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The equivalence problem of curves in a Riemannian manifold

M. Castrillón López · V. Fernández Mateos · J. Muñoz Masqué

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Abstract The equivalence problem of curves with values in a Riemannian manifold is solved. The domain of validity of Frenet's theorem is shown to be the spaces of constant curvature. For a general Riemannian manifold, new invariants must thus be added. There are two important generic classes of curves: namely Frenet curves and a new class called curves "in normal position". They coincide in dimensions ≤ 4 only. A sharp bound for asymptotic stability of differential invariants is obtained, the complete systems of invariants are characterized, and a procedure of generation is presented.

Keywords Asymptotic stability · Complete systems of invariants · Congruence · Curvatures of a curve · Differential invariant · Frenet frame · Isometry · Killing vector field · Levi-Civita connection · Normal general position · Riemannian metric

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The second author is deceased (March 20, 2011). A Memorial Seminar was delivered at Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, on May 13, 2011; also see, http://www.mat.ucm.es/geomfis/HomenajeVictor.html. We are proud of having been able to work with such an enthusiastic and generous young researcher as Víctor.

M. Castrillón López (✉)
Departamento de Geometría y Topología Facultad de Matemáticas, UCM, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Avda. Complutense s/n, 28040 Madrid, Spain
e-mail: mcastri@mat.ucm.es

V. Fernández Mateos · J. Muñoz Masqué
Instituto de Seguridad de la Información, CSIC, C/ Serrano 144, 28006 Madrid, Spain
e-mail: victor.fernandez@iec.csic.es

J. Muñoz Masqué
e-mail: jaime@iec.csic.es

1 Introduction

A fundamental problem in Riemannian Geometry is that of equivalence of objects in a determined class, namely to provide a criterion to know whether two given objects in this class are congruent under isometries or not. Below, this problem is solved in full generality for the simplest case: That of curves with values in a Riemannian manifold.

For the Euclidean space $\mathbb{R}^m$, the equivalence problem is solved by virtue of the Frenet the theorem: Two curves parameterized by the arc length are congruent if and only if they have the same curvatures, $\kappa_1, \ldots, \kappa_{m-1}$; but the domain of validity of Frenet's theorem is too restrictive. In fact, Frenet's theorem classifies curves in a Riemannian manifold (M, g) if and only if it is of constant curvature. In consequence, in spaces of non-constant curvature, new invariants are required (different from curvatures κ_i) to classify curves and, although by means of curvatures a given curve can be reconstructed (see Theorem 3.6), the role of such invariants becomes weaker in spaces of non-constant curvature, even of low dimension. A generic Riemanniannian metric in a compact manifold admits no isometry other than the identity map (cf. [5,6]). Therefore, the difficulty of the equivalence problem is closely related to the size of the isometry group.

Below, the equivalence problem is solved in general in the framework of real analytic manifolds by means of functions that are invariant under the isometry group of the Riemannian manifold. In this way, a general result is stated for their solution in Theorem 4.3: Essentially, it gives a set of invariants that, together with the classical Frenet curvatures, solves the congruence problem; But it has the inconvenience of using a redundant—in principle, infinite—number of invariants (cf. Remark 4.7). The remarks following its proof (in Sect. 4.3) show, however, that this is the best general result expectable, as simple examples (see Example 4.9) make clear that a solution in closed form to the equivalence problem for C^∞ manifolds is not reachable. Furthermore, certain classes of Riemannian manifolds can be characterized by means of their invariants; e.g. symmetric spaces or Lie groups with invariant metrics.

The study of invariants is developed in Sect. 5, where the main questions on such functions are solved for an arbitrary Riemannian manifold: The theorem of asymptotic stability (Theorem 5.4 and Corollary 5.5), the completeness theorem (Theorem 5.10) that allows us to solve the general problem of equivalence by means of a complete system of invariants and the theorem of generation of invariants (Theorem 5.12). An interesting consequence of the generating theorem proves that the ring of invariants can be generated by means of m invariants (where $m = \dim M$) by taking successive total derivatives with respect to t.

In [10] the number of differential invariants with respect to the induced operation of the group G on jet bundles $J^r(\mathbb{R}, G/H)$ of the homogeneous space G/H is calculated without assuming G is the group of isometries of a metric. According to [14, IV, Example 1.3], if the subgroup H is compact, the quotient manifold G/H admits a Riemannian metric left invariant. The converse statement also holds true, as the isotropy subgroup of a point in a Riemannian manifold is compact (cf. [14, I, Corollary 4.8]). Moreover, it should be noted that most part of the results in [10] hold in general, i.e. without assuming the manifold to be Riemannian homogeneous, as shown in Theorem 5.12 and Remark 5.13 below.

In Sect. 3.3, two basic existence theorems for the generic class $\mathcal{F}$ of Frenet curves are stated. The first result (Theorem 3.6) is a generalization to arbitrary Riemannian manifolds of the existence of curves in Euclidean 3-space with given curvature and torsion, but the second one (Theorem 3.7) is completely new.

In addition to Frenet curves, another generic class $\mathcal{N}$ of curves in a Riemannian manifold is introduced in Definition 2.3, which seems to be the natural setting for the statement of the asymptotic stability theorem (Theorem 5.4). The classes $\mathcal{F}$ and $\mathcal{N}$ are compared in detail in Sect. 3.4. As it is proved in Theorem 3.8, if either $\dim M = m \leq 4$ or g is flat at a neighbourhood of x_0, then $\mathcal{F}^{m-1}_{t_0,x_0}(M) = \mathcal{N}^{m-1}_{t_0,x_0}(M)$, $(t_0, x_0) \in \mathbb{R} \times M$. In general, however, both generic sets of curves do not coincide, as shown in the example 3.9.

We want to note the connection of this article with other different frameworks on the techniques of moving frames as those in the interesting papers [22,23] and the references therein. The approach in these references is different and, in particular, is strongly oriented to other general geometric settings dealing with other groups of invariance.

2 General position

2.1 Definitions

Definition 2.1 A smooth curve $\sigma : (a, b) \to M$ taking values into a manifold M endowed with a linear connection ∇ is said to be in *general position up to the order* r, for $1 \leq r \leq m = \dim M$, at $t_0 \in (a, b)$ if the vector fields $T^\sigma, \nabla_{T^\sigma} T^\sigma, \dots, \nabla^{r-1}_{T^\sigma} T^\sigma$ along σ are linearly independent at t_0, where T^σ is the tangent field to σ. The curve σ is in general position up to the order r if it is in general position up to this order for every $t \in (a, b)$.

Geometrically, a curve in general position is as twisted as possible. For example, if (M, g) is of constant curvature, then σ is in general position up to order r at $\sigma(t_0)$ if and only if no neighbourhood $\{\sigma(t) : |t - t_0| < \varepsilon\}$ is contained into an auto-parallel submanifold (cf. [14, VII, Section 8]) of M of dimension $< r$. The condition of being in general position up to first order is none other than an immersion, and hence, it is independent of ∇; but for $r \geq 2$, the condition of being in general position up to order r does depend on ∇.

Lemma 2.2 *Let $\sigma : (a, b) \to M$ be a smooth curve taking values into a manifold M endowed with a linear connection ∇. If $(x^1, \dots, x^m)$ is a normal coordinate system with respect to ∇ centred at $x_0 = \sigma(t_0)$, $a < t_0 < b$, then the tangent vectors*

$$U^{\sigma,k}_{t_0} = \frac{d^k(x^i \circ \sigma)}{dt^k}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\sigma(t_0)} \in T_{\sigma(t_0)}M, \quad k \geq 1, k \in \mathbb{N}, \tag{1}$$

do not depend on the particular normal coordinates chosen.

Proof If $x'^i = a^i_j x^j$, $A = (a^i_j) \in Gl(m, \mathbb{R})$, is another normal coordinate system, then

$$\frac{\partial}{\partial x'^i} = b^h_i \frac{\partial}{\partial x^h}, \quad (b^h_i) = A^{-1},$$

and hence

$$\begin{aligned} \frac{d^k(x'^i \circ \sigma)}{dt^k}(t_0) \left.\frac{\partial}{\partial x'^i}\right|_{\sigma(t_0)} &= \frac{d^k(a^i_j x^j \circ \sigma)}{dt^k}(t_0) b^h_i \left.\frac{\partial}{\partial x^h}\right|_{\sigma(t_0)} \\ &= b^h_i a^i_j \frac{d^k(x^j \circ \sigma)}{dt^k}(t_0) \left.\frac{\partial}{\partial x^h}\right|_{\sigma(t_0)} \\ &= \frac{d^k(x^j \circ \sigma)}{dt^k}(t_0) \left.\frac{\partial}{\partial x^j}\right|_{\sigma(t_0)}. \end{aligned}$$

□

Definition 2.3 A curve σ is said to be in *normal general position up to the order* r at $t_0 \in (a,b)$ if the tangent vectors $U_{t_0}^{\sigma,1}, U_{t_0}^{\sigma,2}, \dots, U_{t_0}^{\sigma,r}$ are linearly independent. The curve σ is in normal general position up to the order r if it is in normal general position up to this order for every $t \in (a,b)$.

2.2 Genericity results

Lemma 2.4 *Let* $(U; x^1, \dots, x^m)$ *be a coordinate open domain in a smooth manifold* M *endowed with a linear connection* ∇. *There exist smooth functions*

$$F^{k,i}\colon J^k(\mathbb{R}, U) \to \mathbb{R}, \qquad k \in \mathbb{N},\ 1 \le i \le m,$$

such that,

$$\left(\nabla^k_{T^\sigma} T^\sigma\right)_t = \left(\frac{d^{k+1}(x^i \circ \sigma)}{dt^{k+1}}(t) + F^{k,i}\left(j^k_t \sigma\right)\right) \left.\frac{\partial}{\partial x^i}\right|_{\sigma(t)}, \tag{2}$$

for every curve $\sigma : \mathbb{R} \to U$ *and every* $t \in \mathbb{R}$, *which are determined as follows:*

$$F^{0,i} = 0, \tag{3}$$

$$F^{1,i} = \sum_{j,h=1}^{m} \Gamma^i_{jh} x^j_1 x^h_1, \tag{4}$$

where Γ^i_{jh} *are the local symbols of* ∇ *in* $(U; x^1, \dots, x^m)$, *and*

$$F^{k,i} = D_t\left(F^{k-1,i}\right) + \sum_{h,j=1}^{m} \Gamma^i_{hj} x^j_1 \left(x^h_k + F^{k-1}_h\right), \quad \forall k \ge 2, \tag{5}$$

$(x^h_l)^{1\le h\le m}_{0\le l\le k}$ *being the coordinates induced by* $(x^i)^m_{i=1}$ *in the* k*-jet bundle, i.e.*

$$x^h_l\left(j^k_t \sigma\right) = \frac{d^l(x^h \circ \sigma)}{dt^l}(t), \quad x^h_0 = x^h, \qquad 0 \le l \le k,\ 1 \le h \le m,$$

and D_t *denotes the "total derivative" with respect to* t, *namely*

$$D_t = \frac{\partial}{\partial t} + \sum_{r=0}^{\infty} x^i_{r+1} \frac{\partial}{\partial x^i_r}.$$

Proposition 2.5 *Let* M *be a smooth manifold of dimension* m *endowed with a linear connection* ∇. *The set of curves in general position up to the order* $r \le m-1$ *is a dense open subset in* $C^\infty(\mathbb{R}, M)$ *with respect to the strong topology.*

Proof By using the formulas (2), it follows that the mapping

$$\begin{aligned} &\Phi^r_\nabla\colon J^r(\mathbb{R}, M) \to \mathbb{R} \times (\oplus^r TM), \\ &\Phi^r_\nabla\left(j^r_{t_0}\sigma\right) = \left(t_0; T^\sigma_{t_0}, (\nabla_{T^\sigma} T^\sigma)_{t_0}, \dots, \left(\nabla^{r-1}_{T^\sigma} T^\sigma\right)_{t_0}\right), \end{aligned} \tag{6}$$

is a diffeomorphism inducing the identity on $J^0(\mathbb{R}, M) = \mathbb{R} \times M$. We set

$$E = \left\{\left(t, X^1, \dots, X^r\right) \in \mathbb{R} \times \left(\oplus^r TM\right) : X^1 \wedge \dots \wedge X^r = 0\right\},$$

for every $1 \leq k \leq r-1$ and for every strictly increasing system of indices $1 \leq i_1 < \cdots < i_k \leq r$ we set

$$E_{i_1,\dots,i_k} = \left\{ \begin{array}{r} \left(t, X^1, \dots, X^r\right) \in \mathbb{R} \times (\oplus^r TM) : X^{i_1} \wedge \cdots \wedge X^{i_k} \neq 0, \\ X^{j_1}, \dots, X^{j_{r-k}} \in \left\langle X^{i_1}, \dots, X^{i_k} \right\rangle, \end{array} \right\}$$

with $j_1 < \cdots < j_{r-k}$ and $\{j_1, \dots, j_{r-k}\} = \{1, 2, \dots, r\} \setminus \{i_1, \dots, i_k\}$, and finally we set $E_0 = \mathbb{R} \times Z$, Z being the zero section in $\oplus^r TM$. Hence

$$E = E_0 \cup \bigcup_{k=1}^{r-1} \bigcup_{i_1 < \cdots < i_k} E_{i_1,\dots,i_k}.$$

