left(\frac{2(n-r)}{r}\right)},$$

where c^2 , $c > 0$, comes from the integration and we must have $y < c^{\frac{r}{n-r}}$. We then have

$$\dot{\lambda} = \frac{y^{\left(\frac{n-2r}{r}\right)}}{\sqrt{c^2 - y^{\frac{2(n-r)}{r}}}},$$

which gives, up to vertical translations, the one parameter family of solutions

$$\lambda(y) = \pm \frac{r}{n-r} \arcsin\left(\frac{y^{\frac{n-r}{r}}}{c}\right), \text{ for a positive constant } c.$$

The function $\lambda(y)$ given by b) is increasing in the interval $(0, c^{\frac{2}{n-2}})$ and is vertical at $y = c^{\frac{2}{n-2}}$. Since the induced metric in the yt -plane is euclidean, a simple computation shows

that the euclidean curvature is finite and strictly positive at this point. If we set $t_0 = \lambda \left(c^{\frac{2}{n-2}} \right)$, we can then glue together the graph $t = \lambda(y)$ with its reflection given by

$$2t_0 - \lambda(y)$$

in order to obtain a horizontal $(r - 1)$ -minimal complete graph invariant by parabolic screw motion defined over $\{(x_1, \dots, x_{n-1}, t) \in \mathbb{H}^{n-1} \times \mathbb{R} \mid 0 \leq t \leq 2t_0\}$. The asymptotic boundary of this example is formed by two parallel hyperplanes. \square

Now, we give a tour on rotational H_r -hypersurfaces. Following the steps of Sect. 5, with the ball model for \mathbb{H}^n , we can see that the r -mean curvature

$$\binom{n}{r} H_r(p) = \sum_{i_1 < \dots < i_r} k_{i_1} \cdots k_{i_r}$$

w.r.t. the upward normal vector is given by

$$nH_r \frac{(\sinh^{n-1}(\rho))}{\cosh^{r-1}(\rho)} = \frac{\partial}{\partial \rho} \left[\sinh^{n-r}(\rho) \left(\frac{\dot{\lambda}^2}{1 + \dot{\lambda}^2} \right)^{r/2} \right]. \tag{22}$$

By setting $I(\xi) = \int_0^\xi \frac{(\sinh^{(n-1)}(s))}{\cosh^{r-1}(s)} ds$ and integrating twice we obtain, up to vertical translation or reflection,

$$\lambda(\rho) = \int_*^\rho \sqrt{\frac{(nH_r I + d)^{2/r}}{(\sinh^{(n-r)}(\xi))^{2/r} - (nH_r I + d)^{2/r}}} d\xi, \tag{23}$$

where the constant d comes from the first integration.

From the computation, we deduce that the term $(nH_r I + d)$ must be positive. We set $p_r(\xi) = (nH_r I + d)^{2/r}$ and $q_r(\xi) = (\sinh^{(n-r)})^{2/r}(\xi) - (nH_r I + d)^{2/r}$, but for the sake of simplicity, we drop the subscript r . We also notice that when $d \neq 0$, λ could not be defined for $\rho = 0$, since, in this case, $\frac{p(0)}{q(0)} = -1$.

We can easily see that the slice $\lambda = \text{constant}$ is a solution of (22) for $H_r = 0$ and any dimension. We also see that putting $n = r$ and $H_r = 0$ in (22) we obtain that the cone $\lambda(\rho) = c\rho$, $c \in \mathbb{R}$, $\rho > 0$ is a $(r - 1)$ -minimal hypersurface.

Now we consider the case $d = 0$, which implies $H_r > 0$ since $(nH_r I + d) > 0$. We then have

$$\lambda(\rho) = \int_0^\rho \sqrt{\frac{(nH_r I)^{2/r}}{(\sinh^{(n-r)})^{2/r}(\xi) - (nH_r I)^{2/r}}} d\xi \tag{24}$$

and the following lemma holds.

Lemma 6.3 *The function λ given by (24) for $H_r > 0$:*

- (i) *is increasing.*
- (ii) *satisfies $\ddot{\lambda} > 0$, for all $\xi > 0$.*
- (iii) *satisfies $\lambda = \frac{(H_r)^{1/r}}{2} \rho^2 + o(\rho^3)$, near 0.*

In particular, the corresponding hypersurface is strictly convex.

Sketch of the Proof We follow step by step the proofs for the case $r = 2$ making appropriate adjustments. We sketch the proof of ii) for completeness. We have

$$\ddot{\lambda} = \frac{1}{r q^2} \left(\frac{q}{p} \right)^{1/2} (nH_r I \cdot \sinh^{n-r}(\xi))^{(2/r)-1} \frac{\sinh^{n-r-1}(\xi)}{\cosh^{r-1}(\xi)} S(\xi),$$

where $S(\xi) = nH_r \sinh^n(\xi) - (n - r) \cosh^r(\xi)nH_rI$. If $n = r$, $S(\xi)$ and therefore $\ddot{\lambda}$ are positive for $\xi > 0$. We now consider the case $n > r$. A computation gives

$$\dot{S}(\xi) = 2nH_r \sinh(\xi) \cosh^{r-1}(\xi)(\sinh^{n-2}(\xi) \cosh^{2-r}(\xi) - (n - r)I).$$

