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On a problem of Huang concerning best constants in Sobolev embeddings

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Abstract Answering a question raised by Y.X. Huang, we prove what follows: if Ω is a bounded smooth domain and $p > 1$, then the mapping $q \mapsto \lambda_q |\Omega|^{\frac{p}{q}}$ is decreasing in $]0, p^*[$ and Lipschitz continuous on compact subsets of $]0, p^*[$, λ_q being the p-th power of the best Sobolev constant for the embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$.

Keywords p-Laplacian · Singular elliptic equations · Sobolev constants

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1 Introduction and main result

The present paper is devoted to the classical Dirichlet problem for the p-Laplacian operator

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1}$$

To Prof. Mario Marino, with gratitude and esteem, on the occasion of his seventieth birthday.

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where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with a boundary $\partial\Omega$ of class $C^{1,\alpha}$ $(0 < \alpha \le 1)$, $p > 1, 0 < q < p^*$ are real numbers (we recall that the Sobolev critical exponent is $p^* = Np/(N-p)$ if $1 < p < N$, $p^* = +\infty$ if $p \ge N$), and $\lambda > 0$ is a parameter.
Problem (1) has been widely investigated, with different results according to the relation between the exponents p and q. For the homogenous case $(p = q)$, we refer the reader to Lê [13] for a detailed description of the eigenvalues and eigenspaces. Ôtani [16] proved that, for $\lambda = 1$, problem (1) admits at least one nonnegative solution for $1 < q < p^*$, and, if Ω is strictly star-shaped, it admits no non-negative, nonzero solutions for $q = p^*$ and no nontrivial solutions for $q > p^*$. Uniqueness of the positive solution with minimal energy for $1 < q < p, \lambda = 1$ was proved by Franzina and Lamberti [8], who reduced problem (1) to a homogeneous one by means of a nonlocal term of the type $\|u\|_p^{q-p}$. Most of the mentioned results are based on variational methods, and they all deal with the case $1 < q < p^*$.
For $0 < q < 1$, problem (1) is singular at 0 and it cannot be directly studied in a variational framework. For the semilinear case $(p = 2)$, existence results for related singular equations were proved by Crandall et al. [3] and by Lazer and McKenna [12], while uniqueness of the solution was examined by Diaz et al. [4]. A bifurcation result for a semilinear equation with a singular perturbation was obtained by Cîrstea et al. [2]. In most cases, the study of singular problems is based on sub- and super-solutions.
Problem (1) is strictly related to the best constant in the Sobolev embeddings. Let us consider the Sobolev space $W_0^{1,p}(\Omega)$ and the Lebesgue space $L^q(\Omega)$, endowed with the norms $\|\nabla u\|_p$, $\|u\|_q$, respectively. By the Sobolev theorem, we have

$$\inf_{u \in W_0^{1,p}(\Omega),\ u \neq 0} \frac{\|\nabla u\|_p^p}{\|u\|_q^p} = \lambda_q \in]0, +\infty[. \tag{2}$$

The constant λ_q was explicitly determined by Talenti [19] for $\Omega = \mathbb{R}^N, q = p^*$, but it is not known in general. It can be proved that for $\lambda = \lambda_q$, problem (1) has a positive smooth solution u_q s.t. $\|u_q\|_q = 1$ and $\|\nabla u_q\|_p = \lambda_q^{\frac{1}{p}}$. In his interesting paper [11], Huang proved that the mapping $q \mapsto \lambda_q$ is continuous in $]1, p[$ and upper semicontinuous in $]p, p^*[$. Later, he put forward the following conjecture: *We suspect that λ_q has some monotonicity with respect to q. In fact, one can prove that, if $|\Omega| \le 1$, then $\lambda_q \le \lambda_r$ if $r < q$* (here and in the sequel, $|\Omega|$ denotes the Lebesgue N-dimensional measure of Ω).
The aim of the present note is to give such question an answer, which turns out to be positive (actually, we will prove much more). Our main result is the following:

Theorem 1 *The mapping $g :]0, p^*[\to]0, +\infty[$, defined by*

$$g(q) = \lambda_q |\Omega|^{\frac{p}{q}} \quad \textit{for all} \quad q \in]0, p^*[,$$

is Lipschitz continuous on compact subsets of $]0, p^*[$ *and decreasing in* $]0, p^*[$.