Moreover, if $U_k(M) \subset \oplus^k TM$ denotes the open subset of all linearly independent systems of k vectors, then the mapping

$$\begin{aligned} & A_{i_1,\dots,i_k} : \mathbb{R}^{1+k(r-k)} \times U_k(M) \to \mathbb{R} \times \left(\oplus^r TM\right), \\ & A_{i_1,\dots,i_k}\left(t; \lambda_1^1, \dots, \lambda_k^1, \dots, \lambda_1^{r-k}, \dots, \lambda_k^{r-k}; X^1, \dots, X^k\right) = \left(t; \bar{X}^1, \dots, \bar{X}^r\right), \\ & \bar{X}^{i_h} = X^h, \quad 1 \leq h \leq k, \\ & \bar{X}^{j_h} = \sum_{i=1}^{k} \lambda_i^h X^i, \, 1 \leq h \leq r-k, \end{aligned}$$

is an injective immersion such that $\operatorname{im}(A_{i_1,\dots,i_k}) = E_{i_1,\dots,i_k}$, and we have

$$\begin{aligned} \operatorname{codim} E_{i_1,\dots,i_k} &= \dim\left(\mathbb{R} \times \left(\oplus^r TM\right)\right) - \dim E_{i_1,\dots,i_k} \\ &= (1+m+rm) - (1+k(r-k)+m+km) \\ &= (m-k)(r-k) \\ &\geq m+1-r, \end{aligned}$$

as the product $(m-k)(r-k)$ takes its minimum value when k takes its maximum value, i.e. $k = r-1$. Accordingly, $Y_{i_1,\dots,i_k} = \left(\Phi_\nabla^r\right)^{-1}\left(E_{i_1,\dots,i_k}\right)$ is a submanifold in $J^r(\mathbb{R}, M)$ of codimension $(m-k)(r-k)$. From Thom's transversality theorem (e.g. see [28, VII, Théorème 4.2]) the set of curves $\sigma : \mathbb{R} \to M$ the r-jet extension of which, $j^r\sigma$, is transversal to $Y_{i_1,\dots,i_k}$ is a residual subset (and hence dense) in $C^\infty(\mathbb{R}, M)$ for the strong topology. For such curves, $(j^r\sigma)^{-1}\left(Y_{i_1,\dots,i_k}\right)$ is a submanifold of the real line of codimension $(m-k)(r-k) \geq m+1-r$. If $r \leq m-1$, then it is only possible if such a submanifold is the empty set. Consequently, for $r \leq m-1$, the following formula holds:

$$\begin{aligned} \mathcal{F}^r &= \left\{\sigma \in C^\infty(\mathbb{R}, M) : j^r\sigma \text{ is transversal to every } Y_{i_1,\dots,i_k}\right\} \\ &= \left\{\sigma \in C^\infty(\mathbb{R}, M) : \left(j^r\sigma\right)(\mathbb{R}) \cap Y = \emptyset\right\}, \end{aligned}$$

where $Y = Y_0 \cup \bigcup_{k=1}^{r-1} \bigcup_{i_1 < \cdots < i_k} Y_{i_1,\dots,i_k}$. Therefore, $\Phi_\nabla^r\left(j^r\sigma(\mathbb{R})\right) \cap E = \emptyset$ if $\sigma \in \mathcal{F}^r$; in other words, σ is a curve in general position up to order r with respect to ∇.

Finally, we prove that $\mathcal{F}^r$ is an open susbet. If d is a complete distance function defining the topology in $J^r(\mathbb{R}, M)$, then for every $\sigma \in \mathcal{F}^r$ the function $\delta_\sigma : \mathbb{R} \to \mathbb{R}^+$, $\delta_\sigma(t) = d\left(j_t^r\sigma, Y\right) > 0$ makes sense as Y is a closed subset and

$$N(\sigma) = \left\{\gamma \in C^\infty(\mathbb{R}, M) : d\left(j_t^r\sigma, j_t^r\gamma\right) < \delta_\sigma(t), \forall t \in \mathbb{R}\right\}$$

is a neigbourhood of σ in the strong topology of order r and hence, also in the strong topology of order ∞. As $\gamma \in N(\sigma)$ implies $\gamma \in \mathcal{F}^r$, we can conclude. □

Remark 2.6 The statement of Proposition 2.5 is the best possible, as the curves in general position up to the order $m = \dim M$ with respect to a linear connection ∇ are not dense in $C^\infty(\mathbb{R}, M)$ for the strong topology, because inflection points are unavoidable. In fact, with the similar notations as in the proof of Proposition 2.5, we set

$$\mathcal{F}^m = \left\{\sigma \in C^\infty(\mathbb{R}, M) : j^m\sigma(\mathbb{R}) \cap Y = \emptyset\right\},$$
$$\bar{\mathcal{F}}^m = \left\{\sigma \in C^\infty(\mathbb{R}, M) : j^m\sigma \text{ is transversal to every } Y_{i_1,\ldots,i_k}\right\}.$$

The set $\bar{\mathcal{F}}^m$ is dense in $C^\infty(\mathbb{R}, M)$ as it is residual and $\mathcal{F}^m$ coincides with the set of curves in general position up to order m. In order to prove that $\mathcal{F}^m$ is not dense, it suffices to obtain an open subset contained in its complementary set. We set

$$Y' = \bigcup_{k=0}^{m-2} \bigcup_{i_1<\cdots<i_k} Y_{i_1,\ldots,i_k}; \qquad Y_i = \left(\Phi^m_\nabla\right)^{-1}(E_i), \quad 1 \le i \le m,$$

where E_i is the set of points $(t, X^1, \ldots, X^m) \in \mathbb{R} \times (\oplus^m TM)$ such that,

(i) $X^1 \wedge \cdots \wedge \widehat{X^i} \wedge \cdots \wedge X^m \neq 0$,
(ii) $X^i \in \left\langle X^1, \ldots, \widehat{X^i}, \ldots, X^m\right\rangle$.

Then, $Y = Y' \cup Y_1 \cup \cdots \cup Y_m$ and $Y_0 = Y_1 \cap \cdots \cap Y_m$ is an open subset in each Y_i; hence, Y_0 is a submanifold of codimension 1 in $J^m(\mathbb{R}, M)$. According to a classical result (see [20, Theorem 6.1]), there exists a curve $\sigma : \mathbb{R} \to M$ such that, 1) $j^m\sigma$ is transversal to Y_0, and 2) $j^m\sigma(\mathbb{R}) \cap Y_0 \neq \emptyset$. Therefore, $j^m\sigma(\mathbb{R}) \cap Y \neq \emptyset$. Moreover, according to [18, Lemma 1, p. 45]], given a neighbourhood U of t, there exists a neigbourhood E_σ of σ in the weak (and hence, in the strong) topology, such that $\tau \in E_\sigma$ implies $j^m\tau$ cuts transversally to Y at some point $t' \in U$. Hence, $\tau \in E_\sigma$ implies $j^m\tau(\mathbb{R}) \cap Y \neq \emptyset$, i.e. $\tau \notin \mathcal{F}^m$ and σ thus possesses a neighbourhood of curves not belonging to $\mathcal{F}^m$.

Proposition 2.7 *Let M be a smooth manifold of dimension m endowed with a linear connection ∇. The set of curves in normal general position up to the order $r \le m-1$ is a dense open subset in $C^\infty(\mathbb{R}, M)$ with respect to the strong topology.*

Proof It is similar to the proof of Proposition 2.5 by using the fact that the mapping

$$\Psi^r_\nabla : J^r(\mathbb{R}, M) \to \mathbb{R} \times (\oplus^r TM),$$
$$\Psi^r_\nabla\left(j^r_{t_0}\sigma\right) = \left(t_0; U^{1,\sigma}_{t_0}, U^{2,\sigma}_{t_0}, \ldots, U^{r,\sigma}_{t_0}\right),$$

is a diffeomorphism over $\mathbb{R} \times M$. □

3 Frenet curves

3.1 A Frenet curve defined

Definition 3.1 A curve $\sigma : (a, b) \to M$ with values into a Riemannian manifold (M, g) is said to be a *Frenet curve* if σ is in general position up to order $m-1$ with respect to the Levi-Civita connection of the metric g.

Proposition 3.2 (Frenet frame, [3,7,8,11,13,25]) *If (M, g) is an oriented connected Riemannian manifold of dimension m and $\sigma : (a, b) \to M$ is a Frenet curve, then there exist unique vector fields $X^\sigma_1, \ldots, X^\sigma_m$ defined along σ and smooth functions $\kappa^\sigma_0, \ldots, \kappa^\sigma_{m-1} : (a, b) \to \mathbb{R}$ with $\kappa^\sigma_j > 0$, $0 \le j \le m-2$, such that,*

(i) $(X_1^\sigma(t), \dots, X_m^\sigma(t))$ *is a positively oriented orthonormal linear frame,* $\forall t \in (a, b)$.

(ii) *The systems* $(X_1^\sigma(t), \dots, X_i^\sigma(t))$, $(T_t^\sigma, (\nabla_{T^\sigma} T^\sigma)_t, \dots, (\nabla_{T^\sigma}^{i-1} T^\sigma)_t)$ *span the same vector subspace, and they are equally oriented for every* $1 \le i \le m-1$ *and every* $t \in (a, b)$.

(iii) *The following formulas hold:*

(a) $T^\sigma = \kappa_0^\sigma X_1$,

(b) $\nabla_{X_1^\sigma} X_1^\sigma = \kappa_1^\sigma X_2^\sigma$,

(c) $\nabla_{X_1^\sigma} X_i^\sigma = -\kappa_{i-1}^\sigma X_{i-1}^\sigma + \kappa_i^\sigma X_{i+1}^\sigma, \quad 2 \le i \le m-1$,

(d) $\nabla_{X_1^\sigma} X_m^\sigma = -\kappa_{m-1}^\sigma X_{m-1}^\sigma$.

Definition 3.3 The frame $(X_1^\sigma, \dots, X_m^\sigma)$ along σ determined by the conditions (i)–(iii) above is called the *Frenet frame* of σ, and the functions $\kappa_0^\sigma, \dots, \kappa_{m-1}^\sigma$ are the *curvatures* of σ.

3.2 Basic formulas

According to the item (ii) of Proposition 3.2, there exist functions $f_{ij}^\sigma \in C^\infty(a, b)$, $1 \le i \le j \le m$, such that,

$$\nabla_{T^\sigma}^{j-1} T^\sigma = \sum_{i=1}^{j} f_{ij}^\sigma X_i^\sigma, \quad 1 \le j \le m, \tag{7}$$

and by using the equations (a)–(d) in the item (iii) above the following recurrence formulas are obtained for these functions:

$$\begin{cases} f_{11}^\sigma = \kappa_0^\sigma, \\ f_{12}^\sigma = \dfrac{df_{11}^\sigma}{dt}, \\ f_{22}^\sigma = f_{11}^\sigma \kappa_0^\sigma \kappa_1^\sigma, \end{cases} \tag{8}$$

$$3 \le j \le m \quad \begin{cases} f_{1j}^\sigma = \dfrac{df_{1,j-1}^\sigma}{dt} - f_{2,j-1}^\sigma \kappa_0^\sigma \kappa_1^\sigma, \\ f_{ij}^\sigma = \dfrac{df_{i,j-1}^\sigma}{dt} - f_{i+1,j-1}^\sigma \kappa_0^\sigma \kappa_i^\sigma + f_{i-1,j-1}^\sigma \kappa_0^\sigma \kappa_{i-1}^\sigma, \\ 2 \le i \le j-2, \\ f_{j-1,j}^\sigma = \dfrac{df_{j-1,j-1}^\sigma}{dt} + f_{j-2,j-1}^\sigma \kappa_0^\sigma \kappa_{j-2}^\sigma, \\ f_{jj}^\sigma = f_{j-1,j-1}^\sigma \kappa_0^\sigma \kappa_{j-1}^\sigma. \end{cases} \tag{9}$$

Proposition 3.4 *If* $\sigma : (a, b) \to M$ *is a Frenet curve in an oriented connected Riemannian manifold* (M, g)*, then*

$$\kappa_0^\sigma = \sqrt{\Delta_1^\sigma},$$

$$\kappa_1^\sigma = \sqrt{\frac{\Delta_2^\sigma}{(\Delta_1^\sigma)^3}},$$

$$\kappa_i^\sigma = \frac{\varepsilon_i \sqrt{\Delta_{i-1}^\sigma \Delta_{i+1}^\sigma}}{\sqrt{\Delta_1^\sigma} \Delta_i^\sigma}, \quad 2 \le i \le m-1,$$

where

$$\begin{cases} \Delta_k^\sigma = \det \left(g \left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma \right) \right)_{i,j=1}^{k}, \\ \varepsilon_i = 1 \text{ for } 2 \le i \le m-2, \text{ and } \varepsilon_{m-1} = \pm 1. \end{cases} \tag{10}$$

Proof The formulas in the statement follow from (7), (8), and (9) taking the identity $\Delta_k^\sigma = (\det(f_{ij}^\sigma)_{i,j=1}^k)^2 = \prod_{i=1}^k (f_{ii}^\sigma)^2$ into account. □

For a smooth curve σ, the property of being a Frenet curve at t depends on $j_t^{m-1}\sigma$ only; hence, for every $t \in \mathbb{R}$, we can speak about the open subset $\mathcal{F}_t^{m-1}(M) \subset J_t^{m-1}(\mathbb{R}, M)$ of Frenet jets. Let

$$f_{ij} \colon (\pi_{m-1}^m)^{-1}(\mathcal{F}^{m-1}(M)) \to \mathbb{R}, \quad 1 \leq i \leq j \leq m,$$

be the mapping defined by $f_{ij}(j_t^m\sigma) = f_{ij}^\sigma(t)$, $\pi_l^k \colon J^k(\mathbb{R}, M) \to J^l(\mathbb{R}, M)$, $k \geq l$, being the canonical projections. Similarly, let

$$\Delta_k \colon J^k(\mathbb{R}, M) \to \mathbb{R}, \quad 1 \leq k \leq m, \tag{11}$$

be the mapping given by $\Delta_k(j_t^k\sigma) = \Delta_k^\sigma(t)$, which is well defined according to the formula (10).