Now, we set $R(\xi) = \sinh^{n-2}(\xi) \cosh^{2-r}(\xi) - (n - r)I$ and we write

$$\dot{S}(\xi) = rnH_r \sinh(\xi) \cosh^{r-1}(\xi)R(\xi).$$

We easily see that

$$\dot{R}(\xi) = (n - 2) \frac{\sinh^{n-3}(\xi)}{\cosh^{r-1}(\xi)} > 0 \text{ for all } \xi > 0.$$

Then, we can see that R , and *a fortiori* S , vanish at $\xi = 0$ and is positive for $\xi > 0$. □

We want to see for which values of $\xi > 0$, the denominator q in (24) is positive, that is, for which values of $\xi > 0$, $\frac{\sinh^{(n-r)}(\xi)}{nH_rI} > 1$. For that, we analyze the sign of

$$g(\xi) := \sinh^{(n-r)}(\xi) - (nH_rI).$$

Differentiating and rearranging, we obtain

$$\dot{g}(\xi) = \frac{\sinh^{n-r-1}(\xi)}{\cosh^{2r-1}(\xi)} nH_r \left[\frac{(n - r)}{nH_r} - \operatorname{tgh}^r(\xi) \right] \tag{25}$$

Similarly to what we have done in Sect. 5, we can see that the sign of g , and *a fortiori* the type of solution, depends on whether the value of H_r is greater or less than $\frac{n-r}{n}$. We have the following proposition.

- Proposition 6.4** (i) For each H_r , $0 < H_r \leq \frac{(n-r)}{n}$, the solution of (24) is an entire rotational strictly convex H_r -graph in $\mathbb{H}^n \times \mathbb{R}$.
 (ii) For each H_r , $H_r > \frac{(n-r)}{n}$, a solution of (24) gives rise to an embedded compact strictly convex rotational H_r -hypersurface in $\mathbb{H}^n \times \mathbb{R}$.

Sketch of the Proof From (25), since $\operatorname{tgh}(\xi) < 1$, we easily see that for the case i), $g > 0$ for $\xi > 0$. Therefore, $q > 0$ for $\xi > 0$ and then $\lambda(\rho)$ is defined for $\rho \in [0, \infty)$. In view of Lemma 6.3, item i) of the proposition is proved.

For the case ii), we first claim that, under our hypothesis, g becomes negative when ξ tends to infinity. This is clear for $r = n$. For $n > r$, this can be seen by observing the asymptotic behavior of $g(\xi)$ when ξ tends to infinity, namely,

$$g(\xi) = \frac{e^{(n-r)\xi}}{2^{n-r}} \left[(1 - e^{-2\xi})^{(n-r)} - \left(\frac{nH_r}{n-r} \right) + O(e^{-2\xi}) \right].$$

Now, we notice that the expression (25) gives that \dot{g} is positive near zero, vanishes at $\xi = \operatorname{arctgh}(\sqrt{\frac{n-r}{nH_r}})$ and becomes negative later. This shows that g , and therefore q , is positive near 0, attains a maximum at $\operatorname{arctgh}(\sqrt{\frac{n-r}{nH_r}})$, has a positive root $\rho_0 > \operatorname{arctgh}(\sqrt{\frac{n-r}{nH_r}})$, and is negative for $\xi > \rho_0$. Then, $\lambda(\rho)$ is defined for $\rho \in [0, \rho_0)$ and at ρ_0 we have $nH_rI(\rho_0) = \sinh^{(n-r)}(\rho_0)$.

Moreover, since $nH_rI(\xi)$ is increasing, we obtain

$$\int_0^{\rho_0} \sqrt{\frac{(nH_rI)^{2/r}}{(\sinh^{(n-r)}(\xi))^{2/r} - (nH_rI)^{2/r}}} d\xi \leq \int_0^{\rho_0} \sqrt{\frac{(\sinh^{(n-r)}(\rho_0))^{2/r}}{(\sinh^{(n-r)}(\xi))^{2/r} - (\sinh^{(n-r)}(\rho_0))^{2/r}}} d\xi$$

and then, by setting $v = \left(\frac{\sinh(\xi)}{\sinh(\rho_0)}\right)^{2/r}$ we see that

$$\int_0^{\rho_0} \sqrt{\frac{(nH_r I)^{2/r}}{(\sinh^{(n-r)}(\xi))^{2/r} - (nH_r I)^{2/r}}} d\xi \leq \int_0^1 \frac{r}{2} v^{-1}(v^{n-r} - 1)^{-1/2} dv.$$

Now, we use that $\int v^{-1}(v^{n-r} - 1)^{-1/2} = \frac{r}{n-r} \arctan(v^{n-r} - 1)^{1/2} + \text{constant}$ to conclude that the integral converges to a finite value $t_0 = \lambda(\rho_0)$ at ρ_0 .

Thus, all this together with Lemma 6.3 gives that for each value of H_r greater than $\frac{n-r}{n}$, we obtain that λ is increasing and strictly convex in the interval $(0, \rho_0)$ and is vertical and assume a finite value at $\rho = \rho_0$.

Since the induced metric in the ρt -plane is euclidean, a simple computation shows that the euclidean curvature is finite and strictly positive at this point. If we set $t_0 = \lambda(\rho_0)$, we can then glue together the graph $t = \lambda(y)$ with its reflection given by

$$2t_0 - \lambda(y)$$

in order to obtain a compact rotational surface. □

We now state a result similar to that of Proposition 5.2, for $r > 2$.