Our result both improves that of Huang and answers the question about monotonicity. Indeed, clearly from Theorem 1 it follows that $q \mapsto \lambda_q$ is continuous on the whole interval $]0, p^*[$. Moreover, if $|\Omega| \le 1$, then for all $0 < r < q < p^*$ we have from $g(r) > g(q)$ that

$$\lambda_q < \lambda_r |\Omega|^{\frac{p(q-r)}{rq}} \le \lambda_r.$$

For a surprising coincidence we became aware, after writing the present paper, that a partial positive answer to the problem raised by Huang had been given by Ercole in his very recent paper [6]. The author deals with properties of the map $q \mapsto \lambda_q$ when $p < N$ and proves that

it is absolutely continuous in $[1, p^*]$. Our result is more general as in our arguments p is any real number in $]1, +\infty[$ and the properties of λ_q are also studied in $]0, 1]$ (for more details see Remark 10).
The proof of Theorem 1 is delivered as follows (see Sect. 3). First, by applying Hölder inequality, we prove that g is Lipschitz continuous on compact subsets of $]0, p^*[$ and non-increasing in $]0, p^*[$. Then, dealing with the more delicate issue of *strict* monotonicity, we split our study in two parts: for $0 < q \leq 1$, by means of sub- and super-solutions, we prove existence and some estimates for a positive solution of class $C^1(\Omega)$ of the singular problem (1), which in turn imply that g is decreasing in $]0, 1]$; for $1 < q < p^*$, by variational methods, we prove the existence of a positive solution with higher regularity $C^1(\overline{\Omega})$, from which, by topological arguments, we deduce that g is decreasing in $]1, p^*[$.
A possible application of Theorem 1 is toward the study of the asymptotic behavior of the pair (λ_q, u_q), which has been addressed by many authors (see for instance Lee [15] and Garcia Azorero and Peral Alonso [9]). Namely we will prove (see Sect. 4) that $\lambda_q \to \lambda_{p^*}$ as $q \to p^*$ (if $p < N$), and that $\lambda_q, \|\nabla u_q\|_p \to +\infty$ as $q \to 0$ (if $|\Omega| < 1$).

2 Preliminaries

In an attempt to make this paper as self-contained as possible, we will recall some well-known results. In the ordered Banach space $W_0^{1,p}(\Omega)$, the positive cone

$$W_+ = \left\{ u \in W_0^{1,p}(\Omega) \ : \ u \geq 0 \text{ a.e. in } \Omega \right\}$$

has empty interior. Instead, in $C_0^1(\overline{\Omega})$ the positive cone

$$C_+ = \left\{ u \in C_0^1(\overline{\Omega}) \ : \ u(x) \geq 0 \quad \text{for all} \quad x \in \Omega \right\}$$

has a nonempty interior, given by

$$\text{int}(C_+) = \left\{ u \in C_0^1(\overline{\Omega}) \ : \ u(x) > 0 \quad \text{for all} \quad x \in \Omega \text{ and } \frac{\partial u}{\partial n}(x) < 0 \quad \text{for all} \quad x \in \partial\Omega \right\}$$

(see Gasiński and Papageorgiou [10, Remark 6.2.10]). We consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{3}$$

where $f : \mathbb{R} \to]0, +\infty[$ is a continuous function. We recall that $\underline{u} \in W^{1,p}(\Omega)$ is a *sub-solution* of problem (3) if $\underline{u} \leq 0$ on $\partial\Omega$ (in the distributional sense) and

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla v \, \mathrm{d}x \leq \int_\Omega f(\underline{u}) v \, \mathrm{d}x \quad \text{for all} \quad v \in W_+,$$

Similarly, $\overline{u}$ is a *super-solution* of (3) if $\overline{u} \geq 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla v \, \mathrm{d}x \geq \int_\Omega f(\overline{u}) v \, \mathrm{d}x \quad \text{for all} \quad v \in W_+.$$

We have the following result for sub- and super-solutions (a slightly rephrased version of Theorem 2.2 of Faria et al. [7]):

Theorem 2 *If the mapping $t \mapsto t^{1-p} f(t)$ is decreasing in $]0, +\infty[$ and $\underline{u}, \overline{u} \in W^{1,p}(\Omega) \cap C^{1,\beta}(\Omega)$ ($0 < \beta \le 1$) are a positive sub-solution and a positive super-solution, respectively, of (3) s.t.*

$$\underline{u}, \overline{u}, \Delta_p \underline{u}, \Delta_p \overline{u}, \underline{u}/\overline{u}, \overline{u}/\underline{u} \in L^\infty(\Omega),$$

then $\underline{u}(x) \le \overline{u}(x)$ for all $x \in \Omega$.