Proposition 3.5 *If* $\sigma \colon (a, b) \to M$, $\bar{\sigma} \colon (a, b) \to \bar{M}$ *are two Frenet curves with values in Riemannian manifolds* (M, g), $(\bar{M}, \bar{g})$, *then* $\left|\kappa_i^\sigma\right| = \left|\kappa_i^{\bar{\sigma}}\right|$, $0 \leq i \leq m-1$, *if and only if,*

$$g\left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma\right) = \bar{g}\left(\bar{\nabla}_{T^{\bar{\sigma}}}^{i-1} T^{\bar{\sigma}}, \bar{\nabla}_{T^{\bar{\sigma}}}^{j-1} T^{\bar{\sigma}}\right), \quad i, j = 1, \ldots, m, \tag{12}$$

∇, $\bar{\nabla}$ *being the Levi-Civita connections associated with* g, $\bar{g}$.

Proof If (12) holds for every $i, j = 1, \ldots, m$, then $\Delta_k^\sigma = \Delta_k^{\bar{\sigma}}$ for $1 \leq k \leq m$. From the formulas (10) in Proposition 3.4 we deduce $\kappa_i^\sigma = \kappa_i^{\bar{\sigma}}$ for $0 \leq i \leq m-2$ and $|\kappa_{m-1}^\sigma| = |\kappa_{m-1}^{\bar{\sigma}}|$. Conversely, if $\left|\kappa_i^\sigma\right| = \left|\kappa_i^{\bar{\sigma}}\right|$, $0 \leq i \leq m-1$, then from the formulas (8) and (9) by recurrence on the subindex of κ_i we obtain $\left|f_{mm}^\sigma\right| = \left|f_{mm}^{\bar{\sigma}}\right|$ and $f_{ij}^\sigma = f_{ij}^{\bar{\sigma}}$ otherwise. Hence, for every $i, j = 1, \ldots, m$, we have

$$\begin{aligned} g\left(\nabla_{T^\sigma}^{i-1} T^\sigma, \nabla_{T^\sigma}^{j-1} T^\sigma T\right) &= \sum_{k=1}^{i}\sum_{l=1}^{j} f_{ki}^\sigma f_{lj}^\sigma \delta_{kl} = \sum_{k=1}^{i}\sum_{l=1}^{j} f_{ki}^{\bar{\sigma}} f_{lj}^{\bar{\sigma}} \delta_{kl} \\ &= \bar{g}\left(\bar{\nabla}_{T^{\bar{\sigma}}}^{i-1} T^{\bar{\sigma}}, \bar{\nabla}_{T^{\bar{\sigma}}}^{j-1} T^{\bar{\sigma}}\right). \end{aligned}$$

□

3.3 Existence theorems

The fundamental theorem for curves in Euclidean 3-space (cf. [26, pp. 29–31]) states that if two smooth functions $\kappa(s) > 0$, $\tau(s)$ are given, then there exists a unique curve for which s is the arc length, κ the curvature, and τ the torsion, the moving trihedron of which at $s = s_0$ coincides with the coordinate axes. The full generalization of this result is as follows:

Theorem 3.6 *Let* (M, g) *be an* m*-dimensional oriented Riemannian manifold and let* $(v_1, \ldots, v_m)$ *be a positively oriented orthonormal basis for* $T_{x_0}M$. *Given functions* $\kappa_j \in C^\infty(t_0 - \delta, t_0 + \delta)$, $0 \leq j \leq m-1$, *with* $\kappa_j > 0$ *for* $0 \leq j \leq m-2$, *there exists* $0 < \varepsilon \leq \delta$ *and a unique Frenet curve* $\sigma \colon (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ *such that,*

(i) $\sigma(t_0) = x_0$,
(ii) $X_j^\sigma(t_0) = v_j$ *for* $1 \leq j \leq m$,
(iii) $\kappa_j^\sigma = \kappa_j$ *for* $0 \leq j \leq m-1$.

Proof Let $(U;\ x^1,\dots,x^m)$ be the normal coordinate system centred at x_0 associated with the orthonormal linear frame $(v_1,\dots,v_m)$ given in the statement, let $p^m_M\colon \oplus^m TM \to M$ be the bundle projection, and let denote by $(x^i, y^j_k),\ i,j,k=1,\dots,m$, the induced coordinate system on $(p^m_M)^{-1}(U)$, i.e.

$$u_j = y^i_j(u)\,\frac{\partial}{\partial x^i}\bigg|_x, \qquad \forall u=(u_1,\dots,u_m)\in \oplus^m T_xM,\ x\in U.$$

First of all, we prove that the Frenet formulas are locally equivalent to a system of first-order ordinary differential equations on the manifold $\oplus^m TM$. In fact, as a computation shows, the formulas (a)–(d) in Proposition 3.2 can be written in local coordinates as follows:

$$\frac{d(x^j\circ\sigma)}{dt} = \kappa^\sigma_0(y^j_1\circ X^\sigma), \tag{13}$$

$$\frac{d(y^j_1\circ X^\sigma)}{dt} = \kappa^\sigma_0\kappa^\sigma_1(y^j_2\circ X^\sigma) - \kappa^\sigma_0(\Gamma^j_{hi}\circ\sigma)(y^h_1\circ X^\sigma)(y^i_1\circ X^\sigma), \tag{14}$$

$$\frac{d(y^c_i\circ X^\sigma)}{dt} = \kappa^\sigma_0[\kappa^\sigma_i(y^c_{i+1}\circ X^\sigma) - \kappa^\sigma_{i-1}(y^c_{i-1}\circ X^\sigma)] -\kappa^\sigma_0\left(\Gamma^c_{ab}\circ\sigma\right)(y^a_1\circ X^\sigma)(y^b_i\circ X^\sigma),\quad 2\le i\le m-1, \tag{15}$$

$$\frac{d(y^c_m\circ X^\sigma)}{dt} = -\kappa^\sigma_0\kappa^\sigma_{m-1}(y^c_{m-1}\circ X^\sigma) - \kappa^\sigma_0\left(\Gamma^c_{ab}\circ\sigma\right)(y^a_1\circ X^\sigma)(y^b_m\circ X^\sigma), \tag{16}$$

where Γ^c_{ab} are the components of the Levi-Civita connection ∇ of g with respect to the coordinate system $(x^h)^m_{h=1}$ and $X^\sigma\colon (a,b)\to\oplus^m TM$, $a<t_0<b$, is the curve given by $X^\sigma(t) = t(X^\sigma_1(t),\dots,X^\sigma_m(t))$, $\forall t\in(a,b)$.

Hence, the functions $x^h\circ\sigma,\ y^i_j\circ X^\sigma\colon (a,b)\to\mathbb{R}$, $h,i,j=1,\dots,m$, are the only solutions to the system (13)–(16) satisfying the initial conditions (i), (ii) in the statement; i.e. $(x^h\circ\sigma)(t_0) = x^h(x_0),\ (y^i_j\circ X^\sigma)(t_0) = y^i_j(v_1,\dots,v_m) = \delta^i_j$.

Conversely, if X^σ and the curvatures κ^σ_{j-1}, $1\le j\le m$, are replaced by an arbitrary smooth curve $X=(X_1,\dots,X_m)\colon (t_0-\delta,t_0+\delta)\to\oplus^m TM$, and the given functions κ_{j-1}, $1\le j\le m$, respectively, with $\sigma = p^m_M\circ X$, into the equations (13)–(16) above, then the following system is obtained:

$$\frac{d(x^j\circ\sigma)}{dt} = \kappa_0(y^j_1\circ X), \tag{17}$$

$$\frac{d(y^j_1\circ X)}{dt} = \kappa_0\kappa_1(y^j_2\circ X) - \kappa_0(\Gamma^j_{hi}\circ\sigma)(y^h_1\circ X)(y^i_1\circ X), \tag{18}$$

$$\frac{d(y^c_i\circ X)}{dt} = \kappa_0[\kappa_i(y^c_{i+1}\circ X) - \kappa_{i-1}(y^c_{i-1}\circ X)] -\kappa_0\left(\Gamma^c_{ab}\circ\sigma\right)(y^a_1\circ X)(y^b_i\circ X),\quad 2\le i\le m-1, \tag{19}$$

$$\frac{d(y^c_m\circ X)}{dt} = -\kappa_0\kappa_{m-1}(y^c_{m-1}\circ X) - \kappa_0\left(\Gamma^c_{ab}\circ\sigma\right)(y^a_1\circ X)(y^b_m\circ X), \tag{20}$$

We claim that the only solution $x^h\circ\sigma,\ y^i_j\circ X\colon (t_0-\varepsilon,t_0+\varepsilon)\to\mathbb{R}$ to the system (17)–(20) satisfying the initial conditions

$$(x^h\circ\sigma)(t_0) = x^h(x_0),\ 1\le h\le m;\quad (y^i_j\circ X)(t_0) = \delta^i_j,\ i,j=1,\dots,m,$$

provides the desired Frenet curve.

First, we observe that from the very definition of (17)–(20), the linear frame $(X_1,\dots,X_m)$ —defined along the curve σ with components $x^h\circ\sigma$—determined by X, i.e. $X_j=(y^i_j\circ X)\frac{\partial}{\partial x^i}$, satisfies the following equations:

$$\begin{cases} T^\sigma = \kappa_0 X_1, \\ \nabla_{X_1}X_1 = \kappa_1 X_2, \\ \nabla_{X_1}X_i = -\kappa_{i-1}X_{i-1}+\kappa_i X_{i+1},\ 2\le i\le m-1, \\ \nabla_{X_1}X_m = -\kappa_{m-1}X_{m-1}. \end{cases} \tag{21}$$

Next, the item (i) in Proposition 3.2 is proved to hold for this linear frame. In fact, the functions $\varphi_{ij}(t)=g(X_i(t),X_j(t))$, $|t-t_0|<\varepsilon,\ 1\le i\le j\le m$, are the only solution to the system

$$\begin{aligned}
\frac{d\varphi_{11}}{dt} &= 2\kappa_0\kappa_1\varphi_{12},\\
\frac{d\varphi_{1j}}{dt} &= \kappa_0\left(\kappa_1\varphi_{2j}-\kappa_{j-1}\varphi_{1,j-1}+\kappa_j\varphi_{1,j+1}\right),\\
&\quad 2\le j\le m-1,\\
\frac{d\varphi_{1m}}{dt} &= \kappa_0\left(\kappa_1\varphi_{2m}-\kappa_{m-1}\varphi_{2,m-1}\right),\\
\frac{d\varphi_{ij}}{dt} &= \kappa_0\left(\kappa_i\varphi_{i+1,j}+\kappa_j\varphi_{i,j+1}-\kappa_{i-1}\varphi_{i-1,j}-\kappa_{j-1}\varphi_{i,j-1}\right),\\
&\quad 2\le i\le j\le m-1,\\
\frac{d\varphi_{im}}{dt} &= \kappa_0\left(\kappa_i\varphi_{i+1,m}-\kappa_{i-1}\varphi_{i-1,m}-\kappa_{m-1}\varphi_{i,m-1}\right),\\
&\quad 2\le i\le m-1,\\
\frac{d\varphi_{mm}}{dt} &= -2\kappa_0\kappa_{m-1}\varphi_{m-1,m},
\end{aligned}$$

such that $\varphi_{ij}(0)=\delta_{ij}$, but Kronecker deltas are readily seen to be also a solution to this system; hence $g(X_i(t),X_j(t))=\delta_{ij}$. By virtue of the assumption, one has $\mathrm{vol}_g(X_1(0),\dots,X_m(0))=\mathrm{vol}_g(v_1,\dots,v_m)=1$, and accordingly, $\mathrm{vol}_g(X_1(t),\dots,X_m(t))=1$ for every $t\in(t_0-\varepsilon,t_0+\varepsilon)$.

Finally, as the curvatures κ^σ_j, $0\le j\le m-1$, are completely determined by the Frenet formulas, it suffices to prove that the linear frame $(X_1,\dots,X_m)$ satisfies the property (ii) of Proposition 3.2, which is equivalent to prove the existence of functions $h_{ij}\in C^\infty(t_0-\varepsilon,t_0+\varepsilon)$, $1\le i\le j\le m$, such that $\nabla^{j-1}_{T^\sigma}T^\sigma=\sum_{i=1}^{j}h_{ij}X_i$ for $1\le j\le m$. If $j=1$, then this formula follows from the first formula in (21) with $h_{11}=\kappa_0$. Hence, we can proceed by recurrence on $j\ge 2$. By applying the operator ∇_{T^σ} to both sides of the equation $\nabla^{j-2}_{T^\sigma}T^\sigma=\sum_{i=1}^{j-1}h_{i,j-1}X_i$, we have $\nabla^{j-1}_{T^\sigma}T^\sigma=\sum_{i=1}^{j-1}((dh_{i,j-1}/dt)X_i+\nabla_{T^\sigma}X_i)$, and the result follows by replacing the term $\nabla_{T^\sigma}X_i=\kappa_0\nabla_{X_1}X_i$ by its expression deduced from the formulas in (21) above. □