Proposition 6.5 *For fixed n and r , the profile curves of the rotational H_r -hypersurfaces obtained by (24) satisfy:*

- (i) *For a fixed ρ , λ increases when H_r increases.*
- (ii) *For $H_r > \frac{n-r}{n}$, we have that ρ_0 tends to infinity when H_r tends to $\frac{n-r}{n}$.*
- (iii) *$\lambda(\rho) \rightarrow 0$ uniformly in compact subsets of $[0, \infty]$ when $H_r \rightarrow 0$. This means that when $H_r \rightarrow 0$, the rotational H_r -hypersurfaces converge to the slice, uniformly in compact subsets of $\mathbb{H}^n \times \{0\}$.*

Before finishing this section, we state an existence and uniqueness result for complete rotational H_r -hypersurfaces.

Theorem 6.6 (i) *For each $0 < H_r \leq \frac{(n-r)}{n}$, there exists, up to translations or reflections, a unique entire rotational H_r -graph in $\mathbb{H}^n \times \mathbb{R}$ and it is strictly convex.*

(ii) *For each $H_r > \frac{(n-r)}{n}$, there exists, up to translations or reflections, a unique embedded compact rotational H_r -hypersurface and it is strictly convex.*

Moreover, in $\mathbb{H}^n \times \mathbb{R}$, an embedded compact rotational H_r -hypersurface must have $H_r > \frac{(n-r)}{n}$ and an entire rotational H_r -graph must have $0 < H_r \leq \frac{(n-r)}{n}$.

Proof As we notice before, a careful analysis of (23) shows that if $d \neq 0$, the solution λ is not defined for $\rho = 0$. Then, in order to obtain, either an entire rotational graph or a compact rotational hypersurface, we must have $d = 0$. The solutions of (24) are then, up to vertical translations or reflections, the ones obtained in Proposition 6.4. □

For future use, for each r , we denote by \mathcal{P}_r **the entire rotational strictly convex H_r -graph with $H_r = \frac{n-r}{n}$** of Proposition 6.4(i) and by \mathcal{Q}_r **the embedded rotational strictly convex H_r -hypersurface with $H_r > \frac{n-r}{n}$** obtained in Proposition 6.4(ii).

7 Barriers for hypersurfaces with prescribed H_r

For $n > r$, the hypersurfaces obtained in Proposition 6.4(i) are complete simply connected hypersurfaces which are entire strictly convex graphs with constant r -mean curvature,

$0 < H_r \leq \frac{n-r}{n}$. The hypersurface obtained in Proposition 6.4(ii) gives an example of constant r -mean curvature, $H_r > \frac{n-r}{n}$, sphere-like hypersurface. They suggest, as mentioned in the introduction, a strong analogy with the constant mean curvature case in $\mathbb{H}^n \times \mathbb{R}$. In this section, we use these hypersurfaces as barrier in order to prove some beautiful geometric results. For that, we use suitable versions of the Maximum Principle extracted from [13, Theorem (1.1)], by F. Fontenele and S. Silva, where the reader can find details and proofs, based on a classical Maximum Principle for elliptic functions. By using (2) and [13, Proposition (3.2) and Lemma (3.3)] we can see that ellipticity, in this case, is equivalent to the definite positiveness of P_{r-1} . The positiveness of P_{r-1} is somehow well explored in the literature and we quote, for instance, [6, Proposition (3.2)] that states the following.

[6, Proposition (3.2)] Let \bar{M}^{n+1} be an $(n + 1)$ -dimensional oriented Riemannian manifold and let M be a connected n -dimensional orientable Riemannian manifold (with or without boundary). Suppose $x : M \rightarrow \bar{M}$ is an isometric immersion with $H_r > 0$ for some $1 \leq r \leq n$. If there exists a convex point $q \in M$, then for all $1 \leq j \leq r - 1$, P_j is positive definite, and the j -mean curvature H_j is positive.

Then, based on [13, Theorems (1.1) and (1.2)], we obtain the following.

Interior Geometric Maximum Principle

Let M_1 and M_2 be oriented connected hypersurfaces of an oriented Riemannian manifold that are tangent at p . Let η_0 be a unitary normal to M_2 and suppose that M_2 is strictly convex at a point q . If $H_r(M_2) \geq H_r(M_1) > 0$ and M_1 remains above M_2 w.r.t. η_0 then $M_1 = M_2$.

Boundary Geometric Maximum Principle

Let M_1 and M_2 be oriented connected hypersurfaces of an oriented Riemannian manifold with boundaries ∂M_1 and ∂M_2 , respectively. Suppose that M_1 and M_2 , as well as ∂M_1 and ∂M_2 , are tangent at $p \in \partial M_1 \cap \partial M_2$. Let η_0 be a unitary normal to M_2 and suppose that M_2 is strictly convex at a point q . If $H_r(M_2) \geq H_r(M_1) > 0$ and M_1 remains above M_2 w.r.t. η_0 then $M_1 = M_2$.

With the Maximum Principles in hand, we can proceed with the geometric results. We start by noticing that the existence of a compact (without boundary) hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with constant $H_r > \frac{n-r}{n}$, does not allow, by the Maximum Principle, the existence of an entire H_r -graph, $H_r > \frac{n-r}{n}$, strictly convex at some point.