We recall a consequence of the strong nonlinear maximum principle (see Vázquez [20, Theorem 5]):

Theorem 3 *If $u \in C_+ \setminus \{0\}$, $\Delta_p u \in L^2_{\mathrm{loc}}(\Omega)$ and $\Delta_p u \le 0$ a.e. in Ω, then $u \in \mathrm{int}(C_+)$.*

We denote by $\hat{u}_1 \in \mathrm{int}(C_+)$ the unique positive solution of the constant right-hand side problem

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{4}$$

3 Proof of the main result

We will split the proof of Theorem 1 in several steps. We begin by achieving some global properties of the mapping g:

Lemma 4 *The mapping g is Lipschitz continuous on compact subsets of $]0, p^*[$ and nonincreasing in $]0, p^*[$.*

Proof By $\partial B_1(0)$, we denote the unit sphere in $W_0^{1,p}(\Omega)$ centered at 0. For all $u \in \partial B_1(0)$, the mapping $q \mapsto \|u\|_q^q$ is at least twice differentiable in $]0, p^*[$ with

$$\frac{\mathrm{d}^2}{\mathrm{d}q^2}\|u\|_q^q = \int\limits_{\{u \neq 0\}} |u|^q (\ln|u|)^2 \, \mathrm{d}x \ge 0 \quad \text{for all} \quad q \in]0, p^*[,$$

So, $q \mapsto \|u\|_q^q$ is convex for all $u \in \partial B_1(0)$. Now set

$$h(q) = \sup_{u \in \partial B_1(0)} \|u\|_q^q \quad \text{for all} \quad q \in]0, p^*[.$$

The mapping $h :]0, p^*[\to]0, +\infty[$ is convex. By (2), it is easily seen that

$$g(q) = (h(q))^{-\frac{p}{q}} |\Omega|^{\frac{p}{q}} = e^{-\frac{p}{q}\log(|\Omega|^{-1}h(q))} \quad \text{for all} \quad q \in]0, p^*[. \tag{5}$$

Now fix a compact set $K \subset]0, p^*[$. Being convex, h is Lipschitz continuous in K. Since $\min_{q \in K}(|\Omega|^{-1}h(q)) > 0$, the mapping $q \mapsto \log(|\Omega|^{-1}h(q))$ is still Lipschitz continuous in K and from (5), we easily deduce the first part of our claim.
Now, let $0 < r < q < p^*$ be real numbers. For all $u \in \partial B_1(0)$, Hölder inequality yields

$$\|u\|_r^r \le \|u\|_q^r |\Omega|^{\frac{q-r}{q}},$$

so we have $h(r) \le h(q)^{\frac{r}{q}} |\Omega|^{\frac{q-r}{q}}$. Using (5), we easily obtain

$$g(r) \ge g(q).$$

Thus, g is nonincreasing in $]0, p^*[$. □

In order to prove *strict* monotonicity of g, we need to consider separately the intervals $]0, 1]$ and $]1, p^*[$. We first focus on the case $0 < q \le 1$. We say $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \lambda \int_\Omega |u|^{q-2}uv \, dx \tag{6}$$

for all $v \in W_0^{1,p}(\Omega)$ with $\mathrm{supp}(v) \subset \Omega$. We have the following existence result:

Lemma 5 *If* $0 < q \le 1$*, then problem (1) with* $\lambda = \lambda_q$ *admits a positive weak solution* $u_q \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$ *s.t.*

(i) $\|\nabla u_q\|_p = \lambda_q^{\frac{1}{p}}$, $\|u_q\|_q = 1$;
(ii) $c_1\hat{u}_1(x) \le u_q(x) \le c_2(1 + \hat{u}_1(x))$ *for all* $x \in \Omega$ $(c_1, c_2 > 0)$;
(iii) *if* $0 < r < q \le 1$*, then* $u_r \ne u_q$ *on a dense subset of* Ω.

Proof Let (ε_n) be a decreasing sequence in $]0, 1[$ s.t. $\varepsilon_n \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$, we consider the nonsingular problem

$$\begin{cases} -\Delta_p u = (u + \varepsilon_n)^{q-1} & \text{in } \Omega \\ u \ge 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} . \tag{7}$$

We set

$$\varphi_n(u) = \frac{\|\nabla u\|_p^p}{p} - \int_\Omega \left[\frac{1}{q}(u^+ + \varepsilon_n)^q - \varepsilon_n^{q-1}u^-\right] dx \text{ for all } \quad u \in W_0^{1,p}(\Omega)$$

(as usual, we denote $t^\pm = \max\{\pm t, 0\}$). It is easily seen that

$$\frac{d}{dt}\left[\frac{1}{q}(t^+ + \varepsilon_n)^q - \varepsilon_n^{q-1}t^-\right] = (t^+ + \varepsilon_n)^{q-1} \text{ for all } \quad t \in \mathbb{R}.$$

So, $\varphi_n \in C^1(W_0^{1,p}(\Omega))$ and for all $u, v \in W_0^{1,p}(\Omega)$ we have

$$\langle \varphi_n'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx - \int_\Omega (u^+ + \varepsilon_n)^{q-1}v \, dx.$$

In particular, if $u \in W_+$ is a critical point of φ_n, then u is a weak solution of (7). The functional φ_n is coercive and sequentially weakly lower semicontinuous in $W_0^{1,p}(\Omega)$, so there exists $u_n \in W_0^{1,p}(\Omega)$ s.t.