Theorem 3.7 *Let (M,g) be an m-dimensional oriented Riemannian manifold. Given a system of functions $\kappa=(\kappa_0,\dots,\kappa_{m-1})$, with $\kappa_j\in C^\infty(t_0-\delta,t_0+\delta)$ for $0\le j\le m-1$ and $\kappa_j>0$ for $0\le j\le m-2$, let $f^\kappa_{ij}\colon(t_0-\delta,t_0+\delta)\to\mathbb{R}$, $1\le i\le j\le m$, be the functions defined by the following recurrence relations:*

$$\begin{cases} f^{\kappa}_{11} = \kappa_0, \\ f^{\kappa}_{12} = \dfrac{df^{\kappa}_{11}}{dt}, \\ f^{\kappa}_{22} = f^{\kappa}_{11}\kappa_0\kappa_1, \end{cases} \tag{22}$$

$$3 \le j \le m \quad \begin{cases} f^{\kappa}_{1j} = \dfrac{df^{\kappa}_{1,j-1}}{dt} - f^{\kappa}_{2,j-1}\kappa_0\kappa_1, \\ f^{\kappa}_{ij} = \dfrac{df^{\kappa}_{i,j-1}}{dt} - f^{\kappa}_{i+1,j-1}\kappa_0\kappa_i + f^{\kappa}_{i-1,j-1}\kappa_0\kappa_{i-1}, \\ 2 \le i \le j-2, \\ f^{\kappa}_{j-1,j} = \dfrac{df^{\kappa}_{j-1,j-1}}{dt} + f^{\kappa}_{j-2,j-1}\kappa_0\kappa_{j-2}, \\ f^{\kappa}_{jj} = f^{\kappa}_{j-1,j-1}\kappa_0\kappa_{j-1}. \end{cases} \tag{23}$$

Let $w_j \in T_{x_0}M$, $1 \le j \le m$, be vectors such that the system $(w_1, \dots, w_{m-1})$ is linearly independent. The necessary and sufficient conditions for a Frenet curve $\sigma\colon (t_0-\varepsilon, t_0+\varepsilon) \to M$, $0 < \varepsilon \le \delta$, to exist such that,

(i) $\sigma(t_0) = x_0$,
(ii) $\left(\nabla^{j-1}_{T^\sigma} T^\sigma\right)(t_0) = w_j$ *for* $1 \le j \le m$,
(iii) $\kappa^{\sigma}_{j-1} = \kappa_{j-1}$ *for* $1 \le j \le m$,

are the following:

$$g\left(w_i, w_j\right) = \sum_{h=1}^{i} f^{\kappa}_{hi}(t_0) f^{\kappa}_{hj}(t_0), \quad 1 \le i \le j \le m. \tag{24}$$

Proof If the curve σ in the statement exists, then $f^{\kappa}_{ij} = f^{\sigma}_{ij}$, where the functions f^{σ}_{ij} are given in the formulas (8) and (9), and from the formulas (7) we have

$$g\left(w_i, w_j\right) = g\left(\nabla^{i-1}_{T^\sigma} T^\sigma, \nabla^{j-1}_{T^\sigma} T^\sigma\right)(t_0) = \sum_{h=1}^{i} f^{\sigma}_{hi}(t_0) f^{\sigma}_{hj}(t_0).$$

Hence, all the conditions (24) are necessary for the curve σ to exist.

Let $(v_1, \dots, v_{m-1})$ be the orthonormal system in $T_{x_0}M$ obtained by applying the Gram-Schmidt process to the system $(w_1, \dots, w_{m-1})$, and let v_m be the only unitary tangent vector orthogonal to $v_1, \dots, v_{m-1}$ for which the basis $(v_1, \dots, v_{m-1}, v_m)$ of $T_{x_0}M$ is positively oriented. According to Theorem 3.6, there exists a Frenet curve $\sigma\colon (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ such that,

a) $\sigma(t_0) = x_0$; b) $X^{\sigma}_j(t_0) = v_j$, $1 \le j \le m$; c) $\kappa^{\sigma}_j = \kappa_j$, $0 \le j \le m-1$.

Hence $f^{\kappa}_{ij} = f^{\sigma}_{ij}$, as follows from the formulas (8), (9), (22), and (23), and from (7) we obtain $(\nabla^{j-1}_{T^\sigma} T^\sigma)(t_0) = \sum_{i=1}^{j} f^{\sigma}_{ij}(t_0) v_i$, $1 \le j \le m$. Consequently, the Gram-Schmidt process applied to $(T^\sigma(t_0), \dots, (\nabla^{m-2}_{T^\sigma} T^\sigma)(t_0))$ also leads one to the orthonormal system $(v_1, \dots, v_{m-1})$. By virtue of (24), we thus have $g(w_i, w_j) = g(\nabla^{i-1}_{T^\sigma} T^\sigma, \nabla^{j-1}_{T^\sigma} T^\sigma)(t_0)$ for $1 \le i \le j \le m$, and we can conclude by simply recalling the following fact: If $(u_1, \dots, u_k)$, $(u'_1, \dots, u'_k)$ are two linearly independent systems such that, 1st) the Gram-Schmidt process applied to $(u_1, \dots, u_k)$, as well as to $(u'_1, \dots, u'_k)$, leads to the same orthonormal system, and 2nd) $g(u_i, u_j) = g(u'_i, u'_j)$ for $1 \le i \le j \le k$, then both systems coincide, i.e. $u_i = u'_i$ for $i = 1, \dots, k$.

Finally, the Frenet frame at t_0 of any Frenet curve $\sigma\colon (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ satisfying (i)–(iii) in the statement coincides with the system $(v_1, \dots, v_m)$, and we can conclude its uniqueness from Theorem 3.6. □

3.4 $\mathcal{F}^{m-1}(M)$ and $\mathcal{N}^{m-1}(M)$

Theorem 3.8 *Let* (M, g) *be a Riemannian manifold. Let* $\mathcal{F}^{m-1}_{t_0,x_0}(M)$ *be the subset of Frenet jets* $j^{m-1}_{t_0}\sigma \in J^{m-1}(\mathbb{R}, M)$ *such that,* $\sigma(t_0) = x_0$. *Let* $\mathcal{N}^{m-1}_{t_0,x_0}(M)$ *be the subset of jets* $j^{m-1}_{t_0}\sigma \in J^{m-1}(\mathbb{R}, M)$ *such that,* i) $\sigma(t_0) = x_0$, ii) *the curve* σ *is normal general position up to the order* $m-1$ *at* t_0*, i.e. the tangent vectors* $U^{\sigma,1}_{t_0}, U^{\sigma,2}_{t_0}, \dots, U^{\sigma,r}_{t_0}$ *defined in* (1) *are linearly independent.*

If either $\dim M = m \leq 4$ *or* g *is a flat metric at a neighbourhood of* x_0*, then* $\mathcal{F}^{m-1}_{t_0,x_0}(M) = \mathcal{N}^{m-1}_{t_0,x_0}(M)$, $\forall t_0 \in \mathbb{R}$, $\forall x_0 \in M$.

In the general case, $\mathcal{F}^{m-1}_{t_0,x_0}(M)\backslash\mathcal{N}^{m-1}_{t_0,x_0}(M)$ *(resp.* $\mathcal{N}^{m-1}_{t_0,x_0}(M)\backslash\mathcal{F}^{m-1}_{t_0,x_0}(M)$*) is not empty but nowhere dense in* $\mathcal{F}^{m-1}_{t_0,x_0}(M)$ *(resp.* $\mathcal{N}^{m-1}_{t_0,x_0}(M)$*).*

Proof If $(U, x^1, \dots, x^m)$ is the normal coordinate system attached to an orthonormal basis for $T_{x_0}M$ with respect to the Levi-Civita connection ∇ of g, then for every smooth curve $(t_0 - \varepsilon, t_0 + \varepsilon) \to M$, $\sigma(t_0) = x_0$, the following formulas hold:

$$T^{\sigma}_{t_0} = U^{\sigma,1}_{t_0}, \quad \left(\nabla_{T^\sigma} T^\sigma\right)_{t_0} = U^{\sigma,2}_{t_0}, \quad \left(\nabla^2_{T^\sigma} T^\sigma\right)_{t_0} = U^{\sigma,3}_{t_0}, \tag{25}$$

and for $r \geq 3$, from the formulas (2), (3), (4), (5), we conclude the existence of a polynomial P^r_i in the values $(\partial^{|I|}\Gamma^h_{jk}/\partial x^i)(x_0)$, $I \in \mathbb{N}^m$, $|I| \leq r-2$, Γ^h_{jk} being the Christoffel symbols of ∇ with respect to the coordinates chosen, and the components $(d^k(x^i \circ \sigma)/dt^k)(t_0)$ (also in such coordinates) of the tangent vectors $U^{\sigma,k}_{t_0}$, $1 \leq k \leq r-1$, defined in (1) such that,

$$\left(\nabla^r_{T^\sigma} T^\sigma\right)_{t_0} = U^{\sigma,r+1}_{t_0} + P^r_i \left.\frac{\partial}{\partial x^i}\right|_{x_0}. \tag{26}$$

As the values $(\partial^{|I|}\Gamma^h_{jk}/\partial x^i)(x_0)$, $1 \leq |I| \leq k$, can be written as a polynomial (e.g. see [9,15]) in the components of the curvature tensor field R^g and its covariant derivatives $\nabla R^g, \dots, \nabla^{k-1}R^g$ at x_0, we conclude that the same holds for P^r_i. For example,

$$\begin{aligned}
\left(\nabla^3_{T^\sigma} T^\sigma\right)_{t_0} &= U^{\sigma,4}_{t_0} + \tfrac{1}{3} R^g_{x_0}\left(T^\sigma_{t_0}, \left(\nabla_{T^\sigma} T^\sigma\right)_{t_0}\right) T^\sigma_{t_0}, \\
\left(\nabla^4_{T^\sigma} T^\sigma\right)_{t_0} &= U^{\sigma,5}_{t_0} + 2\left(\nabla R^g\right)_{x_0}\left(T^\sigma_{t_0}, \left(\nabla_{T^\sigma} T^\sigma\right)_{t_0}, T^\sigma_{t_0}, T^\sigma_{t_0}\right) \\
&\quad + 3R^g_{x_0}\left(T^\sigma_{t_0}, \left(\nabla_{T^\sigma} T^\sigma\right)_{t_0}\right)\left(\nabla_{T^\sigma} T^\sigma\right)_{t_0} + \tfrac{7}{3} R^g_{x_0}\left(T^\sigma_{t_0}, \left(\nabla^2_{T^\sigma} T^\sigma\right)_{t_0}\right) T^\sigma_{t_0}.
\end{aligned}$$

For $m \leq 4$ from the formulas (25), we conclude

$$\mathcal{F}^{m-1}_{t_0,x_0}(M) = \mathcal{N}^{m-1}_{t_0,x_0}(M). \tag{27}$$

Moreover, the equation $(\nabla^r_t T)_{t_0} = U^{\sigma,r+1}_{t_0}$ holds for $0 \leq r \leq m-2$ if and only if, $P^r_i = 0$ for $0 \leq r \leq m-2$, $1 \leq i \leq m$. In particular, this happens when g is flat at a neighbourhood of x_0; hence, the equality (27) also holds in this case. If the tangent vectors $T^\sigma_{t_0}, (\nabla_{T^\sigma} T^\sigma)_{t_0}, \dots, (\nabla^{m-2}_{T^\sigma} T^\sigma)_{t_0}$ are linearly independent, but there exists a non-trivial linear combination, i.e. $0 = \sum_{h=1}^{m-1} \lambda_h U^{\sigma,h}_{t_0}$, then from (26) we deduce

$$\sum_{r=0}^{m-2} \lambda_{r+1} \left\{ \left(\nabla^r_{T^\sigma} T^\sigma\right)_{t_0} - P^r_i \left(\partial/\partial x^i\right)_{x_0} \right\} = 0, \tag{28}$$

which implies that at least one of the vectors $P^r_i \left(\partial/\partial x^i\right)_{x_0}$, $3 \leq r \leq m-2$, does not vanish.

Letting $N_{t_0} = T_{t_0} \times (\nabla_t T)_{t_0} \times \cdots \times (\nabla_t^{m-2} T)_{t_0}$, where $\times$ stands for cross-product, we obtain a basis $(T^\sigma_{t_0}, (\nabla_{T^\sigma} T^\sigma)_{t_0}, \ldots, (\nabla^{m-2}_{T^\sigma} T^\sigma)_{t_0}, N_{t_0})$ for $T_{x_0}M$, and we can write,

$$P_i^r \left(\partial/\partial x^i\right)_{x_0} = \sum_{q=0}^{m-2} \mu_q^r \left(\nabla^q_{T^\sigma} T^\sigma\right)_{t_0} + \mu^r N_{t_0}, \quad 0 \le r \le m-2,$$

for some scalars μ_q^r, μ^r, agreeing that $P_i^r = 0$ for $0 \le r \le 2$; hence $\mu_q^r = \mu^r = 0$ for $0 \le r \le 2$. Then, (28) is equivalent to saying that the homogeneous linear system

$$\sum_{r=0}^{m-2} \mu^r \lambda_{r+1} = 0,$$
$$\sum_{r=0}^{m-2} \left(\mu_q^r - \delta_q^r\right) \lambda_{r+1} = 0, \quad 0 \le q \le m-2,$$

of m equations in the $m-1$ unknowns $\lambda_1, \ldots, \lambda_{m-1}$ admits a non-trivial solution, i.e. the rank of the $m \times (m-1)$ matrix

$$\mu(m) = \begin{pmatrix} 0 & 0 & 0 & \mu^3 & \ldots & \mu^{m-2} \\ -1 & 0 & 0 & \mu_0^3 & \ldots & \mu_0^{m-2} \\ 0 & -1 & 0 & \mu_1^3 & \ldots & \mu_1^{m-2} \\ 0 & 0 & -1 & \mu_2^3 & \ldots & \mu_2^{m-2} \\ 0 & 0 & 0 & \mu_3^3 - 1 & \ldots & \mu_3^{m-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mu_{m-2}^3 & \ldots & \mu_{m-2}^{m-2} - 1 \end{pmatrix}$$

must be $\le m-2$. This condition characterizes $\mathcal{F}^{m-1}_{t_0,x_0}(M) \backslash \mathcal{N}^{m-1}_{t_0,x_0}(M)$. The proof for $\mathcal{N}^{m-1}_{t_0,x_0}(M) \backslash \mathcal{F}^{m-1}_{t_0,x_0}(M)$ is similar. □