Now, we prove a *Convex Hull Lemma*. For the mean curvature case in $\mathbb{H}^n \times \mathbb{R}$, it was proved in [5, 11]. Here, the proof is essentially the same. The real difference is on the version of the Maximum Principle. We sketch it here for completeness.

Let \mathcal{P}_r be as defined above and let $\check{\mathcal{P}}_r$ be the symmetric of \mathcal{P}_r with respect to a horizontal slice. We consider the set Υ_r of hypersurfaces obtained from \mathcal{P}_r or $\check{\mathcal{P}}_r$ by a vertical or a horizontal translation in $\mathbb{H}^n \times \mathbb{R}$. We denote by $\mathcal{C}(S)$ the mean convex side of $S \in \Upsilon_r$. Let K be a compact set in $\mathbb{H}^n \times \mathbb{R}$ and set

$$\mathcal{F}_K^r = \{B \subset \mathbb{H}^n \times \mathbb{R} \mid K \subset B, B = \mathcal{C}(S) \text{ for some } S \in \Upsilon_r\}.$$

Lemma 7.1 (Convex Hull Lemma) *Let M be a compact connected immersion in $\mathbb{H}^n \times \mathbb{R}$, $n > r$, with prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$. Then M is contained in the convex hull of the family $\mathcal{F}_{\partial M}^r$.*

Proof Let $S \in \Upsilon_r$ be such that $\partial M \subset \mathcal{C}(S)$. We will prove that $M \subset \mathcal{C}(S)$ and this will finish the proof. Since M is compact and S is an entire graph, we can obtain a copy of S by a vertical translation that contains M in its mean convex side. Now, we start moving this copy back by vertical translations. Suppose that in this process, S touches M at a point p . Then,

M and S are tangent at p . By hypothesis, the r -mean curvature of M is less than or equal to the r -mean curvature of S and since S is strictly convex, we can use the Maximum Principle to obtain that either M is contained in S or the point of contact belongs to ∂M . In both cases, we could not have $\partial M \subset \mathcal{C}(S)$, then we can come back into the original position without touching M . This proves that $M \subset \mathcal{C}(S)$. \square

As a corollary, we can see that an immersion in $\mathbb{H}^n \times \mathbb{R}$, $n > r$, with prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$ cannot be compact without boundary.

The Convex Hull Lemma also gives *a priori* height estimates for graphs over compact domains with prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$ whose boundary lies in a slice (see Corollary 7.3 below). We wonder if a height estimate could also be obtained for the case $H_r > \frac{n-r}{n}$ and we ask:

Question Would the maximum height of a compact graph with boundary in a slice with $H_r = \text{constant} > \frac{n-r}{n}$ be given by half of the total height of the compact rotational corresponding example?

As far as we know, height estimates for H_r -hypersurfaces in product spaces were first obtained in [6]. Thereafter, it was approached again in [12] for extrinsic curvature in a 3-dimensional product space, in the recent preprint [14], where the authors deal with H_r -hypersurfaces in n -dimensional warped products and for some especial Weingarten surfaces in $M^2(c) \times \mathbb{R}$ in the paper [19].

Before stating the next results we establish some notation. Let $B \subset \mathbb{H}^n \times \{0\}$ be an n -ball and let $S_{B,+}$ and $S_{B,-}$ be the hypersurfaces in Υ_r , $r < n$, passing through the $(n - 1)$ -sphere ∂B and symmetric with respect to the slice $\mathbb{H}^n \times \{0\}$. $S_{B,+}$ is above the slice and $S_{B,-}$ is below the slice.

Let $\phi : M \rightarrow \mathbb{H}^n \times \mathbb{R}$ a connected hypersurface and let $\pi : \mathbb{H}^n \times \mathbb{R} \rightarrow \mathbb{R}$ denote the natural projection. We denote by $h : M \rightarrow \mathbb{R}$ the *height function* of M , that is, $h(p) = \pi(\phi(p))$, we have the following.

Theorem 7.2 *Let M be a compact connected immersion in $\mathbb{H}^n \times \mathbb{R}$ with prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$, $n > r$, such that $\partial M \subset \mathbb{H}^n \times \{0\}$. Let $\Omega \subset \mathbb{H}^n$ be the bounded domain of M such that $\partial M = \partial\Omega$. Then, there exists a constant C (depending on Ω and n) such that $|h(p)| \leq C$ for all $p \in M$.*

Proof Let $B \subset \mathbb{H}^n$ be an n -ball such that $\partial M \subset B$. By the Convex Hull Lemma, $M \subset (\mathcal{C}(S_{B,-}) \cap \mathcal{C}(S_{B,+}))$. \square

Let $\Omega \subset \mathbb{H}^n$ be compact, u be a real function over Ω with $u|_{\partial\Omega} = 0$ and let G be the vertical graph of u in $\mathbb{H}^n \times \mathbb{R}$. As corollaries of the last theorem we obtain *a priori* height and gradient estimates for graphs described in the two following results.