$$\varphi_n(u_n) = \inf_{u \in W_0^{1,p}(\Omega)} \varphi_n(u). \tag{8}$$

It is easily seen that $\varphi_n(u^+) \le \varphi_n(u)$ for all $u \in W_0^{1,p}(\Omega)$, hence we may assume $u_n \in W_+$. So, u_n is a weak solution of (7). Moreover, nonlinear regularity theory (see Gasiński and Papageorgiou [10, Theorem 1.5.5], and Lieberman [15, Theorem 1]) implies $u_n \in C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$. A standard argument shows $u_n \ne 0$. So, we can apply Theorem 3 and obtain $u_n \in \mathrm{int}(C_+)$.

Now we will prove some useful estimates for the function u_n. First, we observe that, for $\rho > 0$ small enough, $\rho\hat{u}_1$ is a sub-solution of (7). Indeed, choose $\rho > 0$ small s.t.

$$\rho^{p-1} < (\rho\hat{u}_1(x) + 1)^{q-1} \quad \text{for all} \quad x \in \Omega.$$

Then, we have for all $v \in W_+$

$$\int_\Omega |\nabla(\rho \hat{u}_1)|^{p-2}\nabla(\rho \hat{u}_1) \cdot \nabla v \, dx = \rho^{p-1} \int_\Omega v \, dx \le \int_\Omega \left(\rho \hat{u}_1 + \varepsilon_n\right)^{q-1} v \, dx.$$

Clearly we can regard u_n as a super-solution of (7). By regularity theory, taking $\beta > 0$ even smaller if necessary, we may assume $\rho \hat{u}_1, u_n \in C^{1,\beta}(\overline{\Omega})$. Moreover, since $u_n \in \mathrm{int}(C_+)$, there exists $r > 0$ s.t. $u_n - r\rho \hat{u}_1 \in \mathrm{int}(C_+)$, in particular

$$u_n(x) - r\rho \hat{u}_1(x) > 0 \quad \text{for all} \quad x \in \Omega,$$

which implies $\rho \hat{u}_1 / u_n \in L^\infty(\Omega)$. Similarly we prove that $u_n/\rho \hat{u}_1 \in L^\infty(\Omega)$. Thus, we apply Theorem 2 and we have

$$\rho \hat{u}_1(x) \le u_n(x) \text{ for all } \quad x \in \Omega \quad (\text{with } \rho > 0 \text{ independent of } n). \tag{9}$$

We set

$$\Omega_n = \{x \in \Omega \, : \, u_n(x) > 1\}.$$

Clearly, $\Omega_n \subset \Omega$ is open with a $C^{1,\beta}$ boundary (due to the regularity of u_n). Without any loss of generality, we may assume that Ω_n is connected and consider the Dirichlet problem:

$$\begin{cases} -\Delta_p u = (u + 1 + \varepsilon_n)^{q-1} & \text{in } \Omega_n \\ u \ge 0 & \text{in } \Omega_n \\ u = 0 & \text{on } \partial\Omega_n \end{cases}. \tag{10}$$

Therefore, $u_n - 1$ is a positive sub-solution of (10). With an argument analogous to that employed above, we prove that $\hat{u}_1$ is a super-solution of (10). An application of Theorem 2 then yields

$$u_n(x) \le \hat{u}_1(x) + 1 \quad \text{for all} \quad x \in \Omega. \tag{11}$$

For all $n \in \mathbb{N}$, we have

$$\|\nabla u_n\|_p^p = \int_\Omega (u_n + \varepsilon_n)^{q-1} u_n \, dx \le \|u_n\|_q^q \le \lambda_q^{-\frac{q}{p}} \|\nabla u_n\|_p^q \text{ (see (2))},$$

hence the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$. Passing to a subsequence, we may assume that there exists $\tilde{u}_q \in W_0^{1,p}(\Omega)$ s.t. $u_n \rightharpoonup \tilde{u}_q$ in $W_0^{1,p}(\Omega)$ and $u_n \to \tilde{u}_q$ in $L^q(\Omega)$. In particular, $u_n(x) \to \tilde{u}_q(x)$ for a.e. $x \in \Omega$, so $\tilde{u}_q \in W_+$.