Example 3.9 For $m = 5$ jets in $\mathcal{F}^4_{t_0,x_0}(M) \backslash \mathcal{N}^4_{t_0,x_0}(M)$ are given by, rk $\mu(5) = 3$. Hence $\mu^3 = 0$, $\mu_3^3 = 1$; $P_i^3 \left(\partial/\partial x^i\right)_{x_0} = \sum_{q=0}^2 \mu_q^3 \left(\nabla^q_{T^\sigma} T^\sigma\right)_{t_0} + \left(\nabla^3_{T^\sigma} T^\sigma\right)_{t_0}$, i.e.

$$R^g_{x_0} \left(T^\sigma_{t_0}, (\nabla_{T^\sigma} T^\sigma)_{t_0}\right) T^\sigma_{t_0} - \left(\nabla^3_{T^\sigma} T^\sigma\right)_{t_0} \in \left\langle T^\sigma_{t_0}, (\nabla_{T^\sigma} T^\sigma)_{t_0}, (\nabla^2_{T^\sigma} T^\sigma)_{t_0} \right\rangle. \tag{29}$$

In addition, assume $(x^i)_{i=1}^5$ is the normal coordinate system defined by the Frenet frame $\left(X_i^\sigma(t_0)\right)_{i=1}^5$. From the formulas (7), (8), and (9), it follows the formula (29) can be reformulated by saying that the tangent vector

$$\begin{aligned}&\left(\kappa_0^\sigma\right)^4 \kappa_1^\sigma R^g_{x_0} \left(X_1^\sigma(t_0), X_2^\sigma(t_0)\right) X_1^\sigma(t_0) - f_{14}^\sigma(t_0) X_1^\sigma(t_0) - f_{24}^\sigma(t_0) X_2^\sigma(t_0) \\ &\quad - f_{34}^\sigma(t_0) X_3^\sigma(t_0) - f_{44}^\sigma(t_0) X_4^\sigma(t_0)\end{aligned}$$

must belong to $\left\langle X_1^\sigma(t_0), X_2^\sigma(t_0), X_3^\sigma(t_0)\right\rangle$, or equivalently,

$$g\left(R^g_{x_0}\left(X_1^\sigma(t_0), X_2^\sigma(t_0)\right) X_1^\sigma(t_0), X_4^\sigma(t_0)\right) = \kappa_2^\sigma(t_0)\,\kappa_3^\sigma(t_0),$$
$$g\left(R^g_{x_0}\left(X_1^\sigma(t_0), X_2^\sigma(t_0)\right) X_1^\sigma(t_0), X_5^\sigma(t_0)\right) = 0.$$

4 The equivalence problem

4.1 Necessary conditions for congruence

Definition 4.1 Two curves $\sigma\colon(a,b)\to(M,g)$, $\bar{\sigma}\colon(a,b)\to(\bar{M},\bar{g})$ with values in two Riemannian manifolds are said to be *congruent* if an open neighbourhood U of the image of σ in M and an isometric embedding $\phi\colon U\to\bar{M}$ exist such that, $\bar{\sigma}=\phi\circ\sigma$. If M and $\bar{M}$ are oriented, then ϕ is assumed to preserve the orientation.

Proposition 4.2 *The curvatures of a Frenet curve with values into an oriented Riemannian manifold (M,g) are invariant by congruence.*

Let (M,g), $(\bar{M},\bar{g})$ be two oriented Riemannian manifolds with associated Levi-Civita connections $\nabla,\bar{\nabla}$, respectively, and let $\sigma\colon(a,b)\to M$, $\bar{\sigma}\colon(a,b)\to\bar{M}$, be two Frenet curves which are congruent under the isometric embedding ϕ. Then, $\phi\cdot X_i^{\sigma}=X_i^{\bar{\sigma}}$, , $\omega_{\sigma}^i=\phi^*\omega_{\bar{\sigma}}^i$, for $1\leq i\leq m$, $\left(\omega_{\sigma}^1,\dots,\omega_{\sigma}^m\right)$, , $\left(\omega_{\bar{\sigma}}^1,\dots,\omega_{\bar{\sigma}}^m\right)$ being the dual coframes of the Frenet frames of σ, $\bar{\sigma}$, respectively. Moreover,

$$\phi_*\left(\nabla^j R\left(X_{i_1}^{\sigma},\dots,X_{i_{j+3}}^{\sigma},\omega_{\sigma}^i\right)\right)(\sigma(t))=\bar{\nabla}^j\bar{R}\left(X_{i_1}^{\bar{\sigma}},\dots,X_{i_{j+3}}^{\bar{\sigma}},\omega_{\bar{\sigma}}^i\right)(\bar{\sigma}(t)),$$

for all $j\in\mathbb{N}$, $t\in(a,b)$, and all systems of indices $i,i_1,\dots,i_{j+3}=1,\dots,m$, where R, $\bar{R}$ are the curvature tensors of (M,g), $(\bar{M},\bar{g})$, respectively.

4.2 General criterion of congruence

Theorem 4.3 *Let (M,g), , $(\bar{M},\bar{g})$ be two oriented connected Riemannian manifolds of class C^{ω} of the same dimension, $m=\dim M=\dim\bar{M}$, with Levi-Civita connections ∇, $\bar{\nabla}$, and let $\sigma\colon(a,b)\to M$, $\bar{\sigma}\colon(a,b)\to\bar{M}$ be two Frenet curves of class C^{ω} with tangent fields T, $\bar{T}$, respectively. If $x_0=\sigma(t_0)$, $\bar{x}_0=\bar{\sigma}(t_0)$, $a<t_0<b$, then σ and $\bar{\sigma}$ are congruent on some neighbourhoods of x_0 and $\bar{x}_0$, respectively, if and only if the following conditions hold:*

(i) *For every $j\in\mathbb{N}$ and every $0\leq i\leq m-1$,*

$$\frac{d^j\kappa_i^{\sigma}}{dt^j}(t_0)=\frac{d^j\kappa_i^{\bar{\sigma}}}{dt^j}(t_0),\tag{30}$$

(ii) *For every $j\in\mathbb{N}$ and all systems of indices $i,i_1,\dots,i_{j+3}=1,\dots,m$, the following formula holds:*

$$\left(\nabla^j R\right)\left(X_{i_1}^{\sigma},\dots,X_{i_{j+3}}^{\sigma},\omega_{\sigma}^i\right)(x_0)=\left(\bar{\nabla}^j\bar{R}\right)\left(X_{i_1}^{\bar{\sigma}},\dots,X_{i_{j+3}}^{\bar{\sigma}},\omega_{\bar{\sigma}}^i\right)(\bar{x}_0),\tag{31}$$

where $\left(\omega_{\sigma}^1,\dots,\omega_{\sigma}^m\right)$, $\left(\omega_{\bar{\sigma}}^1,\dots,\omega_{\bar{\sigma}}^m\right)$ are the dual coframes of the Frenet frames $\left(X_1^{\sigma},\dots,X_m^{\sigma}\right)$, $\left(X_1^{\bar{\sigma}},\dots,X_m^{\bar{\sigma}}\right)$ of σ, $\bar{\sigma}$, and R, $\bar{R}$ are the curvature tensors of (M,g), $(\bar{M},\bar{g})$, respectively.

Proof From Proposition 4.2, the equations (30) and (31) follow. To prove the converse, let $A\colon T_{x_0}M\to T_{\bar{x}_0}\bar{M}$ be the linear isometry given by, $A\left(X_i^{\sigma}(t_0)\right)=X_i^{\bar{\sigma}}(t_0)$, $1\leq i\leq m$. The condition (31) implies that A maps the tensor $(\nabla^j R)_{x_0}$ into the tensor $(\bar{\nabla}^j\bar{R})_{\bar{x}_0}$, for all $j\in\mathbb{N}$. From [14, VI, Theorem 7.2], we conclude that the polar map $\phi\colon U\to\overline{U}$, $\phi=\exp_{\bar{x}_0}\circ A\circ\exp_{x_0}^{-1}$, is an affine isomorphism and from [29, Lemma 2.3.1] it follows that ϕ is an isometry. In order to finish the proof, it suffices to check that $\phi(\sigma(t))=\bar{\sigma}(t)$ for $|t-t_0|<\varepsilon$, and a small enough $\varepsilon>0$. The Frenet curve $\gamma=\phi\circ\sigma\colon(a,b)\to\bar{M}$ satisfies $\gamma(t_0)=\bar{x}_0$,

$X_i^\gamma(t_0) = \phi_*\left(X_i^\sigma(t_0)\right) = X_i^{\bar\sigma}(t_0)$. As the curvatures are of class C^ω, from the condition (30), we deduce $\kappa_j^{\bar\sigma} = \kappa_j^\sigma$, $0 \le j \le m-1$, and since the curvatures are invariant by congruence, we know $\kappa_j^\sigma = \kappa_j^\gamma$; hence $\kappa_j^{\bar\sigma} = \kappa_j^\gamma$, $0 \le j \le m-1$. Taking the formulas (7), (9) and the condition (30) into account for $1 \le j \le m$, we have

$$\begin{aligned} A\left(\left(\nabla_{T^\sigma}^{j-1}T^\sigma\right)_{t_0}\right) &= \sum_{i=1}^m f_{ij}^\sigma(t_0)A\left(X_i^\sigma(t_0)\right) \\ &= \sum_{i=1}^m f_{ij}^{\bar\sigma}(t_0)X_i^{\bar\sigma}(t_0) \\ &= \left(\bar\nabla_{T^{\bar\sigma}}^{j-1}T^{\bar\sigma}\right)_{t_0}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\bar\nabla_{T^\gamma}^{j-1}T^\gamma\right)_{t_0} &= \phi_*\left(\left(\nabla_{T^\sigma}^{j-1}T^\sigma\right)_{t_0}\right) \\ &= A\left(\left(\nabla_{T^\sigma}^{j-1}T^\sigma\right)_{t_0}\right) \\ &= \left(\bar\nabla_{T^{\bar\sigma}}^{j-1}T^{\bar\sigma}\right)_{t_0}, \end{aligned}$$

for $1 \le j \le m$. By applying Theorem 3.6, we conclude $\bar\sigma = \gamma = \phi\circ\sigma$ on $(t_0-\varepsilon, t_0+\varepsilon)$. □

Corollary 4.4 *Let (M, g), $(\bar M, \bar g)$ be two oriented connected Riemannian manifolds of class C^ω of the same dimension, $m = \dim M = \dim \bar M$, and let $\sigma : (a, b) \to M$, $\bar\sigma : (a, b) \to \bar M$ be two Frenet curves, respectively. If $x_0 = \sigma(t_0)$, $\bar x_0 = \bar\sigma(t_0)$, $a < t_0 < b$, then σ and $\bar\sigma$ are congruent on some neighbourhoods U and $\bar U$ of x_0 and $\bar x_0$, respectively if, and only if, the following conditions hold:*

(i) *For every $0 \le i \le m-1$, it holds $\kappa_i^\sigma(t) = \kappa_i^{\bar\sigma}(t)$, for $|t-t_0| < \varepsilon$.*
(ii) *For every $j \in \mathbb{N}$ and every system of indices $i, i_1, \ldots, i_{j+3} \in \{1, \ldots, m\}$,*

$$(\nabla^j R)\left(X_{i_1}^\sigma, \ldots, X_{i_{j+3}}^\sigma, \omega_\sigma^i\right)(x_0) = (\bar\nabla^j \bar R)\left(X_{i_1}^{\bar\sigma}, \ldots, X_{i_{j+3}}^{\bar\sigma}, \omega_{\bar\sigma}^i\right)(\bar x_0).$$

4.3 Remarks on the criterion of congruence

Remark 4.5 The condition (31) of Theorem 4.3 is *not* equivalent to the following:

$$R\left(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i\right)(\sigma(t)) = \bar R\left(X_j^{\bar\sigma}, X_k^{\bar\sigma}, X_l^{\bar\sigma}, \omega_{\bar\sigma}^i\right)(\bar\sigma(t)), \quad |t-t_0| < \varepsilon. \tag{32}$$

Differentiating the left-hand side of (32), we have

$$\begin{aligned} \frac{d}{dt}R\left(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i\right)(\sigma(t)) &= (\nabla_{T^\sigma}R)\left(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i\right)(\sigma(t)) \\ &\quad + R\left(\nabla_{T^\sigma}X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i\right)(\sigma(t)) + \cdots \\ &\quad + R\left(X_j^\sigma, X_k^\sigma, X_l^\sigma, \nabla_{T^\sigma}\omega_\sigma^i\right)(\sigma(t)) \\ &= \kappa_0^\sigma(t)\nabla R\left(X_1^\sigma, X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i\right)(\sigma(t)) + \cdots \end{aligned}$$

As the first argument of ∇R in the formula above is X_1^σ, the function

$$\nabla R\left(X_h^\sigma, X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i\right), \quad h \neq 1,$$

cannot be recovered from $R\left(X_j^\sigma, X_k^\sigma, X_l^\sigma, \omega_\sigma^i\right)(\sigma(t))$. Therefore, the formulas (32) do not imply the formulas (31), although (31) do imply (32) as the manifolds involved are analytic.

Example 4.6 Let us consider the bidimensional torus $\mathbb{T} \subset \mathbb{R}^3$ with implicit equation $(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2)$. On the radius-2 circumference $C = \mathbb{T} \cap \{z = 1\}$, the Gaussian curvature of T vanishes and C is a regular curve of positive constant curvature. The curvature tensor of $\mathbb{R}^2$ vanishes in particular along any curve $C' \subset \mathbb{R}^2$ with the same curvature as C; but C and C' are not congruent since the Gaussian curvature of $\mathbb{T}$ does not vanish at every neighbourhood of a point of C.