Corollary 7.3 *Suppose that G has prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$, $n > r$. Then, there exists a constant C (depending on Ω and n) such that $\max_{p \in \Omega} |u(p)| \leq C$.*

Corollary 7.4 *Suppose that G has prescribed r -mean curvature function $H_r : M \rightarrow (0, \frac{n-r}{n}]$, $n > r$, and that $\Gamma = \partial\Omega$ is connected and has all its principal curvatures (w.r.t. the inner normal vector) greater than 1. Then, there exists a constant C (depending on Ω and n) such that $\max_{p \in \partial\Omega} |\nabla_{\mathbb{H}} u(p)| \leq C$.*

Proof We first notice that the hypothesis on the principal curvatures of Γ imply that we can find a radius R such that, for each $p \in \Gamma$, there is an n -ball B_p of radius R whose boundary is tangent to p at Γ and satisfying $\Gamma \subset B_p$. Now, as in the proof of Theorem 7.2, we have, for each $p, M \subset (\mathcal{C}(S_{B_p,-}) \cap \mathcal{C}(S_{B_p,+}))$ and the result follows. \square

In the sequel, we prove two uniqueness results, but before that, we need to prove a lemma and we recall [20, Proposition (4.1)].

Lemma 7.5 *Let M be a compact H_r -immersion in $\mathbb{H}^n \times \mathbb{R}$, $H_r = k > 0$, such that if $\partial M \neq \emptyset$ we have $\partial M \subset \mathbb{H}^n \times \{0\}$. Then there is an interior point of M where M is strictly convex, that is, M has an elliptic point.*

Proof Since M is compact, there is a $\check{\mathcal{P}}_r \in \Upsilon_r$ such that $M \subset \mathcal{C}(\check{\mathcal{P}}_r)$. Without loss of generality, we can assume that there is a subset of M above the slice $\mathbb{H}^n \times \{0\}$. Set p for the highest point of $\check{\mathcal{P}}_r$. Now, we consider a sequence of rotational strictly convex H_r -hypersurfaces $\check{\mathcal{L}}_i$, tangent to $\check{\mathcal{P}}_r$ at p , such that $H_r(\check{\mathcal{L}}_i) < k$ and $H_r(\check{\mathcal{L}}_i) \rightarrow 0$ (see Proposition 6.5). We start moving p toward its projection on $\mathbb{H}^n \times \{0\}$ along the vertical axis while $i \rightarrow \infty$. By using the behavior of the sequence $\check{\mathcal{L}}_i$ described in Proposition 6.5, we can do this process keeping M in $\mathcal{C}(\check{\mathcal{L}}_i)$. We do this until a translated $\check{\mathcal{L}}_i$ meet M at a point q_0 . It is clear by construction that q_0 is not at ∂M . Then, we obtain that $\check{\mathcal{L}}_i$ contains M in its mean convex side and $\check{\mathcal{L}}_i$ is tangent to M at q_0 . Since $\check{\mathcal{L}}_i$ is strictly convex, we obtain the result. \square

[20, Proposition (4.1)] *Let $S \subset \mathbb{H}^n$ be a finite union of connected, closed and embedded $(n - 1)$ -submanifolds $C_j, j = 1, \dots, k$, such that the bounded domains whose boundary are the C_j are pairwise disjoint. Assume that for any geodesic $\gamma \in \mathbb{H}^n$, there exists a $(n - 1)$ -geodesic plane $\pi_\gamma \subset \mathbb{H}^n$ of symmetry of S which is orthogonal to γ . Then S is a $(n - 1)$ -geodesic sphere of \mathbb{H}^n .*

Now, we prove an Alexandrov-type Theorem for H_r in $\mathbb{H}^n \times \mathbb{R}$.

Theorem 7.6 (Alexandrov Theorem) *Let M be a compact (without boundary) connected embedded hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with constant r -mean curvature function $H_r > 0$ then M is, up to translations, the rotational hypersurface \mathcal{Q}_r .*

Proof By Lemma 7.5, M is strictly convex at a point $p \in M$. Then, the Maximum Principle allow us to use Alexandrov Reflection Method. Let us suppose that q is the highest point of M , say at $t = t_0$. We use Alexandrov Reflection Method reflecting M through horizontal slices $\mathbb{H}^n \times \{t\}$. For each t , we denote the part of M above (respectively, below) $\mathbb{H}^n \times \{t\}$ by M_t^+ (respectively, M_t^-). We denote by M_t^{+*} the reflection of M_t^+ through the slice. For t slightly smaller than t_0 , M_t^+ is a graph of bounded slope over a domain in $\mathbb{H}^n \times \{t\}$ and M_t^{+*} is above M_t^- . We keep doing reflection, for decreasing t , until finding an interior or boundary tangent point for both M_t^- and M_t^{+*} . Then the, interior or boundary, Maximum Principle imply that they coincide and we obtain a slice of symmetry that w.l.g. we suppose is $\mathbb{H}^n \times \{0\}$. With this process, we also obtain that the above and below parts of M w.r.t the slice are vertical graphs.