We shall now prove that $u_n \to \tilde{u}_q$ in $W_0^{1,p}(\Omega)$. First we observe that, since $u_n \rightharpoonup \tilde{u}_q$ in $W_0^{1,p}(\Omega)$,

$$\liminf_n \frac{\|\nabla u_n\|_p^p}{p} \ge \frac{\|\nabla \tilde{u}_q\|_p^p}{p}. \tag{12}$$

We set for all $u \in W_0^{1,p}(\Omega)$

$$\varphi_q(u) = \frac{\|\nabla u\|_p^p}{p} - \frac{\|u\|_q^q}{q},$$

so $\varphi_q : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a continuous (though not differentiable) functional. It is easily seen that

$$\lim_n \varphi_n(u) = \varphi_q(u) \quad \text{for all} \quad u \in W_+. \tag{13}$$

Also, from (8), we have for all $n \in \mathbb{N}$

$$\begin{aligned}\frac{\|\nabla u_n\|_p^p}{p} &= \varphi_n(u_n) + \frac{1}{q}\int_\Omega (u_n+\varepsilon_n)^q \, \mathrm{d}x \\ &\le \varphi_n(\tilde{u}_q) + \frac{1}{q}\int_\Omega (u_n+\varepsilon_n)^q \, \mathrm{d}x.\end{aligned}$$

Hence, by (13) and since $u_n \to \tilde{u}_q$ in $L^q(\Omega)$,

$$\limsup_n \frac{\|\nabla u_n\|_p^p}{p} \le \varphi_q(\tilde{u}_q) + \frac{\|\tilde{u}_q\|_q^q}{q} = \frac{\|\nabla \tilde{u}_q\|_p^p}{p},$$

which, together with (12), gives $\|\nabla u_n\|_p \to \|\nabla \tilde{u}_q\|_p$. This in turn implies $u_n \to \tilde{u}_q$ in $W_0^{1,p}(\Omega)$.

Obviously, (9) and (11) imply

$$\rho \hat{u}_1 \le \tilde{u}_q \le \hat{u}_1 + 1 \text{ a.e. in } \Omega. \tag{14}$$

We prove now that $\tilde{u}_q$ is a weak solution of problem (1) with $\lambda = 1$. Let us fix $v \in W_0^{1,p}(\Omega)$ with $\operatorname{supp}(v) \subset \Omega$. We have for all $n \in \mathbb{N}$

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v \, \mathrm{d}x = \int_\Omega (u_n+\varepsilon_n)^{q-1} v \, \mathrm{d}x.$$

By (9), (11) we can pass to the limit in the above equality as $n \to \infty$ and get

$$\int_\Omega |\nabla \tilde{u}_q|^{p-2}\nabla \tilde{u}_q \cdot \nabla v \, \mathrm{d}x = \int_\Omega \tilde{u}_q^{q-1} v \, \mathrm{d}x. \tag{15}$$

Now we discuss regularity of $\tilde{u}_q$. For any smooth domain Ω' s.t. $\overline{\Omega}' \subset \Omega$, we have from (15)

$$\int_{\Omega'} |\nabla \tilde{u}_q|^{p-2}\nabla \tilde{u}_q \cdot \nabla v \, \mathrm{d}x = \int_{\Omega'} \tilde{u}_q^{q-1} v \, \mathrm{d}x$$

for all $v \in W_0^{1,p}(\Omega)$ with $\operatorname{supp}(v) \subset \Omega'$. By interior regularity theory for degenerate elliptic equations (see Di Benedetto [5, Theorem 2]), we have $\tilde{u}_q \in C^1(\Omega')$, which, as Ω' is arbitrary, implies $\tilde{u}_q \in C^1(\Omega)$.

It is easily seen that

$$\liminf_n \varphi_n(u_n) \ge \varphi_q(\tilde{u}_q). \tag{16}$$

From (8), (13), and (16), we have for all $u \in W_0^{1,p}(\Omega)$

$$\begin{aligned}\varphi_q(u) &= \varphi_q(u^+) + \varphi_q(u^-) \\ &= \lim_n \left(\varphi_n(u^+) + \varphi_n(u^-)\right) \\ &= \lim_n \left[\varphi_n(|u|) + \int_{\{u=0\}} \frac{\varepsilon_n^q}{q} \mathrm{d}x\right] \\ &\ge \liminf_n \varphi_n(u_n) \\ &\ge \varphi_q(\tilde{u}_q).\end{aligned}$$

So,

$$\varphi_q(\tilde{u}_q) = \inf_{u \in W_0^{1,p}(\Omega)} \varphi_q(u). \tag{17}$$

Since u_n solves (7), we have for all $n \in \mathbb{N}$

$$\|\nabla u_n\|_p^p = \int_\Omega (u_n + \varepsilon_n)^{q-1} u_n \, dx.$$

Passing again to the limit as $n \to \infty$, we have

$$\|\nabla \tilde{u}_q\|_p^p = \|\tilde{u}_q\|_q^q. \tag{18}$$

We set

$$T = \left\{ w \in W_0^{1,p}(\Omega) \;:\; w \neq 0, \; \|\nabla w\|_p^p = \|w\|_q^q \right\}.$$

From (17) and (18),

$$\left(\frac{1}{p} - \frac{1}{q}\right) \|\tilde{u}_q\|_q^q \leq \left(\frac{1}{p} - \frac{1}{q}\right) \|w\|_q^q \quad \text{for all} \quad w \in T,$$