Remark 4.7 From Proposition 3.2, we deduce that the Frenet frame of a Frenet curve σ and its dual frame at a point $\sigma(t_0)$ depend on $j_{t_0}^{m-1}\sigma$ only. Hence, for every system of indices $j \in \mathbb{N}$, $i_1, \dots, i_{j+3}, i \in \{1, \dots, m\}$, a function $I^j_{i_1 \dots i_{j+3}, i} \colon \mathcal{F}^{m-1}(M) \to \mathbb{R}$ can be defined on the open subset $\mathcal{F}^{m-1}(M) \subset J^{m-1}(\mathbb{R}, M)$ of the jets of order $m-1$ of Frenet curves with values in M by setting

$$I^j_{i_1 \dots i_{j+3}, i}(j_t^{m-1}\sigma) = (\nabla^j R)\left(X_{i_1}^\sigma, \dots, X_{i_{j+3}}^\sigma, \omega_\sigma^i\right)(\sigma(t)). \tag{33}$$

Similarly, for every $0 \leq i \leq m-1$, a function

$$\varkappa_i \colon (\pi_{m-1}^m)^{-1}\mathcal{F}^{m-1}(M) \subset J^m(\mathbb{R}, M) \to \mathbb{R} \tag{34}$$

can be defined by setting $\varkappa_i(j_t^m\sigma) = \kappa_i^\sigma(t)$.

From Theorem 4.3, it follows that all these functions are invariant by isometry (see the Sect. 5). Since $\dim \mathcal{F}^{m-1}(M) = m^2 + 1$, only a finite number (not greater than $m^2 + 1$) of such functions can be functionally independent generically. Hence, the infinite number of conditions given in (31) can be reduced to a finite number. Nevertheless, it is not easy to determine a bound for the index j, which measures the times one has to differentiate covariantly the curvature tensor.

Remark 4.8 Theorem 4.3 is the most general result we can expect without imposing any additional condition on (M, g) and $(\bar{M}, \bar{g})$ except for the fact of being analytic. This is principally due to the fact that [14, VI, Theorem 7.2] cannot be generalized to non-analytic manifolds, as shown in the next example.

Example 4.9 Let g, $\bar{g}$ be the two Riemannian metrics on $M = \bar{M} = \mathbb{R}^m$, $m \geq 2$, defined by, $g_{ij}(x) = \delta_{ij} + \exp(-|x|^{-2})$, $\bar{g}_{ij}(x) = \delta_{ij}$, respectively; hence, (M, g) is not analytic at the origin. If R is the curvature tensor of (M, g) and ∇ is its associated Levi-Civita connection, then $(\nabla^n R)(0) = 0$ for all $n \in \mathbb{N}$. The identity map $Id \colon T_0 M \to T_{\bar{0}}\bar{M}$ is an isometry, since $g_{ij}(0) = \bar{g}_{ij}(\bar{0}) = \delta_{ij}$. Moreover, $(\nabla^j R)(0) = (\bar{\nabla}^j \bar{R})(\bar{0}) = 0$, where $\bar{R}$ (resp. $\bar{\nabla}$) is the curvature tensor (resp. the Levi-Civita connection) of $\bar{g}$. If there exists an affine isomorphism $\phi \colon U \to \bar{U} = \bar{M}$, defined on normal neighbourhoods of 0, such that $\phi_{*,0} = Id$, then taking [29, Lemma 2.3.1] into account, ϕ must necessarily be an isometry. Hence ϕ maps the tensor $\nabla^j R$ into the tensor $\bar{\nabla}^j \bar{R} = 0$, for all $j \in \mathbb{N}$. Consequently, $\nabla^j R$ must vanish in a normal neighbourhood of 0, but this is not true. In fact, as $g_{ij} = \delta_{ij} + h(|x|)$, with $h(s) = \exp(-s^{-2})$, we have

$$g^{ij} = \delta_{ij} - \frac{h(|x|)}{1 + mh(|x|)}.$$

Following the notation in [14], the Christoffel symbols are,

$$\Gamma_{ij}^{k}=\frac{h'(|x|)}{|x|}\left(x^{i}+x^{j}-x^{k}+\frac{h(|x|)}{1+mh(|x|)}\left(\sum_{a=1}^{m}x^{a}-m(x^{i}+x^{j})\right)\right).$$

If $x_t=(t,\ldots,t)\in\mathbb{R}^m$, $t\neq 0$, then $\Gamma_{ij}^{k}(x_t)=\frac{h'(|x_t|)t}{|x_t|(1+mh(|x_t|))}\neq 0$, and

$$\Gamma_{ii}^{a}(x_t)\,\Gamma_{aj}^{j}(x_t)=\left(\frac{h'(|x_t|)}{|x_t|\,(1+mh(|x_t|))}\right)^{2}t^{2}=\Gamma_{ji}^{a}(x_t)\,\Gamma_{ai}^{j}(x_t)\,.$$

Consequently, $\Gamma_{ii}^{a}(x_t)\,\Gamma_{aj}^{j}(x_t)-\Gamma_{ji}^{a}(x_t)\,\Gamma_{ai}^{j}(x_t)=0$, and hence

$$R_{iji}^{j}(x_t)=\frac{-2h'(|x_t|)}{|x_t|}\neq 0.$$

Thus, R_{iji}^{j} does not vanish at x_t for small enough $t\neq 0$.

5 Differential invariants

5.1 Basic definitions

Let $\mathfrak{I}(M,g)$ be the group of isometries of a complete Riemannian connected manifold (M,g) endowed with its structure of Lie transformation group (cf. [14, VI, Theorem 3.4]) and let $\mathfrak{i}(M,g)$ be its Lie algebra, which is anti-isomorphic to the algebra of Killing vector fields.

Every diffeomorphism $\phi\colon M\to M$ induces a transformation $\phi^{(r)}$ on $J^r(\mathbb{R},M)$ given by $\phi^{(r)}\left(j_t^r\sigma\right)=j_t^r(\phi\circ\sigma)$, and a natural action (on the left) of the group $\mathfrak{I}(M,g)$ on $J^r(\mathbb{R},M)$ can be defined by $\phi\cdot j_t^r\sigma=\phi^{(r)}\left(j_t^r\sigma\right)$. Each $X\in\mathfrak{i}(M,g)$ induces a flow ϕ_t, and its jet prolongation $\phi_t^{(r)}$ determines a flow on $J^r(\mathbb{R},M)$, the infinitesimal generator of which is the vector field denoted by $X^{(r)}\in\mathfrak{X}(J^r(\mathbb{R},M))$. The tangent spaces to the orbits of the action of $\mathfrak{I}(M,g)$ on $J^r(\mathbb{R},M)$ coincide with the fibres of the distribution $\mathfrak{D}^r\subset\mathfrak{X}(J^r(\mathbb{R},M))$ spanned by the vector fields $X^{(r)}$; more precisely, we have

$$T_{j_t^r\sigma}\left(\mathfrak{I}(M,g)\cdot j_t^r\sigma\right)=\mathfrak{D}^r_{j_t^r\sigma}=\left\{X^{(r)}_{j_t^r\sigma}:X\in\mathfrak{i}(M,g)\right\}.$$

Definition 5.1 A smooth function $I\colon J^r(\mathbb{R},M)\to\mathbb{R}$ is said to be an *invariant* of order r (cf. [1,7, 4.1], [16]) if, $I\circ\phi^{(r)}=I$, $\forall\phi\in\mathfrak{I}(M,g)$. A first integral $f\colon J^r(\mathbb{R},M)\to\mathbb{R}$ of the distribution $\mathfrak{D}^r$ is called a *differential invariant* of order r; i.e. $X^{(r)}(f)=0$, $\forall X\in\mathfrak{i}(M,g)$.

Remark 5.2 A differential invariant is an invariant with respect to the connected component of the identity $\mathfrak{I}^0(M,g)$ in $\mathfrak{I}(M,g)$.

Lemma 5.3 (cf. [21]) *The distribution $\mathfrak{D}^r$ is involutive, and its rank is locally constant on a dense open subset $\mathcal{U}^r\subseteq J^r(\mathbb{R},M)$. If N_r denotes the maximal number of functionally independent differential invariants of order $r\geq 0$, then*

$$\begin{aligned}N_r&=\dim J^r(\mathbb{R},M)-\operatorname{rk}\mathfrak{D}^r\big|_{\mathcal{U}^r}\\&=m(r+1)+1-\operatorname{rk}\mathfrak{D}^r\big|_{\mathcal{U}^r}\,.\end{aligned}\qquad(35)$$

Proof $\mathfrak{D}^r$ is involutive as $[X_1^{(r)}, X_2^{(r)}] = [X_1, X_2]^{(r)}$, $\forall X_1, X_2 \in \mathfrak{i}(M, g)$. Let $\mathcal{U}^r$ be the subset defined as follows: A point $\xi = j_t^r \sigma \in J^r(\mathbb{R}, M)$ belongs to $\mathcal{U}^r$ if and only if ξ admits an open neigbourhood N_ξ such that $\dim \mathfrak{D}^r_{\xi'} = \dim \mathfrak{D}^r_\xi$ for every $\xi' \in N_\xi$. As $N_\xi \subseteq \mathcal{U}^r$, it follows that $\mathcal{U}^r$ is an open subset, which is non-empty as the dimension of the fibres of $\mathfrak{D}^r$ is uniformly bounded and hence, $\mathcal{U}^r$ contains the points ξ for which $\dim \mathfrak{D}^r_\xi = \max_{\xi' \in J^r(\mathbb{R},M)} \dim \mathfrak{D}^r_{\xi'} = d$. In fact, if this equation holds, then there exists an open neighbourhood N_ξ of ξ such that the dimension of the fibres of $\mathfrak{D}^r$ over the points $\xi' \in N_\xi$ is at least d, as if $(X_i^{(r)})_\xi$, $1 \leq i \leq d$, is a basis for $\mathfrak{D}^r_\xi$, then the vector fields $(X_i^{(r)})$ are linearly independent at each point of an open neigbourhood and hence, they are also a basis, d being the maximal value of the dimension of the fibres of $\mathfrak{D}^r$. From the very definition of $\mathcal{U}^r$ we thus conclude that $N_\xi \subseteq \mathcal{U}^r$. The same argument proves that the rank of $\mathfrak{D}^r$ is locally constant over $\mathcal{U}^r$. Next, we prove that $\mathcal{U}^r$ is dense. If $O \subset J^r(\mathbb{R}, M)$ is a non-empty open subset, then there exists $\xi \in O$ such that $\dim \mathfrak{D}^r_\xi = \max_{\xi' \in O} \dim \mathfrak{D}^r_{\xi'}$ and we can conclude as above. The last part of the statement follows directly from the Frobenius theorem. □

If f is a differential invariant of order r, then $D_t(f)$ is a differential invariant of order $r+1$. This fact follows from the formula $X^{(r+1)} \circ D_t = D_t \circ X^{(r)}$ for every $X \in \mathfrak{i}(M, g)$, which, in its turn, follows from the formula

$$X^{(r)} = \sum_{j=0}^{r} (D_t)^j (f^i) \frac{\partial}{\partial x^i_j}, \tag{36}$$

for every $X \in \mathfrak{X}(M)$ with local expression

$$X = f^i \frac{\partial}{\partial x^i}, \quad f^i \in C^\infty(M). \tag{37}$$

If $\pi^k_l \colon J^k(\mathbb{R}, M) \to J^l(\mathbb{R}, M)$ is the canonical projection for $k > l$, then

$$(\pi^r_{r-1})_* X^{(r)} = X^{(r-1)}, \quad \forall X \in \mathfrak{X}(M),$$

and the following exact sequence defines the subdistribution $\mathfrak{D}^{r,r-1}$:

$$0 \to \mathfrak{D}^{r,r-1}_{j_t^r \sigma} \to \mathfrak{D}^r_{j_t^r \sigma} \xrightarrow{(\pi^r_{r-1})_*} \mathfrak{D}^{r-1}_{j_t^{r-1} \sigma} \to 0, \qquad \forall j_t^r \sigma \in J^r(\mathbb{R}, M). \tag{38}$$

5.2 Stability

Theorem 5.4 *Let (M, g) be a complete Riemannian connected manifold and let $\sigma \colon (a, b) \to M$ be a smooth curve such that $j_{t_0}^{m-1} \sigma \in \mathcal{N}^{m-1}(M)$, $a \leq t_0 \leq b$, with the same notations as in* Sect. 3.4. *If $X \in \mathfrak{i}(M, g)$ is a Killing vector field such that $X^{(m-1)}_{j_{t_0}^{m-1} \sigma} = 0$, $m = \dim M$, then $X = 0$.*

Proof Let $x_0 = \sigma(t_0)$ and let $U \subset T_{x_0}(M)$ be an open neighbourhood of the origin on which the exponential mapping $\exp \colon T_{x_0}(M) \to M$ is a diffeomorphism onto its image. Let $(X_j)_{j=1}^m$ be a g-orthonormal basis for $T_{x_0} M$ with dual basis $(w^i)_{i=1}^m$, $w^i \in T^*_{x_0}(M)$ (i.e. $w^i(X_j) = \delta^i_j$) and let $x^i = w^i \circ (\exp|_U)^{-1}$, $1 \leq i \leq m$, be the corresponding normal coordinate system.