Now we fix a geodesic γ in $\mathbb{H}^n \times \{0\}$ that passes through the origin, and we apply Alexandrov Reflection Method with vertical geodesic n -planes orthogonal to γ . Similarly to what we have done above, we obtain a geodesic n -plane φ_γ of symmetry of M which is orthogonal to γ . φ_γ divides then M in two symmetric parts that are horizontal graphs, with respect to γ , over φ_γ . We do the same process for each γ . For each copy of \mathbb{H}^n at height t , and for each γ , let $\pi_\gamma = \varphi_\gamma \cap (\mathbb{H}^n \times \{t\})$. Then, we are able to use [20, Proposition (4.1)]

above for each t , in order to conclude that M is a rotational hypersurface homeomorphic to an n -sphere. We obtain \mathcal{Q}_r , up to translations. □

Theorem 7.7 *Let M be a compact connected embedded hypersurface in $\mathbb{H}^n \times \mathbb{R}$ such that $\partial M \subset \mathbb{H}^n \times \{0\}$, with constant r -mean curvature function satisfying $0 < H_r \leq \frac{n-r}{n}$, $n > r$. Suppose that $\Gamma = \partial M$ is connected and has all its principal curvatures (w.r.t. the inner normal vector) greater than 1. Let $\Omega \subset \mathbb{H}^n$ be the bounded domain such that $\Gamma = \partial\Omega$. Then M is a vertical graph over Ω . Moreover, if Γ is an $(n - 1)$ -sphere bounding the n -ball B then $M = S_{B,+}$ or $M = S_{B,-}$.*

Proof Since $H_r > 0$, M cannot be the slice. Let us suppose that there exists a part of M above the slice $\mathbb{H}^n \times \{0\}$. By Lemma 7.5, M is strictly convex at a point $p \in M$, and we can therefore use the Maximum Principle. Now, as in Corollary 7.4, for each $q \in \Gamma$, let B_q be the ball of radius R tangent to q at Γ and satisfying $\Gamma \subset B_q$. Since $M \subset (\mathcal{C}(S_{B_q,-}) \cap \mathcal{C}(S_{B_q,+}))$ the intersection of the vertical cylinder over $\partial\Omega$ with $M \setminus \partial M$ is empty. Let $\tilde{\Omega}$ be any fixed vertical copy of Ω , below the slice $\mathbb{H}^n \times \{0\}$, such that $\tilde{\Omega} \cup M = \emptyset$. Let Σ be the piece of the vertical cylinder over Γ bounded by Γ and $\tilde{\Omega}$. Then $M \cup \Sigma \cup \tilde{\Omega}$ is an orientable homological boundary of a $(n+1)$ -dimensional chain in $\mathbb{H}^n \times \mathbb{R}$. We choose the inwards normal to $M \cap \Sigma \cap \tilde{\Omega}$, and then, the normal to M is downward pointing.

Suppose that M is not a graph. Then, the vertical line over a point w of Ω intersects M in, at least, two points. Set w_2 for the highest point of M in this vertical line and w_1 for the lowest. Consider a vertical translate \tilde{M} above M such that $M \cap \tilde{M} = \emptyset$. Let \tilde{w}_1 and \tilde{w}_2 be the corresponding points in \tilde{M} . Now we vertically translate \tilde{M} down and we stop when we find a first point of contact or when $\tilde{w}_1 = w_2$. In the latter, $\tilde{w}_1 = w_2$ will be the first point of contact. In both situations, the Maximum Principle would imply that M and the translated copy should be equal. This gives a contradiction since we would not have reached the initial position.

If M has no part above the slice, we proceed in an analogous way with the lowest point of M . This completes the proof of the first part.

Now we suppose that Γ is an $(n - 1)$ -sphere bounding the n -ball B . Let us fix a geodesic γ in $\mathbb{H} \times \{0\}$ that passes through the origin and we apply Alexandrov Reflection Method with vertical geodesic n -planes orthogonal to γ . By applying Alexandrov reflection Method with vertical geodesic n -planes we obtain, by using [20, Proposition (4.1)] that M is part of a rotational hypersurface. Then, M is completely above or below the slice $\mathbb{H}^n \times \{0\}$. Let us suppose that it is above. By translating $S_{B,+}$ upwards and downwards suitably and by using the Maximum Principle we conclude that M is above and below $S_{B,+}$. Then they should coincide. The same happens with $S_{B,-}$ if M is below the slice. □

8 Appendix: r -Mean curvature for vertical graphs

Here, we generalize some results of Sect. 3 and we use the same notation of that section. We recall that $M = M_g$ denotes a Riemannian n -manifold with metric g and that we consider on $\bar{M} = M \times \mathbb{R}$ the product metric $\langle \cdot, \cdot \rangle = g + dt^2$. $G = \{(p, u(p)) \in M \times \mathbb{R} \mid p \in \Omega\}$ is the vertical graph of u , $W = \sqrt{1 + |\nabla_g u|_g^2}$ and we choose the orientation given by the upward unit normal.

Now, we set

$$\tau_r(v_1, v_2) = \text{trace}(z \rightarrow P_{r-1}R_g(v_1, z)v_2)$$

and we define inductively,

$$\begin{aligned}
 J_1(v_1, v_2) &= \tau_1(v_1, v_2) = \text{Ric}_g(v, v_2), \\
 J_s(v_1, v_2) &= \tau_s(v_1, v_2) - J_{s-1}(Av_1, v_2).
 \end{aligned}$$

Then, similar to Proposition 3.1, we have the following.

Proposition 8.1 *The $(r+1)$ -mean curvature H_{r+1} of the graph G is given by*

$$(r + 1)S_{r+1} = (r + 1) \binom{n + 1}{r} H_r = \text{div}_g \left(P_r \frac{\nabla_g u}{W} \right) + J_r \left(\frac{\nabla_g u}{W}, \frac{\nabla_g u}{W} \right), \tag{26}$$

where div_g means the divergence in M .