So, we have

$$\|\tilde{u}_q\|_q^q = \sup_{w \in T} \|w\|_q^q = \sup_{u \in W_0^{1,p}(\Omega),\; u \neq 0} \left(\frac{\|u\|_q}{\|\nabla u\|_p}\right)^{\frac{pq}{p-q}} = \lambda_q^{\frac{q}{q-p}}.$$

Finally, set $u_q = \tilde{u}_q / \|\tilde{u}_q\|_q$. Clearly $u_q \in C^1(\Omega)$ is a weak solution of (1) with $\lambda = \lambda_q$. Moreover, $\|u_q\|_q = 1$ and by (18) we also have $\|\nabla u_q\|_p = \lambda_q^{\frac{1}{p}}$, so u_q satisfies (i). By (14), we can find $c_1, c_2 > 0$ s.t. (ii) holds. Finally, we prove (iii): let $0 < r < q \leq 1$ be real numbers. We will prove that $u_r \neq u_q$ on a dense subset of Ω, arguing by contradiction. Assume that $u_r = u_q$ in A, where $A \subset \Omega$ is a nonempty open set. For any $v \in W_0^{1,p}(\Omega)$ s.t. $\mathrm{supp}(v) \subset A$, we have by (6)

$$\lambda_q \int u_q^{q-1} v \, dx = \lambda_r \int_A u_q^{r-1} v \, dx,$$

which implies $u_q(x) = (\lambda_r/\lambda_q)^{\frac{1}{q-r}}$ for every $x \in A$, hence $\Delta_p u_q = 0$ a.e. in A, a contradiction. So the proof is concluded. □

Now we can prove strict monotonicity of g in $]0, 1]$:

Lemma 6 *The mapping g is decreasing in* $]0, 1]$.

Proof We know from Lemma 4 that g is nonincreasing. We argue by contradiction, assuming that there exist $0 < r < q \leq 1$ s.t. $g(r) = g(q)$, hence

$$\lambda_r = \lambda_q |\Omega|^{\frac{p}{q} - \frac{p}{r}}.$$

By Lemma 5 (and rescaling), there exist $\check{u}_r, \check{u}_q \in \partial B_1(0)$, $\check{u}_r \neq \check{u}_q$, s.t. $\|\check{u}_r\|_r = \lambda_r^{-\frac{1}{p}}$ and $\|\check{u}_q\|_q = \lambda_q^{-\frac{1}{p}}$. We apply Hölder inequality and the above equality to get

$$\lambda_q^{-\frac{r}{p}} |\Omega|^{1-\frac{r}{q}} = \|\check{u}_r\|_r^r \leq \|\check{u}_r\|_q^r |\Omega|^{1-\frac{r}{q}} \leq \lambda_q^{-\frac{r}{p}} |\Omega|^{1-\frac{r}{q}},$$

so $\|\check{u}_r\|_q = \lambda_q^{-\frac{1}{p}}$. By concavity of the functional $u \mapsto \|u\|_q^q$ (recall that $q \leq 1$), we have

$$\left\| \frac{\check{u}_r + \check{u}_q}{2} \right\|_q^q \geq \lambda_q^{-\frac{q}{p}}.$$

Setting

$$\check{w} = \frac{(\check{u}_r + \check{u}_q)/2}{\|\nabla(\check{u}_r + \check{u}_q)/2\|_p},$$

we obviously have $\check{w} \in \partial B_1(0)$ and

$$\|\check{w}\|_q > \left\| \frac{\check{u}_r + \check{u}_q}{2} \right\|_q \geq \lambda_q^{-\frac{1}{p}},$$

against (2). □

We now turn to the case $1 < q < p^*$: in this case problem (1) can be treated via purely variational methods.

Lemma 7 *If* $1 < q < p^*$, *then problem (1) with* $\lambda = \lambda_q$ *admits a solution* $u_q \in \mathrm{int}(C_+)$ *s.t.*

(i) $\|\nabla u_q\|_p = \lambda_q^{\frac{1}{p}}$ *and* $\|u_q\|_q = 1$;
(ii) if $1 < r < q < p^*$, *then* $u_r \neq u_q$ *on a dense subset of* Ω.