If $\phi \colon M \to M$ is an affine transformation of the Levi-Civita connection of g (in particular, if ϕ is an isometry of g) leaving the point x_0 invariant, then (cf. [14, VI, Proposition 1.1]), $\phi \circ \exp = \exp \circ \phi_*$, $\phi_* \colon T_{x_0}(M) \to T_{x_0}(M)$ being the Jacobian mapping at x_0. Hence

$x^i \circ \phi = (w^i \circ \phi_*) \circ (\exp|_U)^{-1}$. If $\phi_*(X_j) = a^i_j X_i$, then $w^i \circ \phi_* = a^i_h w^h$ and hence, $x^i \circ \phi = a^i_h x^h$. In particular, if ϕ_τ is the flow of a Killing vector field X locally given as in (37), then

$$x^i \circ \phi_\tau = a^i_h(\tau) x^h, \quad (a^i_h(\tau))^m_{h,i=1} \in O(m), \quad \forall \tau \in \mathbb{R},$$

$$f^i = b^i_h x^h, \qquad b^i_h = \frac{da^i_h}{d\tau}(0), \qquad B = (b^i_h)^m_{h,i=1} \in \mathfrak{so}(m).$$

According to (36), the assumption on X in the statement is equivalent to saying

$$\frac{d^k(f^i \circ \sigma)}{dt^k}(t_0) = 0, \quad 1 \leq i \leq m, \; 0 \leq k \leq m-1,$$

or equivalently, $b^i_h (d^k(x^h \circ \sigma)/dt^k)(t_0) = 0$, i.e. $B(U^{\sigma,k}_{t_0}) = 0$, $1 \leq k \leq m-1$, the tangent vectors $U^{\sigma,k}_{t_0}$ being defined in the formula (1). As B is skew-symmetric, we also have $B(U^{\sigma,1}_{t_0} \times \cdots \times U^{\sigma,m-1}_{t_0}) = 0$. Therefore, $B = 0$. □

Corollary 5.5 *On a complete Riemannian connected manifold* (M, g)*, the order of asymptotic stability (in this context, the order from which all invariants functionally depend on invariants of smaller order and their derivatives, cf.* [2,16]*) is* $\leq m$*. Accordingly,* $N_r = (r+1)m + 1 - \dim \mathfrak{i}(M, g)$*,* $\forall r \geq m-1$.

Proof This follows from the previous theorem and the exact sequence (38), taking account of the fact that $X^{(m-1)}_{j^{m-1}_t \sigma} = 0$ if and only if $X^{(m)}_{j^m_t \sigma} \in \mathfrak{D}^{m,m-1}_{j^m_t \sigma}$. □

Corollary 5.6 *On a complete Riemannian connected manifold* (M, g)*, the distribution* $\mathfrak{D}^{m-1}$ *takes its maximal rank on* $\mathcal{N}^{m-1}(M)$.

Proof If $j^{m-1}_t \sigma \in \mathcal{N}^{m-1}(M)$, then from Theorem 5.4 it follows that the linear map $\mathfrak{i}(M, g) \to \mathfrak{D}^{m-1}_{j^{m-1}_t \sigma}$, $X \mapsto X^{(m-1)}_{j^{m-1}_t \sigma}$, is an isomorphism. □

5.3 Completeness

Let a Lie group G act on a manifold N. If the quotient manifold $q\colon N \to N/G$ exists, then the image of the mapping $q^*\colon C^\infty(N/G) \to C^\infty(N)$, $f \mapsto f \circ q$, is the subalgebra of G-invariant functions, namely $C^\infty(N)^G = q^* C^\infty(N/G)$. Below, we are concerned with the case $G = \mathcal{I}^0(M, g)$ acting on $N = J^r(\mathbb{R}, M)$ as defined at the beginning of the Sect. 5.1.

Definition 5.7 Let $O^r \subseteq J^r(\mathbb{R}, M)$ be an invariant open subset under the natural action of the group $\mathcal{I}^0(M, g)$. A system of invariant functions $I_i\colon O^r \to \mathbb{R}$, $1 \leq i \leq \nu$, is said to be *complete* if the equations $I_i(j^r_{t_0}\sigma) = I_i(j^r_{t_0}\sigma')$, $1 \leq i \leq \nu$, $j^r_{t_0}\sigma$, $j^r_{t_0}\sigma' \in O^r$, imply that σ and σ' are congruent on a neighbourhood of t_0.

Remark 5.8 If M is simply connected, then the isometry transforming σ onto σ' on a neighbourhood of t_0 can uniquely be extended to a global isometry of M (see [14, VI, Corollary 6.4]), which also transforms the whole image of σ onto that of σ' if both curves are analytic.

Proposition 5.9 *Let* ∇ *be a linear connection on* M*. The mapping* Φ^r_∇ *defined in the formula (6) makes the diagram*

$$\begin{array}{ccc} J^r(\mathbb{R}, M) & \xrightarrow{\Phi^r_\nabla} & \mathbb{R} \times \oplus^r TM \\ \phi^{(r)} \downarrow & & \downarrow (1_{\mathbb{R}}, \oplus^r \phi_*) \\ J^r(\mathbb{R}, M) & \xrightarrow{\Phi^r_{\phi\cdot\nabla}} & \mathbb{R} \times \oplus^r TM \end{array}$$

commutative for every $\phi \in \mathrm{Diff}(M)$.

Proof The proof is a consequence of the formula (2) and Lemma 3.6, taking the definition of $\phi \cdot \nabla$ into account. □

Theorem 5.10 *Let (M, g) be a complete oriented connected Riemannian manifold of class C^ω. If*

$$I_i : (\pi^r_{m-1})^{-1}\mathcal{F}^{m-1}(M) \to \mathbb{R}, \quad r \geq m,\ 1 \leq i \leq \nu, \tag{39}$$

is a complete system of invariants, then there exists a dense open subset O^r in $(\pi^r_{m-1})^{-1}\mathcal{F}^{m-1}(M)$ such that $I_i|_{O^r}$, $1 \leq i \leq \nu$, generate the ring of differential invariants under the group $\mathcal{I}^0(M, g)$ on an open neighbourhood $N^r \subseteq O^r$ of every point $j^r_t\sigma \in O^r$, i.e.

$$C^\infty\left(N^r\right)^{\mathcal{I}^0(M,g)} = \left(I_1|_{N^r}, \ldots, I_\nu|_{N^r}\right)^* C^\infty(\mathbb{R}^\nu).$$

Conversely, if a system of functions as in (39) locally generates the ring of invariants over a dense subset $\tilde{O}^r \subseteq (\pi^r_{m-1})^{-1}\mathcal{F}^{m-1}(M)$, then it is complete.

Proof According to Theorem 5.4, we can confine ourselves to prove the statement for $r = m$. First of all, we prove that the quotient manifold

$$q^m : (\pi^m_{m-1})^{-1}\mathcal{F}^{m-1}(M) \to (\pi^m_{m-1})^{-1}\mathcal{F}^{m-1}(M)/\mathcal{I}^0(M, g) = Q^m$$

exists. To this end, by applying [17, Theorem 9.16], we only need to prove that the following two conditions hold:

1. The isotropy subgroup $\mathcal{I}^0(M, g)_{j^m_t\sigma}$ reduces to the identity map of M for every $j^m_t\sigma$ in $(\pi^m_{m-1})^{-1}\mathcal{F}^{m-1}(M)$.
2. $\mathcal{I}^0(M, g)$ acts properly on $(\pi^m_{m-1})^{-1}\mathcal{F}^{m-1}(M)$.

The image of $(\pi^m_{m-1})^{-1}(\mathcal{F}^{m-1})$ by the diffeomorphism Φ^m_∇ is equal to the subset $U^m \subset \mathbb{R} \times \oplus^m TM$ of elements $(t, X_1, \ldots, X_m)$ such that $(X_1, \ldots, X_{m-1})$ are linearly independent tangent vectors. From Proposition 5.9, we deduce that an isometry ϕ belongs to the isotropy subgroup $\mathcal{I}^0(M, g)_{j^m_t\sigma}$ of a point $j^m_t\sigma$ in $(\pi^r_{m-1})^{-1}\mathcal{F}^{m-1}(M)$, if and only $(1_\mathbb{R}, \oplus^m\phi_*(\sigma(t)))$ belongs to the isotropy subgroup of the point $\Phi^m_\nabla(j^m_t\sigma) = (t, X_1, \ldots, X_m) \in U^m$. Hence $\phi = Id_M$ and consequently, $\mathcal{I}^0(M, g)$ acts freely on $(\pi^m_{m-1})^{-1}(\mathcal{F}^{m-1})$, thus proving the first item above.

Moreover, if g_1 is the Sasakian metric induced by g on TM (e.g. see [4, 1.K], [12, section 7], [31, IV, section 1]), then $\mathcal{I}^0(M, g)$ acts by isometries of the metric on $J^m(\mathbb{R}, M)$ given by

$$\mathbf{g}^m = \left(\Phi^m_\nabla\right)^*\left(dt^2 + \sum_{i=1}^m (\mathrm{pr}_i)^* g_1\right),$$

where $\mathrm{pr}_i : \mathbb{R} \times \oplus^m TM \to TM$ is the projection $\mathrm{pr}_i(t, X_1, \ldots, X_m) = X_i$, and the image of the mapping $\mathcal{I}^0(M, g) \to \mathcal{I}^0(J^m(\mathbb{R}, M), \mathbf{g}^m)$, $\phi \mapsto \phi^{(m)}$, is closed as it is defined by the following closed conditions:

$$\varphi^*(t) = t,\ (j^m\sigma)^*(\varphi^*\omega) = 0, \quad \varphi \in \mathcal{I}^0(J^m(\mathbb{R}, M), \mathbf{g}^m),$$

for every $\sigma \in C^\infty(\mathbb{R}, M)$ and every contact 1-form ω on $J^m(\mathbb{R}, M)$, by virtue of [30, Theorem 3.1]. From [24, 5.2.4. Proposition], we conclude the second item above.

The invariant functions $I_i\colon (\pi^m_{m-1})^{-1}\mathcal{F}^{m-1}(M) \to \mathbb{R}$, $1 \le i \le \nu$, induce smooth functions on the quotient manifold, $\bar{I}_i\colon Q^m \to \mathbb{R}$. As $I_1, \dots, I_\nu$ is a complete system of invariants, the mapping $\Upsilon\colon Q^m \to \mathbb{R}^\nu$ whose components are $\bar{I}_1, \dots, \bar{I}_\nu$, is injective.

The same argument as in the proof of Lemma 5.3 states the following property: If $\phi\colon N \to N'$ is an smooth mapping, then the subset of the points $x \in N$ for which there exists an open neighbourhood $U(x) \subseteq N$ such that $\phi|_{U(x)}$ is a mapping of constant rank, is a dense open subset in N. Hence, an injective smooth map $\phi\colon N \to N'$ is an immersion on a dense open subset in N (cf. [17, Theorem 7.15-(b)]). By applying this result to Υ, we conclude the existence of a dense open subset $\bar{O}^m \subseteq Q^m$ such that $\Upsilon|_{\bar{O}^m}$ is an injective immersion. Hence, for every $q^m(j^m_t\sigma) \in \bar{O}^m$, there exists a system of coordinates on Q^m defined on an open neighbourhood of $q^m(j^m_t\sigma)$ constituted by some functions $\bar{I}_{i_1}, \dots, \bar{I}_{i_k}$, $k = \dim Q^m$. As $(q^m)^*C^\infty(Q^m)$ can be identified to the ring of differential invariants, we can take $O^m = (q^m)^{-1}(\bar{O}^m)$.

Conversely, if $j^m_{t_0}\sigma, j^m_{t_0}\sigma' \in \tilde{O}^m$ are such that $q^m(j^m_{t_0}\sigma') \neq q^m(j^m_{t_0}\sigma)$, then there exists $\rho \in C^\infty(q^m\tilde{O}^m)$ satisfying $\rho(q^m(j^m_{t_0}\sigma)) = 0$, $\rho(q^m(j^m_{t_0}\sigma')) = 1$. As $\rho \circ q^m$ is an invariant function on $\tilde{O}^m$ by virtue of the hypothesis, there exists $f \in C^\infty(\mathbb{R}^\nu)$ such that $\rho \circ (q^m) = f \circ \left(I_1|_{\tilde{O}^m}, \dots, I_\nu|_{\tilde{O}^m}\right)$. Hence, an index i must exists for which $I_i(j^m_{t_0}\sigma) \neq I_i(j^m_{t_0}\sigma')$, thus proving that $I_1, \cdots, I_\nu$ is a complete system of invariants. □

Remark 5.11 If the injective immersion $\Upsilon\colon \bar{O}^m \to \mathbb{R}^\nu$ is a closed map, then one has

$$C^\infty\left(O^m\right)^{\mathcal{I}^0(M,g)} = (I_1, \dots, I_\nu)^*\, C^\infty(\mathbb{R}^\nu).$$

5.4 Generating complete systems of invariants

Theorem 5.12 *For every $r \in \mathbb{N}$, let k_r be the maximal number of generically functionally independent r-th order invariants not belonging to the closed—in the C^∞ topology—subalgebra generated by the invariants of order $< r$, and their derivatives with respect to the operator D_t. Then*

$$k_r = N_r - 1 - \sum_{i=0}^{r-1}(r+1-i)k_i, \tag{40}$$

$$m = \sum_{i=0}^{m} k_i. \tag{41}$$

Hence, for every complete Riemannian manifold of dimension m, there exist m generically independent invariants generating a complete system of invariant functions by adding their derivatives with respect to D_t for every order $r \le m$. Moreover, $k_r = 0$, $\forall r > m$.