Proof By (1) and (4) we have

$$(r + 1)S_{r+1} = \text{trace} \left(z \rightarrow P_r \nabla_z^g \left(\frac{\nabla_g u}{W} \right) \right).$$

Claim:

$$\text{trace} \left(z \rightarrow P_r \nabla_z^g v \right) = \text{trace} \left(z \rightarrow \nabla_z^g (P_r v) \right) + J_r \left(v, \frac{\nabla_g u}{W} \right). \tag{27}$$

The proposition will be proved by taking $v = \frac{\nabla_g u}{W}$ in (27). We prove (27) by induction. For $r = 1$, it was proved before, see Eq. (6). We assume that it is true for $r - 1$.

Let $p \in M$ and let $\{v_i\}_i$ be an orthonormal basis in a neighborhood of p in M which is geodesic at p , that is, such that $\nabla_{v_j}^g v_i(p) = 0$. Let $v = \sum_i a_i v_i$. As in Proposition 3.1, the proof will be completed if we prove (27) for $v = v_i$, for some i . The left hand side vanishes and then we have to prove that

$$\text{trace} \left(z \rightarrow \nabla_z^g (P_r v_i) \right) (p) = -J_r \left(v_i, \frac{\nabla_g u}{W} \right) (p).$$

Since $P_r = S_r I - P_{r-1} A$, this holds provided that

$$\text{trace} \left(z \rightarrow \nabla_z^g (P_{r-1} A(v_i)) \right) (p) = \text{trace} \left(z \rightarrow \nabla_z^g (S_r(v_i)) \right) (p) + J_r \left(v_i, \frac{\nabla_g u}{W} \right) (p). \tag{28}$$

We use that (27) is true for $r - 1$, (7) and the definition of $J_s(v)$ to obtain

$$\begin{aligned}
 \text{trace} \left(z \rightarrow \nabla_z^g (P_{r-1} A(v_i)) \right) (p) &= \text{trace} \left(z \rightarrow P_{r-1} \nabla_z^g (Av_i) \right) (p) - J_{r-1} \left(Av_i, \frac{\nabla_g u}{W} \right) (p) \\
 &= \text{trace} \left(z \rightarrow P_{r-1} \nabla_{v_i}^g (Az) \right) (p) - \text{trace} \left(z \rightarrow P_{r-1} R_g(z, v_i) \frac{\nabla_g u}{W} \right) \\
 &\quad - J_{r-1} \left(Av_i, \frac{\nabla_g u}{W} \right) (p) \\
 &= \text{trace} \left(z \rightarrow P_{r-1} \nabla_{v_i}^g (Az) \right) (p) + \text{trace} \left(z \rightarrow P_{r-1} R_g(v_i, z) \frac{\nabla_g u}{W} \right) \\
 &\quad - J_{r-1} \left(Av_i, \frac{\nabla_g u}{W} \right) (p) \\
 &= \text{trace} \left(z \rightarrow P_{r-1} \nabla_{v_i}^g (Az) \right) (p) + J_r \left(v_i, \frac{\nabla_g u}{W} \right) (p).
 \end{aligned}$$

We complete the proof of the claim by using that $\text{trace} \left(z \rightarrow P_{r-1} \nabla_{v_i}^g (Az) \right) (p) = v(S_r)$ (see [8, Formula (7)]) and that $\text{trace} \left(z \rightarrow \nabla_z^g (S_r(v_i)) \right) (p) = v_i(S_r)$ in order to obtain (28). \square

Proposition 8.2 *If M_g has constant curvature c , then we have*

$$\tau_{r+1}(v, z) = c \left((n - r) S_r I - P_r \right) v, z \rangle_g$$

Proof Let e_i be the principal directions and set $v = \sum a_j e_j$ and $z = \sum b_l e_l$. We recall that P_r is self-adjoint and we set m_i for its eigenvalues. We have

$$\begin{aligned} \tau_{r+1}(v, z) &= \sum_i \langle P_r R_g(v, e_i)z, e_i \rangle \\ &= \sum_{i,j,l} m_i a_j b_l \langle R_g(e_j, e_i)e_l, e_i \rangle = \sum_{i,j} m_i a_j b_j \langle R_g(e_j, e_i)e_j, e_i \rangle \\ &= c \sum_i m_i \sum_{j \neq i} a_j b_j = c \sum_i m_i (\langle v, z \rangle - a_i b_i) \\ &= c \sum_i m_i \langle v, z \rangle - c \sum_i m_i a_i b_i = c \operatorname{trace}(P_r) \langle v, z \rangle - c \langle P_r v, z \rangle \\ &= c(n-r)S_r \langle v, z \rangle - c \langle P_r v, z \rangle = c \langle (n-r)S_r I - P_r \rangle v, z \end{aligned}$$

where in the last equality we use (3). □

Using the last proposition and the inductive definition of J_r , we obtain

Corollary 8.3 *If M_g has constant curvature c , then we have*

$$J_r(v, z) = c(n-r) \langle P_{r-1} v, z \rangle_g$$

We, now, consider the particular case where $M^n = \mathbb{H}^n \subset \mathbb{R}^n$ and we denote by (x_1, x_2, \dots, x_n) the (euclidean) coordinates of M . On \mathbb{H}^n , we consider the metric given by

$$g = \frac{1}{F^2} \langle \cdot, \cdot \rangle,$$

where $\langle \cdot, \cdot \rangle = (dx_1^2 + \dots + dx_n^2)$ is the euclidean metric. For this case, Proposition 8.1 reads as follows. The proof is the same of Proposition 3.1.