Proof Set

$$\partial B_1^q(0) = \left\{ u \in W_0^{1,p}(\Omega) \ : \ \|u\|_q^q = 1 \right\}.$$

The set $\partial B_1^q(0)$ is sequentially weakly closed in $W_0^{1,p}(\Omega)$. We can rephrase (2) as follows:

$$\lambda_q = \inf_{u \in \partial B_1^q(0)} \|\nabla u\|_p^p. \tag{19}$$

By standard variational arguments, there exists $u_q \in \partial B_1^q(0)$ s.t. $\|\nabla u_q\|_p^p = \lambda_q$. We may assume $u_q \in W_+$ (otherwise we pass to $|u_q|$). Lagrange multipliers theory on Finsler manifolds (see for instance Perera et al. [17], p. 65) implies that there exists $\mu \in \mathbb{R} \setminus \{0\}$ s.t.

$$\int_\Omega |\nabla u_q|^{p-2} \nabla u_q \cdot \nabla v \, dx = \mu \int_\Omega u_q^{q-1} v \, dx \quad \text{for all} \quad v \in W_0^{1,p}(\Omega).$$

Taking $v = u_q$, we have

$$\lambda_q = \|\nabla u_q\|_p^p = \mu \|u_q\|_q^q = \mu.$$

So, u_q is a weak solution of (1) with $\lambda = \lambda_q$ and satisfies (i). Nonlinear regularity theory (see Gasiński and Papageorgiou [10, Theorem 1.5.5], and Lieberman [15, Theorem 1]) implies $u_q \in C_+$. Besides, since $u_q \in \partial B_1^q(0)$, we have $u_q \neq 0$. So, we can apply Theorem 3 and obtain $u_q \in \mathrm{int}(C_+)$.
As in Lemma 5, we can achieve (ii). □

We complete the proof of Theorem 1 by introducing the following result:

Lemma 8 *The mapping* g *is decreasing in* $]1, p^*[$.

Proof Arguing by contradiction, we assume that there exist $1 < r < q < p^*$ s.t. $g(r) = g(q)$. Applying Lemma 7 (and rescaling), we deduce that there exists $\check{u}_r \in \partial B_1(0) \cap \mathrm{int}(C_+)$ s.t.

$$\|\check{u}_r\|_r^r = \lambda_r^{-\frac{r}{p}} = \sup_{u \in \partial B_1(0)} \|u\|_r^r.$$

By Lagrange multipliers theory, there exists $\mu \in \mathbb{R} \setminus \{0\}$ s.t.

$$\int_\Omega |\nabla \check{u}_r|^{p-2} \nabla \check{u}_r \cdot \nabla v \, \mathrm{d}x = \mu \int_\Omega \check{u}_r^{r-1} v \, \mathrm{d}x \quad \text{for all} \quad v \in W_0^{1,p}(\Omega).$$

Since $g(r) = g(q)$, arguing as in Lemma 6, we get

$$\|\check{u}_r\|_q^q = \lambda_q^{-\frac{q}{p}} = \sup_{u \in \partial B_1(0)} \|u\|_q^q.$$

Hence, there exists $\nu \in \mathbb{R} \setminus \{0\}$ s.t.

$$\int_\Omega |\nabla \check{u}_r|^{p-2} \nabla \check{u}_r \cdot \nabla v \, \mathrm{d}x = \nu \int_\Omega \check{u}_r^{q-1} v \, \mathrm{d}x \quad \text{for all} \quad v \in W_0^{1,p}(\Omega).$$

This implies

$$\check{u}_r(x) = \left(\frac{\mu}{\nu}\right)^{\frac{1}{q-r}} \quad \text{for all} \quad x \in \Omega,$$

hence $\Delta_p \check{u}_r = 0$ a.e. in Ω, a contradiction. □

Now, Theorem 1 is proved simply patching together Lemmas 4 (which implies, in particular, that g is continuous in $]0, p^*[$), 6 and 8.

4 Further results

In this final section, we will examine the asymptotic behavior of the mapping g as q approaches either 0 or p^*. First, we prove that (if $p < N$) g admits a continuous extension to $]0, p^*]$. According to the definition of g, we set

$$g(p^*) = \lambda_{p^*} |\Omega|^{\frac{p}{p^*}}.$$

Theorem 9 *If $p < N$, then $g :]0, p^*] \to \mathbb{R}$ is absolutely continuous on compact subsets of $]0, p^*]$ and decreasing in $]0, p^*]$*

Proof First, we prove that

$$\lim_{q \to p^*} g(q) = \lambda_{p^*} |\Omega|^{\frac{p}{p^*}}. \tag{20}$$

Let us fix $p < q < p^*$. By Lemma 7, there exists $u_q \in \mathrm{int}(C_+)$ s.t. $\|\nabla u_q\|_p^p = \lambda_q$ and $\|u_q\|_q^q = 1$. We define $\varphi_q : W_0^{1,p}(\Omega) \to \mathbb{R}$ as in Lemma 5. This time we have $\varphi_q \in C^1(W_0^{1,p}(\Omega))$, and all critical points of φ_q are weak solutions of (1) with $\lambda = 1$. It is easily seen that $\varphi_q(0) = 0$ and 0 is a strict local minimizer of φ_q. Besides, for all $\delta > 0$ we have

$$\varphi_q(\delta u_q) = \frac{\lambda_q \delta^p}{p} - \frac{\delta^q}{q},$$

so $\varphi_q(\delta u_q) \to -\infty$ as $\delta \to +\infty$. So, we take $\delta > 1$ s.t. $\varphi_q(\delta u_q) < 0$ and set

$$\Gamma = \left\{\gamma \in C([0,\delta], W_0^{1,p}(\Omega)) \ : \ \gamma(0) = 0,\ \gamma(\delta) = \delta u_q\right\},$$
$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,\delta]} \varphi_q(\gamma(t)) > 0.$$