Proof Let $t, I^0_i \in C^\infty(M)$, $1 \le i \le k_0 \le m$, $N_0 = 1 + k_0$, be a maximal system of invariant functions of order zero. (If (M, g) is not homogeneous, then there exist zero-order differential invariants independent of t; because of this the proof must start on this order.) Therefore, the rank of the Jacobian matrix $\mathcal{J}^0(I^0_1, \dots, I^0_{k_0}) = \left(\partial I^0_i/\partial x^j\right)^{1\le i\le k_0}_{1\le j\le m}$ must be maximal; namely $\operatorname{rk} \mathcal{J}^0(I^0_1, \dots, I^0_{k_0}) = k_0$. Moreover, one has

$$\frac{\partial(D_t f)}{\partial x^i_{r+1}} = \frac{\partial f}{\partial x^i_r}, \quad \forall f \in C^\infty(J^r(\mathbb{R}, M)),$$

as $\left[\partial/\partial x^i_{r+1}, D_t\right] = \partial/\partial x^i_r$, $\forall r \in \mathbb{N}$. Hence, the Jacobian matrix of the functions $I^0_1, \dots, I^0_{k_0}, D_t I^0_1, \dots, D_t I^0_{k_0}$ on $J^1(\mathbb{R}, M)$ is of the form

$$\mathcal{J}^1(I_1^0,\ldots,I_{k_0}^0,D_tI_1^0,\ldots,D_tI_{k_0}^0)=\begin{pmatrix}\left(\frac{\partial I_i^0}{\partial x^j}\right) & 0\\ \star & \left(\frac{\partial I_i^0}{\partial x^j}\right)\end{pmatrix},$$

and $\operatorname{rk}\mathcal{J}^1(I_1^0,\ldots,I_{k_0}^0,D_tI_1^0,\ldots,D_tI_{k_0}^0)=2k_0\le N_1-1$. We can thus complete the previous system with $k_1=N^1-1-2k_0$ new functionally independent invariants $I_1^1,\ldots,I_{k_1}^1$, in such a way that the Jacobian matrix of the full system is as follows:

$$\mathcal{J}^1(I_1^0,\ldots,I_{k_0}^0,D_tI_1^0,\ldots,D_tI_{k_0}^0,I_1^1,\ldots,I_{k_1}^1)=\begin{pmatrix}\left(\frac{\partial I_i^0}{\partial x^j}\right) & 0\\ \star & \left(\frac{\partial I_i^0}{\partial x^j}\right)\\ \star & \left(\frac{\partial I_i^1}{\partial x_1^j}\right)\end{pmatrix},$$

with $\operatorname{rk}\mathcal{J}^1(I_1^0,\ldots,I_{k_0}^0,D_tI_1^0,\ldots,D_tI_{k_0}^0,I_1^1,\ldots,I_{k_1}^1)=2k_0+k_1=N_1-1$. Let us consider the second-order invariants

$$\begin{aligned}&I_1^0,\ldots,I_{k_0}^0,\\&D_tI_1^0,\ldots,D_tI_{k_0}^0,I_1^1,\ldots,I_{k_1}^1,\\&D_t^2I_1^0,\ldots,D_t^2I_{k_0}^0,D_tI_1^1,\ldots,D_tI_{k_1}^1,\end{aligned}$$

the Jacobian matrix of which is

$$\begin{pmatrix}\left(\frac{\partial I_i^0}{\partial x^j}\right) & 0 & 0\\ \star & \left(\frac{\partial I_i^0}{\partial x^j}\right) & 0\\ \star & \left(\frac{\partial I_i^1}{\partial x_1^j}\right) & 0\\ \star & \star & \left(\frac{\partial I_i^0}{\partial x^j}\right)\\ \star & \star & \left(\frac{\partial I_i^1}{\partial x_1^j}\right)\end{pmatrix}$$

and its rank is equal to $3k_0+2k_1$. Hence, we need to choose k_2 new second-order invariants $I_1^2,\ldots,I_{k_2}^2$, with $k_2=N_2-1-(3k_0+2k_1)$, such that the matrix

$$\begin{pmatrix}\left(\frac{\partial I_i^0}{\partial x^j}\right) & 0 & 0\\ \star & \left(\frac{\partial I_i^0}{\partial x^j}\right) & 0\\ \star & \left(\frac{\partial I_i^1}{\partial x_1^j}\right) & 0\\ \star & \star & \left(\frac{\partial I_i^0}{\partial x^j}\right)\\ \star & \star & \left(\frac{\partial I_i^1}{\partial x_1^j}\right)\\ \star & \star & \left(\frac{\partial I_i^2}{\partial x_2^j}\right)\end{pmatrix}$$

is of maximal rank, i.e. $3k_0 + 2k_1 + k_2 = N_2 - 1$. Proceeding step by step in the same way, we conclude that the formula (40) in the statement holds true. Moreover, from this formula, we obtain $N_r - N_{r-1} = \sum_{i=0}^{r} k_i$. According to Corollary 5.5, we have $N_r = (r+1)m + 1 - \dim \mathfrak{i}(M, g)$, $\forall r \geq m-1$, and then $\sum_{i=0}^{m} k_i = N_m - N_{m-1} = m$, thus proving the formula (41) in the statement and finishing the proof. □

Remark 5.13 Following the same notations as in [10], let $H_{r+1} = \mathcal{I}^0(M, g)_{j_t^r \sigma}$ be the isotropy subgroup of a point $j_t^r \sigma$ belonging to the open subset $\mathcal{U}^r$ where the distribution $\mathfrak{D}^r$ is of constant rank (see Lemma 5.3). In [10], the following formula is mentioned: $k_r = \dim H_{r-1} + \dim H_{r+1} - 2 \dim H_r$ in the homogeneous case, i.e. $M = G/H$. Nevertheless, this formula holds on an arbitrary Riemannian manifold, and it is an easy consequence of the formulas (35) and (40). In fact, one has, $\dim H_{r+1} = \dim \mathcal{I}^0(M, g) - \mathrm{rk}\, \mathfrak{D}^r|_{\mathcal{U}^r}$. Hence

$$\dim H_{r-1} + \dim H_{r+1} - 2 \dim H_r = N_{r-2} + N_r - 2N_{r-1} = k_r.$$

6 Congruence on symmetric manifolds

Theorem 6.1 *Let (M, g), $(\bar{M}, \bar{g})$ be two locally symmetric Riemannian manifolds of the same dimension, $m = \dim M = \dim \bar{M}$, and let $\sigma : (a, b) \to M$, $\bar{\sigma} : (a, b) \to \bar{M}$ be two Frenet curves. If $x_0 = \sigma(t_0)$, $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, then σ and $\bar{\sigma}$ are congruent on some neigbourhoods U and $\bar{U}$ of x_0 and $\bar{x}_0$, respectively if, and only if, the following conditions hold:*

1. *For every $j \in \mathbb{N}$ and every $0 \leq i \leq m-1$,*

$$\kappa_i^{\sigma}(t) = \kappa_i^{\bar{\sigma}}(t), \quad |t - t_0| < \varepsilon. \tag{42}$$

2. *For every $i, j, k, l = 1, \ldots, m$,*

$$R(X_i^{\sigma}, X_j^{\sigma}, X_k^{\sigma}, \omega_{\sigma}^l)(x_0) = \bar{R}(X_i^{\bar{\sigma}}, X_j^{\bar{\sigma}}, X_k^{\bar{\sigma}}, \omega_{\bar{\sigma}}^l)(\bar{x}_0)), \tag{43}$$

$(X_1^{\sigma}, \ldots, X_m^{\sigma})$, $(X_1^{\bar{\sigma}}, \ldots, X_m^{\bar{\sigma}})$ being the Frenet frames of σ, $\bar{\sigma}$, with dual coframes $(\omega_{\sigma}^1, \ldots, \omega_{\sigma}^m)$, $(\omega_{\bar{\sigma}}^1, \ldots, \omega_{\bar{\sigma}}^m)$, and R, $\bar{R}$ the curvature tensors of (M, g), $(\bar{M}, \bar{g})$, respectively.

Theorem 6.2 *Let (M, g) be an arbitrary Riemannian manifold verifying the following property: Two Frenet curves $\sigma, \bar{\sigma} : (a, b) \to M$, $\sigma(t_0) = \bar{\sigma}(t_0) = x_0$, are congruent on some neighbourhood of x_0 (preserving the orientation if* dim M *is even and reversing the orientation if* dim M *is odd) if and only if the conditions* (42) *and* (43) *of* Theorem *6.1 hold. Then, (M, g) is locally symmetric.*

Proof Let us fix an orientation on a neighbourhood of $x_0 \in M$, and let $(v_i)_{i=1}^m$ be a positive orthonormal basis of $T_{x_0}M$. Let $\kappa_j \in C^{\infty}(t_0 - \delta, t_0 + \delta)$, $0 \leq j \leq m-1$, $\delta > 0$, be functions such that $\kappa_j > 0$ for $0 \leq j \leq m-2$. From Theorem 3.6, we know there exist two Frenet curves $\sigma, \bar{\sigma} : (t_0 - \varepsilon, t_0 + \varepsilon) \to M$, $0 < \varepsilon < \delta$, such that $\sigma(t_0) = \bar{\sigma}(t_0) = x_0$ and $X_i^{\sigma}(t_0) = -X_i^{\bar{\sigma}}(t_0) = v_i$ (hence $\omega_{\sigma}^i(t_0) = -\omega_{\bar{\sigma}}^i(t_0)$) for $1 \leq i \leq m$, with the same curvatures κ_j, $0 \leq j \leq m-1$. If dim M is odd, we considered the opposite orientation to construct $\bar{\sigma}$. According to this choice of orientations constructing the Frenet curves σ and $\bar{\sigma}$, for every $i, j, k, l = 1, \ldots, m$, we have

$$\begin{aligned} R\left(X_i^{\sigma}, X_j^{\sigma}, X_k^{\sigma}, \omega_{\sigma}^l\right)(x_0) &= R\left(-X_i^{\sigma}, -X_j^{\sigma}, -X_k^{\sigma}, -\omega_{\sigma}^l\right)(x_0) \\ &= R\left(X_i^{\bar{\sigma}}, X_j^{\bar{\sigma}}, X_k^{\bar{\sigma}}, \omega_{\bar{\sigma}}^l\right)(x_0). \end{aligned}$$

From the hypothesis, an open neighbourhood U of the image of σ and an isometric embedding $\phi\colon U \to M$ (leaving x_0 fixed) exist such that $\bar{\sigma} = \phi \circ \sigma$. Moreover $\phi_*(X_i^\sigma(t_0)) = X_i^{\bar{\sigma}}(t_0) = -X_i^\sigma(t_0),\ 1 \le i \le m$. Thus, $\phi_* = -Id_{T_{x_0}M}$. Since $x_0 \in M$ is arbitrary, we can conclude.

Example 6.3 For $M = \mathbb{C}P^n$, from the formula for the curvature tensor in [14, XI, p. 277], the Riemann curvature reads

$$R_4\left(X_i, X_j, X_k, X_l\right) = \tfrac{c}{4}\left\{\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il} + g\left(X_j, JX_l\right)g\left(JX_k, X_i\right) - g\left(X_j, JX_k\right)g\left(JX_l, X_i\right) + 2g\left(X_k, JX_l\right)g\left(JX_j, X_i\right)\right\},$$

J being the canonical complex structure. In particular,

$$R_4\left(X_i, X_j, X_i, X_j\right) = \tfrac{c}{4}\left(1 - \delta_{ij} + 3\omega\left(X_i, X_j\right)^2\right),$$

where ω is the canonical Kähler 2-form in $\mathbb{C}P^n$. Therefore, if we define the functions $\varpi_{ij}\colon (\pi^m_{m-1})^{-1}\mathcal{F}^{m-1}(M) \subset J^m(\mathbb{R}, M) \to \mathbb{R},\ 1 \le i < j \le m$, as

$$\varpi_{ij}(j^m_{t_0}\sigma) = \omega(X_i^\sigma(t_0), X_j^\sigma(t_0)),$$

the family $\{\varpi_{ij}\}_{i<j}$ together with $\{\varkappa_i\}_{i=0}^{m-1}$ constitute a complete system of differential invariants.

In particular, the corresponding result for congruence on constant curvature manifolds can be stated as follows:

Theorem 6.4 *Two Frenet curves $\sigma, \bar{\sigma}\colon (a, b) \to (M, g)$ taking values in an oriented Riemannian manifold of constant curvature are congruent on some neigbourhoods U and $\bar{U}$ of $x_0 = \sigma(t_0)$ and $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, respectively, if and only, $\kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t)$ for $0 \le i \le m-1$ and small enough $|t - t_0|$. Conversely, if on an oriented Riemannian manifold (M, g) two arbitrary Frenet curves $\sigma, \bar{\sigma}\colon (a, b) \to (M, g)$ are congruent on some neighbourhoods of $x_0 = \sigma(t_0)$, $\bar{x}_0 = \bar{\sigma}(t_0)$, $a < t_0 < b$, if and only if $\kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t)$ for $0 \le i \le m-1$ and small enough $|t - t_0|$, then (M, g) is a manifold of constant curvature.*

Proof On a manifold of constant curvature k one has ([14, V. Corollary 2.3]): $R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)$; hence

$$R\left(X_i^\sigma, X_j^\sigma, X_k^\sigma, \omega_\sigma^l\right) = k\left(\delta_{jk}\delta_i^l - \delta_{ik}\delta_j^l\right).$$

It follows that on a manifold of constant curvature the equation (43) holds identically. The first part of the statement thus follows from Theorem 6.1.

Let $(v_1, \ldots, v_m)$, $(w_1, \ldots, w_m)$ be two positively oriented orthonormal bases in $T_{x_0}M$.

From Theorem 3.6, there exist two Frenet curves $\sigma, \bar{\sigma}\colon (t_0 - \varepsilon, t_0 + \varepsilon) \to M$, $\varepsilon > 0$, such that,

(i) $\sigma(t_0) = \bar{\sigma}(t_0) = x_0$,
(ii) $X_i^\sigma(t_0) = v_i$, $X_i^{\bar{\sigma}}(t_0) = w_i$, $1 \le i \le m$,
(iii) $\kappa_i^\sigma(t) = \kappa_i^{\bar{\sigma}}(t)$, $0 \le i \le m-1$, , $|t - t_0| < \varepsilon$.

By virtue of the hypothesis, there exists an isometry ϕ defined on a neighbourhood of x_0, fixing x_0, such that $\phi \circ \sigma = \bar{\sigma}$. Hence $\phi_*(v_i) = w_i$, $1 \le i \le m$, and accordingly M is an isotropic manifold (e.g. see [27]) and therefore of constant curvature. □

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