Proposition 8.4 *The $(r+1)$ -mean curvature of the graph of u when $M = \mathbb{H}^n$ is as above is given by*

$$(r+1)S_{r+1} = F^2 \operatorname{div} \left(P_r \frac{\nabla u}{W} \right) + \frac{(2-n)F \langle P_r \nabla u, \nabla F \rangle}{W} - (n-r)F^2 \langle P_{r-1} \left(\frac{\nabla u}{W} \right), \frac{\nabla u}{W} \rangle. \tag{29}$$

Here, div and ∇ denote quantities in the euclidean metric.

The last equation is elliptic if P_r is positive definite.

References

1. Alexandrov, A.D.: Uniqueness theorems for surfaces in the large. V, Vestnik, Leningrad Univ. **13** (1958). English translation: AMS Transl. **21**, 412–416 (1962)
2. Aledo, J.A., Espinar, J.M., Gálvez, J.A.: Height estimates for surfaces with positive constant mean curvature in $M^2 \times \mathbb{R}$. III. J. Math. **52**(1), 203–211 (2008)
3. Barbosa, J.L., Colares, A.G.: Stability of hypersurfaces with constant r -mean curvature. Ann. Global Anal. Geom. **15**, 277–297 (1997)
4. Bérard, P., Sa Earp, R.: Minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$, total curvature and index. [arXiv:0808.3838v3](https://arxiv.org/abs/0808.3838v3) [math.DG]
5. Bérard, P., Sa Earp, R.: Examples of H -hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and geometric applications. Mat. Contemp. **34**, 19–51 (2008)
6. Cheng, X., Rosenberg, H.: Embedded positive constant r -mean curvature hypersurfaces in $M \times \mathbb{R}$. An. Acad. Bras. de Ciências. **77**(2), 183–199 (2005)

7. Colares, A.G., Palmas, O.: Addendum to “Complete rotation hypersurfaces with H_k constant in space forms. *Bull. Braz. Math. Soc., New Series* **39**(1), 11–20 (2008)
8. Elbert, M.F.: On complete graphs with negative r -mean curvature. *Proc. Am. Math. Soc.* **128**(5), 1443–1450 (2000)
9. Elbert, M.F.: Constant positive 2-mean curvature hypersurfaces. III. *J. Math.* **46**(1), 247–267 (2002)
10. Elbert, M.F., Sa Earp, R.: All solutions of the CMC-equation in $\mathbb{H}^n \times \mathbb{R}$ invariant by parabolic screw motion. *Annali di Mat. Pura e App* **193**(1), 103–114 (2014)
11. Elbert, M.F., Nelli, B., Sa Earp, R.: Existence of vertical ends of mean curvature $1/2$ in $\mathbb{H}^n \times \mathbb{R}$. *Trans. Am. Math. Soc.* **364**(3), 1179–1191 (2012)
12. Espinar, J.M., Gálvez, J.A., Rosenberg, H.: Complete surfaces with positive extrinsic curvature in product spaces. *Comment. Math. Helv.* **84**(2), 351–386 (2009)
13. Fontenele, F., Silva, S.: A tangency principle and applications. III. *J. Math.* **54**, 213–228 (2001)
14. García-Martínez, S.C., Impera, D., Rigoli, M.: A sharp height estimate for compact hypersurfaces with constant k -mean curvature in warped product spaces. [arXiv:1205.5628v2](https://arxiv.org/abs/1205.5628v2) [math.DG]
15. Korevaar, N.J.: Sphere theorems via Alexandrov for constant Weingarten curvature hypersurfaces. Appendix to a Note of A. Ros. *J. Diff. Geom.* **27**, 221–223 (1988)
16. Leite, M.L.: Rotational hypersurfaces of space forms with constant scalar curvature. *Manuscr. Math.* **67**, 285–304 (1990)
17. Leite, M.L.: The tangency principle for hypersurfaces with null intermediate curvature. XI Escola de Geometria Diferencial-UFF Brazil (2000)
18. Montiel, S., Ros, A.: The Alexandrov theorem for higher order mean curvatures. *Differential Geometry—A symposium in honour of Manfredo do Carmo. Pitman Survey in Pure and Appl. Math.* **52**, 279–296 (1991)
19. Morabito, F.: Height Estimate for special Weingarten surfaces of elliptic type in $M^2(c) \times \mathbb{R}$. *Proc. Am. Math. Soc. Ser B* **1**, 14–22 (2014)
20. Nelli, B., Sa Earp, R., Toubiana, E.: Maximum principle and symmetry for minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze.* doi:[10.2422/2036-2145.201211-004](https://doi.org/10.2422/2036-2145.201211-004) (2014)
21. Palmas, O.: Complete rotation hypersurfaces with H_k constant in space forms. *Bull. Braz. Math. Soc.* **30**(2), 139–161 (1999)
22. Schoen, R., Yau, S.T.: *Lectures on Differential Geometry.* International Press, Boston (1994)
23. Sa Earp, R., Toubiana, E.: Minimal graphs in $\mathbb{H}^n \times \mathbb{R}$ and \mathbb{R}^{n+1} . *Annales de l’Institut Fourier* **60**(7), 2373–2402 (2010)