By the mountain pass theorem of Ambrosetti and Rabinowitz [1], there exists a critical point $\overline{u}_q \in W_0^{1,p}(\Omega)$ of φ_q s.t. $\varphi_q(\overline{u}_q) = c$. In particular, we have

$$\|\nabla \overline{u}_q\|_p^p = \|\overline{u}_q\|_q^q > 0. \tag{21}$$

We determine the value in (21). By (2), we have

$$\lambda_q^{\frac{1}{p}} \le \frac{\|\nabla \overline{u}_q\|_p}{\|\overline{u}_q\|_q} = \|\nabla \overline{u}_q\|_p^{\frac{q-p}{q}}.$$

Moreover, straightforward computation and (21) lead to

$$\begin{aligned}\left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla \overline{u}_q\|_p^p &= \varphi_q(\overline{u}_q)\\ &\le \max_{t\in[0,\delta]} \varphi_q(tu_q)\\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\lambda_q^{\frac{q}{q-p}}.\end{aligned}$$

So, we perfect (21) getting

$$\|\nabla \overline{u}_q\|_p^p = \|\overline{u}_q\|_q^q = \lambda_q^{\frac{q}{q-p}}. \tag{22}$$

By Garcia Azorero and Peral Alonso [9, Lemma 5], we have

$$\lim_{q\to p^*} \varphi_q(\overline{u}_q) = \frac{\lambda_{p^*}^{\frac{N}{p}}}{N},$$

which, together with (22), yields

$$\lim_{q\to p^*} \lambda_q = \lim_{q\to p^*} \left[\frac{pq}{q-p}\varphi_q(\overline{u}_q)\right]^{\frac{q-p}{q}} = \lambda_{p^*}.$$

Thus, we get (20).

So, g extends to a continuous, decreasing mapping on $]0, p^*]$, which we still denote g. We already know from Lemma 4 that, for all $0 < a < b < p^*$, g is Lipschitz (in particular, absolutely continuous) in $[a, b]$. Moreover, there exists g' almost everywhere in $]0, p^*[$ and for all $0 < a < b < p^*$ we have

$$\int_a^b g'(q)dq = g(b) - g(a). \tag{23}$$

In fact, we can pass to the limit in (23) as $b \to p^*$. Indeed, the left-hand side tends to $\int_a^{p^*} g'(q)dq$ by basic results in measure theory, while the right-hand side tends to $g(p^*)-g(a)$

by (20). So we get

$$\int_a^{p^*} g'(q)dq = g(p^*) - g(a).$$

By classical results in real analysis (see, for instance, Royden and Fitzpatrick [18, Section 6.5,Corollary 12]), g turns out to be absolutely continuous in $[a, p^*]$. This concludes the proof. □

Remark 10 In [6], Ercole, assuming $p < N$, proves that the map $q \mapsto \lambda_q$ is of bounded variation in $[1, p^*]$ (this follows from the monotonicity of g), Lipschitz continuous in any closed interval of the type $[1, p^* - \varepsilon]$ for $\varepsilon > 0$ and left-side continuous at $q = p^*$. Combining these properties, the author obtains that λ_q is absolutely continuous on $[1, p^*]$. The techniques adopted in [6] rely on some formula which describes the dependence of the Rayleigh quotient with respect to the parameter q. Our result extends those of [6] in a twofold sense: p is allowed to be also greater or equal than N, and when $p < N$, λ_q is absolutely continuous in a bigger interval than $[1, p^*]$.

As said in the Introduction, we can apply our results to the study of asymptotic behavior of λ_q and u_q, in the spirit of Lee [14]:

Corollary 11 *If* $|\Omega| < 1$*, then*

$$\lim_{q\to 0} \lambda_q = \lim_{q\to 0} \|\nabla u_q\|_p = +\infty.$$

Proof We clearly have $|\Omega|^{\frac{p}{q}} \to 0$ as $q \to 0$. By Theorem 1

$$\lim_{q\to 0} g(q) = \sup_{0<q<p^*} g(q) > 0,$$

from which the thesis immediately follows. □

We end our study by presenting an open problem: *if* $p \geq N$, *what happens when* $q \to +\infty$? Perhaps the properties of the mapping g can be used, as in Corollary 11, to answer such question.

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